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ON THE PROFINITE HOMOTOPY TYPE OF LOG SCHEMES

DAVID CARCHEDI, SARAH SCHEROTZKE, NICOLÒ SIBILLA, AND MATTIA TALPO

Abstract. We complete the program, initiated in [8], to compare the many different possible definitions of the underlying homotopy type of a log scheme. We show that, up to profinite completion, they all yield the same result, and thus arrive at an unambiguous definition of the profinite homotopy type of a log scheme. Specifically, in [8], we define this to be the profinite étale homotopy type of the infinite root stack, and show that, over $\mathbb{C}$, this agrees up to profinite completion with the Kato-Nakayama space. Other possible candidates are the profinite shape of the Kummer étale site $X_{k\acute{e}t}$, or of the representable étale site of $\sqrt[\infty]{X}$. Our main result is that all of these notions agree, and moreover the *profinite* étale homotopy type of the infinite root stack is not sensitive to whether or not it is viewed as a pro-system in stacks, or as an actual stack (by taking the limit of the pro-system). We furthermore show that in the log regular setting, all these notions also agree with the étale homotopy type of the classical locus X^{triv} (up to an appropriate completion). We deduce that, over an arbitrary locally Noetherian base, the étale homotopy type of $\mathbb{G}_m^N$ agrees with that of $B\boldsymbol{\mu}_\infty^N$ up to completion.

1. Introduction

The category of logarithmic (log) schemes is an enlargement of the category of schemes. Initially designed for applications to arithmetic geometry, log geometry has proved to be an invaluable tool in a broad array of mathematical areas, including algebraic and symplectic geometry, mirror symmetry and homotopy theory.

In this paper we complete the program we initiated in [8], and aimed at charting the topology of log schemes. We obtain comparison results linking all different models of the underlying homotopy type of a log scheme. Additionally, we relate this to the étale homotopy type of the classical locus.

1.1. The étale homotopy type.

If X is a classical scheme, there are several ways to extract from it topological information. If X is a complex scheme of finite type, we can consider its analytification $X^{an} = X(\mathbb{C})$. Over a general base, the closest approximation to the underlying topological space of X is given by the *étale homotopy type* of Artin–Mazur and Friedlander. From a modern perspective, this is an instance of the general formalism of the *shape* of a topos, which is applied in this case to the small *étale topos* of X. When these notions overlap, we have comparison results ensuring information flow across these different perspectives. Over $\mathbb{C}$, the comparison between the étale homotopy type and the analytification is a generalization of Riemann's existence theorem. These comparison theorems are both computationally and conceptually powerful: they show that there is, at

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bottom, only one meaningful notion of the underlying topology of a scheme, and that there is a variety of different techniques that we can deploy to study it.

In this article, we describe the different models for the underlying homotopy type of a log scheme, and prove comparison results between them. Log geometry is a powerful formalism, but teasing out the underlying geometric information of log schemes is not easy. One reason is that the definition of a log structure combines geometric and combinatorial data, which might keep track for instance of the combinatorics of the compactifying divisors, or of the singularities of the central fiber of a degeneration. To get around this issue, several more classically geometric objects have been designed to capture the geometric properties of log schemes: this includes the *Kato–Nakayama space*, a beautiful construction available over $\mathbb{C}$, that attaches to a log scheme a topological space (which is homeomorphic to a topological manifold with corners, under log regular assumptions); the *Kummer étale topos*, which is the category of sheaves on a log analogue of the small étale site; and the *infinite root stack*, due to Talpo and Vistoli [23], which is a limit of tame algebraic stacks.

It turns out that, at a topological level, all these different incarnations of the geometry of a log scheme yield equivalent objects. This is our main theorem. As in the classical case, our comparison result does not have purely a conceptual import: it is also a valuable computational tool. In the next section we give a more detailed account of our main result. Then in section 1.4 we will explain applications of our work, some of which will appear in a future companion article.

1.2. **The main result.** Many of our proofs will depend on delicate descent arguments. Thus it will be essential for us to work with a sufficiently flexible formalism, capable of keeping track of all the higher homotopies involved in descent statements. Therefore, it will be indispensable for us to work within the framework of ∞-categories. Moreover, to deal with higher categorical descent statements, we will need the framework of ∞-topoi. We will study sheaves of spaces on various sites: contrary to classical references we will work with *∞-topoi of sheaves of spaces, or ∞-groupoids*, rather than just with sheaves of sets.

Recall that if $\mathcal{E}$ is a ∞-topos, the *shape* of $\mathcal{E}$, denoted by $\mathit{Shape}(\mathcal{E})$, is, in a precise sense, the best approximation of $\mathcal{E}$ by a pro-space. We remark that in the case of a scheme, the shape of the small étale ∞-topos is the modern formulation of the étale homotopy type. As we will explain below, the notion of shape is also one of the key tools for defining the underlying homotopy type of log schemes.

In order to give a more precise account of our work, let us start by introducing in some more detail the cast of characters which will play a role in our comparison result. The *Kummer étale site* of a log scheme X is the analogue in the log setting of the classical étale site. In the divisorial case, Kummer étale maps are maps which restrict to ordinary étale maps away from the divisor, but can be tamely ramified along it. This definition is well aligned with the general logarithmic philosophy, which roughly gives us a way to regard objects that develop singular behavior along the log locus as being still smooth. We denote by $\mathrm{Sh}\,(X_{k\acute{e}t})$ the Kummer étale topos of X. Since the shape of the étale topos of a classical scheme is precisely its étale homotopy type, the shape of $\mathrm{Sh}\,(X_{k\acute{e}t})$ is a natural candidate for the notion of the underlying homotopy type of a log scheme X.

The *infinite root stack* of X is an inverse limit of the root construction along the support of the log structure. It was introduced by Talpo–Vistoli in [23]. It can be regarded both as a stack, or as a pro-object in stacks, by viewing it as a formal limit of finite root stacks. We are ultimately interested in shapes, and the two point of views on the infinite root stack will yield genuinely different answers in general. One of the results in this paper however shows that the difference however is, in a precise sense, mild, as it is erased after profinite completion. There is a third possible way of extracting an underlying homotopy type from a log scheme which emerges naturally from work of Talpo and Vistoli, namely the shape of the topos of sheaves on the representable étale site of the infinite root stack.

Finally over $\mathbb{C}$, Kato and Nakayama gave a recipe to attach to a locally finite-type log scheme X an actual topological space X^{log}, called the Kato–Nakayama (KN) space. In the divisorial case, the KN space is homeomorphic to the bordification of the complement of the divisor; equivalently, it is homeomorphic to the real oriented blow-up of X along the divisor. Over $\mathbb{C}$, the KN space provides another possible definition of the underlying homotopy type of a log scheme.

Summarizing the above discussion, we obtain five distinct possible definitions of the underlying homotopy type of a log schemes. We list them below:

(1) The shape of the ∞-topos $\mathrm{Sh}\left(X_{k\acute{e}t}\right)$.
(2) The étale homotopy type of the ∞-root stack $\sqrt[\infty]{X}$.
(3) The étale homotopy type of the ∞-root stack $\sqrt[\infty]{X}^{pro}$ regarded as a pro-object.
(4) The shape of the ∞-topos of étale representable maps to $\sqrt[\infty]{X}$, $\mathrm{Sh}\left(\sqrt[\infty]{X}^{rep}_{\acute{e}t}\right)$.
(5) Over $\mathbb{C}$, the homotopy type of the Kato-Nakayama space of X.

Our main result completes the program initiated in [8] by comparing these different homotopy types. We state the general comparison result as the following statement.

Theorem A (Theorem 5.14, Corollary 6.15)**.** *Let X be a log scheme.*

(1) *The homotopy types (1) , (2) , (3) and (4) are all equivalent* up to profinite completion.
(2) *Over $\mathbb{C}$, they are also all equivalent to (5) up to profinite completion.*

Theorem A combines our results in this article, with previous results obtained by us and our collaborators. Namely, the comparison between (1) and (4) follows from work of Talpo–Vistoli, and holds before passing to profinite completions; over $\mathbb{C}$ the comparison between (5) and (3) was the main result of our previous work [8]. The comparison between (1) and (5) over $\mathbb{C}$ was recently proven independently and with a direct proof by Piotr Achinger, and will appear in upcoming joint work with the fourth-named author [1].

Remark. It is worth mentioning that the comparison between (4) and (2) is *NOT* a tautology since $\sqrt[\infty]{X}$ is not Deligne-Mumford— it is not even algebraic! The proof is in fact quite involved.

The heavy lifting done in this paper is in establishing the comparison between (2), (3) and (4). This is far from obvious, and requires grappling with delicate technical issues. The necessary arguments make up the bulk of the present paper. Section 4 contains some preliminary results on descent for Betti stacks. Namely, we prove that for V any π-finite space, its étale Betti stack $\Delta^{\acute{e}t}(V)$ satisfies fpqc-descent. This has the following interesting consequence, which as far as the authors are aware has not been observed before. Let $\mathcal{X}$ be a *higher stack*, i.e. a sheaf of spaces over the of affine schemes equipped with the étale topology. We can define the étale (resp. fpqc) shape of $\mathcal{X}$, as the shape of an étale (resp. fpqc) topos of $\mathcal{X}$. Defining the fpqc shape properly requires some extra care, as fpqc covers of a scheme are not a small family: we refer the reader to Section 4 for a fuller discussion of these aspects.

Theorem B (Corollary 4.12)**.** *The étale shape and the fpqc shape of $\mathcal{X}$ agree up to profinite completion.*

We remark that Theorem B seems to be new also for schemes.

Leveraging this, in Section 5.1 we prove that (2) and (3) are equivalent after profinite completion; the comparison between (3) and (4) is established in section 6. We stress that, before completion, the constructions (2), (3) and (4) yield genuinely different pro-spaces.

1.3. **The homotopy type of the classical locus.** If X is a log scheme, its *classical locus* is the largest open subscheme where the log structure is trivial. We denote it by X^{triv}. Log structures appear naturally when working with compactifications. If $X \supset U$ is a sufficiently well-behaved compactification of an open variety U, log geometry allows us to recover information on U by working relative to the compactifying divisor: X is equipped with a natural log structure which keeps track of the divisor at infinity, and has the property that $X^{triv} = U$. A central organizing principle in the area is that, under suitable assumptions, log invariants of X should coincide with the corresponding classical invariants of U. As an early instance of this circle of ideas, we should mention Grothendieck's theorem stating that the cohomology of differential forms with log poles along a divisor computes the de Rham cohomology of the complement of the divisor.

In this paper we focus on what is, in a precise sense, the most fundamental topological invariant of a log scheme X: namely, its (profinite) homotopy type. By Theorem A this is a well-defined notion, as up to profinite completion all different constructions converge to yield the same answer. To differentiate it from the classical étale homotopy type, we will refer to it as the (profinite) *log homotopy type* of X. We state our second main result below. We assume that X is *log regular*, which is a generalization of regularity for log schemes: the result cannot be expected to hold under weaker hypotheses.[1]. We remark that our theorem builds on recent results contained in Berner's thesis [3].

Theorem C (Corollary 7.5)**.** *Let X be a log regular log scheme.*

1. *Let ℓ be a prime which is invertible on X. Then the ℓ-profinite completion of the log homotopy type of X is equivalent to the ℓ-profinite completion of the étale homotopy type of X^{triv}.*
2. *In characteristic zero, the profinite completion of the log homotopy type of X is equivalent to the profinite completion of the étale homotopy type of X^{triv}.*

Over $\mathbb{C}$, item (2) also follows directly from the comparison between the Kummer étale topos and the Kato–Nakayama space that was established in [1].

[1]Here and elsewhere we also make further standard mild assumptions on X, we refer the reader to the main text for a complete account of this. We remark however that we do not work over a field, but over a general base scheme

Leveraging all of the comparison results, we are able to prove the following theorem comparing the étale homotopy type of $\mathbb{G}_m$ with that of $B\boldsymbol{\mu}_\infty$.

Theorem C (Theorem 7.7)**.** *Let S be a locally Noetherian scheme, and denote by $\boldsymbol{\mu}_\infty$ the affine group scheme over S*

$$\varprojlim_n \boldsymbol{\mu}_n.$$

Let ℓ be a prime invertible on S, and N any non-negative integer. Then there is an equivalence of ℓ-profinite spaces

$$\Pi_\infty^{\acute{e}t}\left(\mathbb{G}_m^N\right)^\wedge_\ell \simeq \Pi_\infty^{\acute{e}t}\left(B\boldsymbol{\mu}_\infty^N\right)^\wedge_\ell.$$

Moreover, in characteristic zero, this holds up to profinite completion.

1.4. **Applications and future work.** The étale homotopy type is a very rich invariant of schemes. Both the étale fundamental group, and the étale cohomology of locally constant sheaves, can be computed from the étale homotopy type. In fact, under suitable assumptions, all *topological* invariants of a scheme should be encoded, up to completion, in its étale homotopy type. As a prominent example of this kind of thinking, let us mention the beautiful work of Friedlander which recasts étale K-theory in terms of the topological K-theory of the étale homotopy type [9].

One of the motivations behind the present project is to transfer this perspective to the logarithmic setting. In order to do so, it is necessary first of all to gain a finer understanding of the underlying homotopy type of a log scheme. In this article we accomplish this first step via our Theorem A and C. In a companion article, we hope to pursue applications of our results to the logarithmic version of étale K-theory, which was introduced by Nizioł in [20].

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2. Conventions

To deal with set theoretic technicalities rigorously, we fix three uncountable Grothendieck universes $\mathcal{U} \in \mathcal{V} \in \mathcal{W}$. Elements of the set $\mathcal{U}$ will be called ($\mathcal{U}$-)small sets, elements of $\mathcal{V}$ will be called large sets (or $\mathcal{V}$-small sets), and elements of $\mathcal{W}$ will be called very large sets (or $\mathcal{W}$-small sets). By convention, $\mathbf{Set}$, $\widehat{\mathbf{Set}}$, and $\widehat{\widehat{\mathbf{Set}}}$ will denote the categories of small sets, large sets, and very large sets respectively.

We will be working in the framework of ∞-category theory. By an ∞-category, we mean an $(\infty,1)$-category. Concretely, these can be modeled by quasicategories, a.k.a. inner Kan complexes, hence in particular, simplicial sets. To speak of large ∞-categories, such as the ∞-category of small ∞-categories, we will need to use large simplicial sets, i.e. objects of the form

$$X_\bullet : \Delta^{op} \to \widehat{\mathbf{Set}}.$$

We will denote the ∞-category of small ∞-categories by $\mathbf{Cat}_\infty$. We will also work with ∞-categories which which are only $\mathcal{W}$-small, such as the ∞-category of large ∞-category, and the ∞-category $\mathfrak{Top}_\infty$ of ∞-topoi.

We will be generally following the established notation and terminology of [16], unless otherwise explained. Some small departures are that we denote the ∞-category of (small) spaces (∞-groupoids) by $\mathcal{S}\mathrm{pc}$ rather than $\mathcal{S}$, and we denote the ∞-category of spectra by **Spt**.

3. Preliminaries

3.1. **The ∞-category of pro-spaces and its localizations.** We will also assume the reader is familiar with the salient points of the theory of pro-objects in ∞-categories. For more details on this theory see e.g. [17, Appendix E], [14, Section 3], [2] and [7, Section 2.1]. Here we merely give a rapid overview.

Given an ∞-category $\mathscr{C}$, the ∞-category $\operatorname{Pro}(\mathscr{C})$ of pro-objects is defined by a universal property, making rigorous the idea that objects of $\operatorname{Pro}(\mathscr{C})$ are *formal cofiltered limits* of objects in $\mathscr{C}$.

Definition 3.1. The ∞-category $\operatorname{Pro}(\mathscr{C})$ has small cofiltered limits, is equipped with a fully faithful functor

$$j : \mathscr{C} \to \operatorname{Pro}(\mathscr{C}) \tag{1}$$

and if $\mathscr{D}$ is an ∞-category admitting small cofiltered limits, then the precomposition with j induces an equivalence of ∞-categories:

$$\operatorname{Fun}_{cofilt}(\operatorname{Pro}(\mathscr{C}), \mathscr{D}) \to \operatorname{Fun}(\mathscr{C}, \mathscr{D})$$

where $\operatorname{Fun}_{cofilt}(\operatorname{Pro}(\mathscr{C}), \mathscr{D})$ is the full subcategory of $\operatorname{Fun}(\operatorname{Pro}(\mathscr{C}), \mathscr{D})$ on those functors that preserve small cofiltered limits.

We will make continual use of the following two propositions:

Proposition 3.2. [14, Proposition 3.1.6] *Suppose that $\mathscr{C}$ is accessible and has finite limits, then $\operatorname{Pro}(\mathscr{C})$ is equivalent to the full subcategory of $\operatorname{Fun}(\mathscr{C}, \mathcal{S}\mathrm{pc})^{op}$ on those functors which are left exact and accessible.*

Proposition 3.3. *Let $f : \mathscr{D} \to \mathscr{C}$ be a functor between accessible ∞-categories with finite limits, and suppose that f is left exact, then the functor*

$$\operatorname{Pro}(f) : \operatorname{Pro}(\mathscr{D}) \to \operatorname{Pro}(\mathscr{C})$$

has a left adjoint

$$f^* : \operatorname{Pro}(\mathscr{C}) \to \operatorname{Pro}(\mathscr{D}),$$

given by restriction along f.

Definition 3.4. The ∞-category of **pro-spaces** is the ∞-category $\operatorname{Pro}(\mathcal{S}\mathrm{pc})$.

Note that, given any full subcategory $\mathscr{C} \subseteq \mathcal{S}\mathrm{pc}$ closed under finite limits and retracts, it follows from Proposition 3.3 that the canonical inclusion

$$\mathrm{Pro}\left(\mathscr{C}\right) \hookrightarrow \mathrm{Pro}\left(\mathcal{S}\mathrm{pc}\right)$$

has a left adjoint $\widehat{(\,\cdot\,)}_{\mathscr{C}}$.

Definition 3.5. A space X is called **π-finite** if it has finitely many connected components, and finitely many non-trivial homotopy groups, all of which are finite. We denote by $\mathcal{S}\mathrm{pc}^{\pi} \subset \mathcal{S}\mathrm{pc}$ the full subcategory of π-finite spaces. The ∞-category $\mathrm{Pro}\left(\mathcal{S}\mathrm{pc}^{\pi}\right)$ is the ∞-category of **profinite spaces**, and is denoted by $\mathrm{Prof}\left(\mathcal{S}\mathrm{pc}\right)$. We denote the left adjoint $\widehat{(\,\cdot\,)}_{\mathcal{S}\mathrm{pc}^{\pi}}$ simply by $\widehat{(\,\cdot\,)}$ and call it the **profinite completion** functor.

Definition 3.6. Fix a prime ℓ. A space X is called **ℓ-finite** if it is π-finite and all its homotopy groups are finite ℓ-groups. The ∞-category $\mathrm{Pro}\left(\mathcal{S}\mathrm{pc}^{\pi-\ell}\right)$ is the ∞-category of **ℓ-profinite spaces**, and is denoted by $\mathrm{Prof}_{\ell}\left(\mathcal{S}\mathrm{pc}\right)$. We denote the left adjoint $\widehat{(\,\cdot\,)}_{\mathcal{S}\mathrm{pc}^{\pi-\ell}}$ by $(\,\cdot\,)^{\wedge}_{\ell}$ and call it the **ℓ-profinite completion** functor.

Proposition 3.7. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a map in $\mathrm{Pro}\left(\mathcal{S}\mathrm{pc}\right)$, then f induces an equivalence*

$$\mathcal{X}^{\wedge}_{\ell} \to \mathcal{Y}^{\wedge}_{\ell}$$

if and only if for all n,

$$\mathcal{X}\left(K\left(\mathbb{Z}/\ell, n\right)\right) \to \mathcal{Y}\left(K\left(\mathbb{Z}/\ell, n\right)\right)$$

is an equivalence.

Proof. This follows immediately from [2, Corollary 7.3.7]. □

Definition 3.8. Denote by $\mathcal{S}\mathrm{pc}_{<\infty}$ the full subcategory of $\mathcal{S}\mathrm{pc}$ on those spaces which have only finitely many non-trivial homotopy groups. The ∞-category $\mathrm{Pro}\left(\mathcal{S}\mathrm{pc}_{<\infty}\right)$ is called the ∞-category of **pro-truncated spaces**. We denote this ∞-category by $\mathrm{Pro}_{<\infty}\left(\mathcal{S}\mathrm{pc}\right)$ and the associated localization functor by $(\,\cdot\,)^{\wedge}_{<\infty}$.

3.2. **Shape theory of ∞-topoi.** Recall that an **∞-topos** is an ∞-category $\mathcal{E}$ which arises as an accessible left exact localization of an ∞-category of presheaves of spaces $\mathrm{Psh}\left(\mathscr{C}\right)$ on a small ∞-category $\mathscr{C}$. In other words, $\mathcal{E}$ is accessible, and there is a fully faithful inclusion

$$i : \mathcal{E} \hookrightarrow \mathrm{Psh}\left(\mathscr{C}\right)$$

which has a left adjoint a, and this left adjoint moreover is left exact, i.e. preserves finite limits. The prototypical examples arise by equipping $\mathscr{C}$ with a Grothendieck topology, and letting $\mathcal{E}$ be the full subcategory on those presheaves that satisfy descent with respect to Čech-covers, or hyperdescent.

A **geometric morphism** $f : \mathcal{E} \to \mathcal{F}$ between ∞-topoi is a pair of adjoint functors

$$f^* \dashv f_*,$$

such that f^* is left exact. These are the 1-morphisms in the ∞-category $\mathfrak{Top}_{\infty}$ of ∞-topoi.

Example 3.9. The ∞-topos $\mathcal{S}\mathrm{pc}$ of spaces is a terminal object in $\mathfrak{Top}_\infty$. This follows since a colimit preserving functor $f^* : \mathcal{S}\mathrm{pc} \to \mathcal{E}$ with $\mathcal{E}$ cocomplete is completely determined by where it sends the one-point space, and moreover has a right adjoint f_*. Furthermore, if f^* is left exact, it must send the one-point space to a terminal object in $\mathcal{E}$, since the one-point space is terminal in $\mathcal{S}\mathrm{pc}$. The essentially unique geometric morphism

$$\mathcal{E} \to \mathcal{S}\mathrm{pc}$$

is denoted by $\Delta_{\mathcal{E}} \dashv \Gamma_{\mathcal{E}}$. Concretely,

$$\Gamma_{\mathcal{E}}(E) \simeq \mathbb{M}\mathrm{ap}_{\mathcal{E}}(1, E)$$

where we denote by 1 a terminal object in $\mathcal{E}$. If $\mathcal{E}$ arises as sheaves on a site, then $\Delta_{\mathcal{E}}(X)$ is the constant sheaf with values $X \in \mathcal{S}\mathrm{pc}$, i.e. the sheafification of the constant presheaf.

Given a space $X \in \mathcal{S}\mathrm{pc}$, there is a canonical equivalence $\mathcal{S}\mathrm{pc}/X \simeq \mathrm{Psh}(X)$, and there is an induced fully faithful functor [16, Remark 6.3.5.10, Theorem 6.3.5.13, and Proposition 6.3.4.1]

$$\mathcal{S}\mathrm{pc}/\cdot : \mathcal{S}\mathrm{pc} \hookrightarrow \mathfrak{Top}_\infty.$$

By the universal property of $\mathrm{Pro}(\mathcal{S}\mathrm{pc})$ this extends to a unique cofiltered limit preserving functor

$$\mathcal{S}\mathrm{pc}/\cdot : \mathrm{Pro}(\mathcal{S}\mathrm{pc}) \to \mathfrak{Top}_\infty.$$

Remark 3.10. The above functor is not fully faithful, but its restriction to $\mathrm{Prof}(\mathcal{S}\mathrm{pc})$ is by [17, Appendix E.2].

By [16, Remark 7.1.6.15], this functor has a left adjoint

$$\mathit{Shape} : \mathfrak{Top}_\infty \to \mathrm{Pro}(\mathcal{S}\mathrm{pc}).$$

In fact, it has a relatively simple description: to describe a pro-space, it suffices to give a left exact accessible functor from $\mathcal{S}\mathrm{pc}$ to itself. The functor *Shape* sends an ∞-topos $\mathcal{E}$ to the functor

$$\mathcal{S}\mathrm{pc} \xrightarrow{\Delta_{\mathcal{E}}} \mathcal{E} \xrightarrow{\Gamma_{\mathcal{E}}} \mathcal{S}\mathrm{pc}.$$

Definition 3.11. Let $\mathcal{E}$ be an ∞-topos, then the pro-space

$$\mathit{Shape}(\mathcal{E})$$

is called the **shape** of $\mathcal{E}$.

Applying natural completion functors to the shape functor gives natural variants. For example, the profinite completion of $\mathit{Shape}(\mathcal{E})$ is referred to as the *profinite shape* of $\mathcal{E}$, etc.

The shape of an ∞-topos is, in a precise sense, its best approximation by a pro-space. It serves as a suitable notion of underlying homotopy type of an ∞-topos. For example, if a *topological* space T has the homotopy type of a CW-complex, it follows from [15, Remark A.1.4] that the shape of its ∞-topos of sheaves of spaces $\mathrm{Sh}(T)$ is that of the underlying homotopy type of T. For further geometric intuition about this functor, we refer the reader to [7, Section 2.2.1].

3.3. **Étale homotopy types.** In this paper we will work with higher stacks on classical schemes.

For X any scheme, its *étale homotopy type* is the pro-space

$$\mathit{Shape}\left(\mathrm{Sh}\left(X_{\acute{e}t}\right)\right)$$

where $\mathrm{Sh}\left(X_{\acute{e}t}\right)$ is the ∞-category of sheaves of spaces on the small étale site of X.

In [6], the first author extends this definition to arbitrary sheaves of spaces on the large étale site. We give a rapid recollection:

Let us take $\mathbf{Aff}'$ to be a suitable *small* subcategory of affine schemes, closed under finite limits and étale morphisms, and containing the empty scheme. Then there is a unique colimit preserving functor

$$\mathrm{Sh}\left(\left(\,\cdot\,\right)_{\acute{e}t}\right) : \mathrm{Sh}\left(\mathbf{Aff}', \acute{e}t\right) \to \mathfrak{Top}_{\infty}$$

which sends each affine scheme S to the topos of sheaves of spaces on its small étale site. Concretely, for an arbitrary sheaf of spaces $\mathcal{X}$ on $\left(\mathbf{Aff}', \acute{e}t\right)$, the ∞-topos $\mathrm{Sh}\left(\mathcal{X}_{\acute{e}t}\right)$ is the colimit

$$\varinjlim_{T \to \mathcal{X}} \mathrm{Sh}\left(T_{\acute{e}t}\right),$$

computed in $\mathfrak{Top}_{\infty}$, where the colimit ranges over all maps from affine schemes (in $\mathbf{Aff}'$) into $\mathcal{X}$. In other words, $\mathrm{Sh}\left(\mathcal{X}_{\acute{e}t}\right)$ is the left Kan extension of the assignment mapping an affine scheme in $\mathbf{Aff}'$ to its small étale topos.

