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Holography, localization of information and subregions

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Abstract

Locality is undoubtedly an integral part of both quantum theory and general relativity. On the other hand, a holographic theory like the AdS/CFT implies that the bulk quantum gravitational degrees of freedom are encoded at spatial infinity in the boundary theory. Even though this statement is a claim at the non perturbative level, there are still remnants of this property in the perturbative limits of quantum gravity. This is primarily due to the gravitational Gauss law, which prevents us from defining strictly local operators. Since including gravity in the description *is* requiring the theory to be invariant under coordinate transformations, the physical operators need to be diffeomorphism invariant. This condition, implemented by Gauss law, demands the operators be dressed to the boundary and include a gravitational version of the Wilson line extending all the way to infinity, therefore demands them to be non-local. Towards resolving this tension, we propose candidate operators that bypass this requirement and are local and diffeomorphism invariant at the same time, in the AdS/CFT context. These operators still satisfy a version of gravitational Gauss law, as they are interpreted to be dressed with respect to the features of the states. Therefore, the states these operators are defined on are states that break the symmetries of the theory and have 'features'. These states are in general high energy states with large variance and correspond to non trivial semiclassical geometries in the bulk. This proposal will also help resolve paradoxes raised concerning the island proposal. In addition, this enables one to discuss subregions, their associated subsystems and localization of information more concretely in perturbative quantum gravity.

In the second part, we will be mostly concerned with a bulk sub region called an AdS-Rindler wedge. We will use what is called the Petz map, borrowed from the quantum information and quantum computing community, to explicitly reconstruct this bulk subregion from its boundary dual subregion. This agrees with the earlier conjecture on the bulk subregion reconstruction and the proposal that, due to the quantum error correcting nature of gravity, the Petz map can be used to reconstruct the entanglement wedge. In addition, we study the algebra of operators in the AdS Rindler wedge precisely, both in the bulk and the boundary dual. Using the crossed product construction and a novel method of renormalizing the Ryu Takayanagi surface, we show how including gravitational corrections modifies the algebra to a more manageable one where we can define density matrices and von Neumann entropy. Finally, we study a particular representation of algebra of operators in general backgrounds, called the covariant representation, in situations where gravitational interactions are present. This representation will illuminate what the crossed product construction is in physical terms.

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To my beloved parents and my little sister,
for their love and endless encouragement.

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1
Introduction

In this chapter we set the ground and motivation for the following chapters that constitute the main content of the thesis. In each chapter, relevant parts of this introductory story are discussed in more detail.

One of the most important discoveries in the past three decades in the field of high energy particle physics and quantum gravity is arguably the AdS/CFT correspondence [Maldacena, 1998]. The first example of the correspondence was discovered after studying the low energy limit of systems of D-branes in string theory. Following the conjectured equivalence of D_p branes and black p branes, a stack of N D3 branes in a type IIB string theory in ten dimensions can be taken as backreacting and producing a geometry that asymptotes the $\mathrm{AdS}_5 \times \mathbb{S}^5$ spacetime close to the branes in the large N limit, where the radius of curvature of the AdS geometry is of the order of $N^{1/4}$ in Planck units. On the other hand, the region close to the D3 branes at low energies can be described by a $3+1$ dimensional $\mathcal{N} = 4$ super Yang Mills (SYM) theory with $SU(N)$ gauge group; while the gauge coupling constant is given by $g_{YM}^2 = 4\pi g_s$, where g_s is the string coupling constant. While the former description is more precise in the large N limit, the latter is appropriate for any N. Therefore, since in the region where the two theories are valid, they should be equivalent and since the latter description makes sense for any N, it was proposed that type IIB string theory on $\mathrm{AdS}_5 \times \mathbb{S}^5$ spacetime is dual to the $\mathcal{N} = 4$ SYM theory with $SU(N)$ gauge group in $3+1$ dimensions, for any N and g [Maldacena, 1998, Witten, 1998, Aharony et al., 2000].

Following this particular duality, many more examples were studied leading to similar conjectures between theories in several dimensions, some of which are [Aharony et al., 2000, Aharony et al., 2008, Klebanov & Polyakov, 2002, Sezgin & Sundell, 2002, Petkou, 2003, Leigh & Petkou, 2003, Das & Jevicki, 2003, de Mello Koch et al., 2011, Gaberdiel & Gopakumar, 2011, Maldacena et al., 1997, Minasian et al., 2000, Kiritsis & Niarchos, 2011, Kiritsis, 2006, Aharony et al., 2006, Kiritsis & Niarchos, 2008].

The AdS/CFT correspondence can thus be broadly stated as *an equivalence between a theory of quantum gravity in an asyptotically $AdS_{d+1} \times M$ geometry (the bulk), where M is* *some compact manifold, and a conformal field theory on* $\mathbb{R} \times \mathbb{S}^{d-1}$ *(the boundary).*

The most obvious possible application of this correspondence is to better understand theories of quantum gravity from the study of a standard relativistic conformal quantum field theory. In particular, an exploration into how a quantum gravity theory with extra dimensions emerges from the boundary conformal field theory. However, it is better to look at this task from the different layers of complexities that arose while studying the several realizations of the correspondence. Specifically, we can ask how a full theory of quantum gravity emerges from a boundary CFT, where there may be no limit where the bulk theory simplifies to any theory that we have a grasp on at this moment. This question nonetheless is quite complicated, mainly because we do not know what such a quantum gravitational bulk theory is, and even though there are certain approaches that are trying to precisely understand what the relationship between a certain full theory quantum gravity and the boundary CFT is [Gopakumar, 2004b, Gopakumar, 2004c, Gopakumar, 2004a, Gopakumar, 2005, Aharony et al., 2007, Aharony & Komargodski, 2008, Eberhardt et al., 2020, Eberhardt et al., 2019, Gopakumar & Mazenc, 2022], it will not be the focus of our discussion here. We will also include in this category the cases where the bulk theory is not given by a perturbative worldsheet theory, like M theory backgrounds or when the boundary is a theory like the $O(N)$ vector model.

A lower level of complexity for the correspondence is when there is a limit where the bulk theory becomes a *classical* string theory on a possibly highly curved backgrounds or a higher spin gravity theory. In this limit, we will have a large number of light bulk degrees of freedom corresponding, for example, to the stringy modes for the classical string theory. To be more precise, in the aforementioned example, such a bulk theory is dual to the boundary

$SU(N), \mathcal{N} = 4$ SYM theory in the large N limit with a small or finite 't Hooft coupling, $\lambda = g_{YM}^2 N$, which is the effective coupling constant of the large N theory.

An even simpler case is when the bulk theory is just a quantum field theory on a curved space time which is asymptotically AdS$\times M$, where M is some compact manifold with large size compared with the Planck or string scale, with only few bulk light modes. We recover this limit if for example we consider the large N, $\mathcal{N} = 4$ SYM theory in the large 't Hooft limit. As has already been implied, in both of the latter two cases, the strict $N \to \infty$ limit treats gravity classically that is the background geometry is taken to be fixed[1]. However, perturbative gravitational interactions are included as we back away from the $1/N = 0$ limit and add perturbative $1/N$ corrections to the boundary theory[2].

We consider the last two cases of the bulk theory that are discussed above, including the $1/N$ perturbative corrections, and collectively call them the semiclassical bulk theory and focus on the emergence of this semiclassical bulk theory from the boundary conformal field theory. We will differentiate between the two cases later in the manuscript.

1.1 The emergence of the semiclassical bulk

The essential property of the semiclassical bulk theory as defined above is that the ratio of the Plank length to the effective radius of curvature of the AdS geometry goes to zero. In addition, it is expected that this theory is only a small sub sector of the full theory of quantum gravity, after all it is just a perturbation theory around some (possibly highly curved) geometry. Therefore, in the boundary we consider families of CFTs that are characterized

[1]Note that free gravitons are also included in the theory, the essential point is that nothing backreacts and modifies the geometry.

[2]One has to deal with a slight complexity that in general, bulk non gravitational interactions between the light modes also emerge as $1/N$ corrections are added.

by some parameter, g, which in the limit $g \to 0$ we recover the free field limit of the bulk. On the other hand, note that $g = 1/\sqrt{c}$ is a universal coupling constant in conformal perturbation theory, where c is the central charge of the theory. This coupling arises from the interaction of the fields with the stress energy tensor and directly follows from the Ward identities of the conformal symmetries. As a result, as we take the limit $g = 1/\sqrt{c} \to 0$, the stress energy tensor can be decoupled from other conformal primaries and the n-point correlators of the CFT can be seen to factorize. In fact, there is a sub sector of the CFT that emerges in this limit where the 'confined' degrees of freedom do not satisfy any field equation of motion but obey the Wick factorization of free fields. Such theories are called generalized free field theories and, with the appropriate $1/\sqrt{c} \to 0$ limit, are low lying sectors of holographic CFTs. In particular, we assume that in the bulk, the number of light modes does not depend on the effective gravitational coupling (which is proportional to the ratio of the Plank length to the AdS radius), thus stays finite in the semiclassical limit. Therefore, the appropriate limit of the boundary is taking $1/\sqrt{c} \to 0$ while keeping the number or the conformal dimension of the (gauge invariant or confined) degrees of freedom, sometimes called single trace operators, independent of c.

In summary, we expect a semiclassical bulk dual theory to emerge from a CFT with the following properties [Heemskerk et al., 2009, El-Showk & Papadodimas, 2012a],

- It has large central charge i.e. many degrees of freedom.
- It has a small number of operators of low conformal dimension.
- The correlators of the low lying operators factorize.

Again in the example mentioned at the beginning of this chapter, this limit corresponds to the large N limit while the difference between the two cases of the semiclassical theory comes from taking finite λ or infinite λ. This implies a further distinction within the generalized free field theory, where there are theories with a parametric gap in the conformal dimension of supergravity fields and the 'closed string modes' (or those corresponding to single trace operators of spin less than 2 and higher than 2) *and* those with no such gap [Heemskerk et al., 2009]. The former corresponds in bulk to perturbative quantum gravity in the sense that we start out with a quantum field theory on a curved space time, then add perturbative gravitational corrections; while the latter is associated with, for instance, a perturbative string theory or Vasiliev theory. On the other hand, the elements of the generalized free field theory will be linear combinations and derivatives of the single trace operators and their products.

Coming back to considering generalized free fields in general, we can try to see how the semiclassical bulk theory emerges. The first hint of this emergence is the fact that the free energy of the generalized free fields is the same as the free energy for a thermal gas of free particles in AdS, which can be shown as long as we take the appropriate limit i,e. keep the number of the degrees of freedom independent of c, while taking c to infinity. We will comment later in the chapter about when number of operators or energy of states is $O(c)$.

As was already mentioned, the generalized free fields do not satisfy any equations of motion in the CFT even though their correlators factorize in the large c limit. This implies that they form the Fock space of states that are in the representation of the conformal group $SO(d, 2)$, which presumably can be superimposed. This might be a bit confusing as there is no equation of motion for these fields. But the linearity of the fields can be made explicit by considering an auxiliary spacetime in one higher dimension than the CFT. In particular, if the spacetime coordinates of the boundary is (t, Ω), then we consider a spacetime (t, r, Ω) with the AdS metric,

$$ds^2 = -(1+r^2)dt^2 + \frac{dr^2}{(1+r^2)} + r^2 d\Omega_{d-1}^2 \tag{1.1}$$

with $t \in (-\infty, \infty)$ and $r \in [0, \infty)$. Then consider the field,

$$\Phi(t, r, \Omega) = \int dt' \, d\Omega'_{d-1} K(t, r, \Omega; t', \Omega') \mathcal{O}(t', \Omega') \,. \tag{1.2}$$

where $\mathcal{O}$ is a generalized free field with conformal dimension Δ and the kernel K satisfies,

$$(\Box_{d+1} - m^2) K(t, r, \Omega; t', \Omega') = 0 \tag{1.3}$$

where $\Box_{d+1}$ is the Laplacian in the AdS spacetime and Δ is related to m as

$$\Delta = \frac{d}{2} + \sqrt{\frac{d^2}{4} + m^2} \tag{1.4}$$

This reconstruction method is well known by the name HKLL reconstruction [Bena, 2000, Hamilton et al., 2006d, Hamilton et al., 2006c, Hamilton et al., 2007b]. Note that Φ is a boundary operator that is non local because of the smearing with the above kernel. But because of (1.3), it can be seen that it satisfies a linear wave equation in a one higher dimension. Then the large c factorization would guarantee that this field is equivalent to a free massive field in AdS spacetime. It can also be checked that commutators of Φ's with a spacelike separation in higher dimensional space is zero thus (1.2) can be interpreted as a local bulk field. This procedure can also be done for the boundary stress energy tensor (present for any CFT) which would give a spin 2 field, therefore, the bulk theory is actually gravitational. From the boundary perspective this may seem a nice way to reorganize the generalized free fields, but it also shows the emergence of a semiclassical bulk theory, with semi local bulk fields in general[3], from this low lying sector. This statement can be even more confirmed if we check the conformal bootstrap condition the generalized free fields are supposed to satisfy. Studying the relations between the boundary conformal partial waves (CPWs) will naturally lead to what we can interpret as Witten diagrams in the bulk supergravity theory [El-Showk & Papadodimas, 2012a].

1.2 Subregion-subregion duality

The above analysis shows the emergence of the bulk theory over the full spacetime from the generalized free field sector of the boundary CFT. But, we can also ask if there is any sense in which a given bulk subregion can be considered dual to a certain boundary subregion. With the introduction of a subregion of a bulk Cauchy surface called the *entanglement wedge*, this question was addressed for 'semi finite' bulk subregions which are subregions that extend all the way to the boundary of the spacetime, therefore, part of their boundary is also part of the full spacetime boundary. The entanglement wedge, a, for a given boundary subregion, A, is defined to be a bulk subregion that is bounded by A and a co-dimension 2 surface called the quantum extremal surface, χ_A. The quantum extremal surface on the other hand is defined to be a surface with $\partial\chi_A = \partial A$ and homologous to A such that,

$$\mathcal{S}(\rho_A) = \mathcal{S}(\rho_a) + Tr(\rho_a \mathcal{A}_{loc}) \tag{1.5}$$

[3] This is because, in general the bulk theory can also be a classical string theory.

is extremized. To leading order in the gravitational coupling G, $\mathcal{A}_{loc} = \frac{Area(\chi_A)}{4G}$. This definition is in increasing amount of precision given in [Ryu & Takayanagi, 2006, Hubeny et al., 2007, Faulkner et al., 2013, Engelhardt & Wall, 2015]. The density matrix ρ_A is the boundary CFT density matrix associated with the subregion A, while ρ_a is a density matrix in the semiclassical bulk theory. The above formula seems to imply that the information contained in the boundary region A is the same as the one that is in the entanglement wedge. This statement was made more manifest by a proof given by [Jafferis et al., 2016a, Dong et al., 2016a], that the relative entropy of two states in A is the same as the relative entropy of states in the entanglement wedge to all orders in $1/N$,

$$\mathcal{S}(\rho_A||\sigma_A) = \mathcal{S}(\rho_a||\sigma_a). \tag{1.6}$$

Since relative entropy is a measure of distinguishability of states, (1.6) is true only if the states ρ_A and σ_A contain the same information about the bulk as ρ_a and σ_a.

There is another way to state the above property which goes by the name, reconstruction theorem [Dong et al., 2016a]. This states that the two subregions, A and its entanglement wedge, are dual in the sense that any operator in the entanglement wedge can be reconstructed from the operators in boundary subregion A. The simplest case where this can be done explicitly is when the entanglement wedge is entirely causally connected to the A. In this scenario, we can use a generalization of the method used in deriving (1.2). This generalization, discussed in detail in chapter 4, gives the bulk fluctuations in the entanglement wedge in terms of generalized free fields in the subregion A.

$\star$ But the entanglement wedge in general includes regions that are not causally connected to the boundary subregion. In this case we can not use the HKLL reconstruction to reproduce the operators in the entanglement wedge. To this end, people have proposed an operation, borrowed from the quantum computing community, called the Petz map to reconstruct the bulk operators. Chapter 4 is concerned with the explicit application of this Petz map to reconstruct entanglement wedge operators from the boundary subregion operators.

This discussion also applies to the region compliment to A, with the corresponding bulk subregion being, the complement of a.

But a more precise way to put the above subregion-subregion duality is in terms of causal diamonds of the subregions [Leutheusser & Liu, 2022]. In the bulk, we started out by saying that a is a subregion in the bulk Cauchy surface. But this is in fact equivalent to the causal diamond of the subregions for relativistic field theories. On the other hand, as can be seen from (1.2), the boundary smearing to represent the bulk operators involves integral in both space and time directions. Both the HKLL reconstruction in the boundary subregions and Petz map reconstruction involve operators in the causal diamond of the boundary subregion, not just the boundary spacelike subregion.

What about subregions that are not connected to the spacetime boundary? For such subregions, the compliment, an annular region surrounding the subregion in the center, is necessarily connected to the boundary, in fact it seems like it is connected to the full boundary subregion. At this point, it is important to think of subregion-subregion duality as a duality between causal wedges and not just a duality at the level of the Cauchy surface. Therefore, we should consider the boundary of the causal wedge of the annular region, which would correspond to a time band in the boundary CFT to be the boundary dual, see figure 2.1. There are several issues associated with this time band and operators in this time band which will be discussed in detail in chapter 3. This on the other hand implies that the bulk subregion that is not connected to the boundary is supposed to be dual to the complement of the time band at the boundary.

Coming back to the emergence of the semiclassical bulk theory from the boundary, we can consider adding $1/\sqrt{c}$ corrections to the boundary theory which would correspond to adding perturbative gravitational corrections in the bulk theory. This leads to corrections to (1.2) [Kabat et al., 2011, Heemskerk et al., 2012], which were derived by requiring the bulk field to commute with spacelike separated single trace operators inside the correlators. This leads to corrections that involve multi trace operators (depending on the type of interaction that is turned on) and in general we have,

$$\Phi(t,r,\Omega) = \int_{\text{bdry}} K(t,r,\Omega;t',\Omega')\mathcal{O}(t',\Omega') + \frac{1}{N}\int\int_{\text{bdry}} K^{(1)}\mathcal{O}(t_1,\Omega_1)\mathcal{O}(t_2,\Omega_2) + O(1/N^2) \tag{1.7}$$

where $K^{(1)}$ is another smearing function that depends on (t,r,Ω), (t_1,Ω_1) and (t_2,Ω_2) and $1/\sqrt{c} \sim 1/N$. This will make the operator Φ commute with all the spacelike separated single trace operators and enable us to define a bulk field looks local even in the presence of perturbative gravitational interactions. But this field is not quite local, as was noted by the authors in [Kabat et al., 2011], it is impossible to remove the commutator of this operator with the Hamiltonian (using the same methods discussed above). Since in gravitational theories, the Hamiltonian can be written as a boundary integral, this implies that the HKLL operator is rather 'highly non local', not commuting with an operator that is localized at spatial infinity.

> ⋆ The central goal of the second chapter is to remove this non locality of the HKLL operator and define a local diffieomorphism invariant operators. This on the other hand has implications on locality and localization of information in AdS/CFT and perturbative quantum gravity in general. The subtleties concerning these issues in classical gravity, quantum field theories and perturbative quantum gravity are also discussed in detail in this chapter.

An issue we just mentioned and may need a bit elaboration is *localization of information*. The simple question we would want to answer is,

> *Is it possible to hide information at the center of a given spacetime from a space-like separated observer far away?*

To appreciate the non triviality of the question, we discuss it is addressed in the context a relativistic quantum field theory.

1.3 Localization of information in quantum field theory

As one would naively expect, localization of information has a lot to do with locality and local operators. But, Localization of information in QFT needs understanding of localized states rather just local (localized) operators. The need for this distinction is the following. Let's take a compact spherical subregion of the initial value surface, which we call the 'base', and consider its causal diamond, U_1, then if one considers the action of operators localized U_1, denoted by $\mathcal{A}(U_1)$, on the vacuum, one can generate the full Hilbert space of the theory[4] This is due to what is called the Reeh Schlider property of the vacuum of QFT

[4]In fact, one generates the vacuum sector of the Hilbert space.

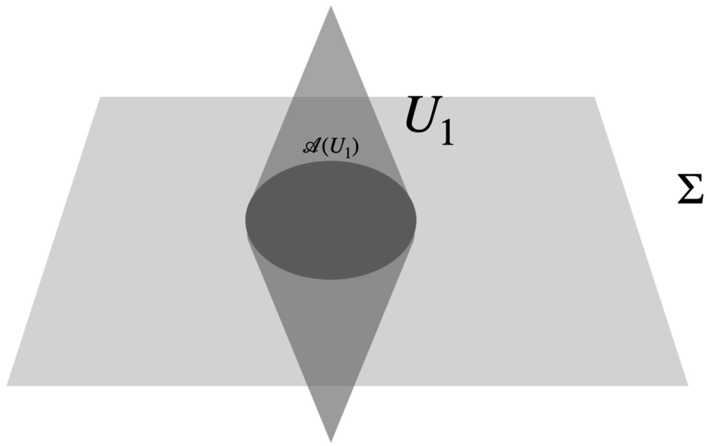


Figure 1.1: A spherical compact subregion of the Cauchy surface, Σ, has an associated causal diamond, U_1, a region of spacetime that can influence it and be influenced by it. In this spacetime region, we can consider the algebra of local operators signified by $\mathcal{A}(U_1)$.

(check [Witten, 2018] for a for a detailed introduction that is more accessible to a physicist).

A better proposal would be to act on the vacuum with *unitary operators* in $\mathcal{A}(U_1)$ and not the full algebra. For these states, measurements in the compliment of $\mathcal{A}(U_1)$ will be the same as measurements in the vacuum,

$$\langle\Psi| A^{'} |\Psi\rangle = \langle\Omega| A^{'} |\Omega\rangle \tag{1.8}$$

where, $|\Psi\rangle = W|\Omega\rangle$ and $W \in \mathcal{A}(U_1)$ with $W^* W = 1$ while $A^{'} \in \mathcal{A}^{'}(U_1)$.

Even though each of these states can be considered as states that are *strictly localized* in U_1, their linear combination is not necessarily a localized state for the simple reason that a sum of unitary operators is not again a unitary operator in general. In fact, since any state can be approximated well enough by $\mathcal{A}(U_1)\,|\Omega\rangle$, the linear span of states like $W\,|\Omega\rangle$ is the again the full Hilbert space. But this is not a problem, it only tells us that strictly localized states do not form a subspace and we should only talk about sets of states rather than a space of localized states.

Nonetheless, it seems like there is some tension between the Reeh Schlider property and localized states in QFT. But we should understand that the origin of this property is the non vanishing and decaying correlation present between far away observables in the physical states of the theory. This correlation decays with the separation of the observables and it implies that the norm of the states localized far away from U_1, should be smaller and smaller. A crude estimate would give an exponential decay for the correlations for a 'delocalization' or separation of order of a few Compton lengths. Following this property, we can define what we call *essentially localized* states, with a small amount of delocalization [Haag & Swieca, 1965].

Therefore we can relax the earlier condition of considering only unitary operators to construct localized states and take the set of states,

$$\mathcal{A}^1(U_1)\,|\Omega\rangle \tag{1.9}$$

where $\mathcal{A}^1(U_1)$ corresponds to operators in $\mathcal{A}(U_1)$ but with unit norm or less[5]. We then define a quantity,

$$b_A = \frac{1}{|A\,|\Omega\rangle\,|}, \tag{1.10}$$

$A \in \mathcal{A}^1(U_1)$. It can be seen that b_A gets bigger and bigger for highly delocalized states,

[5] The norm of an operator A is defined to be the maximum value of $\frac{|A|\Psi\rangle|}{||\Psi\rangle|}$, for $|\Psi\rangle \in \mathcal{H}$

in fact, it is shown in [Haag & Swieca, 1965] that the above set of states can be considered to be essentially localized for $b_A \leq b$ for some fixed b, which determines the degree of delocalization we can tolerate. The previously discussed strictly localized states correspond to $b = 1$.

But if one is interested in finding well behaved localized states, it is reasonable to expect that they will also have bounded energy. Therefore, we project out the states with energy more than some maximum value E_{max} in the above set (or put a smooth cutoff rather than a sharp cutoff). Then these states will be localized in a finite region of space and have bounded energy. Classically, this corresponds to states in a finite region of the phase space, Γ, thus it is reasonable to expect that these localized states in QFT form a compact subset of the full Hilbert space, namely there is Γ/h^3 of them, where h is the Planck's constant.

Since the norm of states gets smaller and smaller for more and more delocalized states, one can consider the set $\mathcal{N} = P_{E_{max}}\mathcal{A}^1(U_1)\,|\Omega\rangle$, and the analogy with the classical theory requires it to be a compact subset of the Hilbert space for any sensible quantum theory. The operator $P_{E_{max}}$ is a projection operator that projects into states with energy E_{max} or less. More precisely, for any small number $\varepsilon > 0$, one can find finite dimensional Hilbert space $\mathcal{H}_{d(\varepsilon)} \subset \mathcal{H}$, such that states in $\mathcal{N}$ that are orthogonal to $\mathcal{H}_{d(\varepsilon)}$ have norm less than ε. In other words, for any small positive number ε, all the states in $\mathcal{N}$ with higher norm than ε can be considered to be states in some compact finite dimensional Hilbert space $\mathcal{H}_{d(\varepsilon)}$. Physically reasonable quantum field theories are expected to satisfy this condition but, interestingly, not theories like generalized free field theories[6] [Haag & Swieca, 1965].

Therefore we can say we can localize information if we can have a set like $\mathcal{N}$ in our the-

[6]More precisely, generalized free fields with continuous mass spectrum.

ory. We can consider a smooth cutoff in the energy and define the set [Buchholz & Wichmann, 1986],

$$\mathcal{N}_\beta = e^{-\beta H}\mathcal{A}^1(U_1)\,|\Omega\rangle \tag{1.11}$$

and the 'compactness' condition becomes,

$$\mathcal{N}_\beta \subset T_\beta \mathcal{H}^1 \tag{1.12}$$

for a positive operator T_β with finite trace where $\mathcal{H}^1$ corresponds to the states in $\mathcal{H}$ with unit norm or less. The minimum value for the traces of such operators T_β is called the nuclearity index and is bounded by,

$$\nu_{\beta,r} < e^{cr^3\beta^{-n}} \tag{1.13}$$

for small β and large size of the base of U_1, r. The coefficients c and n are some positive constants. The 'compactness' condition is called the nuclearity condition and the set $\mathcal{N}_\beta$ is called a nuclear set. Thus, more generally, the localization of information is possible in a given quantum field theory, if the nuclearity condition is satisfied in other words if a nuclear set exists. This property on the other hand implies to what is called the split property (see section 2.2.2) which associates a notion of subsystem in to the subregion U_1.

If we can define algebra of operators like $\mathcal{A}^1(U_1)$ in the bulk theory in the presence of gravitational interactions, we can argue more or less rigorously for a localization of information in perturbative quantum gravity along the same lines. One of the main goals of the second chapter is to analyze and argue for this possibility in the presence of gravity.

1.4 Algebra of operators

Another facet of bulk reconstruction that has received a renewed attention in recent years relates to considering the algebraic structure of the operators in the bulk and the boundary theories. One of the first important applications of the properties of the algebra of operators in the semiclassical theory in AdS/CFT context was by Papadodimas and Raju, in their use of Tomita Takesaki theory to reconstruct the black hole interior [Papadodimas & Raju, 2013, Papadodimas & Raju, 2014b, Papadodimas & Raju, 2014a]. However, much of the recent interest follows from the work of Liu and Leutheusser, [Leutheusser & Liu, 2021a, Leutheusser & Liu, 2021b] where they realized that the algebra of operators exterior to the eternal black hole in AdS is what is called a type III_1 von Neumann algebra. This algebra has a natural time like evolution operator that can be naturally associated with an observer falling into the black hole. As noticed by Witten and et. al., [Witten, 2021b, Chandrasekaran et al., 2022a, Chandrasekaran et al., 2022b, Penington & Witten, 2023] a careful treatment of the addition of perturbative gravity into this algebra, would result in a rigorous Lorentzian derivation of the generalized entropy of a black hole.

But first we have to discuss how the black hole geometry emerges from the boundary CFT. We were careful to take a specific limit when we discussed the emergence of the bulk in section 1.1. In particular, we assumed the dimension of the operators and/or the number of the operators do not depend on c as we take the $c \to \infty$ limit. This corresponds to states with energy of $O(c)$ by state operator correspondence. Note that these states are states whose degeneracy diverge in the large c limit, since c 'counts' the degrees of freedom of the CFT and can be considered as the black hole microstates of the bulk theory. Recall that in the bulk, the limit $c \to \infty$ corresponds to the $\hbar \to 0$ limit and it follows from Bekenstein and Hawking that the black hole entropy diverges in this limit. This is a hint that this sector of the theory does not have a nice Hilbert space description in the strict large c limit, as the number of microstates diverge [Schlenker & Witten, 2022]. However, we can still describe the quantum fluctuations on a given black hole microstate, or a linear combination of microstates in some energy band ΔE around a given energy $E \sim O(c)$. In fact any coherent bulk state in the AdS space is an example of such a state, check chapter 3. The emergence of the semiclassical theory on this background in the given in a more or less the same way as discussed in section 1.1, except that the generalized free fields are to be uplifted into the bulk with a different kernel, a kernel that propagates in the corresponding new bulk geometry.

An interesting example of such a geometry is the eternal black hole in AdS at some temperature $1/\beta$. This geometry is conjectured to be dual to the thermofield double state of two thermally entangled CFTs above the Hawking-Page temperature [Maldacena, 2003]. One justification for this duality is that there is a phase transition as we increase the temperature of the CFTs above the Hawking-Page temperature and, in the same sense as the previous paragraph, the entropy becomes c dependent and divergent in the large c limit. As discussed in [El-Showk & Papadodimas, 2012a] this indicates the appearance of a black hole in the bulk. Nevertheless, the semiclassical bulk will be described by the generalized free fields of the boundary CFTs and for instance the scalar bulk fluctuations will be given by (1.2) where now the kernel solves (1.3) on the eternal black hole background. This is the bulk geometry that was studied by Liu and Leutheusser, leading to a proposal for the infalling observer's evolution operator.

In a quantum field theory on a curved spacetime, we can define an algebra of operators associated to a given subregion, for instance one of the two exteriors to the eternal black hole or a subregion that is extend to the boundary in AdS like the one considered earlier. The operators form a von Neumann algebra of type III_1 (check chapter 4). This algebra is a difficult type of von Neumann algebra in that there are no microstates, no density matrices and no entropy that we can associate to the states of the theory. There is even no Hilbert space we can associate solely with the subregion we are describing. Technically this means, the algebra of operators for this region does not act irreducibly on the Hilbert space. The physical states are maps from the algebra of operators to complex numbers that are identified with the expectation values of the operators. However, this algebra seems similar to the $O(c)$ sector of the CFT in the strict large c limit, for instance there are no microstates. There is an appropriate Hilbert space construction we can perform for these types of algebras in the absence of microstates that goes by the name, Gelfand–Naimark–Segal (GNS) construction. As was shown by [Leutheusser & Liu, 2021b], with the appropriate renormalization of the generalized free fields, we can identify this Hilbert space with the semiclassical Hilbert space of eternal black hole background. Therefore, this seems to be a natural way to capture the large c limit of the black hole microstates.

In the several works mentioned earlier, it was shown that adding gravitational interactions to the subregions will lead to a less complicated algebra of operators called type II von Neumann algebra. For this algebra, even though microstates still do not exist, we can define density matrices associated with the states a von Neumann entropy.

> ⋆ The goal of chapter 4 is to provide such an example, in addition to black holes and deSitter spacetimes, where the type of the von Neumann algebra of

operators gets modified to a more mangable one as we add gravitation interactions. A semi infinite subregion, which is a subregion that extends all the way to the boundary, in AdS called an AdS-Rindler wedge is considered. This spacetime is quite similar to the BTZ black hole and can be considered its generalization with no singularity. This analysis has also led to a novel renormlaization method of the Ryu-Takayanagi surface.

This modification to an easier type of von Neumann algerba was possible due to what is called the crossed product construction [Takesaki, 1973, Witten, 2021b]. Adding gravitational interactions to the theory will translate to applying the crossed product construction to the type III_1 algebra associated with the bulk subregions. In the several examples considered up till now, this procedure has led to either a type II_1 or a type II_∞ von Neumann algebras. However, this construction is quite technical and it is important to grasp what we are doing physically.

⋆ To this end, the last chapter is concerned with physically elaborating what the crossed product algebra is in quantum gravity. We use what is called the covariant representation of an algebra and discuss it in general backgrounds. We will use the fact that it is in a one to one correspondence with the representations of the crossed product algebra to elucidate the physical meaning of the cross product algebra.

This thesis will constitute the following publications,

- "*Holography and Localization of Information in Quantum Gravity*",Eyoab Bahiru, Alexandre Belin, Kyriakos Papadodimas, Gabor Sarosi, Niloofar Vardian, arXiv: 2301.08753 [hep-th]
- "*Explicit reconstruction of the entanglement wedge via the Petz map*",Eyoab Bahiru, Niloofar Vardian, arXiv: 2210.00602 [hep-th]
- "*Algebra of operators in an AdS-Rindler wedge*", Eyoab Bahiru, arXiv:2208.04258 [hep-th]
- "*Algebras and their Covariant representations in quantum gravity*", Eyoab Bahiru, arXiv: 2308.14166 [hep-th]

In addition, the author has produced the following works during the PhD, which are not included in the thesis.

- "*The centaur-algebra of observables*",Sergio E. Aguilar-Gutierrez, Eyoab Bahiru, Ricardo Espíndola, arXiv:2307.04233 [hep-th]
- "*State-dressed local operators in the AdS/CFT correspondence*",Eyoab Bahiru, Alexandre Belin, Kyriakos Papadodimas, Gabor Sarosi, Niloofar Vardian, arXiv: 2209.06845 [hep-th]

Locality... reflects our ability to construct the "big picture" like a jigsaw puzzle, starting with a description of the most basic interactions ...

Mario Livio

2

Holography and Localization of Information in Quantum Gravity

This chapter consists of the paper [Bahiru et al., 2023] written in collaboration with Alexandre Belin, Kyriakos Papadodimas, Gabor Sarosi and Niloofar Vardian. The original abstract is as follows:

Within the AdS/CFT correspondence, we identify a class of CFT operators which represent diff-invariant and approximately local observables in the gravitational dual. Provided that the bulk state breaks all asymptotic symmetries, we show that these operators commute to all orders in 1/N with asymptotic charges, thus resolving an apparent tension between locality in perturbative quantum gravity and the gravitational Gauss law. The interpretation of these observables is that they are not gravitationally dressed with respect to the boundary, but instead to features of the state. We also provide evidence that there are bulk observables whose commutator vanishes to all orders in 1/N with the entire algebra of single-trace operators defined in a space-like separated time-band. This implies that in a large N holographic CFT, the algebra generated by single-trace operators in a short-enough time- band has a non-trivial commutant when acting on states which break the symmetries. It also implies that information deep in the interior of the bulk is invisible to single-trace correlators in the time-band and hence that it is possible to localize information in perturbative quantum gravity.

2.1 Introduction

It is generally believed that in quantum gravity, space-time locality is an emergent notion which becomes accurate and useful in certain limits of the underlying theory. This perspective is realized in the AdS/CFT correspondence [Maldacena, 1998]: bulk locality becomes precise in the large N, strong coupling limit and when probing the theory with simple enough operators. Moreover, a large number of proposals aiming to resolve the black hole information paradox rely on a certain amount of non-locality ['t Hooft, 1985, Susskind et al., 1993, Giddings, 2013, Bousso, 2013, Papadodimas & Raju, 2013, Verlinde & Verlinde, 2013, Maldacena & Susskind, 2013, Penington, 2020a, Almheiri et al., 2020, Almheiri et al., 2019, Penington et al., 2022, Laddha et al., 2021]. A natural question is to understand whether non-local features of quantum gravity are visible only in the non-perturbative regime, or whether remnants of non-locality are also visible at the perturbative level.

Even in classical general relativity it is not entirely straightforward to formulate the concept of locality, as it is non-trivial to define local observables. Physical observables need to be diff-invariant and, in order for them to also be local, they have to be associated to points in space-time which have to be specified in a diff-invariant way. If the space-time has a boundary, a standard approach is to define points relationally with respect to the boundary or by completely fixing the gauge. We say that these observables are *gravitationally dressed* with respect to the boundary. However, the resulting observables, while diff-invariant, are not strictly localized and have non-vanishing Poisson brackets at space-like separation. A particular aspect of this difficulty is related to the gravitational Gauss law: in gravitational theories defined with asymptotically flat or AdS boundary conditions, the Hamiltonian, and other asymptotic symmetry charges, are boundary terms. Acting with a candidate local, diff-invariant observable in the interior of space will generally change the energy of the state, which is immediately measurable at space-like separation due to Gauss's law.

Despite these difficulties, at the classical level, there are ways of defining local and diff-invariant observables in the neighborhood of a state, provided that the state is sufficiently complicated. A class of such observables introduced a long time ago [Komar, 1958, Bergmann

& Komar, 1960, DeWitt, 1962] will be reviewed in sub-section 2.2.3, see also [Giddings et al., 2006, Marolf, 2015, Khavkine, 2015] for more recent discussions. These observables respect the causal structure of the underlying space-time, in the sense that their Poisson brackets at space-like separation vanish. In particular, provided that the state we are considering is complicated enough, the action of these observables is not visible by the boundary Hamiltonian, as these observables only rearrange energy in the interior of space. The price we have to pay is that these observables are not defined globally on the phase space of solutions. They have desired properties only for certain states.

A natural question is to what extent can such local diff-invariant observables be defined at the quantum level. As mentioned above, we do not expect to be able to find exactly local diff-invariant observables at the non-perturbative level, however it may be possible to do so in perturbation theory. This question is important in order to be able to quantify departures from locality in quantum gravity and to understand if there is a way to generalize the structure of algebras of observables of quantum field theory to situations where gravity is included perturbatively.

It is useful to formulate these questions in the context of the AdS/CFT correspondence. We consider a CFT state $|\Psi_0\rangle$ that is dual to a semi-classical asymptotically AdS$_{d+1}$ geometry in global coordinates and a short time-band near the boundary as shown in Fig. 2.1. We consider the algebra $\mathcal{A}$ of observables in semi-classical gravity which are localized in this time band. This algebra includes the Hamiltonian and other asymptotic charges. From the point of view of the dual CFT, it is natural to identify the algebra $\mathcal{A}$ with the algebra generated by single-trace operators localized in this time-band, we will call it the "single-trace algebra". The expectation is that the single-trace algebra $\mathcal{A}$ corresponds to the causal wedge of the time-band [Banerjee et al., 2016][1]. Notice that here we have causal-wedge reconstruction and not entanglement wedge reconstruction, as we are looking only at the single-trace subalgebra. In the CFT the notion of a time-band algebra only makes sense at large N, since large N generates a natural hierarchy between operators that are small combinations of single-trace operators and arbitrarily complicated operators. For finite N there is no such hierarchy and the time-slice axiom would imply that $\mathcal{A}$ is the full CFT algebra[2]. Algebras of single-trace operators in holographic CFTs have been discussed in [Papadodimas & Raju, 2013, ?, Papadodimas & Raju, 2014a] and more recently in [Leutheusser & Liu, 2021a, Leutheusser & Liu, 2021b, Witten, 2021b, Chandrasekaran et al., 2022b, Bahiru, 2022, Leutheusser & Liu, 2022].

If the time-band is short enough, then there is a region in the bulk which is space-like with respect to the time-band. We will refer to this region as the "diamond"[3]. If we were able to define diff-invariant observables localized in the diamond, they should commute with the algebra $\mathcal{A}$. As already mentioned, the question is non-trivial as these observables must be gravitationally dressed and if we use the boundary to dress them, then they will not commute with $\mathcal{A}$. For example, it appears that since the Hamiltonian H is an element of $\mathcal{A}$ it would be able to detect any excitation added in the interior of the diamond using the gravitational Gauss law. To summarize, the question we want to examine:

[1] A different approach for studying time-bands based on gravitational entropy and minimal surfaces was initiated in [Balasubramanian et al., 2013, Balasubramanian et al., 2014, Myers et al., 2014, Headrick et al., 2014b]. It would be interesting to understand possible connections between those ideas and the results presented in this paper.

[2] We do not include in the single-trace algebra elements like $e^{iHt}\mathcal{O}(t = 0, x)e^{-iHt}$ with $t = O(N^0)$ and large enough to exit the time-band. Such "precursor" operators are complicated from the point of view of operators in the time-band and go beyond the semi-classical description.

[3] For now we assume that the state has simple topology and there are no black hole horizons in the interior.

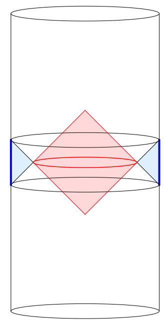


Figure 2.1: The single-trace operators localized in the time band $t \in (-\varepsilon, \varepsilon) \times \mathbb{S}^{d-1}$ (dark blue region on boundary) form an algebra $\mathcal{A}$ which is conjectured to be dual to the causal wedge of the region (light blue). If the state $|\Psi_0\rangle$ of the system breaks all symmetries, then the causal diamond in the middle (light red), which is spacelike separated from the time-band, corresponds to the commutant $\mathcal{A}'$ of the algebra $\mathcal{A}$ when acting on the code subspace of the state $|\Psi_0\rangle$.

> *Does the algebra $\mathcal{A}$, when acting on the state $|\Psi_0\rangle$ and small perturbations around it, have a non-trivial commutant in the $1/N$ expansion?*

As we will discuss later, we need to refine the question by demanding that the commutant acts non-trivially within the code-subspace of the state, in order to avoid obvious but uninteresting constructions[4]. We emphasize that we do not expect the algebra to have a commutant at finite N [Banerjee et al., 2016].

A closely related question is that of localization of information. According to AdS/CFT the quantum state of the CFT at any moment in time contains the full information of the bulk. In particular, if we had considered the *full* algebra of all operators in the time-band, as opposed to the algebra generated by few (relative to N) single-trace operators, then we would be able to reconstruct the interior of the diamond. Suppose however, that we only have access to the algebra $\mathcal{A}$ of single-trace operators in the time band. Can we then

[4] For example, for a complicated state with energy of $O(N^2)$, a unitary which rotates the phase of a single energy eigenstate will have commutators of $O(e^{-N^2})$ with all elements of $\mathcal{A}$. However, this would not be an interesting example, as this operator is generally ”invisible” from the bulk point of view and does not create excitations inside the diamond.

reconstruct the information of whatever is hidden inside the diamond? This can also be rephrased as follows:

> *Given a state $|\Psi_0\rangle$, can we find another state $|\Psi_0\rangle'$ such that the correlators of the single-trace algebra $\mathcal{A}$ in the time-band, evaluated on these two states agree to all orders in $1/N$, but correlators of single-trace operators differ at $O(N^0)$ outside the time-band?*

The intuition here is that we want to find a state $|\Psi_0\rangle'$ which contains an additional excitation relative to $|\Psi_0\rangle$ in the interior of the diamond which becomes visible by single-trace operators only after a light-ray has reached the boundary i.e. in the future or past of the time-band. If the algebra $\mathcal{A}$ had a commutant then we could take $|\Psi_0\rangle' = U(A')|\Psi_0\rangle$ for some unitary U built out of operators A' in the commutant.

We will provide evidence that the answer to the two aforementioned questions is positive, provided that the state $|\Psi_0\rangle$ is complicated enough. The reasoning was first outlined in [Bahiru et al., 2022]. In this paper we extend the construction in a few ways and provide additional arguments and examples.

Standard approaches to bulk reconstruction lead to observables which are relationally defined with respect to the boundary. This is the case for the HKLL reconstruction [Banks et al., 1998, Bena, 2000, Hamilton et al., 2006d, Hamilton et al., 2006c, Hamilton et al., 2007b, Hamilton et al., 2008b, Heemskerk et al., 2012], as well as approaches based on the Petz map [Cotler et al., 2019a, Chen et al., 2020a] or modular reconstruction [Jafferis et al., 2016a, Faulkner & Lewkowycz, 2017a], as they all require some sort of boundary dressing.

For concreteness we start with a standard HKLL operator given by

$$\Phi(t, r, \Omega) = \int_{\text{bdry}} dt' \, d\Omega'_{d-1} K(t, r, \Omega; t', \Omega') \mathcal{O}(t', \Omega') \,. \tag{2.1}$$

Here K is a particular Green's function which depends on the background metric. Implicit in this expression is a gauge-fixing scheme in a particular coordinate system, which is uniquely determined by making use of the boundary. If we pick the point (t, r, Ω) to be in the diamond, the operator 2.1 commutes with all single-trace operators in the time band at large N. At subleading orders multi-trace corrections need to be added to 2.1 to ensure vanishing commutators. However the commutator with the Hamiltonian and other asymptotic charges, which is nonzero at order $1/N$, cannot generally be corrected by multi-trace corrections. The physical reason is that the operator 2.1 is gravitationally dressed with respect to the boundary. The non-vanishing commutator with H appears to be an obstacle in identifying 2.1 as an element of the commutant of $\mathcal{A}$ [Giddings & Kinsella, 2018, Donnelly & Giddings, 2016].

In this paper we present a way to find operators which commute with the asymptotic charges to all orders in $1/N$, while at the same time create excitations in the interior of the diamond similar to those of the HKLL operator. These operators can be defined provided the state $|\Psi_0\rangle$ that we are considering breaks all asymptotic symmetries. These operators correspond to observables gravitationally dressed with respect to features of the state.

A crucial starting observation is that if a state $|\Psi_0\rangle$ is dual to a bulk geometry which breaks the asymptotic symmetries, then the overlap

$$\langle \Psi_0 | U(g) | \Psi_0 \rangle \qquad g \in SO(2, d) \,, \tag{2.2}$$

is generally exponentially small, of order $O(e^{-aN^2})$ with $\mathrm{Re}(a) > 0$ provided that the element g is sufficiently far from the identity[5]. Here $SO(2,d)$ represents the asymptotic symmetry group of AdS$_{d+1}$. We will quantify this statement more precisely in the later sections. In fact, we will provide evidence that if we introduce the code subspace around the state $|\Psi_0\rangle$, defined as

$$\mathcal{H}_0 = \mathrm{span}\{|\Psi_0\rangle, \mathcal{O}(t,\Omega)|\Psi_0\rangle, ..., \mathcal{O}_1(t_1,\Omega_1)...\mathcal{O}_n(t_n,\Omega_n)|\Psi_0\rangle\}\,, \tag{2.3}$$

and similarly $\mathcal{H}_g$ for the state $U(g)|\Psi_0\rangle$ then any inner product between unit normalized states of $\mathcal{H}_0, \mathcal{H}_g$ will also be of order $O(e^{-aN^2})$.

Starting with a standard HKLL operator Φ we consider the operator

$$\widehat{\Phi} = c\int_B d\mu(g) U(g) P_0 \Phi P_0 U(g)^{-1}\,, \tag{2.4}$$

where P_0 denotes the projector on 2.3 and $d\mu(g)$ is the Haar measure on $SO(2,d)$ and B is a reasonably sized neighborhood of $SO(2,d)$ around the identity. The overall normalization constant c will be specified later. The main claim, which will be discussed in section 2.4, is that operators 2.4 have the desired properties: their commutators with the asymptotic symmetry charges Q of $SO(2,d)$ are exponentially small

$$[Q,\hat{\Phi}] = O(e^{-N^2})\,, \tag{2.5}$$

when acting on the code subspace, while at the same time, the *leading* large-N action of $\hat{\Phi}$

[5] But not too far. The state may return to itself in compact directions of the conformal group or approximately back to itself due to Poincare recurrences.

on the code subspace 2.3 is the same as that of the corresponding HKLL operator Φ, that is

$$\langle\Psi_1|\hat{\Phi}|\Psi_2\rangle = \langle\Psi_1|\Phi|\Psi_2\rangle + O(1/N) \qquad\qquad \forall\ |\Psi_1\rangle, |\Psi_2\rangle \in \mathcal{H}_0\,. \tag{2.6}$$

The interpretation is that by performing the integral 2.4 we have removed the gravitational dressing of the operators from the boundary and moved it over to the state. This is only possible on states where 2.2 decays sufficiently fast.

The operators 2.4 have vanishing commutators with the asymptotic charges to all orders in $1/N$. This demonstrates that the apparent obstacle to identifying a commutant due to Gauss's law can be overcome. In order to find a true commutant we need to ensure vanishing commutators to all orders in $1/N$ with all single-trace operators in the time-band algebra. It would be interesting to explore whether a formula achieving this goal and similar to 2.4 can be derived, possibly by integrating over the unitary orbits generated by $\mathcal{A}$.

We provide an alternative formal argument supporting the idea that the algebra $\mathcal{A}$ has a nontrivial commutant when acting on the code subspace $\mathcal{H}_{code}$ of a complicated state $|\Psi_0\rangle$. To see that we consider an operator $\hat{\Phi}$ *defined* by

$$\hat{\Phi}A|\Psi_0\rangle = A\Phi|\Psi_0\rangle \qquad \forall A \in \mathcal{A}\,, \tag{2.7}$$

where again Φ is a standard HKLL operator. This represents a set of linear equations, one for each $A \in \mathcal{A}$, which define the action of $\hat{\Phi}$ on $\mathcal{H}_0$. A sufficient condition for the consistency of these equations is that for all non-vanishing operators $A \in \mathcal{A}$ we have $A|\Psi_0\rangle \neq 0$. In section 2.5 we provide evidence that this is true in the $1/N$ expansion. Given that these equations are consistent, we will show in section 2.5 that the operators $\hat{\Phi}$ defined by 2.7

obey the following properties: i) by construction they commute with operators in $\mathcal{A}$ and ii) to leading order at large N act like HKLL operators. This provides evidence that the algebra $\mathcal{A}$ has a commutant in the $1/N$ expansion. As mentioned earlier, a commutant is not expected at finite N. Indeed, at finite N it is possible to find complicated operators in the time-band which annihilate the state $|\Psi_0\rangle$ and equations 2.7 do not have a consistent solution.

If we take the state $|\Psi_0\rangle$ to be the vacuum, i.e. empty AdS, then the previous construction fails: since the vacuum is invariant under the asymptotic symmetries we no longer have the decay of 2.2 and 2.6 fails. Also 2.7 fails because there are operators in the time-band, in particular H, which annihilate the state. We emphasize that this failure is not a limitation of our particular construction. Instead the interpretation of this failure is that since empty AdS has no bulk features, the only way to specify a point in the bulk is by dressing it to the boundary. Hence any bulk diff-invariant operators acting around the vacuum will not commute with the asymptotic charges [Giddings & Kinsella, 2018, Donnelly & Giddings, 2016]. This can also be seen from the fact that even classically, the local diff-invariant observables cannot be defined properly in the vacuum.

We emphasize that the results of this paper do not contradict the claim of [Chowdhury et al., 2022] that *specifically* for perturbative states around empty AdS, it is possible to reconstruct the state from correlators in the time-band. However we notice that *interesting* states, that is, states which have bulk observers capable of performing physical experiments, are expected to be of the form where the symmetries are broken and the construction presented in this paper can be applied.

If the state $|\Psi_0\rangle$ corresponds to a black hole state, and if the variance of the asymptotic charges scales like N^2 [6] we find that using the operators 2.4 we can create excitations behind the horizon which cannot be detected by correlators of single-trace operators in the $1/N$ expansion. Understanding how to diagnose these excitations from a CFT calculation remains an outstanding open problem. We emphasize that this does not contradict the fact that, generally, excitations created by unitaries on top of typical states with small energy spread can be detected by single-trace correlators [?, Papadodimas & Raju, 2016, ?]. Such states with small energy spread are those for which our construction cannot be applied.

The operators we identify provide evidence supporting the idea that locality is respected in perturbative quantum gravity and that information can be localized in subregions at the level of perturbation theory, provided that the underlying state is sufficiently complicated. It also suggests that it should be possible to associate algebras of observables to subregions. However these observables have certain features of state-dependence, since both 2.4 and 2.7 give operators which are defined only on the code-subspace of the original state $|\Psi_0\rangle$. It is certainly possible to extend the domain of definition of our operators by combining together code subspaces of sufficiently different states, each one of which must break the asymptotic symmetries, thus partly eliminating the state-dependence of the operators. However the number of these states must not be too large, otherwise the small overlaps between the code subspaces start to accumulate and modify the correlators. This becomes particularly relevant for black hole states, where we do not expect to have operators with the desired properties defined globally for most microstates and some genuine state-dependence is expected.

The plan of the paper is as follows: in section 2.2 we review background material about

[6] For example, this is true for black hole states with energy spread similar to the canonical ensemble.

various aspects of locality in field theory and gravity. In section 2.3 we describe the setup in AdS/CFT and study the decay of the inner product 2.2. In section 2.4 we introduce the operators 2.4 and discuss their basic properties. In section 2.5 we provide an alternative argument for the existence of a commutant based on equations 2.7. In section 2.6 we consider various examples. In section 2.7 we consider aspects of our operators in the presence of black holes. Finally we close with a discussion of open problems in 2.8.

2.2 Aspects of locality in field theory and gravity

In this section, mostly addressed to non-experts, we review some background necessary to explore the question of localizing information in different regions of space. A closely related question is the association of algebras of observables to subregions and the factorization of the Hilbert space. We start with non-gravitational field theories, where a non-dynamical background space-time can be used in order to define sub-regions and their causal relations, and then we consider the additional complications when gravity is taken into account.

In relativistic theories we expect that signals and information cannot travel faster than light. We then want to address the following question: consider an initial space-like slice Σ and divide it into a compact subregion D and its complement D'. We denote by $J(D')$ the domain of dependence of D'. The question is the following: is it possible to modify the state[7] in region D without affecting the state in $J(D')$. If the answer is positive then an observer initially in D', and confined to move in $J(D')$, cannot reconstruct information about the interior of D. Then we say that information can be localized.

[7] Either classical state, or quantum density matrix.