Definition 3.12. The **étale homotopy type** $\Pi_{\infty}^{\acute{e}t}\left(\mathcal{X}\right)$ of $\mathcal{X}$ is the shape of the ∞-topos $\mathrm{Sh}\left(\mathcal{X}_{\acute{e}t}\right)$.

Since it involves a colimit, the above definition is a bit opaque for stacks which are not schemes (or at least Deligne-Mumford stacks). However, there is a much simpler description given also in [6], which we now recall.

Definition 3.13. Let V be a space in $\mathcal{S}\mathrm{pc}$. Denote by $\Delta^{\acute{e}t}\left(V\right)$ its constant stack on $\left(\mathbf{Aff}', \acute{e}t\right)$. This stack is called the **Betti stack** associated to V.

Theorem 3.14. [6, Theorem 2.40] *For a sheaf of spaces $\mathcal{X}$ on $\left(\mathbf{Aff}', \acute{e}t\right)$, its étale homotopy type is given by*

$$\begin{aligned} \Pi_{\infty}^{\acute{e}t}\left(\mathcal{X}\right) : \mathcal{S}\mathrm{pc} &\to \mathcal{S}\mathrm{pc} \\ V &\mapsto \mathbb{M}\mathrm{ap}\left(\mathcal{X}, \Delta^{\acute{e}t}\left(V\right)\right). \end{aligned}$$

Remark 3.15. Let $S_{\mathcal{E}} : \mathcal{S}\mathrm{pc} \to \mathcal{S}\mathrm{pc}$ be the shape of an ∞-topos $\mathcal{E}$. The profinite homotopy type of $\mathcal{E}$ can be seen as the functor $\mathcal{S}\mathrm{pc}^{\pi} \to \mathcal{S}\mathrm{pc}$ obtained by precomposing $S_{\mathcal{E}}$ with the inclusion $\mathcal{S}\mathrm{pc}^{\pi} \to \mathcal{S}\mathrm{pc}$.

3.4. **Higher Deligne-Mumford Stacks.** Recall that for a scheme X, its small étale site $X_{\acute{e}t}$ is the category of étale maps $\mathrm{Spec}\, A \to X$, with A a commutative ring, equipped with the étale topology. There is a canonical sheaf of rings $\mathcal{O}_X$ on $X_{\acute{e}t}$ which assigns to such an object the ring A, and the stalks are strictly Henselian. This makes $\left(\mathrm{Sh}\left(X_{\acute{e}t}\right), \mathcal{O}_X\right)$ into a strictly Henselian ringed ∞-topos. The collection of strictly Henselian ringed ∞-topoi forms an ∞-category $\mathfrak{Top}_{\infty}^{Hens.}$, where the 1-morphisms are maps of locally ringed ∞-topoi; essentially the maps on stalks are maps of local rings. See [13, Section 1, Section 2.6] for details.

Definition 3.16. Let $(\mathcal{E}, \mathcal{O}_{\mathcal{E}})$ be a strictly Henselian ringed ∞-topos. It is **Deligne-Mumford** if there exists a set of objects $(E_\alpha)_\alpha$ in $\mathcal{E}$ such that

$$\coprod_\alpha E_\alpha \to 1$$

is an epimorphism and such that for all α,

$$(\mathcal{E}/E_\alpha, \mathcal{O}_{\mathcal{E}}|_{E_\alpha})$$

is equivalent to $(\mathrm{Sh}(X_{\acute{e}t}), \mathcal{O}_X)$ for some scheme X (depending on α).

Remark 3.17. Let X be any scheme and suppose that $\mathcal{F} \in \mathrm{Sh}(X_{\acute{e}t})$ is a sheaf. Then, one can write $\mathcal{F}$ as a colimit of representables, and hence by [16, Lemma 6.2.3.13], there is an effective epimorphism

$$u : \coprod_\alpha U_\alpha \to \mathcal{F}$$

with each $U_\alpha \in A_{\acute{e}t}$, i.e. U_α is an étale map Spec $(B_\alpha) \to X$. The morphism u can equivalently be viewed as an object in the slice topos $\mathrm{Sh}(X_{\acute{e}t})/\mathcal{F}$, whose canonical morphism to 1 is an effective epimorphism. Moreover, one has that

$$\begin{aligned} (\mathrm{Sh}(X_{\acute{e}t})/\mathcal{F})/u &\simeq \mathrm{Sh}(X_{\acute{e}t})/\coprod_\alpha U_\alpha \\ &\simeq \mathrm{Sh}\left(\left(\coprod_\alpha \mathrm{Spec}\ (B_\alpha)\right)_{\acute{e}t}\right). \end{aligned}$$

One concludes that $(\mathrm{Sh}(X_{\acute{e}t})/\mathcal{F}, \mathcal{O}_X|_{\mathcal{F}})$ is Deligne-Mumford.

More generally, given any Deligne-Mumford ∞-topos $(\mathcal{E}, \mathcal{O}_{\mathcal{E}})$ and E an object of $\mathcal{E}$, then $(\mathcal{E}/E, \mathcal{O}_{\mathcal{E}}|_E)$ is Deligne-Mumford. To see this, let $\coprod_\alpha U_\alpha \to 1$ be an effective epimorphism in $\mathcal{E}$ such that for each α,

$$(\mathcal{E}/U_\alpha, \mathcal{O}_{\mathcal{E}}|_{U_\alpha}) \simeq (\mathrm{Sh}((X_\alpha)_{\acute{e}t}), \mathcal{O}_{X_\alpha}),$$

for some scheme X_α. Then

$$\varphi : \coprod_\alpha (U_\alpha \times E) \to E$$

is an effective epimorphism, and for each α,

$$\begin{aligned} (\mathcal{E}/E)/\varphi_\alpha &\simeq \mathcal{E}/U_\alpha \times E \\ &\simeq (\mathcal{E}/U_\alpha)/pr_\alpha, \end{aligned}$$

where $pr_\alpha : U_\alpha \times E \to U_\alpha$ is the canonical projection. But $(\mathcal{E}/U_\alpha)/pr_\alpha$ is equivalent to a slice topos of $\mathrm{Sh}(X_\alpha)$, so is Deligne-Mumford. Notice that moreover the canonical geometric morphism $\mathcal{E}/E \to \mathcal{E}$ is also a morphism

$$(\mathcal{E}/E, \mathcal{O}_{\mathcal{E}}|_E) \to (\mathcal{E}, \mathcal{O}_{\mathcal{E}})$$

of Deligne-Mumford ∞-topoi.

Definition 3.18. A morphism $(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \to (\mathcal{E}, \mathcal{O}_{\mathcal{E}})$ between Deligne-Mumford ∞-topoi is **étale** if it is equivalent to a canonical one of the form

$$(\mathcal{E}/E, \mathcal{O}_{\mathcal{E}}|_E) \to (\mathcal{E}, \mathcal{O}_{\mathcal{E}}).$$

Definition 3.19. A sheaf of spaces $\mathcal{X}$ on $(\mathbf{Aff}, \acute{e}t)$ is **Deligne-Mumford** (or a higher Deligne-Mumford stack) if it is equivalent to the functor of points of a Deligne-Mumford strictly Henselian ringed ∞-topos $(\mathcal{E}, \mathcal{O}_{\mathcal{E}})$, i.e. if $\mathcal{X}$ is equivalent to a functor of the form

$$S \in \mathbf{Aff} \mapsto \mathbb{M}\mathrm{ap}_{\mathfrak{Top}_{\infty}^{Hens.}} \left(\left(\mathrm{Sh}\left(S_{\acute{e}t}\right), \mathcal{O}_S \right), \left(\mathcal{E}, \mathcal{O}_{\mathcal{E}} \right) \right).$$

A morphism of higher Deligne-Mumford stacks is **étale** if it arises from an étale morphism of Deligne-Mumford ∞-topoi.

Remark 3.20. The functor of points construction is fully faithful on Deligne-Mumford ∞-topoi. Moreover, the functor of points of $\left(\mathrm{Sh}\left(X_{\acute{e}t}\right), \mathcal{O}_X\right)$, with X a scheme, is the usual functor of points of X. See [13, Theorem 2.4.1].

Remark 3.21. If X is a scheme and $\mathcal{F} \in \mathrm{Sh}\left(X_{\acute{e}t}\right)$ is such that

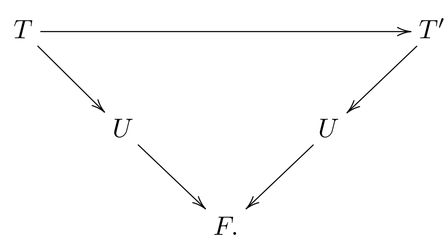
$$\left(\mathrm{Sh}\left(X_{\acute{e}t}\right)/\mathcal{F}, \mathcal{O}_X|_{\mathcal{F}}\right) \simeq \left(\mathrm{Sh}\left(Y_{\acute{e}t}\right), \mathcal{O}_Y\right),$$

for some scheme Y, then $\mathcal{F}$ is represented by an étale morphism of schemes $Y \to X$ in the usual sense.

We will show in what sense this notion of Deligne-Mumford subsumes and generalizes the traditional definition. Firstly, let us explain how algebraic spaces are Deligne-Mumford in this sense. There are many definitions of algebraic space, but we will take the most flexible one. That is, suppose that F is a sheaf on the big étale site (with values in sets). We will say that F is an **algebraic space** if there exists an effective epimorphism $U \to F$ with U a scheme, which is representably étale, that is for any $T \to F$ with T a scheme, $U \times_F T \to T$ is an étale morphism of schemes. (Notice we do not demand any separation conditions on the diagonal of F). In this case, F is the colimit of the Čech nerve of the $U \to F$ (by definition of effective epimorphism). But the Čech nerve is the nerve of a groupoid object in schemes:

$$U \times_F U \rightrightarrows U,$$

for which $U \times_F U \to U \times U$ is an étale equivalence relation. (Conversely, given an étale equivalence relation $R \to U \times U$, $R \rightrightarrows U$ is a groupoid object, whose associated stack is in fact a sheaf.) Consider now the small étale site of F, $F_{\acute{e}t}$. One description is to define its objects as étale morphisms $T \to U$ with T an affine scheme, but restrict the morphisms to be those which commute over the epimorphism $U \to F$, rather than those that commute over U itself:



Equipping it with the Grothendieck topology generated by étale covering families, we can consider the ∞-topos $\mathrm{Sh}\left(F_{\acute{e}t}\right)$. It has a canonical sheaf of rings $\mathcal{O}_F$ which assigns $\mathrm{Spec}\,(A) \to U$ the ring A. There is a canonical object of $F_{\acute{e}t}$, namely $id_U : U \to U$. The slice category $F_{\acute{e}t}/id_U$ is canonicaly equivalent to $U_{\acute{e}t}$, and so we have

$$\mathrm{Sh}\left(F_{\acute{e}t}\right) / \left(id_U\right) \simeq \mathrm{Sh}\left(U_{\acute{e}t}\right).$$

This equivalence also respects the structure sheaves. Moreover, every object $T \to U$ of $F_{ét}$ factors through id_U, hence the canonical map $id_U \to 1$ in $\mathrm{Sh}(F_{ét})$ is an effective epimorphism. Hence,

$$(\mathrm{Sh}(F_{ét}), \mathcal{O}_F)$$

is Deligne-Mumford. Once can show, using a descent argument, that the functor of points recovers F.

More generally, this notion of higher Deligne-Mumford stack subsumes what one would classically call a Deligne-Mumford stack, i.e. a stack $\mathcal{X}$ of groupoids with an epimorphism $X \to \mathcal{X}$ from an algebraic space, such that for any $T \to \mathcal{X}$ with T a scheme, $X \times_{\mathcal{X}} T \to T$ is representable by an étale morphism of algebraic spaces. Indeed, the entire proof above carries through, as nowhere was it used that $X \times_{\mathcal{X}} X \to X \times X$ was an equivalence relation; one needs only ask for the above triangles to commute up to a specified 2-morphism. For more detail, and an alternative argument, see [13, Theorem 2.6.18].

Another important class of examples of stacks on the big étale site that are higher Deligne-Mumford in this sense are so-called Betti stacks. Given a space $V \in \mathcal{S}\mathrm{pc}$, the étale sheafification of the constant presheaf $\Delta(V)$ with value V, is denoted $\Delta^{ét}(V)$, and called its **Betti stack**. Indeed, consider instead the restriction of $\Delta^{ét}(V)$ to the small étale site of Spec $(\mathbb{Z})$, and denote it by $\Delta^{ét}(V)_{\mathbb{Z}}$. Then one can identify $\Delta^{ét}(V)$ with the functor of points of

$$\left(\mathrm{Sh}(\mathrm{Spec}\ (\mathbb{Z})_{ét}) / \Delta^{ét}(V)_{\mathbb{Z}}, \mathcal{O}_{\mathbb{Z}}|_{\Delta^{ét}(V)_{\mathbb{Z}}}\right).$$

If one works over another base scheme, say S, then the base-change is still Deligne-Mumford.

Definition 3.22. Let $\mathcal{X}$ be Deligne-Mumford. Then its **small étale site** $\mathcal{X}_{ét}$ is the ∞-category of (not-necessarily representable) étale maps Spec $A \to \mathcal{X}$, equipped with the induced étale topology.

Remark 3.23. Let $\mathcal{X}$ be a Deligne-Mumford stack represented by $\mathcal{E}$, a strictly Henselian Deligne-Mumford ringed ∞-topos. By [6, Lemma 2.25], we have an identification

$$\mathrm{Sh}(\mathcal{X}_{ét}) \simeq \mathcal{E},$$

where $\mathcal{E}$ is the underlying ∞-topos of the Deligne-Mumford stack whose functor of points is $\mathcal{X}$.

Proposition 3.24. *Let $\mathcal{X}$ be Deligne-Mumford. Then the ∞-category $\mathcal{X}_{ét}^{\mathfrak{DM}_{rep}}$ of (not-necessarily representable) étale maps over $\mathcal{X}$ (with domain any other higher Deligne-Mumford stack) is equivalent to* $\mathrm{Sh}(\mathcal{X}_{ét})$.

Proof. This follows immediately from the previous remark and [16, Remark 6.3.5.10] □

In particular, for any scheme T, we have that $\mathrm{Sh}(T_{ét})$ can be identified with the ∞-category $\mathfrak{DM}_T^{ét}$ of étale maps $\mathcal{X} \to T$, with $\mathcal{X}$ a higher Deligne-Mumford stack.

Finally, we remark that the subcategory of $\mathrm{Sh}(T_{ét})$ on the 0-truncated objects, i.e. the topos sheaves of sets over the small étale site, is equivalent as a category to the category of algebraic spaces étale over T, as these are those Deligne-Mumford stacks étale over T whose functor of points are sheaves of sets.

3.5. **Brief review on some notions in log geometry.** We assume that the reader is familiar with the basic notions of logarithmic geometry (see for example [8, Appendix]). We include a brief review of some of the objects that play an important role in this paper.

Let X be a fine saturated log scheme, with a Deligne-Faltings log structure $L\colon A \to \mathrm{Div}_X$. Recall that this means that A is an étale sheaf of monoids on X, and L is a symmetric monoidal functor with trivial kernel to the stack Div_X over X, of pairs (L, s) of line bundles with a section. One gets back the more usual notion of log scheme "à la Kato" by constructing the sheaf of monoids M_X as the fibered product of the map L and the canonical $\mathcal{O}_X \to \mathrm{Div}_X$ sending a section f of $\mathcal{O}_X$ to the pair $(\mathcal{O}_X, f)$, together with the induced homomorphism $\alpha\colon M_X \to \mathcal{O}_X$ - see [5, Theorem 3.6] for details. In particular, the sheaf A in the Deligne-Faltings language is the same as the sheaf $\overline{M_X}$ in Kato's notation.

Fix now a natural number $n \in \mathbb{N}$.

Definition 3.25. The n-th root stack $\sqrt[n]{X}$ is the stack on the category of schemes over X, that assigns to a scheme $f\colon T \to X$ the groupoid of Deligne-Faltings structures $f^*\frac{1}{n}A \to \mathrm{Div}_X$ extending $f^*L\colon f^*A \to \mathrm{Div}_T$. We denote also by $\sqrt[n]{X}$ the underlying stack on all schemes. By definition, there is a canonical forgetful map

$$\sqrt[n]{X} \to X.$$

If $n \mid m$ there is a natural map $\sqrt[m]{X} \to \sqrt[n]{X}$, given by restricting along the inclusion $\frac{1}{n}A \to \frac{1}{m}A$. The assignment $n \mapsto \sqrt[n]{X}$ assembles into a pro-object in stacks, where $\mathbb{N}$ is ordered by divisibility.

Definition 3.26. We denote the corresponding pro-object by $\sqrt[\infty]{X}_{pro}$. The actual limit in stacks is called the **infinite root stack** of X and is denoted by $\sqrt[\infty]{X}$.

Remark 3.27. In [8], we refer to the pro-object as the infinite root stack instead.

If the log structure of X is coherent (in particular if X is fine and saturated), the stacks $\sqrt[n]{X}$ are algebraic. More precisely, if $X \to \operatorname{Spec}\mathbb{Z}[P]$ is a Kato chart, where P is a fine saturated sharp monoid, then the n-th root stack $\sqrt[n]{X}$ is isomorphic to the quotient stack $[X \times_{\operatorname{Spec}\mathbb{Z}[P]} \operatorname{Spec}\mathbb{Z}[\frac{1}{n}P]/\boldsymbol{\mu}_n(P)]$, where $\boldsymbol{\mu}_n(P)$ is the Cartier dual of the cokernel of the map $P^{\mathrm{gp}} \to \frac{1}{n}P^{\mathrm{gp}}$, acting on the product $X \times_{\operatorname{Spec}\mathbb{Z}[P]} \operatorname{Spec}\mathbb{Z}[\frac{1}{n}P]$ via the trivial action on X and the natural action on the second factor.

A limit version of this construction gives an fpqc atlas for the infinite root stack (which is not algebraic). Indeed, again if $X \to \operatorname{Spec}\mathbb{Z}[P]$ is a Kato chart, and if we denote by $P_{\mathbb{Q}}$ the rational cone spanned by P inside $P^{\mathrm{gp}} \otimes \mathbb{Q}$, then the infinite root stack $\sqrt[\infty]{X}$ is isomorphic to the global quotient $[X \times_{\operatorname{Spec}\mathbb{Z}[P]} \operatorname{Spec}\mathbb{Z}[P_{\mathbb{Q}}]/\boldsymbol{\mu}_\infty(P)]$, where $\boldsymbol{\mu}_\infty(P) = \varprojlim_n \boldsymbol{\mu}_n(P)$.

In general, there exists an étale cover $\{X_i \to X\}$ where the X_i admit Kato charts $X_i \to \operatorname{Spec}\mathbb{Z}[P_i]$, and this provides an fpqc atlas

$$\bigsqcup_i X_i \times_{\operatorname{Spec}\mathbb{Z}[P_i]} \operatorname{Spec}\mathbb{Z}[(P_i)_{\mathbb{Q}}] \to \sqrt[\infty]{X}.$$

Sheaves on an appropriately defined "small étale site" of the infinite root stack of a fine saturated log scheme X are the same as sheaves on the Kummer étale site of X (a natural generalization of the small étale site of a scheme), as we recall now.

Definition 3.28. We denote by $\mathrm{Sh}\left(\sqrt[\infty]{X}^{rep}_{\acute{e}t}\right)$ the ∞-topos of sheaves of spaces on the site of maps $\mathcal{Y} \to \sqrt[\infty]{X}$ of stacks which are étale and representable by an algebraic space, with the induced étale topology.

We recall the notion of Kummer étale maps of log schemes.

Definition 3.29. Let $f : X \to Y$ be a morphism of fine saturated log schemes whose underlying map of schemes is of finite presentation.

- f is **log étale** if étale locally on X and Y there exists a chart
$$(P \to M_X(X), Q \to M_Y(Y), P \to Q)$$
of f such that
$$Y \to X \times_{\mathrm{Spec}\,(\mathbb{Z}[P])} \mathrm{Spec}\,\mathbb{Z}\,[Q]$$
and
$$\mathrm{Spec}\,(\mathcal{O}_Y\,[Q^{\mathrm{gp}}]) \to \mathrm{Spec}\,(\mathcal{O}_Y\,[P^{\mathrm{gp}}])$$
are étale maps of schemes.
- f is **Kummer** if the induced map $f^{-1}\overline{M_Y} \to \overline{M_X}$ is injective, and for every geometric point $\overline{x}$ of X, every element of the stalk $\overline{M}_{X,\overline{x}}$ has a multiple which lies in the image of $\overline{M}_{Y,f(\overline{x})} \to \overline{M}_{X,\overline{x}}$
- f is called **Kummer étale** if it is both log étale and Kummer.

Definition 3.30. Let X be a fine saturated log scheme. Its **Kummer étale site**, denoted by $X_{k\acute{e}t}$ is the subcategory of log schemes over X on the Kummer étale morphisms, with jointly surjective Kummer étale morphisms as covers.

Theorem 3.31. *There is a canonical equivalence of ∞-categories*
$$\mathrm{Sh}\,(X_{k\acute{e}t}) \simeq \mathrm{Sh}\left(\sqrt[\infty]{X}^{rep}_{\acute{e}t}\right).$$

Proof. Since both sites have finite limits, this follows immediately from [23, Theorem 6.21] and the proof of [16, Proposition 6.4.5.7]. □

We also recall the notion of log regularity, a generalization of regularity to the logarithmic setup.

Definition 3.32 ([19, Definition 2.2])**.** A fine saturated locally Noetherian log scheme X is **log regular** if for every point $x \in X$ the ring $\mathcal{O}_{X,\overline{x}}/I_{\overline{x}}\mathcal{O}_{X,\overline{x}}$ is regular, and
$$\dim(\mathcal{O}_{X,\overline{x}}) = \dim(\mathcal{O}_{X,\overline{x}}/I_{\overline{x}}\mathcal{O}_{X,\overline{x}}) + \mathrm{rk}(\overline{M_X})^{\mathrm{gp}}_{\overline{x}}$$
where $\overline{x}$ is a geometric point lying over x, $\mathcal{O}_{X,\overline{x}}$ denotes the stalk of the structure sheaf for the étale topology, and $I_{\overline{x}} = M_{X,\overline{x}} \setminus \mathcal{O}^{\times}_{X,\overline{x}}$.

Given a fine saturated log scheme X, there are various natural ways to construct a pro-space which is a candidate for the "underlying homotopy type" of X:

1) the étale homotopy type of the ∞-root stack $\sqrt[\infty]{X}$,
2) the étale homotopy type of the ∞-root stack $\sqrt[\infty]{X}_{pro}$ regarded as a pro-object,
3) the shape of the ∞-topos $\mathrm{Sh}\,(X_{k\acute{e}t})$,
4) the shape of the ∞-topos $\mathrm{Sh}\left(\sqrt[\infty]{X}^{rep}_{\acute{e}t}\right)$,
5) for a log scheme locally of finite type over $\mathbb{C}$, the profinite completion of its Kato-Nakayama space X_{log},

and in the log regular case, also

6) the profinite étale homotopy type of the trivial locus X^{triv}.

One of the main goals of this paper is to show that, up to profinite completion, $1)-4)$ agree. Note that [8, Theorem 7.3] implies that 2) and 5) agree when working over $\mathbb{C}$.

Remark 3.33. Note that Theorem 3.31 immediately implies that 3) and 4) agree, even before profinite completion.

Remark 3.34. Although 1) and 2) may seem as though they are practically the same, they are subtly different. Moreover, the proof that they agree after profinite completion is very technical and involved. Furthermore, in the proof of [3, Theorem 4.36] Berner implicitly assumes, without proof, that 4) and 2) gives the same profinite homotopy type. We will show that this is indeed true, but it is very far from obvious.

Expanding on results of Berner [3], we will show furthermore in Section 7, that if X is log regular and in characteristic zero, then 6) also agrees with $1) - 4)$ and in arbitrary characteristic the result is still true after completing away from the residue characteristic. See Corollary 7.5.

4. fpqc descent for Betti stacks

In this section we will prove a technical result needed to establish that 1) and 2) above are the same after profinite completion. Namely, we will prove that the Betti stack $\Delta^{\acute{e}t}(V)$ for V a π-finite space satisfies fpqc descent. We will do this over a fixed base scheme S.

Proposition 4.1. *Let G be a finite group. Then its classifying stack of (étale) torsors BG satisfies fpqc descent.*

Proof. This follows from the well known fact that any algebraic stack with quasi-affine diagonal satisfies fpqc descent, see Proposition 3.3.6 of [21]. For a proof see also [22, TAG 0GRH]. □

We denote by **Sch** the 1-category of schemes. Let X be an object in **Sch**. There is a canonical geometric morphism

$$\lambda : \mathrm{Sh}(X_{\acute{e}t}) \to \mathrm{Sh}(\mathbf{Sch}/X, \acute{e}t)$$

such that the inverse image functor λ^* is given simply by restriction. It in fact has a *left* adjoint $\lambda_!$. Identifying $\mathrm{Sh}(X_{\acute{e}t})$ with the ∞-category $\mathfrak{DM}^{\acute{e}t}_X$ of ∞-Deligne Mumford stacks étale over X, and using the canonical identification

$$\mathrm{Sh}(\mathbf{Sch}/X, \acute{e}t) \simeq \mathrm{Sh}(\mathbf{Sch}, \acute{e}t)/X,$$

$\lambda_!$ sends an étale map $\mathcal{Y} \to X$ from a Deligne-Mumford stack simply to itself, as an object of $\mathrm{Sh}(\mathbf{Sch}, \acute{e}t)/X$. Since étale maps are stable under pullback, we conclude that $\lambda_!$ preserves fibered products. It moreover preserves the terminal object, and therefore is left exact. We conclude that there is a well-defined geometric morphism in the opposite direction

$$\tau : \mathrm{Sh}(\mathbf{Sch}/X, \acute{e}t) \to \mathrm{Sh}(X_{\acute{e}t})$$

with $\tau^* = \lambda_!$ and $\tau_* = \lambda^*$.

Remark 4.2. Unwinding definitions, we see that if G is a sheaf on the small étale site of X, then $\tau^*(G)$ is the sheaf

$$(f : Y \to X) \mapsto \Gamma_{Y_{\acute{e}t}}(f^*G).$$

Lemma 4.3. *Let X be an affine scheme. Let A be a torsion abelian sheaf on the small étale site of X, and*

$$K(\tau^*(A), n) : (\mathbf{Sch}/X)^{op} \to \mathcal{S}\mathrm{pc}$$

the étale sheaf of spaces given by the Eilenberg-Maclane construction applied to $\tau^*(A)$. Then $K(\tau^*(A), n)$ satisfies fpqc descent.

Proof. Unwinding the definitions, $K(\tau^*(A), n)$ is the sheafification of the presheaf of spaces that assigns to a morphism $f : Y \to X$ the space

$$K(\tau^*(A)(f), n) \simeq K(\Gamma_{Y_{\acute{e}t}}(f^*A), n),$$

i.e. one should apply the Eilenberg-Maclane space construction object-wise to τ^*A and then sheafify.

Since the inclusion $j : \mathbf{Ab} \hookrightarrow \mathbf{Ch}$ of the category of abelian groups into the ∞-category of unbounded chain complexes of abelian groups does not preserve limits, $j \circ \tau^*A$ may fail to be an étale (homotopy) sheaf of chain complexes. Its étale sheafification is the familiar right derived functor

$$\begin{array}{rcl} \mathcal{F}_A : (\mathbf{Sch}/X)^{op} & \to & \mathbf{Ch}_{\leq 0} \\ (f : Y \to X) & \mapsto & R\Gamma(Y_{\acute{e}t}, f^*A). \end{array}$$

Note that $\mathbf{Ch}_{\leq 0}$ includes $\mathbf{Ab} = \mathbf{Ch}^{\heartsuit}$ and is closed under limits, hence the sheafification of $j \circ \tau^*A$ is indeed an étale sheaf with values in non-positively graded chain complexes.

By [4, Proposition 5.2], the functor $\mathcal{F}_A$ is not only an étale sheaf, but also a sheaf with respect to the v-topology. Since every standard fpqc cover is a v-cover, we conclude it is also a sheaf with respect to the fpqc topology. The ∞-category $\mathbf{Ch}$ of unbounded chain complexes is stable and equivalent to that of $H\mathbb{Z}$-module spectra. The underlying sheaf of coconnective spectra of $\mathcal{F}_A$ is simply $H\mathcal{F}_A$— the Eilenberg Maclane spectrum of $\mathcal{F}_A$. Since the inclusion of coconnective spectra

$$\mathbf{Spt}_{\leq 0} \hookrightarrow \mathbf{Spt}$$

into all spectra preserves limits [15, 1.2.1.6], it follows that this is indeed a sheaf with values in $\mathbf{Spt}$. Étale sheafification induces a left exact colimit preserving functor

$$a : \mathrm{PSh}(\mathbf{Sch}/X, \mathbf{Spt}) \to \mathrm{Sh}_{\acute{e}t}(\mathbf{Sch}/X, \mathbf{Spt})$$

which commutes with suspension. Since this localization is obtained from the analogous one for presheaves of spaces by tensoring with $\mathbf{Spt}$ in the ∞-category of presentable ∞-categories, we abuse notation and write a for the sheafification of presheaves with values in spectra or in spaces.

Hence

$$\begin{array}{rcl} a(\Sigma^n j \circ \tau^*A) & \simeq & \Sigma^n a(j \circ \tau^*A) \\ & \simeq & \Sigma^n \mathcal{F}_A. \end{array}$$

Consider $\Omega^\infty \circ \Sigma^n H\mathcal{F}_A$, defined by sending f to the space $\Omega^\infty \Sigma^n (H\mathcal{F}_A(f))$. Since Ω^∞ preserves limits, $\Omega^\infty \circ \Sigma^n H\mathcal{F}_A$ satisfies fpqc descent. Also, by the above, it is the same as $\Omega^{\infty} \circ a\left(\Sigma^n j \circ \tau^* A\right)$. But we have a natural equivalence

$$\Omega^{\infty} \circ a\left(\Sigma^n j \circ \tau^* A\right) \simeq a\left(\Omega^{\infty} \Sigma^n j \circ \tau^* A\right),$$

and the presheaf of spaces $\Omega^{\infty} \Sigma^n j \circ \tau^* A$ assigns to a morphism $f : Y \to X$ the space $K\left(\tau^*\left(A\right)\left(f\right), n\right)$. It follows thus that

$$\Omega^{\infty} \circ a\left(\Sigma^n j \circ \tau^* A\right) \simeq K\left(\tau^* A, n\right).$$

Since $\Omega^{\infty} \circ a\left(\Sigma^n j \circ \tau^* A\right)$ satisfies fpqc descent, we are done. □

Corollary 4.4. *Let A be a finite abelian group. Then the Betti stack $\Delta^{\acute{e}t}\left(K\left(A, n\right)\right)$ satisfies fpqc descent.*

Proof. Firstly note that $\Delta^{\acute{e}t}\left(K\left(A, n\right)\right)$ satisfies fpqc descent on $\mathbf{Sch}_S$ if and only if for all $g : X \to S$ with X affine, $g^* \Delta^{\acute{e}t}\left(K\left(A, n\right)\right)$ satisfies fpqc descent on the large site of affine schemes over X. But

$$g^* \Delta^{\acute{e}t}\left(K\left(A, n\right)\right) \simeq K\left(g^* \Delta^{\acute{e}t}\left(A\right), n\right),$$

and $g^* \Delta^{\acute{e}t}\left(A\right)$ can be canonically identified with $\tau^* \Delta^{\acute{e}t}_{sm}\left(A\right)$, where $\Delta^{\acute{e}t}_{sm}\left(A\right)$ is the constant stack on the small étale site of X, so we are done by Lemma 4.3. □

Denote by $\iota\mathcal{S}\mathrm{pc}$ the maximal Kan subcomplex of the ∞-category $\mathcal{S}\mathrm{pc}$, i.e. the (large) ∞-groupoid of spaces and equivalences. Fix an abelian group A. Denote by $B\mathbf{Aut}\left(K\left(A, n\right)\right)$ the full subcategory of $\iota\mathcal{S}\mathrm{pc}$ spanned by the single object $K\left(A, n\right)$. This is a small ∞-groupoid, and hence canonically identified with a space in $\mathcal{S}\mathrm{pc}$. (Concretely it is the space of self homotopy equivalences of $K\left(A, n\right)$.)

Lemma 4.5. *For any finite abelian group A, $\Delta^{\acute{e}t}\left(B\mathbf{Aut}\left(K\left(A, n\right)\right)\right)$ satisfies fpqc descent.*

Proof. Recall from Section 2, that we fix two Grothendieck universes $\mathcal{U} \in \mathcal{V}$. For this proof, we will be considering sheaves with values in $\mathcal{V}$-small spaces, which are ∞-topoi in the universe $\mathcal{V}$. In particular, fpqc-sheafification of a $\mathcal{U}$-small presheaf of spaces on $\mathbf{Sch}$ exists, but may result in a $\mathcal{V}$-small sheaf of spaces, i.e. a sheaf of large spaces. For any $\mathcal{U}$-small étale sheaf H, denote by aH its fpqc-sheafification, viewed as a large presheaf of spaces. It suffices to prove that for all affine schemes X, the canonical map

$$\mathbb{M}\mathrm{ap}\left(X, \Delta^{\acute{e}t}\left(B\mathbf{Aut}\left(K\left(A, n\right)\right)\right)\right) \to \mathbb{M}\mathrm{ap}\left(X, a\Delta^{\acute{e}t}\left(B\mathbf{Aut}\left(K\left(A, n\right)\right)\right)\right)$$

is an equivalence. Note that by [6, p. 31], it follows that there is a canonical map

$$\psi : B\mathbf{Aut}\left(K\left(A, n\right)\right) \to B\mathbf{Aut}\left(A\right)$$

identifying $B\mathbf{Aut}\left(A\right)$ with the 1-truncation. Notice that by Proposition 4.1, we have that

$$a\Delta^{\acute{e}t}\left(B\mathbf{Aut}\left(A\right)\right) \simeq \Delta^{\acute{e}t}\left(B\mathbf{Aut}\left(A\right)\right).$$

Therefore, it suffices to prove that for any

$$\omega : * \to \mathbb{M}\mathrm{ap}\left(X, \Delta^{\acute{e}t}\left(B\mathbf{Aut}\left(A\right)\right)\right) \simeq \mathbb{M}\mathrm{ap}\left(X, a\Delta^{\acute{e}t}\left(B\mathbf{Aut}\left(A\right)\right)\right),$$

the induced map between the homotopy fibers of $\mathbb{M}\mathrm{ap}\left(X, \Delta^{\acute{e}t}\left(\psi\right)\right)$ and $\mathbb{M}\mathrm{ap}\left(X, a\Delta^{\acute{e}t}\left(\psi\right)\right)$ is an equivalence. Denote the homotopy fiber of the former by $F^{\acute{e}t}_{\omega}$, and the latter by $F^{\mathrm{fpqc}}_{\omega}$. By [16, Proposition 5.5.5.12], we have a canonical identification

$$F^{\acute{e}t}_{\omega} \simeq \mathbb{M}\mathrm{ap}_{\mathrm{Sh}(\mathbf{Sch}, \acute{e}t)/\Delta^{\acute{e}t}(B\mathbf{Aut}(A))}\left(\omega, \Delta^{\acute{e}t}\left(\psi\right)\right).$$

Consider the map

$$\omega : X \to \Delta^{ét}\left(B\mathbf{Aut}\left(A\right)\right).$$

It classifies a locally constant étale sheaf $\mathcal{F}_\omega$ of abelian groups on X with coefficients in A. Explicitly, the canonical functor

$$B\mathbf{Aut}\left(A\right) \to \mathbf{Ab}$$

corresponds to an abelian sheaf $\mathcal{F}_A$ on $\mathcal{S}\mathrm{pc}/B\mathbf{Aut}\left(A\right)$ and ω corresponds to a geometric morphism

$$\overline{\omega} : \mathrm{Sh}\left(\mathbf{Sch}, ét\right)/X \to \mathcal{S}\mathrm{pc}/B\mathbf{Aut}\left(A\right),$$

and

$$\mathcal{F}_\omega = \overline{\omega}^* \mathcal{F}_A.$$

Indeed, for any geometric morphism $f : \mathcal{E} \to \mathcal{F}$ of ∞-topoi, and any object $F \in \mathcal{F}$, by [16, Remark 6.3.5.8], the following is a pullback diagram in the ∞-category $\mathfrak{Top}_\infty$ of ∞-topoi:

$$\begin{array}{ccc} \mathcal{E}/f^*F & \longrightarrow & \mathcal{F}/F \\ \downarrow & & \downarrow \\ \mathcal{E} & \xrightarrow[f]{} & \mathcal{F}. \end{array}$$

Applying this to the terminal ∞-topos $\mathcal{S}\mathrm{pc}$, and letting f be the unique geometic morphism

$$\mathcal{E} \to \mathcal{S}\mathrm{pc},$$

we have that the square in the below diagram is a pullback diagram:

$$\begin{array}{ccccc} & & \mathcal{E}/\Delta_{\mathcal{E}}\mathcal{G} & \longrightarrow & \mathcal{S}\mathrm{pc}/\mathcal{G} \\ & \nearrow & \downarrow & & \downarrow \\ \mathcal{E}/E & \longrightarrow & \mathcal{E} & \longrightarrow & \mathcal{S}\mathrm{pc}. \end{array}$$

Since by [16, 6.3.5.10], $\mathcal{E}$ is canonically equivalent to the ∞-category of étale geometric morphisms over $\mathcal{E}$, the space of maps from E to $\Delta_{\mathcal{E}}\mathcal{G}$ in $\mathcal{E}$ is equivalent to the space of lifts as in the dotted arrow above. But since we have a pullback diagram and $\mathcal{S}\mathrm{pc}$ is terminal, this is equivalent to the space of geometric morphisms $\mathcal{E}/E \to \mathcal{S}\mathrm{pc}/\mathcal{G}$. Letting $\mathcal{E} = \mathrm{Sh}\left(\mathbf{Sch}, et\right)$, $E = X$ and $\mathcal{G} = B\mathbf{Aut}\left(A\right)$, yields the result. Moreover, by [6, Theorem 2.40], it follows that

$$\mathcal{F}_\omega \simeq \tau^* \mathcal{F}_{\omega'},$$

for $\mathcal{F}_{\omega'}$ an abelian sheaf on the small étale site of X. By the proof of [6, Proposition 4.11], we have a canonical identification

$$F_\omega^{ét} \simeq \Gamma_X\left(K\left(\mathcal{F}_\omega, n+1\right)\right).$$

By an analogous argument, we have

$$F_\omega^{\mathrm{fpqc}} \simeq \Gamma_X\left(aK\left(\mathcal{F}_\omega, n+1\right)\right).$$

Since $\mathcal{F}_\omega$ is the pull back from the small étale topos of a torsion abelian sheaf, the result now follows from Lemma 4.3. □

Proposition 4.6. *For any connected π-finite space V, the Betti stack $\Delta^{ét}\left(V\right)$ satisfies fpqc descent.*

Proof. We prove this by induction on homotopy dimension. Suppose that the Betti stack of all connected $(n-1)$-truncated π-finite spaces satisfies fpqc descent. Let V be an n-truncated connected π-finite space. Then there is a Cartesian diagram in $\mathcal{S}\mathrm{pc}$

$$\begin{array}{ccc} V & \longrightarrow & B\mathbf{Aut}\left(\pi_n\left(V\right)\right) \\ \downarrow & & \downarrow \\ V_{n-1} & \longrightarrow & B\mathbf{Aut}\left(K\left(\pi_n\left(V\right), n\right)\right) \end{array}$$

where V_{n-1} is the $(n-1)$-truncation of V. Since $\Delta^{\acute{e}t}$ preserves finite limits, we have

$$\Delta^{\acute{e}t}\left(V\right) \simeq \Delta^{\acute{e}t}\left(V_{n-1}\right) \times_{\Delta^{\acute{e}t}\left(B\mathbf{Aut}\left(K\left(\pi_n\left(V\right)\right)\right)\right)} \Delta^{\acute{e}t}\left(B\mathbf{Aut}\left(\pi_n\left(V\right)\right)\right).$$

Since fpqc-sheaves are stable under limits, the result now follows from the inductive hypothesis, Proposition 4.1, and Lemma 4.5. □

Definition 4.7. Let $\mathscr{C}$ denote the following Grothendieck pre-topology on **Sch**. A finite family of maps in **Sch**

$$\left(f_i : X_i \to X\right)_{i=1}^n$$

is a $\mathscr{C}$**-cover** if the canonical map

$$\coprod_i X_i \to X$$

is an isomorphism. We call the corresponding Grothendieck topology the **coproduct topology**. Denote the corresponding sheafification functor by c.

Remark 4.8. Note that every $\mathscr{C}$-cover can be written as a finite composition of 2-term $\mathscr{C}$-covers. Moreover, by [22, TAG 02WN], every 2-term $\mathscr{C}$-cover is isomorphic to the inclusion of two complementary clopen subschemes.

Let $OpCl : \mathbf{Sch}^{op} \to \mathbf{Set}$ be the presheaf sending a scheme to the set of its clopen subsets.

Lemma 4.9. *Let J be a Grothendieck topology on* **Sch** *for which every $\mathscr{C}$-cover is a cover, for which open closed inclusions have J-descent, and such that $OpCl$ is a J-sheaf. Let F_1 and F_2 be two J-sheaves of spaces. Then the coproduct of F_1 and F_2 as $\mathscr{C}$-sheaves and as J-sheaves coincide.*

Proof. Let F_1 and F_2 be J-sheaves and let $F_1 \coprod F_2$ denote the coproduct in presheaves. Since any J-sheaf is a $\mathscr{C}$-sheaf, it suffices to prove that the $\mathscr{C}$-sheafification $c\left(F_1 \coprod F_2\right)$ is a J-sheaf. Since sheafification is a transfinite composition of the plus-construction [16, Section 6.5.3], it suffices to prove that the $\mathscr{C}$-plus construction $\left(F_1 \coprod F_2\right)^+$ is a J-sheaf (since then it is also a $\mathscr{C}$-sheaf and coincides with the $\mathscr{C}$-sheafification).

To compute the value of the $\mathscr{C}$-plus construction of $F_1 \coprod F_2$ on a scheme U, we need to consider finite disjoint unions of clopen subsets whose union is U. Since the pairwise intersections of disjoint clopen subsets are empty, the Čech nerve simplifies and we can write the $\mathscr{C}$-plus construction as

$$\left(F_1 \coprod F_2\right)^+ (U) = \varinjlim_{\{U_\alpha\}_{\alpha \in I}} \prod_{\alpha \in I} \left(F_1 \coprod F_2\right) (U_\alpha)$$

where the colimit runs over all $\mathcal{C}$-covers $\{U_\alpha\}_{\alpha \in I}$ of X.

For a general $\mathscr{C}$-cover $\{U_\alpha\}_{\alpha\in I}$ we can rewrite

$$\prod_{\alpha\in I}\left(F_1\coprod F_2\right)(U_\alpha)$$

by distributing the product over the disjoint union. Hence we obtain that

$$\begin{aligned}\prod_{\alpha\in I}\left(F_1\coprod F_2\right)(U_\alpha) &\simeq \coprod_{I_1,\, I_2}\Big(\big(\prod_{i\in I_1}F_1(U_i)\big)\times\big(\prod_{i\in I_2}F_2(U_i)\big)\Big)\\ &\simeq \coprod_{I_1,\, I_2}F_1\big(\bigcup_{i\in I_1}U_i\big)\times F_2\big(\bigcup_{i\in I_2}U_i\big)\end{aligned}$$

where I_1, I_2 run over all partitions of I into two disjoint subsets: $I_1\cup I_2=I$ and $I_1\cap I_2=\varnothing$. The last equivalence uses that F_1 and F_2 are $\mathscr{C}$-sheaves. Hence, by the fact that F_1 and F_2 are sheaves with respect to $\mathscr{C}$, we can now simplify the $\mathscr{C}$-plus construction to

$$\left(F_1\coprod F_2\right)^+(X)=\coprod_{Z\in OpCl(X)}F_1(Z)\times F_2(X-Z).$$

Note that in particular, this implies that

$$\left(*\coprod *\right)^+ = OpCl.$$

Consider the map in presheaves

$$F_1\coprod F_2\to *\coprod *$$

mapping F_1 to the first copy of the trivial sheaf, and F_2 to the second one. After applying the plus construction, we get an induced map

$$\left(F_1\coprod F_2\right)^+\to\left(*\coprod *\right)^+=OpCl.$$

Now let $(f_\alpha : U_\alpha\to U)_\alpha$ be a J-cover of a scheme U. To prove the lemma it suffices to show that the canonical map

$$\left(F_1\coprod F_2\right)^+(U)\to\varprojlim\left[\prod_\alpha\left(F_1\coprod F_2\right)^+(U_\alpha)\rightrightarrows\prod_{\alpha,\beta}\left(F_1\coprod F_2\right)^+(U_{\alpha\beta})\,\substack{\to\\\to\\\to}\,\cdots\right]$$

is an equivalence. The natural commutative diagram of augmented cosimplicial objects induces a commutative square

$$\begin{array}{ccc}(F_1\coprod F_2)^+(U) & \longrightarrow & \varprojlim\left[\prod_\alpha(F_1\coprod F_2)^+(U_\alpha)\rightrightarrows\prod_{\alpha,\beta}(F_1\coprod F_2)^+(U_{\alpha\beta})\,\substack{\to\\\to\\\to}\,\cdots\right]\\ \downarrow & & \downarrow \qquad\qquad \downarrow\\ (*\coprod *)^+(U) & \xrightarrow{\sim} & \varprojlim\left[\prod_\alpha(*\coprod *)^+(U_\alpha)\rightrightarrows\prod_{\alpha,\beta}(*\coprod *)^+(U_{\alpha\beta})\,\substack{\to\\\to\\\to}\,\cdots\right].\end{array}$$

Notice also that $(*\coprod *)^+(U)=OpCl(U)$, so by our assumption that $OpCl$ is a J-sheaf, the bottom horizontal arrow is an equivalence. Hence, there is an induced commutative triangle

$$\begin{array}{ccc} (F_1 \coprod F_2)^+ (U) & \longrightarrow & \varprojlim \left[\prod_\alpha (F_1 \coprod F_2)^+ (U_\alpha) \rightrightarrows \prod_{\alpha,\beta} (F_1 \coprod F_2)^+ (U_{\alpha\beta}) \Rrightarrow \cdots \right] \\ & \searrow \quad \swarrow & \\ & OpCl(U). & \end{array}$$

To show that the horizontal map is an equivalence, it suffices to prove the induced map over each homotopy fiber over $OpCl(U)$ is. Let

$$Z \in OpCl(U) = \left(* \coprod *\right)^+ (U)$$

Since

$$\left(F_1 \coprod F_2\right)^+ (X) = \coprod_{Z \in OpCl(X)} F_1 (Z) \times F_2 (X - Z),$$

the fiber of $(F_1 \coprod F_2)^+ \to OpCl$ over Z is equivalent to $F_1(Z) \times F_2(U - Z)$, and since limits commute with taking fibers, the induced map over Z-fibers can be rewritten as

$$\begin{aligned} F_1(Z) \times F_2(U-Z) \quad &\to \quad \varprojlim \left[\prod_\alpha F_1 (U_\alpha \times_U Z) \times \prod_\alpha F_2 (U_\alpha \times_U (U - Z)) \rightrightarrows \cdots \right] \\ &\simeq \quad \varprojlim \left[\prod_\alpha F_1 (U_\alpha \times_U Z) \rightrightarrows \cdots \right] \times \varprojlim \left[\prod_\alpha F_2 (U_\alpha \times_U (U - Z)) \rightrightarrows \cdots \right] \end{aligned}$$

and the induced map can be identified as the cartesian product of the two canonical descent maps. But these are equivalences since F_1 and F_2 are J-sheaves. This completes the proof. □

Corollary 4.10. *If F_1 and F_2 are fpqc sheaves of spaces on* **Sch**, *then their coproduct as fpqc sheaves and étale sheaves coincide.*

Proof. Since $OpCl$ is an fpqc sheaf, and every $\mathscr{C}$ cover is an étale cover, this follows immediately from Lemma 4.9. □

Theorem 4.11. *Let V be a π-finite space. Then its Betti stack $\Delta^{ét}(V)$ satisfies fpqc descent.*

Proof. Since $\Delta^{ét}(V)$ is a finite coproduct of Betti stacks of connected π-finite spaces, this follows immediately from Proposition 4.6 and Corollary 4.10. □

Recall that the étale homotopy type of a scheme is defined as the shape of its small étale topos. Defining in a similar way a fpqc homotopy type is possible, but requires some extra care. As in Section 2, we fix two Grothendieck universes $\mathcal{U} \in \mathcal{V}$, where $\mathcal{U}$ is the universe of small sets we place ourselves in from the outset. If X is an affine scheme, then the collection of fpqc covers of X is not $\mathcal{U}$-small, but only $\mathcal{V}$-small. Thus one can define the $\mathcal{V}$-small fpqc topos of X, which we denote

$$\mathrm{Sh}_{\mathcal{V}} (X_{\mathrm{fpqc}})$$

If $\mathcal{X}$ is a sheaf of spaces, we define its $\mathcal{V}$-small fpqc topos via a left Kan extension from affine schemes, exactly as we defined the small étale site of $\mathcal{X}$ in Section 3.3.

Now we can define the fpqc shape of $\mathcal{X}$ as

$$\mathit{Shape}\left(\mathrm{Sh}_{\mathcal{V}}\left(X_{\mathrm{fpqc}}\right)\right)$$

Note that by construction this will give rise to a pro-object in large, i.e. $\mathcal{V}$-small, spaces, which we denote $\mathcal{S}\mathrm{pc}_{\mathcal{V}}$. Thus the fpqc homotopy type of $\mathcal{X}$ by construction is a left exact $\mathcal{V}$-accessible functor

$$\Pi_\infty^{\mathrm{fpqc}}(\mathcal{X}) := \mathit{Shape}\left(\mathrm{Sh}_{\mathcal{V}}\left(X_{\mathrm{fpqc}}\right)\right) : \mathcal{S}\mathrm{pc}_{\mathcal{V}} \to \mathcal{S}\mathrm{pc}_{\mathcal{V}}$$

The fpqc and the étale homotopy type live in different worlds in terms of size, and this makes it difficult to compare them directly. The situation is different however if we want to compare instead their profinite completions. Note that indeed although the category of spaces changes as we change the universe, the full subcategory of π-finite spaces stays the same. Thus the profinite completion of the fpqc homotopy type can be computed as the composite

$$\widehat{\Pi}_\infty^{\mathrm{fpqc}}(\mathcal{X}) : \mathcal{S}\mathrm{pc}^{\pi} \longrightarrow \mathcal{S}\mathrm{pc}_{\mathcal{V}} \xrightarrow{\quad \Pi_\infty^{\mathrm{fpqc}}(\mathcal{X}) \quad} \mathcal{S}\mathrm{pc}_{\mathcal{V}}$$

It turns out that in fact $\widehat{\Pi}_\infty^{\mathrm{fpqc}}(\mathcal{X})$ corestricts to $\mathcal{U}$-small spaces, therefore is in fact a $\mathcal{U}$-small profinite space. It is therefore comparable to the profinite completion of the étale homotopy type. After Theorem 4.11, it is easy to see that the two are in fact equivalent.

Corollary 4.12. *For a sheaf of spaces $\mathcal{X}$ on $\left(\mathbf{Aff}', \acute{e}t\right)$, the étale homotopy type of $\mathcal{X}$ and the fpqc homotopy type of $\mathcal{X}$ agree up to profinite completion.*

Proof. Let V be a π-finite space. Denote by $\Delta^{\mathrm{fpqc}}(V)$ the sheafification of the constant presheaf with values V with respect to the fpqc topology. By adapting [6, Theorem 2.40] one can easily show that for a sheaf of spaces $\mathcal{X}$ on $\left(\mathbf{Aff}', \acute{e}t\right)$, the profinite completion of its fpqc homotopy type

$$\widehat{\Pi}_\infty^{\mathrm{fpqc}}(\mathcal{X}) : \mathcal{S}\mathrm{pc}^{\pi} \to \mathcal{S}\mathrm{pc}_{\mathcal{V}}$$

is equivalent to the functor given by the following

$$\begin{array}{rcl} \mathcal{S}\mathrm{pc}^{\pi} & \to & \mathcal{S}\mathrm{pc}_{\mathcal{V}} \\ V & \mapsto & \mathbb{M}\mathrm{ap}\left(\mathcal{X}, \Delta^{\mathrm{fpqc}}\left(V\right)\right) \end{array}$$

We know by the previous Theorem 4.11 that the Betti stack $\Delta^{\acute{e}t}\left(V\right)$ satisfies fpqc descent if V is π-finite, thus if V is a π-finite space

$$\Delta^{\mathrm{fpqc}}\left(V\right) \simeq \Delta^{\acute{e}t}\left(V\right)$$

Thus in particular we have that

$$\mathbb{M}\mathrm{ap}\left(\mathcal{X}, \Delta^{\mathrm{fpqc}}\left(V\right)\right) \in \mathcal{S}\mathrm{pc} \subset \mathcal{S}\mathrm{pc}_{\mathcal{V}}$$

and further there is an equivalence natural in $V \in \mathcal{S}\mathrm{pc}^{\pi}$

$$\mathbb{M}\mathrm{ap}\left(\mathcal{X}, \Delta^{\acute{e}t}\left(V\right)\right) \simeq \mathbb{M}\mathrm{ap}\left(\mathcal{X}, \Delta^{\mathrm{fpqc}}\left(V\right)\right) \in \mathcal{S}\mathrm{pc}$$

This shows that there is an equivalence of functors

$$\widehat{\Pi}_\infty^{\mathrm{fpqc}}(\mathcal{X}) \simeq \widehat{\Pi}_\infty^{\acute{e}t}(\mathcal{X}) : \mathcal{S}\mathrm{pc}^{\pi} \to \mathcal{S}\mathrm{pc}$$

which is what we wanted to prove. $\square$

5. Shape comparison

5.1. **To pro or not to pro.** Fix a fine saturated log scheme X over a base scheme S. In this section we prove that the étale homotopy type of the infinite root stack of X is the same when it is viewed as a pro-object, or as an actual fpqc-stack, at least after profinite completion.