2.2.1 Classical field theories

At the classical level this question can be addressed by studying the initial value problem: we specify initial data $\mathcal{C}$ on a spacelike slice Σ and then look for a solution in the entire space-time, or at least a neighborhood of the slice Σ, compatible with the initial data. The initial data will typically include the values and time-derivatives of various fields of the theory. The theories we will be considering have gauge invariance. One of the implications is that the existence of a solution is guaranteed only if the initial data satisfy certain *constraints*. In relativistic field theories theories the dynamical equations are hyperbolic, which ensures that signals propagate forward from Σ at most at the speed of light. On the other hand the constraint equations for initial data are of elliptic nature. This makes the question of being able to specify the initial data independently in region D and its complement D' non-trivial. It is thus convenient to divide the question formulated above in two steps:

- **A. Localized preparation of states:** for given initial data $\mathcal{C}_1$ on Σ satisfying the constraints, to what extent can we deform to other initial data $\mathcal{C}_2$, also satisfying the constraints, such that $\mathcal{C}_1, \mathcal{C}_2$ agree on D', possibly up to a gauge transformation, but differ essentially[8] on D?

- **B. No super-luminal propagation:** suppose we are given two initial data $\mathcal{C}_1, \mathcal{C}_2$ which satisfy the constraints, which agree on D' and differ on D. We then want to show that the two corresponding solutions agree on $J(D')$, possibly up to a gauge transformation.

We will return to the classical problem in theories with gauge invariance in the following

[8] i.e. cannot be matched by a gauge transformation on D.

subsections. For now we briefly consider the simplest example of a free Klein-Gordon field in flat space obeying $\Box\varphi = m^2\varphi$. We consider initial data on the slice Σ corresponding to $t = 0$. The initial data on this slice are parametrized by $\mathcal{C} = \{\varphi(t = 0, x), \partial_0\varphi(t = 0, x)\}$. In this case condition A mentioned earlier is clearly satisfied: the initial data do not need to obey any constraint, so we can simply select the functions $\varphi, \partial_0\varphi$ to have any smooth profile with features strictly localized in D. Notice that this requires the use of non-analytic initial data. Condition B is also satisfied, see [Wald, 1984] for a basic review.[9]

2.2.2 Localization of information in QFT

In non-gravitational QFT we can associate algebras of observables to space-time regions [Streater & Wightman, 1989, Haag & Kastler, 1964, Haag, 1992b]. Locality is exact, and is expressed by the condition that algebras corresponding to space-like separated regions commute. An analogue of the initial value problem in QFT is expressed by the condition of *primitive causality* or relatedly the *time-slice axiom* which postulates that the only operators commuting with the algebra generated by operators in a time-band are proportional to the identity. Moreover a local version of these statements postulates that the algebra of operators in a subregion coincides with the algebra of operators in the causal domain of dependence of the subregion [Haag & Schroer, 1962a].

An intuitive way to see that that information can be localized in QFT is as follows: suppose $|\Psi_0\rangle$ is a state in the Hilbert space of the QFT. Consider a unitary operator U_D constructed out of observables localized in D and the new state $|\Psi\rangle = U_D|\Psi_0\rangle$. The unitary

[9] In the case of non-relativistic theories, for example the heat equation, which is first order in time and hence not hyperbolic, we are able to specify the initial data in subregions independently but the speed of propagation is unbounded. Hence the heat equation obeys condition A but not B.

U_D modifies the state by creating an excitation in region D which encodes the desired information in that region. For any observation $\mathcal{O}_{D'}$ in region D', and more generally in $J(D')$, we have

$$\langle\Psi|\mathcal{O}_{D'}|\Psi\rangle = \langle\Psi_0|U_D^\dagger\mathcal{O}_{D'}U_D|\Psi_0\rangle = \langle\Psi_0|\mathcal{O}_{D'}|\Psi_0\rangle\,, \tag{2.8}$$

where we used $[U_D, \mathcal{O}_{D'}] = 0$. Hence states $|\Psi\rangle, |\Psi_0\rangle$ are indistinguishable by measurements in $J(D')$ and the excitation created by U_D in D is invisible in $J(D')$.

Comments on the split property

More generally we would like to know whether it is possible to independently specify the quantum state in space-like separated regions. The question is non-trivial since in most quantum states these regions will be entangled. It is believed that, as long as the regions in question are separated by a finite buffer region, then the answer should be positive. This is related to the *split property* of quantum field theory [Haag, 1992b, Roos, 1970, Buchholz, 1974, Doplicher & Longo, 1984].

The split property can be defined as follows: consider the causal diamond whose base is a ball D_1 and the corresponding operator algebra $\mathcal{A}_{D_1}$. Consider a slightly larger ball D_2, containing D_1, with corresponding operator algebra $\mathcal{A}_{D_2}$ in its causal diamond. The split property is satisfied if we can find a type I von Neumann algebra of operators $\mathcal{N}$ such that $\mathcal{A}_{D_1} \subset \mathcal{N} \subset \mathcal{A}_{D_2}$. It has been shown that quantum field theories with a reasonable thermodynamic behavior, as expressed in terms of nuclearity conditions (see [Haag, 1992b] for an introduction), satisfy the split property. Using the algebra $\mathcal{N}$ we can have strict localization of quantum information which is completely inaccessible from $J(D_2')$.

Equivalently, the split property can be defined by the existence of a state $|\varphi\rangle$ which is cyclic and separating for the algebra $\mathcal{A}_{D_1 \cup D_2'}$ and such that

$$\langle \varphi | a\, b | \varphi \rangle = \langle 0 | a | 0 \rangle \langle 0 | b | 0 \rangle \qquad \forall\; a \in \mathcal{A}_{D_1}, b \in \mathcal{A}_{D_2'}\,, \tag{2.9}$$

where $|0\rangle$ is the Minkowski vacuum and D_2' denotes the complement of D_2. In the state $|\varphi\rangle$, the mutual information between regions D_1 and D_2' is vanishing. Such a state is not uniquely defined, since for any unitary $U \in \mathcal{A}_{(D_1' \cap D_2)}$ a state of the form $U|\varphi\rangle$ will also satisfy 2.9.

Starting with a split state $|\varphi\rangle$ we can construct more general states by exciting the two regions D_1 and D_2' acting with localized operators in the corresponding algebras. Since there is no entanglement between D_1 and D_2' in the split state $|\varphi\rangle$ the two algebras act independently and we can arbitrarily approximate an excited state in D_1 and another state in D_2'.

An interesting question is to estimate the energy of a split state. We do not expect a split state to be an energy eigenstate, so in general it will have non-vanishing energy variance. Here we provide some very heuristic arguments about the expectation value of the energy. As a starting point, let us consider a CFT on $\mathbb{R}^{1,d-1}$ with coordinates $x^0, x^1, ..., x^{d-1}$. We define two regions to be the causal domains of two slightly displaced Rindler wedges with bases $x^0 = 0, x^1 < -\varepsilon$ and $x^0 = 0, x^1 > \varepsilon$ respectively. The two wedges are separated by the buffer region $-\varepsilon < x^1 < \varepsilon$. In this case the total energy of the split state will be infinite due to the infinite planar extension of the regions in the transverse directions. However, we expect to have a finite energy per unit area $\mathcal{E}$. Since we are dealing with a CFT then the only scale in the problem is the size ε of the buffer region. Hence by dimensional analysis the energy per unit area will scale like $\mathcal{E} = \frac{s}{\varepsilon^{d-1}}$ where s is a constant depending on the CFT. If we now consider a more general compact region D_1 of typical size R, which is separated by a small buffer region of typical size ε from D_2' then we would expect that a split state with respect to D_1, D_2' will have energy which in the $\varepsilon \to 0$ limit will scale like

$$E = s\frac{A_{(\partial D_1)}}{\varepsilon^{d-1}} + O(\frac{\varepsilon}{R})\,, \tag{2.10}$$

where $A_{(\partial D_1)}$ is the area of the boundary of D_1. This is a heuristic estimate and it would be interesting to investigate it more carefully.

As mentioned above, this is the expectation value of the energy and it would be interesting to understand the spectral decomposition of a split state in the energy basis. Notice that a split state does not respect the Reeh-Schlieder property with respect to the algebra $\mathcal{A}_{D_1}$ This implies in particular that the split state should have non-compact support in energy, since otherwise the Reeh-Schlieder property would have to hold for D_1, see for example [Witten, 2018].

Subtleties with gauge invariance

Consider U(1) gauge theory minimally coupled to a charged scalar with Lagrangian $\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - (D_\mu\varphi)^* D^\mu\varphi\ ,\ D_\mu\varphi = \partial_\mu\varphi - igA_\mu\varphi$. The system has $U(1)$ gauge invariance $A_\mu \to A_\mu + \partial_\mu\Lambda$, $\varphi \to e^{ig\Lambda}\varphi$. The dynamical equations are

$$\partial^\nu F_{\mu\nu} = ig(\varphi\partial_\mu\varphi^* - \varphi^*\partial_\mu\varphi) - 2g^2 A_\mu\varphi^*\varphi \tag{2.11}$$

$$\Box\varphi = ig(\partial_\mu A^\mu)\varphi + 2igA^\mu\partial_\mu\varphi + g^2 A_\mu A^\mu\varphi\,. \tag{2.12}$$

In this case the initial data are $\mathcal{C} = \{A_\mu(t=0,x), \partial_0 A_\mu(t=0,x), \varphi(t=0,x), \partial_0\varphi(t=0,x)\}$. Here we encounter the subtleties mentioned for gauge systems: initial data related by a gauge transformation are physically equivalent and initial data are admissible (i.e. lead to a solution) only if the obey a constraint, the Gauss law, which is the $\mu = 0$ component of the first equation in 2.11

$$\partial^i(\partial_0 A_i - \partial_i A_0) = ig(\varphi\partial_0\varphi^* - \varphi^*\partial_0\varphi) - 2g^2 A_0\varphi^*\varphi\,. \tag{2.13}$$

We now revisit the two properties mentioned in subsection 2.2.1. The fact that the dynamical part of 2.11 obey condition B follows from general properties of hyperbolic equations of this type. Let us now examine question A in this theory. From 2.13 we see that if we try to deform the initial data in region D, then we may be forced to change them in D' too. For example if we turn on a profile for the scalar in region D with total non-zero charge, then the gauge field has to be turned on in region D'. The Gauss law constraint 2.13 is of the familiar form $\nabla\cdot\vec{E} = \rho$. This imposes the constraint that $\oint_{\partial D}\vec{E}\cdot d\vec{S} = Q_D$.

However it is clear that once we make sure that the initial data in $\mathcal{D}'$ are compatible with the Gauss constraint from the total charge Q_D enclosed in D, there are many ways of rearranging the initial data in region D keeping those in D' fixed. In other words there are deformations of the constraint equation 2.13, which are not gauge-equivalent, and which have compact support localized in D. This means that theory under consideration obeys condition A.

Moving on to the quantum theory, we can consider $U(1)$ gauge theory weakly coupled to matter. As in the classical theory the total charge Q enclosed in a region can be measured on its boundary and the total charge of the entire state can be measured at space-like infinity. At the quantum level we can get information not only about the expectation value of the charge but all the higher moments

$$\langle\Psi|Q^n|\Psi\rangle \qquad , \qquad n = 1, 2, ... \tag{2.14}$$

To proceed it is useful to consider observables in this theory. Physical observables must be gauge invariant. In a $U(1)$ gauge theory there are several examples of such observables which are also local, for example local operators constructed out of $F_{\mu\nu}(x)$ or $\varphi^*(x)\varphi(x)$. Other interesting gauge invariant operators which are not completely local, but can be contained in compact regions are closed Wilson loops $e^{ig\oint_C A_\mu dx^\mu}$ or bilocals of the form $\varphi^*(x)e^{ig\int_{C,x}^{y} A_\mu dx^\mu}\varphi(y)$. All these operators are neutral and do not change the electric charge of the region D, if they are entirely contained in D. We can use such operators localized in region D to construct unitaries U_D which can be used to modify the state inside D leaving all correlators outside invariant, as in 2.8. So information can be localized in this theory if we work with neutral operators.

But what if we want to create an excitation in region D which has non-zero charge? We already know from the classical problem that it will not be possible to add a charge in D without affecting the exterior due to Gauss law 2.13. The same is true at the quantum level. A charged operator φ in D is not gauge invariant. It can be made gauge invariant by dressing it with a Wilson line extending all the way to infinity. We can think of the Wilson line as a localized tube of electric flux ensuring that Gauss law is satisfied. It may be energetically better to smear the Wilson line in a spherically symmetric configuration. The important point is that the dressed operator $\Phi(x) = e^{ig\int_\infty^x A_\mu dx^\mu}\varphi(x)$ is no longer a local operator, though it is gauge invariant. If we act with a unitary made out of this operator, we will modify correlators outside D and 2.8 will fail. This means that the addition of the charge in D can be detected immediately outside. This is not surprising, as the same thing is already visible at the classical level.

However, looking a bit more carefully, we run into certain somewhat surprising features of the quantum theory. Suppose we have several charged fields φ_i, labeled by a flavor index i, with the same electric charge. We construct the corresponding dressed operators $\Phi_i(x) = e^{ig\int_\infty^x A_\mu dx^\mu}\varphi_i(x)$, using some particular prescription for the Wilson line. These obey

$$[Q, \Phi_i(x)] = g\,\Phi_i(x)\,, \tag{2.15}$$

where $Q = \int_{\mathbb{S}^2_\infty} *F$ is the charge operator which can be measured at space-like infinity. Suppose the point $x = 0$ is inside D. We create a charged excitation of type i in region D by acting on $|0\rangle$ with a unitary $U_i = e^{i\varepsilon\Phi_i(0)}$. Then we study correlators in region D' in the state $U_i|0\rangle$ in perturbation theory. Consider a correlator of Q and $\Phi_j(x)$ in region D'.

$$\langle 0|U_i^\dagger \Phi_j(x) Q U_i|0\rangle = \langle 0|\Phi_j(x)|0\rangle + i\varepsilon\langle 0|[\Phi_j(x)Q, \Phi_i(0)]|0\rangle + \mathcal{O}(\varepsilon^2)\,, \tag{2.16}$$

where to leading order in the perturbative expansion the second term is

$$\langle 0|[\Phi_j(x)Q, \Phi_i(0)]|0\rangle = g\langle 0|\Phi_j(x)\Phi_i(0)|0\rangle \propto \delta_{ij}\,. \tag{2.17}$$

Hence by measuring correlators of all $\varphi_j(x)$ and Q in $\overline{\mathcal{D}}$ it seems that we can detect not only the presence of a charge in D, which is expected by Gauss's law, but we can even identify the flavor of the charged particle, i.e. the value of the index i in the interior of D. A similar argument in the gravitational case was discussed in [?, ?] for black hole states and in [Chowdhury et al., 2021] around empty AdS.

The reason we were able to get information beyond the total charge in D is that in the vacuum the fields have non-trivial entanglement, on which the non-vanishing 2-point function 2.17 depends. When we act with the unitary containing the Wilson line, the Wilson line disturbs the pattern of entanglement in such a way that it breaks the symmetry between the fields φ_i and we can detect from D' the flavor of the excitation in D

This suggests a way to avoid the issue and succeed in hiding the flavor of charge in D. We start with the analogue of a split state in the U(1) gauge theory, see the discussion in [Donnelly & Giddings, 2017], and then create the charged excitation in D by acting with the same unitary. In that case there is no entanglement between D and D' and hence 2.17 will vanish making it impossible to tell from measurements in D' what is the type of charged excitation in D. (A more mundane way to hide the charge is to add ”screening charges” in the buffer region, but here we want to discuss how information can be localized even though a Wilson line extends all the way to infinity.) This requires creating the charged excitation on top of the split state, with typical energy scaling like 2.10, rather than the ground state.

2.2.3 Classical and Quantum Gravity

First we notice that in non-perturbative quantum gravity we do not expect to be able to localize information in space: holography and AdS/CFT suggest that the fundamental degrees of freedom in quantum gravity are not local, but rather lie at the boundary. Moreover there is strong evidence that an ingredient towards the resolution of the black hole information paradox is that the naive factorization of the Hilbert space in space-like separated subregions may not be true in the underlying theory of quantum gravity.

On the other hand at the classical level in General Relativity we do have an exact notion of locality and information can be localized, as we will discuss below. An interesting question, which is the main focus of this paper, is to understand the fate of locality at the level of perturbative quantum gravity.

On the initial value problem of general relativity

In General Relativity the initial value problem is formulated by starting with a spacelike slice Σ and specifying the data $\mathcal{C} = (h_{ab}, K_{ab})$ where h_{ab} is the intrinsic metric and K_{ab} the extrinsic curvature of Σ. If we have matter then the values of the fields and their normal derivatives need to be specified. Initial data related by spatial diffeomorphisms on the slice Σ are gauge-equivalent and have to be physically identified. In general relativity there is one more subtlety: even if we have two initial data on the slice Σ which are not related by a spatial diffeomorphism, they may still correspond to the same physical solution in space-time. This is related to the freedom of choosing the initial slice Σ in space-time and diffeomorphism invariance in full space-time.

Admissible initial data, which can be extended into a solution of the Einstein equations must obey the following constraints

$$R + (K^a_a)^2 - K_{ab}K^{ab} = 16\pi G\rho \tag{2.18}$$

$$\nabla^a K_{ab} - \nabla_b K^c_c = -8\pi G J_b \,, \tag{2.19}$$

where R is the Ricci scalar of h_{ab} on Σ, the covariant derivatives are with respect to h_{ab} on Σ, n^a is the unit normal to Σ and $\rho = T_{ab}n^a n^b$ and $J_b = -h^c_b T_{ca} n^a$.

We now want to address the question of localization of information in classical general relativity, as formulated in subsection 2.2.1. A theorem, see for example [Hawking & Ellis, 2011, Wald, 1984], settles question B for pure general relativity: if we have two admissible initial data which agree, up to spatial diffeomorphism, on a part D' of Σ, then the corresponding solutions will agree, up to a space-time diff, on the development of D'. This continues to be true in the presence of matter provided certain reasonable conditions are satisfied. This shows that in general relativity signals propagate at most at the speed of light: if we modify the initial data only in the region D, then the signals will propagate in the causal future of D.

Then we come to question A, that of localizing information on compact regions on Σ: to what extent is it possible to find two initial data satisfying the constraints 2.18, 2.19, which agree on D' but differ on D? (Here we need to keep in mind that even if the initial data differ on D they may correspond to the same solution in space-time, as they may correspond to two different choices of the slice Σ in the same space-time solution.) The equations 2.18 and 2.19 are non-linear and of elliptic nature, though underdetermined. Understanding the space of solutions of the constraint equations is an interesting problem which has been studied extensively in the literature. Here we summarize some relevant points:

1. **Gravitational Gauss law:** in asymptotically flat or AdS space-times, the energy and other conserved charges are defined at space-like infinity. The constraints of general relativity relate these asymptotic charges to contributions from excitations in the interior of space-time. For example, in the Newtonian limit the constraint equations

reduce to the gravitational analogue of Gauss's law

$$\Box\varphi = 4\pi G\rho\,.$$

As in electromagnetism this implies that the initial data in region D' know about the total mass enclosed in D.

2. **Existence of localized deformations:** it is possible to find many solutions of the constraint equations which look the same in the domain D' but differ on D. For example, if we restrict our attention to spherically symmetric solutions, Birkhoff's theorem implies that there is a large number of solutions of 2.18 and 2.19 which all look like the Schwarzschild metric of mass M in D' but differ in D. Examples include static, interior, star-like geometries supported by matter or more generally spherically symmetric, time-dependent collapsing geometries of mass M. More generally, it has been shown [Corvino & Schoen, 2006] that under reasonable conditions a compact patch D of a solution of the constraints 2.18 and 2.19 can be glued to a boosted, Kerr solution in D' of appropriate mass, angular momentum, momentum and center of mass position. The existence of a large number of solutions, which all look exactly the same in D' demonstrates that it is possible to localize information in classical general relativity.

3. **Comments on the vacuum:** For asymptotically AdS geometries, if a solution looks like empty AdS in D' we assume that D is compact so D' includes the region near space-like infinity, then it is guaranteed to be empty AdS in D as well. In other words, starting with the vacuum it is not possible to modify the initial data in D into a new

solution, without at the same time modifying the solution in D'.

Diff-invariant observables in classical GR

We now consider the question of defining local diff-invariant observables in gravity. This is a long-standing problem which is subtle even at the classical level. Let us consider general relativity, possibly coupled to other fields, defined with certain asymptotic boundary conditions at infinity (for example asymptotically flat or AdS) or on a closed manifold of fixed topology. We denote by $\overline{\mathcal{X}}$ the space of solutions of the equations of motion, in any possible coordinate system. On this space we have the action of the group Diff of diffeomorphisms[10]. Solutions related by a diffeomorphism are physically identified and we introduce

$$\mathcal{X} = \overline{\mathcal{X}}/\mathrm{Diff}\,. \tag{2.20}$$

We can think of a diff-invariant observable as a function which has definite values on points of $\mathcal{X}$. However, we do not demand an observable to be necessarily defined on the entire space of solutions $\mathcal{X}$. Instead we will allow observables to possibly have a limited domain of definition. Hence a diff-invariant observable is a map

$$A : U \subset \mathcal{X} \to \mathbb{R}\,, \tag{2.21}$$

where U is an open subset of $\mathcal{X}$. Such observables can also be expressed as functions on $\overline{\mathcal{X}}$ which must obey $\overline{A}(s) = \overline{A}(f_* s)$, where s denotes a solution in some coordinate system and f_* the action of a diffeomorphism.

[10] If the space-time is non-compact along space we only consider *small* diffeomorphism, i.e. those which become trivial fast enough at infinity.

In order for a diff-invariant observable to be local we need to impose additional conditions. To formulate these conditions it is useful to introduce the Peierls bracket $\{A, B\}$ between two diff-invariant observables [Peierls, 1952], which is a covariant generalization of the Poisson bracket. To compute the value of $\{A, B\}$ we consider a modification of the action as $S \to S + \varepsilon A$ and compute the difference of the first order change of observable B on the perturbed solutions with advanced $(+)$ and retarded $(-)$ boundary conditions. The Peierls bracket is defined as[11]

$$\{A, B\} = \delta_A^- B - \delta_A^+ B\,. \tag{2.22}$$

It can be shown that the Peierls bracket has similar properties as the Poisson bracket, for example linearity, antisymmetry and the Jacobi identity, and in fact coincides with the Poisson bracket if a Hamiltonian formalism is introduced. One of the advantages of the Peierls bracket is that we do not need to pass to the Hamiltonian formalism which is somewhat complicated due to the constraints. Notice that to define the Peierls bracket of two observables A, B they must have a common domain of definition on $\mathcal{X}$ and the bracket will be generally a non-trivial function on this overlap.

We would like to define diff-invariant observables which can be associated to *points* in space-time with the property that if two such observables are associated to space-like separated points the corresponding Peierls bracket must vanish. The difficulty in doing this is that in order to define an observable we need to define it at least in an open neighborhood around a state as in 2.21, so we need some prescription for following "the same point", on

[11] The first order solutions are not unique due to diffeomorphism invariance, however the ambiguity drops out when computing the change of the diff-invariant observable B.

which the candidate diff-invariant observable will be localized, as we move on the space of solutions $\mathcal{X}$. General covariance implies that there is no canonical way to keep track of the point as we change the state.

If the space-time has a well-defined boundary we can find prescriptions which select a point in space-time for each solution in $\mathcal{X}$ *relationally* with respect to the boundary. For example in AdS we can define a diff-invariant observable which *seems* to be localized at a point by considering a radial geodesic at right angle from a specific point on the boundary, moving a fixed regularized distance along it and measuring the value of a scalar quantity, for example a scalar field or a scalar combination of the curvature, at the resulting point. This gives a map from the space of solutions $\mathcal{X}$ to $\mathbb{R}$, so it is a diff-invariant observable which could potentially be local. Notice however that the location of the resulting point depends on the entire geometry along the geodesic, all the way from the boundary. Changing the metric anywhere along this geodesic will move the resulting point. Hence the value of the observable will not strictly depend on local data near the point. Similarly, if we act with one of the asymptotic symmetries the boundary starting point will move and also the resulting bulk point will move. This implies that the Peierls brackets of this candidate observable with the boundary charges, or other observables along the geodesic will be non-zero, even though these regions are space-like separated. Hence this relational observable is not really local.

Another way to to define candidate local diff-invariant observables is to consider a complete gauge fixing scheme. Then observables in the particular gauge labeled by a space-time coordinates are automatically diff-invariant. However they will generally have non-local Peierls brackets, since the assignment of a coordinate value to a point in space-time in the particular gauge, will generally depend on the solution everywhere.

Additional difficulties arise in space-times without boundaries, for example in de Sitter space. A boundary is an (asymptotic) part of the spacetime where gravity is not dynamical anymore. This is why we can for example anchor geodesics to the boundary, and define relational diff-invariant observables. Without a boundary, there is no part of the space-time where gravity is turned off, and consequently no place to anchor geodesics.

State-dressed observables

If we consider a solution that is sufficiently complicated it is possible to specify points, and hence define local diff-invariant observables, by using *features of the state*. We emphasize that these observables will not have all the desired properties over the entire space of solutions $\mathcal{X}$, so these observables have certain aspects of state-dependence as discussed around 2.21. One approach based on this idea was studied by DeWitt [DeWitt, 1962], building on [Komar, 1958, Bergmann & Komar, 1960]. For a D-dimensional space-time we start by identifying D scalar quantities Z^a, $a = 1, \ldots, D$. These can be combinations of curvature invariants and other scalars formed by the fields of the theory. We could try to fix a coordinate system by using these D-scalars as coordinates. We can use this intuition to introduce candidate local diff-invariant observables of the form

$$\varphi(Z_0^a) = \int d^D x \, \varphi(x) \, \delta^D(Z^a - Z_0^a) \det \frac{\partial Z}{\partial x} \,. \tag{2.23}$$

Here Z^a are the D scalar quantities introduced above and φ is any other scalar combination of the fundamental fields of the theory. Similar constructions can be done for fields with tensor indices.

Some comments are in order:

1. For a general space-time which is in-homogenous, and for certain choices of the values Z_0^a, the delta function in 2.23 will click on a finite number of points, so the quantity above is well-defined and finite. In symmetric space-times it will either not click at all, hence the observable will be zero, or an infinite number of times so the observable will be ill-defined. This shows that 2.23 is a quantity which is defined only on part of the phase space. This is in accordance with our expectation that state-dressed observables have to be state-dependent 2.21.

2. Suppose that the observable 2.23 is well defined on a state s and a neighborhood U of the space of solutions $\mathcal{X}$ around it. It is clear that, at least at the classical level, this observable is diff-invariant, i.e. a well defined map $\varphi(Z_0^a) : U \subset \mathcal{X} \to \mathbb{R}$ and hence a good observable according to the definition 2.21.

3. One can show that under certain conditions, observables 2.23 are also *local*. If we have a state s on which two such observables $\varphi(Z_A^a), \varphi(Z_B^b)$ are well defined, with the property that the delta functions click at single points A, B and that these points are space-like separated with respect to the metric of s, then the corresponding observables have vanishing Peierls brackets $\{\varphi(Z_A^a), \varphi(Z_B^b)\} = 0$, see [Dewitt, 1999] for a review. This follows from the causality properties of linearized Green's functions appearing in 2.22 around the solution s. Notice that if two points A, B are spacelike separated on a solution s, then there is a small enough neighborhood of s in which they remain space-like separated. Hence their Peierls bracket will vanish in this entire neighborhood.

4. This shows that, as long as we accept that observables may be defined only locally on the phase space of solutions, it is possible to find local, diff-invariant observables in classical general relativity around states which are complicated enough. These are also the interesting states, i.e. those containing bulk observers who want to study physics in their environment.

5. Similar ideas are useful in cosmology, where the value of a scalar field can be used as as clock [Page & Wootters, 1983, Kuchar, 2011, Isham, 1993].

The next question is whether it is possible to define similar observables at the quantum level. Aspects of this question were discussed in [Giddings et al., 2006] and [Marolf, 2015], where it was argued that there is a quantum version of these observables which retain their locality properties to all orders in the $\hbar$ expansion, even though they are not expected to be local at the non-perturbative level. Various difficulties are encountered at the quantum level including the question of the renormalization of the composite operators 2.23, establishing diffeomorphism invariance at the quantum level and the role of Poincare recurrences which will generally introduce infinite copies where the delta function will have support [Marolf, 2015]. In this paper we provide support in favor of this conjecture by finding observables with certain similarities in spirit to 2.23 directly in CFT language. This has the advantage that any object built directly in the CFT is by construction diff-invariant.

A time-band in AdS

We now specialize to a setup that will allow us to make contact with AdS/CFT. We consider geometries that are asymptotically AdS_{d+1} and we consider a short time-band $\mathcal{T}_{-\varepsilon,\varepsilon}$ on the boundary in global coordinates, defined as the set of points $(-\varepsilon, +\varepsilon) \times \mathbb{S}^{d-1}$, $\varepsilon > 0$, where the first interval refers to the time coordinate t. Near the boundary we can select a Fefferman-Graham coordinate system where the fields, for example the metric and a scalar of mass m^2, have the behavior

$$ds^2 = \frac{dr^2}{r^2} + r^2(-dt^2 + d\Omega_{d-1}^2) + r^{2-d} g_{\mu\nu}(r,x)\ dx^\mu dx^\nu \qquad g_{\mu\nu}(r,x) = g^{(0)}_{\mu\nu}(x) + g^{(2)}_{\mu\nu}(x) r^{-2} + \ldots \tag{2.24}$$

$$\varphi = r^{-\Delta}(\varphi^{(0)}(x) + \varphi^{(2)}(x) r^{-2} + \ldots)\,, \tag{2.25}$$

where $x = (t, \Omega_{d-1})$ and $\Delta = \frac{d}{2} + \sqrt{\frac{d^2}{4} + m^2}$. Here we consider normalizable states so the growing modes, which would be dual to sources in the CFT, are set to zero[12]. The Fefferman-Graham coefficients $g^{(0)}_{\mu\nu}(x)$, $\varphi^{(0)}(x)$ are diff-invariant observables and are labelled by boundary coordinates[13]. This set of observables includes the asymptotic charges, for example the ADM Hamiltonian can be computed as

$$H = \frac{1}{\text{const}} \int_{\mathbb{S}^{d-1}} d\Omega^{d-1} g^{(0)}_{00}(x)\,. \tag{2.26}$$

We focus on these Fefferman-Graham observables restricted in the time band $\mathcal{T}_{-\varepsilon,\varepsilon}$. This set of observables is closed under Peierls brackets and form a Poisson algebra $\mathcal{A}$. Notice that in this algebra we do not include observables which would be finite distance under Poisson flow, otherwise flowing by finite distance with H would take us out of the time-band, see also the discussion in [Marolf, 2009].

[12] We only assume that the sources are zero in the time band $\mathcal{T}$, they could be turned on in the far past in order to prepare a state.

[13] The subleading coefficients are fixed by the equations of motion in terms of the leading ones.

Starting with the classical theory, we ask whether we can find observables localized deep in the interior of AdS which are space-like with respect to the time-band and which have vanishing Peierls brackets with observables in the time-band algebra $\mathcal{A}$. These candidate observables are to be defined as in (2.21), in particular they need to be defined on a neighborhood $U \subset \mathcal{X}$ of a solution $s \in U$ and not necessarily on the entire space of solutions $\mathcal{X}$.

It is clear that observables defined relationally with respect to the boundary, or with a gauge fixing condition which makes use of the boundary, do not satisfy these conditions. Due to their gravitational Wilson lines they will have non-vanishing Peierls brackets with the Hamiltonian and other charges on the boundary [Giddings & Kinsella, 2018, Donnelly & Giddings, 2016]. Such observables generally change the energy of the state, which due to the gravitational Gauss law can be measured in the time band $\mathcal{T}_{-\varepsilon,\varepsilon}$ by 2.26. Another point of view is that such observables identify a point in the bulk, and in particular *a moment in time*, relationally with respect to the boundary. Thus an infinitesimal motion in time of the starting point on the boundary is translated via the relational prescription into an infinitesimal time motion of the corresponding bulk point. Then the Peierls bracket of the candidate bulk observable with H generates time-derivatives of the point in the bulk and is non-vanishing.

The discussion of the previous subsection implies that if we start with an asymptotically AdS_{d+1} solution s of the bulk equations which is complicated enough, then we can define diff-invariant observables of the form 2.23 in a neighborhood of s so that they have vanishing Peierls bracket with all elements of the time-band algebra $\mathcal{A}$ including charges like the Hamiltonian 2.26. Such observables do not change the total energy of the state but instead they rearrange the energy, ”absorbing” from the background solution the amount of energy they themselves create. These observables select a point in the bulk, and a moment in time, by using features of the state.

In what follows we will provide evidence that the same conclusions are true in perturbative quantum gravity. We will proceed by translating the question in CFT language and using the AdS/CFT correspondence.

2.3 Holographic setup

In this paper, we will study the question of locality in quantum gravity in the context of the AdS/CFT correspondence. A question we would like to understand is how certain bulk subregions are encoded in the boundary CFT. There are cases where this is well understood. For example, the bulk dual of a boundary subregion is known as the entanglement wedge, which is the bulk region extending between the boundary subregions and the relevant Ryu-Takayanagi surface extending in the bulk [Ryu & Takayanagi, 2006]. This correspondence between parts of the boundary and bulk is known as subregion-subregion duality [Czech et al., 2012, Almheiri et al., 2015a, Jafferis et al., 2016a], and it is worthwhile to mention that in general, the entanglement wedge of a boundary subregion is much larger than its causal wedge (the part of the bulk contained by lightrays shot from the causal developments of the boundary subregion).

Subregion-subregion duality and entanglement wedge reconstruction utilizes the organization and entanglement of CFT degrees of freedom organized spatially. We will be interested in rather different bulk subregions, which lie deep down in the bulk and never extend to the boundary CFT. What is the CFT dual of a causal diamond located deep near the center of AdS? The answer to this question remains elusive, and in particular it is understood that in general, these bulk regions do not correspond to the entanglement wedge of any boundary subregion. There have been previous attempts to understand the CFT mapping of such regions, see for example [Balasubramanian et al., 2013, Balasubramanian et al., 2014, Myers et al., 2014, Headrick et al., 2014b] which attempt to assign a meaning to the entropy of a general closed codimension-2 spatial curve in AdS. Here we will follow a different approach by focusing on the algebra of single-trace operators [Banerjee et al., 2016].

We will start by reviewing some basic but relevant features of AdS/CFT, before turning to a discussion of the class of states that we will be considering throughout this paper and their salient properties.

2.3.1 Gravitional states in AdS, large diffeomorphisms and asymptotic symmetries

We will be interested in gravitational solutions which are asymptotically AdS_{d+1}. We have in mind an embedding in a top-down setup with a holographic dual CFT, like $\mathcal{N} = 4$ SYM at strong coupling, on $\mathbb{S}^3 \times \mathbb{R}$ and the N-scaling we indicate in most of the paper refers to this theory. However for most of the discussion the details of the embedding in string theory, the extra fields, as well as the presence of a compact internal manifold are not important unless explicitly stated.

Solutions to the bulk equations of motion can be thought of as *states* in the dual CFT. If we think of a bulk geometry described by a Penrose diagram, the diagram really represents the entire time-history of the state. We can take the state to live at $t = 0$ on a boundary

Cauchy slice, and the portion of the geometry relevant to describing the state is an initial data surface given by a bulk Cauchy slice (or the Wheeler-de Witt patch associated to the boundary Cauchy slice). To view these geometries as states of the dual CFT, it is important that the bulk fields have a fall-off corresponding to normalizable modes with vanishing CFT sources.[14]

We want to consider semi-classical solutions with non-trivial bulk geometries, i.e. where backreaction is strong. The corresponding CFT states $|\Psi_0\rangle$, which we take to be pure, have large energies which scale as

$$\langle \Psi_0 | H | \Psi_0 \rangle \sim \mathcal{O}(N^2) \,, \tag{2.27}$$

and as we will see, they will generally also have an energy variance of the same order. We will also consider perturbative excitations of the quantum fields on top of the background geometry. These excitations add/subtract quantum particles which change the energy by an $O(N^0)$ amount, and whose backreaction on the geometry is thus generally small.

Geometries of this type will often be macroscopically time-dependent, such that the initial data on a bulk Cauchy slice changes as we perform time-evolution of the state. This has consequences for the variance of the energy, as we will now see. Any state $|\Psi_0\rangle$ can be expanded in the basis of CFT energy eigenstates as

$$|\Psi_0\rangle = \sum_i c_i |E_i\rangle \,. \tag{2.28}$$

The time-dependence of the bulk geometry implies that such states will have energy vari-

[14]If these states are prepared by a Euclidean path-integral [Skenderis & van Rees, 2008, Botta-Cantcheff et al., 2016, Marolf et al., 2018, Belin et al., 2019], sources can be turned on in the Euclidean past which prepares the state, but it is important that they vanish as $t_E \to 0$ for the geometries to be interpreted as states in the undeformed CFT.

ance

$$(\Delta H)^2 \equiv \langle \Psi_0 | H^2 | \Psi_0 \rangle - \langle \Psi_0 | H | \Psi_0 \rangle^2 \sim \mathcal{O}(N^2)\,. \tag{2.29}$$

To see this, consider the inequality

$$\frac{1}{2} |\langle [H, A] \rangle| = \frac{1}{2} \langle \partial_t A \rangle \leq \Delta H \cdot \Delta A\,, \tag{2.30}$$

where in the first equality we assumed that the operator A is not explicitly time-dependent. Then we have

$$\Delta H \geq \frac{1}{2} \frac{\langle \partial_t A \rangle}{\Delta A} \sim O(N)\,, \tag{2.31}$$

where we have used large N factorization for the operator A. This shows that provided there is macroscopic time-dependence (the classical vev of A changes at leading order), the variance of the energy scales at least as N^2.[15] Some bulk geometries we will consider are macroscopically time-dependent, but only inside the horizon. In this case, we cannot use the argument above, but we still expect the variance to be of order N^2. It is interesting to ask whether the variance is a quantity that can be extracted from the semi-classical geometry alone. In general, we expect that the quantum state of the fields in the bulk is important as well. We discuss this further in Appendix 2.9.

There are various types of explicit constructions of states of this kind. There are states prepared by Euclidean path integral with sources for single-trace operators [Skenderis & van Rees, 2008, Botta-Cantcheff et al., 2016, Marolf et al., 2018, Belin et al., 2019]. These states should be interpreted as coherent states of the quantum gravitational dual, which

[15] Note that if the variance is parametrically larger than $O(N^2)$, the state may no longer have a good semi-classical interpretation. An example would be a superposition of black holes of different masses.

are labelled by phase-space points corresponding to initial data[16]. There are also states prepared by a boundary state of the CFT, further evolved by some amount of Euclidean time [Kourkoulou & Maldacena, 2017, Almheiri et al., 2018a, Cooper et al., 2019a, Miyaji et al., 2021]. The bulk interpretation of these states is that they correspond to black hole geometries with End-of-the-World branes sitting behind the horizon. This is an example where the bulk geometry is macroscopically time-dependent, but only behind the horizon. Similarly, for two-dimensional CFTs, we can construct pure states by performing the path integral over a surface of higher topology, for example half a genus-2 surface, see [Marolf & Wien, 2018]. These geometries are also macroscopically time-dependent behind the horizon, but instead of having a brane behind the horizon, they have topology. Finally, it is worth noting that there are semi-classical geometries that also preserve supersymmetry, the most famous of which are the LLM geometries [Lin et al., 2004]. In these cases, one can obtain a better understanding of the dual CFT states. We will come back to these geometries in section 2.6.

As usual in gravity, we should identify solutions which are related by *small diffeomorphisms*, i.e. diffeomorphisms that vanish near the AdS boundary. There is also a class of large diffeomorphisms, which are compatible with the boundary conditions imposed in the definition of our theory of AdS gravity. This set of diffeomorphisms forms what is called the *asymptotic symmetry group*. In the case of $\mathrm{AdS}_{d+1}, d \geq 3$ this is the conformal group $SO(2,d)$, while for $d = 2$ it gets enhanced to the Virasoro group [Brown & Henneaux, 1986]. When acting on a given bulk solution these large diffeomorphisms will generally

[16] It appears that one may not construct arbitrary initial data this way, see [Belin & Withers, 2020]. This will not affect our construction and for states prepared by a Euclidean path integral, we should simply keep in mind that we have access to a restricted class of initial data.

transform the geometry into a new state, which is physically distinguished from the previous one, unless of course the original state happens to be invariant under the symmetry. We will later also discuss solutions with two asymptotic boundaries, such as the eternal black hole in AdS, in which case the asymptotic symmetry group is larger. Let us now discuss the various elements of the asymptotic group/conformal group:

- **Time translations:** One particular class of states we will discuss are those with semiclassical time-dependence in the bulk, for example a state corresponding to the gravitational collapse of a star. In this case large diffeomorphisms corresponding to asymptotic time translations transform the state as $|\Psi_0\rangle \to e^{-iHt}|\Psi_0\rangle$. The initial data corresponding to $|\Psi_0\rangle$ is not the same as that of $e^{-iHt}|\Psi_0\rangle$. Our end goal will be to provide local operators whose gravitational dressing is done towards a feature of the state. If the state is time-dependent then we can select a moment in time by using the features of the state, as opposed to the boundary time coordinate. On the other hand if the state is static, then the only way to identify a moment in time is by dressing to the boundary. This is why it will be important for us to consider time-dependent states.

- **SO(d) rotations:** If the state breaks $SO(d)$, then asymptotic rotations transform it to a new state. In this case we can use the features of the state to identify the angular location of a point. On the other hand, if the state is $SO(d)$ invariant it will generally not be possible and at best we can obtain an operator smeared over the bulk angular coordinates, or alternatively we can fix the angular location by dressing to the boundary.

- **AdS boosts:** The Lorentzian conformal group acting on $\mathbb{S}^{d-1} \times \mathbb{R}$ has another $2d$ generators which correspond to boosts in various directions. These can be realized as d non-independent copies of an $SL(2, \mathbb{R})$ algebra, see for example [Freivogel et al., 2012]. Any state with finite energy cannot be annihilated by Hermitian combinations of these generators, which we show in Appendix 2.10. The only state which is annihilated by these generators is the global vacuum and any other state will necessarily transform under the action of these boosts[17]. Therefore, in any non-trivial state, we can fix the radial position of an operator without referring to the boundary.

In a top-down setup, the gravity dual may have an internal manifold, like the S^5 in the context of $\mathcal{N} = 4$ SYM. In such cases, we would need to break the R-symmetry to localize a bulk operator in the internal space. In this paper, we will mostly restrict to a bottom up construction without an internal manifold but it would be an interesting generalization.

2.3.2 Locality in AdS

We are now ready to discuss locality in quantum gravity with asymptotically AdS boundary conditions. We would like to understand whether one can define local observables and whether we can localize information deep in the center of the AdS.

The presence of the AdS boundary allows us to define one natural class of diff-invariant observables: The fields in AdS can be expanded in a Fefferman-Graham expansion. The coefficients of this expansion are themselves diff-invariant observables, which are dressed to the boundary since the Fefferman-Graham gauge is chosen with respect to the boundary. Let us call these observables FG-observables. For example, the AdM Hamiltonian is one

[17] States with infinite energy like the AdS-Rindler vacuum could also potentially be annihilated by some boost generators.

particular observable in this class. In perturbative quantum gravity, we can also consider the expectation values of these observables as well as their higher-point correlation functions. As we will discuss below, if we want to stay within the regime which can be described by semi-classical gravity we may need to restrict the complexity of the correlators (for example the number of operator insertions in the correlation function). We emphasize again that all these observables are dressed with respect to the boundary. In particular, they will generally not commute with the Hamiltonian or the other charges described in the previous section.

The question we would like to address is the following. If we start with a state with a semi-classical geometric description, is there a way to modify the state in the interior of AdS, without modifying any of the correlators of FG-observables localized in a short time-band of the boundary? If the answer is yes, this means we can localize information since an observer living near the boundary will have no way to know whether or not we modified the state. Rather than trying to come up with bulk objects that achieve this goal, we will address this question directly in the dual CFT. This has the following advantage: any object built out of CFT degrees of freedom is necessarily diff-invariant and non-perturbatively well defined. Provided the object acts in the right away, we can be assured that the construction is fully consistent.

2.3.3 The CFT description and the time band algebra

Consider a large N holographic CFT which is dual to semi-classical general relativity coupled to matter fields. In the large N limit, we can define the algebra $\mathcal{A}$ generated by single-trace operators in a time-band $\mathcal{D}_{t_1,t_2}$, where we allow products of single-trace operators where the number of factors is arbitrary but scales like $O(N^0)$.[18] This was originally discussed in [Banerjee et al., 2016], inspired by the earlier work [Papadodimas & Raju, 2013, Papadodimas & Raju, 2014a, ?]. In [Banerjee et al., 2016] it was proposed that the algebra $\mathcal{A}$ can be thought of as being dual to the causal wedge of the region $\mathcal{D}_{t_1,t_2}$ in the bulk (see Fig. 2.1). This picture also suggests that the algebra $\mathcal{A}$ has a commutant which can be identified with a spacelike-separated causal diamond in the interior. Algebras of this type have received attention recently [Leutheusser & Liu, 2021a, Leutheusser & Liu, 2021b, Witten, 2021b, Chandrasekaran et al., 2022b, Leutheusser & Liu, 2022].

The work [Banerjee et al., 2016] studied this setup for states which are small perturbations around the AdS vacuum. The geometry of AdS is homogeneous and featureless since it is a maximally symmetric space. As already discussed in the previous section, this makes the definition of local diff-invariant observables challenging. We would like to revisit the time-band algebra, this time in cases where the bulk state has features, which in particular are time-dependent. This means the state must be highly excited as can be seen for example from its energy 2.27.

At infinite N the problem can be understood in terms of QFT on a curved and in general time-dependent background. In particular, gravitational backreaction of the quantum fields can be ignored and one does not need to talk about gravitational dressing, which is a form of backreaction. In this case, the existence of the commutant is obvious because we are in a QFT situation. Note that if the Hamiltonian (which is always an element of the time band algebra) is normalized appropriately[19], its commutator with the other single-

[18] Notice that at finite N the algebra in a time-band would be the same as the full algebra. In the large N limit, a natural hierarchy emerges between "small products" of single-trace operators and the rest of the algebra, which allows us to consider the notion of a time-band algebra.

[19] A useful normalization is $h = \frac{1}{N}(H - \langle\Psi_0|H|\Psi_0\rangle)$, which ensures that $\langle\Psi_0|h^2|\Psi_0\rangle \sim O(N^0)$.

trace operators is suppressed by $1/N$ and thus vanishes when N is infinite.

At the level of $1/N$ corrections, the existence of the commutant is less obvious. Backreaction must now be taken into account and the gravitational Gauss law can spoil the commutator between H and the other operators of the time-band algebra. For example, the standard way to write bulk fields in terms of CFT operators is the HKLL construction [Banks et al., 1998, Bena, 2000, Hamilton et al., 2006d, Hamilton et al., 2006c, Hamilton et al., 2007b, Hamilton et al., 2008b, Heemskerk et al., 2012]

$$\Phi(t, r, \Omega) = \int_{\text{bdry}} dt' \, d\Omega'_{d-1} K(t, r, \Omega; t', \Omega') \mathcal{O}(t', \Omega') \,, \tag{2.32}$$

where K is related to a Green's function of the Klein-Gordon operator on the appropriate bulk geometry. This operator is defined purely within the CFT so it is manifestly diff-invariant. To leading order at large N, it acts as a bulk field and commutes with other bulk fields at spacelike separation. Notice however that in order to define the kernel K we have to choose a coordinate system in the bulk, which often is taken using Fefferman-Graham gauge. As we already mentioned, this gauge choice is defined by making use of the asymptotic boundary, and an HKLL operator is thus dressed to the boundary. Because of this, the commutator between an HKLL operator and the Hamiltonian will not vanish at subleading orders in the $1/N$ expansion.

The physical origin of this effect is the gravitational Gauss law: acting with (2.32) will generally create or destroy a particle in the bulk, thus changing the energy of the state, which can be immediately measured at spacelike infinity by H. One can try to correct the HKLL operators at higher orders in $1/N$ by mixing it with other single- and multi-trace operators, see [Kabat et al., 2011, Kabat & Lifschytz, 2013, Heemskerk et al., 2012], but the commutator with the Hamiltonian is universal and generally cannot be removed in this way. It is also possible to think about the dressing in terms of (smeared) gravitational Wilson lines connecting the bulk operator to the boundary, which make it diff-invariant at the price of making it non-local [Anand et al., 2018, Castro et al., 2018, Chen et al., 2019, Giddings, 2019]. The commutator with H is nonzero because H picks up the contribution of the Wilson line.

This raises the question of whether the algebra $\mathcal{A}$ still has a commutant at subleading orders in $1/N$. The main goal of this paper is to provide evidence for the existence of such a commutant. We will do so by identifying a class of operators that are gravitationally dressed with respect to *features of the state*, rather than dressed to the boundary. In particular, these operators will have vanishing commutators with the Hamiltonian, to all orders in $1/N$. In this paper, we will focus mostly on ensuring that bulk operators have a vanishing commutator with the Hamiltonian (and the other charges), but it would be important to extend our construction to all single-trace operators in $\mathcal{D}_{t_1,t_2}$. We given an alternative argument for the existence of a commutatant to all orders in $1/N$ in section 2.5.

The existence of a commutant for $\mathcal{A}$ in $1/N$ perturbation theory would imply that information can be localized in regions of the bulk and is not visible from the boundary at the level of perturbative quantum gravity[20]. We are now ready to formulate the concrete goal that we will achieve in this paper.

[20]See [Marolf, 2009,Donnelly & Giddings, 2017,Bousso et al., 2018,Donnelly & Giddings, 2018,Jacobson & Nguyen, 2019, Giddings, 2020, Chowdhury et al., 2021, Chowdhury et al., 2022, Giddings, 2022] for other discussions of localization of information in perturbative quantum gravity, with varying conclusions.

2.3.4 Formulating the main goal

Our goal is to improve the locality properties of (2.32) by moving the gravitational dressing from the boundary to the state. From a technical point of view, we will find CFT operators $\widehat{\Phi}$ which obey two properties:

1. $[Q_i, \widehat{\Phi}] = 0$ to all orders in $1/N$, for all asymptotic charges $Q_i \in SO(2, d)$.

2. The correlators of $\widehat{\Phi}$ agree with those of Φ_{HKLL} to leading order in the large N expansion, on the code subspace of $|\Psi_0\rangle$.

In taking the large N limit it is important to track how various effects scale with N. As we will see, our new operators $\widehat{\Phi}$ have vanishing commutator with Q_i to all orders in the $1/N$ expansion, but have a non-vanishing commutator at the level of e^{-N^2} corrections.

In what follows we will first focus on ensuring a vanishing commutator of $\hat{\Phi}$ with the Hamiltonian H to all orders in $1/N$ and then discuss the generalization to the other charges in $SO(2, d)$.

As we will see, our construction will not work for $|\Psi_0\rangle = |0\rangle$. Technically, this is because the vacuum does not comply with the properties (2.27) and (2.29). Physically, it is because the AdS vacuum has no feature that we can use to attach the dressing of our local operator. Note that this is in line with the results of [Chowdhury et al., 2022], where a protocol to reconstruct the bulk state from correlators in the time-band was discussed.

2.3.5 Time-shifted states and return probability

We will now present the main technical tool that will enable us to define state-dressed operators: the return probability. Let us start with a state $|\Psi_0\rangle$ satisfying the properties (2.27) and (2.29). We define the following one-parameter family of states

$$|\Psi_T\rangle = e^{-iTH}|\Psi_0\rangle \qquad T \in \mathbb{R}\,. \tag{2.33}$$

In the bulk, the states $|\Psi_T\rangle$ are related to $|\Psi_0\rangle$ by a large diffeomorphism, i.e. one that does not vanish near the boundary and induces a boundary time-translation. It is important to emphasize that they are *different* quantum states, even though they are related by a symmetry. If we think about the phase space of gravity in AdS, the family of states correspond to different phase space points, just like a particle moves on phase space as a function of time in classical mechanics. From the bulk perspective, if $|\Psi_0\rangle$ was a coherent state, we can also think of $|\Psi_T\rangle$ as coherent states.

We would now like to consider the overlap of such states. In particular, we would like to study the overlap

$$\langle\Psi_0|\Psi_T\rangle\,. \tag{2.34}$$

Thinking of these states as coherent states is useful to gain intuition about such overlaps. For the simple harmonic oscillator, the overlap of two coherent states is

$$\langle\alpha|\beta\rangle = e^{-\frac{1}{\hbar}f(\alpha,\beta)}\,, \tag{2.35}$$

for a very simple quadratic function f. For states on the gravitational phase space, recalling that$\hbar \sim G_N \sim 1/N^2$, we thus expect

$$\langle\Psi_0|\Psi_T\rangle = e^{-N^2 f_0(T)}\,, \tag{2.36}$$

for a function f_0 whose real part is positive. In the gravitational setting, it is not straightforward to directly compute $f_0(T)$ from the phase space information, see [Papadodimas & Raju, 2016] for a discussion on nearby states. There is a general way to compute $f_0(T)$ based on a Euclidean preparation of the states [Belin et al., 2019], but it requires some effort (in particular solving the non-linear Einstein equations). The computation of $f_0(T)$ directly from the information on an initial data slice, which specifies the point on phase-space, is an interesting problem.[21]

It is also instructive to think about the overlap from a microscopic point of view. In the CFT, the overlap is given by

$$\langle\Psi_0|\Psi_T\rangle = \sum_i |c_i|^2 e^{-iTE_i} \,. \tag{2.37}$$

Note that there are $e^{S(E)}$ terms here, each of size $e^{-S(E)}$. The suppression (2.36) must therefore come from the summation over a large number of phases.