It turns out that this statement is local in the étale topology of X, so we can first assume that X is affine and has a global Kato chart $X \to \operatorname{Spec} \mathbb{Z}[P]$ where P is a fine saturated monoid. In this case each root stack (including the limit) is a global quotient of affine schemes.

Suppose the log-structure on X is given by a global Kato chart

$$f : X \to \operatorname{Spec}\left(\mathbb{Z}\left[P\right]\right).$$

Recall that in this case, for each finite n, there is a canonical atlas

$$X \times_{\operatorname{Spec}\mathbb{Z}[P]} \operatorname{Spec}\mathbb{Z}[\frac{1}{n}P] \to \sqrt[n]{X}.$$

Moreover, the limit

$$\varprojlim_n X \times_{\operatorname{Spec}\mathbb{Z}[P]} \operatorname{Spec}\mathbb{Z}[\frac{1}{n}P]$$

serves as a (fpqc) atlas of the *infinite* root stack $\varprojlim_n \sqrt[n]{X}$, when regarded as an actual stack, rather than a pro-object. In particular, for each finite n, the atlas above is finitely presented and affine over X, whereas for $n = \infty$ it is only affine. An important step in comparing the étale homotopy type of $\varprojlim_n \sqrt[n]{X}$ when viewed as a pro-object v.s. as viewed as an actual stack is determining which stacks are insensitive to the difference above. By this we mean stacks $\mathcal{F}$ such that for any pro-object $\varprojlim_\alpha Y_\alpha \in \operatorname{Pro}\left(\mathbf{Aff}^{\mathbf{ft}}_X\right)$ the canonical map

$$\mathbb{M}\mathrm{ap}_{\operatorname{Pro}\left(\operatorname{Sh}\left(\mathbf{Aff}^{\mathbf{ft}}_X, \acute{e}t\right)\right)}\left(\varprojlim_\alpha Y_\alpha, \mathcal{F}\right) \longrightarrow \mathbb{M}\mathrm{ap}_{\operatorname{Sh}(\mathbf{Aff}_X, \acute{e}t)}\left(\varprojlim_\alpha Y_\alpha, \mathcal{F}\right),$$

is an equivalence, where $\mathbf{Aff}^{\mathbf{ft}}_X$ is the category of schemes $T \to X$ affine and finitely presented over X, and where the limit on the left hand side is taken in the pro-category. By definition, this means that

$$\mathbb{M}\mathrm{ap}_{\operatorname{Pro}\left(\operatorname{Sh}\left(\mathbf{Aff}^{\mathbf{ft}}_X, \acute{e}t\right)\right)}\left(\varprojlim_\alpha Y_\alpha, \mathcal{F}\right) \simeq \underset{\alpha}{\varinjlim}\, \mathbb{M}\mathrm{ap}_{\operatorname{Pro}(\operatorname{Sh}(\mathbf{Aff}_X, \acute{e}t))}\left(Y_\alpha, \mathcal{F}\right).$$

Note that if $X = \operatorname{Spec}(A)$, then $\mathbf{Aff}^{\mathbf{ft}}_X \simeq \left(A\text{-}\mathbf{Alg}^{\mathbf{fp}}\right)^{op}$, the opposite category of finitely presented A-algebras.

Henceforth, we will fix a base scheme S. The application we have in mind is of course the case that S is our log scheme X. We write $\mathbf{Sch}_S$ for schemes over S. Recall that by an affine S-scheme, we mean an affine map $\operatorname{Spec}(A) \to S$. We denote the corresponding category of affine S-schemes by $\mathbf{Aff}_S$.

The following is a well-known definition ([22, Tag 07XK]):

Definition 5.1. A functor $F : \mathbf{Aff}_S^{op} \to \mathcal{S}\mathrm{pc}$ is **limit-preserving** if for all cofiltered limits $X = \varprojlim_n X_n$ of affine S-schemes, the canonical map

$$\varinjlim_n F(X_n) \to F(X)$$

is an equivalence.

Remark 5.2. By the Yoneda lemma, this is precisely the condition that

$$\varinjlim_n \mathbb{M}\mathrm{ap}(X_n, F) \to \mathbb{M}\mathrm{ap}(X, F)$$

is an equivalence, as discussed above.

We will now give a complete categorical classification of such limit preserving functors.

Note that we replaced $\mathbf{Sch}_S$ with $\mathbf{Aff}_S$ merely as a set-theoretic convenience for what follows. Denote by

$$f : \mathbf{Aff}_S^{\mathbf{ft}} \hookrightarrow \mathbf{Aff}_S$$

the canonical inclusion of affine S-schemes of finite presentation. Then f induces an equivalence of categories

$$\mathrm{Pro}\left(\mathbf{Aff}_S^{\mathbf{ft}}\right) \simeq \mathbf{Aff}_S.$$

This is equivalent to the observation that any affine S-scheme is a cofiltered limit $\varprojlim_\alpha Y_\alpha$ of affine S-schemes of finite presentation, and for any other $Z \in \mathbf{Aff}_S^{\mathbf{ft}}$,

$$\mathrm{Hom}(Y, Z) \cong \varinjlim_\alpha (Y_\alpha, Z).$$

Let us work in the Grothendieck universe $\mathcal{V}$ of large sets, so that $\mathbf{Aff}_S$ is $\mathcal{V}$-small. The restriction functor

$$f^* : \widehat{\mathrm{Psh}}(\mathbf{Aff}_S) \to \widehat{\mathrm{Psh}}\left(\mathbf{Aff}_S^{\mathbf{ft}}\right)$$

has both a left and right adjoint:

$$f_! \dashv f^* \dashv f_*,$$

given by global left and right Kan extension respectively. That is, e.g.,

$$f_!(\mathcal{F}) \simeq \mathrm{Lan}_{f^{op}} \mathcal{F}.$$

The functor $f_!$, being colimit preserving, has a much easier description as the unique colimit preserving functor

$$f_! : \widehat{\mathrm{Psh}}\left(\mathbf{Aff}_S^{\mathbf{ft}}\right) \to \widehat{\mathrm{Psh}}(\mathbf{Aff}_S)$$

such that for all $Z \in \mathbf{Aff}_S^{\mathbf{ft}}$,

$$f_! y(Z) \simeq y(f(Z)),$$

where y denotes the Yoneda embedding. For any $\mathcal{F}$, the unit map $\mathcal{F} \to f^* f_! \mathcal{F}$ is readily seen to be an equivalence. In fact, since $f^* f_!$ and the identity functor are colimit preserving, given that

$$y^* : \mathrm{Fun}^L\left(\widehat{\mathrm{Psh}}\left(\mathbf{Aff}_S^{\mathbf{ft}}\right), \widehat{\mathrm{Psh}}\left(\mathbf{Aff}_S^{\mathbf{ft}}\right)\right) \to \mathrm{Fun}\left(\mathbf{Aff}_S^{\mathbf{ft}}, \widehat{\mathrm{Psh}}\left(\mathbf{Aff}_S^{\mathbf{ft}}\right)\right)$$

is an equivalence of ∞-categories (Proposition [16, 6.2.3.20]), where L denotes colimit-preserving functors, it suffices to verify this on representables, where it follows immediately from the fact that f is fully-faithful. As the unit of the adjunction $f_! \dashv f^*$ is an equivalence, we conclude that $f_!$ is fully faithful.

The upshot is, we have a colimit-preserving fully faithful embedding

$$f_! : \widehat{\mathrm{Psh}}\left(\mathbf{Aff}^{\mathbf{ft}}_S\right) \to \widehat{\mathrm{Psh}}\left(\mathbf{Aff}_S\right)$$

whose essential image is a coreflective subcategory. We claim this subcategory is nothing but the (large) limit preserving presheaves.

Theorem 5.3. *$\mathcal{F} \in \widehat{\mathrm{Psh}}\left(\mathbf{Aff}_S\right)$ is limit-preserving if and only if $\mathcal{F}$ is in the image of $f_!$.*

Proof. Fix a cofiltered system $Y_\bullet : \mathcal{J} \to \mathbf{Aff}_S$ of affine S-schemes, and let $Y := \varprojlim_{i \in \mathcal{J}} Y_j$ in $\mathbf{Aff}_S$. For each i, define a functor

$$\mathsf{ev}_{Y_i} : \widehat{\mathrm{Psh}}\left(\mathbf{Aff}_S\right) \to \widehat{\mathcal{S}\mathrm{pc}}$$

by the rule $\mathcal{F} \mapsto \mathcal{F}\left(Y_i\right)$. Since colimits in presheaves are computed object-wise, each of these functors is colimit preserving, so each is an object of $\mathrm{Fun}^L\left(\widehat{\mathrm{Psh}}\left(\mathbf{Aff}_S\right), \widehat{\mathcal{S}\mathrm{pc}}\right)$, where the L denotes colimit-preserving. Since

$$\mathrm{Fun}^L\left(\widehat{\mathrm{Psh}}\left(\mathbf{Aff}_S\right), \widehat{\mathcal{S}\mathrm{pc}}\right) \simeq \mathrm{Fun}\left(\mathbf{Aff}_S, \widehat{\mathcal{S}\mathrm{pc}}\right),$$

this ∞-category is cocomplete. Denote by Q the colimit

$$Q := \varinjlim_{i \in \mathcal{J}^{op}} \mathsf{ev}_{Y_i}$$

of these functors. Note that we also have a colimit preserving functor ev_Y given by evaluation at Y. There is furthermore a canonical morphism

$$\varphi : Q \Rightarrow \mathsf{ev}_Y.$$

We claim that

$$\varphi \circ f_! : Q \circ f_! \Rightarrow \mathsf{ev} \circ f_!$$

is an equivalence. As each of these functors is colimit preserving, it suffices to prove that the components of $\varphi \circ f_!$ along representables is a equivalence. Let $Z \in \mathbf{Aff}^{\mathbf{ft}}_S$, then we have

$$\begin{aligned}
\mathsf{ev}_Y(Z) &\simeq \mathbb{M}\mathrm{ap}_{\widehat{\mathrm{Psh}}(\mathbf{Aff}_S)}\left(y(Y), y(f(Z))\right)\\
&\simeq \mathbb{M}\mathrm{ap}_{\mathbf{Aff}_S}\left(\underset{i\in\mathcal{J}}{\varprojlim} Y_i, f(Z)\right)\\
&\simeq \mathbb{M}\mathrm{ap}_{\mathrm{Pro}\left(\mathbf{Aff}^{\mathbf{ft}}_S\right)}\left(\underset{i\in\mathcal{J}}{\varprojlim} Y_i, Z\right)\\
&\simeq \underset{i\in\mathcal{J}^{op}}{\varinjlim} \mathbb{M}\mathrm{ap}_{\mathbf{Aff}^{\mathbf{ft}}}(Y_i, Z)\\
&\simeq \underset{i\in\mathcal{J}^{op}}{\varinjlim} y(Z)(Y_i)\\
&\simeq \underset{i\in\mathcal{J}^{op}}{\varinjlim} (\mathsf{ev}_{Y_i})(y(f(Z)))\\
&\simeq (f_! \circ Q)(y(Z)).
\end{aligned}$$

This proves that for any presheaf $\mathcal{F} \in \widehat{Psh}\left(\mathbf{Aff}^{\mathbf{ft}}_S\right)$, $f_!\mathcal{F}$ is limit-preserving. Conversely, suppose that $\mathcal{G} \in \widehat{\mathrm{Psh}}(\mathbf{Aff}_S)$ is limit-preserving. Notice that

$$f_! f^* \mathcal{G} \simeq \underset{f(T)\to\mathcal{G}}{\varinjlim} y(f(T)),$$

with T ranging over $\mathbf{Aff}^{\mathbf{ft}}_S$. Let $X \in \mathbf{Aff}_S$, and write $X = \underset{i\in\mathcal{J}}{\varprojlim} X_i$ as a cofiltered limit of affine S-schemes, then

$$\begin{aligned}
f_! f^* \mathcal{G}(X) &\simeq \underset{f(T)\to\mathcal{G}}{\varinjlim} \mathbb{M}\mathrm{ap}_{\mathbf{Aff}_S}(X, f(T))\\
&\simeq \underset{f(T)\to\mathcal{G}}{\varinjlim}\ \underset{i\in\mathcal{J}^{op}}{\varinjlim} \mathbb{M}\mathrm{ap}_{\mathbf{Aff}_S}(X_i, f(T))\\
&\simeq \underset{i\in\mathcal{J}^{op}}{\varinjlim}\ \underset{f(T)\to\mathcal{G}}{\varinjlim} \mathbb{M}\mathrm{ap}_{\mathbf{Aff}_S}(X_i, f(T))\\
&\simeq \underset{i\in\mathcal{J}^{op}}{\varinjlim} \mathcal{G}(X_i)\\
&\simeq \mathcal{G}(X),
\end{aligned}$$

with the last equivalence following from the fact that $\mathcal{G}$ is limit-preserving. Hence $\mathcal{G}$ is in the essential image of $f_!$. □

Lemma 5.4. *Let $\mathcal{G} \in \widehat{\mathrm{Sh}}\left(\mathbf{Aff}^{\mathbf{ft}}_S, ét\right)$, and denote by i the inclusion of sheaves into presheaves. If $\mathcal{G}$ is n-truncated for finite n, then $f_! i(\mathcal{G})$ is an étale sheaf on $\mathbf{Aff}_S$.*

Proof. Suppose that $X \in \mathbf{Aff}_S$ and that $\mathcal{V} = (V_j \to X)_j$ is an étale cover of X. Write $X = \underset{i\in\mathcal{J}}{\varprojlim} X_i$, as a cofiltered limit of affine S-schemes of finite presentation. By [22, Lemma 34.13.2.], every étale cover of X is obtained via base-change from an étale cover of X_i for some i, so there is an étale cover $\mathcal{U}_i = (U_j \to X_i)_j$ of X_i, such that $\mathcal{V} = (V_j \times_{X_i} X \to X)_j$. Notice moreover that for each j,

$$V_j \times_{X_i} X \simeq \underset{l\geq i}{\varprojlim} V_j \times_{X_i} X_l.$$

Let $S_{\mathcal{V}}$ be the covering sieve of X corresponding to this cover, and let $\mathcal{F}$ be any limit-preserving presheaf. Then we have:

$$\begin{aligned}
\mathbb{M}\mathrm{ap}\left(S_{\mathcal{V}},\mathcal{F}\right) &\simeq \varprojlim\left[\prod_j \mathcal{F}\left(X\times_{X_i}V_i^j\right)\rightrightarrows\prod_{j,k}\mathcal{F}\left(X\times_{X_i}V_i^{jk}\right)\overrightarrow{\rightrightarrows}\ldots\right]\\
&\simeq \varprojlim\left[\prod_j \mathcal{F}\left(\varprojlim_{l\geq i}X_l\times_{X_i}V_i^j\right)\rightrightarrows\prod_{j,k}\mathcal{F}\left(\varprojlim_{l\geq i}X_l\times_{X_i}V_i^{jk}\right)\overrightarrow{\rightrightarrows}\ldots\right]\\
&\simeq \varprojlim\left[\prod_j \varinjlim_{l\geq i}\mathcal{F}\left(X_l\times_{X_i}V_i^j\right)\rightrightarrows\prod_{j,k}\varinjlim_{l\geq i}\mathcal{F}\left(X_l\times_{X_i}V_i^{jk}\right)\overrightarrow{\rightrightarrows}\ldots\right],
\end{aligned}$$

where the last equivalence follows from the fact that $\mathcal{F}$ is limit-preserving. Now suppose that $\mathcal{F}=f_!i\left(\mathcal{G}\right)$. Then, since each of the fibered products is a finite type S-scheme, the last line becomes

$$\varprojlim\left[\prod_j \varinjlim_{l\geq i}\mathcal{G}\left(X_l\times_{X_i}V_i^j\right)\rightrightarrows\prod_{j,k}\varinjlim_{l\geq i}\mathcal{G}\left(X_l\times_{X_i}V_i^{jk}\right)\overrightarrow{\rightrightarrows}\ldots\right].$$

Moreover, since $\mathcal{G}$ is n-truncated for some finite n, by [8, Lemma 2.21], the above limit can be truncated to the first $(n+1)$-terms, and as such can be replaced with a finite limit. Since filtered colimits commute with finite limits, the expression can be rewritten as

$$\mathbb{M}\mathrm{ap}\left(S_{\mathcal{V}},f_!i\mathcal{G}\right)\simeq\varinjlim_{l\geq i}\varprojlim\left[\prod_j \mathcal{G}\left(X_l\times_{X_i}V_i^j\right)\rightrightarrows\prod_{j,k}\mathcal{G}\left(X_l\times_{X_i}V_i^{jk}\right)\overrightarrow{\rightrightarrows}\ldots\right].$$

But $\mathcal{G}$ is a sheaf, so this is equivalent to

$$\begin{aligned}
\mathbb{M}\mathrm{ap}\left(S_{\mathcal{V}},f_!i\mathcal{G}\right) &\simeq \varinjlim_{l\geq i}\mathcal{G}\left(X_l\right)\\
&\simeq \varinjlim_{l\in\mathcal{J}}\mathrm{Map}_{\widehat{\mathrm{Psh}}\left(\mathbf{Aff}^{\mathbf{ft}}_S\right)}\left(y\left(X_l\right),i\mathcal{G}\right)\\
&\simeq \varinjlim_{l\in\mathcal{J}^{op}}\mathrm{Map}_{\widehat{\mathrm{Psh}}\left(\mathbf{Aff}_S\right)}\left(f_!y\left(X_l\right),f_!i\mathcal{G}\right)\\
&\simeq \varinjlim_{l\in\mathcal{J}^{op}}\mathrm{Map}_{\widehat{\mathrm{Psh}}\left(\mathbf{Aff}_S\right)}\left(y\left(f\left(X_l\right)\right),f_!i\mathcal{G}\right)\\
&\simeq \varinjlim_{l\in\mathcal{J}^{op}}f_!i\mathcal{G}\left(X_l\right)\\
&\simeq f_!i\mathcal{G}\left(X\right),
\end{aligned}$$

with the last equivalence following from the fact that any presheaf in the essential image of $f_!$ is limit-preserving. □

Consider the left Kan extension

$$\begin{array}{ccc}
\widehat{\mathrm{Psh}}\left(\mathbf{Aff}^{\mathbf{ft}}_S,\acute{e}t\right) & \overset{\tilde{F}}{\dashrightarrow} & \widehat{\mathrm{Sh}}\left(\mathbf{Aff}_S,\acute{e}t\right),\\
\uparrow{\scriptstyle y} & & \uparrow{\scriptstyle y}\\
\mathbf{Aff}^{\mathbf{ft}}_S & \xrightarrow{\ f\ } & \mathbf{Aff}_S
\end{array}$$

this time into sheaves instead of presheaves. Since it is the unique colimit preserving functor that agrees on representables with the functor

$$\mathbf{Aff}_S^{\mathbf{ft}} \xrightarrow{f} \mathbf{Aff}_S \hookrightarrow \widehat{\mathrm{Sh}}\left(\mathbf{Aff}_S, \acute{e}t\right),$$

we must have it is $a \circ f_!$, where a denotes étale sheafification. Notice moreover that the restriction functor

$$f^* : \widehat{\mathrm{Psh}}\left(\mathbf{Aff}_S\right) \to \widehat{\mathrm{Psh}}\left(\mathbf{Aff}_S^{\mathbf{ft}}\right)$$

restricts to a functor

$$f^* : \widehat{\mathrm{Sh}}\left(\mathbf{Aff}_S, \acute{e}t\right) \to \widehat{\mathrm{Sh}}\left(\mathbf{Aff}_S^{\mathbf{ft}}, \acute{e}t\right).$$

It follows that $a \circ f_! \circ i$ is a colimit preserving functor

$$\widehat{\mathrm{Sh}}\left(\mathbf{Aff}_S^{\mathbf{ft}}, \acute{e}t\right) \to \widehat{\mathrm{Sh}}\left(\mathbf{Aff}_S, \acute{e}t\right)$$

which is left adjoint to the restriction of f^*. By the same reasoning as for $f_!$, $a \circ f_! \circ i$ is fully faithful. The same argument goes through for sheaves with values in n-groupoids for any finite n.

Definition 5.5. Let $\mathcal{X}$ be an étale sheaf on $\mathbf{Aff}_S$. We say that $\mathcal{X}$ is **locally of finite presentation** if there exists an effective epimorphism

$$\coprod_i X_i \to \mathcal{X},$$

with each X_i an affine S-scheme of finite presentation. Denote by $\mathfrak{St}_S^{\mathbf{lfp}}$ the ∞-category of $\mathcal{U}$-small stacks locally of finite presentation over S.

Proposition 5.6. *$\mathfrak{St}_S^{\mathbf{lfp}}$ is closed under $\mathcal{U}$-small colimits in* $\mathrm{Sh}\left(\mathbf{Aff}_S, \acute{e}t\right)$.

Proof. Let $F : \mathcal{J} \to \mathfrak{St}_S^{\mathbf{lfp}}$ and compute the colimit of F in $\mathrm{Sh}\left(\mathbf{Aff}_S, \acute{e}t\right)$. Then by [16, Lemma 6.2.3.13],

$$\coprod_{i \in \mathcal{J}} F(i) \to \varinjlim F$$

is an effective epimorphism. Since each $F(i)$ admits an effective epimorphism

$$\coprod_\alpha X_\alpha^i \to F(i)$$

with each X^i in $\mathbf{Aff}_S^{\mathbf{ft}}$, we have that

$$\coprod_{\alpha, i} X_\alpha^i \to \varinjlim F$$

is an effective epimorphism, and we conclude that $\varinjlim F$ is locally of finite presentation. □

Corollary 5.7. *If $V \in \mathcal{S}\mathrm{pc}$, its étale Betti stack $\Delta^{\acute{e}t}(V)$ is locally of finite presentation.*

Proof. The terminal object $*$ is clearly locally of finite presentation, and every Betti stack is a colimit of $*$. □

Let $\tau_{\leq n}\mathfrak{St}_S^{\mathbf{lfp}}$ denote the full subcategory of $\mathfrak{St}_S^{\mathbf{lfp}}$ on the n-truncated stacks. Denote by

$$w : \mathbf{Aff}_S^{\mathbf{ft}} \hookrightarrow \tau_{\leq n}\mathfrak{St}_S^{\mathbf{lfp}}$$

be the canonical inclusion. Then

$$w_* : \widehat{\mathrm{Psh}}_n\left(\mathbf{Aff}_S^{\mathbf{ft}}\right) \to \widehat{\mathrm{Psh}}_n\left(\tau_{\leq n}\mathfrak{St}_S^{\mathbf{lfp}}\right)$$

is fully faithful with left-exact left adjoint w^*, where the subscript n denotes presheaves in n-groupoids. Hence, $\widehat{\mathrm{Sh}}_n\left(\mathbf{Aff}_S^{\mathbf{ft}}, \acute{e}t\right)$ is a further left exact localization of $\widehat{\mathrm{Psh}}_n\left(\tau_{\leq n}\mathfrak{St}_S^{\mathbf{lfp}}\right)$. As this is an n-topos, this is a topological localization corresponding to a Grothendieck topology. Unwinding the definitions, a collection of maps

$$(\mathcal{X}_i \to \mathcal{X})_i$$

is a covering family if and only if for any map $T \to \mathcal{X}$ with $T \in \mathbf{Aff}_S^{\mathbf{ft}}$,

$$\coprod_i T \times_{\mathcal{X}} \mathcal{X}_i \to T$$

admits sections étale locally on T. But this is the same as asking for

$$\coprod_i \mathcal{X}_i \to \mathcal{X}$$

to be an effective epimorphism. Hence, letting $\mathcal{E}$ denote the Grothendieck topology on $\tau_{\leq n}\mathfrak{St}_S^{\mathbf{lfp}}$ generated by jointly epimorphic families, we have that w^* restricts to an equivalence

$$w^* : \widehat{\mathrm{Sh}}_n\left(\tau_{\leq n}\mathfrak{St}_S^{\mathbf{lfp}}, \mathcal{E}\right) \to \widehat{\mathrm{Sh}}_n\left(\mathbf{Aff}_S^{\mathbf{ft}}, \acute{e}t\right).$$

By definition of an effective epimorphism, every object of $\tau_{\leq n}\mathfrak{St}^{\mathbf{lfp}}$ is the colimit of the Čech nerve of any $\mathcal{E}$-cover of it, hence the Grothendieck topology $\mathcal{E}$ is subcanonical. Since $\tau_{\leq n}\mathfrak{St}^{\mathbf{lfp}}$ is $\mathcal{U}$-small, we can consider the left Kan extension of the inclusion

$$g : \tau_{\leq n}\mathfrak{St}_S^{\mathbf{lfp}} \hookrightarrow \widehat{\mathrm{Sh}}_n\left(\mathbf{Aff}_S, \acute{e}t\right)$$

along the Yoneda embedding

$$y : \tau_{\leq n}\mathfrak{St}_S^{\mathbf{lfp}} \hookrightarrow \widehat{\mathrm{Psh}}_n\left(\mathfrak{St}_n^{\mathbf{lfp}}\right).$$

Its right adjoint lands in $\mathcal{E}$-sheaves since g preserves colimits of Čech nerves of effective epimorphisms. It follows that the composite

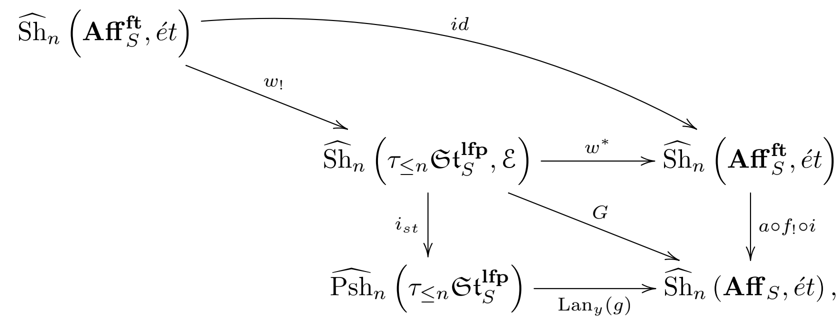
$$\mathrm{Lan}_y\, g \circ i_{st} : \widehat{\mathrm{Sh}}_n\left(\tau_{\leq n}\mathfrak{St}_S^{\mathbf{lfp}}, \mathcal{E}\right) \to \widehat{\mathrm{Sh}}_n\left(\mathbf{Aff}_S, \acute{e}t\right)$$

is colimit preserving, where

$$i_{st} : \widehat{\mathrm{Sh}}_n\left(\tau_{\leq n}\mathfrak{St}_S^{\mathbf{lfp}}, \mathcal{E}\right) \hookrightarrow \widehat{\mathrm{Psh}}_n\left(\tau_{\leq n}\mathfrak{St}_S^{\mathbf{lfp}}\right)$$

is the canonical inclusion.