If the bulk state has no periodicities in time, we expect the real part of $f_0(T)$ to increase as we increase T. However, this increase will not continue forever. We will shortly give an estimate of the time-average of (2.37), and argue that the decay will saturate at some point. Physically, the non-trivial overlaps (2.37) imply that it is not correct to think that all the states $|\Psi_T\rangle$ are independent, see also [Papadodimas & Raju, 2015, Papadodimas & Raju, 2016, Chakravarty, 2021] for related discussions. In particular, even if the bulk state is not macroscopically periodic, there will still be a microscopic periodicity of the state due to Poincare recurrences, that will happen at very large $T \sim \mathcal{O}(e^{e^{N^2}})$. Throughout this paper,

[21] Similarly, we do not know of a gravitational argument that guarantees that the real part of $f_0(T)$ is positive, which must be the case if the geometries have a state interpretation in the dual CFT. We comment on this further in the discussion.

we will be interested in much earlier time scales so it will be sufficient for us to treat the states $|\Psi_T\rangle$ as quasi-orthogonal since all overlaps will be exponentially small.

We will also need to define the notion of code subspace. Starting with the state $|\Psi_0\rangle$ we define the code subspace as

$$\mathcal{H}_0 = \text{span}\{|\Psi_0\rangle, \mathcal{O}(t,\Omega)|\Psi_0\rangle, ..., \mathcal{O}_1(t_1,\Omega_1)...\mathcal{O}_n(t_n,\Omega_n)|\Psi_0\rangle\}\,, \tag{2.38}$$

generated by acting on $|\Psi_0\rangle$ with a small number ($n \ll N$) of single-trace operators[22]. It will also be useful to define the projector P_0 on this subspace. Similarly, a code subspace can be defined for each of the time-shifted states

$$\mathcal{H}_T = \text{span}\{|\Psi\rangle_T, \mathcal{O}(t,\Omega)|\Psi_T\rangle, ..., \mathcal{O}_1(t_1,\Omega_1)...\mathcal{O}_n(t_n,\Omega_n)|\Psi_T\rangle\}\,, \tag{2.39}$$

with the corresponding projector P_T. The projectors P_0 and P_T are simply related by time-evolution, i.e. we have

$$P_T = e^{-iTH}P_0 e^{iTH}\,, \tag{2.40}$$

and in particular, we emphasize again that $P_T \neq P_0$. In what follows, it will be convenient to work with real quantities rather than the overlap (2.34), and we are now ready to define the return probability.

[22]To be precise, we should also give a small smearing to the single-trace operators in order to avoid UV divergences of operator insertions at coincident points. We will leave it as implicit in what follows.

2.3.6 The return probability

We now ready to examine the T-dependence of the overlap (2.37) in more detail. As explained above, it is more convenient to work with a real quantity so let us define *the return probability*

$$R(T) := |\langle\Psi_0|e^{-iTH}|\Psi_0\rangle|^2 \,. \tag{2.41}$$

It is similar to the spectral form factor (the two coincide when $|\Psi_0\rangle = |\mathrm{TFD}\rangle$ and $H = H_L + H_R$). Recently, the spectral form factor has been extensively discussed in connection to the black hole information paradox and quantum chaos, see for example [Cotler et al., 2017]. The time-scales of interest in that context are again late times such as $t \sim e^{N^2}$ (note this is much shorter than the Poincare recurrence time which is doubly exponential). Here again, we will be interested in much earlier time-scales.

In general, it is difficult to compute (2.41). As we mentioned above, the overlaps can be computed from time-shifted coherent states in gravity but the best known technology to do so uses the Euclidean path integral and involves solving the non-linear Einstein's equations. Nevertheless, we can compute the very early time dependence using large N factorization. We present this calculation in Appendix 2.11. At early times, we have

$$R(T) = e^{-(\Delta H)^2 T^2} \,, \tag{2.42}$$

which is generally valid for times up to $T \sim O(N^{-1})$. For the purposes of this paper, we want to understand how the return probability behaves at time-scales $T \sim \mathcal{O}(1)$. Here, the decay does not follow from large N factorization and it is in general not an easy task to compute it.

In Appendix 2.11, we review that for the TFD state, the return probability (which is the spectral form factor) decays as

$$R_{\text{TFD}}(T) = e^{-N^2 f_{\text{TFD}}(T)}, \tag{2.43}$$

where $f_{\text{TFD}}(T)$ is $O(N^0)$ and for early times $T \sim O(N^0) \ll \beta$ behaves like $f_{\text{TFD}}(T) \approx \alpha T^2$, where α is an $O(N^0)$ constant which depends on the temperature. This is an extremely fast decay, much faster than thermalization where the prefactor in the exponent is of order N^0, and shows that thermofield double states at different times orthogonalize exponentially fast.

We expect similar behaviour for many other semi-classically time-dependent states, that is for timescales of $T \sim \mathcal{O}(1)$, we expect

$$R(T) \sim e^{-N^2 \tilde{f}_0(T)}\,, \tag{2.44}$$

for a positive and $O(N^0)$ function $\tilde{f}_0(T)$ which depends on the state $|\Psi_0\rangle$. We expect that for small T the function $\tilde{f}_0(T)$ starts quadratically, as in (2.42). Note that this fast decay is not even a consequence of quantum chaos, as it can occur at weak coupling or even in free theories, provided they have a large number of degrees of freedom (see [Chen, 2022] for a study of this question in weakly coupled $\mathcal{N} = 4$ SYM). The difference between a free theory and a holographic one will manifest itself in the time-scale during which the exponentially small overlap remains valid. For free $\mathcal{N} = 4$ SYM, the spectrum is integer spaced and so the return probability will be periodic with period 2π, while in a chaotic theory it will take doubly exponentially long for the signal to return to unity.

The average late-time value of the signal is also highly dependent on whether the theory is chaotic or not. For a system with no degeneracies,[23]

$$\overline{R} = \lim_{t_* \to \infty} \frac{1}{2t_*} \int_{-t_*}^{t_*} dT\, R(T) = \sum_i |c_i|^4 \,. \tag{2.45}$$

For the type of states we are considering, i.e. those with a large energy variance, this is exponentially small, and scales as $e^{-\alpha' N^2}$, where α' is an $O(1)$ constant which depends on the particular $|\Psi_0\rangle$ we have picked. This value is often referred to as the plateau, especially in the context of the spectral form factor.

Between the initial decay (2.43) and the plateau (2.45), there can be other regimes, which are particularly interesting in connection to quantum chaos [Shenker & Stanford, 2014, Saad et al., 2018]. For example, in the spectral form factor, the plateau is preceded by a ramp where the signal grows linearly. These effects will not be important for the present work, as we will only consider $\mathcal{O}(1)$ timescales. The crucial point we will exploit throughout the paper is that the signal is already exponentailly small in N^2 at those timescales.

The overlap (2.34) obeys the property

$$\langle \Psi_{t_0} | \Psi_{t_0+T} \rangle = \langle \Psi_0 | \Psi_T \rangle \,. \tag{2.46}$$

This may appear trivial, but it means that even if the bulk geometry appears to be static at the semi-classical level, the return probability may still decay following (2.43) if the state had a period of manifest bulk time-dependence in the far past. Said differently, the variance

[23] Systems like $\mathcal{N} = 4$ SYM will have degeneracies due to superconformal symmetry. For example, for every primary, there are towers of descendants with degenerate energy levels. Nevertheless, the number of degenerate states is exponentially smaller than the number of all states, at least in the high-energy sector of the theory, so the degeneracy only contributes a subleading effect.

in energy which determines the decay is unchanged under time-evolution, so even if the 1-point functions have stabilized, the variance remains large. This observation is particularly relevant in the case of a black hole formed by gravitational collapse.

The exponential decay (2.43) can be extended to more general correlators of the form $\langle\Psi_0|\mathcal{O}(t_1)\ldots\mathcal{O}(t_n)|\Psi_T\rangle$, where $\mathcal{O}$ are single-trace operators. We expect

$$\langle\Psi_0|\mathcal{O}(t_1)\ldots\mathcal{O}(t_n)|\Psi_T\rangle = F(T)\langle\Psi_0|\Psi_T\rangle\,, \tag{2.47}$$

where $F(T)$ is finite in the large N limit and satisfies

$$F(0) = \langle\Psi_0|\mathcal{O}(t_1)\ldots\mathcal{O}(t_n)|\Psi_0\rangle \qquad , \qquad \frac{d^kF(T)}{dT^k}|_{T=0} = O(N^0)\,. \tag{2.48}$$

To see the exponential decay we write (2.47) as

$$\langle\Psi_0|\mathcal{O}(t_1)\ldots\mathcal{O}(t_n)|\Psi_T\rangle = \frac{\langle\Psi_0|\mathcal{O}(t_1)\ldots\mathcal{O}(t_n)|\Psi_T\rangle}{\langle\Psi_0|\Psi_T\rangle}\langle\Psi_0|\Psi_T\rangle\,. \tag{2.49}$$

The second term in this product is really responsible for the decay of the correlator. The first term is hard to evaluate from first principles, but in holography its meaning is clearer. In the bulk theory, it is computed by computing a correlation function on a background dictated by the Euclidean path integral with different sources on the northern and soutern hemisphere (corresponding to $|\Psi_0\rangle$ and $|\Psi_T\rangle$, respectively). This correlator is $\mathcal{O}(1)$ and a smooth function of the background, which will generally change slowly with T, so we expect its time derivatives not to scale with N as indicated in (2.49). We check this statement in a few examples in section 2.6.

To sum up, any state in the code subspace (2.38) has an exponentially small overlap with any state in the code subspace (2.39). This can be summarized by the relation

$$R_{code}(T) = \frac{1}{d_{code}} Tr[P_T P_0] = \mathcal{O}(e^{-N^2 \tilde{f}(T)}) \tag{2.50}$$

where d_{code} is the dimensionality of the code subspace, and for the time-scales we have discussed. The decay (2.50) can be used in combination with other useful inequalities. For example, for a Hermitian operator $\mathcal{O}$ with eigenvalues λ_i, and if $[P_0, \mathcal{O}] = 0$, we have $|\langle\Psi_0|\mathcal{O}|\Psi_T\rangle|^2 \leq \sqrt{\mathrm{Tr}[\mathcal{O}^4]}\sqrt{\mathrm{Tr}[P_T P_0]}$ and $|\langle\Psi_0|\mathcal{O}|\Psi_T\rangle|^2 \leq \max(\lambda_i^2)\,\mathrm{Tr}[P_T P_0]$.

2.3.7 Other asymptotic charges

More generally we can consider the change of the state by large diffeomorphisms corresponding to the other asymptotic symmetries of the theory, in the case of AdS_{d+1} the conformal group $SO(2,d)$ with the generators we discussed in section 2.3.1. This leads us to define a natural generalization of the return probability

$$R(g) = |\langle\Psi_0|U(g)|\Psi_0\rangle|^2 \qquad , \qquad g \in SO(2,d)\,, \tag{2.51}$$

where $U(g)$ is the unitary realizing the conformal transformation of the CFT on $\mathbb{S}^{d-1} \times$ time.

What can we expect for these overlaps? To start, let us suppose the state $|\Psi_0\rangle$ breaks rotational $SO(d)$ symmetry at the classical level. By this, we mean that bulk dual geometry breaks the symmetry, which would be the case for some spherically asymmetric lump of matter. Take J to be the angular momentum generator, then we expect that the variance of

J will be of $O(N^2)$ for such a state. Hence we expect that for small values of a rotation angle φ dual to J we will have

$$R(\varphi) = e^{-(\Delta J)^2\varphi^2} = e^{-\kappa N^2 \varphi^2}\,, \tag{2.52}$$

for $\kappa \sim \mathcal{O}(1)$. For more general angles, we expect

$$R(\varphi) = e^{-N^2 f_{\rm rot}(\varphi)}\,. \tag{2.53}$$

However, because angular momentum is quantized, we have

$$R(\varphi + 2\pi) = R(\varphi)\,, \tag{2.54}$$

hence the function $f_{\rm rot}(\varphi)$ has period 2π. In this direction of the conformal group the return probability has a very short Poincare recurrence equal to 2π.

All in all we find that as we increase φ away from 0 the return probability $R(\varphi)$ very quickly dips down to exponentially small values and stays there until the Poincare recurrence at $\varphi = 2\pi$. As we see from (2.53), for any fixed φ which is in the range $(0, 2\pi)$, we have $R(\varphi)$ being exponentially small in the large N limit.

Of course if the state respects spherical symmetry then the return probability will not decay in the corresponding $SO(d)$ directions. It is worthwhile to discuss several distinct scenarios. In the simplest case, the state preserves the symmetry and is thus annihilated by the generators of rotations. The second simplest situation is the case where the symmetry is manifestly broken at the classical level (for example an asymmetric lump of matter). In this case, the breaking of the symmetry is manifest, and would be visible in the 1-point function of single-trace operators. There are also more subtle situations where the state breaks the symmetry classically in the bulk, but this may be invisible in the 1-point functions. An example of this are states by prepared by the path integral on higher genus surfaces in $d = 2$, and have topology behind the horizon [Marolf & Wien, 2018].[24]

Finally as discussed in section 2.3.1, we expect that semi-classical states also break the other conformal symmetries. We can get some intuition by considering a state dual to a conformal primary of dimension Δ. In this case the return probability along one of the conformal boost directions is determined by a group theoretic computation

$$R(s) = |\langle\Delta|e^{-isK}|\Delta\rangle|^2 = \left(\frac{1}{\cosh^2 s}\right)^{2\Delta} . \tag{2.55}$$

For primary states with $\Delta \sim O(N^2)$, we get exponential decay of the form $e^{-N^2 f(s)}$ for any non-zero s. Notice that for the conformal boosts we do not expect any Poincare recurrence for large s, which in the case of primaries is obvious from the formula above, since such a transformation monotonically increases the energy of the state.

In the case of AdS_3 the asymptotic symmetry group is enhanced to Virasoro and similar statements hold for the flow of the state under more general large diffeomorphisms generated by $L_n, \overline{L}_n$.

To summarize, if we start with a state $|\Psi_0\rangle$ which breaks all conformal symmetries at the level of the semi-classical geometry we expect that $R(g)$ defined in (2.51) will decay exponentially fast in all directions away from the identity element on the conformal group manifold.

[24]The thermofield double also has this property. It breaks rotational symmetry of each CFT individually, but the breaking is invisible in 1-point functions. It would be interesting to understand if this type of breaking always requires a horizon.

We are now in a position to introduce operators $\hat{\Phi}$ which satisfy the two properties described in section 2.3.4, namely their commutator with the Hamiltonian and other asymptotic charges is zero to all orders in the $1/N$ expansion and they act like HKLL operators to leading order at large N on the code subspaces $\{\mathcal{H}_T, \;\; T \in (-t_\star, t_\star)\}$. Here t_* is an order one (i.e. N^0)) time of our choice. We define the HKLL operator Φ, (2.32), in the $N \to \infty$ limit. In this limit the bulk is described by a quantum field theory on a curved spacetime and code subspaces for different T will be strictly orthogonal to one another. In addition, Φ is a local bulk operator which commutes with all the boundary single-trace operators in the time band algebra, including the appropriately normalized Hamiltonian [Kabat et al., 2011, Hamilton et al., 2006c]. But it will no longer be commuting once $1/N$ corrections are included. In particular, we will have

$$[\Phi, \frac{H - \langle H \rangle}{N}] = O(1/N) \neq 0 \,. \tag{2.56}$$

Again, the physical reason behind this is that (2.32) is a diff-invariant operator that is dressed to the boundary. Note that for the naive HKLL operator (2.32), the commutator with other single-trace operators will also be non-zero at order $O(1/N)$. For almost all single-trace operators, this can be removed order by order in $1/N$ by adding the appropriate corrections to Φ [Kabat et al., 2011]. However, these modifications will not be able to remove the non-vanishing commutator with the Hamiltonian (2.56). Thus, to remove the gravitational dressing to the boundary CFT, a more sophisticated procedure is required.

We start by focusing on setting the commutator with the Hamiltonian to zero and discuss the extension to other asymptotic charges later. To this end, we introduce the following operator[25]

$$\widehat{\Phi} = c \int_{-t_*}^{t_*} dT\, e^{-iTH} P_0 \Phi P_0 e^{iTH}\,, \tag{2.57}$$

where t_* is an $O(N^0)$ timescale of our choice, and c is an overall normalization constant

$$c^{-1} = \int_{-t_*}^{t_*} dT \langle \Upsilon_0 | P_T | \Upsilon_0 \rangle\,. \tag{2.58}$$

As we will see, the projector P_0 will be key and will make $\widehat{\Phi}$ act appropriately on the code subspaces. The range $(-t_\star, t_\star)$ determines the set of code subspaces on which $\widehat{\Phi}$ acts in the desired fashion, and ultimately cannot be taken to be bigger than the time range where the exponential decay of the return probability (2.43) is valid. To make the operator (2.57) have the desired properties on as many states as possible, we can take this range to be the time range where the return probability decays exponentially, though this is not strictly necessary and a t_* of $O(N^0)$ is sufficient. We also provide an alternative presentation of the operators in subsection 2.4.4. In the following subsections, we will study the action of these operators in the relevant code subspaces, and will be particularly interested in their commutator with the Hamiltonian.

[25] Recall that P_0 is the projector on the code subspace of $|\Upsilon_0\rangle$, and thus $[\Phi, P_0] = 0$ in that code subspace. Therefore, we could have defined operators with the same action on the code subspace as 2.57, using a single projector on the left (or right) of Φ. Even though the resulting operators would act in the same way on the relevant code subspace, the operators would not be exactly identical: they would have additional non-zero matrix elements associated to subspaces orthogonal to $\mathcal{H}_0$.

2.4.1 Vanishing commutator with H to all orders in $1/N$

We now show that the operator (2.57) has vanishing commutator with H to all orders in $1/N$. We start by rewriting the commutator as

$$[H, \widehat{\Phi}] = -i\frac{d}{ds}\left(e^{isH}\widehat{\Phi}e^{-isH}\right)\Big|_{s=0}, \tag{2.59}$$

and performing a change of variables, we find

$$\begin{aligned} [H, \widehat{\Phi}] &= -i\frac{d}{ds}\left(c\int_{-t_*-s}^{t_*-s} dT\, e^{-iTH}P_0\Phi P_0 e^{iTH}\right)\Big|_{s=0} \\ &= ic(P_{t_*}\Phi_{t_*}P_{t_*} - P_{-t_*}\Phi_{-t_*}P_{-t_*})\,, \end{aligned} \tag{2.60}$$

where we defined $\Phi_{t_*} = e^{-iHt_*}\Phi e^{iHt_*}$. Using the decay of the return probability through (2.50), we see that the commutator inserted inside a correlator of a small number of single-trace operators and evaluated on the state $|\Psi_T\rangle$ will give an exponentially small answer, since each of the two terms in (2.60) give exponentially small numbers. This is valid for any T as long as $|T| < t_\star$ and $|T| - t_\star \sim O(N^0)$. Thus,

$$[H, \widehat{\Phi}] = O(e^{-\gamma N^2})\,, \tag{2.61}$$

where γ is positive and $O(N^0)$, proving property 1, defined in subsection 2.3.4, for these operators. Note (2.61) is true for our set of code subspaces with T constrained as above, but not for all states. For example, the commutator is not exponentially suppressed in the state $|\Psi_{t_*}\rangle$.

2.4.2 Similar action as HKLL operators

A vanishing commutator with the Hamiltonian is necessary but not sufficient. There are many CFT operators that commute with the Hamiltonian up to exponentially small corrections in N^2, but they will not have the same effect as acting with a local bulk operator, see for example footnote 4. Therefore, we also need to show that the operator $\hat{\Phi}$ behaves in the same way as the HKLL operator (2.32) to leading order at large N inside correlation functions of single-trace operators. For that we consider

$$\langle\Psi_0|\mathcal{O}...\widehat{\Phi}...\mathcal{O}|\Psi_0\rangle = c\int_{-t_*}^{t_*} dT\,\langle\Psi_0|\mathcal{O}...e^{-iTH}P_0\Phi P_0 e^{iTH}...\mathcal{O}|\Psi_0\rangle \tag{2.62}$$

$$= c\int_{-t_*}^{t_*} dT\,\langle\Psi_0|\mathcal{O}...P_0P_T(e^{-iTH}\Phi e^{iTH})P_TP_0...\mathcal{O}|\Psi_0\rangle. \tag{2.63}$$

In the last line, we have inserted two projectors P_0, which we are free to do since the correlators is evaluated in the state $|\Psi_0\rangle$. The integrand above corresponds to $\mathrm{Tr}P_TP_0$, up to some operator insertions that do not affect its general structure. From 2.50 we see that the integrand will be exponentially suppressed as $|T|$ increases (and is not $O(1/N)$) because of the exponentially small overlap of the code subspaces. We can thus evaluate the integral by a saddle-point method controlled by the large N limit. The dominant contribution comes from $T = 0$. One might worry about the possibility of rapidly oscillating phases, such as the one in $\langle\Psi_0|\Psi_T\rangle$ displacing the location of the saddle point. Notice however that from 2.47,2.48 it follows that such rapidly oscillating phases cancel between the bra and ket contribution. Using 2.47 and 2.58 we have

$$\langle \Psi_0 | \mathcal{O} ... \widehat{\Phi} ... \mathcal{O} | \Psi_0 \rangle = \langle \Psi_0 | \mathcal{O} ... \Phi ... \mathcal{O} | \Psi_0 \rangle + O(1/N), \tag{2.64}$$

as desired. The $1/N$ corrections can be thought of coming from corrections to the leading saddle-point, and would be sensitive to the more detailed form of $F(T)$ in (2.47).

Notice that if we apply the operator $\widehat{\Phi}$ to one of the time-shifted states, then as long as $|T| < t_*$, we find

$$\langle \Psi_T | \mathcal{O} ... \widehat{\Phi} ... \mathcal{O} | \Psi_T \rangle = \langle \Psi_T | \mathcal{O} ... (e^{-iTH} \Phi e^{iTH}) ... \mathcal{O} | \Psi_T \rangle + O(1/N) \tag{2.65}$$

Thus in the code subspace $\mathcal{H}_T$, $\hat{\Phi}$ acts as $e^{-iTH} \Phi e^{iTH}$ to leading order at large N. To make this more manifest, we can also write (2.57) as

$$\widehat{\Phi} = c \int_{-t_*}^{t_*} dT \, P_T (e^{-iTH} \Phi e^{iTH}) P_T. \tag{2.66}$$

Since we have shown that, to leading order at large N, $\hat{\Phi}$ and Φ have the same matrix elements on the entire code subspace it follows that higher point functions of $\hat{\Phi}$ will also agree at large N with those of Φ. Consider for instance,

$$\hat{\Phi}_i = c \int_{-t_*}^{t_*} dT \; e^{-iTH} P_0 \Phi_i P_0 e^{iTH} \tag{2.67}$$

where $\Phi_i \equiv \Phi(x_i)$ is an HKLL operator located at a certain spacetime point x_i, then in the large N limit

$$\begin{aligned}\langle\Psi_0|\mathcal{O}...\widehat{\Phi}_1\widehat{\Phi}_2...\widehat{\Phi}_n...\mathcal{O}|\Psi_0\rangle =&c^n\int_{-t_*}^{t_*}dT_1...dT_n\,\langle\Psi_0|\mathcal{O}...P_{T_1}(e^{-iT_1H}\Phi_1e^{iT_1H})P_{T_1}P_{T_2}\\ &(e^{-iT_2H}\Phi_2e^{iT_2H})P_{T_2}...P_{T_n}(e^{-iT_nH}\Phi_ne^{iT_nH})P_{T_n}...\mathcal{O}|\Psi_0\rangle\\ &\approx\langle\Psi_0|\mathcal{O}...\Phi_1\Phi_2...\Phi_n...\mathcal{O}|\Psi_0\rangle\,.\end{aligned} \tag{2.68}$$

In addition, this implies that the commutator of $\hat{\Phi}_i$'s is the same as that of HKLL operators in the large N limit. Two operators, $\hat{\Phi}(x_i)$ and $\hat{\Phi}(x_j)$, will have zero commutator at spacelike separated points whereas they have $O(1)$ commutator if they are timelike-separated. This is true even though these operators do not translate under commutation with the boundary Hamiltonian, up to exponentially small corrections in N. Nevertheless, they still have bulk space-time labels and preserve the causal properties of HKLL operators in the large N limit.

2.4.3 Interpretation and comments

We have just seen that to leading order in the large N limit, the operator (2.57) acts like the HKLL operator (2.32) in the appropriate code subspace. However, it commutes with H to all orders in $1/N$. The existence of these operators provides strong evidence that the algebra of single-trace operators in a short time band can have a non-trivial commutant when acting on time-dependent states of high energy.

The vanishing of the commutator with H should be interpreted as (2.57) being gravitationally dressed not with respect to the boundary, but instead with respect to features of the bulk state, in particular its time-dependence. This can be seen by the fact that $\hat{\Phi}$ acts differently on different states. On the time-shifted states $|\Psi_T\rangle$ and their code subspaces, it acts as $e^{-iTH}\Phi e^{iTH}$. For example, imagine that in the state $|\Psi_0\rangle$ we have a supernova explosion taking place at $t = 0$ and we chose the operator (2.32) so that it acts right next to the explosion. In the state $|\Psi_T\rangle$ the explosion obviously takes place at $t = -T$. From equation (2.65), we can see that the operator $\widehat{\Phi}$ will act again right next to the supernova explosion, even though the supernova is now at $t = -T$. Therefore, one and the same operator $\widehat{\Phi}$ knows how to always act at the correct moment (right next to the explosion) for the entire family of states $|\Psi_T\rangle$, as long as $|T| < t_*$. The finiteness of t_* indicates that there is still some residual boundary dressing, which however is not visible in pertubation theory[26].

The property of being dressed with respect to features of the state is also present in the local observables one defines in general relativity, discussed in section 2.2.3. These state dressed observables are defined at points where a set of D scalars, like the Ricci scalar or $\mathcal{R}_{\mu\nu\rho\sigma}\mathcal{R}^{\mu\nu\rho\sigma}$ where $\mathcal{R}_{\mu\nu\rho\sigma}$ is the Riemann tensor, 'click' with a certain set of numbers. The observables are labeled by these values and they are evaluated precisely where the scalars take those values in each state. Locality of these observables requires them to be defined only in some neighbourhood of a classical solution. In the same spirit, the operators discussed in this section are also local for a certain family of code subspaces, see section 2.4.1.

As mentioned earlier, if the spacetime is so symmetric that the scalars take the same values throughout the spacetime, then these classical observables are not well defined. Since every point in the spacetime is physically equivalent, it is reasonable that local observables are ill defined for these solutions. For this reason, the observables are state dependent. Sim-

[26] Similar remarks were made in [Marolf, 2015] for the DeWitt observables in AdS.

ilarly, it is not possible to apply the same logic discussed in the previous subsections to empty AdS, or other static states, as there are no time-dependent features in the bulk that can be used as a 'clock' to define a moment in time where the operator acts. Technically, the return probability for such states does not exhibit the rapid decay (2.36). We thus see a nice parallel between the classical and quantum situations.

The definition of our operator gives a bulk operator which is dressed with respect to features of the state, but in an implicit manner. Our construction does not permit us to extract the details of the dressing. Going back to our example of a supernova explosion, one might guess that the dressing is with respect to the supernova and that one could in principle define a gravitational Wilson line between the operator and the supernova. But what if the state described instead two supernovas exploding at the same or different times? To which explosion would our operator be dressed to? The construction does not give a definite answer, and the way to address this question would be to enlarge the set of code subspaces on which our operator correctly acts. For example, if our operator did not move under the time-translation of one of two supernovae, we would say that it is dressed to the other one. We hope to return to this question in the future, but see subsection 2.7.3 for some related remarks.

2.4.4 A similarity transformation

We briefly mention a variant of operators with properties similar to those of (2.57). We first define the shifted Hamiltonian[27]

$$\hat{H} = H - \langle \Psi_0 | H | \Psi_0 \rangle \mathbb{I} \,. \tag{2.69}$$

[27] This shift is useful in order to avoid rapidly oscillating phases in the discussion below.

Then we introduce

$$V = \frac{c}{\sqrt{2}} \int_{-t_*}^{t_*} dT e^{-i\hat{H}T} P_0 \,, \tag{2.70}$$

with c given in (2.58). We have

$$VV^\dagger = \frac{c^2}{2} \int_{-t_*}^{t_*} dT \int_{-t_*}^{t_*} dT' e^{-i\hat{H}T} P_0 e^{i\hat{H}T'} \,, \tag{2.71}$$

where we used $P_0^2 = P_0$. Following arguments similar to those of the previous subsection, we find that to leading order at large N, and when computing the matrix elements of (2.71) within the code subspace, the two integrals in (2.71) can be computed by a saddle point method, where the dominant saddle is $T = T' = 0$. We then find that in this class of states and at large N

$$VV^\dagger \simeq \mathbb{I}, \quad \text{and} \quad V^\dagger V \simeq \mathbb{I} \,. \tag{2.72}$$

in the sense that, within the code subspace V behaves like a unitary, up to $1/N$ corrections.

Then we start with a boundary-dressed operator Φ and define

$$\hat{\Phi} = V \Phi V^\dagger \,. \tag{2.73}$$

Following similar arguments as before we can show that the operator (2.73) satisfies properties 1 and 2 of subsection 2.3.4. To check the commutator of $\hat{\Phi}$ with H. We write

$$\begin{aligned} [H, \hat{\Phi}] &= -i \frac{d}{ds} \left(e^{i\hat{H}s} V \Phi V^\dagger e^{-i\hat{H}s} \right) |_{s=0} \\ &= -i \frac{d}{ds} \frac{c^2}{2} \Big(\int_{-t_*-s}^{t_*-s} dT e^{-i\hat{H}T} \Big) P_0 \Phi P_0 \Big(\int_{-t_*-s}^{t_*-s} dT' e^{i\hat{H}T'} \Big) |_{s=0} \,, \end{aligned} \tag{2.74}$$

which again localizes on boundary terms and is thus exponentially suppressed.

Second, to show that the leading large N correlators of $\hat{\Phi}$ are the same as those of Φ we follow exactly the same reasoning as in the previous subsection, but now we will have two time-integrals. Each one of these time integrals will lead to a sharply suppressed Gaussian around $T = T' = 0$ and can be evaluated by saddle-point at large N, reproducing the desired result.

2.4.5 Other asymptotic charges

More generally we need to make (2.32) commute with all boundary symmetry generators corresponding to asymptotic symmetries. For asymptotically AdS_{d+1} space-times this is the conformal group $SO(2, d)$ and we consider a generalization of the form

$$\widehat{\Phi} = c \int_B d\mu(g) U(g) P_0 \Phi P_0 U(g)^{-1}, \tag{2.75}$$

where now

$$c^{-1} = \int_B d\mu(g) \langle \Psi_0 | U(g) P_0 U(g)^{-1} | \Psi_0 \rangle \,. \tag{2.76}$$

Above, $d\mu(g)$ is the Haar measure on $SO(2, d)$ and B is a reasonably sized connected submanifold of $SO(2, d)$ containing the identity. The commutator with conformal generators will then be given by operators in the code subspace of states $U(g_*)|\Psi_0\rangle$, where g_* lies on the boundary ∂B. For the construction to work in this generalization we must make sure that the overlaps

$$R(g) = |\langle \Psi_0 | U(g) | \Psi_0 \rangle|^2, \tag{2.77}$$

decay exponentially in the geodesic distance of g from the identity. As discussed in subsection 2.3.7 we expect this to be true for states which break all symmetries at the semiclassical level[28]. The quantity $R(g)$ is an interesting generalization of the return probability (2.41) that would be interesting to study further.

2.5 A more general argument for the commutant

The operators (2.75) constructed in the previous section commute with the asymptotic charges to all orders in $1/N$, however they commute with the other single-trace operators in the time-band generally only to leading order in $1/N$. To identify a commutant for the time-band algebra $\mathcal{A}$, the operators (2.75) have to be improved. In this short section we outline a somewhat different argument suggesting that it is indeed possible to find a commutant to all orders in $1/N$. We caution the reader that the argument that follows is based on certain assumptions which seem physically plausible, but for which a rigorous proof is still lacking. A more careful treatment for the existence of a commutant (as well as a mathematically precise definition of the time-band algebra in the first place) would be desirable.

Let us start with a standard HKLL operator Φ. We also introduce the notation $q_i = \frac{Q_i - \langle Q_i \rangle}{N}$ for where Q_i denotes any of the asymptotic $SO(2,d)$ charges and $\mathcal{O}_j$ a general single-trace operator in the time-band. Our goal is to find an operator $\hat{\Phi}$ which has the following properties:

1. $[\hat{\Phi}, q_i] = 0$ and $[\hat{\Phi}, \mathcal{O}_j] = 0$ for all $q_i \in SO(2,d)$ and $\mathcal{O}_j \in \mathcal{A}$, to all orders in $1/N$.

2. To *leading order* at large N the correlators of $\hat{\Phi}$ with $q_j, \mathcal{O}_i$ must be the same as those

[28] For compact symmetries, such as rotations, $R(g)$ will have recurrences every 2π. Hence along the compact directions we take $g_* \sim O(1) < 2\pi$.

of Φ. In particular this means that for single-trace operators $\mathcal{O}_i$ outside the time-band we generally expect $[\mathcal{O}_i, \hat{\Phi}] = O(N^0)$.

The first condition is obvious. The second condition is necessary in order to ensure that the operator $\hat{\Phi}$ acts in the expected way, at least to leading order at large N, and creates particles that can be detected with an $O(1)$ effect by operators outside the time-band when light rays from the diamond hit the boundary.

Here we remark that in order for the two conditions to be mutually consistent, it is important that we impose the second condition only to leading order at large N. The point is that $[q_i, \Phi] = O(1/N)$ hence when looking at leading order correlators it is indeed consistent to demand simultaneously that i) $\hat{\Phi}$ commutes with q_i and that ii) $\hat{\Phi}$ acts like Φ. However, when moving on to subleading corrections we have a non-vanishing commutator $[q_i, \Phi]$ hence we cannot impose both conditions at the same time. We choose to impose that our operators $\hat{\Phi}$ continue to commute with q_i to all orders in $1/N$, but we allow their correlators to depart from those of Φ_i at subleading orders in $1/N$.

We now define the desired operators $\hat{\Phi}$ by specifying how they act on the code subspace $\mathcal{H}_0$. Earlier we defined the code subspace as the space generated by acting on $|\Psi_0\rangle$ with single-trace operators, which are not necessarily restricted in the time-band. However, by an analogue of the Reeh-Schlieder theorem[29] we expect that for reasonable bulk states $|\Psi_0\rangle$ the code subspace $\mathcal{H}_0$ can also be generated by acting on $|\Psi_0\rangle$ with only elements of the time-band algebra $\mathcal{A}$

$$\mathcal{H}_0 = \mathrm{span}\{\mathcal{A}|\Psi_0\rangle\}\,. \tag{2.78}$$

[29] This was discussed in [Banerjee et al., 2016] for the case of empty AdS and at large N. We believe that a similar result should hold for more general heavy states and even when taking $1/N$ corrections into account, but it would be interesting to develop a more careful proof.

We now define the action of the the operator $\hat{\Phi}$ on the code subspace by the following conditions

$$\hat{\Phi}A|\Psi_0\rangle = A\Phi|\Psi_0\rangle\,, \qquad \forall A \in \mathcal{A}\,. \tag{2.79}$$

This set of linear equations, one for every element of the small algebra $\mathcal{A}$, defines the action of $\hat{\Phi}$ on the code subspace, in a way which satisfies the desired properties as we will see below.

Notice that these equations can also be represented as follows: we first select a basis of linearly independent elements A_i of the algebra $\mathcal{A}$. then we define the matrix of 2-point functions

$$g_{ij} = \langle\Psi_0|A_i^\dagger A_j|\Psi_0\rangle\,. \tag{2.80}$$

From (2.78), it follows that the set of states $|i\rangle = A_i|\Psi_0\rangle$ form a (possibly over-complete) basis of the code subspace. Since $\hat{\Phi}$ is an operator on the code subspace it can be written as

$$\hat{\Phi} = K^{ij}|i\rangle\langle j| = K^{ij}A_i|\Psi_0\rangle\langle\Psi_0|A_j^\dagger\,. \tag{2.81}$$

for an appropriate choice of K^{ij}. To find the matrix K, we start with the desired relation (2.79) written as

$$\hat{\Phi}A_l|\Psi_0\rangle = A_l\Phi|\Psi_0\rangle\,, \tag{2.82}$$

then we replace $\hat{\Phi}$ with (2.81) and multiply from the left with $\langle\Psi_0|A_k^\dagger$ to get

$$g_{jl}\,g_{ki}\,K^{ij} = \langle\Psi_0|A_k^\dagger A_l\Phi|\Psi_0\rangle\,. \tag{2.83}$$

If the set of states $|i\rangle = A_i|\Psi_0\rangle$ are linearly independent then the matrix g_{ij} is positive definite and invertible. In that case we can solve for K as

$$K_{ij} = g^{ik} g^{jl} \langle \Psi_0 | A_k^\dagger A_l \Phi | \Psi_0 \rangle \,, \tag{2.84}$$

where $g^{ij} g_{jk} = \delta^i_k$. When (2.84) is replaced in expression (2.81), we find an explicit solution of the desired equation (2.79).

We emphasize that the necessary ingredient to arrive at (2.84) was the linear independence of the states $A_i|\Psi_0\rangle$, which is equivalent to the statement that there is no non-vanishing operator in $\mathcal{A}$ which annihilates the state $|\Psi_0\rangle$. We discuss this condition in the following subsection.

2.5.1 On the consistency of the defining equations

Before checking that the operators $\hat{\Phi}$ defined by (2.79), or equivalently via (2.81),(2.84), have the desired properties, we need to check that equations (2.79) are self-consistent linear equations. The only possible source of inconsistency is the following: if there was an element $A \neq 0$ of the time-band algebra $\mathcal{A}$ such that $A|\Psi_0\rangle = 0$, this could potentially be a problem since we would then have $A|\Psi_0\rangle = 0$, while in general $A\Phi|\Psi_0\rangle \neq 0$. Then the equation (2.79) would imply $0 = A|\Psi_0\rangle = A\Phi|\Psi_0\rangle \neq 0$ which is a contradiction. Relatedly, g_{ij} defined in (2.80) would not be invertible and we would not be able to get to (2.84).

We will now show that this situation does not arise, that is

$$A|\Psi_0\rangle \neq 0 \qquad \forall A \in \mathcal{A} \;,\; A \neq 0 \,. \tag{2.85}$$

We will prove this by first proving that at large N (2.85) is true and then we will argue that $1/N$ corrections cannot change the conclusion.

We have been working under the assumption that the time-band is short enough, which means that in the bulk there will be a region which is space-like relative to the time band. In the large N limit, where gravitational backreaction is turned off, operators inside that region (for example usual HKLL operators) commute with all elements of the algebra $\mathcal{A}$, including the appropriately normalized asymptotic charges q_i. Hence, in the large N limit the algebra $\mathcal{A}$ has a non-trivial commutant $\mathcal{A}'$. We want to argue that this commutant continues to exist when $1/N$ corrections are taken into account, provided that the state $|\Psi_0\rangle$ has non-vanishing variance of $O(N^2)$ under the asymptotic charges.

Assuming that at large N the theory in the bulk behaves like usual QFT on a curved background, we expect that an analogue of the Reeh-Schlieder theorem will hold for the commutant $\mathcal{A}'$, which means that we can generate the code subspace $\mathcal{H}_0$ by acting on $|\Psi_0\rangle$ with elements of $\mathcal{A}'$.

Suppose now that there was an element A of the time-band algebra $\mathcal{A}$ which annihilated the state $|\Psi_0\rangle$. Then for any element $a' \in \mathcal{A}'$ we have

$$Aa'|\Psi_0\rangle = a'A|\Psi_0\rangle = 0\,. \tag{2.86}$$

Since states of the form $a'|\Psi_0\rangle$ generate $\mathcal{H}_0$ we conclude that the operator A has vanishing matrix elements in $\mathcal{H}_0$ at large N. From this we can not immediately conclude that $A = 0$ as an operator when $1/N$ corrections are included. For example, for $|\Psi_0\rangle = |0\rangle$ the normalized $SO(2,d)$ generators $q_i = \frac{Q_i}{N}$ have vanishing matrix elements at large N, since they annihilate $|0\rangle$ and commute with all other operators. However they are non-vanishing operators at order $1/N$. If A is a non-vanishing operator which has vanishing matrix elements at large N on $\mathcal{H}_0$ then it means that it acts as a central element at large N. Here we make an additional assumption, that the only central elements are the $SO(2,d)$ generators q_i and their functions[30]. Since, by assumption, the state $|\Psi_0\rangle$ has non-trivial variance under these generators, we conclude that it cannot be annihilated by a non-trival A.

Let us assume now that we have a state of the form $A|\Psi_0\rangle$ which has finite (i.e. $O(N^0)$) positive norm at large N. Including $1/N$ corrections will generally modify the norm of this state, but it will do so by corrections suppressed by powers of $1/N$. Since the previous argument established that the leading large N norm of the state $A|\Psi_0\rangle$ is a finite positive number, perturbative $1/N$ corrections cannot make it vanish. Hence we expect property (2.85) to be true to all orders in $1/N$ perturbation theory.

We emphasize that the fact that we cannot annihilate the state by the time-band algebra $\mathcal{A}$ relies on the fact that we have restricted our attention to small products of single-trace operators. As discussed in a related context [?, Banerjee et al., 2016], if we consider the *full* algebra of operators in the time-band we can find sufficiently complicated combinations which can annihilate the state[31].

Finally, as should be clear from the above, if the state $|\Psi_0\rangle$ has very small or vanishing variance in the asymptotic charges then (2.85) fails and it is not possible to define operators obeying (2.79).

[30] We believe this assumption to be quite weak, but it would be interesting to prove it more thoroughly.

[31] For example, consider a state $|\Psi\rangle$ with $\langle\Psi_0|\Psi\rangle = 0$. Then the (complicated) operator $|\Psi\rangle\langle\Psi|$ annihilates $|\Psi_0\rangle$.

2.5.2 Proof that $\hat{\Phi}$ has the desired properties

Having established that equations (2.79) are consistent, we argue that the operator $\hat{\Phi}$ has the desired properties.

First it is obvious by (2.79) that the operator $\hat{\Phi}$ has vanishing commutators with elements of $\mathcal{A}$. To see that consider $A_1 \in A$ and a general state in the code subspace which can be written as $A_2|\Psi_0\rangle$, with $A_2 \in \mathcal{A}$. Then we have

$$[\hat{\Phi}, A_1]A_2|\Psi_0\rangle = \hat{\Phi}(A_1A_2)|\Psi_0\rangle - A_1(\hat{\Phi}A_2|\Psi_0\rangle) = A_1A_2\Phi|\Psi_0\rangle - A_1A_2\Phi|\Psi_0\rangle = 0\,, \tag{2.87}$$

where in the second equality we used (2.79). Since this is true for all A_2, we find

$$[\hat{\Phi}, A_1] = 0 \qquad \forall A_1 \in \mathcal{A}\,, \tag{2.88}$$

where it should be understood that this equation holds on the relevant code subspace.

Second, we will show that *to leading order at large* N, the operator $\hat{\Phi}$ acts like the HKLL operator Φ. To see this, consider an arbitrary matrix element on the code subspace. Two general states of the code subspace can be written as $A_1|\Psi_0\rangle, A_2|\Psi_0\rangle$. Then we have

$$\langle\Psi_0|A_1^\dagger\hat{\Phi}A_2|\Psi_0\rangle = \langle\Psi_0|A_1^\dagger A_2\Phi|\Psi_0\rangle = \langle\Psi_0|A_1^\dagger \Phi A_2|\Psi_0\rangle + \langle\Psi_0|A_1^\dagger[\Phi, A_2]|\Psi_0\rangle\,. \tag{2.89}$$

In the first equality we used (2.79). Now, the operator A_2 is some combination of single-trace operators in the time band, as well as the normalized $SO(2,d)$ generators q_i. All of these operators have commutators with Φ which are suppressed by powers of $1/N$. Hence the last term in the equation above is suppressed. All in all, we find

$$\langle \Psi_0 | A_1^\dagger \hat{\Phi} A_2 | \Psi_0 \rangle = \langle \Psi_0 | A_1^\dagger \Phi A_2 | \Psi_0 \rangle + O(1/N)\,, \tag{2.90}$$

which establishes the desired result. This ensures that large N correlators of $\hat{\Phi}$ are the same as Φ.

We emphasize that the operators defined in this section are not exactly the same as the operators (2.57) discussed earlier. For example, unlike (2.57) the operators (2.79) were defined to act only on the code subspace $\mathcal{H}_0$ of $|\Psi_0\rangle$ and not on the code subspace $\mathcal{H}_T$ for $T = O(N^0)$. Also, the commutator of (2.57) with the Hamiltonian is of order e^{-N^2} while it is exactly zero, within the code subspace, for the operators (2.79).

2.6 Examples

In this section we consider various examples. Our primary focus will be on examining the validity of equations (2.44), (2.47),(2.48), on which the construction of our operators relies.

2.6.1 Coherent states

In general, we are interested in time-dependent semi-classical geometries. Many of these states can be thought of as bulk coherent states. We will discuss the overlap of these states closely following [Belin et al., 2019]. In the CFT, these states are prepared by a Euclidean path integral

$$|\Psi\rangle = Te^{-\int_{t_E<0} dt_E d^{d-1}x \; \varphi_b(t_E,x)\mathcal{O}(t_E,x)}\,|0\rangle\;, \tag{2.91}$$

where $\mathcal{O}$ is a single-trace operator dual to a supergravity field, and the source is scaled appropriately so that it leads to states with non-trivial gravitational backreaction, i.e. the expectation value of the energy and variance of this state will scale like (2.27) and (2.29).

In the large N limit the overlap of two such states can be computed by a Euclidean gravitational path integration which in the semi-classical limit can be approximated by a saddle point computation. For example, the norm of the state is

$$\langle \Psi | \Psi \rangle \approx e^{-I_{grav}(\lambda_b)} , \tag{2.92}$$

where λ_b is the following boundary condition for the bulk field

$$\lambda_b = \begin{cases} \varphi_b(t_E, x), \ t_E < 0 \\ \varphi_b^\star(-t_E, x), \ t_E > 0 , \end{cases} \tag{2.93}$$

and $I_{grav}(\lambda_b)$ is the on-shell gravitational action in the presence of the sources specified above.

Generalizing to two states $|\Psi_1\rangle$ and $|\Psi_2\rangle$, the normalized inner product between them is

$$\mathcal{R} = \frac{|\langle \Psi_1 | \Psi_2 \rangle|^2}{\langle \Psi_1 | \Psi_1 \rangle \langle \Psi_2 | \Psi_2 \rangle} , \tag{2.94}$$

which at large N can be computed by a supergravity saddle-point computation

$$\mathcal{R} \approx \exp\left[-2\,\mathrm{Re}(I_{grav}(\tilde{\lambda})) + I_{grav}(\lambda_1) + I_{grav}(\lambda_2) \right] , \tag{2.95}$$

where the supergravity solutions have the boundary sources $\tilde{\lambda}$, λ_1 and λ_2 which take the following form

$$\tilde{\lambda} = \begin{cases} \varphi_2(t_E, x), \ t_E < 0 \\ \\ \varphi_1^\star(-t_E, x), \ t_E > 0, \end{cases} \qquad \lambda_i = \begin{cases} \varphi_i(t_E, x), \ t_E < 0 \\ \\ \varphi_i^\star(-t_E, x), \ t_E > 0, \end{cases} \tag{2.96}$$

where $i = 1, 2$[32].

Notice that in each of the terms of (2.95), the gravitational on-shell action is proportional to $\frac{1}{G_N} \sim N^2$. Since quantum mechanically we need $\mathcal{R} \leq 1$, we find that the following inequality has to be satisfied

$$2\,\mathrm{Re}(I_{grav}(\tilde{\lambda})) \geq I_{grav}(\lambda_1) + I_{grav}(\lambda_2)\,, \tag{2.97}$$

for the on-shell value of solutions of the Einstein plus matter equations, for any choice of sources of the form (2.96). If the two sources are different, we expect a strict inequality. It would be interesting to explore this inequality directly from the gravitational point of view. We discuss this further in the discussion.

We now move on to the computation of the return probability for states of the form (2.91) after a small (not N-dependent) time evolution. That is, we take the time-evolved state, $|\Psi(T)\rangle = e^{-iHT}|\Psi\rangle$, and consider the following quantity

$$R(T) = \frac{|\langle\Psi(0)|\Psi(T)\rangle|^2}{\langle\Psi(0)|\Psi(0)\rangle\langle\Psi(T)|\Psi(T)\rangle}\,. \tag{2.98}$$

[32]The sources $\varphi_2(t_E, x)$ and $\varphi_1^\star(-t_E, x)$ should decay sufficiently fast at the $t = 0$ surface such that the states are normalizable. This also implies that the bra and ket prepations of different states can be smoothly glued to each other.

To apply the general formalism described above, we need to analyze how the Euclidean sources φ_0 preparing the state $|\Psi(0)\rangle$ need to be modified to φ_T, in order to prepare $|\Psi(T)\rangle$. From a technical point of view computing φ_T in terms of φ_0 is not straightforward, as it requires a solution of the Einstein equations. Nevertheless, we can in principle compute the return probability using (2.94) and (2.95) with a modified source

$$\tilde{\lambda} = \begin{cases} \varphi_T(t_E, x), \;\; t_E < 0 \\ \varphi_0^\star(-t_E, x), \;\; t_E > 0, \end{cases} \qquad \lambda_T = \begin{cases} \varphi_T(t_E, x), \;\; t_E < 0 \\ \varphi_T^\star(-t_E, x), \;\; t_E > 0\,. \end{cases} \tag{2.99}$$

Thus we get

$$R(T) = \exp\left[-2\,\mathrm{Re}(I_{grav}(\tilde{\lambda})) + I_{grav}(\lambda_0) + I_{grav}(\lambda_t)\right], \tag{2.100}$$

and this is exponentially suppressed in the semi-classical limit because of the $1/G_N \sim N^2$ coefficient in the gravitational action and the condition (2.97).

2.6.2 Thermofield double state

We now consider the thermofield double state

$$|\mathrm{TFD}\rangle = \frac{1}{\sqrt{Z(\beta)}} \sum_n e^{-\frac{\beta E_n}{2}} \, |E_n\rangle_L \otimes |E_n\rangle_R \;, \tag{2.101}$$

where the $|E_n\rangle$'s are the energy eigenstates and $Z(\beta)$ is the partition function at inverse temperature β. In the strong coupling limit, for temperatures below the Hawking-Page temperature, the state is dual to two entangled thermal AdS geometries, while for temperatures higher than the Hawking-Page temperature, it is expected to be dual to the eternal black hole in AdS [Maldacena, 2003]. This geometry has two asymptotically AdS boundaries, on the "left" and the "right", hence the asymptotic symmetry group is $SO(2,d)_L \times SO(2,d)_R$. The state (2.101) is invariant under certain combinations of the asymptotic charges, for example we have

$$(H_R - H_L)\left|\text{TFD}\right\rangle = 0 \qquad \text{but} \qquad (H_R + H_R)\left|\text{TFD}\right\rangle \neq 0 \tag{2.102}$$

and similarly for the other charges. In this case we can generalize the return probability to include all possible large diffeomorphisms on the two sides

$$R(g_1, g_2) = |\left\langle\text{TFD}\right| U_L(g_L)\, U_R(g_R) \left|\text{TFD}\right\rangle|^2 \qquad , \qquad g_{L/R} \in SO(2,d)_{L/R}\,. \tag{2.103}$$

In this case we expect $R(g_L, g_R)$ to rapidly decay along certain directions but remain constant along others due to the symmetries of the state (2.101).

In what follows we focus on a particular class of deformations, corresponding to evolving with $H_L + H_R$. This gives what is usually called the spectral form factor (SFF) defined as

$$R(t) = |\langle\text{TFD}|e^{-i\frac{T}{2}(H_L+H_R)}|\text{TFD}\rangle|^2 = \left|\frac{Z(\beta+iT)}{Z(\beta)}\right|^2 , \tag{2.104}$$

which was introduced in the context of the eternal AdS black hole in [Papadodimas & Raju, 2015] and studied in detail in [Cotler et al., 2017].

We are interested in studying (2.104) above the Hawking-Page temperature for small times, i.e, $T \sim O(1)$. One way to proceed is by computing $Z(\beta)$ and then analytically continuing $\beta \to \beta + iT$. If we are above the Hawking-Page temperature $Z(\beta)$ can be estimated by the Euclidean AdS-Schwarzschild black hole saddle point

$$Z(\beta) \approx e^{-I_{BH}(\beta)}\,, \tag{2.105}$$

where $I_{BH}(\beta)$ is the on-shell action on the Euclidean black hole background. For example, we find

$$I_{BH}(\beta) = -\frac{\pi^2}{2G_N\beta}\ \ (\text{for } AdS_3) \qquad I_{BH}(\beta) = \frac{\beta}{G_N}g(r_H)\ \ (\text{for } AdS_5)\,, \tag{2.106}$$

where we have set the AdS radius $\ell_{\text{AdS}} = 1$ and r_H is the horizon radius, while

$$g(r_H) = \frac{V_3}{8\pi}(-r_H^4 + r_H^2) \tag{2.107}$$

where V_3 is the dimensionless volume associated with the metric on a unit sphere. For the AdS_5 case, $r_H \approx \pi/\beta$ for small real β. A detailed discussion of the action can be found in [Emparan et al., 1999]. The central charge of the CFT_2 is $c = 3/2G_N$ and the rank for the gauge group of the dual four dimensional $SU(N)$ $\mathcal{N} = 4$ super Yang Mills theory is given by $N^2 = \pi/2G_N$.

For small T the complexified partition function $Z(\beta + iT)$ will be given in terms of the analytic continuation of the above actions. Thus for $T \ll \beta$, one gets the following for AdS_3,

$$R(T) \approx e^{-\frac{2\pi^2}{\beta^3}c\,T^2}\,, \tag{2.108}$$

which is exponentially small in the large central charge limit[33]. Similarly for AdS_5, we find that $Z(\beta) \sim e^{\frac{\pi N^2}{\beta^3}}$ in the high temperature limit. Again for $T \ll \beta$, we have

$$R(T) \approx e^{-\frac{12\pi}{\beta^5} N^2 T^2}, \tag{2.109}$$

As T becomes larger and approaches $T \sim \beta$, the dominant saddle point will no longer be the black hole, as the analytically continued action can start to compete with thermal AdS. In addition the analytically continued black hole saddle point corresponds to a geometry with a complex metric, and as $T \sim O(\beta)$ this metric becomes 'unallowable' according to the criteria of [Kontsevich & Segal, 2021], see also [Witten, 2021a]. Interestingly, thermal AdS becomes the dominant saddle point before the metric becomes not allowable [Chen, 2022].

An exponential decay of $R(T)$ in N is to be expected even when $T \sim \beta$, since in this case the thermal AdS saddle dominates and, $|Z(\beta + iT)|^2 \sim e^{\tilde{g}(T)/\beta^3}$ where $\tilde{g}$ is $O(N^0)$ periodic function of time. Thus, the numerator of (2.104) $|Z(\beta{+}iT)|^2$ is N^0 while the denominator is $O(e^{N^2})$ leading to an exponentially suppressed $R(T)$.

2.6.3 Weakly coupled, large N gauge theories

It is interesting to consider the behavior of the SFF at small, or even vanishing 't Hooft coupling λ. In this case the bulk dual is stringy and moreover at $\lambda = 0$, the spectrum of the dual CFT is (half)-integer-spaced and thus not chaotic at all. Nevertheless the decay (2.44) is still valid for a certain time-scale, even in the free theory. This was discussed in detail in

[33] There will be additional terms suppressed in T^2/β^2 which will not affect the exponential decay in the large c limit as long as t is smaller than β.

[Chen, 2022]. For concreteness, we consider the partition function of free $\mathcal{N}=4$ SYM on $\mathbb{S}^3\times\mathbb{R}$, where the sphere has unit radius. It has the form [Sundborg, 2000, Aharony et al., 2004]

$$Z(\beta)=\int \mathcal{D}U\, e^{\sum_R\sum_m^\infty \frac{1}{m}z_m^R(\beta)\chi_R(U^m)}\,, \tag{2.110}$$

where $\mathcal{D}U$ is the invariant Haar measure on the gauge group normalized to one, χ_R is character in the representation R and

$$z_m^R(\beta)=\sum_{R_{i,B}=R}e^{-m\beta E_i}+(-1)^{m+1}\sum_{R_{i,F}=R}e^{-m\beta E_i}\,, \tag{2.111}$$

where the first sum is over bosonic states and the sum in the second term is over fermionic states.

The behavior of the SFF $\left|\frac{Z(\beta+iT)}{Z(\beta)}\right|^2$, as well as of the microcanonical analogue $Y_{E,\Delta E}(T)$, based on the analytic continuation of (2.110) was discussed in [Chen, 2022].

Even at $\lambda=0$ the SFF obeys (2.44), though in this case the Poincare recurrence time is very short, i.e. 4π.[34] While in this limit the bulk theory does not admit a semiclassical gravitational description, we could still apply the procedure (2.57) to identify operators with vanishing commutators with the Hamiltonian to all orders in $1/N$, though now they do not have a nice bulk interpretation.[35] In doing so, we would need to be careful to take t_* to be a short $O(1)$ time-scale which is less than 4π.

Here we notice that similar results have been derived for the analytically continued superconformal index [Choi et al., 2022], which can be thought of as the SFF for 2-sided

[34]In our conventions conformal dimensions in the free theory are half-integers.

[35]To start with, the HKLL procedure cannot be implemented at subleading orders in $1/N$ due to the many stringy fields present in the bulk. Therefore, the issue of non-commutativity with the Hamiltonian does not stand out like it does in the case of Einstein gravity.

eternal supersymmetric AdS black holes.

2.6.4 Perturbative states around empty AdS

We now briefly discuss the return probability for perturbative states around empty AdS. We want to consider states which have a large number of particles, but still small enough so that we can ignore gravitational backreation. We can get some useful estimates by considering a thermal gas of particles in AdS_{d+1}. These are dual to a gas generated by single-trace operators in the CFT. Suppose we have low-lying single-trace operators with conformal dimension Δ_i. For simplicity we consider only scalars and we take the radius of AdS_{d+1} to be 1. Then the partition function of single-particle states $z(\beta)$ and the multi-trace Fock-space partition function are respectively

$$z(\beta) = \sum_i \frac{e^{-\beta\Delta_i}}{(1-e^{-\beta})^d} \qquad , \qquad Z(\beta) = \exp\left[\sum_{n=1}^{\infty} \frac{1}{n} z(n\beta)\right] . \tag{2.112}$$

It is now straightforward to do the analytic continuation

$$Z(\beta + iT) = \exp\left\{\sum_{n=1}^{\infty} \sum_i \frac{e^{-(n\beta+inT)\Delta_i}}{(1-e^{-n\beta+inT})^d}\right\} . \tag{2.113}$$

For scalar BPS operators dual to SUGRA modes, Δ_i is integer. Then it is obvious that the SFF $R(T) = \left|\frac{Z(\beta+iT)}{Z(\beta)}\right|^2$ has periodicity $T = T + 2\pi$, as expected. What we want to estimate is the decay rate of the SFF at early times, and how close to 0 the SFF drops between the recurrences.

First we notice that the partition function factorizes to a product over Δ_i. Hence we can study the behavior of a given Δ_i and we drop the sum over i. If we first take the small β

limit, before analytically continuing, we find

$$Z(\beta) \sim \exp\left[\zeta(d+1)\frac{1}{\beta^d}\right] . \tag{2.114}$$

Using this approximation we find that for early times

$$R(T) \sim e^{-\frac{d(d+1)\zeta(d+1)}{\beta^{d+2}}T^2} . \tag{2.115}$$

As expected the decay is controlled by the variance of H. Of course if we use the high temperature approximation (2.114) to perform the analytic continuation, then we do not see the recurrences. At high temperature the SFF starts decaying quite rapidly, stays close to zero for a while and then goes back to 1 every $T = 2\pi \times$ integer. To find an estimate of how closely it approaches zero it is convenient to evaluate it at $T = \pi$. Suppose that the conformal dimension is an even integer. Then we find

$$R(\pi) = \frac{\exp\left[2\sum_{n=1}^{\infty}\frac{1}{n}\frac{e^{-n\beta\Delta}}{(1-(-1)^n e^{-n\beta})^d}\right]}{\exp\left[2\sum_{n=1}^{\infty}\frac{1}{n}\frac{e^{-n\beta\Delta}}{(1-e^{-n\beta})^d}\right]} \sim e^{-(2-2^{-d})\zeta(d+1)\frac{1}{\beta^d}-\frac{1}{2^d}\log\frac{\beta\Delta}{2}} \tag{2.116}$$

So we see significant suppression at small β, though of course, the suppression does not scale like e^{-N^2}.

We expect a similar qualitative behavior for $R(T)$ for generic pure states of similar energy as the states studied above (namely high energy states whose energy scales as $O(N^0)$): they will have recurrences every 2π, but the return probability will quickly decay to small values for $0 < t < 2\pi$. If we use (2.57) for such states, with $t_* \sim O(1) < \pi$, then the commutator with H will be suppressed by a factor of the order of (2.116) rather than e^{-N^2}. Note that this is not good enough, since the commutator we are trying to cancel is $\mathcal{O}(1/N)$, which in the large N limit is much smaller than the suppression controlled by (2.116).

2.6.5 LLM geometries

An interesting class of semiclassical states with $\mathrm{AdS}_5 \times S^5$ asymptotics in type IIB supergravity are the LLM geometries [Lin et al., 2004]. These are dual to $\frac{1}{2}$-BPS states in $\mathcal{N} = 4$ SYM. While these geometries do not break all of the asymptotic symmetries, they do provide a useful toy model where we can study in detail the behavior of the return probability as a function of time.

The $\frac{1}{2}$-BPS states in $\mathcal{N} = 4$ SYM on $S^3 \times R$ are states that preserve 16 of the 32 supersymmetries of the theory in addition to the bosonic symmetries $SO(4) \times SO(4) \times R$ where R corresponds to the Hamiltonian $H - \hat{J}$ where H is the Hamiltonian and $\hat{J}$ an R-symmetry generator. These states correspond to operators that lie in the $(0, J, 0)$ representation of the $SU(4) \sim SO(6)$ R-symmetry and they saturate a unitarity bound for their conformal dimension. It is illuminating to consider the $\mathcal{N} = 1$ vector and three chiral multiplet decomposition of the $\mathcal{N} = 4$ theory. In this case the scalars of the chiral multiplets are organized into $Z^j = \varphi^j + i\varphi^{j+3}$, where $j = 1, 2, 3$, which are in the adjoint representation. We will focus on $j = 1$ from now on without loss of generality. Then the states we are interested in correspond, via the state-operator map, to single-trace operators of the form $\mathrm{Tr}(Z^{n_i})$, as well as multi-trace operators of the form $\Pi_i(\mathrm{Tr}(Z^{n_i}))^{r_i}$ [Corley et al., 2002, Lin et al., 2004, Berenstein, 2004].

Since these operators saturate the unitarity bound $\Delta = J$, they correspond to the lowest Kaluza-Klein mode of Z on S^3. This mode has a harmonic oscillator potential due to its conformal coupling to the curvature of S^3. Thus we are interested in gauge invariant states of the matrix Z in a harmonic potential [Berenstein, 2004]. The ground state, corresponding to empty AdS, is given by a Gaussian wave function

$$\Psi_{vac} = Ce^{-\frac{1}{2}N^2 \mathrm{tr}(Z^2)}\,, \tag{2.117}$$

where $C = (\pi/N)^{-N^2/4}$ and we introduce the notation $\mathrm{tr}Z = \frac{1}{N}\ \mathrm{Tr}Z = \frac{1}{N}\sum_i^N v_{ii}$. Fluctuations with operators with $\Delta = J \ll N$ will be small excitations around the ground state. As discussed in [Yaffe, 1982] excitations with $\Delta = J \sim N^2$ will be other coherent states which are given by

$$\Psi = C[J(Z)]^{1/2} e^{-N^2 \mathrm{tr}(\frac{1}{2}\varphi(Z)^2 - i\psi(Z))}\,, \tag{2.118}$$

parameterized by two functions $\varphi(Z)$ and $\psi(Z)$ which are monotonically increasing and arbitrary functions of Z, respectively. $J[Z]$ is the Jacobian given by $\det\left[\partial\varphi(Z)_{ij}/\partial Z_{kl}\right]$.

It is well known that one can describe such a system by N fermions in a harmonic potential [Brézin et al., 1978]. In the large N limit, states in such a system can be thought of as droplets in a two dimensional phase space, where for example a circular droplet corresponds to the ground state of the system [Brézin et al., 1978, Jevicki & Sakita, 1980, Shapiro, 1981]. The precise connection between the functions $\varphi(Z)$ and $\psi(Z)$ and the droplet picture on the phase space will be discussed in the next subsection.

In the bulk, the LLM solutions correspond to 10 dimensional geometries of asymptotically $AdS_5 \times S^5$ spacetimes, see appendix 2.12, that are completely determined by a function z on a two dimensional surface. In particular, specifying whether z takes value $1/2$ or $-1/2$

at each point on this plane completely specifies the full bulk solution. This is in parallel with the two dimensional fermionic phase space mentioned earlier where the fermion takes occupation number 1 (black) or 0 (white) at each point in the phase space, giving droplet of a given shape. For instance, in the fermionic picture the ground state is a circular droplet of a certain radius, say r_0. It corresponds in the bulk is to the empty $AdS_5 \times S^5$.

Fluctuations with operators of $\Delta = J \ll N$ correspond to having ripples in the edge of the circular droplet and corresponds to having gravitons propagating in the $AdS_5 \times S^5$ background. While operators of energy $\Delta = J \sim N$ correspond to giant gravitons in the bulk. Operators with $\Delta = J \sim N^2$ correspond to other bulk geometries and different shapes of droplets in the fermionic phase space [Lin et al., 2004, Berenstein & Miller, 2017, Berenstein & Miller, 2018]. The geometries will not be time translation invariant (rotational invariant in the fermionic picture) in general[36], but they are invariant under $t \to t + 2\pi$.

The goal here is to consider a certain geometry that breaks time translation invariance and compute its return probability for short time scales. In the fermionic picture this corresponds to a droplet that breaks the rotational invariance, an ellipse for instance. In the matrix quantum mechanics picture it is easy to compute the return probability, evolving (2.118) with the quadratic Hamiltonian and computing the square of the inner product. But first, we need to review the dictionary between the two pictures.

[36]There are also static configurations, concentric circles for example [Lin et al., 2004]

The way the matrix quantum mechanics picture and the fermionic picture are related will be obvious once we diagonalize the matrix Z and express it in terms of the eigenvalues (μ_i), where the Jacobian becomes

$$J(Z) = \prod_i^N \varphi'(\mu_i) \prod_{i \neq j} \frac{\varphi(\mu_i) - \varphi(\mu_j)}{\mu_i - \mu_j}, \tag{2.119}$$

which is 1 for the vacuum. In the large N limit, the Gaussian measure $dZ \exp(-N^2 \mathrm{tr}(Z^2))$ will reduce to the well known Wigner semi-circle distribution for the density of eigenvalues [Wigner, 1959],

$$d\varrho(\mu) = \frac{1}{\pi}(2-\mu^2)^{1/2}\Theta(2-\mu^2)d\mu. \tag{2.120}$$

Let us now introduce new variables to parameterize the coherent states in the large N limit, $w(\mu) := d\varrho(\varphi(\mu))/d\mu$ which is the density of eigenvalues and $v(\mu) := \psi(\mu)$. These parameters are canonical conjugates of one another[37], that is their Poisson bracket is the Dirac delta function. In the large N limit, the appropriately renormalized Hamiltonian (h_{cl}) can also be written in terms of w and $v' = dv/d\mu$ and thus an action can be written for these variables [Yaffe, 1982, Dhar et al., 1993a, Dhar et al., 1992, Dhar et al., 1993b]. In particular,

$$h_{cl} = \frac{1}{2}\int d\mu\, w(\mu)(v'(\mu)^2 + \frac{\pi^3}{3}w(\mu)^2 + \mu^2). \tag{2.121}$$

Coming back to the two dimensional phase space picture, we consider a blob centered at the origin. We assume the horizontal direction (x-axis) represents the q variable of the phase

[37]Note that the two variables are not totally independent and $w(\mu)$ has to satisfy a constraint, in particular $\int d\mu\, w(\mu) = 1$.

space, which we take to be the eigenvalues (μ). Consider a vertical line crossing the blob. Assuming that the blob has a simple geometry without folds, this vertical line intersects the boundary of the blob twice. We parametrize these points by $p_{\pm}(\mu)$ respectively. Then, the density of eigenvalues for any μ is proportional to $(p_+ - p_-)(\mu)$. Computing the kinetic energy of fermions for a given $d\mu$ by integrating $p^2/2$ from $p = p_-$ to $p = p_+$ and matching this to the kinetic part of (2.121), we get

$$p_{\pm} = \pm \pi w + v^{'}. \tag{2.122}$$

This has also been mentioned in the context of $c = 1$ string theory in [Polchinski, 1991, Ginsparg & Moore, 1993, Das, 1992, Das, 2004]. Note that for the vacuum (i.e. the empty $\mathrm{AdS}_5 \times S^5$ geometry), $p_{\pm} = \pm(2-\mu^2)^{1/2}\Theta(2-\mu^2)$ and $v' = 0$.

Since we are looking for a time dependent geometry, we need a blob in fermion phase space that breaks the rotational symmetry. The simplest non trivial modification of (2.122) is to take v to be quadratic[38]. In this case we have

$$p_+(\mu) = (2-\mu^2)^{1/2}\Theta(2-\mu^2) + 2\mu. \tag{2.123}$$

This can be seen to be half of a tilted ellipse, which combined with an appropriate p_- gives the full elliptic blob. This will evolve non trivially under rotation and the corresponding geometry will be a time dependent one. This geometry, together with the five form, can be found using the mapping discussed earlier, by first solving for $z(x_1, x_2, y)$ then inserting it into (2.223), (2.224), (2.225) and (2.226).

[38] Translated circle blobs will not correspond to physical geometries when the gauge group is $SU(N)$, since the centre of the blob is fixed by imposing the condition $\mathrm{Tr}(Z)$=0.

Now we proceed with the computation of the return probability for this state. We go back to (2.118) and consider a state $\Upsilon(0)$ with $\varphi = Z$ and $\psi = v = Z^2$ and after evolving it, compute the overlap

$$\langle \Upsilon(0) | \Upsilon(T) \rangle = \int dZ \ \Upsilon(Z, 0)^* \Upsilon(Z, T) \tag{2.124}$$

The state we are interested has the form

$$\Upsilon(Z, 0) = \left(\frac{\pi}{N}\right)^{-N^2/4} e^{-\frac{1}{2}N^2(1-2i)\mathrm{tr}(Z^2)} = \prod_{i,j} \phi(\nu_{ij}) \, , \text{where } \phi(\nu) = \left(\frac{\pi}{N}\right)^{-1/4} e^{-\frac{N}{2}(1-2i)\nu^2} \, . \tag{2.125}$$

Since we are dealing with matrix quantum mechanics with a quadratic potential, each matrix element evolves independently and governed by the usual harmonic oscillator propagator

$$\phi(\nu, T) = \int d\nu' K(\nu', \nu, T) \phi(\nu') \, , \tag{2.126}$$

where[39]

$$K(\nu', \nu, T) = \sqrt{\frac{N}{2\pi i \sin T}} \exp\left[\frac{iN}{2\sin T}((\nu^2 + (\nu')^2)\cos T - 2\nu\nu')\right] \, . \tag{2.127}$$

for $t < \pi$. We can then compute the overlap

$$\langle \Upsilon(0) | \Upsilon(T) \rangle = [z(T)]^{N^2}, \qquad z(T) = \int d\nu \ \phi^\star(\nu, 0) \phi(\nu, T) \tag{2.128}$$

[39]One has to be careful in choosing the correct branch while taking the square root, but this will not affect the final result we are interested in.

Following (2.126) we find

$$\phi(\nu, T) = \left(\frac{N}{\pi\mathcal{X}}\right)^{1/4} e^{-N\mathcal{Y}\nu^2} \text{ and,} \qquad \Psi(Z, T) = \left(\frac{N}{\pi\mathcal{X}}\right)^{-N^2/4} e^{-N^2\mathcal{Y}\mathrm{tr}(Z^2)}\,, \tag{2.129}$$

where $\mathcal{X}$ and $\mathcal{Y}$ are periodic functions of time given by,

$$\begin{aligned} \mathcal{X}(T) &= (\cos T + (2+i)\sin T)^2 \\ \mathcal{Y}(T) &= \frac{1}{2}\left(\frac{(1-2i)\,\cos T + i\,\sin T}{(i+2)\,\sin T + \cos T}\right)\,. \end{aligned} \tag{2.130}$$

Thus,

$$z(T) = \int d\nu\, \phi^\star(\nu, 0)\phi(\nu, T) = \mathcal{A}^{1/2}\,, \tag{2.131}$$

where

$$\mathcal{A} = \frac{1}{3i\,\sin T + \cos T}\,. \tag{2.132}$$

It can be checked that $z(T)$ is 1 when $T = 0$ and

$$|z(T)|^2 = |\mathcal{A}| = \left(\frac{1}{9\,\sin^2 T + \cos^2 T}\right)^{1/2}\,. \tag{2.133}$$

Since $|z(T)|^2 \le 1$, $R(T)$ is an exponentially decaying function in the large N limit, for small times, but a periodic function in $T = \pi$. That is

$$R(T) = |\langle\Psi(0)|\Psi(T)\rangle|^2 = e^{-N^2 F(T)}\,, \tag{2.134}$$

where the function $F(T) = -2\log|z(T)|$ is zero at $T = 0$, and increases to the local maximum $F(T = \pi/2) = \log 9$ and goes back to zero at $T = \pi$. Thus in the time scales we are interested in, in particular $T < \pi/2$, the square of the inner product (2.124) which is the return probability of a given LLM semi classical geometry in the large N limit, is exponentially suppressed in N^2 as expected. Note that the return probability is periodic in π, which is due to the symmetry of the particular state considered. In general, the period will be 2π.

We can also compute the overlap of states in different code subspaces built upon $\Upsilon(Z,0)$ and $\Upsilon(Z,T)$. The simplest is the inner product of the states $\Upsilon(Z,0)$ and $\mathrm{Tr}(Z^{2n})\Upsilon(Z,T)$ which can be written as

$$\begin{aligned}\langle\Upsilon(0)|\,\mathrm{tr}(Z^{2n})\,|\Upsilon(T)\rangle &\equiv \int dZ\;\Upsilon^\star(Z,0)\mathrm{tr}(Z^{2n})\Upsilon(Z,T)\\ &= \left(\frac{\pi}{N\mathcal{X}^{1/2}}\right)^{-N^2/2}\int dZ\;\mathrm{tr}(Z^{2n})\,e^{-\frac{\mathcal{S}}{2}N^2\;\mathrm{tr}(Z^2)}\end{aligned} \tag{2.135}$$

where $\mathcal{S} = (1+2i) + 2\mathcal{Y}$. Following (2.49), we can rewrite the above integral as

$$\langle\Upsilon(0)|\,\mathrm{tr}(Z^{2n})\,|\Upsilon(T)\rangle = \langle\Upsilon(0)|\Upsilon(T)\rangle\,\frac{\int dZ\;\mathrm{tr}(Z^{2n})\,e^{-\frac{\mathcal{S}}{2}N^2\;\mathrm{tr}(Z^2)}}{\int dZ\;\;e^{-\frac{\mathcal{S}}{2}N^2\;\mathrm{tr}(Z^2)}} \tag{2.136}$$

The second factor corresponds to an expectation value in a Gaussian matrix model. Keeping only planar diagrams at large N we find

$$\langle\Upsilon(0)|\,\mathrm{tr}(Z^{2n})\,|\Upsilon(T)\rangle \simeq \langle\Upsilon(0)|\Upsilon(T)\rangle\,\frac{C_n}{\mathcal{S}^n} \tag{2.137}$$

where $C_n = \frac{1}{n+1}\binom{2n}{n}$ are the Catalan numbers.

Similarly, for multi-trace operators the overlap can be computed and using large N factorization we get

$$\langle\Psi(0)|\prod_i^k \mathrm{tr}(Z^{2n_i})\,|\Psi(T)\rangle \simeq \langle\Psi(0)|\Psi(T)\rangle\,\frac{\prod_i^k C_{n_i}}{\mathcal{S}^n}, \tag{2.138}$$

where $n = n_1 + \ldots + n_k$.

Thus, as long as n does not scale with N the correlator will still be exponentially suppressed, otherwise the periodic coefficient can spoil the exponentially decaying behaviour. This is to be expected since in such cases the dimension of the multi-trace operators will be order N and they will not be just small fluctuations of the background and can, in principle, evolve the state back in time to $T = 0$. In any case, our code subspace is constructed by the action of multitrace operators whose dimension is finite in the large N limit, i.e, n is an $O(1)$ number.

2.6.6 Kourkoulou-Maldacena states in SYK model

The SYK model is a quantum mechanical model of N Majorana fermions interacting with random interactions which is given by the Hamiltonian

$$H = \sum_{iklm} j_{iklm}\,\psi_i\psi_k\psi_l\psi_m\,, \tag{2.139}$$

where ψ_i are the Majorana fermions $\{\psi_i, \psi_j\} = \delta_{ij}$, and the coupling j_{iklm} has drawn from the distribution

$$P(j_{iklm}) \sim \exp\left(-N^3 j_{iklm}^2/12J^2\right)\,, \tag{2.140}$$

leading to disorder average of

$$\overline{j_{iklm}} = 0, \qquad \overline{j^2_{iklm}} = \frac{3! J^2}{N^3}. \tag{2.141}$$

In a particular realization of the couplings, we consider pure states which are obtained by using the Jordan-Wigner transformation and combining pairs of Majorana fermions into qubit like operators and choosing states with definite eigenvalues for the σ_3 components of all qubits. These states are denoted by $|B_s\rangle$, where $s = (s_1, s_2, \ldots, s_{N/2})$ with $s_k = \pm 1$, and they satisfy the relations below

$$S_k \, |B_s\rangle = s_k \, |B_s\rangle \,, \tag{2.142}$$

where $S_k = \sigma_3^k/2 \equiv 2i\,\psi^{2k-1}\psi^{2k}$ is the spin operator. By choosing all possible combinations of the $\{s_k\}$'s we get a basis of the Hilbert space whose dimension is $2^{N/2}$ (N is an even integer number). We further evolve these states over some distance l in Euclidean time in order to get low energy states $|B_{s,l}\rangle = e^{-lH}\,|B_s\rangle$ which we will refer to as Kourkoulou-Maldacena (KM) states. To stay in the low-energy regime where the SYK model exhibits conformal invariance we take $1 \ll lJ \ll N$ [Kourkoulou & Maldacena, 2017].

As discussed in [Kourkoulou & Maldacena, 2017] the KM states can be thought of as a toy model of pure black hole microstates which are out of equilibrium and which contain excitations behind the horizon. Hence they are states which exhibit time-dependence and our general formalism should be applicable. We start by discussing the behavior of the return probability for these states.

Analytical computation of the return probability at large N

We start with the normalization of the KM states. In the large N limit, due to the approximate $O(N)$ symmetry of the theory it can be shown [Kourkoulou & Maldacena, 2017] that

$$\langle B_{s,l} | B_{s,l} \rangle = \langle B_s | e^{-2lH} | B_s \rangle = 2^{-N/2} Z(\beta) \,, \tag{2.143}$$

where $\beta = 2l$ [Kourkoulou & Maldacena, 2017]. The return probability then in the large N limit is given by

$$R(T) = \Big| \frac{\langle B_{s,l} | e^{-iHT} | B_{s,l} \rangle}{\langle B_{s,l} | B_{s,l} \rangle} \Big|^2 = \Big| \frac{Z(\beta + iT)}{Z(\beta)} \Big|^2 . \tag{2.144}$$

In a low temperature expansion, the partition function can be estimated [Maldacena & Stanford, 2016] using the Schwarzian approximation to be

$$Z(\beta) \propto \frac{e^{2\sqrt{2}\pi^2 \alpha_S \frac{N}{\beta J}}}{(\beta J)^{3/2}} . \tag{2.145}$$

Using (2.145) we find for the return probability

$$R(T) = \frac{1}{(1 + \frac{T^2}{\beta^2})^{3/2}} e^{-(4\sqrt{2}\pi^2 \alpha_S \frac{N}{J\beta^3})T^2} \,, \tag{2.146}$$

which is compatible with (2.44), after we take into account the different N-dependence in the SYK model vs $\mathcal{N} = 4$ SYM.

We can now try to test the more general decay of the inner product between states in time-shifted code subspaces (2.47). Let us denote the unit-normalized KM states as

$$|\widehat{B}_{s,l}\rangle = \frac{|B_{s,l}\rangle}{\sqrt{\langle B_{s,l}|B_{s,l}\rangle}}\,, \tag{2.147}$$

and denote their time-dependence as $|\hat{B}_{s,l}(T)\rangle = e^{-iHT}|\widehat{B}_{s,l}\rangle$. We consider an operator $A(t)$ which is a simple combination of the fermions, so that the state $A(t)|\hat{B}_{s,l}\rangle$ is in the code subspace. Then we write

$$\langle\widehat{B}_{s,l}(0)|A(t)|\widehat{B}_{s,l}(T)\rangle = \langle\widehat{B}_{s,l}(0)|\widehat{B}_{s,l}(T)\rangle \times \frac{\langle B_{s,l}(0)|A(t)|B_{s,l}(T)\rangle}{\langle B_{s,l}(0)|B_{s,l}(T)\rangle}\,. \tag{2.148}$$

Let us focus on the last ratio. We can rewrite it as

$$\frac{\langle B_{s,l}(0)|A(t)|B_{s,l}(T)\rangle}{\langle B_{s,l}(0)|B_{s,l}(T)\rangle} = \frac{\langle B_s|e^{-(l+i\frac{T}{2})H}A(t-\frac{T}{2})e^{-(l+i\frac{T}{2})H}|B_s\rangle}{\langle B_s|e^{-(l+i\frac{T}{2})H}e^{-(l+i\frac{T}{2})H}|B_s\rangle}\,, \tag{2.149}$$

which depends holomorphically on $l+i\frac{T}{2}$, so we can evaluate if by analytic continuation. All in all we find

$$\langle\widehat{B}_{s,l}(0)|A(t)|\widehat{B}_{s,l}(T)\rangle = \langle\widehat{B}_{s,l}(0)|\widehat{B}_{s,l}(T)\rangle \times \left[\langle\hat{B}_{s,l}(0)|A(t-\frac{T}{2})|\hat{B}_{s,l}(0)\rangle\right]_{l\to l+i\frac{T}{2}}\,. \tag{2.150}$$

At large N and for flip-invariant operators [Kourkoulou & Maldacena, 2017] we can also write this as

$$\langle\widehat{B}_{s,l}(0)|A(t)|\widehat{B}_{s,l}(T)\rangle = \langle\widehat{B}_{s,l}(0)|\widehat{B}_{s,l}(T)\rangle \times \langle A(t-\frac{T}{2})\rangle_\beta|_{\beta\to\beta+iT}\,, \tag{2.151}$$

where in the last term we first compute the thermal 1-point function $\langle A(t-\frac{T}{2})\rangle_\beta$ as a function of β and then analytically continue β.

As an example, we consider the case where $A = \psi^k(t)\psi^k(t')$ (no summation over k implied). Following [Kourkoulou & Maldacena, 2017] we have for real time and large N

$$\langle \widehat{B}_{s,l}(0)|\psi^k(t)\psi^k(t')\ |\widehat{B}_{s,l}(0)\rangle = G_\beta(t-t')\,, \tag{2.152}$$

where, for $t > t'$, we have

$$G_\beta(t-t') = \frac{\pi^{1/4}}{\sqrt{2\beta J}}\frac{e^{-i\pi/4}}{\sqrt{\sinh[\pi(t-i\varepsilon)/\beta]}}, \tag{2.153}$$

Therefore, using (2.150) we get

$$\langle \widehat{B}_{s,l}(0)|\psi^k(t)\psi^k(t')|\widehat{B}_{s,l}(T)\rangle = \langle \widehat{B}_{s,l}(0)|\widehat{B}_{s,l}(T)\rangle\ G_{\beta+iT}(t-t')\,, \tag{2.154}$$

where the last term can be computed as the analytic continuation of (2.153).

Similarly for $A = \psi^{2k-1}(t)\psi^{2k}(t')S_k$ we have [Kourkoulou & Maldacena, 2017]

$$\langle \widehat{B}_{s,l}(0)|\psi^{2k-1}(t)\psi^{2k}(t')S_k|\widehat{B}_{s,l}(0)\rangle = -2is_kG_\beta(t)G_\beta(t') + O(1/N), \tag{2.155}$$

hence

$$\langle \widehat{B}_{s,l}(0)|\psi^{2k-1}(t)\psi^{2k}(t')S_k|\widehat{B}_{s,l}(T)\rangle = \langle \widehat{B}_{s,l}(0)|\widehat{B}_{s,l}(T)\rangle\times \tag{2.156}$$

$$\times\left[-2is_kG_{\beta+iT}(t-\frac{T}{2})G_{\beta+iT}(t'-\frac{T}{2}) + O(1/N)\right]. \tag{2.157}$$

The examples (2.154) and (2.156) are consistent with our general expectations, see (2.47) and (2.48).

Some numerical checks

In this subsection we perform some simple numerical checks of 2.47 and 2.50, as well as the behavior of the operators 2.57 for KM states in the SYK model. The first step is to select an appropriate value for the inverse temperature $\beta = 2l$. The early time decay of the return probability is

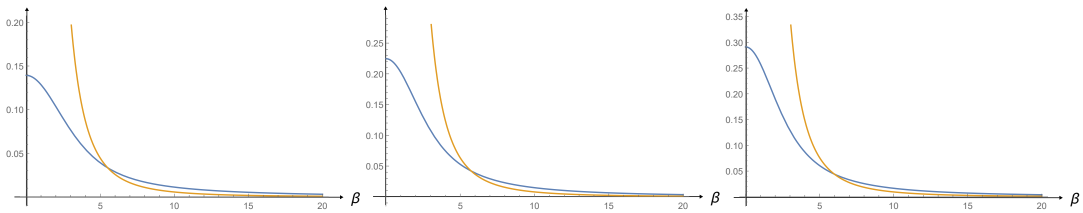
$$R(T) = e^{-\Delta H^2 T^2} . \tag{2.158}$$

Earlier we used the Schwarzian approximation to compute the partition function 2.145 from which we can also get the variance

$$\Delta H^2 = 4\sqrt{2}\pi^2 \alpha_S \frac{N}{\beta^3} = 0.396 \frac{N}{\beta^3} . \tag{2.159}$$

We compare this result with a numerical computation of the variance ΔH^2 for a KM state constructed from $|B_s\rangle = |+ - - ... -\rangle$. This is shown in Figure 3.2. In Figure 2.3, we show the value of the plateau for the KM state, as defined in 2.45 for various values of N and β. For the range of values of N we are interested in, we can take the inverse temperature to be $\beta = 5$, which is the value we will use in what follows.

In Figure 2.4 we can see the return probability as a function of t for different values of N for the corresponding KM state. As discussed in subsection 2.3.6, we expect that the overlap between any state in the code subspace at $t = 0$ will and the one at $t = T$ will also



(a) N = 14 **(b)** N = 20 **(c)** N = 24

Figure 2.2: The blue lines are the numerical results for the variance of Hamiltonian as a function of β while the yellow ones are the Schwarzian approximation $\Delta H^2 = 0.396N/\beta^3$.

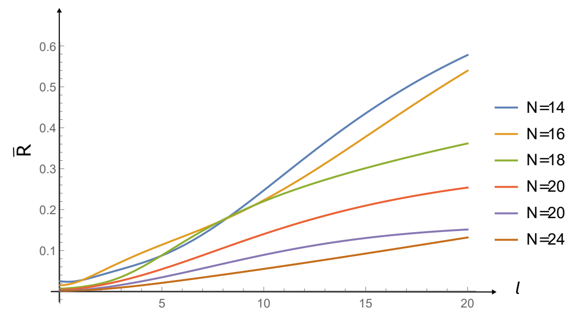


Figure 2.3: The plateau height $\bar{R}$ as a function of $l = \beta/2$.

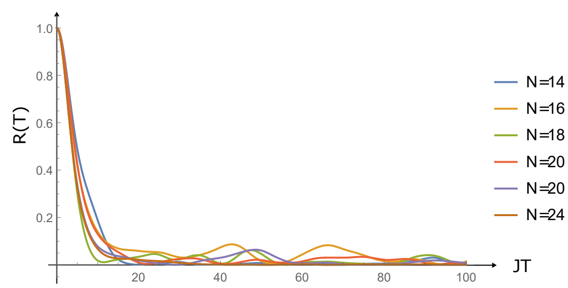


Figure 2.4: Return probability as a function of T for different values of N

decay exponentially fast. We can encode the overlap between all such pairs of states by

$$R_{code}(T) = \frac{1}{d_{code}} \mathrm{Tr}[P_T P_0] \,. \tag{2.160}$$

For the numerical computation we need to make some choice about the code subspace. One condition is that the dimension d_{code} of the code subspace should satisfy $d_{code} \ll 2^{N/2}$.

As an example, and for the purpose of the numerical computation, we can define the code subspace as

$$\mathcal{H}_{code} = \text{span}\{\mathcal{O}_1^{i_1}...\mathcal{O}_k^{i_k}|B_s\rangle; i_j = 0, 1\} , \tag{2.161}$$

for some choice of the operators $\mathcal{O}_i$. Here $D_{code} = 2^k$ the value of k should be such that $D \ll 2^{N/2}$. Note that the states in (2.161) are generally not orthonormal but it is easy to write a projector on the code subspace in terms of elements of this basis, see [Bahiru & Vardian, 2022] for a related discussion.

In Fig. 2.5, we see plots of the behavior of $R_{code}(T)$ as a function of time for some specific choices of such a code subspace:

- **a :** the dimension of the code subspace is $D = 8$ and the operators are chosen to be

$$\mathcal{O}_1 = \psi_1(t = 0), \qquad \mathcal{O}_2 = \psi_1(t = 0.1), \qquad \mathcal{O}_3 = \psi_1(t = 0.5).$$

- **b :** the dimension of the code subspace is $D = 8$ and the operators are chosen to b

$$\mathcal{O}_1 = \psi_1(t = 0), \qquad \mathcal{O}_2 = \psi_1(t = 0.1), \qquad \mathcal{O}_3 = h.$$

- **c :** the dimension of the code subspace is $D = 16$ and the operators are chosen to be

$$\mathcal{O}_1 = \psi_1(t = 0), \quad \mathcal{O}_2 = \psi_1(t = 0.1), \quad \mathcal{O}_3 = \psi_1(t = 0.5), \quad \mathcal{O}_4 = \psi_1(t = 1).$$

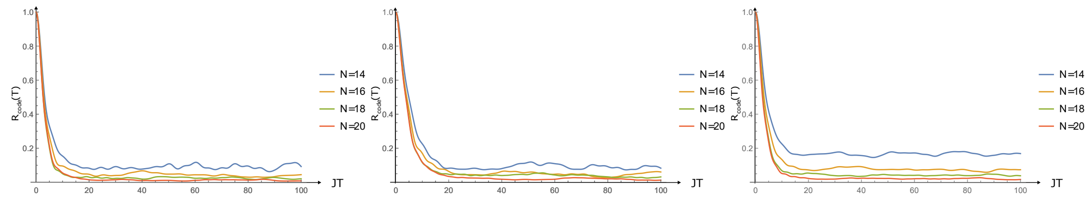


Figure 2.5: $R_{code}(T)$ as a function of T for three different examples of codesubspaces in the form of (2.161).

where in case (b) the operator h is the normalized Hamiltonian

$$h = \frac{1}{\sqrt{N}}(H - \langle H \rangle). \tag{2.162}$$

We finally check that the operator (2.57) has similar correlators as the boundary-dressed operator. We take the code subspace as

$$\mathcal{H}_{code} = \text{span}\{|B_s\rangle, \mathcal{O}_1|B_s\rangle, ...\mathcal{O}_k|B_s\rangle, h|B_s\rangle, h\mathcal{O}_1|B_s\rangle, ...h\mathcal{O}_k|B_s\rangle\}, \tag{2.163}$$

where the dimension of the code subspace is $d_{code} = 2(k+1) \ll 2^{N/2}$. In Fig. 2.6, we plot the result for the case of $k = 5$ and where the operators chosen to be

$$\mathcal{O}_1 = \psi_1(t=0), \quad \mathcal{O}_2 = \psi_1(t=2), \quad \mathcal{O}_3 = \psi_1(t=4) \quad \mathcal{O}_4 = \psi_1(t=6), \quad \mathcal{O}_5 = \psi_1(t=8)$$

for $N = 20$ ($d_{code} = 12 \ll 2^{10}$) are plotted. One can see from Fig.2.6b that the state-dressed operator for ψ_3 has approximately the same correlation function as the original one.

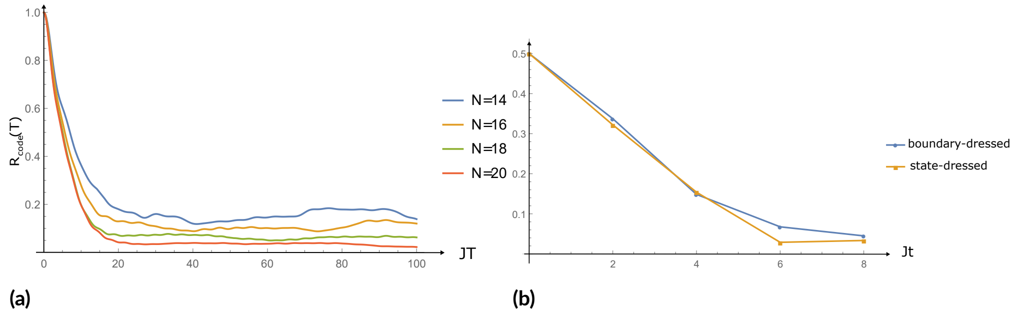


Figure 2.6: Results for the code subspace (2.163). (a) $R_{code}(T)$ as a function of T. (b) The blue line is $\langle\psi_3(0)\psi_3(t)\rangle$ as a function of t, while in the case of the yellow line, $\psi_3(0)$ is replaced by the dressed operator obtained from our proposal. Here N=20.

2.6.7 Holographic boundary states

The KM states discussed in the previous section can be thought of as certain a-typical black hole microstates in the context of SYK/AdS_2. Interesting analogs in higher dimensional examples of AdS/CFT can be found by considering boundary states in CFTs [Almheiri et al., 2018b, Takayanagi, 2011, Karch & Randall, 2001]. A boundary state characterizes boundary conditions which can be imposed on a boundary of space-time on which the CFT lives. For each allowed boundary condition, we can evolve the state along the Euclidean time to suppress the high-energy contributions and obtain a state of finite energy which is called a regularized boundary state of the CFT.

For holographic theories, the CFT path integral maps onto the gravity path integral. Therefore, we will be able to make use of the AdS/CFT correspondence to deduce the corresponding geometries if we can choose a state for which we can understand a gravity prescription for dealing with the boundary condition at the initial Euclidean time. As discussed in [Cooper et al., 2019b], we can describe boundary states by starting with the TFD state of two CFTs labeled by L and R

$$|\text{TFD}(\beta/2)\rangle = \frac{1}{Z}\sum_i e^{-\beta E_i/4} |E_i\rangle_L \otimes |E_i\rangle_R \,, \tag{2.164}$$

and then project the TFD state onto some particular pure state $|B\rangle$ of the left CFT. As a result, we obtain a pure state of the right CFT given by

$$|\Psi_{B,\beta}\rangle = \frac{1}{Z}\sum_i e^{-\beta E_i/4} \langle B\,|E_i\rangle\,|E_i\rangle \,. \tag{2.165}$$

If the temperature is high enough, the TFD state is dual to the maximally extended AdS-Schwarzschild black hole in the bulk. The geometry which is dual to these regularized boundary states is expected to contain a significant portion of the left asymptotic region. Therefore, in a holographic CFT, this class of regularized boundary states can be regarded as microstates of a single-sided black hole. These black hole microstates can be thought of as black holes with end of the world (EOW) branes on the left side.[40] Generally the EOW brane configuration is time-dependent at the macroscopic level. Hence these are states with energy and energy variance compatible with (2.27) and (2.29), so we expect to be able to apply our construction and define operators (2.57). As we will discuss in the next section, one way to think of them is that the gravitational dressing has been moved over to the EOW brane.

[40] Proving from first principles that boundary states dual to EOW branes exist is far from trivial. It has been investigated from a bootstrap perspective in [Reeves et al., 2021], where it was suggested that such boundary states must be extremely fine-tuned. In [Belin et al., 2022], the full classification of boundary states in large N symmetric orbifolds was carried out, and typical boundary states are not of this form.

Computation of the return probability and correlators

First we define unit-normalized boundary states

$$|\widehat{B}_a(0)\rangle = \frac{e^{-\frac{\beta H}{4}}\,|B_a\rangle}{\sqrt{\langle B_a|\,e^{-\frac{\beta H}{2}}\,|B_a\rangle}}\,. \tag{2.166}$$

Then we want to show that return probability of a boundary state

$$R(T) = |\langle\widehat{B}_a(0)\,|\widehat{B}_a(T)\rangle\,|^2\,, \tag{2.167}$$

decays exponentially fast at early time. For boundary states in holographic 2d CFTs we have (3.31)

$$G(\beta) = \langle B_a|e^{-\frac{\beta H}{2}}|B_a\rangle \simeq e^{\frac{\pi^2 c}{6\beta}}\,. \tag{2.168}$$

where we have taken the CFT to be defined on a spatial circle of length 2π. For small T we have

$$R(T) = \frac{|G(\beta+2iT)|^2}{|G(\beta)|^2} \simeq e^{-\frac{4\pi^2 c}{3\beta^3}T^2}\,. \tag{2.169}$$

The energy variance of the boundary state can be easily computed from (2.168) and we find

$$\Delta H^2 = \langle H^2\rangle - \langle H\rangle^2 = \frac{4\pi^2 c}{3\beta^3}\,, \tag{2.170}$$

so the initial decay (2.169) is, not surprisingly, consistent with (2.42), (2.44) and (2.170).

In higher dimensional cases we can read from (2.241)

$$G(\beta) = e^{\frac{\alpha_d}{\beta^{d-1}}}, \tag{2.171}$$

thus

$$R(T) = \frac{|G(\beta + 2iT)|^2}{|G(\beta)|^2} \simeq \exp\left[-\frac{\alpha_d}{\beta^{d+1}} 4d(d-1)T^2\right]. \tag{2.172}$$

We can again check that

$$\Delta H^2 = \langle H^2 \rangle - \langle H \rangle^2 = \frac{\alpha_d}{\beta^{d+1}} 4d(d-1), \tag{2.173}$$

which is compatible with (2.172).

We now proceed with checking that the other states in the code subspace around a boundary state are orthogonal to the time evolved code subspace. Consider for example the state $\mathcal{O}(t,x)|\widehat{B}_a\rangle$. Following similar reasoning as in subsection 2.6.6 we can show that

$$|\langle \widehat{B}_a(0)|\mathcal{O}(t,x)|\widehat{B}_a(T)\rangle|^2 = \langle \widehat{B}_a(0)|\mathcal{O}(t,x)|\widehat{B}_a(T)\rangle \, \langle \mathcal{O}(t-\frac{T}{2}, x)\rangle_{\beta \to \beta + 2iT}. \tag{2.174}$$

where $\langle \widehat{B}_a(0)|\mathcal{O}(t,x)|\widehat{B}_a(T)\rangle = \frac{G_a(I,\beta+2iT)}{G_a(I,\beta)}$.

More generally

$$\begin{aligned} &\langle \widehat{B}_a(0)|\mathcal{O}(t_1,x_1)\mathcal{O}(t_2,x_2)...\mathcal{O}(t_n,x_n)|\widehat{B}_a(T)\rangle \\ &= \langle \widehat{B}_a(0)|\mathcal{O}(t,x)|\widehat{B}_a(T)\rangle\langle \mathcal{O}(t_1-\frac{T}{2},x_1)\mathcal{O}(t_2-\frac{T}{2},x_2)...\mathcal{O}(t_n-\frac{T}{2},x_n)\rangle_{\beta\to\beta+2iT}. \end{aligned} \tag{2.175}$$

Thus, as long as the analytical continuation of the correlation function in β does not introduce any surprising N-dependent factors we will get the expected behavior (2.48). We now check this condition for low-point functions in 2d boundary states.

Here we assume that for a holographic CFT, and if we are working in the large N limit, the 1-point function of light conformal primaries can be computed by a method of images. Then for a 1-point function of a scalar primary $\mathcal{O}$ with dimension Δ on a boundary state we have

$$\langle \widehat{B}_a(0)|\mathcal{O}(t,x)|\widehat{B}_a(0)\rangle = \frac{A_{\mathcal{O}}}{(\frac{\beta}{\pi}\cosh[\frac{2\pi}{\beta}t])^{\Delta}}. \tag{2.176}$$

for some constant $A_{\mathcal{O}}$ which depends on the boundary state a and the operator $\mathcal{O}$. After the analytic continuation necessary for (2.174) we find

$$\langle O(t-\frac{T}{2},x)\rangle_{\beta\to\beta+2iT} = \frac{A_{\mathcal{O}}}{(\frac{(\beta+2iT)}{\pi}\cosh[\frac{2\pi}{(\beta+2iT)}(t-\frac{T}{2})])^{\Delta}}. \tag{2.177}$$

Hence we notice that the results (2.174),(2.177) are consistent with our general expectations (2.47),(2.48).

We can also check 2-point functions, which we can compute in the large N limit. First we compute the 2-point function on the boundary state, using the method of images

$$\begin{aligned}&\langle \widehat{B}_a(0)|\mathcal{O}(t_1,x_1)\mathcal{O}(t_2,x_2)|\widehat{B}_a(0)\rangle = \\ &\sum_{n=-\infty}^{+\infty}\frac{1}{\left|\frac{\beta}{\pi}\sinh\left(\frac{\pi}{\beta}[(x_1-x_2+2\pi n)-(t_1-t_2)]\right)\right|^{2\Delta}} \pm \frac{1}{\left|\frac{\beta}{\pi}\cosh\left(\frac{\pi}{\beta}[(x_1-x_2+2\pi n)-(t_1+t_2)]\right)\right|^{2\Delta}},\end{aligned} \tag{2.178}$$

After the analytic continuation necessary for (2.175) we find from (2.178) that we do not notice any unexpected behavior of this part of the correlator as T increases, so the result (2.175) is dominated by the decay of the return probability, and is consistent with our expectations (2.47),(2.48).

2.7 Black Hole microstates

One question which is particularly interesting is whether we can apply our construction to black hole microstates. We have already mentioned in section 2.3.1 that there are various classes of black hole microstates, some of which have macroscopic time dependence and some of which do not. We will now discuss these various cases in more detail and interpret our operators for these types of states.

2.7.1 States with macroscopic time-dependence

We will start with the simplest situation: states with macroscopic time-dependence. This can be visible outside the horizon, for example black holes in the presence of infalling matter. Alternatively it can be that the geometry appears to be static outside the horizon but there is no corresponding Killing isometry in the interior. As the first case is more straightforward, we focus on the second case. Two examples of such states are boundary states of the CFT, corresponding to end-of-the-world branes inside the horizon, which have already been discussed in the previous section. A second example is states prepared by the Euclidean path integral on some surface of higher topology. The dual geometries have topology behind the horizon, and are often referred to as geons [Louko & Marolf, 1999, Guica & Ross, 2015, Marolf & Wien, 2018]. It is worth re-emphasizing that both of these states are usually prepared by the Euclidean path integral and are in fact very a-typical states, even if the CFT 1-point functions are very close to those in a thermal state (or said differently, even if the classical geometry is exactly that of a black hole outside the horizon).

Both of these examples involve pure states $|\Psi_0\rangle$ that have a large energy variance, of order N^2, such that the return probability will decay as (2.43). We can thus apply our construction to build local operators that are not dressed to the boundary CFT. The interpretation is that the operators are dressed with respect to the time-dependence of the interior. Consider for example the genus-2 geon in $d = 2$, which is prepared by the Euclidean path integral on half of a genus-2 surface [Maxfield et al., 2016, Marolf & Wien, 2018]. Microscopically, the state can be described by

$$|\Psi_0\rangle \sim \sum_{i,j} C_{iij} e^{-E_i \beta_i/2 - E_j \beta_j} |E_j\rangle \ , \tag{2.179}$$

where $\sim$ indicates that we have not been careful about the parametrization of the genus-2 surface, but $\beta_{i,j}$ are related to the moduli of the surface. The un-normalized overlap of this state corresponds to a genus-2 partition function in the dumbbell channel, where β_j parametrizes the length of the two handles, and β_i parametrizes the length of the neck between them.

It is not straightforward to write down a metric that covers the entire space-time of such states. Outside the horizon whose size is controlled by β_j, they look exactly like the BTZ geometry. Inside the horizon, they have macroscopic time-dependence. A nice coordinate patch that covers the Wheeler-de Witt patch of the $t = 0$ slice of the geometry can be written down in a very simple form

$$ds^2 = -dt^2 + \cos^2 t \, d\Sigma_2^2 \, , \tag{2.180}$$

where $d\Sigma_2^2$ is the constant negative curvature metric on half of a genus-2 surface. This coordinate patch covers the entire $t = 0$ slice of the geometry, which is precisely half of a genus-2 surface. The neck corresponds to the horizon, and there is topology (one handle) behind the horizon. From this metric, we explicitly see the time dependence of the geometry, even if a metric for the full spacetime is hard to write down. The interpretation of our operator is that the dressing is to the time-dependence of the geometry that sits inside the horizon. For end-of-the-world brane geometries, the situation is similar and the operator is dressed to the end-of-the-world brane.

2.7.2 Typical states

The question we would now like to ask is whether our prescription works in typical black hole microstates. Contrary to states with end-of-the-world branes or topology behind the horizon, it seems reasonable to expect that typical states should also look like the thermal state a finite distance inside the black hole (see for example [de Boer et al., 2019, De Boer et al., 2020]).

Whether or not our prescription works depends on the definition of a typical black hole microstate, and in particular on the energy spread we are choosing. One possibility is to define typical states using an ensemble of energy eigenstates with spread $\mathcal{O}(N^0)$ in energy (recall that there are still e^S with $S \sim O(N^2)$ states in this energy band). In that situation, our prescription does not work, as the variance of energy is $\mathcal{O}(N^0)$ and the return probability will not decay fast enough. Another possibility is to consider typical states with an energy spread similar to that of the canonical ensemble, that is

$$(\Delta E)^2 \sim \mathcal{O}(N^2)\,. \tag{2.181}$$

For such states, the return probability will decay following the behaviour (2.43). Therefore, we can follow our prescription and define the operators in the same way and they will satisfy the two properties of commuting with the Hamiltonian to all orders in $1/N$ and acting like HKLL operators to leading order at large N.

While these operators are certainly diff-invariant, since they are operators defined in the CFT, the bulk interpretation of their gravitational dressing on typical black hole microstates is not entirely clear. When the gravitational configurations are macroscopically time-dependent, our operators are dressed with respect to the features of the geometry. The typical states are still time-dependent, but only microscopically, as it seems plausible to assume that macroscopically they are featureless. In some sense our operators are dressed to the microscopic time-dependence of the state (the phases of the c_i in (2.28)), but it is unclear exactly what that means in the bulk.