Consider the following diagram, which we claim commutes:



where $G := \mathrm{Lan}_y(g) \circ i_{st}$. Since G is colimit preserving, as is w^* and $a \circ f_! \circ i$, to show that

$$G \simeq a \circ f_! \circ i \circ w^*,$$

it suffices to show that they agree on representables. However, the counit

$$w_! w^* \overset{\sim}{\Longrightarrow} id$$

is an equivalence, so it suffices to show that

$$G \circ w_! \simeq a \circ f_! \circ i,$$

and we are reduced to checking this on affine S-schems of finite type, as these are the representables. However, this is clear, by definition, since both functors send an affine S-scheme of finite type to its representable sheaf on $\mathbf{Aff}_S$. In particular, the essential image of $\mathrm{Lan}_y\, g \circ i_{st}$ and $a \circ f_! \circ i$ (where we are abusing notation by restricting the latter to n-truncated sheaves) agree. Since $\mathrm{Lan}_y\, g \circ i_{st}$ restricted to representables is just the inclusion

$$g : \tau_{\leq n}\mathfrak{St}^{\mathbf{lfp}}_S \hookrightarrow \widehat{\mathrm{Sh}}\left(\mathbf{Aff}_S, ét\right),$$

we conclude that $\tau_{\leq n}\mathfrak{St}^{\mathbf{lfp}}_S$ is contained in the essential image of $a \circ f_! \circ i$. However, by Lemma 5.4, $f_! \circ i$ (restricted to n-sheaves) is already included in $\widehat{\mathrm{Sh}}_n\left(\mathbf{Aff}_S, ét\right)$, so the sheafification functor a is superfluous. The following is an immediate corollary.

Corollary 5.8. *Any n-truncated stack on $\mathbf{Aff}_S$ locally of finite presentation is limit-preserving.*

Theorem 5.9. *A $\mathcal{U}$-small n-truncated étale sheaf $\mathcal{F}$ on $\mathbf{Aff}_S$ is limit-preserving precisely if it is locally of finite presentation.*

Proof. By Corollary 5.8, every $\mathcal{U}$-small n-truncated étale sheaf locally of finite presentation is limit-preserving. Conversely, suppose that $\mathcal{F}$ is a $\mathcal{U}$-small n-truncated limit preserving étale sheaf. Then it is in the essential image of $f_!$, so $\mathcal{F} \simeq f_! f^* \mathcal{F}$. Notice that

$$\int\limits_{\mathbf{Aff}^{\mathbf{ft}}} f^* \mathcal{F}$$

is $\mathcal{U}$-small since $\mathcal{F}$ is. It follows that $f_! f^* \mathcal{F}$ (and hence $\mathcal{F}$) is a $\mathcal{U}$-small colimit of affine S-schemes of finite presentation, in presheaves. However, since $\mathcal{F}$ is a sheaf, as is each of these affine S-schemes of finite presentation in the above diagram, $\mathcal{F}$ remains a $\mathcal{U}$-small colimit of affine S-schemes of finite presentation, with the colimit being taken in sheaves. Hence, $\mathcal{F}$ is locally of finite presentation itself, by Proposition 5.6. $\square$

Corollary 5.10. *For every n-truncated space V, $\Delta^{\acute{e}t}(V)$ is limit-preserving.*

The following corollary is immediate:

Corollary 5.11. *Let $Z = \varprojlim_{\alpha} Z_\alpha$ be a cofiltered limit of affine S-schemes. Then there is an equivalence of pro-truncated spaces*

$$\widehat{\Pi}^{\acute{e}t}_{\infty}(Z)_{<\infty} \simeq \varprojlim_{\alpha} \widehat{\Pi}^{\acute{e}t}_{\infty}(Z_\alpha)_{<\infty},$$

and therefore also an equivalence of profinite spaces

$$\widehat{\Pi}^{\acute{e}t}_{\infty}(Z) \simeq \varprojlim_{\alpha} \widehat{\Pi}^{\acute{e}t}_{\infty}(Z_\alpha),$$

Proposition 5.12. *Suppose that X is a fine saturated affine log scheme over S with a global Kato chart $X \to \operatorname{Spec}\mathbb{Z}[P]$. Let $\mathcal{X}$ be a limit preserving fpqc sheaf of spaces which is k-truncated for some $k < \infty$. Then the natural map*

$$\varinjlim_{n} \operatorname{Map}\left(\sqrt[n]{X}, \mathcal{X}\right) \to \operatorname{Map}\left(\sqrt[\infty]{X}, \mathcal{X}\right)$$

is an equivalence of spaces.

Proof. For every $n \in \mathbb{N}$, let U_n denote the fibered product $X \times_{\operatorname{Spec}\mathbb{Z}[P]} \operatorname{Spec}\mathbb{Z}[\frac{1}{n}P]$, and let U_∞ be the inverse limit, that is isomorphic to $X \times_{\operatorname{Spec}\mathbb{Z}[P]} \operatorname{Spec}\mathbb{Z}[P_{\mathbb{Q}}]$. Moreover, let $G_n = \boldsymbol{\mu}_n(P)$ denote the Cartier dual of the cokernel of the natural inclusion $P \to \frac{1}{n}P$, and let $G_\infty = \boldsymbol{\mu}_\infty(P)$ be the inverse limit. Recall that we have equivalences

$$\sqrt[n]{X} \simeq [U_n/G_n]$$

for the natural action of G_n on U_n (including $n = \infty$, if we use the fpqc topology), and these are compatible with the projections between root stacks and the quotient stacks. Notice that this implies that in the $(k+1,1)$-category of fpqc sheaves of k-groupoids,

$$\sqrt[n]{X} \simeq \varinjlim_{l \in \Delta^{op}_{\leq k}} \left(U_n \times G_n^l\right).$$

We have the following string of natural equivalences

$$\begin{aligned}
\operatorname{Map}\left(\sqrt[\infty]{X}, \mathcal{X}\right) &\simeq \varprojlim_{m \in \Delta_{\leq k}} \operatorname{Map}\left(U_\infty \times G_\infty^m, \mathcal{X}\right) \\
&\simeq \varprojlim_{m \in \Delta_{\leq k}} \mathcal{X}\left(U_\infty \times G_\infty^m\right) \\
&\simeq \varprojlim_{m \in \Delta_{\leq k}} \mathcal{X}\left(\varprojlim_{n} \left(U_n \times G_n^m\right)\right) \\
&\simeq \varprojlim_{m \in \Delta_{\leq k}} \varinjlim_{n} \mathcal{X}\left(U_n \times G_n^m\right)
\end{aligned}$$

Since the above colimit is a filtered colimit, it commutes with finite limits, so we have further that:

$$\begin{aligned}\mathbb{M}\mathrm{ap}\left(\sqrt[\infty]{X},\mathcal{X}\right) &\simeq \varinjlim_{n} \varprojlim_{m\in\Delta_{\leq k}} \mathbb{M}\mathrm{ap}\left(U_n\times G_n^m,\mathcal{X}\right)\\ &\simeq \varinjlim_{n} \mathbb{M}\mathrm{ap}\left(\varinjlim_{m\in\Delta_{\leq k}^{op}} \left(U_n\times G_n^m\right),\mathcal{X}\right)\\ &\simeq \varinjlim_{n} \mathbb{M}\mathrm{ap}\left(\sqrt[n]{X},\mathcal{X}\right).\end{aligned}$$

□

Lemma 5.13. *Suppose that X is a fine saturated affine log scheme over S with a global Kato chart. Let $\sqrt[\infty]{X}_{pro}$ denote the infinite root stack viewed as a pro-object and $\sqrt[\infty]{X}$ the actual limit of this pro-object, which is an fpqc-stack of groupoids. Then the canonical map*

$$\widehat{\Pi}_{\infty}^{\acute{e}t}\left(\sqrt[\infty]{X}_{pro}\right)\to\widehat{\Pi}_{\infty}^{\acute{e}t}\left(\sqrt[\infty]{X}\right)$$

between their profinite étale homotopy types is an equivalence.

Proof. By definition, $\widehat{\Pi}_{\infty}^{\acute{e}t}\left(\sqrt[\infty]{X}_{pro}\right)$ is the cofiltered limit in the ∞-category of profinite spaces

$$\varprojlim_{n} \widehat{\Pi}_{\infty}^{\acute{e}t}\left(\sqrt[n]{X}\right).$$

Recall that the ∞-category $\mathrm{Prof}(\mathcal{S}\mathrm{pc})$ is the opposite of the full-subcategory of $\mathrm{Fun}(\mathcal{S}\mathrm{pc}^{\pi},\mathcal{S}\mathrm{pc})$, on those functors

$$\mathcal{S}\mathrm{pc}^{\pi}\to\mathcal{S}\mathrm{pc}$$

from π-finite spaces to spaces which are accessible and preserve finite limits. In particular, filtered limits in $\mathrm{Prof}(\mathcal{S}\mathrm{pc})$ are computed as the filtered colimit in the functor category. So we have, as a functor, for all π-finite spaces V,

$$\begin{aligned}\widehat{\Pi}_{\infty}^{\acute{e}t}\left(\sqrt[\infty]{X}_{pro}\right)(V) &\simeq \varinjlim_{n} \widehat{\Pi}_{\infty}^{\acute{e}t}\left(\sqrt[n]{X}\right)(V)\\ &\simeq \varinjlim_{n} \Pi_{\infty}^{\acute{e}t}\left(\sqrt[n]{X}\right)(V)\end{aligned}$$

But, by [6, Theorem 2.40], we have

$$\Pi_{\infty}^{\acute{e}t}\left(\sqrt[n]{X}\right)(V)\simeq \mathbb{M}\mathrm{ap}_{\mathrm{Sh}(\mathbf{Sch}_S,\acute{e}t)}\left(\sqrt[n]{X},\Delta^{\acute{e}t}(V)\right),$$

for all n including $n=\infty$. But by Theorem 4.11 and Corollary 5.10, $\Delta^{\acute{e}t}(V)$ is a limit-preserving fpqc-sheaf, and since every π-finite spaces is k-truncated for some $k<\infty$, the result now follows from Proposition 5.12. □

The next result is a globalization of Lemma 5.13 to general fine saturated log schemes, which are not necessarily affine. It shows that two of the alternative definitions of the profinite homotopy type of a log scheme which we described in Section 1.2 of the Introduction do indeed coincide.

Theorem 5.14. *Let X be a fine saturated log scheme over S. Let $\sqrt[\infty]{X}_{pro}$ denote the infinite root stack viewed as a pro-object and $\sqrt[\infty]{X}$ the actual limit of this pro-object, which is an fpqc-stack of groupoids. Then the canonical map*

$$\widehat{\Pi}_{\infty}^{\acute{e}t}\left(\sqrt[\infty]{X}_{pro}\right) \to \widehat{\Pi}_{\infty}^{\acute{e}t}\left(\sqrt[\infty]{X}\right)$$

between their profinitely completed étale homotopy types is an equivalence.

Proof. Firstly, notice that since we are working with the profinitely completed étale homotopy type, we may work with hypersheaves rather than Čech sheaves, since the difference between the shape of an ∞-topos and its hypercompletion is erased by profinite completion. See e.g. [6, Proposition 2.16]. Choose an étale hypercover

$$U_{\bullet} : \Delta^{op} \to \mathbb{H}\mathrm{ypSh}\left(\mathbf{Sch}_S\right)/X$$

of X such that each U_k is the coproduct of fine saturated affine log schemes each of which admits a global Kato chart

$$U_k = \coprod_{\alpha \in I_k} X_{\alpha,k}.$$

Consider the pro-object $p : \sqrt[\infty]{X}_{pro} \to X$, which may be viewed as a functor

$$\mathbb{N} \to \mathbb{H}\mathrm{yp}\,\mathrm{Sh}\left(\mathbf{Sch}_S\right)/X.$$

Then, for each n including $n = \infty$

$$\left(p_n\right)^* U_{\bullet}$$

is a hypercover of $\sqrt[n]{X}$. By a completely analogous argument to the proof of [8, Lemma 6.1], we have that

$$\widehat{\Pi}_{\infty}^{\acute{e}t}\left(\sqrt[\infty]{X}_{pro}\right) \simeq \underset{k \in \Delta^{op}}{\varinjlim\mathrm{colim}}\left[\widehat{\Pi}_{\infty}^{\acute{e}t}\left(\varprojlim_{n} (p_n)^* U_k\right)\right].$$

Notice however that for each n and k we have canonical identifications

$$\left(p_n\right)^*\left(U_k\right) \simeq \coprod_{\alpha \in I_k}\left(\sqrt[n]{X} \times_X X_{\alpha,k}\right) \simeq \coprod_{\alpha \in I_k} \sqrt[n]{X_{\alpha,k}}.$$

By Corollary A.14, we therefore have that for each k,

$$\widehat{\Pi}_{\infty}^{\acute{e}t}\left(\varprojlim_{n} (p_n)^* U^k\right) \simeq \coprod_{\alpha \in I_k} \widehat{\Pi}_{\infty}^{\acute{e}t}\left(\sqrt[\infty]{X_{\alpha,k}}_{pro}\right).$$

Since each $X_{\alpha,k}$ is affine with a global Kato chart, by Lemma 5.13, we have for each k and each α

$$\widehat{\Pi}_{\infty}^{\acute{e}t}\left(\sqrt[\infty]{X_{\alpha,k}}_{pro}\right) \simeq \widehat{\Pi}_{\infty}^{\acute{e}t}\left(\sqrt[\infty]{X_{\alpha,k}}\right)$$

and hence

$$\begin{aligned}
\widehat{\Pi}^{\acute{e}t}_{\infty}\left(\sqrt[\infty]{X}_{pro}\right) &\simeq \underset{k\in\Delta^{op}}{\underrightarrow{\operatorname{colim}}} \coprod_{\alpha\in I_k} \widehat{\Pi}^{\acute{e}t}_{\infty}\left(\sqrt[\infty]{X_{\alpha,k}}\right)\\
&\simeq \underset{k\in\Delta^{op}}{\underrightarrow{\operatorname{colim}}} \coprod_{\alpha\in I_k} \widehat{\Pi}^{\acute{e}t}_{\infty}\left(\sqrt[\infty]{X}\times_X X_{\alpha,k}\right)\\
&\simeq \widehat{\Pi}^{\acute{e}t}_{\infty}\left(\underset{k\in\Delta^{op}}{\underrightarrow{\operatorname{colim}}} \coprod_{\alpha\in I_k} \left(\sqrt[\infty]{X}\times_X X_{\alpha,k}\right)\right)\\
&\simeq \widehat{\Pi}^{\acute{e}t}_{\infty}\left(\underset{k\in\Delta^{op}}{\underrightarrow{\operatorname{colim}}} \coprod_{\alpha\in I_k} (p_\infty)^*(U_k)\right)\\
&\simeq \widehat{\Pi}^{\acute{e}t}_{\infty}\left(\sqrt[\infty]{X}\right).
\end{aligned}$$

□

6. Shape comparison with the Kummer étale topos

In this section we accomplish another piece of our general comparison result. Namely, we show that the shape of the Kummer étale topos is equivalent, after profinite completion (in fact after protruncation), to the étale homotopy type of the infinite root stack. The infinite root stack is a stack of groupoids (1-truncated spaces) on the big étale site. For a general sheaf of spaces $\mathcal{X}$ on the big étale site, its étale homotopy type is defined as the shape of its small étale topos. The latter topos defined abstractly via left Kan extension, and thus expresses the small étale topos as a colimit of étale topois of affine schemes. However for sheaves of groupoids, we show that the small étale ∞-topos can be described fairly explicitly. Namely, we show it is equivalent as an ∞-category to the subcategory of the slice ∞-topos over $\mathcal{X}$ whose objects are maps $\mathcal{Y}\to\mathcal{X}$ which are étale and representable by higher Deligne-Mumford stacks.

There is however an alternative definition of the étale homotopy type of $\mathcal{X}$, which clearly coincides with the previous one whenever $\mathcal{X}$ is a scheme. Namely, since algebraic spaces are a special case of Deligne-Mumford stacks, we could look at subcategory of the above small étale ∞-topos on the maps $\mathcal{Y}\to\mathcal{X}$ which are étale and *representable by an algebraic space.* This subcategory carries a natural Grothendieck topology, and we can define the étale homotopy type as the shape of the associated ∞-topos. In principle this would give rise to a different pro-space, which is an alternative encoding of the étale homotopy theory of $\mathcal{X}$. In this section we show that, though different in general, this pro-space is equivalent to the standard étale homotopy type *after pro-truncation* whenever $\mathcal{X}$ is a classical stack, i.e. a sheaf of groupoids. It turns out that in log geometry the shape of the Kummer étale topos captures precisely this latter, less standard, notion of the étale shape of the infinite root stack. Thus as a consequence of our general comparison result we obtain that the shape of the Kummer étale topos is equivalent to the étale homotopy type of the infinite root stack, which is the main result of this section.

Fix a stack $\mathcal{X}$ such that there exists a small subcategory $\mathbf{Aff}'$ of affine schemes, closed under finite limits and étale morphisms, and containing the empty scheme, such that $\mathcal{X}$ is determined by its restriction to $\mathbf{Aff}'$. More precisely, if

$$\lambda : \mathbf{Aff}' \hookrightarrow \mathbf{Aff}$$

is the fully faithful inclusion, we have that

$$\mathcal{X} \simeq \lambda_! \lambda^* \mathcal{X},$$

where $\lambda_! \dashv \lambda^*$. The point of this is to find a *small* subsite of **Aff** over which $\mathcal{X}$ can be faithfully represented by its functor of points. We say in this case that $\mathcal{X}$ is *prolonged* from the small subsite $\mathbf{Aff}'$. Most stacks of interest are prolonged from a small subsite of **Aff**, for example any stack locally of finite type is prolonged from subsite of affine schemes of finite type. Also, any algebraic stack, locally of finite type or not, is prolonged from a small subsite. Indeed, if $\mathcal{G}$ is a groupoid object in algebraic spaces representing $\mathcal{X}$, then $\mathcal{G}_0$ and $\mathcal{G}_1$ are algebraic spaces, and we can find an étale cover of each of these algebraic spaces by affine schemes. If we take $\mathbf{Aff}'$ be the collection of affine schemes we used to form these étale covers, and close it under the above operations, then $\mathcal{X}$ is equivalent to the left Kan extension of its restriction to $\mathbf{Aff}'$. In what follows, we will be fixing a fixed small subsite $\mathbf{Aff}'$ once and for all, with the understanding that we can enlarge $\mathbf{Aff}'$ whenever necessary to include a site of definition for a given stack $\mathcal{X}$.

Definition 6.1. Let $\mathcal{X} \in \mathrm{Sh}\left(\mathbf{Aff}', \acute{e}t\right)$. Denote by $\mathcal{X}_{\acute{e}t}^{\mathfrak{DM}_{rep}}$ the full subcategory of $\mathrm{Sh}\left(\mathbf{Aff}', \acute{e}t\right)/\mathcal{X}$ on those maps $\mathcal{Y} \to \mathcal{X}$ such that for any map $f : T \to \mathcal{X}$ from a scheme, $\mathcal{Y} \times_{\mathcal{X}} T \to T$ is representable by a higher Deligne-Mumford stack étale over T.

Lemma 6.2. *For all $\mathcal{X}$ as above, there is a canonical equivalence of ∞-categories*

$$\mathcal{X}_{\acute{e}t}^{\mathfrak{DM}_{rep}} \simeq \varprojlim_{T \to \mathcal{X}} T_{\acute{e}t}^{\mathfrak{DM}_{rep}},$$

where the limit ranges over $\mathbf{Aff}'/\mathcal{X}$.

Proof. Consider the functor

$$\begin{array}{rcl} \chi : \mathrm{Sh}\left(\mathbf{Aff}', \acute{e}t\right) & \to & \mathfrak{Top}_\infty \\ \mathcal{Y} & \mapsto & \mathrm{Sh}\left(\mathbf{Aff}', \acute{e}t\right)/\mathcal{Y}. \end{array}$$

By [16, Proposition 6.3.5.14], χ preserves small colimits. Note that by the Yoneda lemma,

$$\mathcal{X} \simeq \varinjlim_{T \to \mathcal{X}} T,$$

where the colimit ranges over $\mathbf{Aff}'/\mathcal{X}$. Thus, there is an equivalence in $\mathfrak{Top}_\infty$

$$\mathrm{Sh}\left(\mathbf{Aff}', \acute{e}t\right)/\mathcal{X} \simeq \varinjlim_{T \to \mathcal{X}} \mathrm{Sh}\left(\mathbf{Aff}', \acute{e}t\right)/T.$$

Moreover, by [16, Proposition 6.3.2.3], this colimit is computed by taking the limits of the underlying ∞-categories in the opposite direction, using the inverse-image functors. In other words the canonical map

$$\theta : \mathrm{Sh}\left(\mathbf{Aff}', \acute{e}t\right)/\mathcal{X} \to \varprojlim_{T \to \mathcal{X}} \mathrm{Sh}\left(\mathbf{Aff}', \acute{e}t\right)/T$$

is an equivalence in $\widehat{\mathbf{Cat}}_\infty$— the (very large) ∞-category of large ∞-categories.[2]

Since each inverse image functor in the diagram induced by χ has a left adjoint, θ has a left adjoint τ, which is moreover an equivalence since θ is. Indeed, [10,

[2]Here we mean the ∞-category of $\mathcal{V}$-small ∞-categories, where $\mathcal{V}$ is our fixed universe of large sets.

Corollary 1.3 and Theorem B] give a very nice explicit description of this left adjoint τ. It is given by

$$\lim_{\overleftarrow{T\to\mathcal{X}}} \operatorname{Sh}\left(\mathbf{Aff}', \acute{e}t\right)/T \xrightarrow{\rho} \operatorname{Fun}\left(\mathbf{Aff}'/\mathcal{X}, \operatorname{Sh}\left(\mathbf{Aff}', \acute{e}t\right)/\mathcal{X}\right) \xrightarrow{\overrightarrow{\text{colim}}} \operatorname{Sh}\left(\mathbf{Aff}', \acute{e}t\right)/\mathcal{X},$$

where for an object $P \in \lim_{\overleftarrow{T\to\mathcal{X}}} \operatorname{Sh}\left(\mathbf{Aff}', \acute{e}t\right)/T$,

$$\begin{aligned} \rho(P) : \mathbf{Aff}'/\mathcal{X} &\to \operatorname{Sh}\left(\mathbf{Aff}', \acute{e}t\right)/\mathcal{X} \\ f : T \to \mathcal{X} &\mapsto f_!\left(\pi_f(P)\right) = P_f \to T \xrightarrow{f} \mathcal{X}, \end{aligned}$$

where for each $f : T \to \mathcal{X}$ in $\mathbf{Aff}'/\mathcal{X}$,

$$\pi_f : \lim_{\overleftarrow{T'\to\mathcal{X}}} \operatorname{Sh}\left(\mathbf{Aff}', \acute{e}t\right)/T' \to \operatorname{Sh}\left(\mathbf{Aff}', \acute{e}t\right)/T,$$

is the component of the limiting cone

$$\pi : \Delta_{\lim_{\overleftarrow{T'\to\mathcal{X}}} \operatorname{Sh}(\mathbf{Aff}', \acute{e}t)/T'} \Rightarrow \chi$$

along f, and the object $\pi_f(P)$ in the slice ∞-category $\operatorname{Sh}\left(\mathbf{Aff}', \acute{e}t\right)/T$ has domain P_f. In other words, we have

$$\tau(P) \simeq \underset{\overrightarrow{f:T\to\mathcal{X}}}{\text{colim}}\, \pi_f(P).$$

Notice moreover that the functor θ satisfies that property that for all $f : T \to \mathcal{X}$ in $\mathbf{Aff}'/\mathcal{X}$,

$$\pi_f \circ \theta \simeq f^*,$$

where

$$f^* : \operatorname{Sh}\left(\mathbf{Aff}', \acute{e}t\right)/\mathcal{X} \to \operatorname{Sh}\left(\mathbf{Aff}', \acute{e}t\right)/T$$

is the functor given by pulling back along f, i.e. $\mathcal{Y} \to \mathcal{X}$ is sent to $T \times_{\mathcal{X}} \mathcal{Y} \to T$.