Notice however, that if we start with a particular typical pure state $|\Psi_0\rangle$ and act with a unitary made out of the operator (2.57), associated to that state, then the predictions for what an infalling observer jumping into the black hole will see are unambiguous. For example, the operators (2.57) will generally create an excitation in the bulk and the location in time relative to that of the infalling observer who jumps from the boundary at a particular boundary time, can be unambiguously computed for each state $|\Psi_0\rangle$ and corresponding operators (2.57). We emphasize that for this interpretation it is important to remember that the operators (2.57) are state-dependent and cannot generally be promoted to a single operator which acts in a specific way globally on most typical states.

We briefly comment on black hole interior reconstruction. Suppose we start with a typical black hole microstate with energy spread of order (2.29). If we assume that the interior geometry contains part of the left asymptotic region, then the possibility of removing the dressing of the operators implies that we can deform the state behind the horizon by creating some particles there, in such a way that these excitations cannot be detected from the boundary CFT by the measurement of single-trace correlators, including the Hamiltonian, in the $1/N$ expansion. This was also discussed in [Harlow, 2014, de Boer et al., 2022]. We emphasize that this does not contradict the statements made in [?, de Boer et al., 2019, De Boer et al., 2020] that for typical states with microcanonical energy spread, it is impossible to add excitations without affecting single-trace correlators.

2.7.3 Two entangled CFTs

Similar considerations apply to geometries with two asymptotically AdS regions. Consider two non-interacting CFTs with total Hamiltonian $H = H_L + H_R$. We take the full system to be in a pure state $|\Psi_0\rangle$ which may be entangled, but we will assume the pattern of entanglement is generic. In particular, we *do not* consider states like the thermofield-double which have a very fine-tuned structure of entanglement. We can imagine the state $|\Psi_0\rangle$ to be, for example, $U_L\,|\mathrm{TFD}\rangle$, where U_L is a complicated random unitary acting on the left CFT. In this case we can consider the following generalization of our construction. Let us consider the 2-parameter family of time-shifted states

$$e^{-i(T_L H_L + T_R H_R)}|\Psi_0\rangle.$$

We start with an HKLL operator Φ dressed with respect the to left system, which commutes with H_R but not H_L. We now consider the following generalization of the operators (2.57)

$$\widehat{\Phi} = c \int dT_L dT_R e^{-i(T_L H_L + T_R H_R)} P_0 \Phi P_0 e^{i(T_L H_L + T_R H_R)} \tag{2.182}$$

using $P_0 = P_0^L \otimes P_0^R$ and $[\Phi, P_0^R] = 0$ then

$$\widehat{\Phi} = c \int dT_L e^{-iT_L H_L} P_0^L \Phi P_0^L e^{iT_L H_L} \otimes \int dT_R P_{T_R}^R \tag{2.183}$$

The resulting operator commutes with both H_L and H_R on the relevant code subspaces. In this case, the operator is not dressed with respect to the overall time-dependence of the full system, but rather to the time dependence of the "left" subsystem.

There are states with special entanglement pattern such as the TFD state, which was already discussed in section 2.6.2. The generalized return amplitude $\langle \Psi_0 | e^{-i(H_L T_L + H_r T_R)} | \Psi_0 \rangle$ which is a function of T_L and T_R does not decay in all directions for these special states. For example, in the TFD state it is constant along the line $T_L = -T_R$. In those cases we cannot set both commutators with H_L, H_R to zero. So we can move the dressing from one side to another if we wish to, but there it is always dressed to one of the boundaries. This happens because the TFD state has a symmetry, it is annihilated by $H_L - H_R$.

2.7.4 Island discussion

Our prescription is also useful to resolve some paradoxes in the context of black hole evaporation and islands. Consider a setup where a holographic CFT is coupled to a bath such that the bulk description is given by an evaporating black hole. After the Page time, a non-trivial quantum extremal surface appears in the bulk delimiting an island, i.e. a part of the interior of the black hole that is encoded in the bath degrees of freedom rather than in those of the CFT [Penington, 2020b, Almheiri et al., 2019].

There is an apparent tension in this context related to gravitational dressing [Geng et al., 2022]. If we create an excitation in the island by acting with a local operator φ_{island}, where does the gravitational dressing go? It appears that the only place for the dressing to go is the boundary CFT. But this implies that the local operator will have the property

$$[\varphi_{\text{island}}, H_{\text{CFT}}] \neq 0\,. \tag{2.184}$$

But this seems to be inconsistent, because since the operator is in the island, it should be reconstructable from the bath degrees of freedom, and commute with the CFT degrees of freedom.

Our operators provide a way out of this paradox. We can apply our prescription above in terms of two entangled systems with a generic pattern of entanglement (there is a subtlety here since the bath and CFT are actually coupled rather than non-interacting, but we can treat this interaction as weak). In that case, even if we did start with an operator that had a non-trivial commutator (2.184), we would engineer a new operator which commutes with H_{CFT} up to exponentially small corrections. This new operator is now dressed with respect to the radiation, rather than the boundary CFT.

The interpretation of the dressing is similar to that of the typical states. While it would be tempting to imagine dressing the operator to the quantum extremal surface, the bulk geometry only has extremely slow time-dependence so it is unclear if time-dependent features of the geometry are sharp enough to dress with respect to them. It appears that that the dressing is towards the microscopic time-dependence of the radiation. The story becomes less subtle if we consider a doubly holographic model (see for example [Almheiri et al., 2020, Chen et al., 2020c]). In that case, the dressing to the bath can be directly geometrized in the higher-dimensional geometry. Our operators can perhaps be thought as a counter-part of the operators in the doubly-holographic setup, but in cases where the dressing cannot be so easily geometrized.

Finally, we would like to clarify the distinction between reconstruction and dressing. To make things simple, let us consider the TFD state and consider an HKLL operator on the left ϕ_L. This operator is dressed to the left CFT. Now we run our protocol, and as explained above, we can move the dressing to the right. The operator $\hat{\phi}_L$ now commutes with H_L but no longer with H_R [Papadodimas & Raju, 2015]. This *does not* mean that it can be reconstructed from the right degrees of freedom, but that it can be detected from the right CFT via the Gauss law tail. It is still mostly built from the left CFT degrees of freedom, only its dressing has been pushed to the right.

2.8 Discussion

In this paper, we have investigated whether information can be localized in perturbative quantum gravity, in the context of the AdS/CFT correspondence. The challenge at hand is to construct local diff-invariant operators that are not dressed to the boundary where the CFT lives. We have presented evidence that such operators exist, at least around high energy states with a large energy variance. Such states include semi-classical geometries with features that break the symmetries of the dual CFT and for such states, local operators can be dressed to the features of the state. We have argued that there exist CFT operators that commute with all single-trace operators in a narrow time-band to all orders in the $1/N$ expansion, including the Hamiltonian and other charges that generate conformal transformations, while at the same time act like standard HKLL operators to leading order at large N.

We have presented an explicit construction of such operators, and checked that they commute with the Hamiltonian to all orders in the $1/N$ expansion, and act like HKLL operators to leading order. Technically the construction of such operators is made possible due to the fact that different semi-classical states have exponentially small overlap. We have also discussed a generalization of our operators that would commute with all boundary charges of the conformal group. Moreover, we presented a definition of operators that commutes with all single-trace operators, not just conserved charges. The construction of these operators is slightly less explicit, and we define them by specifying their action on the code subspace around a semi-classical geometry. We argue that such operators commute with all single-trace operators in a narrow CFT time-band, while also acting like HKLL operators to leading order at large N. Acting with such operators creates excitations that are completely invisible to CFT correlation functions in a narrow time-band, even if they become accessible at later times when a lightray from the location where the bulk excitation was created reaches the boundary. This suggests that information can be lozalized in perturbative quantum gravity, to all orders in G_N perturbation theory. We conclude with some open questions that we raised along the way.

2.8.1 The variance of the energy from semi-classical gravity

A quantity that played a primordial role throughout the paper is the variance of energy, which controls the early time decay of the return probability through (2.42). One question that would be interesting to understand better is how we can compute the variance $\langle\Psi_0|\Delta H^2|\Psi_0\rangle$ from semi-classical gravity. In appendix 2.9 we give an example that we can change the $O(N^2)$ coefficient of the variance of the Hamiltonian without changing the semi-classical geometry. This implies that the variance of the energy is not just a property of the geometry, but also of the quantum state of the fields on top of that geometry. Of course, if the metric changes as a function of time, this puts a bound on the variance through (2.31). This suggests that if we start with some time-dependent semi-classical geometry with a matter QFT state with large variance, it should not be possible to change the state in a way to make the variance decrease to $\mathcal{O}(1)$ without changing the metric towards a time-independent solution. The mechanism by which this would happen is unclear, and it would be interesting to pursue it further.

On a related note, we can ask how we can quantize the bulk mode associated to the Hamiltonian directly in gravity. The expectation value of the Hamiltonian is extracted through the fall-off of the metric near the AdS boundary, as is standard in AdS/CFT, but this does not capture its quantum 2-point function. If one computes the stress-tensor connected 2-point function on the geometry, takes the relevant components and performs the spatial integrals, one should obtain the variance. It would be desirable to have a more direct representation of the variance in terms of the the bulk wavefunction of the non-propagating s-wave mode of the graviton and also understand from this point of view the lower bound on the variance for time-dependent geometries.

2.8.2 Gravitational proof for the decay of the return probability

A central part of this paper was played by the decay of the return probability. The physical interpretation of this decay for a semi-classical time-dependent geometry is that it computes (the square) of an overlap between two distinct geometries, namely the original one and the time-evolved one. The general expectation is that the overlap of two distinct coherent states should be given by

$$\langle \lambda_1 | \lambda_2 \rangle \sim e^{-N^2 f(\lambda_1, \lambda_2)} , \qquad (2.185)$$

where f is some $\mathcal{O}(1)$ function whose real part is positive (we have assumed that the states $|\lambda_{1,2}\rangle$ are normalized). The intuition is that N^2 plays the role of $1/\hbar$ which controls the overlap of coherent states, and from a gravitational stand-point, the on-shell action of any geometry will be proportional to $1/G_N$. However, this gravitational argument does not necessarily imply that the real part of f is positive, which is required by reflection positivity of the CFT dual. As we have seen in (2.97), interpreting geometries as quantum states implies constraints on various on-shell actions.

It would be interesting to understand this problem directly in gravity. Can reflection positivity be proven directly at the level of the gravitational path integral? This requires proving (2.97) directly in gravity. A possible way to prove this is the following: we consider two states λ_1 and λ_2 with fixed sources, and their associated geometries contributing to the overlaps $\langle \lambda_{1,2} | \lambda_{1,2} \rangle$, with geometries g_1 and g_2 and on-shell actions I_1 and I_2. We start by considering a gravitational configuration which is half of g_1 (say the northern hemi-ball) and half of g_2 (the southern hemi-ball). This configuration has action

$$I_{\text{tot}} = \frac{I_1 + I_2}{2} \,. \tag{2.186}$$

Note that the geometry is off-shell at the gluing surface between g_1 and g_2, and there could be another contribution I_{junction} to the action coming from the gluing, which we will not include for now. To find the smooth saddle-point geometry, we need to let this geometry relax by modifying its configuration near the junction. One may be able to prove that this smoothing of the glued geometry comes with a definite sign in the action, therefore proving (2.97). It would be interesting to pursue this idea.

2.8.3 Microscopically time-dependent states

We have seen that for any state with large energy and large energy variance, we can find bulk local operators who commute with the time-band algebra. The interpretation of these operators is that they are dressed with respect to features of the state (in particular the time-depdence of the state), rather than to the boundary CFT. This intuitive picture is clear when the state describes a semi-classical geometry that is macroscopically time-dependent, as the time-dependence can be seen directly from the background metric which has features with respect to which we can attach a gravitational dressing.

As we have discussed, our prescription also works for typical states with energy variance of $\mathcal{O}(N^2)$. In that context, the interpretation of the dressing is less clear. The dual geometry is not macroscopically time-dependent. We can declare that the operator is dressed with respect to the microscopic time-dependence, but it is unclear what that means. It would be interesting to have a better physical understanding of the dressing for such type of states.

We hope to return to this question in the future.

It is also important to note that our operators are state-dependent, even outside the horizon. For a given typical state, we can use our construction to find the state-dressed local operator. However, if we now pick a different typical state then the operator will not act in the desired fashion. In this sense, our operators are similar to mirror operators [Papadodimas & Raju, 2013], but they can live outside the horizon. Nevertheless, we wish to emphasize again that independently of questions surrounding the interpretation of these operators, an important message of this paper is that these operators exist and that *states* created by acting on the corresponding typical state with unitaries built from these state-dressed operators have identical correlators of single-trace operators in a narrow time-band in the $1/N$ expansion as the original state. Moreover, this can be done around any typical state once the state has been fixed.

2.8.4 Microcanonical states and small energy variance

There are also typical states with a small energy variance, of $\mathcal{O}(N^0)$. For example, when one refers to the microcanonical ensemble, one often has in mind picking a state with spread in energy which is $\mathcal{O}(N^0)$. For such states, the return probability does not decay to values which are exponentially small in N^2 after an order one time, which means we cannot use our construction to define state-dressed operators. The variance of the energy is a very coarse way to define how time-dependent a state is, and for states with energy variance of size $\mathcal{O}(N^0)$, the state is not time-dependent enough to dress operators to it. Of course, all these states look macroscopically time independent, and all the information is in the microscopic phases of the state. It would be interesting to study this further, and have a better physical picture of whether one can find state-dressed operators to these small variance states.

It is worthing mentioning that if the variance is $\mathcal{O}(N^c)$ for any $0 < c < 2$, our prescription does work. For typical states, this is some kind of intermediate regime between canonical states and microcanonical states. For coherent states that are macroscopically time-dependent, this situation would occur if the profile of the fields are not $\mathcal{O}(1)$, but rather scale with some positive power of G_N. In that case, backreaction is small, but the return probability still decays. It would be interesting to understand these regimes better, they interpolate between coherent states of the bulk quantum fields propagating on a frozen AdS background, and semi-classical geometries with a non-trivial metric.

2.8.5 The AdS vacuum and low-energy states

For low-energy states like the AdS vacuum or states with an $\mathcal{O}(N^0)$ energy above it, our construction does not work. Therefore, the results of this paper do not contradict the claims of [Chowdhury et al., 2022], that for perturbative excitations on top of the AdS vacuum one can reconstruct the state directly from the time-band. Technically, this happens because the return probability does not decay to exponentially small values for such states. Physically, states like the AdS vacuum have no features to which we can dress operators, so the only possible diff-invariant way to specify a point is with relation to the boundary. Even classically, there are no diff-invariant local observables in classical general relativity for the case of vacuum AdS. It thus appears that the failure of constructing approximately local diff-invariant operators around the AdS vacuum happens because of the special nature of the state, rather than a fundamental obstruction due to the non-locality of quantum gravity.

For excited states on top of the AdS vacuum, it is less obvious why local diff-invariant states cannot be constructed. One may imagine that if the VEV of a scalar field has a quantum lump in some region of space-time, we could dress an operator to the location of this lump. Technically, we see that at least our operators cannot achieve this goal. It would be interesting to have a more physical understanding of why it is not possible to dress operators to quantum profiles, rather than semi-classical ones. As we have seen in the previous subsection, it is not completely related to backreaction. If we consider a coherent state on top of vacuum AdS corresponding to a source which scales as $N^{1/4}$, the return probability would decay fast enough for our construction to work, even if backreaction can be neglected. Note however that such a state is not really part of the low-energy EFT on top of vacuum AdS, since it has energy that scales with some fractional power of the Planck scale. It would be interesting to understand this better.

2.9 Appendix: Changing the variance of H

We would like to understand whether the variance of the energy is accessible within semi-classical gravity, simply from the geometry, or whether it requires more knowledge and in particular, the knowledge of the bulk quantum state for the fields propagating on the background. As we will see, knowledge of the quantum state seems to be required to extract the variance.

The quantity we would like to compute is

$$\langle\Psi_0|H^2|\Psi_0\rangle - \langle\Psi_0|H|\Psi_0\rangle^2 \equiv \langle\Psi_0|H^2|\Psi_0\rangle_c \; . \tag{2.187}$$

This is a connected correlation function in holography, which usually would be compute from the 2-point function of the associated propagating fields on the relevant background. This 2-point function is sensitive both to the geometry and to the bulk quantum state of the propagating fields. However, here the situation is more subtle, because we are not studying the local correlation function of an operator, but rather the 2-point function of the spatial integral of a local operator. In this particular case, the situation is a lot more confusing because the dual bulk field would be the s-wave graviton, which is not a propagating degree of freedom in gravity.

So what computes this variance? We will not be able to answer this question, and we believe it to be an interesting open problem which we hope to return to in the future. Nevertheless, we will study some particular states that should be interpreted as adding an s-wave graviton in the bulk. Even though this mode doesn't propagate, we will see that adding it can affect the CFT variance. We will consider two type of deformations of the thermofield double (TFD) state, both of which are related to adding an integrated stress-tensor operator on the cylinder that prepares the TFD state. Let us start with some basics. We consider the TFD state

$$|\mathrm{TFD}\rangle = \frac{1}{\sqrt{Z}} \sum_i e^{-\beta E_i/2} |E_i\rangle |E_i\rangle \ . \tag{2.188}$$

We assume that the partition function has the usual large N behavior

$$Z(\beta) = \exp\left[N^2 \left(F_0(\beta) + \frac{1}{N^2} F_1(\beta) + \ldots\right)\right] \ , \tag{2.189}$$

from which we can compute

$$\langle H^n \rangle_\beta = (-1)^n \frac{1}{Z}\frac{d^n}{d\beta^n} Z\,. \tag{2.190}$$

where H is H_L or H_R. We have

$$\begin{aligned} \langle \mathrm{TFD}|\, H \,|\mathrm{TFD}\rangle &= \langle H\rangle_\beta = -N^2 F_0' - F_1'\,, & (2.191)\\ \langle \mathrm{TFD}|\, H^2 \,|\mathrm{TFD}\rangle - \langle \mathrm{TFD}|\, H \,|\mathrm{TFD}\rangle^2 &= \langle H^2\rangle_{\beta,c} \equiv \langle H^2\rangle_\beta - \langle H\rangle_\beta^2\,. & (2.192)\end{aligned}$$

We have

$$\langle H^2\rangle_{\beta,c} = N^2 F_0'' + F_1''\,. \tag{2.193}$$

Now, consider the following state

$$|\psi\rangle = H\,|\mathrm{TFD}\rangle\,. \tag{2.194}$$

We now have

$$\langle \psi|\psi\rangle = \langle H^2\rangle_\beta \tag{2.195}$$

Let us now see how the energy and variance of the state have evolved. We have

$$\frac{\langle \psi|\, H \,|\psi\rangle}{\langle \psi|\psi\rangle} = \frac{\langle \mathrm{TFD}|\, H^3 \,|\mathrm{TFD}\rangle}{\langle \mathrm{TFD}|\, H^2 \,|\mathrm{TFD}\rangle} = \frac{\langle H\rangle_\beta^3 + 3\,\langle H^2\rangle_{\beta,c}\,\langle H\rangle_\beta + \langle H^3\rangle_{\beta,c}}{\langle H\rangle_\beta^2 + \langle H^2\rangle_{\beta,c}}\,, \tag{2.196}$$

where we defined

$$\langle H^3\rangle_{\beta,c} \equiv \langle H^3\rangle_\beta - 3\,\langle H^2\rangle_{\beta,c}\,\langle H\rangle_\beta - \langle H\rangle_\beta^3\,. \tag{2.197}$$

Large N factorization implies that we can expand this answer and we find

$$\begin{aligned}
\frac{\langle\psi|\,H\,|\psi\rangle}{\langle\psi|\psi\rangle} &= \langle H\rangle_\beta + 2\frac{\langle H^2\rangle_{\beta,c}}{\langle H\rangle_\beta} + \cdots \\
&= -N^2F_0' - F_1' - 2\frac{F_0''}{F_0'} + \cdots \,. \qquad (2.198)
\end{aligned}$$

We see that we obtain the TFD answer, up to a correction term, which is of size N^0. This means we have not changed the geometry classically, but only added a quantum particle on top of the TFD state. Similarly, one can compute

$$\begin{aligned}
\frac{\langle\psi|\,H^2\,|\psi\rangle}{\langle\psi|\psi\rangle} - \left(\frac{\langle\psi|\,H\,|\psi\rangle}{\langle\psi|\psi\rangle}\right)^2 &= \frac{\langle H\rangle_\beta^4 + 6\,\langle H^2\rangle_{\beta,c}\,\langle H\rangle_\beta^2 + \cdots}{\langle H\rangle_\beta^2 + \langle H^2\rangle_{\beta,c}} - \left(\langle H\rangle_\beta^2 + 4\,\langle H^2\rangle_{\beta,c} + \cdots\right) \\
&= \langle H^2\rangle_{\beta,c} + \cdots \\
&= N^2F_0'' + \cdots \qquad (2.199)
\end{aligned}$$

We see that that the energy has changed at N^0, but the variance has not changed at order N^2, only at order N^0. So this state modifies both the variance and the energy at subleading order compared to the TFD. We will now build a state that modifies the energy at subleading order, but the variance at leading order compared to the TFD.

Consider the state

$$|\varphi\rangle = (H - \langle H\rangle_\beta)\,|\mathrm{TFD}\rangle \,. \qquad (2.200)$$

We now have

$$\langle\varphi|\varphi\rangle = \langle H^2\rangle_{\beta,c} \,, \qquad (2.201)$$

and we can now compute the energy in this state:

$$\frac{\langle\varphi|\,H\,|\varphi\rangle}{\langle\varphi|\varphi\rangle}=\frac{\langle H^3\rangle_\beta-2\,\langle H^2\rangle_\beta\,\langle H\rangle_\beta+\langle H\rangle_\beta^3}{\langle H^2\rangle_{\beta,c}}=\langle H\rangle_\beta+\frac{\langle H^3\rangle_{\beta,c}}{\langle H^2\rangle_{\beta,c}}=-N^2F_0-F_1-2\frac{F_0'''}{F_0'}+\cdots\,. \tag{2.202}$$

We see that this state modifies again the energy only at order N^0, and in a slightly different way than the previous state. In a similar way, we compute the variance and find

$$\begin{aligned}\frac{\langle\varphi|\,H^2\,|\varphi\rangle}{\langle\varphi|\varphi\rangle}-\left(\frac{\langle\varphi|\,H\,|\varphi\rangle}{\langle\varphi|\varphi\rangle}\right)^2 &= \langle H\rangle_\beta^2+3\,\langle H\rangle_{\beta,c}^2+\frac{2\,\langle H^3\rangle_{\beta,c}\,\langle H\rangle_\beta+\langle H^4\rangle_{\beta,c}}{\langle H^2\rangle_{\beta,c}}-\left(\langle H\rangle_\beta+\frac{\langle H^3\rangle_{\beta,c}}{\langle H^2\rangle_{\beta,c}}\right)^2\\ &= 3\,\langle H^2\rangle_{\beta,c}+\frac{\langle H^4\rangle_{\beta,c}}{\langle H^2\rangle_{\beta,c}}-\left(\frac{\langle H^3\rangle_{\beta,c}}{\langle H^2\rangle_{\beta,c}}\right)^2\\ &= 3N^2F_0''+\frac{3(F_0')^2F_1'-(F_0''')^2+F_0'F_0''''}{(F_0')^2}+\ldots\end{aligned} \tag{2.203}$$

One can see that the change in the variance is order N^2 (it is three times the variance of the TFD state), so this is a modification of the variance at the order we were looking for.

From this, we can conclude that the semi-classical geometry is not enough to extract the variance of the energy. The quantum state of the bulk fields is equally important. For the state $|\varphi\rangle$, we have the same leading large N properties, but a different quantum state for the graviton. The fact that it is the s-wave of the graviton that enters is still puzzling, and it would be interesting how to propertly quantize this non-propagating degree of freedom. We leave this for the future.

2.10 Appendix: Boosts in global AdS

As we have discussed in section 2.3, the conformal generators on the d-dimensional cylinder $\mathbb{R} \times S^{d-1}$ organize themselves as time-translations, rotations, and $2d$ remaining generators which correspond to boosts in the dual AdS geometry. The goal of this section is to discuss whether there exist states that can preserve the boost symmetry. As we have seen throughout the paper, symmetries that are broken by semi-classical states allow us to specify bulk points by dressing the location of a bulk point to the feature of the state that breaks the symmetry. It is important to understand which symmetries are broken, and which symmetries can be preserved by semi-classical states. For time translations and rotations, this is straightforward, but it is somewhat more subtle for boosts, which is the purpose of this section.

The $2d$ boost generators can be realized as d non-independent copies of $SL(2,\mathbb{R})$ [Freivogel et al., 2012]. For simplicity, we will study the case of AdS_3, but the higher dimensional versions follow in a straight forward manner. In $d = 2$, the two copies of $SL(2,\mathbb{R})$ are well-known and correspond to the left and right moving sectors of conformal transformation. The generators are given by L_{-1}, L_0, L_1 and $\bar{L}_{-1}, \bar{L}_0, \bar{L}_1$. Time-translations and rotations are obtained by the combinations

$$H = L_0 + \bar{L}_0 \,, \qquad J = L_0 - \bar{L}_0 \,. \tag{2.204}$$

The four residual generators correspond to boosts in AdS_3. For explicit expressions, see [Maldacena & Strominger, 1998]. We would now like to analyze whether non-trivial states can be annihilated by these boosts. As a starting point, notice that there are obviously CFT states which are annilitated by L_{-1} and $\bar{L}_{-1}$: primary states. However, we would like to consider generators that can be exponentiated to norm-preserving group elements. This means the generators should be Hermitian. The generators L_{-1} and $\bar{L}_{-1}$ do not satisfy this property. However, we can assemble them into the combinations

$$L_+ = L_{-1} + L_1 \qquad , \qquad L_- = i(L_{-1} - L_1) \tag{2.205}$$

Using that $L_{-1}^\dagger = L_1$, we see that $L_\pm$ are hermitian operators and can thus be exponentiated to form unitaries.

The question we would like to ask is whether there are states in the Hilbert space that are eigenstates of $L_\pm$. We will see that the only finite energy eigenstates of these operators are those where the left-moving part of the CFT is in the vacuum. To see this, we consider the commutator

$$[L_+, L_-] = 4iL_0 \tag{2.206}$$

Suppose now that $|\psi\rangle$ is a normalizable eigenstate of —say— L_+. Computing the expectation value of this equation we find

$$\langle\psi|L_0|\psi\rangle = 0 \tag{2.207}$$

From the positivity of the energy spectrum this is possible only if $L_0|\psi\rangle = 0$. The only states with this property are states where the left moving sector of the CFT is in the vacuum.

Non-trivial states will thus break boost invariance, which can be use to specify the radial location of an operator. For the construction of operators presented in this paper, this would require considering the states obtained by acting with the unitary operators on semi-classical states $|\psi_0\rangle$ as

$$e^{-i\gamma L_{\pm}} |\psi_0\rangle \ , \tag{2.208}$$

and studying the generalized return probability

$$R(\gamma) \equiv \left| \langle \psi_0 | \, e^{-i\gamma L_{\pm}} \, |\psi_0\rangle \right|^2 \ . \tag{2.209}$$

These return probabilities have not been studied but for semi-classical states, it is natural to expect them to be exponentially small for $\gamma \sim \mathcal{O}(1)$.

2.11 Appendix: Early time decay of the return probability

We wish to estimate the early time decay of the return probability (2.41). We will see that at very early times, namely $t \sim \frac{1}{N}$, we can find the decay purely from large N factorization. We will first recall a general property of coherent state overlaps which follows from large N factorization, and then adapt the situation slightly to the return probability.

2.11.1 Overlap of coherent states and large N factorization

Coherent states of quantum gravity in AdS/CFT can be described by states prepared by a Euclidean path integral with sources turned on for single-trace operators. These states are thus given by

$$|\lambda\rangle = e^{\int_{x_0<0} dx^d \lambda(x)\mathcal{O}(x)} \, |0\rangle \ , \tag{2.210}$$

where we have not written the appropriate time-ordering which is left implicit. We will now show that the overlap is given by

$$\langle\lambda_1|\lambda_2\rangle = e^{\int_{\mathbb{R}^d} \lambda_1^*(y)\lambda_2(x)\langle\mathcal{O}(y)\mathcal{O}(x)\rangle} + \mathcal{O}(1/N)\,, \tag{2.211}$$

where it should be understood that y is integrated over the upper half plane while x is integrated over the lower half plane.

We can explicitly expand out the integrals of the bra and the ket states, and use large N factorization: this implies that the operators should be paired up and contracted using Wick's theorem, up to $1/N$ corrections. At a given power in the source, we will have a term of the form

$$\left(\int dxdy\right)^k \frac{1}{(k!)^2}\lambda_1^*(y)^k\lambda_2(x)^k \langle 0|\,\mathcal{O}^k(y)\mathcal{O}^k(x)\,|0\rangle\ . \tag{2.212}$$

We can now apply Wick's theorem and find

$$\left(\int dxdy\right)^k \frac{1}{(k!)^2}\lambda_1^*(y)^k\lambda_2(x)^k \langle 0|\,\mathcal{O}^k(y)\mathcal{O}^k(x)\,|0\rangle = \frac{1}{k!}\left(\int dxdy\lambda_1^*(y)\lambda_2(x)\,\langle 0|\,\mathcal{O}(y)\mathcal{O}(x)\,|0\rangle)\right)^k\,,$$

which we can re-exponentiate to find (2.211). Note that we have not written the normalization of the states, which takes care of the Wick contraction between any two operators living both in the lower half plane, or upper half plane. Similarly, terms which have a different powers of upper and lower operators do not give contributions to leading order at large N because we cannot pair the operators and use Wick's theorem.

For this to work, we have implicitly assumed that $\lambda \sim \mathcal{O}(N^0)$. To see this, note that the connected correlation functions of higher-point operators are suppressed by $1/N$, but also have more sources than lower-point functions. If we scale the sources as $\lambda \sim N^{1/2}$, which is the correct scaling to induce $\mathcal{O}(1)$ back-reaction on the dual spacetime[41], we have to be more careful, as some of the terms we dropped involving connected correlators will be the same size as the Wick contractions. For example, we have

$$\lambda_1^*(y)\lambda_2(x)\,\langle\mathcal{O}(y)\mathcal{O}(x)\rangle \quad\sim\quad N^2 \tag{2.213}$$

$$(\lambda_1^*(y)\lambda_2(x))^2\,\langle\mathcal{O}(y)\mathcal{O}(y)\mathcal{O}(x)\mathcal{O}(x)\rangle_c \quad\sim\quad N^2\,. \tag{2.214}$$

This means that we cannot truncate to the sector of Wick contraction, and we must resum the entire expansion. Note however that the contributions corresponding to loop diagrams in AdS are still suppressed by $1/N$, so we are resumming tree-level diagrams to build the backreacted geometry.

The upshot of this analysis is that we can use large-N factorization to easily compute the overlap of coherent states, but only if the sources are $\mathcal{O}(1)$, in which case the exponent in the exponential is also $\mathcal{O}(1)$. If we try to make the sources scale with N, the exponent will be of order N^2 and then infinitely many contributions must be resummed. We will now apply this logic to the return probability.

2.11.2 The return probability

We can now apply the same logic as above, taking the operator e^{-iHT} to be seen as an imaginary Euclidean source for the Hamiltonian (which is the integral of the stress-tensor). We want to compute

$$R(T) = \langle\Psi_0|e^{-iHT}|\Psi_0\rangle\langle\Psi_0|e^{iHT}|\Psi_0\rangle\,. \tag{2.215}$$

[41] For operators that have unit 2-point function.

Applying the logic above, we would find that to leading order we have

$$R(T) = e^{-iT\langle\Psi_0|H_0|\Psi_0\rangle} e^{iT\langle\Psi_0|H_0|\Psi_0\rangle} = 1 + \mathcal{O}(1/N)\,. \tag{2.216}$$

So we see that the candidate leading term vanishes, and we must go to the next order. This is due to the nature of the return probability, which is a square of overlaps. A quick expansion of the exponentials shows that at order T^2, we have

$$T^2\Big(-\langle\Psi_0|H^2|\Psi_0\rangle + \big(\langle\Psi_0|H|\Psi_0\rangle\big)^2\Big) = -T^2\Delta H^2\,. \tag{2.217}$$

For reasons similar to those explained above, this term can be exponentiated such that we find

$$R(T) = e^{-T^2\Delta H^2} + \mathcal{O}(1/N)\,. \tag{2.218}$$

As in the previous section, we can only trust this approximation if the exponent is $\mathcal{O}(1)$. Because we are considering states that have $\Delta H \sim N^2$, we see that we can trust this exponential decay of the return probability for time-scales up to $t \sim 1/N$.

For larger time-scales, it may still hold, but it cannot be justified based solely on large N factorization. It is instructive to consider the case of the thermofield double state and the spectral form factor, as we already discussed in section 2.6.2. For simplicity, we set $d = 2$ where we have

$$Z(\beta) = e^{\frac{c}{12}\frac{4\pi^2}{\beta}}\,. \tag{2.219}$$

The spectral form factor then gives

$$R(T) = e^{\frac{\pi^2 c}{3}\left(\frac{1}{\beta+IT}+\frac{1}{\beta-iT}\right)} = e^{\frac{2\pi^2 c}{3}\frac{\beta}{\beta^2+T^2}} \, . \tag{2.220}$$

We can expand this expression in T, as long as $T \ll \beta$, to find

$$R(T) \approx Z(\beta)^2 e^{-\frac{2\pi^2 c}{3}\frac{T^2}{\beta^3}} \, . \tag{2.221}$$

We find the exponential decay that goes like T^2. What is important is that even though T must be much smaller than β, it is allowed to scale as N^0. This cannot be justified solely from large N factorization, but still holds in this particular context. We expect the return probability to satisfy this property for holographic states more generally.

2.12 Appendix: LLM solutions in the bulk

The LLM geometries correspond to solutions of type IIB supergravity with symmetry $SO(4) \times SO(4) \times R$. We assume the axion and dilaton are constant and the IIB three forms are vanishing. We introduce coordinates $x^\mu = (t, y, x_1, x_2)$ and Ω_3, $\tilde{\Omega}_3$ for two 3-spheres corresponding to the $SO(4)$ isometries. We parametrize the five form as

$$F_5 = F_{\mu\nu} dx^\mu \wedge dx^\nu \wedge d\Omega_3 + \tilde{F}_{\mu\nu} dx^\mu \wedge dx^\nu \wedge d\tilde{\Omega}_3 \, , \tag{2.222}$$

where the self duality of the five form implies that the two forms F and $\widetilde{F}$ are dual to each other.

After demanding that the geometry preserves the Killing spinor in the presence of the five form, we arrive at the following solution for the $\frac{1}{2}$-BPS bulk states [Lin et al., 2004]

$$ds^2 = -\frac{(dt+V_i dx^i)^2}{h^2} + h^2(dy^2 + dx^i dx^i) + y e^G d\Omega_3^2 + \frac{y}{e^G} d\tilde{\Omega}_3^2\,, \tag{2.223}$$

where every function in the metric is expressed in terms of a function $z(x_1, x_2, y)$ and we defined $z = \frac{1}{2}\tanh\, G$, $h^{-2} = 2y\, \cosh\, G$, and

$$y\partial_y V_i = \varepsilon_{ij}\partial_j z, \qquad\qquad y(\partial_i V_j - \partial_j V_i) = \varepsilon_{ij}\partial_y z\,. \tag{2.224}$$

For the forms $F, \tilde{F}$ we have

$$F = dB_t \wedge (dt+V) + B_t dV + d\hat{B}, \qquad \tilde{F} = d\tilde{B}_t \wedge (dt+V) + \tilde{B}_t dV + d\hat{\tilde{B}}\,, \tag{2.225}$$

where $B_t = -\frac{1}{4}y^2 e^{2G}$ and $\tilde{B}_t = -\frac{1}{4}y^2 e^{-2G}$. On the other hand,

$$d\hat{B} = -\frac{1}{4}y^3 \star_3 d(\frac{z+2}{y^2}), \qquad\qquad d\hat{\tilde{B}} = -\frac{1}{4}y^3 \star_3 d(\frac{z-2}{y^2})\,, \tag{2.226}$$

where $\star_3$ is the epislon symbol in the flat three dimensions.

The only free function, z, is constrained to solve the equation,

$$\partial_i\partial_j z + y\partial_y(\frac{\partial_y z}{y}) = 0\,. \tag{2.227}$$

We focus our attention on the plane $y = 0$. Since the product of the radii of the two 3-spheres is y, there will be a conical singularity at $y = 0$ unless the function z has a special behaviour.

Let's consider the case where R_1 is kept finite, i.e, $e^{-G} \to 0$ as $y \to 0$. Thus, one has, $z \sim 1/2 - e^{-2G} + \dots$. If one assumes that $z = 1/2$ at $y = 0$, then one gets the expansion, $z \sim 1/2 - y^2 f(x_1, x_2) + \dots$ for some positive function f, with our boundary conditions. Thus, $e^{-G} \sim yc(x_1, x_2) + \dots$ and $h^2 \sim c(x_1, x_2) + \dots$. Therefore, close to $y = 0$, the part of the metric involving R_2 will look like,

$$h^2 dy^2 + R_2 d\tilde{\Omega}_3^2 \approx c(dy^2 + y^2 d\tilde{\Omega}_3^2) \,. \tag{2.228}$$

Thus the conical singularity is resolved. In the case where R_2 is kept fixed, the same argument goes through but now with the condition that $z = -1/2$ at $y = 0$.

With these boundary values of z at $y = 0$ as a source, one can solve the Laplace equation[42] (2.227) and compute $z(x_1, x_2, y)$. In addition, V_i can also be expressed in terms of an integral of $z(x_1, x_2, 0)$ over the two dimensional space.

2.13 Appendix: Notes on boundary states

Some useful references for this section are [Cardy, 2004, Miyaji et al., 2015, Guo, 2018, Miyaji et al., 2021].

2.13.1 Boundary states in 2D CFT

Boundary states in a 2d CFT need to satisfy [Cardy, 2004]

$$(L_n - \tilde{L}_n) |B\rangle = 0. \tag{2.229}$$

[42] More precisely, it is a Laplace equation for z/y^2.

In any Verma module, one can find a simple solution to these conditions as

$$|I_h\rangle = \sum_{\vec{k}} |\vec{k}, h\rangle_L \otimes |\vec{k}, h\rangle_R \,, \tag{2.230}$$

where $|\vec{k}, h\rangle_L$ is a linear combination of Virasoro descendants of the primary state $|h\rangle$ characterized by an infinite dimensional vector $\vec{k} = (k_1, k_2, ...)$ with non-negative integer components. We identify these states by starting with descendants of the form

$$...L_{-n}^{K_n}...L_{-1}^{K_1}\,|h\rangle_L \,. \tag{2.231}$$

and forming an orthonormal basis selected such that ${}_L\langle\vec{k}, h|\vec{k}', h\rangle_L = \delta_{\vec{k},\vec{k}'}$.

The state $|I_h\rangle$ is called the Ishibashi state for the primary state $|h\rangle_L$, where the states $|\vec{k}, h\rangle$ are the descendant on top of the primary labeled by h. It can be seen easily that

$$L_n|I_h\rangle = \tilde{L}_n|I_h\rangle \,. \tag{2.232}$$

It is clear that the Ishibashi states have maximal entanglement between the left-moving and right-moving sectors. Linear combinations of the Ishibashi states satisfy the constraint (3.38) as well.

Physical boundary sates are given by special linear combinations of Ishibashi states which are called Cardy states

$$|B_a\rangle = \sum_h C_{a,h}\,|I_h\rangle \ . \tag{2.233}$$

Physical boundary states should satisfy a consistency condition of the partition function on a finite cylinder related to open-closed duality [Cardy, 2004].

The Cardy states are singular because the norm of the Ishibashi states is divergent. One can define regularized boundary states by evolving in Euclidean time as

$$|B_{a,\beta}\rangle = e^{-\frac{\beta}{4}H_c}\,|B_a\rangle\,, \tag{2.234}$$

where β is a positive constant and $H_c = L_0 + \tilde{L}_0 - \frac{c}{12}$. Since $[L_0 - \tilde{L}_0, H_c] = 0$, the state (2.234) is still space-translational invariant on the circle, but it is time-dependent.

Ishibashi states are orthogonal to each other. The amplitude of Euclidean time evolution by $\beta/2$ between two such states is computed as

$$\langle I_k|e^{-\beta H_c/2}|I_l\rangle = \delta_{kl}\chi_k(e^{-\beta/2})\,. \tag{2.235}$$

χ_k is the character for the primary k. On the other hand, the Cardy states are not orthogonal to each other but satisfy the open-closed duality relation as follows

$$\langle B_a|e^{-\frac{\beta}{2}H_c}|B_b\rangle = \sum_k N^{(k)}_{a,b}\mathrm{Tr}_k[e^{-\frac{4\pi^2}{\beta}H_o}] \tag{2.236}$$

where $H_o = L_o - \frac{c}{24}$ denotes the Hamiltonian in the dual channel, characterized by the boundary conditions a, b. On the right hand side, $\mathrm{Tr}_k[\ldots]$ denotes a trace in the sector associated to a primary k as well as its descendants. Moreover, $N^{(k)}_{a,b}$ counts the degeneracy of sectors which belong to the primary k with boundary conditions a and b.

In the high temperature limit $\beta \to 0$, we find that

$$\langle B_a|e^{-\frac{\beta}{2}H_c}|B_b\rangle \simeq N^{(k_m)}_{a,b} e^{-\frac{4\pi^2}{\beta}(h^{(min)}_{a,b}-\frac{c}{24})}\,, \tag{2.237}$$

where k_m is the lightest primary among those satisfy $N^{(k_m)}_{a,b} \neq 0$, whose conformal dimension is denoted as $h^{(min)}_{a,b}$.

We can estimate the inner products between two normalized boundary states in this limit as

$$\langle\psi_a|e^{-\frac{\beta}{2}H_c}|\psi_b\rangle = \frac{\langle B_a|e^{-\frac{\beta}{2}H_c}|B_b\rangle}{\sqrt{\langle B_a|e^{-\frac{\beta}{2}H_c}|B_a\rangle\langle B_b|e^{-\frac{\beta}{2}H_c}|B_b\rangle}} \simeq \delta_{a,b} + N^{(k_m)}_{a,b} e^{-\frac{4\pi^2}{\beta}h^{(min)}_{a,b}}\,. \tag{2.238}$$

Note that $N^{(0)}_{a,a} = 1$. In this way, a large gap in the open string channel leads to a large exponential suppression of off-diagonal elements of inner products.

In holographic BCFT, the inner product between two boundary states can be computed by evaluating the gravity action on the dual background. When we consider the gravity dual of a cylinder, there are two candidates of classical gravity solutions depending on whether the end of the word brane is connected or disconnected which are called connected and disconnected solutions. When we consider the overlap for an identical boundary condition a, then both the connected and disconnected solution are allowed. In the limit $\beta \to 0$, the connected solution is favored and one can find that

$$\langle B_a|e^{-\frac{\beta}{2}H_c}|B_a\rangle \simeq e^{\frac{\pi^2 c}{6\beta}}\,. \tag{2.239}$$

We will use it later to calculate the return probability for boundary states. In addition to it, one can find the inner product between two boundary states with different boundary conditions. In this case, only the disconnected solutions are allowed and

$$\langle B_a | e^{-\frac{\beta}{2}H_c} | B_b \rangle \simeq e^{\frac{c\beta}{12} + S^{(a)}_{bdy} + S^{(b)}_{bdy}}, \tag{2.240}$$

where $S^{(i)}_{bdy}$, $i = a, b$ are the boundary entropies [Miyaji et al., 2021].

2.13.2 Boundary states in higher dimensions

One can generalize to higher dimensions and define a boundary state $|B_a\rangle$ as a state associated to a $(d-1)$-dimensional boundary in d-dimensional CFT [Fujita et al., 2011, Miyaji et al., 2021]. Taking the boundary to be a torus $\mathbb{T}^{d-1}$, the inner product between two boundary states in a holographic BCFT can be computed as a partition function on a d-dimensional open manifold $I_{\beta/2} \times \mathbb{T}^{d-1}$ where $I_{\beta/2}$ is a length $\beta/2$ interval. As in the 2d case, there are two bulk solutions, a connected and a disconnected one. In the $\beta \to 0$ limit the connected solution is dominant and one can find the inner product between two identical boundary states using the gravity solution as

$$\langle B_a | e^{-\frac{\beta}{2}H_c} | B_a \rangle_{con} \simeq e^{\alpha_d/\beta^{d-1}}, \tag{2.241}$$

where

$$\alpha_d = (4\zeta(T))^d \frac{R^{d-1}}{16 G_N} L^{d-1}, \tag{2.242}$$

where R is the AdS radius, L is the length of the compactified spatial directions and $\zeta(T)$ is a function of tension which is defined when $T < 0$ as

$$\zeta(T) \equiv \frac{\Gamma(1/d)\Gamma(1/2)}{\Gamma(1/d+1/2)} \frac{R|T|}{d(d-1)} (1 - \frac{R^2T^2}{(d-1)^2})^{1/d-1/2} F(1, 1/d, 1/2+1/d; 1 - \frac{R^2T^2}{(d-1)^2}) , \tag{2.243}$$

and when $T > 0, \zeta(T) = \frac{2\pi}{d} - \zeta(-T)$. The tension takes values in the range $|T| < \frac{d-1}{R}$. For $d > 2, \zeta(T)$ non-trivially depends on T and there is an upper bound of the tension $T < T_*$ which $T_* > 0$ and $\zeta(T_*) = 0$ [Miyaji et al., 2021].

2.13.3 Correlation functions in BCFTs

Let us first start with the simplest case where the CFT is defined on the upper half plane and the boundary state $|B\rangle$ is placed along the real axis. We consider the 1-point function of a local operator placed at z in the upper half plane. In the case of a CFT on the plane, the 1-point function of a primary operator in the vacuum is required to vanish by the symmetries. These are partly broken in a BCFT. The remaining symmetries constraint the 1-point function to have the form

$$\langle \mathcal{O}(z) \rangle_{\text{UHP}} = \frac{A_{\mathcal{O}}}{(2\,\text{Im}(z))^{\Delta}} , \tag{2.244}$$

where $A_{\mathcal{O}}$ is determined by the details of the theory and the precise boundary state in question. One could think of this as the boundary providing a source for the operator $\mathcal{O}$.

The 2-point function of a primary operator in a BCFT is more complicated than the case with no boundaries where it is exactly fixed by the symmetries. Non-trivial information about the operator content and OPE coefficients is necessary to compute the 2-point function exactly in a BCFT. We assume that for large N holographic CFTs the large N 2-

point function takes the form

$$\langle \mathcal{O}(z_1)\mathcal{O}(z_2)\rangle_{UHP} = \langle \mathcal{O}(z_1)\rangle_{UHP}\langle \mathcal{O}(z_2)\rangle_{UHP} + \langle \mathcal{O}(z_1)\mathcal{O}(z_2)\rangle \pm \langle \mathcal{O}(z_1)\mathcal{O}(z_2^*)\rangle \,, \tag{2.245}$$

where

$$\langle \mathcal{O}(z_1)\mathcal{O}(z_2)\rangle = \frac{1}{|z_1 - z_2|^{2\Delta}}\,, \tag{2.246}$$

where the contribution from an image insertion placed at z_2^*. The sign of the last term is governed by the boundary conditions, being either Dirichlet $(-)$ or Neumann $(+)$.

Mapping the z coordinate to a new coordinate w by

$$w \to z = \exp(2\pi w/\beta + i2\pi/4)\,, \tag{2.247}$$

we can map the upper half plane to the a strip of width $\beta/2$, where the positive (negative) real axis is mapped to the lower (upper) edge of the strip.

Since primary operators continue to transform in the usual way, the correlation functions now transform to

$$\begin{aligned} \langle \mathcal{O}(w)\rangle_{\text{strip}} &= \frac{A_{\mathcal{O}}}{(\frac{\beta}{\pi}\cos[\frac{2\pi}{\beta}\tau])^{\Delta}} \\ \langle \mathcal{O}(w_1)\mathcal{O}(w_2)\rangle^{\text{connected}}_{\text{strip}} &= \frac{1}{|\frac{\beta}{\pi}\sinh[\frac{\pi}{\beta}(w_1 - w_2)]|^{2\Delta}} \pm \frac{1}{|\frac{\beta}{\pi}\cosh[\frac{\pi}{\beta}(w_1 - \bar{w}_2)]|^{2\Delta}}\,, \end{aligned} \tag{2.248}$$

where the second line is only the connected piece of the large N 2-point function [Almheiri et al., 2018b]. Higher order correlation function can be found through large N factorization.

Correlation functions on a state defined on a circle by

$$|B_\beta\rangle = e^{-\beta H/4}\,|B\rangle\,, \tag{2.249}$$

can be thought of as correlation function on a cylinder of width $\beta/2$ where the boundary state is placed on both sides. We can instead consider a strip of width $\beta/2$, from $\tau = -\beta/4$ to $\tau = \beta/4$ with periodicity $x \sim x + R$. We choose $R = 2\pi$ for simplicity from now on. In large N holographic CFTs correlation functions on the cylinder can be found from the correlation function on the strip using the method of images

$$\langle O(w_1)O(w_2)\rangle^{\text{connected}}_{\text{cylinder}} = \sum_{n=0}^{\infty}\langle O(w_1+2\pi n)O(w_2)\rangle^{\text{connected}}_{\text{strip}}\,. \tag{2.250}$$

It from [qu]bit. Otherwise put, every it—every particle, every field of force, even the spacetime continuum itself—derives its function, its meaning, its very existence entirely—even if in some contexts indirectly—from the apparatus-elicited answers to yes or no questions, binary choices [or anything in between], [qu]bits.

updated quote of John Wheeler

3

Explicit reconstruction of the entanglement wedge via the Petz map

This chapter consists of the paper [Bahiru & Vardian, 2022] written in collaboration with Niloofar Vardian. The original abstract is as follows:

We revisit entanglement wedge reconstruction in AdS/CFT using the Petz recovery channel. In the case of a spherical region on the boundary, we show that the Petz map reproduces the AdS-Rindler HKLL reconstruction. Moreover, for a generic subregion of the boundary, we could obtain the same boundary representation of a local bulk field lies in the entanglement wedge as the one proposed earlier in [Jafferis et al., 2016b, Faulkner & Lewkowycz, 2017b] using properties of the modular flow.

3.1 Introduction

An important question in the AdS/CFT correspondence is that of *subregion duality*: is it possible to associate regions of the bulk to specific regions of the boundary CFT?

Given a spacelike region A in the CFT, one can associate two candidate dual regions in the bulk to it. One is the *causal wedge* of region A. This is constructed by first considering the boundary domain of dependence $\mathcal{D}(A)$ of A and then considering all bulk points which are both in the causal future and causal past of $\mathcal{D}(A)$. The other one is the *entanglement wedge* $\mathcal{E}(A)$ of the region A, defined as the bulk domain of dependence of a bulk spacelike surface whose boundary is the union of A and the Ryu-Takayanagi surface [Ryu & Takayanagi, 2006] associated to A [1]. The entanglement wedge contains the causal wedge, and in general, it is larger than the causal wedge [Headrick et al., 2014a, Wall, 2014]. [2]

[1] When the bulk geometry is time-dependent, one has to take what is called HRT surface [Hubeny et al., 2007], generalization of Ryu-Takayanagi surface. Later, considering quantum corrections to HRT surfaces led to the conjecture of quantum extremal surfaces [Engelhardt & Wall, 2015, Dong & Lewkowycz, 2018], which should be used at higher orders in perturbation theory.

[2] The simplest case that it can happen is when we consider two disconnected regions in the boundary.

It is believed that the entire entanglement wedge of a given boundary region A can be reconstructed from region A. This means that bulk operators acting inside the entanglement wedge can be expressed in terms of CFT operators in the region A. This idea of *entanglement wedge reconstruction* (EWR) has been introduced and developed in various works [Czech et al., 2012, Headrick et al., 2014a, Wall, 2014, Jafferis et al., 2016b, Dong et al., 2016b, Cotler et al., 2019b]. Important evidence in favor of EWR was given in [Jafferis et al., 2016b], where it was argued that the relative entropy of two states in the entanglement wedge is the same as that in the corresponding boundary region. Using this argument, the authors in [Dong et al., 2016b] could prove that for large class of states the bulk region dual to a given region of the boundary is its entanglement wedge[3]. If EWR holds, then it follows that the causal wedge of A can also be reconstructed from A, as it is generally smaller than the entanglement wedge.

Causal wedge reconstruction is well understood at large N. Using the bulk equations of motion and the fact that bulk fields asymptote to CFT local operators near the boundary of AdS (also called the extrapolate dictionary), it is possible to express bulk operators in the causal wedge in terms of smeared single-trace operators in the causal domain of the boundary region A. This approach is called HKLL reconstruction which has been introduced in seminal of papers [Banks et al., 1998, Bena, 2000, Hamilton et al., 2006b, Hamilton et al., 2006a, Hamilton et al., 2007a, Hamilton et al., 2008a]. A simple example of a subregion where this can be worked out explicitly is the case of the AdS-Rindler wedge, where via an HKLL approach one can express bulk operators in the AdS-Rindler subregion in terms of

When such regions are big enough the entanglement wedge is bigger than the causal wedge.

[3]For subtleties involving the entanglement wedge, see [Hayden & Penington, 2019, Akers & Penington, 2021].

CFT operators on the corresponding boundary subregion. It is generally believed that this procedure can be extended to all orders in $1/N$, by adding multi-trace corrections [Kabat & Lifschytz, 2014].

On the other hand EWR is more subtle. This is especially true in cases where the entanglement wedge is larger than the causal wedge. The arguments mentioned above in support of EWR are somewhat formal and do not provide us with a concrete representation of bulk operators lie in the entanglement wedge in terms of CFT operators.