Let

$$p : \int_{\mathbf{Aff}'/\mathcal{X}} \chi \to \mathbf{Aff}'/\mathcal{X}$$

denote the Cartesian fibration classified by the functor χ. Objects of $\int_{\mathbf{Aff}'/\mathcal{X}} \chi$ may be identified with pairs $(f : T \to \mathcal{X}, \mathcal{G})$, with $\mathcal{G} \to T \in \operatorname{Sh}\left(\mathbf{Aff}'\right)/T$. For each affine scheme T, $T_{\acute{e}t}^{\mathfrak{DM}_{rep}}$ is the full subcategory of $\operatorname{Sh}\left(\mathbf{Aff}', \acute{e}t\right)/T$ spanned by maps which are representable by a higher Deligne-Mumford stack étale over T. Denote by $\mathcal{D} \subseteq \int_{\mathbf{Aff}'/\mathcal{X}} \chi$ the full subcategory of $\int_{\mathbf{Aff}'/\mathcal{X}} \chi$ on those pairs $(f : T \to \mathcal{X}, \mathcal{G})$, with $\mathcal{G} \in T_{\acute{e}t}^{\mathfrak{DM}_{rep}}$. For every morphism $\varphi : T \to T'$ in $\mathbf{Aff}'$, the functor

$$\varphi^* : \operatorname{Sh}\left(\mathbf{Aff}'\right)/T' \to \operatorname{Sh}\left(\mathbf{Aff}'\right)/T$$

carries $T'^{\mathfrak{DM}_{rep}}_{\acute{e}t}$ to $T_{\acute{e}t}^{\mathfrak{DM}_{rep}}$. It follows that $p|_{\mathcal{D}}$ is the cartesian fibration classified by the functor

$$f : T \to \mathcal{X} \mapsto T_{\acute{e}t}^{\mathfrak{DM}_{rep}},$$

and the inclusion

$$\mathcal{D} \hookrightarrow \int_{\mathbf{Aff}'/\mathcal{X}} \chi$$

preserves and reflects Cartesian edges. Hence, the ∞-category of Cartesian sections of $p|_{\mathcal{D}}$ is canonically a full subcategory of the ∞-category of Cartesian sections of p. Since ∞-categories of Cartesian sections are models for the ∞-limit of ∞-categories, we conclude that there is a canonical realization of $\varprojlim_{T\to\mathcal{X}} T_{\acute{e}t}^{\mathfrak{DM}_{rep}}$ as a full subcategory of $\varprojlim_{T\to\mathcal{X}} \operatorname{Sh}\left(\mathbf{Aff}', \acute{e}t\right)/T$. Moroever, an object $P \in \varprojlim_{T\to\mathcal{X}} \operatorname{Sh}\left(\mathbf{Aff}', \acute{e}t\right)/T$ is in the full subcategory $\varprojlim_{T\to\mathcal{X}} T_{\acute{e}t}^{\mathfrak{DM}_{rep}}$ if and only if for all $f : T \to \mathcal{X}$ in $\mathbf{Aff}'/\mathcal{X}$, $\pi_f(P) \in T_{\acute{e}t}^{\mathfrak{DM}_{rep}}$.

It follows from the above discussion that θ restricts to a functor

$$\theta : \mathcal{X}_{\acute{e}t}^{\mathfrak{DM}_{rep}} \to \varprojlim_{T\to\mathcal{X}} T_{\acute{e}t}^{\mathfrak{DM}_{rep}}.$$

We claim that the left adjoint functor τ also restricts to a functor in the opposite direction. Note that this would imply the lemma, since then the restriction of τ would be the inverse of the restriction of θ.

Let us prove the claim. The claim is an object-wise condition, so it suffices to prove that for any $P \in \varprojlim_{T\to\mathcal{X}} T_{\acute{e}t}^{\mathfrak{DM}_{rep}}$, we must have $\tau(P) \in \mathcal{X}_{\acute{e}t}^{\mathfrak{DM}_{rep}}$. Let $f : T \to \mathcal{X}$. We want to show that $T \times_{\mathcal{X}} \tau(P) \to T$ in in $T_{\acute{e}t}^{\mathfrak{DM}_{rep}}$. By the fact that π is a cone for χ and θ is the morphism induced by π, we have that

$$T \times_{\mathcal{X}} \tau(P) = f^*\tau(P) \simeq \pi_f \theta(\tau(P)).$$

But $\theta\tau P \simeq P$, and so

$$T \times_{\mathcal{X}} \tau(P) \simeq \pi_f P \in T_{\acute{e}t}^{\mathfrak{DM}_{rep}}.$$

This implies that $\tau(P) \in \mathcal{X}_{\acute{e}t}^{\mathfrak{DM}_{rep}}$, completing the proof. □

Corollary 6.3. *For all $\mathcal{X}$ as in Lemma 6.2, the ∞-category $\mathcal{X}_{\acute{e}t}^{\mathfrak{DM}_{rep}}$ is an ∞-topos, and in $\mathfrak{Top}_\infty$, we have*

$$\mathcal{X}_{\acute{e}t}^{\mathfrak{DM}_{rep}} = \varinjlim_{T\to\mathcal{X}} \operatorname{Sh}(T_{\acute{e}t}),$$

where $\operatorname{Sh}(T_{\acute{e}t})$ is the small étale ∞-topos of the affine scheme T.

Proof. For all T, there is a canonical equivalence of ∞-categories

$$\operatorname{Sh}(T_{\acute{e}t}) \simeq T_{\acute{e}t}^{\mathfrak{DM}_{rep}},$$

so each ∞-category $T_{\acute{e}t}^{\mathfrak{DM}_{rep}}$ is an ∞-topos. By [16, Proposition 6.3.2.3], the ∞-category $\mathfrak{Top}_\infty$ of ∞-topoi has small colimits and they are computed as limits of the underlying ∞-categories *in the opposite direction,* using the inverse-image functors. Therefore, the underlying ∞-category of the above colimit of ∞-topoi is

$$\varprojlim_{T\to\mathcal{X}} \operatorname{Sh}(T_{\acute{e}t}).$$

Let $\varphi : T \to T'$ be a morphism in $\mathbf{Aff}'$, and let $\mathcal{F}$ be a sheaf in $\operatorname{Sh}(T'_{\acute{e}t})$. Then we have a pullback diagram of Deligne-Mumford stacks (modelled as strictly Henselian ringed ∞-topoi) of the form:

$$\begin{array}{ccc} \operatorname{Sh}(T_{\acute{e}t})/\varphi^*\mathcal{F} & \longrightarrow & \operatorname{Sh}(T'_{\acute{e}t})/\mathcal{F} \\ \downarrow & & \downarrow \\ \operatorname{Sh}(T_{\acute{e}t}) & \xrightarrow[\varphi]{} & \operatorname{Sh}(T'_{\acute{e}t}) \end{array},$$

and hence a pullback diagram in $\mathrm{Sh}\left(\mathbf{Aff}'\right)$ of Deligne-Mumford stacks

$$\begin{array}{ccc} T\times_{T'}\tilde{\mathcal{F}} & \longrightarrow & \tilde{\mathcal{F}} \quad . \\ \downarrow & & \downarrow \\ T & \xrightarrow[\varphi]{} & T', \end{array}$$

where $\tilde{\mathcal{F}} \to T'$ is the functor of points of the étale morphism $\mathrm{Sh}\left(T'_{\acute{e}t}\right)/\mathcal{F} \to \mathrm{Sh}\left(T'_{\acute{e}t}\right)$ of structured ∞-topoi. It follows that there is a canonical identification

$$\varprojlim_{T\to\mathcal{X}} \mathrm{Sh}\left(T_{\acute{e}t}\right) \simeq \varprojlim_{T\to\mathcal{X}} T_{\acute{e}t}^{\mathfrak{DM}_{rep}}.$$

The result now follows from Lemma 6.2.

□

Remark 6.4. Using the definition of the small étale topos of a stack given in Definition 3.12 (see also [6, Definition 2.27]) we see that the ∞-topos $\mathcal{X}_{\acute{e}t}^{\mathfrak{DM}_{rep}}$ is equivalent to $\mathrm{Sh}\left(\mathcal{X}_{\acute{e}t}\right)$, and hence the shape of $\mathcal{X}_{\acute{e}t}^{\mathfrak{DM}_{rep}}$ is precisely the étale homotopy type of $\mathcal{X}$.

Lemma 6.5. *Let*

$$F : \mathcal{J} \to \mathcal{X}_{\acute{e}t}^{\mathfrak{DM}_{rep}}$$

be any functor from a small category. Then a cocone ρ

$$\begin{array}{ccc} \mathcal{J}^{\triangleright} & \xrightarrow{\rho} & \mathcal{X}_{\acute{e}t}^{\mathfrak{DM}_{rep}} \\ \uparrow & \nearrow_{F} & \\ \mathcal{J} & & \end{array}$$

is colimiting if and only if for every $f : T \to \mathcal{X}$ *in* $\mathbf{Aff}'/\mathcal{X}$,

$$\begin{array}{ccc} \mathcal{J}^{\triangleright} & \xrightarrow{f^*\rho} & \mathrm{Sh}\left(T_{\acute{e}t}\right) \\ \uparrow & \nearrow_{f^*F} & \\ \mathcal{J} & & \end{array}$$

is.

Proof. This holds true in both $\mathcal{X}_{\acute{e}t}^{\mathfrak{DM}_{rep}}$ and in $\mathrm{Sh}\left(\mathbf{Aff}', \acute{e}t\right)/\mathcal{X}$ (with each $\mathrm{Sh}\left(T_{\acute{e}t}\right)$ replaced with $\mathrm{Sh}\left(\mathbf{Aff}', \acute{e}t\right)/T$ in that case) and the proof is completely analogous. In either situation, the statement follows from $f : T \to \mathcal{X}$ being colimit preserving and jointly conservative.

□

Corollary 6.6. *A morphism* $\varphi : \mathcal{Z} \to \mathcal{Y}$ *in* $\mathcal{X}_{\acute{e}t}^{\mathfrak{DM}_{rep}}$ *is an effective epimorphism if and only if for all* $f : T \to \mathcal{X}$ *in* $\mathbf{Aff}'/\mathcal{X}$, $f^*\varphi : f^*\mathcal{Z} \to f^*\mathcal{Y}$ *is an effective epimorphism in* $\mathrm{Sh}\left(T_{\acute{e}t}\right)$.

Proof. This follows formally from Lemma 6.5 together with the fact that each inverse image functor f^* preseves pullbacks. □

Definition 6.7. Let $\mathcal{X}_{\acute{e}t}^{rep}$ denote the subcategory of $\mathcal{X}_{\acute{e}t}^{\mathfrak{DM}_{rep}}$ spanned by those étale maps $\mathcal{Y} \to \mathcal{X}$ which are representable by algebraic spaces (with no separation conditions).

Proposition 6.8. *The subcategory*

$$\mathcal{X}_{\acute{e}t}^{rep} \hookrightarrow \mathcal{X}_{\acute{e}t}^{\mathfrak{DM}_{rep}}$$

is precisely the 0*-truncated objects. Moreover, as a* 1*-category*

$$\mathcal{X}_{\acute{e}t}^{rep} \simeq \varprojlim_{T \to \mathcal{X}} \operatorname{Sh}\left(T_{\acute{e}t}, \mathbf{Set}\right).$$

Proof. By Corollary 6.3, $\mathcal{X}_{\acute{e}t}^{\mathfrak{DM}_{rep}}$ is an ∞-topos, so its subcategory of 0-truncatd objects can be identified with its 1-localic reflection. That is to say, the inclusion of the $(2,1)$-category of topoi into the $(\infty,1)$-category of ∞-topoi admits a left adjoint L, given by taking the full subcategory on the 0-truncated objects. Since L is a left adjoint, we have

$$\begin{aligned} L\left(\mathcal{X}_{\acute{e}t}^{\mathfrak{DM}_{rep}}\right) &\simeq L\left(\varinjlim_{T \to \mathcal{X}} \operatorname{Sh}\left(T_{\acute{e}t}\right)\right) \\ &\simeq \varinjlim_{T \to \mathcal{X}} L\left(\operatorname{Sh}\left(T_{\acute{e}t}\right)\right) \\ &\simeq \varinjlim_{T \to \mathcal{X}} \operatorname{Sh}\left(T_{\acute{e}t}, \mathbf{Set}\right), \end{aligned}$$

where the colimit is in the $(2,1)$-category of topoi. This colimit is computed by taking the limit of the corresponding diagram of categories and inverse-image functors, so we have as a 1-category

$$\mathcal{X}_{\acute{e}t}^{rep} \simeq \varprojlim_{T \to \mathcal{X}} \operatorname{Sh}\left(T_{\acute{e}t}, \mathbf{Set}\right).$$

By the proof of Lemma 6.2, we can conclude that $\mathcal{Y} \to \mathcal{X}$ in $\mathcal{X}_{\acute{e}t}^{\mathfrak{DM}_{rep}}$ is 0-truncated if and only if for all $f : T \to \mathcal{X}$, $f^*\mathcal{Y}$ is a 0-truncated sheaf on $T_{\acute{e}t}$. But by [13, Definition 2.6.4], this is the same as $f^*\mathcal{Y}$ being represented by an algebraic space étale over T. □

Lemma 6.9. *Let $\mathcal{X}$ be a stack of groupoids on a* 1*-categorical Grothendieck site* $(\mathscr{C}, \mathfrak{J})$. *Then if E is any sheaf of sets, and $f : E \to \mathcal{X}$ is any morphism, then f is a* 0*-truncated object of the ∞-topos* $\operatorname{Sh}(\mathscr{C}, \mathfrak{J})/\mathcal{X}$.

Proof. This follows from the general fact that if X is a 1-truncated object in an ∞-category $\mathscr{D}$, and $f : E \to X$ is a morphism from a 0-truncated object, that f is 0-truncated in the slice ∞-category $\mathscr{D}/X$. To see this, if $g : C \to X$ is any other object of $\mathscr{D}/X$, it suffices to prove that $\mathbb{M}\mathrm{ap}_{\mathscr{D}/X}(g, f)$ is 0-truncated, i.e. has no non-trivial homotopy groups beyond π_0. Notice however that we have a pullback diagram of spaces

$$\begin{array}{ccc} \mathbb{M}\mathrm{ap}_{\mathscr{D}/X}(g,f) & \longrightarrow & \mathbb{M}\mathrm{ap}(C,E) \\ \downarrow & & \downarrow f_* \\ * & \xrightarrow{\;g\;} & \mathbb{M}\mathrm{ap}(C,X), \end{array}$$

and since $\mathbb{M}\mathrm{ap}\,(C,E)$ is a set, so we can write

$$\mathbb{M}\mathrm{ap}\,(C,E) \simeq \coprod_{a\in\mathbb{M}\mathrm{ap}(C,E)} *,$$

and hence

$$\mathbb{M}\mathrm{ap}_{\mathscr{D}/X}\,(g,f) \simeq \coprod_{a\in\mathbb{M}\mathrm{ap}(C,E)} \left(* \times_{\mathbb{M}\mathrm{ap}(C,X)} *\right).$$

Each space $* \times_{\mathbb{M}\mathrm{ap}(C,X)} *$ is either empty or equivalent to the based loop space $\Omega_*\mathbb{M}\mathrm{ap}\,(C,X)$, for some basepoint. Since $\mathbb{M}\mathrm{ap}\,(C,X)$, is a groupoid, it has no non-trivial homotopy groups above π_1, hence each loop space is discrete. It follows that $\mathbb{M}\mathrm{ap}_{\mathscr{D}/X}\,(g,f)$ is discrete as well. □

Lemma 6.10. *Suppose that $\mathcal{X}$ is 1-truncated, i.e. a stack of groupoids. Let $\mathcal{Y}\to\mathcal{X}$ be any object in $\mathcal{X}^{\mathfrak{DM}_{rep}}_{\acute{e}t}$. Then there exists an object $Y\to\mathcal{X}$ in $\mathcal{X}^{rep}_{\acute{e}t}$ and an effective epimorphism $\pi : Y\to\mathcal{Y}$ in $\mathcal{X}^{\mathfrak{DM}_{rep}}_{\acute{e}t}$.*

Proof. Consider the two functors

$$\begin{array}{rcl}\chi : \left(\mathbf{Aff}'/\mathcal{X}\right)^{op} & \longrightarrow & \widehat{\mathbf{Cat}}_\infty\\ f:T\to\mathcal{X} & \mapsto & \mathrm{Sh}\left(\mathbf{Aff}',\acute{e}t\right)/T\end{array}$$

and

$$\begin{array}{rcl}\chi^{\acute{e}t} : \left(\mathbf{Aff}'/\mathcal{X}\right)^{op} & \longrightarrow & \widehat{\mathbf{Cat}}_\infty\\ f:T\to\mathcal{X} & \mapsto & \mathrm{Sh}\left(T_{\acute{e}t}\right).\end{array}$$

There is a canonical natural transformation $J^* : \chi\to\chi^{\acute{e}t}$ whose component along $f : T\to\mathcal{X}$ restricts a sheaf on $\mathbf{Aff}'/T$ to the subcategory $T_{\acute{e}t}$ of the étale maps. There is hence a well-defined induced functor, which we abusively write as

$$J^* : \mathrm{Sh}\left(\mathbf{Aff}',\acute{e}t\right)/\mathcal{X} \simeq \varprojlim_{f:T\to\mathcal{X}} \mathrm{Sh}\left(\mathbf{Aff}'/T,\acute{e}t\right) \to \varprojlim_{f:T\to\mathcal{X}} \mathrm{Sh}\left(T_{\acute{e}t}\right) \simeq \mathcal{X}^{\mathfrak{DM}_{rep}}.$$

Note that by Proposition 6.8, for any sheaf of sets $\mathcal{F}$ on $\mathbf{Aff}'/\mathcal{X}$, $J^*\mathcal{F}$ is in $\mathcal{X}^{rep}_{\acute{e}t}$. In particular, by Lemma 6.9, this holds for any representable sheaf on $\mathbf{Aff}'/\mathcal{X}$.

Denote by $A_\mathcal{X}$ the full subcategory of $\mathcal{X}^{rep}_{\acute{e}t}$ spanned by $J^*\,(f : T\to\mathcal{X}) =: \mathscr{S}^f$ for some $f : T\to\mathcal{X}$ in $\mathbf{Aff}'/\mathcal{X}$. Then $A_\mathcal{X}$ has a set of objects. Let K denote the set of 1-morphisms $\varphi : \mathscr{S}^f\to\mathcal{Y}$, with $f : T\to\mathcal{X}$ varying over $\mathbf{Aff}'/\mathcal{X}$. Let

$$\pi_\mathcal{Y} : W_\mathcal{Y} := \coprod_{(\varphi:\mathscr{S}^g\to\mathcal{Y})\in K} \mathscr{S}^g\to\mathcal{Y},$$

where $\pi_\mathcal{Y}$ is the obvious map. Then for any $f : T\to\mathcal{X}$ in $\mathbf{Aff}'/\mathcal{X}$, $f^*W_\mathcal{Y}$ is a coproduct of algebraic spaces, hence an algebraic space, so $W_\mathcal{Y}$ is in $\mathcal{X}^{rep}_{\acute{e}t}$. We claim that $\pi_\mathcal{Y}$ is an effective epimorphism.

By Corollary 6.6, it suffices to prove that for any $f : T\to\mathcal{X}$ in $\mathbf{Aff}'/\mathcal{X}$,

$$f^*\pi_\mathcal{Y} : f^*W_\mathcal{Y}\to f^*\mathcal{Y}$$

is an effective epimorphism in $\mathrm{Sh}\,(T_{\acute{e}t})$. By definition, $f^*J^*\,(f : T\to\mathcal{X})$ is the étale sheaf which assigns an étale map $T'\to T$ in $\mathbf{Aff}'$, the set

$$\mathbb{M}\mathrm{ap}_\mathcal{X}\,(T'\to T\to\mathcal{X}, f : T\to\mathcal{X}) \simeq \mathbb{M}\mathrm{ap}_{\mathrm{Sh}(\mathbf{Aff}'/T,\acute{e}t)}\,(T'\to T, T\times_\mathcal{X} T\to T).$$

In particular, $f^*J^*\,(f : T\to\mathcal{X})$ has a canonical global section corresponding to the diagonal map $T\to T\times_\mathcal{X} T$. Hence, for any étale map $T'\to T$ in $T_{\acute{e}t}$, up to isomorphism, any morphism $T' \to f^*\mathcal{Y}$ factors through some map $f^*\mathscr{S}^f \to f^*\mathcal{Y}$. Since such a map is a summand of $f^*W_{\mathscr{Y}} \to \mathcal{Y}$, $f^*\pi\mathcal{Y}$ is an epimorphism. □

Corollary 6.11. *If $\pi : \mathcal{Y} \to \mathcal{X}$ is any object in $\mathcal{X}_{\acute{e}t}^{\mathfrak{DM}_{rep}}$, then there exists a simplicial object*

$$U^\bullet : \Delta^{op} \to \mathcal{X}_{\acute{e}t}^{rep}$$

which is a hypercover of π. In particular, if π is a hypercomplete sheaf, then π is a colimit of objects in $\mathcal{X}_{\acute{e}t}^{rep}$.

Lemma 6.12. *Let $\mathscr{C}$ be a full subcategory of an ∞-topos $\mathcal{E}$ which is closed under finite limits and is strongly generating. Then $\mathcal{E}$ is a left exact localization of* $\mathrm{Psh}(\mathscr{C})$.

Proof. Recall that strongly generating means that if $i : \mathscr{C} \hookrightarrow \mathcal{E}$ is the fully faithful inclusion, then the composite

$$\mathcal{E} \xrightarrow{i^* \circ y_{\mathcal{E}}} \mathrm{Psh}(\mathscr{C})$$

is fully faithful. This composite is right adjoint to $\mathrm{Lan}_{y_{\mathscr{C}}} i$, and hence $\mathcal{E}$ is exhibited as a localization of $\mathrm{Psh}(\mathscr{C})$. By [16, Proposition 6.1.5.2], the left adjoint $\mathrm{Lan}_{y_{\mathscr{C}}} i$ is moreover left exact. □

Theorem 6.13. *Let $\mathcal{X}$ be any sheaf of groupoids on* $\mathbf{Aff}'$. *Equip $\mathcal{X}_{\acute{e}t}^{rep}$ with the Grothendieck pretopology generated by families of jointly surjective effective epimorphisms. Let $\mathrm{Sh}(\mathcal{X}_{\acute{e}t}^{rep})$ be the subcategory of the functor category $\mathrm{Fun}\left((\mathcal{X}_{\acute{e}t}^{rep})^{op}, \mathcal{S}\mathrm{pc}\right)$ on those presheaves which satisfy hyperdescent with respect to the above Grothendieck topology. Then $\mathrm{Sh}(\mathcal{X}_{\acute{e}t}^{rep})$ is an ∞-topos and there is a geometric morphism of ∞-topoi*

$$\mathcal{X}_{\acute{e}t}^{\mathfrak{DM}_{rep}} \to \mathrm{Sh}(\mathcal{X}_{\acute{e}t}^{rep}),$$

which induces an equivalence between their respective hypercompletions.

Proof. Let $q : \mathscr{C} \hookrightarrow \mathcal{X}_{\acute{e}t}^{rep}$ be a small subcategory of the topos $\mathcal{X}_{\acute{e}t}^{rep}$ which is strongly generating, and which has finite limits. Then there is a subcanonical Grothendieck topology J on $\mathscr{C}$ such that

$$\mathrm{Sh}_J(\mathscr{C}, \mathbf{Set}) \simeq \mathcal{X}_{\acute{e}t}^{rep}. \tag{2}$$

Moreover, it is a standard fact that for any 1-topos $\mathcal{E}$, a functor $F : \mathcal{E}^{op} \to \mathbf{Set}$ satisfies descent with respect to covers by jointly surjective epimorphisms if and only if F is in fact representable, hence there is a canonical identification

$$\mathcal{X}_{\acute{e}t}^{rep} \simeq \mathrm{Sh}(\mathcal{X}_{\acute{e}t}^{rep}, \mathbf{Set}),$$

with the epimorphism Grothendieck topology implicit. Under this identification, the equivalence (2) is the restriction of

$$q^* : \mathrm{Fun}\left((\mathcal{X}_{\acute{e}t}^{rep})^{op}, \mathbf{Set}\right) \to \mathrm{Fun}(\mathscr{C}^{op}, \mathbf{Set}).$$

Let $\mathcal{V}$ denote the Grothendieck universe of large sets. The canonical inclusion q moreover induces a geometric morphism between ∞-topoi (in the universe $\mathcal{V}$) of presheaves of $\mathcal{V}$-small spaces (i.e. large spaces) $\widehat{\mathrm{Psh}}(\mathscr{C}) \to \widehat{\mathrm{Psh}}(\mathcal{X}_{\acute{e}t}^{rep})$, which by abuse of notation we will denote by the pair (q_*, q^*). Since q is fully faithful, so is q_*, and hence this exhibits $\widehat{\mathrm{Psh}}(\mathscr{C})$ as a left-exactly localization of $\widehat{\mathrm{Psh}}(\mathcal{X}_{\acute{e}t}^{rep})$. This localization is automatically topological since both ∞-topoi are 1-localic. Notice that the ∞-category of sheaves of $\mathcal{V}$-small spaces $\widehat{\mathrm{Sh}}_J(\mathscr{C})$ is a further topological localization. By composition, this exhibits $\widehat{\mathrm{Sh}}_J(\mathscr{C})$ as a topological localization of $\widehat{\mathrm{Psh}}(\mathfrak{X}^{rep}_{\acute{e}t})$. Thus $\widehat{\mathrm{Sh}}_J(\mathscr{C})$ is equivalent to sheaves of larges spaces over $\mathfrak{X}^{rep}_{\acute{e}t}$ for some Grothendieck topology. Unwinding the definitions, a family of maps $(X_i \to X)_i$ in $\mathfrak{X}^{rep}_{\acute{e}t}$ is a cover if and only if the sieve obtained as the colimit of the Čech-nerve of

$$\coprod_i y(X_i) \to X_i$$

becomes a covering J-covering sieve after applying q^*. This is the same as asking that

$$\coprod_i C \times_{X_i} X \to C$$

admits sections J-locally on C, but since $\mathscr{C}$ is a site for $\mathfrak{X}^{rep}_{\acute{e}t}$, this is true if and only if

$$\coprod_i X_i \to X$$

is an effective epimorphism. Thus the ∞-category $\widehat{\mathrm{Sh}}(\mathfrak{X}^{rep}_{\acute{e}t})$ of sheaves of $\mathcal{V}$-small spaces on $\mathfrak{X}^{rep}_{\acute{e}t}$ with respect to the effective epimorphism topology is in fact an ∞-topos in the universe $\mathcal{V}$, and (abusing notation)

$$q^* : \widehat{\mathrm{Sh}}(\mathfrak{X}^{rep}_{\acute{e}t}) \xrightarrow{\sim} \widehat{\mathrm{Sh}}_J(\mathscr{C})$$

is an equivalence. Moreover, since q^* is the restriction functor, it further restricts to an equivalence between their subcategories of $\mathcal{U}$-small objects, i.e. we have an equivalence

$$\mathrm{Sh}(\mathfrak{X}^{rep}_{\acute{e}t}) \xrightarrow{\sim} \mathrm{Sh}(\mathscr{C}).$$

In particular, we conclude that $\mathrm{Sh}(\mathfrak{X}^{rep}_{\acute{e}t})$ is an ∞-topos (which is moreover 1-localic).

Consider the left Kan extension along the Yoneda embedding

$$\begin{array}{ccc} \mathrm{Psh}(\mathscr{C}) & \xrightarrow{\mathrm{Lan}_y(i \circ q)} & \mathfrak{X}^{\mathfrak{DM}_{rep}}_{\acute{e}t} \\ {\scriptstyle y}\uparrow & & \uparrow{\scriptstyle i} \\ \mathscr{C} & \xrightarrow[q]{} & \mathfrak{X}^{rep}_{\acute{e}t}, \end{array}$$

which is colimit preserving. Since $\mathscr{C}$ is closed under finite limits, q preserves finite limits, and the inclusion

$$i : \mathfrak{X}^{rep}_{\acute{e}t} \hookrightarrow \mathfrak{X}^{\mathfrak{DM}_{rep}}_{\acute{e}t}$$

does as well, so by [16, Proposition 6.1.5.2], $\mathrm{Lan}_y(i \circ q)$ is left exact. It moreover has a canonical right adjoint R, which by the Yoneda lemma is easily seen to be given by the formula:

$$R(\pi : \mathcal{Y} \to \mathfrak{X})(c) \simeq \mathbb{M}\mathrm{ap}(iq(c), \pi).$$

We claim that for all such $\pi : \mathcal{Y} \to \mathfrak{X}$ in $\mathfrak{X}^{\mathfrak{DM}_{rep}}_{\acute{e}t}$, $R(\pi)$ is a J-sheaf.

To see why this claim is true, let $\mathcal{U} = (c_i \to c)$ be a J-cover of c in $\mathscr{C}$. Consider the Čech nerve of

$$\coprod_i y(c_i) \to y(c),$$

$$\check{\mathrm{C}} : \Delta^{op} \to \mathrm{Psh}(\mathscr{C}).$$

Its colimit

$$S_{\mathcal{U}} := \varinjlim_{n\in\Delta^{op}} \check{\mathrm{C}}^n$$

is a subobject of $y(c)$, and hence 0-truncated, since $y(c)$ is. Since $\mathrm{Lan}_y(i\circ q)$ is left exact, it preserves 0-truncated objects by Proposition [16, 5.5.6.16], so $\mathrm{Lan}_y(i\circ q)(S_{\mathcal{U}})$ is 0-truncated. Notice that $i : \mathcal{X}^{rep}_{\acute{e}t} \hookrightarrow \mathcal{X}^{\mathfrak{D}\mathfrak{M}_{rep}}_{\acute{e}t}$ is precisely the inclusion of the 0-truncated objects, and hence has a left adjoint $\tau_{\leq 0}$. Since $\mathrm{Lan}_y(i\circ q)(S_{\mathcal{U}})$ is 0-truncated, we have

$$\mathrm{Lan}_y(i\circ q)(S_{\mathcal{U}}) \simeq i\circ\tau_{\leq 0}\,\mathrm{Lan}_y(i\circ q)(S_{\mathcal{U}}).$$

But

$$\begin{aligned}\tau_{\leq 0}\,\mathrm{Lan}_y(i\circ q)(S_{\mathcal{U}}) &\simeq \varinjlim_{n\in\Delta^{op}} q\left(\check{\mathrm{C}}^n\right)\\ &\simeq q(c),\end{aligned}$$

so

$$\mathrm{Lan}_y(i\circ q)(S_{\mathcal{U}}) \simeq iq(c).$$

Hence,

$$\begin{aligned}\mathbb{M}\mathrm{ap}(S_{\mathcal{U}}, R(\pi)) &\simeq \mathbb{M}\mathrm{ap}(\mathrm{Lan}_y(i\circ q)(S_{\mathcal{U}}), \pi)\\ &\simeq \mathbb{M}\mathrm{ap}(iq(c), \pi)\\ &\simeq \mathbb{M}\mathrm{ap}(y(c), R(\pi))\end{aligned}$$

which implies that $R(\pi)$ is a sheaf. It follows that $F^* := \mathrm{Lan}_y(i\circ q)(S_{\mathcal{U}})$ restricts to a left exact colimit preserving functor

$$F^* : \mathrm{Shv}(\mathscr{C}) \to \mathcal{X}^{\mathfrak{D}\mathfrak{M}_{rep}}_{\acute{e}t},$$

and hence constitutes a geometric morphism

$$F : \mathcal{X}^{\mathfrak{D}\mathfrak{M}_{rep}}_{\acute{e}t} \to \mathrm{Shv}(\mathscr{C}, J) \simeq \mathrm{Shv}(\mathcal{X}^{rep}_{\acute{e}t}).$$

The direct image functor of this geometric morphism, F_*, is simply the functor R we described above. We claim that F_* in fact preserves colimits. Firstly, notice that it follows from Lemma 6.5 that the canonical inclusion

$$\mathcal{X}^{\mathfrak{D}\mathfrak{M}_{rep}}_{\acute{e}t} \simeq \varprojlim_{f:T\to\mathcal{X}} T^{\mathfrak{D}\mathfrak{M}_{rep}}_{\acute{e}t} \hookrightarrow \varprojlim_{f:T\to\mathcal{X}} \mathrm{Sh}\left(\mathbf{Aff'}, \acute{e}t\right)/T \simeq \mathrm{Sh}\left(\mathbf{Aff'}, \acute{e}t\right)/\mathcal{X}$$

preserves colimits. Hence, if $g : \mathcal{Z}\to\mathcal{X}$ is in $\mathcal{X}^{rep}_{\acute{e}t}$, and if $\pi : \mathcal{Y}\to\mathcal{X}$ is $\varinjlim_{\alpha}(\pi_\alpha : \mathcal{Y}_\alpha \to \mathcal{X})$ in $\mathcal{X}^{\mathfrak{D}\mathfrak{M}_{rep}}_{\acute{e}t}$, then the space

$$F_*(\pi)(g)$$

can be identified with the space of maps in $\mathrm{Sh}\left(\mathbf{Aff'}\right)/\mathcal{X}$ from g to π. This is equivalent to global sections in the ∞-topos $\mathrm{Sh}\left(\mathbf{Aff'}, \acute{e}t\right)/\mathcal{Z}$ of the colimit of $g^*\mathcal{Y}_\alpha \to \mathcal{Z}$. Notice that

$$\mathrm{Sh}\left(\mathbf{Aff'}, \acute{e}t\right)/\mathcal{Z} \simeq \left(\mathrm{Sh}\left(\mathbf{Aff'}, \acute{e}t\right)/\mathcal{X}\right)/g,$$

and g is 0-truncated since for each $f : T\to\mathcal{X}$, f^*g is 0-truncated. So $\mathcal{Z}$ is 1-truncated, and we can conclude that $\mathrm{Sh}\left(\mathbf{Aff'}, \acute{e}t\right)/\mathcal{Z}$ is 1-localic, since

$$\mathrm{Sh}\left(\mathbf{Aff'}, \acute{e}t\right)/\mathcal{Z} \simeq \mathrm{Sh}\left(\mathbf{Aff'}/\mathcal{Z}, \acute{e}t\right).$$

So, the colimit $\underset{\alpha}{g^*\pi_\alpha}$ can be computed first at the level of presheaves on $\mathbf{Aff}'/\mathcal{Z}$, and then sheafified.

Similarly, the space $\left(\underset{\overrightarrow{\alpha}}{\operatorname{colim}} F_*(\pi_\alpha)\right)(g)$ can be identified with global sections of the colimit $\underset{\overrightarrow{\alpha}}{\operatorname{colim}}\, g^*\pi_\alpha$, this time in $\operatorname{Sh}(\mathcal{Z}^{rep}_{\acute{e}t})$. However, the covers of the terminal object of $\mathcal{Z}^{rep}_{\acute{e}t}$ are in canonical bijection with the covers of the terminal object in $\mathbf{Aff}'/\mathcal{Z}$, so the global sections coincide.

By abuse of notation, denote the corresponding geometric morphism between hypercompletions by

$$F : \widehat{\mathcal{X}^{\mathfrak{DM}_{rep}}_{\acute{e}t}} \to \mathbb{H}\operatorname{Sh}(\mathscr{C}, J) \simeq \mathbb{H}\operatorname{Sh}(\mathcal{X}^{rep}_{\acute{e}t}).$$

By a completely analogous proof as before, we have that F_* preserves colimits. We claim that the unit and co-unit of the adjunction $F^* \dashv F_*$ are equivalences.

Notice that for any $c, d \in \mathscr{C}$,

$$\begin{aligned} F_*F^*(y(c))(d) &\simeq F_*(iq(c))(d) \\ &\simeq \operatorname{Map}_{\mathcal{X}^{\mathfrak{DM}_{rep}}_{\acute{e}t}}(iq(d), iq(c)) \\ &\simeq \operatorname{Map}_{\mathscr{C}}(d, c) \\ &\simeq y(c)(d) \end{aligned}$$

So, the component of the unit $id_{\mathbb{H}\operatorname{Sh}(\mathscr{C},J)} \to F_*F^*$ is an equivalence along representables. However, all functors involved are colimit preserving, and the induced functor

$$y^* : \operatorname{Fun}^L(\mathbb{H}\operatorname{Sh}(\mathscr{C}, J), \mathbb{H}\operatorname{Sh}(\mathscr{C}, J)) \to \operatorname{Fun}(\mathscr{C}, \mathbb{H}\operatorname{Sh}(\mathscr{C}, J))$$

is fully faithful (Proposition [16, 6.2.3.20]), so we conclude that the unit is an equivalence globally.

It suffices to prove that the counit

$$\varepsilon : F^*F_* \Rightarrow id$$

is an equivalence. Let $\mathscr{D}$ be the full subcategory of $\widehat{\mathcal{X}^{\mathfrak{DM}_{rep}}_{\acute{e}t}}$ on those objects $\pi : \mathcal{Y} \to \mathcal{X}$ for which the co-unit ε_π is an equivalence. Since both the domain and codomain of ε are colimit preserving functors, the category $\mathscr{D}$ is closed under colimits in $\widehat{\mathcal{X}^{\mathfrak{DM}_{rep}}_{\acute{e}t}}$. Consider now an object $\lambda : \mathscr{S} \to \mathcal{X}$ of $\mathcal{X}^{rep}_{\acute{e}t}$. Then we have that $i(\lambda) \simeq F^*(\lambda)$. Moreover, as for any adjunction, we have a commutative diagram

$$\begin{array}{ccc} F^*\lambda & \xrightarrow{F^*(\eta_\lambda)} & F^*F_*F^*\lambda \\ & \underset{id}{\searrow} & \downarrow{\scriptstyle \varepsilon_{F^*\lambda}} \\ & & F^*\lambda, \end{array}$$

where η is the unit of the adjunction $F^* \dashv F_*$. Since this unit is an equivalence, it follows that the component of the co-unit ε along any object of the form $i(\lambda)$ is an equivalence. Hence $\mathscr{D}$ contains $\mathcal{X}^{rep}_{\acute{e}t}$. However, by Corollary 6.11, every object of $\widehat{\mathcal{X}^{\mathfrak{DM}_{rep}}_{\acute{e}t}}$ can be expressed as a colimit of objects from $\mathcal{X}^{rep}_{\acute{e}t}$. Since $\mathscr{D}$ is closed under colimits, we must have that $\mathscr{D} = \widehat{\mathcal{X}_{\acute{e}t}^{\mathfrak{DM}_{rep}}}$, which means that the co-unit is an equivalence globally. We conclude that $F^* \dashv F_*$ is an adjoint equivalence. ∎

Corollary 6.14. *Let $\mathcal{X}$ be a sheaf of groupoids on a small site of affine schemes. Then the pro-truncated shape of* $\operatorname{Sh}(\mathcal{X}_{\acute{e}t}^{rep})$ *is equivalent to the pro-truncation of the étale homotopy type of $\mathcal{X}$.*

Proof. This is an immediate consequence of Theorem 6.13. Indeed, from that it follows that the pro-truncated shape of $\operatorname{Sh}(\mathcal{X}_{\acute{e}t}^{rep})$ is equivalent to the pro-truncated shape of $\mathcal{X}_{\acute{e}t}^{\mathfrak{DM}_{rep}}$. But the shape of the latter is by definition the étale homotopy type of $\mathcal{X}$. ∎

Corollary 6.15. *Let X be a fine saturated log scheme. Then the pro-truncated shape of* $\operatorname{Sh}\left(\sqrt[\infty]{X}_{\acute{e}t}^{rep}\right)$ *is equivalent to the pro-truncation of the étale homotopy type of $\sqrt[\infty]{X}$. In particular, the pro-truncated shape of the Kummer étale topos agrees with the pro-truncation of the étale homotopy type of the infinite root stack.*

Proof. As $\sqrt[\infty]{X}$ is a sheaf of groupoids, this follows from Corollary 6.14. The second part of the statement follows from Theorem 3.31. ∎

7. Shape in the log regular case

In this section we will give yet another description of the profinite homotopy type of a fine saturated log scheme, in the log regular case. We will then use this to relate the étale homotopy type of $\mathbb{G}_m$ to that of $B\boldsymbol{\mu}_\infty$, where recall that $\boldsymbol{\mu}_\infty = \varprojlim_n \boldsymbol{\mu}_n$ is the inverse limit of the group schemes of n-th roots of unity. We will work with a log regular locally Noetherian log scheme X over a locally Noetherian base S.

We will be using the following recent result:

Theorem 7.1. [3, Theorem 4.34] *Let X be a log regular locally Noetherian log scheme. Then for any locally constant sheaf A of finite abelian groups with orders invertible on X, the inclusion $i : X^{triv} \to X$ induces isomorphisms in sheaf cohomology groups*

$$H^*(X_{k\acute{e}t}, A) \cong H^*_{\acute{e}t}\left(X^{triv}, i^*A\right).$$

Moreover, i induces an isomorphism for any geometric point $x \in X^{triv}$ on pro-ℓ-completions for any prime number ℓ invertible on X

$$\pi_1^{k\acute{e}t}(X, x)^\wedge_\ell \cong \pi_1^{\acute{e}t}\left(X^{triv}, x\right)^\wedge_\ell.$$

More generally, the category of Kummer étale covers of X is equivalent to the category of étale covers of X^{triv} which extend to tamely ramified covers of X.

Lemma 7.2. *If $S = \operatorname{Spec} k$ for k a field of characteristic zero, then any étale cover of X^{triv} extends to a tamely ramified cover of X.*

Proof. Let $D = X \setminus X^{triv}$. Let $Y \to X^{triv}$ be an étale cover. If X is log regular, the rings $\mathcal{O}_{X,D_i}$ are discrete valuation rings for any irreducible component D_i of D, and therefore we can make sense of the ramification of $Y \to X^{triv}$ over D (see [22, Section 53.31]). Furthermore, in characteristic zero, all étale maps $Y \to X^{triv}$ are tamely ramified over D: this follows directly from the definition of tame ramification. Hence by [11, Proposition B.7], $Y \to X^{triv}$ extends to a Kummer étale map $Y' \to X$, which is in particular tamely ramified. ∎

Lemma 7.3. *Let ℓ be a prime invertible on X, and let G be a finite group whose order is relatively prime to ℓ. Then there is an equivalence of categories between G-torsors on X^{triv} and G-torsors in the Kummer étale topos.*

Proof. We use the notations from the previous proof: namely we set $D := X \setminus X^{triv}$ and we let D_i be the irreducible components of D. We observe first that if

$$Y \to X^{triv}$$

is a G-torsor, then it is tamely ramified over D. Recall from [22, 0BSE] that, in order to check this, we consider the fiber product

$$\begin{array}{ccc} \operatorname{Spec} L_{D_i} & \longrightarrow & Y \\ \downarrow & & \downarrow \\ \operatorname{Spec} K_{X,D_i} & \longrightarrow & X^{triv} \end{array}$$

where:

- K_{X,D_i} is the function field of the component containing D_i,
- the map $\operatorname{Spec} K_{X,D_i} \to X^{triv}$ is given by the projection

$$\operatorname{Spec} K_{X,D_i} \cong \operatorname{Spec} \mathcal{O}_{X,D_i} \times_X X^{triv} \to X^{triv}.$$

Now, L_{D_i} is a finite K_{X,D_i}-algebra: it splits as a product of finite field extensions

$$\operatorname{Spec} L_{D_i} \cong \coprod_{j \in J} \operatorname{Spec} L_{D_i,j},$$

where J is a finite set. Further $\operatorname{Spec} L_{D_i} \to \operatorname{Spec} K_{X,D_i}$ is a G-torsor, since it is the base change of a G-torsor. Thus G acts transitively on the components $\operatorname{Spec} L_{D_i,j}$ and this implies that all components are isomorphic, and each map $\operatorname{Spec} L_{D_i,j} \to \operatorname{Spec} K_{X,D_i}$ is Galois for a quotient G' of G. By [22, Tag 09E3] all ramification indexes of $\operatorname{Spec} L_{D_i,j} \to \operatorname{Spec} K_{X,D_i}$ are equal, and thus in particular they divide $|G'|$. Since $|G'|$ divides $|G|$, we conclude that all ramification indexes are coprime to ℓ, and this is exactly the definition of tame ramification over D.

By [11, Proposition B7] we have an equivalence of categories Φ between:

(1) the category $\mathcal{C}_1$ of étale coverings of X^{triv} which are tamely ramified over D, and
(2) the category $\mathcal{C}_2$ of Kummer étale coverings of X.

Note that $\mathcal{C}_1$ and $\mathcal{C}_2$ have finite limits. We will denote by $- \times_{\mathcal{C}_i} -$, $\;i = 1, 2$, the Cartesian product in $\mathcal{C}_1$ and in $\mathcal{C}_2$: it corresponds respectively to the fiber product of schemes over X^{triv}, and to the fiber product of fine and saturated log schemes over X.

Consider the group objects

$$\mathcal{G}_1 := X^{triv} \times_{\mathcal{C}_1} G \to X^{triv} \in \mathcal{C}_1, \quad \mathcal{G}_2 := X \times_{\mathcal{C}_2} G \to X \in \mathcal{C}_2.$$

Note that G-torsors in $\mathcal{C}_1$ can be described purely categorically: they are objects $P \in \mathcal{C}_1$ with an action of $\mathcal{G}_1$ such that the natural map $P \times_{\mathcal{C}_1} \mathcal{G}_1 \to P \times_{\mathcal{C}_1} P$ is an isomorphism. The analogous description holds for Kummer étale G-torsors in $\mathcal{C}_2$, in terms of $\mathcal{G}_2$. Since Φ is an equivalence, it will yield an equivalence between the category of G-torsors in $\mathcal{C}_1$ (i.e. G-torsors on X^{triv} that are tamely ramified over D) and the category of G-torsors in $\mathcal{C}_2$. We showed above that all G-torsors are in fact automatically tamely ramified over D, and this concludes the proof. □

Remark 7.4. In characteristic zero, the above holds for any finite group G.

Corollary 7.5. *Let ℓ be a prime invertible on X, then i induces an equivalences*

$$Shape\left(X_{k\acute{e}t}\right)^{\wedge}_{\ell} \simeq \Pi^{\acute{e}t}_{\infty}\left(X^{triv}\right)^{\wedge}_{\ell}.$$

Moreover, in characteristic zero, there is a profinite homotopy equivalence

$$\widehat{Shape}\left(X_{k\acute{e}t}\right) \simeq \widehat{\Pi}^{\acute{e}t}_{\infty}\left(X^{triv}\right).$$

Proof. Using Theorem 7.1 and Lemma 7.3, the proof is completely analogous to [6, Proposition 4.11]. The statement in characteristic zero then holds by the same argument, using Remark 7.4. □

Remark 7.6. In fact, the above Corollary holds for a less drastic localization than ℓ-completion. If $\mathscr{C}$ is the smallest subcategory of $\mathcal{S}\mathrm{pc}$ closed under finite limits and retracts containing the subcategories $a) - c)$ of [7, Theorem 3.25], then the above holds up to $(\,\cdot\,)^{\wedge}_{\mathscr{C}}$-localization, by the same argument. Furthermore, the proof of the next theorem carries over for $(\,\cdot\,)^{\wedge}_{\mathscr{C}}$-localization as well. We expect it actually holds for completion with respect to all spaces V whose Betti stack $\Delta^{\acute{e}t}(V)$ in S-schemes is $\mathbb{A}^1$-invariant.

Corollary 7.5 is a powerful result, with considerable computational consequences. To illustrate this point, we show that when applied to the simplest possible class of log schemes (namely affine spaces equipped with the toric log structure), Corollary 7.5 recovers and generalizes an interesting calculation of the étale homotopy type of split algebraic tori which was proved in [12, Proposition 6.3].

Theorem 7.7. *Let S be a locally Noetherian scheme, and denote by $\boldsymbol{\mu}_{\infty}$ the affine group scheme over S*

$$\varprojlim_{n} \boldsymbol{\mu}_n.$$

Let ℓ be a prime invertible on S, and N any non-negative integer. Then there is an equivalence of ℓ-profinite spaces

$$\Pi^{\acute{e}t}_{\infty}\left(\mathbb{G}^N_m\right)^{\wedge}_{\ell} \simeq \Pi^{\acute{e}t}_{\infty}\left(B\boldsymbol{\mu}^N_{\infty}\right)^{\wedge}_{\ell}.$$

Moreover, in characteristic zero, this holds up to profinite completion.

Remark 7.8. [12, Proposition 6.3] establishes this result in the special case that

$$S = \operatorname{Spec} k,$$

for k a separably closed field.

Proof. Consider $X = \mathbb{A}^N_S$ with its canonical log structure coming from the identification $X = S \times \operatorname{Spec} \mathbb{Z}[P]$, with $P = \mathbb{N}^N$. Then X is $\mathbb{A}^1$-contractible. So, for Z any S-scheme, denoting by $L_{\mathbb{A}^1}$ the left adjoint to the inclusion of $\mathbb{A}^1$-invariant étale sheaves of spaces into the ∞-category of all étale sheaves of spaces, we have that

$$L_{\mathbb{A}^1}(X \times Z) \simeq L_{\mathbb{A}^1}(Z).$$

Note that the Betti stack $\Delta_{K(\mathbb{Z}/\ell,n)}$ is $\mathbb{A}^1$-invariant for any n by [18, Corollary VI.4.20]. Hence

$$\begin{aligned}
\mathbb{M}\mathrm{ap}\left(X \times Z, \Delta_{K(\mathbb{Z}/\ell,n)}\right) &\simeq \mathbb{M}\mathrm{ap}\left(L_{\mathbb{A}^1}(X \times Z), \Delta_{K(\mathbb{Z}/\ell,n)}\right) \\
&\simeq \mathbb{M}\mathrm{ap}\left(L_{\mathbb{A}^1}(Z), \Delta_{K(\mathbb{Z}/\ell,n)}\right) \\
&\simeq \mathbb{M}\mathrm{ap}\left(Z, \Delta_{K(\mathbb{Z}/\ell,n)}\right).
\end{aligned}$$

It follows from Proposition 3.7 that for any S-scheme Z, the projection map

$$X \times Z \to Z$$

induces an equivalence between ℓ-profinite étale homotopy types

$$\Pi_\infty^{\acute{e}t}\left(X \times Z\right)_\ell^\wedge \simeq \Pi_\infty^{\acute{e}t}\left(Z\right)_\ell^\wedge.$$

Notice that [7, Theorem 3.25] implies that in characteristic zero, the Betti stack of any π-finite space is $\mathbb{A}^1$-invariant, so the analogous result for profinite completion holds.

Observe now that there is an equivalence

$$\sqrt[n]{X} \simeq \left[X/\boldsymbol{\mu}_n^N\right].$$

Since $\Pi_\infty^{\acute{e}t}$ preserves colimits, we have

$$\Pi_\infty^{\acute{e}t}\left(\sqrt[n]{X}\right) \simeq \underset{k \in \Delta^{op}}{\underrightarrow{\operatorname{colim}}}\, \Pi_\infty^{\acute{e}t}\left(\left(\boldsymbol{\mu}_n^N\right)^k \times X\right).$$

Let V be a $\mathbb{A}^1$-invariant space, and assume that V is m-truncated for some finite m (e.g. an ℓ-finite space or, in the characteristic zero case, a π-finite space). Then, since X is $\mathbb{A}^1$-contractible, we have

$$\begin{aligned}
\Pi_\infty^{\acute{e}t}\left(\sqrt[n]{X}\right)(V) &\simeq \operatorname{Map}\left(\Pi_\infty^{\acute{e}t}\left(\sqrt[n]{X}\right), j(V)\right) \\
&\simeq \operatorname{Map}\left(\underset{k \in \Delta^{op}}{\underrightarrow{\operatorname{colim}}}\, \Pi_\infty^{\acute{e}t}\left(\left(\boldsymbol{\mu}_n^N\right)^k \times X\right), j(V)\right) \\
&\simeq \underset{k \in \Delta^{op}}{\underleftarrow{\lim}}\, \Pi_\infty^{\acute{e}t}\left(\left(\boldsymbol{\mu}_n^N\right)^k \times X\right)(V) \\
&\simeq \underset{k \in \Delta^{op}}{\underleftarrow{\lim}}\, \Pi_\infty^{\acute{e}t}\left(\left(\boldsymbol{\mu}_n^N\right)^k\right)(V) \\
&\simeq \underset{k \in \Delta^{op}_{\leq m}}{\underleftarrow{\lim}}\, \Pi_\infty^{\acute{e}t}\left(\left(\boldsymbol{\mu}_n^N\right)^k\right)(V),
\end{aligned}$$

where the last equivalence follows from [8, Lemma 2.21]. So, by Theorem 5.14 we have

$$\begin{aligned}
\Pi_\infty^{\acute{e}t}\left(\sqrt[\infty]{X}\right)(V) &\simeq \underset{n}{\underrightarrow{\operatorname{colim}}}\, \Pi_\infty^{\acute{e}t}\left(\sqrt[n]{X}\right)(V)\\
&\simeq \underset{n}{\underrightarrow{\operatorname{colim}}} \underset{k\in\Delta^{op}_{\leq m}}{\underleftarrow{\lim}} \Pi_\infty^{\acute{e}t}\left(\left(\boldsymbol{\mu}_n^N\right)^k\right)(V)\\
&\simeq \underset{k\in\Delta^{op}_{\leq m}}{\underleftarrow{\lim}} \underset{n}{\underrightarrow{\operatorname{colim}}}\, \Pi_\infty^{\acute{e}t}\left(\left(\boldsymbol{\mu}_n^N\right)^k\right)(V)\\
&\simeq \underset{k\in\Delta^{op}_{\leq m}}{\underleftarrow{\lim}} \Pi_\infty^{\acute{e}t}\left(\left(\boldsymbol{\mu}_\infty^N\right)^k\right)(V)\\
&\simeq \underset{k\in\Delta^{op}}{\underleftarrow{\lim}} \Pi_\infty^{\acute{e}t}\left(\left(\boldsymbol{\mu}_\infty^N\right)^k\right)(V)\\
&\simeq \Pi_\infty^{\acute{e}t}\left(\underset{k\in\Delta^{op}}{\underrightarrow{\operatorname{colim}}}\left(\boldsymbol{\mu}_\infty^N\right)^k\right)(V)\\
&\simeq \Pi_\infty^{\acute{e}t}\left(B\boldsymbol{\mu}_\infty\right)(V),
\end{aligned}$$

where the 4^{th} equivalence follows from Corollary 5.11. So, on one hand we have that

$$\Pi_\infty^{\acute{e}t}\left(\sqrt[\infty]{X}\right)^\wedge_\ell \simeq \Pi_\infty^{\acute{e}t}\left(B\boldsymbol{\mu}_\infty\right)^\wedge_\ell,$$

and in the characteristic zero case

$$\widehat{\Pi}_\infty^{\acute{e}t}\left(\sqrt[\infty]{X}\right) \simeq \widehat{\Pi}_\infty^{\acute{e}t}\left(B\boldsymbol{\mu}_\infty\right).$$

On the other hand, letting $U := X^{triv}$, we have

$$U \cong \mathbb{G}_m^N.$$

So by Corollary 7.5, we then have

$$\begin{aligned}
\Pi_\infty^{\acute{e}t}\left(\mathbb{G}_m^N\right)^\wedge_\ell &\simeq \Pi_\infty^{\acute{e}t}(U)^\wedge_\ell\\
&\simeq \mathit{Shape}\left(X_{k\acute{e}t}\right)^\wedge_\ell\\
&\simeq \Pi_\infty^{\acute{e}t}\left(\sqrt[\infty]{X}\right)^\wedge_\ell\\
&\simeq \Pi_\infty^{\acute{e}t}\left(B\boldsymbol{\mu}_\infty^N\right)^\wedge_\ell,
\end{aligned}$$

and similarly in the characteristic zero case, but up to profinite completion instead of ℓ-profinite completion. □

Appendix A. Appendix on Profinite Spaces

In this appendix, we work out some technical results about profinite spaces which enable us to make local-to-global arguments with them using hypercovers.