A general way to approach the problem of EWR is via the Petz map [Petz, 1986, Petz, 1988]: if a quantum channel between two Hilbert spaces preserves relative entropy then it can be reversed. In situations when the relative entropy is only approximately preserved, there is no exact reversibility but one can also use the twirled Petz map [Cotler et al., 2019b] as an approximation. In our case, the quantum channel is the map from the entanglement wedge of a given boundary region to the region itself. As was argued in [Cotler et al., 2019b], starting with an isometry that maps the entire bulk Hilbert space to the entire boundary Hilbert space, such as the one related to the global HKLL reconstruction, one can find the explicit form of the quantum channel we are interested in. Then, the dual of the corresponding Petz recovery channel, called Petz map, allows us - in principle - to express operators in the entanglement wedge in terms of operators in the region A. Moreover, considering $1/N$ corrections, it was argued in [Chen et al., 2020b] that the Petz map is still good enough in finite-dimensional code subspaces whose dimensions do not grow exponentially in N, as long as the error is non perturbatively small.

However, the expression resulting from the Petz map for the bulk operator in terms of boundary data is still somewhat abstract. The relevant formulae involve a projector on the code subspace and taking a trace over part of the CFT which in practice, it is not easy to treat. Until now, there are no examples where the Petz map expression has been computed in detail in terms of simple CFT quantities.

In this paper, we make two advances towards a better understanding of the EWR via the Petz map. First, we demonstrate how, when applied to an AdS-Rindler wedge and working at large N, the Petz formula reproduces the standard AdS-Rindler HKLL reconstruction. While this has been generally assumed to be true, to our knowledge it has not been explicitly demonstrated[4]. Second, we consider more general subregions, where the entanglement wedge may even be larger than the causal wedge, and we apply the Petz formula and show that it reproduces results on EWR which were earlier conjectured in [Jafferis et al., 2016b] and also derived in [Faulkner & Lewkowycz, 2017b] using arguments based on modular flow. The crucial point in our work is using the Reeh-Schlieder theorem in QFT. Based on this theorem, we can define the code subspace by acting just with operator algebra of the specified subregion on the corresponding semi-classical state.

The plan of the paper is as follows: in section 2, we review some basic aspects of bulk reconstruction using the HKLL approach. In section 3, we introduce the concept of a quantum channel, we discuss conditions for its reversibility and introduce the Petz map. In section 4, we review the proposal of [Cotler et al., 2019b], of how the Petz map can be used for EWR. In section 5, we show how in the case of the AdS-Rindler wedge the Petz map reproduces the more standard HKLL AdS-Rindler reconstruction. In section 6, we apply the Petz map to more general entanglement wedges.

[4] In [Cotler et al., 2019b], the authors used the twirled Petz map (check subsection 3.3.2) to reconstruct AdS-Rindler wedge in a very restrictive case when the code subspace contains only the vacuum and one particle state.

3.2 Bulk reconstruction in AdS/CFT

According to the AdS/CFT correspondence, a holographic CFT on $\mathbb{R} \times \mathbb{S}^{d-1}$ can be interpreted as a theory of quantum gravity in an asymptotically $AdS_{d+1} \times M$ spacetime, where M is some compact manifold. Usually, this involves taking a large N limit in the CFT and bulk interactions are suppressed by powers of $1/N$. Thus, to leading order at large N, the bulk quantum theory consists of free fields.

The correspondence also involves an identification between fields in the bulk and operators in the boundary CFT. For example, the CFT operator dual to a bulk scalar field φ is a scalar primary O with conformal dimension Δ related to the mass of the field φ by $\Delta = d/2 + \sqrt{m^2 + d^2/4}$ and the extrapolate dictionary defines O as the dual of φ at infinity. For simplicity, in the following, we will just focus on scalar fields and discuss the identification with the dual CFT operator O at large N.

First, on the bulk side of the duality, we start with AdS_{d+1} in global coordinates (t, ρ, Ω) which is described with the metric below

$$ds^2 = \frac{1}{\cos^2(\rho)}\big(- dt^2 + d\rho^2 + \sin^2(\rho) d\Omega^2_{d-1}\big). \tag{3.1}$$

Consider a scalar field on the AdS_{d+1} background with the action

$$S = \int d^{d+1}x \sqrt{-g}\, \frac{1}{2} \left(g^{\mu\nu} \nabla_\mu \varphi \nabla_\nu \varphi - m^2 \varphi^2 \right) \tag{3.2}$$

and corresponding equation of motion

$$(\Box - m^2)\varphi = 0. \tag{3.3}$$

This equation has to be supplemented with normalizable boundary conditions at infinity, which implies that near the AdS boundary $\rho = \frac{\pi}{2}$, the field has to decay as $\varphi \sim (\cos\rho)^{\Delta}$. With these boundary conditions at infinity and demanding regularity in the interior, we find a basis of solutions for (3.3) denoted as $f_{nlm}(t, \rho, \Omega)$ which is labeled by the quantum numbers n, l and m, where $n \in \{0, 1, 2, ...\}$, l is the total angular momentum of the corresponding mode and m is related to the other angular quantum numbers needed to specify a mode. These modes are proportional to

$$f_{nlm}(t, \rho, \Omega) \propto e^{-iE_{nl}t} Y_{lm}(\Omega) \sin^l(\rho) \cos^{\Delta}(\rho) P_n^{(l+d/2-1, \Delta-d/2)}(\cos 2\rho) \tag{3.4}$$

while

$$E_{nl} = \Delta + 2n + l, \tag{3.5}$$

and $\Delta = d/2 + \sqrt{m^2 + d^2/4}$ is the conformal dimension of the dual CFT$_d$ operator O.

To quantize the scalar field, we associate an annihilation operator a_{nlm} to each mode f_{nlm} with normalized commutation relation

$$[a_{nlm}, a^{\dagger}_{n'l'm'}] = \delta_{nn'}\delta_{ll'}\delta_{mm'}. \tag{3.6}$$

The quantized free scalar field in AdS_{d+1} is given by

$$\varphi(t,\rho,\Omega)=\sum_{nlm} f_{nlm}(t,\rho,\Omega)a_{nlm}+f^{*}_{nlm}(t,\rho,\Omega)a^{\dagger}_{nlm}. \tag{3.7}$$

The modes $f_{nlm}(t,\rho,\Omega)$ should be normalized in such a way that the field φ obeys the canonical commutation relation. To find the correct normalization factor we consider the Klein-Gordon inner product defined on a Cauchy surface Σ. If we assume that t direction is orthogonal to Σ, for every two functions φ_1 and φ_2, it is defined as

$$\langle\varphi_1,\varphi_2\rangle_{KG}\equiv i\int_{\Sigma} d^d x\,\sqrt{-g}\,g^{tt}\left(\varphi_1^{*}\nabla_t\varphi_2-\varphi_2\nabla_t\varphi_1^{*}\right). \tag{3.8}$$

If both φ_1 and φ_2 obey the equation of motion, the integral above defines a conserved inner product in t. In particular, it says that if we normalize the modes f_{nlm} at some time such that $\langle f_{nlm}, f_{n'l'm'}\rangle = \delta_{nn'}\delta_{ll'}\delta_{mm'}$ and $\langle f_{nlm}, f^{\dagger}_{n'l'm'}\rangle = 0$, they will remain normalized also at later times [Kaplan, 2016]. Following these steps, in the end, one can write the modes explicitly as

$$f_{nlm}(t,\rho,\Omega)=\frac{1}{N_{nlm}}e^{-i(\Delta+2n+l)t}\,Y_{lm}(\Omega)\sin^{l}(\rho)\cos^{\Delta}(\rho)P_n^{(l+d/2-1,\Delta-d/2)}(\cos 2\rho) \tag{3.9}$$

where

$$N_{nlm}=\sqrt{\frac{\Gamma(n+l+d/2)\Gamma(n+\Delta-d/2+1)}{n!\Gamma(n+l+\Delta)}}. \tag{3.10}$$

The conformal boundary of AdS_{d+1} is the cylinder $\mathbb{R}\times\mathbb{S}^{d-1}$ which in terms of the global coordinates we obtain by taking $\rho\to\pi/2$ limit. We can use the coordinate t and Ω

to parametrize the boundary theory with metric

$$ds^2 = -dt^2 + d\Omega_{d-1}^2. \tag{3.11}$$

In the boundary, using the state-operator correspondence in the CFT, the formula (3.5) has a nice interpretation. The state created by the $n = l = 0$ creation operator is identified with the state in the CFT that is produced by inserting the single-trace primary operator O dual to φ into the center of the Euclidean path integral and other excited states come from inserting its descendants.

More generally, to leading order at large N, the Fourier modes O_{nlm} of the single trace primary operator and a_{nlm} for the mode f_{nlm} are the same up to apriori arbitrary constant M_{nlm}.The extrapolate dictionary in the global coordinates is given by

$$O(t, \Omega) = \lim_{\rho \to \pi/2} \frac{1}{\cos^{\Delta} \rho}\, \varphi(t, \rho, \Omega). \tag{3.12}$$

As a result, we can define the CFT operator $\hat{O}_{nlm} = \frac{1}{M_{nlm}} O_{nlm}$ which is identified with the bulk operator

$$\hat{O}_{nlm} = a_{nlm}. \tag{3.13}$$

As we will see later, this allows us to write a CFT expression for a local bulk field at any point in the bulk.

The single trace primary operator O has a mode expansion on $\mathbb{R} \times \mathbb{S}^{d-1}$ as

$$O(t, \Omega) = \sum_{nlm} g_{nlm}(t, \Omega) O_{nlm} + g^*_{nlm}(t, \Omega) O^{\dagger}_{nlm}. \tag{3.14}$$

Following 3.7, we have $g_{nlm} = \frac{1}{M_{nlm}} \lim_{\rho\to\pi/2} \frac{1}{\cos^\Delta \rho} f_{nlm}(t, \rho, \Omega)$. Thus M_{nlm} can be chosen so that mode functions g_{nlm} are orthonormal.

At the large N limit, since we have a free theory in the bulk, all correlators can be reduced to products of 2-point functions by Wick contractions. Therefore, on the boundary side, we already know all the n-point functions of the operator O by taking the spacetime points to the boundary in the expression we found for the bulk and using the extrapolate dictionary (3.12), we have

$$\langle O(x_1)O(x_2)\rangle \propto \frac{1}{(x_1 - x_2)^{2\Delta}} \tag{3.15}$$

$$\langle O(x_1)O(x_2)...O(x_{2n})\rangle = \langle O(x_1)O(x_2)\rangle...\langle O(x_{2n-1})O(x_{2n})\rangle + \textit{permutations}. \tag{3.16}$$

Although the correlation functions of O factorize to the product of 2-point functions, the scalar primary operator is not really a free scalar field. In a CFT in d spacetime dimensions, the condition that a scalar operator is free, i.e. $\nabla^2 O = 0$, is equivalent to the fact that its conformal dimension is $\Delta = d/2 - 1$. Therefore, as the conformal dimension for the scalar primary operator O in a holographic CFT is $\Delta = d/2 + \sqrt{m^2 + d^2/4}$, it is actually not a free scalar theory on the boundary. For the free scalar primaries, the factorization is a consequence of the equation of motion. More generally, the scalar fields with $\Delta \geq d/2 - 1$ are called *generalized free fields* (GFFs) [Greenberg, 1961, Duetsch & Rehren, 2003, El-Showk & Papadodimas, 2012b] if their correlators take the form of Eqs. (3.15) and (3.16). However, because they do not obey the linear equation of motion, we can not describe them in terms of a local free lagrangian in the spacetime background in which the CFT lives. The reason that such fields can be called free is that their Hilbert space has a Fock space structure that is the Hilbert space of a free theory. Nevertheless, such a CFT, extrapolated to high temperatures, has the wrong thermodynamic properties, and therefore it is inconsistent by itself. For the operators with large conformal dimensions, the spectrum can not have the structure of a freely generated Fock space. One way to solve the problem is that think about the GFF as the low-conformal dimension sector of a much larger CFT with a large central charge while all the additional states correspond to the black hole microstates in the bulk [El-Showk & Papadodimas, 2012b].

As a result, we observe that at the large N limit, a free scalar field in pure AdS can be identified as GFF of the boundary.

3.2.1 HKLL reconsruction method: mode sum approach

The study of bulk reconstruction, usually called HKLL, was developed by Hamilton, Kabat, Lifschytz and Lowe in a series of papers [Banks et al., 1998, Bena, 2000, Hamilton et al., 2006b, Hamilton et al., 2006a, Hamilton et al., 2007a, Hamilton et al., 2008a]. They attempt to reconstruct the operators of the bulk gravitational theory in the non-interacting regime from the operators of the boundary. Bulk operators are expressed in terms of smeared single trace operators in the CFT as

$$\varphi(X) \quad \longleftrightarrow \quad \int dx\, K(X|x) O(x), \tag{3.17}$$

where the kernel $K(X|x)$ is called *smearing function*. At large N limit, finding the smearing function can be implemented through Fourier expansion. Consider $f_n(X)$ as the set of orthogonal solutions to the Klein-Gordon equation. For simplicity here we denote the set of labels (nlm) discussed in the previous subsection collectively by n. One can quantize the bulk field in terms of creation and annihilation operators as

$$\varphi(X) = \sum_n f_n(X) a_n + h.c. \tag{3.18}$$

Taking the points to the boundary and using the extrapolate dictionary, we get the mode expansion of the boundary operators as

$$O(x) = \sum_n \tilde{g}_n(x) a_n + h.c. \tag{3.19}$$

where $\tilde{g}_n(x) = \lim_{r\to\infty} r^\Delta f_n(r,x)$. If one defines orthonormal boundary mode functions $g_n(x) = \frac{1}{M_n}\tilde{g}_n(x)$, one can invert (3.19) and obtain

$$a_n = \frac{1}{M_n}\int dx O(x) g_n^*(x). \tag{3.20}$$

By plugging it back to (3.18), we reach

$$\varphi(X) = \sum_n \frac{1}{M_n} f_n(X) \int dx\, O(x) \tilde{g}_n^*(x) + h.c.. \tag{3.21}$$

In case we are able to exchange the summation and integration in (3.21), we will get

$$\varphi(X) = \int dx\, K(X|x) O(x), \tag{3.22}$$

where $K(X|x) = \sum_n M_n^{-1} f_n(X)\, \tilde{g}_n^*(x) + h.c.$ is the smearing function [Leichenauer & Rosenhaus, 2013]. We review the HKLL construction for a free scalar field in pure AdS in global and AdS-Rindler coordinates in appendix 3.8.

3.2.2 Bulk reconstruction and subregion duality

As we had in the previous sections, a bulk operator $\varphi(X)$ can be represented as a smearing integral of boundary operators

$$\varphi(X) = \int_{bdy} d^{d-1}xdt\, K(X|t,x)O(t,x) + O(1/N). \tag{3.23}$$

We can use the CFT Hamiltonian to re-express all operators $O(t,x)$ in terms of fields on a Cauchy surface Σ of the boundary by explicitly evolving the operators with the boundary Hamiltonian. Let us consider the pure AdS case and Σ to be the $t=0$ slice

$$\varphi(X) = \int_{bdy} d^{d-1}x\, dt\, K(X|t,x)\, e^{iH_{CFT}t}O(x)e^{-iH_{CFT}t} \tag{3.24}$$

where $O(t=0,x) = O(x)$. In general, operators of the form $e^{iH_{CFT}t}\,O(x)\,e^{-iH_{CFT}t}$ have support on a large part of the slice $\Sigma,\ t=0$. An interesting question in AdS/CFT is whether the CFT representation of $\varphi(X)$ can be localized to a subregion of Σ. Intuitively, it is expected that the boundary support of $\varphi(X)$ shrinks as the operator approaches the boundary. However, one can see from (3.24) that even if we take X to be very close to the boundary, the CFT reconstruction still has support on the entire Σ.

In fact, it is possible to reconstruct bulk operators so that they are supported on smaller regions on the boundary. Consider a spherical subregion R on Σ and the corresponding causal wedge in the bulk. Consider a local field $\varphi(X)$ localized inside this causal wedge.

Then it is possible to represent the bulk field as

$$\varphi(X) = \int_{\mathcal{D}(R)} d\tau d^{d-1}x\, K_R(X|\tau, x) O(\tau, x). \tag{3.25}$$

where the integral is over the domain of dependence $\mathcal{D}(R)$ of R and $K_R(X|\tau, x)$ is a new smearing function called the AdS-Rindler smearing function.

Again we can write it in terms of non-local operators in the Heisenberg picture which evolves with Rindler Hamiltonian H_τ

$$\varphi(X) = \int_{-\infty}^{\infty} d\tau \int_R d^{d-1}x\, K_R(X|x, \tau)\, e^{iH_\tau \tau} O(x) e^{-iH_\tau \tau}. \tag{3.26}$$

The operators $e^{iH_\tau \tau}\, O(x)\, e^{-iH_\tau \tau}$ are again some non-local operators but this time they have support only on region R instead of entire Σ. Therefore, the AdS-Rindler reconstruction provides us a way to localize the representation of $\varphi(X)$ in the boundary. More generally, it suggests the proposal that a given region R on a Cauchy slice of the boundary encodes the bulk data inside the causal wedge of its boundary domain of dependence.

Nevertheless, one can go ahead and look at the Rindler Hamiltonian in (3.26) as the *modular Hamiltonian* of the region R that generates the modular flow of operators on R. For the case of AdS-Rindler, it is just translation in the τ direction. In [Jafferis et al., 2016b], authors showed that the boundary modular flow is dual to the bulk modular flow in the entanglement wedge $\mathcal{E}_R$ and conjectured that operators in the entanglement wedge of the region R are the ones can be constructed on the boundary region R by replacing τ in

(3.26) by the modular parameter s as

$$\varphi(X) = \int_R d^{d-1}x \int_{-\infty}^{\infty} ds\, K_R'(X|x,s) O(x,s) \tag{3.27}$$

for every $X \in \mathcal{E}_R$ where $O(x,s) = e^{iK_R s}\, O(x, s=0)\, e^{-iK_R s}$.

3.3 Background on quantum channels

In this section, we introduce the notion of a *quantum channel* and the conditions under which it is reversible. The central point of this section is the introduction of the *Petz map* (3.37), which will be used in later sections in the context of bulk reconstruction.

3.3.1 Quantum channels

Real systems suffer from unwanted interactions with the outside world that show up as noise in quantum information processing systems. In order to describe such systems, it is useful to introduce the notion of a quantum channel, i.e. a linear map $\mathcal{E} : L(\mathcal{H}) \to L(\mathcal{H})$ which is completely positive and trace-preserving. Here, $L(\mathcal{H})$ denotes the set of linear operators acting on the Hilbert space $\mathcal{H}$.[5] Every quantum channel $\mathcal{E}$ has an operator sum representation in terms of a non-unique set of operators $\{A_i\}$ known as *Kraus operators* [Kraus et al., 1983, Hellwig & Kraus, 1970, Preskill, 1998] such that,

$$\mathcal{E}(\rho) = \sum_i A_i \rho A_i^\dagger \qquad\qquad \sum_i A_i^\dagger A_i = I. \tag{3.28}$$

[5] More generally a quantum channel can be a map between $L(\mathcal{H}_1), L(\mathcal{H}_2)$ for two different Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2$.

They are the most general transformation of a quantum state in an open quantum system.

A natural way to describe the dynamics of an open quantum system is to regard it as arising from an interaction between the system and an environment which together transform under a unitary. After the evolution we perform a partial trace over the environment to obtain the state of the system. For every quantum channel, there exists a model environment starting in an initial state σ_{en} and model dynamics specified by a unitary operator U such that

$$\mathcal{E}(.) = \mathrm{tr}_{en}\big(U(.\otimes\sigma_{en})U^{\dagger}\big), \tag{3.29}$$

which is a version of the *Stinespring dilation* theorem. If $\sigma_{en} = \sum_j \lambda_j \left|j\right\rangle \left\langle j\right|$, one can find the Kraus representation of (3.29) as

$$\mathcal{E}(\rho) = \sum_{j,k} A_{j,k}\,\rho\, A_{j,k}^{\dagger} \tag{3.30}$$

which $A_{j,k} = \sqrt{\lambda_j}\left\langle k\right| U \left|j\right\rangle$ are the Kraus operators. Therefore, we can describe the dynamics of the system by using the operator-sum representation without having to explicitly consider the properties of the environment. One advantage of this Kraus representation is that it is well adapted to describe discrete state change without explicit reference to the passage of time.

3.3.2 Universal recovery channel and the Petz map

A quantum channel $\mathcal{E}$ is called reversible if one can find a *recovery channel* $\mathcal{R} : L(\mathcal{H}) \rightarrow L(\mathcal{H})$ such that

$$\mathcal{R}\circ\mathcal{E}(\rho) = \rho \qquad \forall\rho\in S(\mathcal{H}). \tag{3.31}$$

Most quantum channels, which correspond to open or noisy systems, are not reversible. We will return to the question of reversibility later in this subsection.

In order to do quantum information and communication in the presence of noise, we need *quantum error correction* (QEC) codes. The basic ideas of the theory are inspired by classical information theory, but all the limitations of quantum mechanics have been considered in its formulation. A quantum error correcting code corresponds to selecting an appropriate subspace, called *code subspace*($\mathcal{C}$ or $\mathcal{H}_{code}$) that has the same dimension as the system, of some larger Hilbert space. In the general theory of QEC, the noise model is described by a quantum channel $\mathcal{E}$. The code subspace can be corrected if we can find a recovery channel $\mathcal{R}$, such that for every state ρ whose support lies within $\mathcal{H}_{code}$, the channel can be reversed, i.e.

$$\mathcal{R} \circ \mathcal{E}(\rho) = \rho \qquad \forall \rho = P\rho P \tag{3.32}$$

where P is the projection into the code subspace. One might be interested to consider a physical system instead of a code subspace. In such a case, if we take $V : \mathcal{H}_{system} \to \mathcal{H}$ as the isometry that embeds the $\mathcal{H}_{system}$ into $\mathcal{H}$, we can rewrite (3.31) as the following

$$\mathcal{R} \circ \mathcal{E}(V\rho V^\dagger) = V\rho V^\dagger \qquad \forall \rho \in S(\mathcal{H}_{system}) \tag{3.33}$$

that is equivalent to having $\mathcal{E}'$ and $\mathcal{R}'$ such that $\mathcal{R}' \circ \mathcal{E}'(\rho) = \rho$ where $\mathcal{E}'(.) = \mathcal{E}(V(.)V^\dagger)$ and $\mathcal{R}'(.) = V^\dagger \mathcal{R}(.)V$ [Bény et al., 2009].

Given a quantum channel $\mathcal{E}$, it is useful to consider the Hilbert-Schmidt dual channel which defines a mapping of *observables* rather than of *states*. This is also sometimes referred to as the Heisenberg picture of the channel. The idea is to think of $\mathcal{E}$ as a (discrete) evolution of a state. After the evolution, the result of a measurement of an observable O will be in the form of $\mathrm{tr}(\mathcal{E}(\rho)O)$, where ρ describes the state of the system. As we usually do when going to the Heisenberg picture, we can alternatively formulate the evolution of the system by transforming the operators, requiring to get the same measurement results. For this purpose, we describe the evolution of the observables by the channel $\mathcal{E}^*$ that is called *Hilbert-Schmidt dual map* defined as

$$\mathrm{tr}(\rho\mathcal{E}^*(O)) = \mathrm{tr}(\mathcal{E}(\rho)O) \qquad \forall\rho, O. \tag{3.34}$$

The set of Kraus operators for $\mathcal{E}^*$ is given easily by cyclicity property of trace as $\{A_a^\dagger\}$ instead of $\{A_a\}$, and trace preservation of $\mathcal{E}$ is equivalent to the requirement that $\mathcal{E}^*$ is unital, $\mathcal{E}^*(I) = I$. In the case of QEC, the conservation of a state by $\mathcal{R} \circ \mathcal{E}$ (3.31) implies that in the Heisenberg picture for all the operators $O \in \mathcal{L}(\mathcal{H})$ we have

$$P(\mathcal{R} \circ \mathcal{E})^*(O)P = P\mathcal{E}^* \circ \mathcal{R}^*(O)P = POP. \tag{3.35}$$

We now return to the general question of the reversibility of a quantum channel. This has been studied widely in [Jenčová & Petz, 2006, Mosonyi & Petz, 2004, Petz, 1986, Petz, 1988]. The reversibility of $\mathcal{E}$ is related to the quantum relative entropy of states under the action of $\mathcal{E}$. The relative entropy between two states ρ and σ is defined as $S(\rho||\sigma) = \mathrm{tr}(\rho\log\rho - \rho\log\sigma)$ and it is a measure of distinguishability between two quantum states. The most important theorem related to this quantity known as *monotonicity of relative entropy* or the *data processing inequality*, whose proof is discussed in appendix 3.9 for finite dimensions, states that $S(\rho||\sigma)$ is non-increasing under the action of any quantum channel

$\mathcal{E}$ [Lindblad, 1975, Uhlmann, 1977], i.e.,

$$\mathcal{S}(\rho||\sigma) \geqslant \mathcal{S}(\mathcal{E}(\rho)||\mathcal{E}(\sigma)). \tag{3.36}$$

It has been shown in [Petz, 2003, Hayden et al., 2004] that there exists a quantum channel $\mathcal{R}$ such that for all states $\rho \in S(\mathcal{H})$, $\mathcal{R} \circ \mathcal{E}(\rho) = \rho$ if and only if $\mathcal{S}(\rho||\sigma) = \mathcal{S}(\mathcal{E}(\rho)||\mathcal{E}(\sigma))$ for all $\rho, \sigma \in S(\mathcal{H})$. Moreover, the explicit form of the quantum channel $\mathcal{R}$ for the set of states $\{\mathcal{E}(\rho)|\forall\rho \in S(\mathcal{H})\}$ has been found in [Hayden et al., 2004]. It is given as a function of a reference quantum state $\sigma \in S(\mathcal{H}_A)$ and the channel $\mathcal{E}$ itself as

$$\mathcal{R}(.) = \mathcal{P}_{\sigma,\mathcal{E}}(.) = \sigma^{1/2}\mathcal{E}^*\big(\mathcal{E}(\sigma)^{-1/2}(.)\mathcal{E}(\sigma)^{-1/2}\big)\sigma^{1/2} \tag{3.37}$$

where $\mathcal{E}^*$ is the dual channel of $\mathcal{E}$. $\mathcal{P}_{\sigma,\mathcal{E}}$ is known as *Petz recovery channel*. This result has been also independently obtained by Barnum and Knill in [Barnum & Knill, 2002]. We review a proof of the theorem in the finite-dimensional case in appendix 3.9.

Practically, in most cases, inequality (3.36) will not be saturated hence exact reversibility will not be satisfied. One can ask if there exists an approximate recovery channel that the recovered state be just close to the state ρ, $|\mathcal{R}_\varepsilon \circ \mathcal{E}(\rho) - \rho| < \varepsilon$ where $\mathcal{R}_\varepsilon$ is approximate version of recovery channel. In [Wilde, 2015, Sutter et al., 2016], with two different approaches, it was shown that for any two states ρ and σ and channel $\mathcal{E}$ there exists a recovery channel $\mathcal{R}$ such that $\mathcal{R} \circ \mathcal{E}(\sigma) = \sigma$ and

$$\mathcal{S}(\rho||\sigma) - \mathcal{S}(\mathcal{E}(\rho)||\mathcal{E}(\sigma)) \geqslant -2\log F(\rho, \mathcal{R} \circ \mathcal{E}(\rho)) \tag{3.38}$$

where $F(\rho,\sigma) := \|\sqrt{\rho}\sqrt{\sigma}\|_1$ is the fidelity of ρ and σ that measure the closeness of two quantum states. $F(\rho,\sigma) = 1$ if and only if $\rho = \sigma$, then the inequality of (3.38) will be saturated just in the case of exact reversibility. In [Junge et al., 2018], it was shown that the recovery channel $\mathcal{R}$ that satisfies (3.38) is universal, which means we can always choose a ρ-independent recovery channel. Furthermore, they could find the explicit expression for the universal recovery map $\mathcal{R}_{\sigma,\mathcal{E}}$ as

$$\mathcal{R}_{\sigma,\mathcal{E}}(.) = \int_{\mathbb{R}} dt\, \beta_0(t)\, \sigma^{-it/2} \mathcal{P}_{\sigma,\mathcal{E}}\left(\mathcal{E}(\sigma)^{it/2}(.)\mathcal{E}(\sigma)^{-it/2}\right)\sigma^{-it/2} \tag{3.39}$$

that is called *twirled Petz map* where $\mathcal{P}_{\sigma,\mathcal{E}}$ is the Petz recovery channel given in (3.37) and $\beta_0(t) = \frac{\pi}{2}(\cosh(\pi t)+1)^{-1}$. In the case of exact reversibility, the expression (3.39) is reduced to the Petz recovery channel.

3.4 Entanglement wedge reconstruction via universal recovery channels

In this section, we review the arugments of [Cotler et al., 2019b], on how the Petz map can be used to reconstruct bulk operators in the *entanglement wedge* of a boundary subregion.

3.4.1 Background

Before we proceed we introduce an ingredient that will be useful in what follows. This is the idea of a *code subspace* around a given state. For example, starting with the global AdS vacuum state $|\Omega\rangle$ we define

$$\mathcal{H}_C = span\{|\Omega\rangle, \varphi_i(x)|\Omega\rangle, ..., \varphi_i(x_1)\varphi_j(x_2)|\Omega\rangle, ...\}, \tag{3.40}$$

where the range of i and the number of φ insertions are finite. More generally we can define the code subspace around any semi-classical state. This subspace is the one where low-energy experiments in the bulk can be described and we will study bulk reconstruction within a given code subspace.

The entanglement wedge of a boundary region A is defined as the bulk domain of dependence of any bulk spacelike surface whose boundary is the union of A and the codimension two extremal area surface of minimal area (more precisely, quantum extremal surface) whose boundary is ∂A. It is generally believed that bulk operators inside the entanglement wedge can be reconstructed by operators in the region A on the boundary.

An important ingredient supporting this is the observation of JLMS [Jafferis et al., 2016b] that the relative entropy of two states in the boundary region A is equal to the relative entropy of the two corresponding bulk states in $\mathcal{E}_A$ up to subleading correction.

$$\mathcal{S}(\rho_A||\sigma_A) = \mathcal{S}(\rho_a||\sigma_a) + O(1/N) \tag{3.41}$$

which already suggests that information in the entanglement wedge is contained in region A on the boundary. Using (3.41) arguments in favor of entanglement wedge reconstruction were given in [Dong et al., 2016b].

Assume that the bulk Hilbert space has a decomposition as $\mathcal{H}_{bulk} = \mathcal{H}_a \otimes \mathcal{H}_{\bar{a}}$, while $a = \mathcal{E}_A$. For the cases where the setup is symmetric, like the vacuum sector of the system, the complement region of a in the bulk is also the entanglement wedge of the region $\bar{A}$ in the boundary, $\bar{a} = \mathcal{E}_{\bar{A}}$, so the same argument applies for $\bar{A}$ and $\bar{a}$. In general, the entanglement wedge of a given boundary region A can be bigger than its causal wedge. Finally, the entanglement wedge reconstruction is a statement that says any bulk operator φ_a acting within $\mathcal{H}_a$ can always be represented in the CFT as an operator O_A has support only on $\mathcal{H}_A$.

3.4.2 Entanglement wedge reconstruction with a universal recovery channel

We now discuss entanglement wedge reconstruction in terms of the universal recovery channels described in Sec. 3.3.2, based on [Cotler et al., 2019b].

First, consider the entanglement wedge reconstruction and for simplicity assume both bulk and CFT Hilbert spaces have a tensor decomposition as $\mathcal{H}_{bulk} = \mathcal{H}_a \otimes \mathcal{H}_{\bar{a}}$ and $\mathcal{H}_{CFT} = \mathcal{H}_A \otimes \mathcal{H}_{\bar{A}}$. At large N when the equality between the relative entropy of the states in the entanglement wedge and the boundary region A is exact, i.e.,

$$\mathcal{S}(\rho_A||\sigma_A) = \mathcal{S}(\rho_a||\sigma_a) \tag{3.42}$$

from the discussion in Sec. 3.3.2, one can say that there exists a quantum channel $\mathcal{R}$ which recovers the information in the entanglement wedge from the boundary region A. Using the dual channel $\mathcal{R}^*$ we can map operators on $\mathcal{H}_a$ to operators on $\mathcal{H}_A$ as $O_A = \mathcal{R}^*(\varphi_a)$.

If we assume that there is no black hole in the bulk, the global HKLL reconstruction reviewed in section 2 provides us a map from states of the entire bulk to states of the entire boundary. We can therefore define an isometry of embedding V_{HKLL} that embeds the bulk effective field theory Hilbert space to the CFT Hilbert space $V_{HKLL} : \mathcal{H}_{bulk} \hookrightarrow \mathcal{H}_{CFT}$, which $\mathcal{H}_{code} = V_{HKLL}\, \mathcal{H}_{bulk}\, V^{\dagger}_{HKLL}$.

We now define a quantum channel $\mathcal{E} : S(\mathcal{H}_a) \to S(\mathcal{H}_A)$. Here $S(\mathcal{H}_a)$ denotes the set of possible density matrices in the bulk region a while $S(\mathcal{H}_A)$ is the set of density matrices in the boundary region A. As the entire AdS space is a closed system, the noise model $\mathcal{E}:S(\mathcal{H}_a)\rightarrow S(\mathcal{H}_A)$ can be written in terms of a model environment (3.29) using the global HKLL map. We take the complementary bulk region $\bar{a}$ in a fixed reference state $\sigma_{\bar{a}}$ and then, we can write the quantum channel $\mathcal{E}$ as

$$\mathcal{E}(.)=\mathrm{tr}_{\bar{A}}\big(V_{HKLL}(.\otimes\sigma_{\bar{a}})V^{\dagger}_{HKLL}\big). \tag{3.43}$$

To map the operators, as we had in Sec. 3.3.2, one can go to the Heisenberg picture and write the dual of Petz recovery channel of $\mathcal{E}$ by taking a fixed reduced density matrix on the entanglement wedge σ_a, using expression (3.37), we reach to

$$O_A=\mathcal{R}^*(\varphi_a)=\mathcal{E}(\sigma_a)^{-1/2}\mathcal{E}\big(\sigma_a^{1/2}\varphi_a\sigma_a^{1/2}\big)\mathcal{E}(\sigma_a)^{-1/2}, \tag{3.44}$$

which for the quantum channel (3.43), it will give us

$$O_A=\mathcal{E}(\sigma_a)^{-1/2}\mathrm{tr}_{\bar{A}}\big(V_{HKLL}(\sigma_a^{1/2}\otimes\sigma_{\bar{a}}^{1/2})(\varphi_a\otimes I_{\bar{a}})(\sigma_a^{1/2}\otimes\sigma_{\bar{a}}^{1/2})V^{\dagger}_{HKLL}\big)\mathcal{E}(\sigma_a)^{-1/2}, \tag{3.45}$$

where $\mathcal{E}(\sigma_a)=\mathrm{tr}_{\bar{A}}\big(V_{HKLL}(\sigma_a\otimes\sigma_{\bar{a}})V^{\dagger}_{HKLL}\big)$. If we take both σ_a and $\sigma_{\bar{a}}$ two maximally mixed states or equivalently putting the bulk in the maximally mixed state τ, the map (3.45) will be simplified as

$$O_A=\frac{1}{d_{code}}\tau_A^{-1/2}\mathrm{tr}_{\bar{A}}\big(V_{HKLL}(\varphi_a)V^{\dagger}_{HKLL}\big)\tau_A^{-1/2}, \tag{3.46}$$

where $\tau_A=\frac{1}{d_{code}}\mathrm{tr}_{\bar{A}}P_{code}$. It is good to note here that the condition

$$\langle\varphi_a\rangle_{\rho_{bulk}}=\langle\Phi_{a,HKLL}\rangle_{\rho_{CFT}} \tag{3.47}$$

implies that $V_{HKLL}(\varphi_a)V^{\dagger}_{HKLL} = P_{code}\Phi_{a,HKLL}P_{code}$, and so the bulk operator φ in the entanglement wedge can map to a boundary operator has support only in the region A as

$$O_A = \frac{1}{d_{code}}\tau_A^{-1/2}\mathrm{tr}_{\bar{A}}\left(P_{code}\Phi_{HKLL}P_{code}\right)\tau_A^{-1/2}. \tag{3.48}$$

This is the main result of the section, that we will use in the rest of the paper.

When including $1/N$ corrections, (3.42) will no longer be exact and we do not expect to have an exact recoverability. In that case, we can try to reconstruct the entanglement wedge using the twirled Petz map (3.39). For the maximally mixed state in the code subspace, the mapping is as below

$$O_A = \frac{1}{d_{code}}\int_{\mathbb{R}} dt\,\beta_0(t)\;\tau_A^{-1/2(1-it)}\mathrm{tr}_{\bar{A}}\left(V_{HKLL}(\varphi_a)V^{\dagger}_{HKLL}\right)\tau_A^{-1/2(1+it)} \tag{3.49}$$

which at large N limit gives us the same formula as (3.46) [Cotler et al., 2019b]. It has been argued that for the reconstruction of the entanglement wedge for any finite-dimensional code subspace as well as code subspaces with dimensions that do not grow exponentially fast in N, while the error is non perturbatively small, the ordinary Petz map works well enough [Chen et al., 2020b].

In the large N limit it is possible to take the size of the code subspace to infinity. In that case, the maximally mixed state on the code subspace does not really exist and we would need to use some regulated version of it[6] that we denote by ρ.

One should be careful at this point that the quantum channel in (3.43), which takes as input the reduced density matrix of the entanglement wedge $\rho_a = \mathrm{tr}_{\bar{a}}(\rho)$ and gives

[6]For example, this could be a thermal state, which approximates the maximally mixed state as $T \to \infty$.

as output a state on A, will not generally provide us exactly the same state on A as $\rho_A = \mathrm{tr}_{\bar{A}}\left(V_{HKLL}(\rho) V^\dagger_{HKLL}\right)$ which depends on the state ρ defined on the entire bulk. Only in the case that the bulk reference state itself is a tensor factor of two states in a and $\bar{a}$, like the maximally mixed state, they will give us the same result. However, their difference is controlled by $1/N$: if we say that $|\mathcal{S}(\rho_A||\sigma_A) - \mathcal{S}(\rho_a||\sigma_a)| \leqslant \varepsilon$, then

$$\|\mathcal{E}(\rho_a) - \rho_A\|^2 \leqslant 2\ln 2\mathcal{S}(\mathcal{E}(\rho_a)||\rho_A) \leqslant (2\ln 2)\varepsilon. \tag{3.50}$$

Hence, at large N limit that ε goes to zero and we have the exact reconstruction of the entanglement wedge, one can exchange the $\mathcal{E}(\rho_a)$ and ρ_A. Then, we can introduce a general version of the Petz map in terms of an arbitrary fixed state ρ as [Penington et al., 2019]

$$O_A^{(\rho)} = \rho_A^{-1/2}\mathrm{tr}_{\bar{A}}\left(V_{HKLL}(\rho^{1/2}\varphi_a\rho^{1/2}) V^\dagger_{HKLL}\right)\rho_A^{-1/2}. \tag{3.51}$$

We note here that, for this reconstruction, the only source of the error is not the $1/N$ correction, but rather the entanglement in the state ρ between the inside and outside of the entanglement wedge causes to not recover the original state.

3.5 AdS-Rindler reconstruction and Petz map

As we saw in the previous chapters, a free scalar field in pure AdS is dual to a GFF of the boundary that can be thought of as a sector of a much larger CFT with a large central charge. In addition, Petz map is a tool that comes from the quantum information theory which provides us the CFT representation of the bulk field $\varphi(X)$ that is localized in any region A when the field lies in the entanglement wedge of A. It is given by

$$\Phi_A(X) = \tau_A^{-1/2} \mathrm{tr}_{\bar{A}} \left(P_{code} \Phi_{HKLL}(X) P_{code} \right) \tau_A^{-1/2} \tag{3.52}$$

where we redefine τ_A to the unnormalized maximally mixed state $\tau_A = \mathrm{tr}_{\bar{A}} P_{code}$ and $\Phi_{HKLL}(X)$ is the boundary reconstruction of the field in global coordinates

$$\Phi_{HKLL}(X) = \int_{bdy} dt d\Omega \; K^g(X|t,\Omega) O(t,\Omega) \tag{3.53}$$

that $K^g(X|t,\Omega)$ is the smearing function for the AdS space which in even and odd dimension given by (3.137) and (3.138) respectively. By plugging (3.53) back into (3.52) and considering the linearity of the trace we will get

$$\Phi_A(X) = \int_{bdy} dt d\Omega \; K^g(X|t,\Omega) \; \tau_A^{-1/2} \mathrm{tr}_{\bar{A}} \left(P_{code} O(t,\Omega) P_{code} \right) \tau_A^{-1/2}. \tag{3.54}$$

Therefore, to find $\Phi_A(X)$ we need to deal with terms

$$\mathrm{tr}_{\bar{A}} \left(P_{code} O(t,\Omega) P_{code} \right) \tag{3.55}$$

for every $O(t,\Omega)$. In order to take trace over $\bar{A}$, we need to re-express them in terms of the operators that act just on the Cauchy surface Σ. In other words, we should use the Heisenberg picture and rewrite all $O(t,\Omega)$ in terms of the scalar primaries on Σ by evolving them with boundary Hamiltonian. Let us consider Σ to be $t = 0$ slice. Then, (3.55) can be read off as

$$\mathrm{tr}_{\bar{A}}\left(P_{code}e^{iH_{CFT}t}O(\Omega)e^{-iH_{CFT}t}P_{code}\right) = \mathrm{tr}_{\bar{A}}\left(P_{code}e^{iH_{GFF}t}O(\Omega)e^{-iH_{GFF}t}P_{code}\right). \tag{3.56}$$

Since we project the Heisenberg picture operators on the code subspace, which should be a subspace of the GFF sector of the CFT, the CFT Hamiltonian can be replaced by the Hamiltonian of generalized free theory, which is

$$H_{GFF} = \sum_{nlm} \omega_{nlm} O^{\dagger}_{nlm} O_{nlm}. \tag{3.57}$$

It is important to note that all the operators in (3.56) have support on entire Σ, even when $t = 0$ and $O(x_A)$ is localized in region A, $P_{code}O(x_A)P_{code}$ still can have support on $\bar{A}$.

To do the calculation, it can be more convenient to go to Fourier space. By substituting (3.14) into (3.56) and plugging it back to (3.54) we arrive to

$$\begin{aligned}\Phi_A(X) = \sum_{nlm} G_{nlm}(X)\tau_A^{-1/2}\mathrm{tr}_{\bar{A}}\left(P_{code}O_{nlm}P_{code}\right)\tau_A^{-1/2} \\ + G^*_{nlm}(X)\tau_A^{-1/2}\mathrm{tr}_{\bar{A}}\left(P_{code}O^{\dagger}_{nlm}P_{code}\right)\tau_A^{-1/2}\end{aligned} \tag{3.58}$$

where

$$G_{nlm}(X) = \int_{bdy} dtd\Omega\, K^g(X|t,\Omega)\, g_{nlm}(t,\Omega). \tag{3.59}$$

To go ahead, we need to determine more precisely the setup we want to study and in particular, specify the region A on the boundary. Let us start with a simple case. Take A to be just one hemisphere of Σ, then as a result, $a = \mathcal{E}_A$ is an AdS-Rindler wedge in the bulk which its entanglement wedge coincides with its causal wedge. In the rest of this chapter, we will focus on finding the boundary representation for the operators that lie in the AdS-Rindler wedge. First, we work on rewriting the operator O_{nlm} in terms of the operators that act just on A or $\bar{A}$. We will then define the code subspace in this case and in particular, we will try to find a suitable choice of basis for code subspace to be able to do the calculation. Finally, we will compare our result with the boundary representation of the field one can find from the HKLL procedure in the AdS-Rindler coordinates.

3.5.1 Boundary treatment

The computation of (3.58) involves tracing out the sub-region $\bar{A}$ from the operator $P_{code}O_{nlm}P_{code}$. For this, one needs to choose appropriate basis to express P_{code}, discussed in Sec.3.5.3, and rewrite O_{nlm} in a way that makes the tracing out of complement easier.

Two ways to proceed further are discussed in this and the next section. One way is to use the bulk Bogoliubov transformation between the global and Rindler modes to adopt the same transformation for the global and Rindler boundary modes. Note that the boundary modes, by themselves, do not have Bogoliubov transformation, since they do not satisfy any equation of motion. Thus, to claim such transformation for the boundary modes, one needs to use the AdS-Rinlder reconstruction to relate the bulk Rindler modes to the boundary Rindler modes.

The other way, which will be discussed in this section, is solely a boundary treatment. As will be seen in Sec. 3.5.3, while computing the trace, one is interested in computing matrix elements like $\langle\Psi| O_{nlm} |\Psi\rangle$ where $|\Psi\rangle$ is an a state in the code subspace, where O_{nlm} is a global mode of single trace operators localized in full boundary. One can choose a Cauchy surface, say at $t = t_0$, that includes the bulk operator and after dividing its boundary into two subregions, A or $\bar{A}$, and write single trace operators localized in the two subregions which we call $O(\tau = 0, r)_A$ and $O(\tau = 0, r)_{\bar{A}}$ respectively (in general one has single trace operators in the wedges associated to the two subregions and in the past and future wedges). Given that the subregion A is spherical, one can write the Fourier transform of $O(\tau, r)_A$, the single trace operator in the domain of dependence of A, as

$$O(\tau, x)_A = \int d\omega d\lambda e^{-i\omega\tau} Y_\lambda(x) O_{\omega\lambda,A} + h.c. \tag{3.60}$$

where $Y_\lambda(x)$ is the eignefunction of the Laplacian on a co-dimension one hyperbolic ball in the boundary.

In addition, due to the particle number conservation of AdS/CFT in the large N limit, the action of the normalized operator O_{nlm}/M_{nlm}[7] on any state in the code subspace is given by a linear combination of the single trace operators in domain of dependence of A and $\bar{A}$, on the state after being appropriately normalized. One may expect that similar Rindler modes in the past and future wedges should also contribute but it can be checked, from the boundary, that the modes in the future and past wedges can be expressed in terms of modes in the domain of dependence of A and $\bar{A}$ by the symmetry property of the problem and requiring the correct two point functions in the future and past wedges. Thus one can just replace $P_{code} O_{nlm} P_{code}$ inside the equation by

$$\sum_{I=A,\bar{A}} M_{nlm} \sum_{\omega\lambda} P_{code} \frac{1}{M_{\omega\lambda}} (\chi^1_{nlm,\omega\lambda} O_{\omega\lambda,I} + \chi^2_{nlm,\omega\lambda} O^\dagger_{\omega\lambda,I}) P_{code} \tag{3.61}$$

[7]The normalization can be computed from the two point function of $O^\dagger_{nlm}$ on the vacuum, requiring that $\frac{O^\dagger_{nlm}}{M_{nlm}} |\Omega\rangle$ has norm one.

where $M_{\omega\lambda}$ is the normalization such that $O_{\omega\lambda}/M_{\omega\lambda}\,|\Omega\rangle$ has norm one. The coefficients $\chi^{1,2}_{nlm,\omega\lambda}$ at least have to satisfy

$$\sum_{\omega\lambda}(\chi^{1}_{nlm;\omega\lambda}\chi^{*1}_{nlm;\omega\lambda}+\chi^{1}_{nlm;\omega\lambda}\chi^{*2}_{nlm;\omega\lambda}+\chi^{2}_{nlm;\omega\lambda}\chi^{*1}_{nlm;\omega\lambda}+\chi^{*2}_{nlm;\omega\lambda}\chi^{2}_{nlm;\omega\lambda})=1. \tag{3.62}$$

In the next section, one can see that the coefficients $\chi^{1,2}_{nlm,\omega\lambda}$ are indeed the bulk global to Rindler Bogolibov coefficients.

3.5.2 Bogoliubov coefficients from Rindler mode expansion of bulk field

Now we proceed with the second way of arriving at (3.61). At every Cauchy surface, the Hilbert space of a QFT is constructed as the Fock space obtained from creation and annihilation operators a_k and $a_k^\dagger$, corresponding to the global modes of the field operator which is

$$\varphi(t,x)=\sum_k f_k(t,x)a_k+f_k^*(t,x)a_k^\dagger. \tag{3.63}$$

k is a collection of indices we need to describe the mode. We can use the same approach to find the mode expansion of the field that lies in the region r by directly solving the equation of motion just in this region to find the appropriate wave functions which have support only on r. Let us take the time slice Σ and decompose it into the subregions Σ_r such that $\Sigma_r\cap\Sigma_{r'}=\varnothing$. For all Σ_r, we should first find a coordinate system U_r which cover $\mathcal{D}(\Sigma_r)$. Then, solve the equation of motion on U_r to find the mode expansion of fields on $\mathcal{D}(\Sigma_r)$

$$\varphi(t_r,x_r)=\sum_k f_k^r(t_r,x_r)a_k^r+f_k^{r*}(t_r,x_r)a_k^{r\dagger}. \tag{3.64}$$

The Hilbert space of the QFT restricted to Σ_r is denoted by $\mathcal{H}_r$ and the Hilbert space of the total theory on Σ is naively a tensor product of the subregion Hilbert spaces $\mathcal{H} = \otimes_r \mathcal{H}_r$.

One can expand the field $\varphi(X)$ in global coordinates in terms of subregion mode functions as

$$\varphi(t,x) = \sum_r \sum_k f_k^r(t_r, x_r)\, a_k^r + f_k^{r*}(t_r, x_r)\, a_k^{r\dagger}. \tag{3.65}$$

The point X is labeled in global coordinates and the coordinate system U_r by (t,x) and (t_r, x_r), respectively. As a result, the creation and annihilation operators of the full Hilbert space can be written as a linear combination of subregions mode functions and vice versa, by comparing (3.63) and (3.65) which is a generalized version of Bogoliubov transformation [Kim, 2017].

Let us come back to our problem. To proceed in the Petz map calculation, it can help us if we could find an expression for O_{nlm} in terms of the mode function corresponding to the subregions A and $\bar{A}$. The subtlety here is the point that GFF on the boundary do not obey the equation of motion and hence, the discussion above is not applied to the boundary QFT. However, in AdS/CFT correspondence, the extrapolate dictionary leads us to the identification between some bulk and boundary operators. As a result, we expect that bulk Bogoliubov transformation can help us to find one expression for O_{nlm} as a linear combination of the operators has support only on one subregion.

The boundary Cauchy slice Σ is divided into two hemispheres A and $\bar{A}$. As a result, their entanglement wedges are AdS-Rindler patches in the bulk which both together cover the entire AdS space. To quantize the free fields in the entire AdS space in Rindler coordinates, we need two copies of the creation and annihilation operators that obey the commutation relation

$$[b_{\omega\lambda,I}, b^{\dagger}_{\omega'\lambda',I'}] = (2\pi)^2\delta(\omega-\omega')\delta(\lambda-\lambda')\delta_{II'} \tag{3.66}$$

where the mode functions $b_{\omega\lambda,a}$ and $b_{\omega\lambda,\bar{a}}$ have support only in a and $\bar{a}$ patches respectively.

One can globally expand the bulk field $\varphi(X)$ in terms of these mode functions as

$$\varphi(X) = \sum_{I\in\{a,\bar{a}\}} \int \frac{d\omega}{2\pi}\frac{d\lambda}{2\pi} \left(f_{\omega\lambda,I}(X) b_{\omega\lambda,I} + f^{*}_{\omega\lambda,I}(X) b^{\dagger}_{\omega\lambda,I}\right) \tag{3.67}$$

where $f_{\omega\lambda,I}(X)$ is given by (3.142) if the point X belongs to the patch I, otherwise it vanishes. The global mode a_{nlm} in AdS are related to these mode functions by Bogoliubov coefficients α and β as

$$a_{nlm} = \sum_{I\in\{a,\bar{a}\}} \int d\omega d\lambda \left(\alpha^{I}_{nlm;\omega\lambda} b_{\omega\lambda,I} + \beta^{*I}_{nlm;\omega\lambda} b^{\dagger}_{\omega\lambda,I}\right). \tag{3.68}$$

The commutation relations (3.66) lead to the following constrain on the Bogoliubov coefficients

$$\sum_{I\in\{a,\bar{a}\}} \int d\omega d\lambda \left(\alpha^{I}_{nlm;\omega\lambda}\alpha^{*I}_{n'l'm';\omega'\lambda'} - \beta^{*I}_{nlm;\omega\lambda}\beta^{I}_{n'l'm';\omega'\lambda'}\right) = \delta_{nn'}\delta_{ll'}\delta_{mm'}. \tag{3.69}$$

We can substitute (3.68) in the bulk global mode expansion (3.7) which lead us to the relations

$$\begin{aligned} &\sum_{nlm} f_{nlm}(X)\alpha^{a}_{nlm;\omega\lambda} + f^{*}_{nlm}(X)\beta^{a}_{nlm;\omega\lambda} = 0 \qquad \forall X \in \bar{a} \\ &\sum_{nlm} f_{nlm}(X)\alpha^{\bar{a}}_{nlm;\omega\lambda} + f^{*}_{nlm}(X)\beta^{\bar{a}}_{nlm;\omega\lambda} = 0 \qquad \forall X \in a. \end{aligned} \tag{3.70}$$

We will use them in what follows.

For the case that we are studying, where on the boundary of pure AdS the GFF lives, the mode functions a_{nlm} and $b_{\omega\lambda}$ are identified with the boundary operators given by (3.13) and (3.146) respectively. By plugging them back into (3.68), one can find

$$O_{nlm} = \sum_{I\in\{A,\bar{A}\}} \int d\omega d\lambda \Big(\frac{M_{nlm}}{M_{\omega\lambda}} \alpha^{I}_{nlm;\omega\lambda} O_{\omega\lambda,I} + \frac{M_{nlm}}{M_{\omega\lambda}} \beta^{*I}_{nlm;\omega\lambda} O^{\dagger}_{\omega\lambda,I} \Big). \tag{3.71}$$

while $\alpha^{A}_{nlm;\omega\lambda} = \alpha^{a}_{nlm;\omega\lambda}$, etc .

3.5.3 Appropriate basis for the code subspace

The code subspace has a Fock space structure $\mathcal{H}_{code} = span\{\prod_{nlm}(O^{\dagger}_{nlm})^{i_{nlm}} |\Omega\rangle\}$, where $|\Omega\rangle$ is the global vacuum defined as $O_{nlm}|\Omega\rangle = 0$ for all n, l and m. The powers i_{nlm} are some non-negative integers and we can also put a cut-off on them. In order to compute the Petz map reconstruction of the bulk field $\varphi(X)$ that lies in the AdS-Rindler patch, we need to compute the terms $\mathrm{tr}_{\bar{A}} P_{code}$ and $\mathrm{tr}_{\bar{A}}(P_{code} O_{nlm} P_{code})$.

Before going through the calculation, we need to choose a basis for code subspace. The natural choice one can take is

$$|\{i_{nlm}\}\rangle \propto \prod_{nlm} (O^{\dagger}_{nlm})^{i_{nlm}} |\Omega\rangle \,. \tag{3.72}$$

In this basis, we should calculate the terms of the form

$$\mathrm{tr}_{\bar{A}}\Big((O^{\dagger}_{nlm})^{i} |\Omega\rangle \langle\Omega| (O_{n'l'm'})^{i'} \Big) \tag{3.73}$$

for every arbitrary integers i and i'. One way to deal with trace can be using (3.71). As we know, the action of Rindler modes on $|\Omega\rangle$ in two wedges are related to each other by

$$\begin{aligned} O_{\omega,\lambda;\bar{A}} |\Omega\rangle &= e^{\pi\omega} O^{\dagger}_{\omega,-\lambda;A} |\Omega\rangle \\ O^{\dagger}_{\omega,\lambda;\bar{A}} |\Omega\rangle &= e^{-\pi\omega} O_{\omega,-\lambda;A} |\Omega\rangle \end{aligned} \tag{3.74}$$

while $|\Omega\rangle = \otimes_{\omega,\lambda} |\Omega_{\omega,\lambda}\rangle = \otimes_{\omega,\lambda}\sqrt{1-e^{-2\pi\omega}}\sum_n e^{-\pi\omega n} |n\rangle^{A}_{\omega,\lambda} |\bar{n}\rangle^{\bar{A}}_{\omega,-\lambda}$. As a result, for each choice of i, one can in principle find an operator A_i that has support only on region A such that

$$(O^{\dagger}_{nlm})^i |\Omega\rangle = A_i |\Omega\rangle\,. \tag{3.75}$$

Therefore, (3.73) can be simplified as

$$A_i \mathrm{tr}_{\bar{A}}\big(|\Omega\rangle \langle\Omega| \big) A_i^{\dagger} = A_i \rho_A^{(0)} A_i^{\dagger} \tag{3.76}$$

that is an operator has support only on A, while $\rho_A^{(0)}$ is a thermal density matrix in the region A. Nevertheless, the equation (3.75) is somewhat abstract, and indeed finding an expression for A_i can be difficult.

To find a more convenient basis for the code subspace we can use the *Reeh-Schlieder* theorem for relativistic QFT. Consider a QFT in Minkowski spacetime $\mathcal{M}$ with a Hilbert space $\mathcal{H}$ and the vacuum state denoted by $|\Omega\rangle \in \mathcal{H}$. For a small open set $\mathcal{U} \subset \mathcal{M}$, there is a bounded algebra of local operators $\mathcal{A}_{\mathcal{U}}$ supported in $\mathcal{U}$. The Reeh-Schlider theorem says that every arbitrary state in $\mathcal{H}$ can be approximated by $\mathcal{A}_{\mathcal{U}} |\Omega\rangle$ that means states created by applying elements of the local algebra to the vacuum are not localized to the region $\mathcal{U}$. In other words, the vacuum is a cyclic and separating vector for the field algebra corresponding to any open set $\mathcal{U}$ in Minkowski spacetime. This is the key point in our work that causes the manageability of the Petz map calculation.

We can construct the code subspace using the Reeh-Schlieder theorem to the boundary QFT by acting on the global vacuum with the operator algebra on region A, $\mathcal{H}_{GFF} = \{\mathcal{L}(\mathcal{H}_A)\,|\Omega\rangle\}$. Since one choice of basis for the operator algebra on A is the set of Rindler modes $O_{\omega\lambda;A}$ and $O^\dagger_{\omega\lambda;A}$, one can take a basis for code subspace at large N as

$$|\{j_{\omega,\lambda},\Delta_{\omega,\lambda}\}\rangle = \prod_{\omega,\lambda}(O_{\omega\lambda;A})^{j_{\omega,\lambda}}(O^\dagger_{\omega\lambda;A})^{j_{\omega,\lambda}+\Delta_{\omega,\lambda}}\,|\Omega\rangle \tag{3.77}$$

where $j \in \mathbb{N}$ and $\Delta \in \mathbb{Z}$. [8] As the theory is free, different modes are decoupled and we can rewrite the code subspace basis as

$$|\{j_{\omega,\lambda},\Delta_{\omega,\lambda}\}\rangle = \otimes_{\omega,\lambda}\,|j_{\omega,\lambda},\Delta_{\omega,\lambda}\rangle = \otimes_{\omega,\lambda}(O_{\omega\lambda;A})^{j_{\omega,\lambda}}(O^\dagger_{\omega\lambda;A})^{j_{\omega,\lambda}+\Delta_{\omega,\lambda}}\,|\Omega_{\omega,\lambda}\rangle \tag{3.78}$$

In the following, for simplicity we will just focus on a single mode which the corresponding Hilbert space is $span\{|j,\Delta\rangle = (O_A)^j(O_A^\dagger)^{j+\Delta}\,|\Omega\rangle\}$. In the new basis, instead of (3.73), we should calculate the terms $\mathrm{tr}_{\bar{A}}\,|j,\Delta\rangle\,\langle j',\Delta'|$ which one can simply find as

$$\mathrm{tr}_{\bar{A}}\,|j,\Delta\rangle\,\langle j',\Delta'| = (O_A)^j(O_A^\dagger)^{j+\Delta}\rho_A^{(0)}(O_A)^{j'+\Delta'}(O_A^\dagger)^{j'} \tag{3.79}$$

We should be careful here that although this set of vectors spans the GFF sector of the

[8] More precise statement here is that, since the representation of the vacuum state in terms of the Rindle modes is cyclic and separating with respect to the operator algebra of the Rindler wedge, the vacuum sector of the Hilbert space is isomorphic to the GNS Hilbert space of the operator algebra of the Rindler wedge over the vacuum.

boundary, they are not orthonormal as we have

$$\langle j, \Delta \, | j', \Delta' \rangle = \delta_{\Delta,\Delta'}(1 - e^{-2\pi\omega}) \sum_{n=max\{0,-\Delta\}} e^{-2\omega n} \sqrt{\frac{(n+j+\Delta)!}{(n+\Delta)!}} \sqrt{\frac{(n+j'+\Delta')!}{(n+\Delta')!}} \quad (3.80)$$

which is proportional to $\delta_{\Delta,\Delta'}$ not $\delta_{j,j'}\delta_{\Delta,\Delta'}$. Nevertheless, we can still use this set of vectors as a basis for the code subspace by considering the correct form of the projection on a non-orthonormal basis.

Consider a vector space $V = span\{|v_i\rangle\}$. One can construct the metric tensor for this basis $G = [g_{ij}]$ that by definition $g_{ij} = \langle v_i \, | v_j \rangle$. The inverse metric $G^{-1} = [g^{ij}]$ is defined to be the inverse of the matrix G, so the relations

$$\sum_j g^{ij} g_{jk} = \delta^i_k, \qquad \sum_j g_{ij} g^{jk} = \delta^k_i \quad (3.81)$$

should satisfy and the projection on the subspace $V_I = span\{|v_i\rangle \, , i \in I\}$ is given by

$$P_I = \sum_{i,j \in I} g^{ij} \, |v_i\rangle \, \langle v_j| \, . \quad (3.82)$$

3.5.4 AdS-Rindler wedge reconstruction via Petz map

Now we have all relations we need to find the Petz reconstruction for the fields in the AdS-Rindler patch in the bulk. By plugging (3.71) back into (3.58), we arrive to

$$\Phi_A(X) = \sum_{I\in\{A,\bar{A}\}} \int d\omega d\lambda\, \mathcal{F}_{\omega,\lambda;I}(X)\, \tau_A^{-1/2} \mathrm{tr}_{\bar{A}}(P_{code} O_{\omega\lambda,I} P_{code}) \tau_A^{-1/2}$$
$$+ \mathcal{F}^*_{\omega,\lambda;I}(X)\, \tau_A^{-1/2} \mathrm{tr}_{\bar{A}}(P_{code} O^\dagger_{\omega\lambda,I} P_{code}) \tau_A^{-1/2} \quad (3.83)$$

while

$$\mathcal{F}_{\omega,\lambda;I}(X) = \sum_{nlm} \frac{M_{nlm}}{M_{\omega\lambda}} (G_{nlm}(X)\alpha^I_{nlm;\omega\lambda} + G^*_{nlm}(X)\beta^I_{nlm;\omega\lambda}). \quad (3.84)$$

By comparing the global mode expansion of $\Phi_{HKLL}(X)$ with $\varphi(X)$, one can find that $G_{nlm}(X) = \frac{1}{M_{nlm}} f_{nlm}(X)$. If we substitute it in (3.84), we can find that

$$\mathcal{F}_{\omega,\lambda;I}(X) = \frac{1}{M_{\omega\lambda}} \sum_{nlm} (f_{nlm}(X)\alpha^I_{nlm;\omega\lambda} + f^*_{nlm}(X)\beta^I_{nlm;\omega\lambda}) \quad (3.85)$$

As $\varphi(X)$ lies in the AdS-Rindler wedge homologous to the region A, by using the relations (3.70), we find that $\mathcal{F}_{\omega,\lambda;\bar{A}}(X) = 0$ for all $X \in \mathcal{E}_A$. Therefore, the Petz reconstruction of $\varphi(X)$ gets simplified as

$$\Phi_A(X) = \int d\omega d\lambda\, \mathcal{F}_{\omega,\lambda;A}(X)\, \tau_A^{-1/2} \mathrm{tr}_{\bar{A}}(P_{code} O_{\omega\lambda,A} P_{code}) \tau_A^{-1/2}$$
$$+ \mathcal{F}^*_{\omega,\lambda;A}(X)\, \tau_A^{-1/2} \mathrm{tr}_{\bar{A}}(P_{code} O^\dagger_{\omega\lambda,A} P_{code}) \tau_A^{-1/2}. \quad (3.86)$$

In our basis the projection to the code subspace is

$$P_{code} = \sum_{j,j'} \sum_{\Delta,\Delta'} g^{(j,\Delta);(j',\Delta')} \left|j, \Delta\right\rangle \left\langle j', \Delta'\right| . \tag{3.87}$$

From the inner product between $\{|j, \Delta\rangle\}$, we see that the metric tensor here is block-diagonal while each block labeled by Δ

$$G = \oplus_{\Delta} G_{\Delta} = \oplus_{\Delta}[g_{j,j';\Delta}] \tag{3.88}$$

where $g_{j,j';\Delta} = \langle j, \Delta | j', \Delta\rangle$. As a result, the inverse metric should have the form of

$$G^{-1} = \oplus_{\Delta} G_{\Delta}^{-1} = \oplus_{\Delta}[A^{\Delta}_{j,j'}] \tag{3.89}$$

for some unknown elements $A^{\Delta}_{j,j'}$ which should satisfy the relations below

$$\begin{aligned} \sum_{j'} A^{\Delta}_{j,j'} \langle j', \Delta | j'', \Delta\rangle &= \delta_{j,j''} \\ \sum_{j'} \langle j, \Delta | j', \Delta\rangle A^{\Delta}_{j',j''} &= \delta_{j,j''}. \end{aligned} \tag{3.90}$$

Since $g^{(j,\Delta);(j',\Delta')} = A^{\Delta}_{j,j'} \delta_{\Delta,\Delta'}$, we can write the projection on the code subspace in terms of $A^{\Delta}_{j,j'}$ as

$$P_{code} = \sum_{\Delta} \sum_{j,j'} A^{\Delta}_{j,j'} \left|j, \Delta\right\rangle \left\langle j', \Delta\right| . \tag{3.91}$$

Now, we can use the form of the code subspace projection to find the three terms we need to find the Petz reconstruction of $\varphi(X)$. First, we start with τ_A which is

$$\tau_A = \mathrm{tr}_{\bar{A}} P_{code} = \sum_{\Delta} \sum_{j,j'} A^{\Delta}_{j,j'} \mathrm{tr}_{\bar{A}} |j, \Delta\rangle \langle j', \Delta|$$

$$= \sum_{\Delta} \sum_{j,j'} A^{\Delta}_{j,j'} (O_A)^j (O_A^\dagger)^{j+\Delta} \rho_A^{(0)} (O_A)^{j'+\Delta'} (O_A^\dagger)^{j'}. \quad (3.92)$$

We also need to calculate the terms in the form of

$$\mathrm{tr}_{\bar{A}}\left(P_{code} O P_{code}\right) = \sum_{\Delta,\Delta'} \sum_{j,j'} \sum_{k,k'} A^{\Delta}_{j,k} A^{\Delta'}_{k',j'} \langle k, \Delta| O |k', \Delta'\rangle \, \mathrm{tr}_{\bar{A}} |j, \Delta\rangle \langle j', \Delta'| , \quad (3.93)$$

For O that is O_A or $O_A^\dagger$, we get

$$\begin{aligned} \langle k, \Delta| O_A |k', \Delta'\rangle &= \langle k, \Delta| k' + 1, \Delta' - 1\rangle \\ \langle k, \Delta| O_A^\dagger |k', \Delta'\rangle &= \langle k + 1, \Delta - 1| k', \Delta'\rangle. \end{aligned} \quad (3.94)$$

By using the relations (3.90) and (3.94), one can find that

$$\begin{aligned} \mathrm{tr}_{\bar{A}}\left(P_{code} O_{\omega,\lambda;A} P_{code}\right) &= O_{\omega,\lambda;A} \tau_A \\ \mathrm{tr}_{\bar{A}}\left(P_{code} O^\dagger_{\omega,\lambda;A} P_{code}\right) &= \tau_A O^\dagger_{\omega,\lambda;A}. \end{aligned} \quad (3.95)$$

The operators $O_{\omega,\lambda;A}$ and $O^\dagger_{\omega,\lambda;A}$ have support only on region A and commute with every operator $X_{\bar{A}}$. We can show that here, it is equivalent to say that τ_A^{-1} commute with the $\mathrm{tr}_{\bar{A}}(P_{code} O P_{code})$ for O being $O_{\omega,\lambda;A}$ and $O^\dagger_{\omega,\lambda;A}$. One can conclude that if they commute with

τ_A^{-1}, they commute with $\tau_A^{-1/2}$ as well. Therefore, we reach

$$\begin{aligned}
\tau_A^{-1/2}\mathrm{tr}_{\bar{A}}\left(P_{code}O_{\omega,\lambda;A}P_{code}\right)\tau_A^{-1/2} =& \mathrm{tr}_{\bar{A}}\left(P_{code}O_{\omega,\lambda;A}P_{code}\right)\tau_A^{-1} = O_{\omega,\lambda;A} \\
\tau_A^{-1/2}\mathrm{tr}_{\bar{A}}\left(P_{code}O^\dagger_{\omega,\lambda;A}P_{code}\right)\tau_A^{-1/2} =& \tau_A^{-1}\mathrm{tr}_{\bar{A}}\left(P_{code}O^\dagger_{\omega,\lambda;A}P_{code}\right) = O^\dagger_{\omega,\lambda;A}.
\end{aligned} \tag{3.96}$$

Finally, we find the Petz reconstruction of the bulk field $\varphi(X)$ in the AdS-Rindler wedge as

$$\Phi_A(X) = \int d\omega d\lambda \left(\mathcal{F}_{\omega,\lambda;A}(X)\, O_{\omega\lambda,A} + \mathcal{F}^*_{\omega,\lambda;A}(X)\, O^\dagger_{\omega\lambda,A}\right). \tag{3.97}$$

By substituting (3.145) in (3.97), we will arrive at

$$\Phi_A(X) = \int d\tau dx\, K_{Petz,A}(X|\tau,x)\, O(\tau,x) \tag{3.98}$$

where the smearing function is

$$\begin{aligned}
K_{Petz,A}(X|\tau,x) &= \int d\omega d\lambda\, \mathcal{F}_{\omega,\lambda;A}(X)e^{i\omega\tau}Y^*_\lambda(x) \\
&= \int d\omega d\lambda\, e^{i\omega\tau}Y^*_\lambda(x)\,\frac{1}{M_{\omega,\lambda}}\sum_{nlm}\left(f_{nlm}(X)\alpha^a_{nlm;\omega\lambda} + f^*_{nlm}(X)\beta^a_{nlm;\omega\lambda}\right) \\
&= \int \frac{d\omega}{2\pi}\frac{d\lambda}{2\pi}\,\frac{1}{M_{\omega,\lambda}}f_{\omega\lambda,A}(X)e^{i\omega\tau}Y^*_\lambda(x).
\end{aligned} \tag{3.99}$$

By comparing with (3.148), we see that the result one can find by applying the Petz map in an AdS-Rindler patch is exactly the same as the result of the HKLL procedure in the AdS-Rindler coordinate.

3.6 Entanglement wedge reconstruction and Petz map

In the previous chapter, we used the Petz map to find the CFT reconstruction of a bulk field in the AdS-Rindler wedge. In principle, this approach can be used to reconstruct the entanglement wedge of any region on the boundary explicitly. Let us consider CFT$_d$ in a semi-classical state $|\Psi\rangle$ which is dual to a smooth asymptotically AdS spacetime $\mathcal{M}$. We also assume that there is no black hole in the bulk. Consider a Cauchy surface Σ of the boundary and divide it into an arbitrary region A and its complementary part $\bar{A}$. In the rest, we focus on finding the reconstruction of the entanglement wedge of A via the Petz map.

In the bulk, one can find the global mode expansion of the field φ as

$$\varphi(X) = \sum_n \left(f_n(X)\, a_n + f_n^*(X)\, a_n^\dagger \right) \tag{3.100}$$

where $f_n(X)$ is the solution of the Klein-Gordon equation on $\mathcal{M}$ and a_n is the mode corresponding to it that obeys the usual canonical commutation relations. All the labels needed to define the modes are shown collectively by n. By applying the HKLL method to an appropriate coordinate system that covers the entire bulk, labeled here by (r, t, x), one can find that

$$\Phi_{HKLL}(X) = \int_{bdy} dtdx\, K^g_{\partial\mathcal{M}}(X|t,x) O(t,x) \tag{3.101}$$

where $K^g_{\partial\mathcal{M}}(X|t,x)$ is the global smearing function. As in the AdS-Rindler case, it is convenient to go to the Fourier modes where the single trace primaries have a mode expansion as

$$O(t,x) = \sum_n \left(\tilde{g}_n(t,x)\, \hat{O}_n + \tilde{g}_n^*(t,x)\, \hat{O}_n^\dagger \right) \tag{3.102}$$

where

$$\hat{O}_n = \frac{1}{M_n} \int dtdx\, O(t,x)\, g_n^*(t,x). \tag{3.103}$$

If we choose $\hat{O}_n$ with standard commutation relation, i.e. identified it with a_n, from extrapolate dictionary, we have $\tilde{g}_n(t,x) = \lim_{r\to\infty} r^\Delta f_n(r,x)$, where M_n and g_n are defined as in Sec. 3.2.1. Therefore

$$\Phi_{HKLL}(X) = \sum_n \left(G_{n,\mathcal{M}}(X)\, \hat{O}_n + G_{n,\mathcal{M}}^*(X)\, \hat{O}_n^\dagger \right) \tag{3.104}$$

while

$$G_{n,\mathcal{M}}(X) = \int_{bdy} dtdx\, K^g_{\partial\mathcal{M}}(X|t,x)\tilde{g}_n(t,x). \tag{3.105}$$

By comparing (3.100) and (3.104), one can find that $G_{n,\mathcal{M}}(X) = f_n(X)$.

Let us choose a basis for the operator algebra of the regions A and $\bar{A}$ which we denote them by $\{A_\nu\}$ and $\{\bar{A}_\nu\}$ respectively. In order to find the Petz reconstruction of $\varphi(X)$, we need to write the mode functions $\hat{O}_n$ as a linear combination of $\{A_\nu\}$ and $\{\bar{A}_\nu\}$. If it is in the form of

$$\hat{O}_n = \sum_\nu \alpha^A_{n,\nu}\, A_\nu + \alpha^{\bar{A}}_{n,\nu}\, \bar{A}_\nu. \tag{3.106}$$

The Petz reconstruction of $\varphi(X)$ arrives to

$$\Phi_A(X) = \sum_\nu \mathcal{F}^A_\nu(X) \tau_A^{-1/2} \mathrm{tr}_{\bar{A}}\left(P_{code} A_\nu P_{code}\right) \tau_A^{-1/2} + \mathcal{F}^{\bar{A}*}_\nu(X) \tau_A^{-1/2} \mathrm{tr}_{\bar{A}}\left(P_{code} \bar{A}_\nu P_{code}\right) \tau_A^{-1/2} \tag{3.107}$$

where

$$\mathcal{F}^I_\nu(X) = \sum_n f_n(X)\, \alpha^I_{n,\nu} + f^*_n(X)\, \alpha^{I*}_{n,-\nu} \tag{3.108}$$

for $I \in \{A, \bar{A}\}$.

For a generic choice of basis, the coefficients behind $\tau_A^{-1/2} \mathrm{tr}_{\bar{A}}(P_{code}\bar{A}_\nu P_{code})\tau_A^{-1/2}$ does not vanish like the case of AdS-Rindler in the previous chapter. Hence, we also need to calculate these terms here. Moreover, for a generic case of the basis of the operator algebra, the sets of $\{A_\nu\}$ and $\{\bar{A}_\nu\}$ do not have in general a simple bulk dual, and therefore, we can not find the Bogoliubov coefficients in (3.106) from the bulk theory.

For now, let us assume that we can somehow find the Bogoliubov coefficients in (3.106). Then in order to proceed, similar to the AdS-Rindler case, we can use the Reeh-Schlieder theorem and write the code subspace as

$$\mathcal{H}_{code} = \mathcal{L}(\mathcal{H}_A)\, |\Psi\rangle\, . \tag{3.109}$$

In principle, to find the Petz reconstruction in (3.107), it is needed to know the commutation relation between the operator algebras of the regions A, and rewrite the action of the operator $\bar{A}_\nu$ on the state $|\Psi\rangle$ in terms of the operators in region A on the state, *i.e.*, finding the operator $O_{A,\nu}$ as a function of $\{A_\nu\}$ such that

$$\bar{A}_\nu\, |\Psi\rangle = O_{A,\nu}\, |\Psi\rangle\, . \tag{3.110}$$

But practically, whether or not we can explicitly compute all the terms in (3.107) depends on the basis we take, and for an appropriate choice of it, we will be able to find an explicit expression for the operator $\Phi_A(X)$.

In the following, we will describe an appropriate choice of the sets $\{A_\nu\}$ and $\{\bar{A}_\nu\}$ that by using them, the calculation becomes attainable. We will see that there is a clever choice of basis, the eigenfunctions of the modular Hamiltonian, that the Petz calculation will get drastically simplified.

3.6.1 Petz map and modular flow

In this section, we will focus on a special choice of basis for operator algebra in the region A and $\bar{A}$ that is the eigenfunctions of modular Hamiltonian of the regions.

The modular Hamiltonian of a given region R is defined as $K_R = -\log \rho_R$ where ρ_R is the reduced density matrix of the region R. K_R generates an automorphism for the operator algebra $\mathcal{A}_R$ associated to ρ_R [Haag, 1992a] as

$$A \in \mathcal{A}_R \quad \longrightarrow \quad A_s = e^{iK_R s} A e^{-iK_R s} \in \mathcal{A}_R. \tag{3.111}$$

called *modular flow*. The modular flow originally introduced in the context of the algebraic QFT [Takesaki, 1970, Haag, 1992a, Bratteli & Robinson, 2012, Borchers, 2000, Takesaki et al., 2003] which recently played a key role in using the concepts of quantum information theory in QFT and gravity [Lashkari, 2019, Lashkari, 2021, Sárosi & Ugajin, 2018, Blanco et al., 2018, Casini, 2008, Blanco & Casini, 2013, Faulkner et al., 2016, Balakrishnan et al., 2019, Ceyhan & Faulkner, 2020, Lashkari, 2016, Lashkari et al., 2021, Cardy & Tonni, 2016, Casini et al., 2011, Blanco & Casini, 2013, Jafferis & Suh, 2016, Koeller et al., 2018, Czech et al., 2017, Chen et al., 2018, Belin et al., 2018, Abt & Erdmenger, 2018, Faulkner et al., 2019, Czech et al., 2019, De Boer & Lamprou, 2020, Arias et al., 2020, Erdmenger et al., 2020].

In modular Fourier space, the Fourier transformation of A_s is

$$A_\omega = \int_{-\infty}^{\infty} ds\, e^{is\omega} e^{iK_R s} A e^{-iK_R s}. \tag{3.112}$$

The operators A_ω are the eigenfunctions of modular Hamiltonian $[K_R, A_\omega] = \omega A_\omega$. They also form a basis for operator algebra on region R. Therefore, we can take the eigenfunctions of the modular Hamiltonian of the both regions A and $\bar{A}$ as the basis for the corresponding operator algebras on these regions.

Moreover, as we assume that there is no black hole in the bulk, the entanglement wedge of the complementary part of A in the boundary is the same region as the complementary part of the entanglement wedge of the region A and hence, the union of a and $\bar{a}$ covers the entire Cauchy surface. As a result, we can expand both the bulk and boundary global modes as a linear combination of the modular eigenbasis as

$$\begin{aligned} a_n &= \sum_\omega \alpha^a_{n,\omega} A^a_\omega + \alpha^{\bar{a}}_{n,\omega} A^{\bar{a}}_\omega \\ \hat{O}_n &= \sum_\omega \alpha^A_{n,\omega} A^A_\omega + \alpha^{\bar{A}}_{n,\omega} A^{\bar{A}}_\omega . \end{aligned} \tag{3.113}$$

In such a case, we can use the JLMS statement that relates the modular Hamiltonian of a given boundary region A to the modular Hamiltonian of its entanglement wedge a as

$$K_A = K_a + \frac{Area}{4G} + O(1/N). \tag{3.114}$$

Since the area term in the right hand side of (3.114) is proportional to the identity, both

$K_{A(\bar{A})}$ and $K_{a(\bar{a})}$ have the same spectrum and we can identify their eigenfunctions as

$$\begin{aligned} A_\omega^A &= A_\omega^a \equiv A_\omega \\ A_\omega^{\bar{A}} &= A_\omega^{\bar{a}} \equiv \bar{A}_\omega. \end{aligned} \tag{3.115}$$

Therefore as $\hat{O}_n = a_n$, by comparing (3.113) and (3.115), we see that both $\hat{O}_n$ and a_n have the same Bogoliubov coefficients $\alpha_{n,\omega}^{A(\bar{A})} = \alpha_{n,\omega}^{a(\bar{a})}$. One can replace it into (3.108) and find that when we take the eigenfunctions of modular Hamiltonian as the basis of operator algebra of subregions, by definition

$$\mathcal{F}_\omega^{\bar{A}}(X) = \sum_n f_n(X)\alpha_{n,\omega}^{\bar{a}} + f_n^*(X)\alpha_{n,-\omega}^{\bar{a}*} = 0, \qquad \forall X \in a. \tag{3.116}$$

Therefore, the Petz reconstruction of $\varphi(X)$ in the entanglement wedge of the region A can be read off as

$$\Phi_A(X) = \sum_\omega \mathcal{F}_\omega^A(X)\, \tau_A^{-1/2} \mathrm{tr}_{\bar{A}}\left(P_{code} A_\omega P_{code}\right) \tau_A^{-1/2} = \sum_\omega \mathcal{F}_\omega^A(X)\, A_\omega \tag{3.117}$$

where A_ω is the eigenfunction of K_A and $\mathcal{F}_\omega^A(X)$ is given by

$$\mathcal{F}_\omega^A(X) = \sum_n \left(f_n(X)\alpha_{n,\omega}^a + f_n^*(X)\alpha^{a*} \right) \tag{3.118}$$

which by using (3.105), it can be rewritten in terms of the global smearing function as

$$\mathcal{F}_\omega^A(X) = \int_{bdy} dtdx\, K_{\partial\mathcal{M}}^g(X|t,x) \sum_n \left(\tilde{g}_n(t,x)\alpha_{n,\omega}^A + \tilde{g}_n^*(t,x)\alpha_{n,-\omega}^{A*} \right). \tag{3.119}$$

At this point to write the operator Φ_A more precisely, we should know more about the A_ω themselves. To leading order in AdS/CFT, the bulk field consists of free fields. For free scalar fields on any region R that all correlators are fixed by the two-point function, the density matrix is Gaussian and the modular Hamiltonian is bilinear. Its eigenfunctions can be labeled by ω and X_S where the coordinates X_S corresponds to a codimension 2 surface $S \in R$ on one Cauchy slice [Faulkner & Lewkowycz, 2017b]. Therefore, we have

$$[K_R, \Phi_\omega(X_S)] = \omega \Phi_\omega(X_S) \tag{3.120}$$

where

$$\Phi_\omega(X_S) = \int ds\, e^{is\omega} e^{iK_R s} \varphi(X_S) e^{-iK_R s} \qquad \forall X_S \in S. \tag{3.121}$$

Now, let us consider the free scalar field in the entanglement wedge of the region A and the Cauchy surface as the slice of bulk that intersects with A. One clever choice for S can be A itself. By using (3.114) and the identification on the boundary $\varphi_0(x_A) = O(x_A)$, we can find the modular eigenfunction of K_A as

$$O_\omega(x_A) = \int ds\, e^{is\omega} e^{iK_A s} O(x_A) e^{-iK_A s} \qquad \forall x_A \in A. \tag{3.122}$$

By substituting it in (3.117), we find the Petz reconstruction of $\varphi(X)$ lies in the entanglement wedge of A as

$$\Phi_A(X) = \int_{-\infty}^{\infty} ds \int_A dx_A\, K_{Petz,A}(X|x_A, s)\, e^{iK_A s} O(x_A) e^{-iK_A s} \tag{3.123}$$

where the smearing function is given by

$$K_{Petz,A}(X|x_A,s) = \sum_n \int d\omega\, e^{is\omega}\left(f_n(X)\alpha_n^A(\omega,x_A) + f_n^*(X)\alpha_n^{A*}(-\omega,x_A)\right) \tag{3.124}$$

while $\alpha_n^A(\omega,x_A)$ is the Bogoliubov coefficient between $\hat{O}_n$ and $O_\omega(x_A)$. As we mentioned in (3.2.2), the equation (3.123) has been conjectured in [Jafferis et al., 2016b], and also derived in [Faulkner & Lewkowycz, 2017b] through acting with the modular flow on the extrapolate dictionary. Here, we could again obtain it by using the Petz map formula which is a more generic approach.

As a consistency check, let us calculate (3.124) for the AdS-Rindler wedge. In this patch, the modular parameter is just the Rindler time τ and the modular Hamiltonian is the Rindler Hamiltonian H_τ. To find the smearing function in (3.124), we need to find the Bogoliubov coefficients between $\hat{O}_{nlm}$ and $O_\omega(x_A) = \int d\tau e^{i\omega\tau}O(\tau,x_A)$ which is

$$\begin{aligned}
\alpha^A_{nlm}(\omega,x_A) &= \int d\lambda \frac{1}{M_{\omega,\lambda}} Y_\lambda^*(x_A)\alpha_{nlm;\omega,\lambda} \qquad \forall\omega\geq 0\\
\alpha^A_{nlm}(\omega,x_A) &= \int d\lambda \frac{1}{M_{\omega,\lambda}} Y_\lambda^*(x_A)\beta^*_{nlm;\omega,\lambda} \qquad \forall\omega< 0
\end{aligned} \tag{3.125}$$

By plugging it into (3.124), we get

$$\begin{aligned}
&K_{Petz,A}(X|x_A,\tau)\\
&\quad= \sum_{nlm}\int d\omega e^{i\omega\tau}\int d\lambda\frac{1}{M_{\omega,\lambda}}Y^*(x_A)\left(f_{nlm}(X)\alpha^A_{nlm;\omega,\lambda} + f^*_{nlm}(X)\beta^A_{nlm;\omega,\lambda}\right)\\
&\quad= \int\frac{d\omega}{2\pi}\frac{d\lambda}{2\pi}\frac{1}{M_{\omega,\lambda}}f_{\omega\lambda,A}(X)e^{i\omega\tau}Y_\lambda^*(x)
\end{aligned} \tag{3.126}$$

which is exactly the smearing function that we know from AdS-Rindler wedge reconstruction.

As illustrated, to reconstruct the operator in the interior of the entanglement wedge, we need to learn more about the modular Hamiltonian of general regions in QFTs.

3.7 Discussion

The discussion of EWR in the last section is generic and applies to any desired region on the boundary. In particular, the region can even be disconnected. For example, let us consider the union of two disjoint intervals $A = A_L \cup A_R$, Fig. 3.1, on a Cauchy slice of a 2d holographic CFT dual to AdS_3 in the bulk. If the regions $A_L,\ A_R$ are sufficiently small, the entanglement wedge of A is union of the entanglement wedges of A_L and A_R, denoted by a_L and a_R, individually, i.e. the union of two AdS-Rindler wedges. It is well known [Almheiri et al., 2015b, Headrick, 2010] that as we increase the size of the region A, the extremal surface changes discontinuously and in the new configuration the entanglement wedge of A becomes larger and in particular larger than the causal wedge of A.

An important question is understanding the nature of observables in the region which is in the entanglement wedge, but not the causal wedge. The Petz formula gives in principle a CFT representation of these observables, but their microscopic nature is not understood. To make the question more precise, notice that from the point of view of the bulk there is a well defined Bogoliubov transformation between the bulk global modes and the modes in regions $a_L,\ a_R,\ b_U,\ b_D,\ \mathcal{E}_M$ (see Fig. 3.1) . The modes in $a_R,\ a_L,\ b_U,\ b_D$ can be related to modes of single trace operators in the corresponding boundary regions $A_L,\ A_R,\ B_U,\ B_D$. Entanglement wedge reconstruction and the Petz formula suggests that the modes d, which we take to be localized only in $\mathcal{E}_M$, should also be representable in region $A = A_L \cup A_R$, but

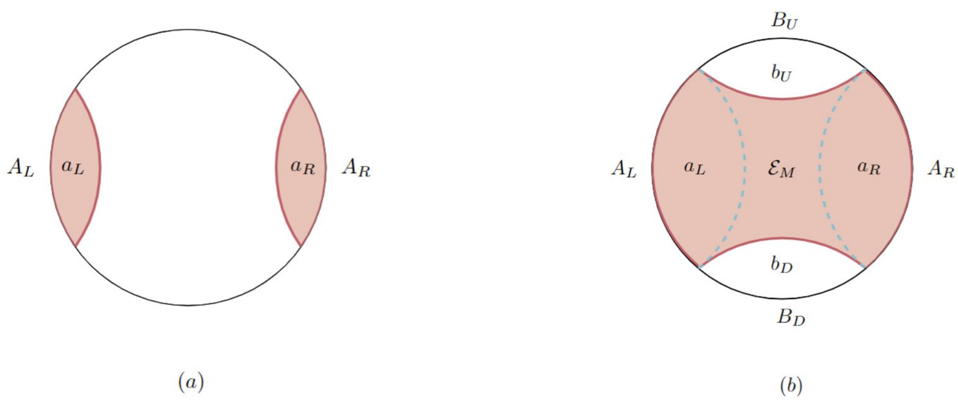


Figure 3.1: The entanglement wedge of a two disjoint intervals $A = A_L \cup A_R$ in AdS$_3$/ CFT$_2$. (a) The entanglement is the region bounded by the boundary region A and the minimal area co-dimension 1 surface in the bulk, with the same boundary as A. Thus $\mathcal{E}_A = a_L \cup a_R$. (b) As one increases the sizes of A_L and A_R, the minimal area surface changes and the entanglement wedge is no longer just $a_L \cup a_R$, rather it is all of the shade region, that is $\mathcal{E}_A = a_L \cup a_R \cup \mathcal{E}_M$.

the nature of these observables remains mysterious. Of course the modes d are precisely the modes which are in the entanglement wedge but not the causal wedge of $A = A_L \cup A_R$.

One possibility is that the d modes in region $\mathcal{E}_M$ are combinations of complicated operators in region A_L and A_R

$$d = \sum_{ij} c_{ij}\, O_i^{A_L} \otimes O_j^{A_R}$$

where $O_i^{A_{L,R}}$ are complicated gauge invariant operators. By *complicated* we mean that they are not single-trace or low-multi-trace operators. In this scenario, while each $O_i^{A_{L,R}}$ by themselves do not behave like GFFs, the particular combination above is expected to behave like a GFF in the large N limit.

Another intriguing possibility is that the modes d are operators which are gauge invariant, but they are made out of constituents in regions $A_L,\ A_R$ which are not separately gauge invariant. This seems natural from the point of view of, for example, the free $O(N)$

model. In that case we have operators like $\sum_{L,R} \varphi^i(x)\varphi^i(y)$, with $x \in A_L$ and $y \in A_R$, which are $O(N)$ invariant but the individual constituents are not, see also discussion in [Mintun et al., 2015].

A difficulty with the second possibility is that in a proper gauge theory one would expect that non-gauge invariant operators in regions $A_L,\ A_R$ have to be connected by Wilson lines which will have to go through the regions B_U or B_D.[9] If the operators in $\mathcal{E}_M$ are actually dual to gauge invariant Wilson line operators with end points in A_L and A_R, this would imply that they cannot strictly commute with all operators in regions B_U and B_D, as generally the Wilson lines can be detected by operators in regions B_U or B_D. This seems to contradict the conventional understanding of EWR, as in the scenario described above the operators on $\mathcal{E}_M$ would not be entirely supported in region A_L, A_R since the Wilson lines are passing through the complementary regions.

It would be interesting to explore whether a particular combination of such Wilson lines connecting the individual non-gauge invariant constituents can be constructed, where commutators of this combination with all simple operators in region B_U, B_D are sufficiently suppressed at large N. This might not directly contradict the arguments supporting EWR. For example, the equality of relative entropies [Jafferis et al., 2016b] has been established at large N and the arguments are not expected to generalize to imply equality including exponentially suppressed corrections[10]. This might suggest a refinement of EWR where bulk operators are *mostly* supported in $A_R \cup A_L$, allowing some form of Wilson lines connecting the two regions.

[9] In the $O(N)$ model the symmetry is global hence no Wilson line is necessary.

[10] For example, at finite N we expect that the bulk geometry is fully quantum and it is not even clear how one can define the entanglement wedge.

In any case, the nature of observables in the entanglement wedge but not the causal wedge, like the operators d in this case, remains somewhat mysterious and further study of their properties is necessary.

3.8 Appendix: HKLL reconstruction in global and Rindler coordinates

We review here the HKLL reconstruction in global and AdS-Rindler coordinates [Hamilton et al., 2006a] where authors constructed the smearing functions based on the mode sum approach.

3.8.1 HKLL reconstruction in global coordinates

Before going through it, one point that one might be interested in is if there is any possibility to find a smearing function that has compact support on the boundary of AdS. In particular, we are interested the smearing function has support only on the points that are spacelike separated from $\varphi(X)$. The HKLL method provides us with a way of reconstruction in the large N limit where the field φ satisfies the free equation of motion. Therefore, the smearing function can be constructed from a suitable Green's function that by definition satisfies

$$(\Box - m^2)G(X|X') = \frac{1}{\sqrt{-g}}\,\delta^{d+1}(X - X'). \tag{3.127}$$

Using the third Green identity, the field φ can be written in global coordinates as

$$\varphi(X') = \int dx\sqrt{-g}\left(\varphi(X)\partial^{\rho}G(X|X') - G(X|X')\partial^{\rho}\varphi(X)\right)\Big|_{\rho=\rho_0} \tag{3.128}$$

where $X = (\rho, x)$, and by sending $\rho_0 \to \pi/2$, one can find the smearing function in (3.128) in terms of the Green's function. For this purpose, let us take the ansatz of Green's function that is non-zero only at spacelike separation

$$G(X|X') = f(\sigma(X|X'))\theta(\sigma(X|X') - 1), \tag{3.129}$$

where σ is an AdS-invariant distant function which in global coordinates is

$$\sigma(X|X') = \frac{cos(t - t') - sin(\rho)sin(\rho')cos(\Omega - \Omega')}{cos(\rho)cos(\rho')} \tag{3.130}$$

and $\Omega - \Omega'$ is the angular separation on the sphere. The points that can be connected by a geodesic necessarily lie in the unit cell $-\pi < t - t' < \pi$. Spacelike separated points are the ones with $\sigma > 1$ that connected by a geodesic proper distance. By plugging back the ansatz (3.129) to (3.127), we can see that $f(\sigma)$ satisfies the AdS wave equation. Then, if we start from the beginning by the ansatz (3.129) [Hamilton et al., 2006b], we can find the smearing function with compact support only at spacelike separated region. We note here that this result has been found in global coordinates and in general, it could not be the case. For example, for odd-dimensional AdS in Poincare coordinates, the smearing function can have support only on the entire boundary.

Now, let us find the smearing function in global coordinates. The exact form of the smearing function depends on the dimension. The scalar field solution can be expanded as a linear combination of independent modes that in global coordinates it is given by (Eq.

3.9). We can split the field into positive and negative frequency components

$$\varphi(X) = \varphi(X)_+ + \varphi(X)_- \tag{3.131}$$

where

$$\varphi(X)_+ = \varphi(X)_-^\dagger = \sum_{nlm} f_{nlm} a_{nlm} \tag{3.132}$$

and for the boundary operator $O(x)$ as well. Since we can use the AdS isometries to bring one point to another one, it is enough to find the smearing function just at one point. In the center ($\rho = 0$), f_{nlm} vanishes for all $l \neq 0$, therefore only the s-waves contributes to the field in the center of Ads which will simplify the calculation drastically. Let us take $a_n = a_{n00} = \hat{O}_{n00}$, we can read a_n in terms of $O(x)$ as

$$a_n = \frac{1}{\pi vol(S^{d-1}) P_n^{(\Delta-d/2,d/2-1))}(1)} \int_{-\pi/2}^{\pi/2} dt \int d\Omega \sqrt{g_\Omega}\, e^{i(2n+\Delta)t} O_+(t,\Omega). \tag{3.133}$$

Plugging it back into the bulk mode expansion, one can find the bulk field at the origin as

$$\varphi(t', \rho' = 0, \Omega') = \int_{-\pi/2}^{\pi/2} dt \int d\Omega \sqrt{g_\Omega}\, K_+(t', \rho' = 0, \Omega' | t, \Omega) O_+(t,\Omega) + h.c. \tag{3.134}$$

where

$$K_+(t', \rho' = 0, \Omega' | t, \Omega) = \frac{1}{\pi vol(S^{d-1})} e^{i\Delta t} F(1, d/2, \Delta - d/2 + 1, e^{-i2t}). \tag{3.135}$$

It is important to know that smearing functions are not necessarily unique. It can be shifted by terms which vanish when integrated against the boundary operators. It can happen in cases that the boundary fields do not involve a complete set of Foureier modes. This freedom enable us to find K_+ which is real and then we can find the kernel such that $K = K_+ = K_-$.

Finally, for an arbitrary bulk point by action of an isometry map, we have

$$\varphi(X) = \int_{x \in bdy} dx\, K(X|x) O(x) \tag{3.136}$$

which for the AdS$_{d+1}$ in even dimension, the smearing function is

$$K_G(X|x) = \frac{\Gamma(\Delta - d/2 + 1)\Gamma(1 - d/2)}{\pi vol(S^{d-1})\Gamma(\Delta - d + 1)}\ lim_{\rho\to\pi/2}(\sigma(X|x)cos\rho)^{\Delta-d}\theta(\sigma(xX|x) - 1) \tag{3.137}$$

and in odd dimension it is given by

$$K_G(X|x) = \frac{2(-1)^{d/2-1}\Gamma(\Delta - d/2 + 1)}{\pi vol(S^{d-1})\Gamma(\Delta - d + 1)\Gamma(d/2)} \lim_{\rho\to\pi/2}(\sigma(X|x)cos\rho)^{\Delta-d}\log(\sigma(X|x)cos\rho)\theta(\sigma(X|x) - 1). \tag{3.138}$$

3.8.2 HKLL reconsreuction in AdS-Rindler coordinates

Consider a CFT Cauchy surface Σ and devide it into two regions A and its complementary part $\bar{A}$. The domain of dependence $\mathcal{D}(A)$ of A which is the set of points on the boundary that every inextendible causal curve that passes through it must also insert A. The causal wedge of a CFT subregion A is defined as

$$\mathcal{C}_A = \mathcal{J}^+[\mathcal{D}(A)] \cap \mathcal{J}^-[\mathcal{D}(A)] \tag{3.139}$$

where $\mathcal{J}^{\pm}[R]$ is the bulk causal future/past of region R in the boundary.

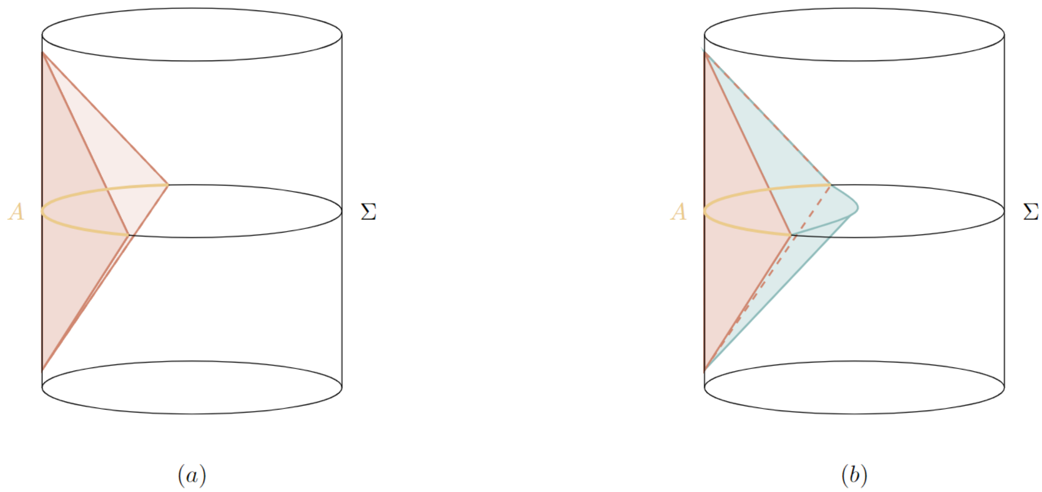


Figure 3.2: (a) Domain of dependence of the spherical region of the boundary. (b) The entanglement wedge of the region A in the bulk which is called AdS-Rindler wedge.

Consider the pure AdS_{d+1} in the bulk. If we take the $t = 0$ slice as the Cauchy surface and A to be one hemisphere of Σ, the causal wedge of A is the region of bulk that referred to as the AdS-Rindler wedge. Although it is naturally associated to a boundary region that covers half of the spatial surface, the patch can be mapped by an isometry to a patch that ends on arbitrary spatial region on the boundary. The coordinate system that covers the AdS-Rindler patch is (r, τ, x) with the metric

$$ds^2 = -(r^2 - 1)d\tau^2 + \frac{dr^2}{r^2 - 1} + r^2 dx^2 \tag{3.140}$$

where x is the set of coordinates on the $(d-1)$ dimensional hyperbolic ball H_{d-1}. We can find the mode expansion of free scalar field in the AdS-Rindler wedge by solving the Klein-

Gordon equation on this background

$$\varphi(r,\tau,x)=\int\frac{d\omega}{2\pi}\frac{d\lambda}{2\pi}\Big(f_{\omega\lambda}(r,\tau,x)b_{\omega\lambda}+f^{*}_{\omega\lambda}(r,\tau,x)b^{\dagger}_{\omega\lambda}\Big) \tag{3.141}$$

where the modes $b_{\omega\lambda}$ satisfy the usual commutation relation and the wave function is in the form of

$$f_{\omega\lambda}(r,\tau,x)=e^{-i\omega\tau}Y_{\lambda}(x)\psi_{\omega\lambda}(r). \tag{3.142}$$

The exact expression for the $\psi_{\omega\lambda}(r)$ in terms of hypergeometric function is [Almheiri et al., 2015b]

$$\begin{aligned}\psi_{\omega\lambda}(r)=M_{\omega\lambda}r^{-\Delta}(1-\frac{1}{r^2})^{-i\omega/2}F\Big(&-\frac{d-2}{4}+\frac{\Delta}{2}-\frac{i\omega}{2}+\frac{1}{2}\sqrt{\frac{(d-2)^2}{4}-\lambda},\\ &-\frac{d-2}{4}+\frac{\Delta}{2}-\frac{i\omega}{2}-\frac{1}{2}\sqrt{\frac{(d-2)^2}{4}-\lambda},\Delta-\frac{d-2}{2},\frac{1}{r^2}\Big)\end{aligned} \tag{3.143}$$

that

$$M_{\omega\lambda}=\frac{1}{\sqrt{2|\omega|}}\frac{\Gamma(-\frac{d-2}{4}+\frac{\Delta}{2}+\frac{i\omega}{2}+\frac{1}{2}\sqrt{\frac{(d-2)^2}{4}-\lambda})\,\Gamma(-\frac{d-2}{4}+\frac{\Delta}{2}+\frac{i\omega}{2}-\frac{1}{2}\sqrt{\frac{(d-2)^2}{4}-\lambda})}{\Gamma(\Delta-\frac{d-2}{2})\,\Gamma(i\omega)}. \tag{3.144}$$

By taking Fourier transformation of the boundary operator $O(\tau,x)=\lim_{r\to\infty}r^{\Delta}\varphi(r,\tau,x)$, we have

$$O_{\omega\lambda}=\int d\tau dx\, e^{i\omega\tau}Y^{*}_{\lambda}(x)O(\tau,x) \tag{3.145}$$

that is in the form of $O_{\omega\lambda}=M_{\omega\lambda}b_{\omega\lambda}$. Therefore, $\hat{O}_{\omega\lambda}=\frac{1}{M_{\omega\lambda}}O_{\omega\lambda}$ is the boundary operator identified with Rindler mode functions

$$\hat{O}_{\omega\lambda} = b_{\omega\lambda}. \tag{3.146}$$

By substituting (3.145) into (3.141) and exchange the order of integration we get

$$\varphi(r,\tau,x) = \int d\tau' dx' \, K(r,\tau,x|\tau',x') O(\tau',x') \tag{3.147}$$

where the smearing function is

$$K_R(r,\tau,x|\tau',x') = \int \frac{d\omega}{2\pi}\frac{d\lambda}{2\pi}\frac{1}{M_{\omega\lambda}} f_{\omega\lambda}(r,\tau,x)\, e^{i\omega\tau'}\, Y^*_\lambda(x'). \tag{3.148}$$

The issue here is that if we substitute the exact expression of $f_{\omega\lambda}(r,\tau,x)$ in (3.148), we find out that the integral does not converge for any choice of bulk and boundary points [Morrison, 2014, Leichenauer & Rosenhaus, 2013, Rey & Rosenhaus, 2014]. In the original paper [Hamilton et al., 2006a], authors argued that they can make the integral convergent by analytically continuation of x coordinates. However, there is still this question that if it is actually well-defined in the physically correct Lorentz signature. The issue was illuminated in [Morrison, 2014] when they gave an interpretation of the divergent smearing function in the context of distribution theory.

3.9 Appendix: Petz Theorem proof in finite dimensions

This appendix closely follows the discussion on [Petz, 2003].

Consider a finite dimensional Hilbert space $\mathcal{H}$ with the Hilbert-Schmidt inner product on the algebra of operators acting on $\mathcal{H}$ as $\langle a, b\rangle = Tr(a^\dagger b)$ for all $a, b \in L(\mathcal{H})$. The action of a modular operator for all $a \in L(\mathcal{H})$ is $\Delta(a) = \sigma a \rho^{-1}$ where ρ and σ are positive definite Hermitian operators in $L(\mathcal{H})$. Consider another Hilbert space $L(\mathcal{K})$ and let $\mathcal{E} : L() \to L(\mathcal{K})$ be a completely positive and trace preserving (CPTP) map, then $\mathcal{E}^*(1) = 1$ for the dual of $\mathcal{E}$ with respect to the Hilbert-Schmidt inner product.

Since $\mathcal{E}^*$ is also completely positive,

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \geq 0 \implies \begin{pmatrix} \mathcal{E}^*(A) & \mathcal{E}^*(B) \\ \mathcal{E}^*(C) & \mathcal{E}^*(D) \end{pmatrix} \geq 0 \tag{3.149}$$

For some $a \in L(\mathcal{H})$,

$$\begin{pmatrix} 1 & a \\ 0 & 0 \end{pmatrix}^* \begin{pmatrix} 1 & a \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & a \\ a^* & a^*a \end{pmatrix} \geq 0, \tag{3.150}$$

then one has $\begin{pmatrix} 1 & \mathcal{E}^*(a) \\ \mathcal{E}^*(a^*) & \mathcal{E}^*(a^*a) \end{pmatrix} \geq 0$. For arbitrary η, one then has

$$\langle \begin{pmatrix} 1 & \mathcal{E}^*(a) \\ \mathcal{E}^*(a^*) & \mathcal{E}^*(a^*a) \end{pmatrix} \begin{pmatrix} -\mathcal{E}^*(a)\eta \\ \eta \end{pmatrix}, \begin{pmatrix} -\mathcal{E}^*(a)\eta \\ \eta \end{pmatrix} \rangle \geq 0. \tag{3.151}$$

After a bit of algebra, one gets

$$(\mathcal{E}^*(a^*)\mathcal{E}^*(a) - \mathcal{E}^*(a^*a))\eta^2 \leq 0 \tag{3.152}$$

which gives the Schwartz inequality for dual map $\mathcal{E}^*$, $\mathcal{E}^*(a^*a) \geq \mathcal{E}^*(a^*)\mathcal{E}^*(a)$.