We start by recalling the following result of Lurie:

Theorem A.1. [17, Theorem E.2.4.1] *The composite*

$$\operatorname{Prof}(\mathcal{S}\mathrm{pc}) \hookrightarrow \operatorname{Pro}(\mathcal{S}\mathrm{pc}) \xrightarrow{\mathcal{S}\mathrm{pc}/(\,\cdot\,)} \mathfrak{Top}_\infty$$

is fully faithful and right adjoint to the profinite shape functor.

Definition A.2. A morphism $f : \mathcal{X} \to \mathcal{Y}$ of profinite spaces is called **étale** if the induced geometric morphism

$$\mathcal{S}\mathrm{pc}/\mathcal{X} \to \mathcal{S}\mathrm{pc}/\mathcal{Y}$$

is an étale geometric morphism. (See [16, Section 6.3.5].)

Lemma A.3. *Let $\mathcal{X}$ and $\mathcal{Y}$ be profinite spaces. Then the coprojection $\mathcal{X} \to \mathcal{X} \coprod \mathcal{Y}$ is étale and (-1)-truncated.*

Proof. Let $\mathcal{X} = \varprojlim_{\alpha} j(X_\alpha)$ and $\mathcal{Y} = \varprojlim_{\beta} j(X_\beta)$ be profinite spaces. Then we have

$$\mathcal{X} \coprod \mathcal{Y} \simeq \varprojlim_{\alpha,\beta} \left(j \left(X_\alpha \coprod Y_\beta \right) \right).$$

As such, we have an identification

$$\mathcal{S}\mathrm{pc}/\left(\mathcal{X} \coprod \mathcal{Y}\right) \simeq \varprojlim_{\alpha,\beta} \mathcal{S}\mathrm{pc}/\left(X_\alpha \coprod Y_\beta\right)$$

in $\mathfrak{Top}_\infty$. However, cofiltered limits in $\mathfrak{Top}_\infty$ are computed at the level of underlying ∞-categories by [16, Theorem 6.3.3.1]. Define an object

$$\overline{\mathcal{X}} \in \varprojlim_{\alpha,\beta} \mathcal{S}\mathrm{pc}/\left(X_\alpha \coprod Y_\beta\right)$$

by

$$\overline{\mathcal{X}} := \left(X_\alpha \to X_\alpha \coprod Y_\beta\right)_{\alpha,\beta}.$$

Notice that each morphism $X_\alpha \to X_\alpha \coprod Y_\beta$ in $\mathcal{S}\mathrm{pc}$ is a monomorphism, i.e. (-1)-truncated, as it is the inclusion of a union of connected components. It follows that we have a canonical identification

$$\begin{aligned}
\left(\mathcal{S}\mathrm{pc}/\left(\mathcal{X} \coprod \mathcal{Y}\right)\right)/\overline{\mathcal{X}} &\simeq \varprojlim_{\alpha,\beta} \mathcal{S}\mathrm{pc}/X_\alpha \\
&\simeq \varprojlim_{\beta} \varprojlim_{\alpha} \mathcal{S}\mathrm{pc}/X_\alpha \\
&\simeq \varprojlim_{\alpha} \mathcal{S}\mathrm{pc}/X_\alpha \\
&\simeq \mathcal{S}\mathrm{pc}/\mathcal{X}.
\end{aligned}$$

Moreover, it is easy to check that the induced morphism

$$\left(\mathcal{S}\mathrm{pc}/\left(\mathcal{X} \coprod \mathcal{Y}\right)\right)/\overline{\mathcal{X}} \to \left(\mathcal{S}\mathrm{pc}/\left(\mathcal{X} \coprod \mathcal{Y}\right)\right)$$

under the equivalence $\left(\mathcal{S}\mathrm{pc}/\left(\mathcal{X} \coprod \mathcal{Y}\right)\right)/\overline{\mathcal{X}} \simeq \mathcal{S}\mathrm{pc}/\mathcal{X}$ is the functor $\mathcal{S}\mathrm{pc}/(\,\cdot\,)$ applied to the coprojection. Hence the coprojection is étale. Notice that, by defining $\overline{\mathcal{Y}}$ analogously, we have that $1 \simeq \overline{\mathcal{X}} \coprod \overline{\mathcal{Y}}$, hence the coprojection is also (-1)-truncated. □

Corollary A.4. *If $\mathcal{X} = \coprod_{\alpha} \mathcal{X}_\alpha$ is a coproduct of profinite spaces, each coprojection $\mathcal{X}_\alpha \to \mathcal{X}$ is étale and (-1)-truncated.*

Proof. Write $\mathcal{Y} := \coprod_{\beta \neq \alpha} \mathcal{X}_\beta$, then $\mathcal{X}_\alpha \to \mathcal{X}_\alpha \coprod \mathcal{Y} = \mathcal{X}$ is étale by Lemma A.3. □

Lemma A.5. *Let $\mathcal{E}$ be an ∞ topos and suppose that*

$$1_{\mathcal{E}} = \coprod_{\alpha} E_{\alpha}.$$

Suppose furthermore that $f : \mathcal{F} \to \mathcal{E}$ is a geometric morphism. Then in $\mathfrak{Top}_{\infty}$,

$$\mathcal{F} \simeq \coprod_{\alpha} \mathcal{F}/f^{*}\left(E_{\alpha}\right).$$

Proof. Recall that

$$\begin{array}{rcl} \chi_{\mathcal{F}} : \mathcal{F} & \to & \mathfrak{Top}_{\infty} \\ F & \mapsto & \mathcal{F}/F \end{array}$$

preserves colimits by [16, Proposition 6.3.5.14], and since so does f^* and f^* is left exact, it follows that

$$\begin{array}{rcl} \coprod_{\alpha} \mathcal{F}/f^{*}\left(E_{\alpha}\right) & \simeq & \mathcal{F}/\left(\coprod_{\alpha} f^{*}\left(E_{\alpha}\right)\right) \\ & \simeq & \mathcal{F}/\left(f^{*}\left(\coprod_{\alpha} E_{\alpha}\right)\right) \\ & \simeq & \mathcal{F}/f^{*}\left(1_{\mathcal{E}}\right) \\ & \simeq & \mathcal{F}/1_{\mathcal{F}} \\ & \simeq & \mathcal{F}. \end{array}$$

□

Corollary A.6. *Let $\mathcal{Y} \to \coprod_{\alpha} \mathcal{X}_{\alpha}$ be a map in* $\operatorname{Prof}(\mathcal{S}\mathrm{pc})$. *Then we have a canonical splitting*

$$\mathcal{Y} \simeq \coprod_{\alpha} \left(\mathcal{Y} \times_{\mathcal{X}} \mathcal{X}_{\alpha}\right).$$

Proof. Using the notation of Lemma A.5, let $\mathcal{E} := \mathcal{S}\mathrm{pc}/\left(\coprod_{\alpha} \mathcal{X}_{\alpha}\right)$, with each E_{α} corresponding to the étale geometric morphism $\mathcal{S}\mathrm{pc}/\mathcal{X}_{\alpha} \to \mathcal{S}\mathrm{pc}/\mathcal{X}$ via Corollary A.4, and let $\mathcal{F} := \mathcal{S}\mathrm{pc}/\mathcal{Y}$.

Recall that, by [16, Remark 6.3.5.8], for each α, the following diagram is Cartesian

$$\begin{array}{ccc} \mathcal{F}/f^{*}\left(E_{\alpha}\right) & \longrightarrow & \mathcal{F} \\ \downarrow & & \downarrow \\ \mathcal{E}/E_{\alpha} & \longrightarrow & \mathcal{E}. \end{array}$$

The result now follows from Lemma A.5, since $\operatorname{Prof}(\mathcal{S}\mathrm{pc})$ is a reflective subcategory of $\mathfrak{Top}_{\infty}$. □

Corollary A.7. *For $\mathcal{X}$, $\mathcal{Y}$, and $\mathcal{Z}$ arbitrary profinite spaces*

$$\mathbb{M}\mathrm{ap}\left(\mathcal{Z}, \mathcal{X} \coprod \mathcal{Y}\right) \simeq \coprod_{\mathcal{Z} = \mathcal{Z}_{\mathcal{X}} \coprod \mathcal{Z}_{\mathcal{Y}}} \left(\mathbb{M}\mathrm{ap}\left(\mathcal{Z}_{\mathcal{X}}, \mathcal{X}\right) \times \mathbb{M}\mathrm{ap}\left(\mathcal{Z}_{\mathcal{Y}}, \mathcal{Y}\right)\right).$$

Proof. This follows immediately from Corollary A.6. □

Lemma A.8. *Suppose that $\mathcal{Z}$, $\mathcal{X}$, and $\mathcal{Y}$ are profinite spaces. Suppose further that there is a decomposition $\mathcal{Z} = \mathcal{Z}_{\mathcal{X}} \coprod \mathcal{Z}_{\mathcal{Y}}$ and maps*

$$f_{\mathcal{X}} : \mathcal{Z}_{\mathcal{X}} \to \mathcal{X}$$

and

$$f_{\mathcal{Y}} : \mathcal{Z}_{\mathcal{Y}} \to \mathcal{Y}$$

and let f be the induced map

$$\mathcal{Z} \to \mathcal{X} \coprod \mathcal{Y}.$$

Then

$$\mathcal{Z}_{\mathcal{X}} \simeq \mathcal{Z} \times_{\mathcal{X} \coprod \mathcal{Y}} \mathcal{X}.$$

Proof. It suffices to prove that there is a pullback diagram in $\mathfrak{Top}_\infty$

$$\begin{array}{ccc} \mathcal{S}\mathrm{pc}/\mathcal{Z}_{\mathcal{X}} & \longrightarrow & \mathcal{S}\mathrm{pc}/\mathcal{X} \\ \downarrow & & \downarrow \\ \mathcal{S}\mathrm{pc}/\mathcal{Z} & \xrightarrow{f} & \mathcal{S}\mathrm{pc}/\left(\mathcal{X} \coprod \mathcal{Y}\right). \end{array}$$

Let $\overline{\mathcal{X}} \in \mathcal{S}\mathrm{pc}/\left(\mathcal{X} \coprod \mathcal{Y}\right)$ be the object corresponding to the étale geometric morphism

$$\mathcal{S}\mathrm{pc}/\mathcal{X} \to \mathcal{S}\mathrm{pc}/\left(\mathcal{X} \coprod \mathcal{Y}\right).$$

Then by [16, Remark 6.3.5.8], we have that

$$(\mathcal{S}\mathrm{pc}/\mathcal{Z}) \times_{\mathcal{S}\mathrm{pc}/(\mathcal{X} \coprod \mathcal{Y})} \mathcal{S}\mathrm{pc}/\mathcal{X} \simeq (\mathcal{S}\mathrm{pc}/\mathcal{Z}) / \left(f^*\left(\overline{\mathcal{X}}\right)\right).$$

It therefore suffices to prove that $f^*\left(\overline{\mathcal{X}}\right) \simeq \overline{\mathcal{Z}_{\mathcal{X}}}$.

Choose level representations $\left(Z_\alpha^{\mathcal{X}} \to X_\alpha\right)_\alpha$ and $\left(Z_\beta^{\mathcal{Y}} \to Y_\beta\right)_\beta$ of $f_{\mathcal{X}}$ and $f_{\mathcal{Y}}$ respectively. Then $\left(Z_\alpha^{\mathcal{X}} \coprod Z_\beta^{\mathcal{Y}} \to X_\alpha \coprod Y_\beta\right)_{\alpha,\beta}$ is a level representation of f. It follows, that in

$$\mathcal{S}\mathrm{pc}/\mathcal{Z} \simeq \varprojlim_{\alpha,\beta} \mathcal{S}\mathrm{pc}/\left(Z_\alpha^{\mathcal{X}} \coprod Z_\beta^{\mathcal{Y}}\right),$$

we have that

$$\begin{aligned} f^*\left(\overline{\mathcal{X}}\right) &\simeq \left(\left(Z_\alpha^{\mathcal{X}} \coprod Z_\beta^{\mathcal{Y}}\right) \times_{X_\alpha \coprod Y_\beta} X_\alpha \to \left(Z_\alpha^{\mathcal{X}} \coprod Z_\beta^{\mathcal{Y}}\right)\right)_{\alpha,\beta} \\ &\simeq \left(Z_\alpha^{\mathcal{X}} \to \left(Z_\alpha^{\mathcal{X}} \coprod Z_\beta^{\mathcal{Y}}\right)\right)_{\alpha,\beta} \\ &\simeq \overline{\mathcal{Z}_{\mathcal{X}}}. \end{aligned}$$

□

Definition A.9. Let $\mathscr{C}$ be an ∞-category with fibered products. We say that **coproducts are universal** in $\mathscr{C}$ if for all $f : D \to C$, the functor

$$\begin{array}{rcl} f^* : \mathscr{C}/C & \to & \mathscr{C}/D \\ (E \to C) & \mapsto & (D \times_C E \to D) \end{array}$$

preserves small coproducts.

Remark A.10. If $\mathscr{C}$ is locally Cartesian closed, then coproducts are universal in $\mathscr{C}$, since f^* then has a right adjoint, e.g. if $\mathscr{C}$ is any ∞-topos.

Proposition A.11. *Let $f : \mathcal{Y} \to \coprod_{\alpha} \mathcal{X}_{\alpha} \simeq \mathcal{X}$ be a morphism in $\mathscr{C}$ and assume that coproducts are universal. Then there is a canonical splitting $\mathcal{Y} \simeq \coprod_{\alpha} (\mathcal{Y} \times_{\mathcal{X}} \mathcal{X}_{\alpha})$ such that f is induced by the family*

$$(\mathcal{Y} \times_{\mathcal{X}} \mathcal{X}_{\alpha} \to \mathcal{X}_{\alpha})_{\alpha}.$$

Proof. Notice that we have a pullback diagram

$$\begin{array}{ccc} f^{*}\left(\coprod_{\alpha} \mathcal{X}_{\alpha}\right) & \xrightarrow{\sim} & \mathcal{Y} \\ \downarrow & & \downarrow f \\ \coprod_{\alpha} \mathcal{X}_{\alpha} & \xrightarrow{\sim} & \mathcal{X} \end{array}$$

Since f^{*} preserves coproducts, we have that

$$\begin{aligned} \mathcal{Y} &\simeq f^{*}\left(\coprod_{\alpha} \mathcal{X}_{\alpha}\right) \\ &\simeq \coprod_{\alpha} f^{*}(\mathcal{X}_{\alpha}) \\ &\simeq \coprod_{\alpha} (\mathcal{Y} \times_{\mathcal{X}} \mathcal{X}_{\alpha}). \end{aligned}$$

□

Remark A.12. Let C be an object of an ∞-category $\mathscr{C}$, then the canonical functor

$$\mathscr{C}/C \to \operatorname{Pro}(\mathscr{C})/j(C)$$

induces a cofiltered limit preserving functor

$$\operatorname{Pro}(\mathscr{C}/C) \to \operatorname{Pro}(\mathscr{C})/j(C).$$

Suppose that coproducts are universal in $\mathscr{C}$ and $\left(\mathcal{Y}^{i} \to \coprod_{\alpha} \mathcal{X}_{\alpha}\right)_{i}$ is a cofiltered system in $\mathscr{C}/\left(\coprod_{\alpha} \mathcal{X}_{\alpha}\right)$. By the above remark, there is a canonical map

$$f : \varprojlim_{i} j(\mathcal{Y}^{i}) \to j\left(\coprod_{\alpha} \mathcal{X}_{\alpha}\right)$$

in $\operatorname{Pro}(\mathscr{C})$. Notice, moreover, that since coproducts are universal, by Proposition A.11, for each i we have a canonical splitting

$$\mathcal{Y}^{i} \simeq \coprod_{\alpha} \mathcal{Y}^{i}_{\alpha}$$

together with canonical maps

$$f_{\alpha} : \mathcal{Y}^{i}_{\alpha} \to \mathcal{X}_{\alpha}$$

inducing f. Suppose furthermore that there is a coproduct preserving functor

$$\Pi : \mathscr{C} \to \operatorname{Prof}(\mathcal{S}\mathrm{pc})$$

which induces a cofiltered limit preserving functor

$$\Pi^{pro} : \operatorname{Pro}(\mathscr{C}) \to \operatorname{Prof}(\mathcal{S}\mathrm{pc}).$$

Theorem A.13. *In the situation above,*

$$\Pi^{pro}\left(\varprojlim_i j\left(\mathcal{Y}^i\right)\right) \simeq \coprod_\alpha \varprojlim_i \Pi\left(\mathcal{Y}^i_\alpha\right).$$

Proof. Let $\mathcal{Z} := \Pi^{pro}\left(\varprojlim_i j\left(\mathcal{Y}^i\right)\right)$. By definition,

$$\Pi^{pro}\left(\varprojlim_i j\left(\mathcal{Y}^i\right)\right) \simeq \varprojlim_i \Pi\left(\coprod_\alpha \mathcal{Y}^i_\alpha\right),$$

and since Π preserves coproducts we have

$$\mathcal{Z} \simeq \varprojlim_i \coprod_\alpha \Pi\left(\mathcal{Y}^i_\alpha\right).$$

Similarly, we have a canonical equivalence

$$\Pi^{pro}\left(j\left(\coprod_\alpha \mathcal{X}_\alpha\right)\right) \simeq \coprod_\alpha \Pi\left(\mathcal{X}_\alpha\right).$$

Since we have a canonical map

$$\Pi(f) : \mathcal{Z} \to \coprod_\alpha \Pi\left(\mathcal{X}_\alpha\right),$$

by Corollary A.6, we have a canonical splitting

$$\mathcal{Z} \simeq \coprod_{\alpha'}\left(\left(\varprojlim_i \coprod_\alpha \Pi\left(\mathcal{Y}^i_\alpha\right)\right) \times_{\coprod\limits_\alpha \Pi(\mathcal{X}_\alpha)} \Pi\left(\mathcal{X}_{\alpha'}\right)\right).$$

For each α' let

$$\mathcal{Z}_{\alpha'} := \left(\left(\varprojlim_i \coprod_\alpha \Pi\left(\mathcal{Y}^i_\alpha\right)\right) \times_{\coprod\limits_\alpha \Pi(\mathcal{X}_\alpha)} \Pi\left(\mathcal{X}_{\alpha'}\right)\right).$$

Since limits commute with limits, we have

$$\mathcal{Z}_{\alpha'} \simeq \varprojlim_i \left(\coprod_\alpha \Pi\left(\mathcal{Y}^i_\alpha\right)\right) \times_{\coprod\limits_\alpha \Pi(\mathcal{X}_\alpha)} \Pi\left(\mathcal{X}_{\alpha'}\right)$$

and by Lemma A.8, for each i we have

$$\left(\coprod_\alpha \Pi\left(\mathcal{Y}^i_\alpha\right)\right) \times_{\coprod\limits_\alpha \Pi(\mathcal{X}_\alpha)} \Pi\left(\mathcal{X}_{\alpha'}\right) \simeq \Pi\left(\mathcal{Y}^i_{\alpha'}\right).$$

Thus $\mathcal{Z}_{\alpha'} \simeq \varprojlim_i \Pi\left(\mathcal{Y}^i_{\alpha'}\right)$, and hence

$$\mathcal{Z} \simeq \coprod_\alpha \varprojlim_i \Pi\left(\mathcal{Y}^i_\alpha\right).$$

□

The following two corollaries are immediate special cases:

Corollary A.14. *Let X be a fine saturated log scheme. Let $(U_\alpha \to X)_\alpha$ be a collection of étale maps. Let $\mathscr{C} = \mathrm{Sh}\left(\mathbf{Aff}', ét\right)$ and $\Pi = \widehat{\Pi}^{ét}_\infty$. Then*

$$\widehat{\Pi}^{ét}_\infty\left(\varprojlim_n j\left(\coprod_\alpha\left(\sqrt[n]{X}\times_X U_\alpha\right)\right)\right)\simeq\coprod_\alpha\varprojlim_n\widehat{\Pi}^{ét}_\infty\left(\sqrt[n]{X}\times_X U_\alpha\right).$$

Corollary A.15. *Let X be a fine saturated log scheme locally of finite type over $\mathbb{C}$. Let $(U_\alpha \hookrightarrow X_{top})_\alpha$ be a collection of open subsets of the analytification X_{top}. Using the notation from* [8]*, let $\mathscr{C} = \mathbb{H}\mathrm{ypSh}_\infty\left(\mathbf{Top}_\mathbb{C}\right)$, and $\Pi = \widehat{\Pi}_\infty$. Then*

$$\widehat{\Pi}_\infty\left(\varprojlim_n j\left(\coprod_\alpha\left(\sqrt[n]{X}_{top}\times_{X_{top}} U_\alpha\right)\right)\right)\simeq\coprod_\alpha\varprojlim_n\widehat{\Pi}_\infty\left(\sqrt[n]{X}_{top}\times_{X_{top}} U_\alpha\right).$$

Remark A.16. The above corollary fixes a small oversight in the proof of [8, Theorem 6.4]. In more detail, the proof there starts by constructing a hypercover $U^\bullet$ of the analytification of the log scheme X_{an}, such that for each n, we have

$$U^n = \coprod_\alpha V_\alpha.$$

In order to justify our reduction of the proof to one such V_α, one needs the above corollary.

References

[1] Piotr Achinger and Mattia Talpo. Betti realization of varieties defined by formal laurent series. *Geometry & Topology*, 25(4):1919–1978, 2021.

[2] Ilan Barnea, Yonatan Harpaz, and Geoffroy Horel. Pro-categories in homotopy theory. *Algebraic & Geometric Topology*, 17(1):567–643, 2017.

[3] Joseph Berner. Shape Theory in Homotopy Theory and Algebraic Geometry. PhD thesis, 2018.

[4] Bhargav Bhatt and Akhil Mathew. The arc-topology. *Duke Mathematical Journal*, 170(9):1899–1988, 2021.

[5] Niels Borne and Angelo Vistoli. Parabolic sheaves on logarithmic schemes. *Advances in Mathematics*, 231(3-4):1327–1363, oct 2012.

[6] David Carchedi. On the étale homotopy types of higher stacks. Preprint, arXiv:1511.07830, 2015.

[7] David Carchedi and Elden Elmanto. Relative étale realizations of motivic spaces and dwyer-friedlander K-theory of noncommutative schemes. Preprint, arXiv:1810.05544, 2018.

[8] David Carchedi, Sarah Scherotzke, Nicolò Sibilla, and Mattia Talpo. Kato-Nakayama spaces, infinite root stacks and the profinite homotopy type of log schemes. *Geom. Topol.*, 21(5):3093–3158, 2017.

[9] Eric M Friedlander. Étale K-theory I: Connections with etale cohomology and algebraic vector bundles. *Inventiones mathematicae*, 60(2):105–134, 1980.

[10] Asaf Horev and Lior Yanovski. On conjugates and adjoint descent. *Topology and its Applications*, 232(1):140–154, 2017.

[11] Yuichiro Hoshi. The exactness of the log homotopy sequence. *Hiroshima Math. J*, pages 61–121, 2009.

[12] Marc Hoyois. The étale symmetric künneth theorem. *Mathematische Zeitschrift*, 304(1):1, 2023.

[13] Jacob Lurie. Derived algebraic geometry V: Structured spaces. Preprint, arXiv:0905.0459, 2009.

[14] Jacob Lurie. Derived algebraic geometry XIII: Rational and p-adic homotopy theory. http://www.math.harvard.edu/ lurie/papers/DAG-XIII.pdf, 2009.

[15] Jacob Lurie. Higher algebra. http://www.math.harvard.edu/ lurie/papers/higheralgebra.pdf, 2009.

[16] Jacob Lurie. *Higher topos theory*, volume 170 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2009.
[17] Jacob Lurie. Spectral algebraic geometry. 2018, http://www.math.harvard.edu/ lurie/papers/SAG-rootfile.pdf.
[18] James S. Milne. *Étale cohomology*, volume 33 of *Princeton Mathematical Series*. Princeton University Press, Princeton, N.J., 1980.
[19] Wiesława Nizioł. Toric singularities: log-blow-ups and global resolutions. *J. Algebraic Geom.*, 15(1):1–29, 2006.
[20] Wiesława Nizioł. K-theory of log-schemes. I. *Doc. Math.*, 13:505–551, 2008.
[21] Anatoly Preygel. An intro to algebraic stacks.
[22] The Stacks Project Authors. Stacks project, 2024.
[23] Mattia Talpo and Angelo Vistoli. Infinite root stacks and quasi-coherent sheaves on logarithmic schemes. *Proc. Lond. Math. Soc. (3)*, 116(5):1187–1243, 2018.

David Carchedi, Department of Mathematical Sciences, George Mason University, 4400 University Drive, MS: 3F2, Exploratory Hall, Fairfax, Virginia 22030, USA
Email address: davidcarchedi@gmail.com

Sarah Scherotzke, Université du Luxembourg, Maison du Nombre, 6, Avenue de la Fonte, L-4364 Esch-sur-Alzette
Email address: sarah.scherotzke@uni.lu

Nicolò Sibilla, SISSA, Via Bonomea, 265, 34136 Trieste TS, Italy
Email address: nsibilla@sissa.it

Mattia Talpo, Department of Mathematics, University of Pisa, Largo Bruno Pontecorvo 5, 56127 Pisa PI, Italy
Email address: mattia.talpo@unipi.it