The relative entropy is defined as,

$$\mathcal{S}(\rho||\sigma) = \mathrm{tr}(\rho(\log\ \rho - \log\ \sigma)) \qquad \text{for } \mathit{supp}\ \rho \subseteq \mathit{supp}\ \sigma. \tag{3.153}$$

otherwise it is defined to be infinite. One can see that $\Delta = \sigma\rho^{-1}$ and define also $\Delta_0 = \mathcal{E}(\sigma)\mathcal{E}(\rho)^{-1}$.[11] If both ρ and σ are invertible, one can write

$$\begin{aligned}\mathcal{S}(\rho||\sigma) &= -\langle\rho^{1/2}, \log\Delta\,\rho^{1/2}\rangle, \\ \mathcal{S}(\mathcal{E}(\rho)||\mathcal{E}(\sigma)) &= -\langle\mathcal{E}(\rho)^{1/2}, \log\Delta_0\ \mathcal{E}(\rho)^{1/2}\rangle\end{aligned} \tag{3.154}$$

To relate the two equations above, we need some kind of relationship between ρ and $\mathcal{E}(\rho)$ and the two modular operators. For some x, the norm of $x\mathcal{E}(\rho)^{1/2}$ is given by

$$\begin{aligned}||x\mathcal{E}(\rho)^{1/2}||^2 &= Tr(\mathcal{E}(\rho)^{1/2}x^*x\mathcal{E}(\rho)^{1/2}) \\ &= Tr(\rho\mathcal{E}^*(x^*x)) \geq Tr(\rho\mathcal{E}^*(x)\mathcal{E}^*(x^*))\end{aligned} \tag{3.155}$$

Thus one has $||x\mathcal{E}(\rho)^{1/2}||^2 \geq ||\mathcal{E}^*(x)\rho^{1/2}||^2$. This means, one can define an operator V such that

$$Vx\mathcal{E}(\rho)^{1/2} = \mathcal{E}^*(x)\rho^{1/2} \text{ and } V^*V \leq 1 \tag{3.156}$$

Considering (3.156) for $x = 1$ and squaring it, one gets $V\mathcal{E}(\rho)V^* = \rho$ and $V^*\rho^{-1}V = \mathcal{E}(\rho)^{-1}$. In addition, it also follows from the previous two formulas that $V^*\Delta V \leq \Delta_0$.

Coming back to the relative entropy in (3.154), a useful way to write logarithm is as the

[11] Here and in what follows, we write $\Delta(1)$ and $\Delta_0(1)$ as Δ and Δ_0 respectively.

integral below

$$-\log\, x = \int_0^\infty \Big((x+t)^{-1} - (1+t)^{-1} \Big)\, dt \tag{3.157}$$

Thus, one can rewrite the relative entropy as

$$\mathcal{S}(\rho||\sigma) = \int_0^\infty \Big(\langle \rho^{1/2}, (\Delta + t)^{-1}\, \rho^{1/2} \rangle - (1+t)^{-1} \Big)\, dt \tag{3.158}$$

similarly for $\mathcal{S}(\mathcal{E}(\rho)||\mathcal{E}(\sigma))$.

Using (3.156) again for $x = 1$, one can see that $\langle \rho^{1/2}, (\Delta{+}t)^{-1}\, \rho^{1/2} \rangle = \langle \mathcal{E}(\rho)^{1/2}, V^*(\Delta + t)^{-1}\; V\mathcal{E}(\rho)^{1/2} \rangle$. But one has the follow relation,

$$V^*(\Delta + t)^{-1} V \geq (\Delta_0 + t)^{-1} \tag{3.159}$$

This is because $V^* \Delta V \leq \Delta_0$ and $(x+t)^{-1}$ is monotonically decreasing function. Thus one gets

$$\langle \rho^{1/2}, (\Delta + t)^{-1}\, \rho^{1/2} \rangle \geq \langle \mathcal{E}(\rho)^{1/2}, (\Delta_0 + t)^{-1}\; \mathcal{E}(\rho)^{1/2} \rangle. \tag{3.160}$$

By substituting it in (3.158), this in turn implies that

$$\mathcal{S}(\rho||\sigma) \geq \mathcal{S}(\mathcal{E}(\rho)||\mathcal{E}(\sigma)) \tag{3.161}$$

for any CPTP map $\mathcal{E}$. This inequality is known in the literature as Ulhmann's theorem of monotonicty of relative entropy.

The interesting case is when the relative entropies are equal, which would imply equality in (3.160) that is

$$\langle \mathcal{E}(\rho)^{1/2}, V^*(\Delta + t)^{-1}\ V\mathcal{E}(\rho)^{1/2}\rangle = \langle \mathcal{E}(\rho)^{1/2}, (\Delta_0 + t)^{-1}\ \mathcal{E}(\rho)^{1/2}\rangle \tag{3.162}$$

where we used (3.156) on the left side. Thus one has, $V^*(\Delta + t)^{-1}\ \rho^{1/2} = (\Delta_0 + t)^{-1}\ \mathcal{E}(\rho)^{1/2}$. Repeating the analysis for another monotonically decreasing function $(x + t)^{-2}$, one gets

$$V^*(\Delta + t)^{-2}\rho^{1/2} = (\Delta_0 + t)^{-2}\ \mathcal{E}(\rho)^{1/2} \tag{3.163}$$

It is then straightforward to show that $||V^*(\Delta + t)^{-1}\rho^{1/2}||^2 = ||(\Delta + t)^{-1}\rho^{1/2}||^2$. For V with property given in (3.156), one can show that

$$\begin{aligned}(\Delta + t)^{-1}\rho^{1/2} &= VV^*(\Delta + t)^{-1}\rho^{1/2} \\ &= V^*(\Delta_0 + t)^{-1}\ \mathcal{E}(\rho)^{1/2}\end{aligned} \tag{3.164}$$

where in the second line we used (3.156) and (3.159). Since the equality (3.163) can be shown to hold for any same negative power of $\Delta+t$ and Δ_0+t, the Stone-Wiestrass theorem implies that

$$Vf(\Delta_0)\mathcal{E}(\rho)^{1/2} = f(\Delta)\rho^{1/2} \tag{3.165}$$

for any continuous function f. Using (3.156), one can rewrite it as $\mathcal{E}^*(f(\Delta_0)) = f(\Delta)$.

3.9.1 Petz Theorem

The theorem states that, for a CPTP map $\mathcal{E}$, $\mathcal{S}(\rho||\sigma) = \mathcal{S}(\mathcal{E}(\rho)||\mathcal{E}(\sigma))$ if and only if there is another CPTP map $\mathcal{P}_{\sigma,\mathcal{E}}$ such that

$$\mathcal{P}_{\sigma,\mathcal{E}} \circ \mathcal{E}(\rho) = \rho, \text{ and } \mathcal{P}_{\sigma,\mathcal{E}} \circ \mathcal{E}(\sigma) = \sigma. \tag{3.166}$$

The proof:

Assume there is some CPTP map $\mathcal{P}_{\sigma,\mathcal{E}}$ such that $\mathcal{P}_{\sigma,\mathcal{E}} \circ \mathcal{E}(\rho) = \rho$ and $\mathcal{P}_{\sigma,\mathcal{E}} \circ \mathcal{E}(\sigma) = \sigma$ then

$$\mathcal{S}(\rho||\sigma) \geq \mathcal{S}(\mathcal{E}(\rho)||\mathcal{E}(\sigma)) \geq \mathcal{S}(\mathcal{P}_{\sigma,\mathcal{E}} \circ \mathcal{E}(\rho)||\mathcal{P}_{\sigma,\mathcal{E}} \circ \mathcal{E}(\sigma)) \tag{3.167}$$

but since $\mathcal{S}(\mathcal{P}_{\sigma,\mathcal{E}} \circ \mathcal{E}(\rho)||\mathcal{P}_{\sigma,\mathcal{E}} \circ \mathcal{E}(\sigma)) = \mathcal{S}(\rho||\sigma)$, the above equation implies that $\mathcal{S}(\rho||\sigma) = \mathcal{S}(\mathcal{E}(\rho)||\mathcal{E}(\sigma))$.

Assume $\mathcal{S}(\rho||\sigma) = \mathcal{S}(\mathcal{E}(\rho)||\mathcal{E}(\sigma))$ then for any continuous function f

$$\mathcal{E}^*(f(\Delta_0)) = f(\Delta). \tag{3.168}$$

Consider $f(\Delta) = (\Delta^*)^{-1/2}(\Delta)^{-1/2} = \sigma^{-1/2}\rho\sigma^{-1/2}$ and $f(\Delta_0) = \mathcal{E}(\sigma)^{-1/2}\mathcal{E}(\rho)\mathcal{E}(\sigma)^{-1/2}$. Then, the above equality after a little algebra gives,

$$\sigma^{1/2}\mathcal{E}^*\left(\mathcal{E}(\sigma)^{-1/2}\mathcal{E}(\rho)\mathcal{E}(\sigma)^{-1/2}\right)\sigma^{1/2} = \rho. \tag{3.169}$$

Thus considering the CPTP map $\mathcal{P}_{\sigma,\mathcal{E}}(.) = \sigma^{1/2}\mathcal{E}^*(\mathcal{E}(\sigma)^{-1/2}(.)\mathcal{E}(\sigma)^{-1/2})\sigma^{1/2}$, the equality $\mathcal{P}_{\sigma,\mathcal{E}} \circ \mathcal{E}(\sigma) = \sigma$ immediately follows and $\mathcal{P}_{\sigma,\mathcal{E}} \circ \mathcal{E}(\rho) = \rho$ follows from the equality of relative entropy.■

This map $\mathcal{P}_{\sigma,\mathcal{E}}$ is usually called the Petz recovery map.

All that the human mind has produced—the brightest in genius, the most persevering in application, has been lavished on the details of the law of gravity.

Charles Babbage

4

Algebra of Operators in an AdS-Rindler Wedge

This chapter consists of the paper [Bahiru, 2022]. The original abstract is as follows:

We discuss the algebra of operators in AdS-Rindler wedge, particularly in AdS_5/CFT_4. We explicitly construct the algebra at N = ∞ limit and discuss its Type III1 nature. We will consider 1/N corrections to the theory and using a novel way of renormalizing the area of Ryu-Takayanagi surface, describe how several divergences can be renormalized and the algebra becomes Type II∞. This will make it possible to associate a density matrix to any state in the Hilbert space and thus a von Neumann entropy.

4.1 Introduction

There has been a recent interest in studying algebra of operators in perturbative limits of quantum gravity in the context of AdS/CFT. Leutheusser and Liu studied the $N = \infty$ limit of the thermofield double state above Hawking-Page temperature [Leutheusser & Liu, 2021b, Leutheusser & Liu, 2021a]. They noted that the algebra of operators exterior to the black hole horizon form what is called Type III_1 von Neumann algebra. From the boundary perspective, the algebra elements are the single trace operators whose thermal one point function is subtracted,

$$\tilde{O} = O - \langle O \rangle.$$

These operators are generalized free fields with $O(1)$ two point functions. Using properties of Type III_1 algebras, they were able to propose a natural time like evolution operator for an observer falling into the black hole.

Witten later proposed to include the Hamiltonian into the algebra [Witten, 2021b]. In the strict $N \to \infty$ limit,

$$[U, \tilde{O}] = 0$$

where U is the properly normalized Hamiltonian with a definite large N limit, $U = \frac{H - \langle H \rangle}{N}$. Thus the algebra at this level is in fact a tensor product of the GFF algebra and the algebra of bounded functions of U, $\mathcal{A}_{\tilde{O}} \otimes \mathcal{A}_U$ and still a Type III_1 algebra.

Another property of Type III_1 von Neumann algebras is that one can extend it to include the outer automorphism of the algebra and changes it to a Type II_∞ von Neumann algebra. This algebra will be acting on a different Hilbert space and does not describe the original system[1]. But, as one includes $1/N$ corrections to the operators exterior to the black hole, it was found that the center mode U will generate an outer automorphism for the algebra $\mathcal{A}_{\tilde{O}}$. Thus the new Type II_∞ algebra in fact describes the algebra of operators as one backs away form strict $N \to \infty$ limit and considers $1/N$ corrections. Entropy can be associated to this algebra of operators and it can be interpreted as the generalized entropy for the black hole. This construction is also done for maximally extended Schwarzschild black hole in flat space-time and deSitter space-time [Chandrasekaran et al., 2022a].

The purpose of this paper is to analyze this construction in the case of AdS-Rindler wedge. This corresponds to a spherical region in the boundary CFT and one needs to analyze single trace operators in this sub-region. They form a Type III_1 algebra and there is a center for these operators. But there is a difference with the previous discussion in that the two point function of the center mode will still be divergent after one naively normalizes it, in a similar way U is defined. In fact the two point function is given by the area of the

[1] Still, by doing an additional crossed product by the dual group of the automorphism to this new algebra, one can get to the tensor product of the original Type III_1 algebra and an algebra of bounded functions on $\mathcal{L}^2(G)$. To study Type III_1 algebras like this was the original purpose of the crossed product construction.

horizon. We will renormalize this area and thus the two point function then we will be able to proceed with the construction that goes parallel to the previous discussions for deSitter and black holes in AdS and flat space-times. This will also provide a new renormalization scheme for the area Ryu-Takayanagi surface of a spherical ball subregion of boundary CFT_d in $d \geq 2$.

4.2 The Type III_1 nature of the AdS-Rindler wedge

Consider two time-like separated points in the boundary CFT, p and q, where p is to the future of q. The intersection of the causal past of p and the causal future of q is defined to be the causal wedge D_q^p. To describe the physics in this diamond, it is enough to know the physical data on a space-like slice in the diamond called the Cauchy surface [2]. This Cauchy slice can be chosen to be finite spherical region at some time slice in the boundary. The causal wedge of D_q^p, what is called the AdS-Rindler wedge, is the bulk region that is causally connected D_q^p. In other words, the fields in this bulk region can be explicitly expressed by evolving the boundary operators in D_q^p in to the bulk using the bulk equations of motion at the time slice [Hamilton et al., 2006d, Hamilton et al., 2006c, Almheiri et al., 2015a]. Since one can think of the causal wedge as a causal diamond in the bulk (with the same time-like separated points but now the causal past and future of p and q respectively, include points also in the bulk, i.e, the diamond includes causal paths from q to p that extend in the bulk), any point in the wedge can be described only using the data on the time slice.

AdS-Rindler wedge is considered as the dual to the boundary causal diamond of a finite spherical region. It is the bulk region that can be reconstructed from this boundary

[2] This is not true in general, for instance in the strict large N limit of the boundary CFT.

sub-region.[3] Since the algebra of operators in a sub-region of a quantum field theory in flat space-time is expected to be Type III_1 [Haag, 1992b], one can conclude that the algebra of operators in the AdS-Rindler wedge is also Type III_1. Without even referring to the boundary, one can see that the algebra should be Type III_1 since in the bulk one has a local quantum field theory on a curved space-time in the strict large N limit. So a sub region like AdS-Rindler wedge will be Type III_1.

4.2.1 Explicit construction of the algebra

Once we have established the type of the algebra of operators in AdS-Rindler, $\tilde{\mathcal{A}}_0$, it follows that the Hilbert space that describes the system will have to be built out of a thermofield double state of the naive Hilbert space one would come up with. The reason for this can be understood if one thinks of the quantum field theory on AdS-Rindler space as a QFT at temperature $T = 1/2\pi$, which arises as a result of the Rindler horizon.

In the usual quantum field theory at zero temperature [Streater & Wightman, 1989], the way to construct a Hilbert space is by first choosing a vector that one is interested in, $|\xi\rangle$[4], and start applying with the modes in the algebra of the system. There will be infinite number of modes, but in most interesting systems it will be enough to act with finite but arbitrary number of modes.[5] This will create for us the pre-Hilbert space. To create the Hilbert space, one requires any Cauchy sequence to converge, i.e, one adds the limit points of any Cauchy sequence,

$$\{|\psi_n\rangle\}_{n\in\mathbb{N}},$$

[3] More precisely, what is dual to a boundary sub-region is the entanglement wedge but in the simple cases where the state is the vacuum, the entanglement wedge and the causal wedge coincide.

[4] It is taken to be the vacuum in most cases.

[5] Physically the intuition is that any experimenter will not be able to act with infinite number operators.

in the pre Hilbert space, this is called its Hilbert space completion.

This Hilbert space, $\mathcal{H}_{[\xi]}$, will not involve states that differ from the original state, $|\xi\rangle$, by the action of infinite number of modes. But with such states, one can create other Hilbert spaces in the same way (by acting with arbitrary but finite number operators and taking the Hilbert space completion). These Hilbert spaces are totally independent[6] and can be treated as different super selection sectors [von Neumann, 1949]. Naively one would think that the full Hilbert space of the system would be the space that includes all the Hilbert spaces whose construction was given just now.

$$\mathcal{H}' = \bigoplus_{[\xi]} \mathcal{H}_{[\xi]}$$

This Hilbert space, called non-separable Hilbert space, has uncountably infinite dimension and is hard to deal with. But, since the super selection rules apply to the 'small Hilbert spaces', it would be enough to consider only these super selection sectors whose dimension is countably infinite.

It is this construction that fails in the case of a QFT at non zero temperature. The reason is thermal fluctuations will take us outside the super selection sectors. Thus, it seems that one necessarily have to deal with the non-separable Hilbert space. A way around this was found by von Neumann [von Neumann, 1940] by using the thermofield double state at temperature T to construct a separable Hilbert space but now the algebra of operators will not act irreducibly on the Hilbert space. Rather than choosing some vector to build the Hilbert space on, now one chooses the thermofield double state at a given temperature and different super selection sectors corresponds to starting with thermofield double states at

[6] The fact that they are independent will necessarily depend on the algebra of operators.

different temperatures. Thus thinking the AdS-Rindler wedge in the strict large N limit, $N=\infty$, as a quantum field theory on a curved space-time at some temperature T, one uses the thermofield double state at temperature $T = 1/2\pi$ to construct the Hilbert space on which the Type III_1 algebra will act.

The algebra $\tilde{\mathcal{A}}_0$ contains bulk fluctuations acting in the AdS-Rindler wedge. Consider the following metric for the AdS-Rindler [Almheiri et al., 2015a],

$$ds^2 = -(r^2-1)dt^2 + \frac{dr^2}{r^2-1} + r^2 dH_{d-1}^2 \tag{4.1}$$

where $dH_{d-1}^2 = d\chi^2 + \chi^2 d\Omega_{d-2}^2$ is the standard metric on the hyperbolic ball H_{d-1}.

A free massless scalar field on this metric is given by,

$$\varphi(t,r,\alpha) = \sum_{\lambda}\int_0^\infty \frac{d\omega d\lambda}{4\pi^2}(f_{\omega,\lambda}(t,r,\alpha)a_{\omega,\lambda} + f^*_{\omega,\lambda}(t,r,\alpha)a^\dagger_{\omega,\lambda}) \tag{4.2}$$

where the sum is over degeneracy for λ and $f(t,r,\alpha)$ solves the Klein-Gordon equation and is of the form,

$$f_{\omega,\lambda}(t,r,\alpha) = e^{i\omega t}Y_\lambda(\alpha)\psi_{\omega,\lambda}(r)$$

and $Y_\lambda(\alpha)$ is eigenfunction of the Laplacian on H_{d-1} and $\psi_{\omega,\lambda}(r)$ is given in terms of hypergeometric functions up to some prefactors that make sure that it has the expected asymptotic behaviour. The elements of $\tilde{\mathcal{A}}_0$, $F(a,a^\dagger)$, are linear combinations of a finite string of operators like

$$(a^\dagger_{\omega_1,\lambda_1})^{i_1}\ldots(a^\dagger_{\omega_r,\lambda_r})^{i_r}(a_{\omega_1,\lambda_1})^{k_1}\ldots(a_{\omega_s,\lambda_s})^{k_s}.$$

with complex coefficients.

Naively, one would start with the vacuum, a state annihilated by all the $a_{\omega,\lambda}$'s and then consider all states that are created by the action of the $a^{\dagger}_{\omega,\lambda}$'s.[7] Considering the vacuum of the full bulk (the AdS vacuum), $|vac\rangle$, one can see that these modes are thermally populated,

$$\langle vac| \, a^{\dagger}_{\omega,\lambda} a_{\omega',\lambda'} \, |vac\rangle = \frac{1}{e^{2\pi\omega} - 1} \delta(\lambda - \lambda') \delta(\omega - \omega') \tag{4.3}$$

But because of the thermal fluctuations, states that can only reached from the AdS-Rindler vacuum by the action of infinite number of $a^{\dagger}_{\omega,\lambda}$'s, should also be included. But, as remarked before, rather than considering the full non-separable Hilbert space, one will proceed to construct the Hilbert space on top of the thermofield double state.

The AdS-Rindler vacuum is $|\tilde{0}\rangle = \bigotimes_{\omega,\lambda} |\tilde{0}\rangle_{\omega\lambda}$ and an arbitrary state created by the action of arbitrary but finite number of $a^{\dagger}_{\omega,\lambda}$'s on each state $|\tilde{0}\rangle_{\omega\lambda}$and take the tensor product over ω and λ,

$$|\tilde{j}\rangle = \bigotimes_{\omega,\lambda} |\tilde{j}_{\omega\lambda}\rangle \,.$$

The Hilbert space $\mathcal{H}_0$ spanned by these states $|\tilde{j}\rangle$ is the non separable Hilbert space. We consider a state (here we assume that we discretized ω and λ and take the continuum limit later when we take n to infinity),

$$|\tilde{\Psi}_n\rangle = \frac{1}{\sqrt{\tilde{Z}_n}} \bigotimes_{\omega,\lambda}^{n} \sum_{j_{\omega\lambda}} e^{-\pi\omega j_{\omega\lambda}} \, |\tilde{j}_{\omega\lambda}\rangle \, |\tilde{j}_{\omega\lambda}\rangle' \tag{4.4}$$

where $\tilde{Z}_n$ is such that $\langle\tilde{\Psi}_n| \, |\tilde{\Psi}_n\rangle = 1$ and the state $|\tilde{j}_{\omega\lambda}\rangle'$ is an element of $\mathcal{H}'_0$, a copy of $\mathcal{H}_0$.

[7] From the full bulk perspective (in contrast to the just the AdS-Rinlder subregion), this vacuum is not well defined, it has a firewall (infinite stress energy tensor) at the entangling surface with the complement region. The reason is that there will be no entanglement between the two complementary regions and the two point functions for this state between the two regions will not have the correct short distance behaviour.

Thus we define the thermofield double state $|\tilde{\Upsilon}\rangle$ as a map from $\tilde{\mathcal{A}}_0$ to $\mathbb{R}$ given by,

$$\langle\tilde{\Upsilon}|\, F(a, a^\dagger)\, |\tilde{\Upsilon}\rangle = \lim_{n\to\infty} \langle\tilde{\Upsilon}_n|\, F(a, a^\dagger)\, |\tilde{\Upsilon}_n\rangle\,. \tag{4.5}$$

[8] Measurements made by an observer restricted to this bulk sub region is thus given by such expectation value. The Hilbert space $\mathcal{H}_\Upsilon \subset \mathcal{H}_0$ is generated by states given by the action of a finite but arbitrary string of a and $a^\dagger$'s on $|\tilde{\Upsilon}\rangle$ and their Hilbert space completion.

The boundary analog of the algebra of operators and Hilbert space construction can be discussed following the extrapolate dictionary [Banks et al., 1998, Harlow & Stanford, 2011],

$$\lim_{r\to\infty} r^\Delta \varphi(r, x) = O(x),$$

Δ being the conformal dimension of the conformal field theory single trace operator O. Then it follows that,

$$O_{\omega\lambda} = \int dt d\alpha\; e^{-i\omega t} Y^\star_\lambda(\alpha) O(t, \alpha) = N_{\omega\lambda} a_{\omega\lambda} \tag{4.6}$$

Then, the non-separable Hilbert space $\mathcal{H}_0$ is spanned by $|j\rangle = \bigotimes_{\omega\lambda} |j_{\omega\lambda}\rangle$ constructed from the vacuum of the boundary subregion, $|0\rangle = \bigotimes_{\omega\lambda} |0\rangle_{\omega\lambda}$, by the action of $j_{\omega\lambda}$ raising operators, $O^\dagger_{\omega\lambda}$.

We also define a thermofield double state $|\Upsilon\rangle$ as a map form $\mathcal{B}_0$, the algebra for single trace operators, to $\mathbb{R}$ exactly like 4.4 and 4.5 but now $a_{\omega\lambda}$ and $a^\dagger_{\omega\lambda}$ are replaced by $O_{\omega\lambda}$ and

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[8] There should also be an upper limit on the sum over $j_{\omega,\lambda}$ which we did not emphasized here. A more careful statement would be to consider $|\tilde{\Upsilon}_n\rangle = \frac{1}{\sqrt{\tilde{Z}_n}} \bigotimes^n_{\omega,\lambda} |\omega, \lambda\rangle$ and define $|\omega, \lambda\rangle_k = \sum^k_{j_{\omega\lambda}} e^{-\pi\omega j_{\omega\lambda}} |\tilde{j}_{\omega\lambda}\rangle\, |\tilde{j}_{\omega\lambda}\rangle'$ and $|\omega, \lambda\rangle$ is defined by $\langle\omega\lambda|\, F(a_{\omega\lambda}, a^\dagger_{\omega\lambda})\, |\omega\lambda\rangle = \lim_{k\to\infty} \langle\omega\lambda|_k\, F(a_{\omega\lambda}, a^\dagger_{\omega\lambda})\, |\omega\lambda\rangle_k$.

$O^\dagger_{\omega\lambda}$.

Single trace operators are gauge invariant operators that can be constructed by considering operators written in terms of trace of the fields and their derivatives with no explicit N dependence in matrix theories. In these theories, where the fields transform in the adjoint representation of, for example $SU(N)$ or $U(N)$, the fields can be normalized such that the action is given with an explicit N factor multiplied by a gauge invariant term with no explicit N dependence. Well known examples include free Yang Mills, $\mathcal{N} = 4$ super Yang Mills and so on. These operators have $O(N)$ one point function and two point functions with a leading $O(N^2)$ disconnected term. Thus to make sense of the large N limit, one needs to consider the subtracted single trace operators, $\tilde{O} = O - \langle O \rangle$. These operators close and are generalized free fields in the large N limit. Thus, since we are studying the $N = \infty$ theory, one should use $\tilde{O}$ to construct $\mathcal{B}_0$ which are operators with a well defined large N limit.

The Hilbert space, $\mathcal{H}_\Psi$ is now constructed from the action of operators in $\mathcal{B}_0$ on $|\Psi\rangle$ and limit points of a Cauchy sequence of states.

The algebra $\mathcal{B}_0$ should also be completed in the weak sense, i.e, if a sequence of expectation values converges, $\lim_{n\to\infty} \langle \Psi | F_n | \Psi \rangle = \langle \Psi | F | \Psi \rangle$ for $F_n \in \mathcal{B}_0$, then one adds the operator F to get the von Neumann algebra $\mathcal{B}$. This is the algebra of single trace operators.

But this algebra of single trace operators is not all there is, there is a conserved charge associated with the conformal Killing vector, χ^μ, that preserve the causal diamond of the spherical boundary sub region and its complement. It is given by an integral that involves the stress energy tensor, χ^μ and the normal vector to the boundary Cauchy surface. Since it is a symmetry it annihilates the full vacuum, $|vac\rangle$. In the bulk this is the boost operator that preserves Rindler horizon. The integral can be naturally divided into a part that acts in the subregion and a part that acts in the complement, $H_\chi = 2\pi(K - \bar{K})$. The operator K is the boost operator of the boundary subregion and the algebra of single trace operators will not capture it. In the strict large N limit, this corresponds in the bulk to the area of the horizon, loosely speaking[9], which is also not captured by the bulk fluctuations.

In the normalization that single trace operator, O, have $O(N)$ one point function and leading disconnected $O(N^2)$ two point function. On the other hand, the stress energy tensor will have the same form as the action, a gauge invariant term multiplied by N. The generator of the conformal transformation that leaves the diamond of the boundary subregion B invariant is given by,

$$K = \int_B T_{\mu\nu}\chi^\mu dB^\nu \tag{4.7}$$

and will also have the explicit N dependence. This explicit N dependence will give K one point and connected two point functions of $O(N^2)$. Thus these quantities will become divergent in the large N limit. In addition, there is also a UV divergence arising from the UV degrees of freedom close to the boundary of the subregion of the boundary CFT. We will discuss this UV divergence later in this section and the next section. A simple subtraction like what we did for the single trace operators will not result in a generalized free field, since the two point function will still be $O(N^2)$. Thus one defines the operator,

$$\tilde{X} = \frac{K - \langle K \rangle}{N} \tag{4.8}$$

Up till now one can draw a direct parallel between the case of large black holes in AdS, with

[9]There are several subtleties concerning the diverging behaviour of the area term which we will discuss in the following sections.

the Hamiltonian being the corresponding K [Witten, 2021b]. The Hamiltonian is also given as an integral of the stress energy tensor and has $O(N^2)$ one point and connected two point function (above Hawking-Page temperature)[10]. The only difference would be in the case of the black hole, there is a finite horizon area. But in the present case, the horizon is of infinite volume. Thus, even though for the Hamiltonian one would be satisfied with 4.8, for the present case there will still be an additional divergence coming from horizon which was not present for the black hole and strictly speaking, the mode $\tilde{X}$ is not yet well defined.

From the bulk perspective, this can be seen from the computation of $\langle K\rangle$ and $\langle K^2\rangle - \langle K\rangle^2$ in [Verlinde & Zurek, 2020]. To leading order in G_N,

$$\langle K\rangle = \langle (K - \langle K\rangle)^2\rangle = \frac{A(\Sigma)}{4G_N} \tag{4.9}$$

where $A(\Sigma)$ is the volume of the Rindler horizon Σ which is a $d-1$ dimensional hyperbolic hyper-surface. Following the normalization 4.8 and using $G_N \sim 1/N^2$ one gets $\langle\tilde{X}\rangle = 0$ and $\langle\tilde{X}^2\rangle \sim A(\Sigma)$. The two point function is infinite since $A(\Sigma)$ is infinite. This is as a result of the boundary UV divergence that was mentioned earlier.

Even though $\tilde{X}$ is well defined $\tilde{X}^2$ is not well defined, it has divergent expectation value. If there was any way to renormalize it and define a new operator X with a well defined large N limit for the one and two point function[11], then one can include to our strict large N limit algebra of operators, the bounded functions of this mode and get the full algebra $\mathcal{A} = \tilde{\mathcal{A}}_0 \otimes \mathcal{A}_X$. This algebra of operators will act on an extended Hilbert space, $\mathcal{H} = \mathcal{H}_\Psi \otimes$

[10] One argument for the Type III$_1$ nature of the AdS-Rindler wedge and also black hole exterior in AdS is the continuous spectrum of the modular Hamiltonian (boost operator) (the Hamiltonian for the black hole case). This continuous spectrum of the Hamiltonian is in fact associated to the appearance of horizon in the bulk [Festuccia & Liu, 2007].

[11] Here we assume large N factorization for X, so it is enough to renormalize the two point function.

$\mathcal{L}^2(\mathbb{R})$ which in general include some entangled state between square intergrable functions of X and states in $\mathcal{H}_\Psi$.

4.2.2 The renormalization of the infinite volume

We now claim that there is a natural renormalization of this additional divergence mentioned in the previous section. We start by noticing 4.9 that the expectation value and the fluctuations of K are given by the area of a $d - 1$ dimensional hyperbolic surface with boundary. A hyperbolic manifold with boundary will have, in general, infinite volume. But there is a canonical renormalization of this volume which would follow from looking at the Einstein-Hilbert action of these manifolds.

Einstein equations imply that the bulk part of the Einstein-Hilbert action is given by the volume of the manifold (up to some prefactors). For a non compact manifold M, this action can be shown to be infinite. But one can consider a finite sub manifold $N \subset M$ and take the action on this sub manifold. In the limit ∂N goes to ∂M, which is at infinity, one can see that the action diverges in terms of local quantities on ∂N which are invariant functions of the induced metric, more specifically the first and second fundamental forms. Thus adding counter terms to this action and renormalizing it will enable one to associate a canonical finite volume to the manifold.[12] In particular, this would imply to a renormalization where one subtracts the divergent terms and then take the limit ∂N going to ∂M. Thus giving a notion of renormalized volume for the hyperbolic manifold M.

[12] In 3 dimensions it is canonical up to the Weyl anomaly of two dimensional CFT. This anomaly is absent in even dimensions.

The action for a manifold M with Euclidean signature is given by,

$$S = -\frac{1}{16\pi G_N}\int_M d^{d-1}x\sqrt{g}(R-2\Lambda) - \frac{1}{8\pi G_N}\int_{\partial M} d^{d-2}x\sqrt{\gamma}K_\gamma \tag{4.10}$$

where Λ is the cosmological constant that is assumed to be negative, g and R are the metric and Ricci scalar on M while γ and K_γ are the induced metric and the mean curvature on its boundary.

The renormalization procedure is discussed in several papers [Henningson & Skenderis, 1998, Krasnov & Schlenker, 2008] but we will revise it here. It essentially involves four steps. The first is to take M to be in the metric,

$$ds^2 = \frac{1}{4\rho^2}d\rho^2 + \frac{h_{ij}}{\rho}dx_i dx_j \tag{4.11}$$

as ρ goes to zero (the boundary), h goes to γ, the boundary metric. Then one solves Einstein's equation for the above metric.

$$\begin{aligned} \rho(2h'' - 2h'h^{-1}h' + Tr(h^{-1}h')h') + R_\gamma - (d-2)h' - Tr(h^{-1}h')h = 0 \\ (h^{-1})^{jk}(D_i h'_{jk} - D_k h'_{ij}) = 0 \\ Tr(h^{-1}h'') - \frac{1}{2}Tr(h^{-1}h'h^{-1}h') = 0 \end{aligned} \tag{4.12}$$

where R_γ is the boundary Ricci scalar and D_i is covariant derivative in the boundary metric while prime is derivative with respect to ρ. We have also used $\Lambda = \frac{-d(d-1)}{2}$.

The important next step is to note that the above equations can be solved by considering the following expansion for h,

$$h = \gamma + \rho\gamma_2 + \rho^2\gamma_4 + \dots \tag{4.13}$$

where for even d, there is an additional term at the $\rho^{d/2}$ order which is $\rho^{d/2}\log\rho\ \kappa_2$. γ_k is given covariantly in terms of γ and contains k derivatives with respect to x_i.

Considering the simplest case of hyperbolic 3 manifolds, i.e, taking the full bulk to be AdS_5, Einstein's equations become $R - 2\Lambda = -4$ and one can see that the action is just the volume of the manifold M[13]. Now we consider the action of a sub manifold $N_\varepsilon \subset M$ that is regularized at $\rho = \varepsilon$ for some cut off $\varepsilon > 0$. The action of N_ε will have both bulk and boundary term, at $\rho = \varepsilon$. The expansion 4.13 implies a similar expansion for $\sqrt{g}$ and the action will be as follows,

$$S = \frac{V(N_\varepsilon)}{4\pi G} = \frac{1}{4\pi G}\int_{\partial N_\varepsilon} d^2x\ \frac{1}{\varepsilon}a_0 + \log\varepsilon\ a_2 + L_{finite}. \tag{4.14}$$

where $a_0 = \sqrt{\gamma}$ and $a_2 = -\frac{1}{4}\sqrt{\gamma}R_\gamma$. Thus adding counter terms to remove the divergent terms and taking the limit $\varepsilon \to 0$, one gets a renormalized volume of the manifold M.

For our present case, this hyperbolic manifold is the horizon of the AdS-Rindler subregion. If we renormalize $\tilde{X}^2$ by subtracting the above divergent terms, then the renormalized mode X is well defined with finite one and two point functions and the algebra given above, $\mathcal{A} = \tilde{\mathcal{A}}_0 \otimes \mathcal{A}_X$ makes sense. In addition, this provides a canonical renormalization of the Ryu-Takayanagi surface for a spherical region in the boundary. Since the renormalization of the volume of hyperbolic manifolds is applicable for any dimensions [Balasubramanian

[13] There are also additional boundary terms but we are going to use local boundary counter terms to renormalize the action and, for 3 manifolds, these boundary terms are cancelled exactly by the counter terms.

& Kraus, 1999, Krasnov & Schlenker, 2008], the renomalization scheme can be used for boundary CFTs in other dimensions.

4.3 Adding gravity and making the algebra Type II_∞

This section closely follows the discussion in [Witten, 2021b] for the eternal black hole in AdS. To move away from the strict large N limit, we will have to include in $\tilde{\mathcal{A}}_0$ linear combinations of single trace operators with coefficients that have asymptotic power expansion in $1/N$ around $1/N = 0$. In addition, one have to add $1/N$ corrections to X. The way to do this is to note that modular Hamiltonian at finite N generates modular time translation within the causal diamond of the boundary sub region.

$$[\frac{K - \langle K \rangle}{N}, O] = \frac{1}{N}\partial_\tau O. \tag{4.15}$$

Thus one adds to X an operator that has well defined large N limit and also has no divergence coming from infinite horizon area, i.e, H_χ. Checking with 4.15, one has a new operator, $X + \frac{H_\chi}{2\pi N}$ that generates time translation for the single trace operators. Any other operator that one can add consistent with 4.15, like operators from the complement region, can be removed by making use of the conjugate operator of X, that is $\Pi = \frac{d}{idX}$. By conjugating the algebra with $e^{i\Pi\hat{O}/N}$, one can remove an additional operator from the complement, like $\hat{O}/N$ [Witten, 2021b].[14] Thus, even at perturbative order in $1/N$, the new mode does not have divergences and is well defined.

The bounded functions for this mode will form the algebra $\mathcal{A}_{X+\frac{H_\chi}{2\pi N}}$. Since this enlarge-

[14]The reason is that these algebras are defined up to conjugation.

ment of the algebra upgrades the center mode to an outer automorphism for $\tilde{\mathcal{A}}_0$, as is well known in the axiomatic quantum field theory literature and very well elaborated in a paper, the full algebra will be a Type II_∞ crossed product algebra $\tilde{\mathcal{A}}_0 \ltimes \mathcal{A}_{X+\frac{H_\chi}{2\pi N}}$. This algebra again will act on the Hilbert space, $\mathcal{H} = \mathcal{H}_\Psi \otimes \mathcal{L}^2(\mathbb{R})$ but now one can associate a density matrix for any state. A certain class of states one can consider are of the form $|\tilde{\Psi}\rangle = |\Psi\rangle \otimes g(X)^{1/2}$. Using such states, a trace can be defined for what are called trace class operators [Witten, 2021b]. An element of the crossed product algebra, $\tilde{O} = \int ds\ O(s)e^{isY}$ where $Y = X + \frac{H_\chi}{2\pi N}$ is trace class if the trace of the operator defined as,

$$Tr(\tilde{O}) = \langle\tilde{\Psi}|\,\tilde{O}\tilde{K}^{-1}\,|\tilde{\Psi}\rangle\,,$$

is finite, where $\tilde{K} = \frac{g(Y)}{e^Y}$. This implies the density matrix for such states is in fact $\tilde{K}$. One can also define a density matrix, ρ_Φ, for a general state $|\tilde{\Phi}\rangle \in \mathcal{H}$. Thus, it is possible to define a von Neumann entropy, $\mathcal{S}_\Phi = -Tr\rho_\Phi \log\rho_\Phi = -\langle\tilde{\Phi}|\log\rho_\Phi|\tilde{\Phi}\rangle$, can be associated to any state in the Hilbert space.

4.4 Discussion

This analysis has been for AdS-Rinlder wedge at perturbative order in $1/N$ around $N = \infty$. But for finite N, the algebra of the AdS-Rindler is expected to be again Type III_1. The reason is from the boundary perspective, one is looking at the algebra of a local region of a quantum field theory which is in general Type III_1. On the other hand, the space-time description of the theory will not be precise for finite value say $N = 243$. Thus the renormalization procedure for the area of the Rindler horizon will make less and less sense as one backs away from the N going to infinity limit. Thus we expect that, to make sense of the theory non perturbatively, one would have to work with $\tilde{X}$ and not the renormalized mode; this restores the Type III_1 nature of the boundary sub region for finite N. But still a more careful analysis has to be done to understand this better.

It would also be interesting to compare the von Neumann entropy given in the previous section to the usual entropy proposed by Hubeny, Rangamani and Takayanagi [Hubeny et al., 2007].

Since the beginning of physics, symmetry considerations have provided us with an extremely powerful and useful tool in our effort to understand nature...

Tsung-Dao Le

5

Algebras and their Covariant representations in quantum gravity

This chapter consists of the paper [Bahiru, 2023]. The original abstract is as follows:

We study a physically motivated representation of an algebra of operators in gravitational and non gravitational theories called the covariant representation of an algebra. This is a representation where the symmetries of the operator algebra are implemented unitarily on the Hilbert space. We emphasize the very close similarity of this representation to the crossed product of an algebra. In fact, as an example of (and sometimes identified with) a covariance algebra, the crossed product of an algebra is in one to one correspondence with the covariant representation of the algebra. This will in turn illuminate physically what the crossed product algebra is in the context of quantum gravity.

5.1 Introduction

A more precise understanding of algebras of operators in several contexts in quantum gravity has been given much attention in recent years. Even though studying the algebra of the operators of the semiclassical physics had already proven to be useful, for instance in the background of a black hole, the Tomita Takesaki theory of the algebra of operators was used in the reconstruction of the interior of a black hole [Papadodimas & Raju, 2013], the renewed interest followed the work of Leutheusser and Liu [Leutheusser & Liu, 2021b, Leutheusser & Liu, 2021a]. They identified the algebra of operators of a CFT that is thermally entangled with another CFT above the Hawking-Page temperature to be a type III_1 von Neumann algebra in the strict large N limit. This identification naturally led them to propose an operator that is associated with an infalling observer and discuss the emergence of time in the eternal black hole background in AdS. This was followed by several works of Witten et. al. [Witten, 2021b, Chandrasekaran et al., 2022a, Chandrasekaran et al., 2022b, Penington & Witten, 2023], where gravitational interactions are added in some limited fashion to the cases where matter does not gravitationally backreact, where the upshot can be summarized to be the Lorentizan derivation of the generalized entropy of the semiclassical states, among others. This is relevant given the role the entropy plays in the understanding of the black hole information paradox and most of the recent applications of quantum information in quantum gravity. Some other directions this analysis proceeded include understanding subregions in several backgrounds, the de Sitter spacetime and Hilbert space, the role of the observer in de Sitter and cosmology in general, progress towards understanding the presence/ absence of the ensemble averaging for black holes in higher dimensions, background independent description of the perturbative quantum gravity to mention few among many [Witten, 2023a, Ali Ahmad & Jefferson, 2023, Jensen et al., 2023, Witten, 2023b, Gomez, 2023, Gomez, 2022, Schlenker & Witten, 2022].

The key role of algebras was noticed even in the early days of quantum field theory in curved spacetime [Wald, 1995]. The reason for this was the fact that the well understood quantization procedures of a classical theory to a quantum theory faces serious problems when applied in curved spacetimes. In essence, there is no unique quantization of the classical theory, rather one arrives at several unitarily inequivalent Hilbert spaces (see 5.3.1 for more discussion) describing possibly 'inequivalent' quantum theories. This is persistent in particular when the background geometries are open with no particular asymptotics (check [Witten, 2021c] to see the discussion of this issue in several geometries). The resolution is to consider the algebra of operators in the algebraic quantum field theory sense instead, where such arbitrariness is not present when we pass from the classical theory to the quantum theory. The issue is quite similar to what we face as we move from special relativity, where there is a preferred frame of reference, to general relativity where any frame of reference is equivalent to any other, thus the theory should be described in a frame independent manner. The algebra of operators for the quantum theory already includes the physical content of all the unitarily inequivalent Hilbert space at once and is the correct way to describe the quantum theory. Thus it is not quite naive to imagine the useful role played by the algebra of operators in different limits of quantum gravity.

This article follows a similar spirit in that, we start with an algebra of operators *then* we consider a particular useful representation of that algebra in different situations. This representation is called the covariant representation [Doplicher et al., 1966, Borchers, 1966, Masamichi Takesaki, 1967] and it includes the representation of the algebra that acts on a certain Hilbert space. But, in addition the symmetries or the automorphisms of the algebra are also implemented as unitary operators on the Hilbert space. For the obvious reason that symmetries are just changes in the our perspective that should not change experimental results, they should be represented by unitary operators in the Hilbert space and this is the only relevant representation for the algebra for physical systems [Borchers, 1969].

Coming back to the recent developments in algebras and quantum gravity, a certain construction called the crossed product construction has been used to go beyond the strict $G_N \to 0$ limit. In most cases, this led to the change in the type of the von Neumann algebra from a type III_1 to type II, where a finite trace and entropy can be defined. But still, it seems that this step is a bit mathematical and not physically clear. On the other hand, the covariant representation of an algebra can be rigorously shown to be in one to one correspondence with what is called a covariance algebra (see appendix 5.5), of which the crossed product algebra is an example [Masamichi Takesaki, 1973]. Thus, in terms of the covariant representation, the crossed product type II algebra can be understood in a physically intuitive way, which is one of the goals the article hopes to accomplish.

We can also imagine a case where the vacuum of the Hilbert space breaks some of the symmetries. These cases will lead to what we call proto quantum gravity Hilbert space which is a Hilbert space not described by a quantum field theory on a curved spacetime even though it arises in the strict $G_N \to 0$ limit. An example of such a Hilbert space was discussed in the [Chandrasekaran et al., 2022b] and dubbed proto holographic black hole by the authors.

In section 5.2, we revise the relevant background discussion about algebra of operators with out gravitational backreaciton. In section 5.3, we discuss what the covariant representation of an algebra is and which algebra precisely we are talking about in the most general cases. Then in the following section, we continue the discussion to proto QG Hilbert spaces and covariant representations for the algebra of operators associated with subregions and an observer's worldline in any spacetime.

5.2 No Backreaction : review

5.2.1 The strict large N limit

Holography implies that the semiclassical physics in AdS is emergent from a low lying sector of the boundary CFT. For a CFT containing a large central charge and operators that factorize, called generalized free fields, there is a reorganization of the small number of these degrees of freedom (compared to the central charge) that reproduces the gravitational theory in AdS [El-Showk & Papadodimas, 2012a, Heemskerk et al., 2009, Heemskerk & Sully, 2010, Fitzpatrick et al., 2011, Penedones, 2011]. In fact this property of generalized free fields, that they obey an equation of motion in a higher dimension even though they do not satisfy any equation of motion in the CFT was well known even before the formulation of AdS/CFT [Haag & Schroer, 1962b]. But generalized free fields are not fully self consistent CFTs by themselves and can only be understood as a small sector of a bigger CFT. This property has been useful in understanding of how gravity encodes information as a quantum error correcting code [Almheiri et al., 2015a], granting the Hilbert space they are acting on, the name the *code subspace*.

Specializing to $\mathcal{N} = 4$ super Yang Mills in $4d$ in the 't Hooft limit, the generalized free fields will be the single trace operators with thermal or vacuum expectation value subtracted. Their correlators factorize into two point functions in the large N limit where $SU(N)$ is the gauge group. We are interested in studying quantum field theory on the eternal black hole in AdS, which is expected to be dual to two thermally entangled CFTs [Maldacena, 2003] above the Hawking Page temperature. Thus we consider the Hilbert space $\mathcal{H} = \mathcal{H}_L \otimes \mathcal{H}_R$, where $\mathcal{H}_L$ and $\mathcal{H}_R$ are the Hilbert spaces of the left and right CFTs. We denote the set of all bounded and linear operators acting on $\mathcal{H}$ by $\mathcal{B}(\mathcal{H})$. The generalized free fields, $\tilde{\mathcal{A}}_{L/R} \subset \mathcal{B}(\mathcal{H})$, acting either on the left or right side form a von Neumann algebra in the strict large N limit (i.e, $1/N$ is set to zero), once their weak closure is taken. The weak closure is a requirement to include any limit point for a Cauchy sequence of matrix elements, i.e, if for $a_n \in \tilde{\mathcal{A}}_{L/R}$, $\lim_{n\to\infty} \langle\psi| a_n |\chi\rangle = \langle\psi| a |\chi\rangle$, for all $|\psi\rangle$ and $|\chi\rangle$ in $\mathcal{H}$, then a must also be in $\tilde{\mathcal{A}}_{L/R}$. If it had not been for the subtraction of the thermal expectation value in the definition the generalized free fields, this algebra $\tilde{\mathcal{A}}_{L/R}$, would have described the entangled system in a background independent way. But following [Leutheusser &

Liu, 2021b, Leutheusser & Liu, 2021a], we will specifically discuss the eternal black hole at a given temperature. What Leutheusser and Liu did was to identify the Hilbert space on the eternal black hole background at some temperature $1/\beta$, $\mathcal{H}_{HH}$, with what is called the GNS Hilbert space built on the thermofield double state of the same temperature, which is distinct from $\mathcal{H}$.

We consider the thermofield double state at inverse temperature β,

$$|\Psi_\beta\rangle = \frac{1}{Z_\beta} \sum_i e^{-\beta E_i/2} |E_i\rangle_R |E_i\rangle_L \tag{5.1}$$

where $|E_i\rangle$ are the energy eigenstates. We can define an inner product on $\tilde{\mathcal{A}}_R$ using $|\Psi_\beta\rangle$ in the large N limit, in particular,

$$\langle a|b\rangle = \lim_{N\to\infty} \langle \Psi_\beta| ab |\Psi_\beta\rangle \ , \qquad \forall a, b \in \tilde{\mathcal{A}}_R \, . \tag{5.2}$$

Note that, if $\lim_{N\to\infty} \langle \Psi_\beta| x^\dagger x |\Psi_\beta\rangle = 0$ for $x \in \tilde{\mathcal{A}}_R$, then it follows the Schwarz inequality that $|a\rangle \ \sim \ |a + x\rangle$. Let X be the set of all operators like x, then we have an equivalence class $\tilde{\mathcal{A}}_R/X$ with an inner product. We can then interpret this set as a pre-Hilbert space, where after we take the Hilbert space completion, becomes a Hilbert space. To take the the Hilbert space completion means to add all the limit points of a Cauchy sequence of states, $\{|\psi_n\rangle\}$ in the pre-Hilbert space. We get the GNS Hilbert space, $\mathcal{H}^\beta_{GNS}$, after we take the

Hilbert space completion of the above equivalence class. Thus we have[1],

$$\mathcal{H}_{HH} \equiv \mathcal{H}^{\beta}_{GNS} \tag{5.3}$$

There is also a representation π_β of the operator algebra $\tilde{\mathcal{A}}_R$ acting on $\mathcal{H}^{\beta}_{GNS}$,

$$\pi_\beta(c)\,|a\rangle = |ca\rangle \ , \qquad \forall c, a \in \tilde{\mathcal{A}}_R \tag{5.4}$$

We will denote this representation by $\mathcal{A}^{\beta}_{R,0}$ (from now on we will only write the superscript β when it is necessary) and it is identified with the bulk fluctuations on right exterior of the eternal black hole geometry at temperature $1/\beta$.

5.2.2 Modifications to $\mathcal{H}_{GNS}$

The first, and perhaps not so severe, modification to the operator algebras and the Hilbert spaces is to include the contribution of the conserved charges acting on the bulk Hilbert spaces $\mathcal{H}_{HH}$ and $\mathcal{H}_{dS}$. The Hilbert space $\mathcal{H}_{dS}$ is constructed like $\mathcal{H}_{HH}$ with the action of the bulk fluctuations but on the background of the Bunch-Davies state of de Sitter. To identify these charges, we look at the symmetry of the 'vacua' of each Hilbert spaces which is seen by an observer on the right exterior. For the eternal black hole the symmetry group observed by each exterior is $G_{HH} = \mathbb{R} \times (Spin(4) \times SU(4)_R)/\mathbb{Z}_2$ while for static patch of deSitter, it is $G_{dS} = \mathbb{R} \times SO(d-1)$. Let's call the generators $Q^\alpha \in \mathbf{g}_{HH}$ and $Q^\beta \in \mathbf{g}_{dS}$. One has to be careful not to naively include these generators to the operator algebras on

[1] This is not exactly correct as will be seen in the sections that follow. The state $\mathcal{H}_{HH}$ is built upon, $|\psi_{HH}\rangle$, should also include a wave function for the right Hamiltonian, which is a delta function.

the Hilbert spaces since they act only in the right exterior. Thus, they do not map a smooth Cauchy surface to a smooth Cauchy surface, in particular, they create a singularity at the horizon. The natural resolution is to impose a brick wall boundary condition close to horizon, renormalize the charges before taking the limit. The same can be accomplished by renormalizing the charges by appropriate power of G_N as we take it to zero. In the case of the eternal black hole the situation is cleaner since we can renormalize the boundary operators such that the conserved charges in the boundary, dual to the bulk charges, take the form $N^2 Tr(L)$ for some operator L with no explicit N dependence. Thus subtracting the thermal one point function and multiplying by $1/N(\sim \sqrt{G_N}$ for $\mathcal{N} = 4$ super Yang Mills in 4 dimensions) will make them finite in the large N limit.

Let's call the renormalized charges $q^\alpha \in \mathbf{g}$ and their bounded functions will act on the corresponding Hilbert spaces. The wave function associated with these modes will take values in $L^2(\mathbf{g})$. Thus, the first extension to the Hilbert spaces will be $\mathcal{H}_{GNS} \otimes L^2(\mathbf{g}_{HH})$ and $\mathcal{H}_{dS} \otimes L^2(\mathbf{g}_{dS})$ while the algebras become $\mathcal{A}_{R,0} \otimes \mathcal{A}_{gHH}$ and $\mathcal{A}_{dS} \otimes \mathcal{A}_{gdS}$, where $\mathcal{A}_{gHH}$ and $\mathcal{A}_{gdS}$ are algebras of bounded functions of the charges of $\mathbf{g}_{HH}$ and $\mathbf{g}_{dS}$ respectively.

5.3 In the presence of gravitational interactions

5.3.1 Algebra of operators on general backgrounds

In this subsection, we discuss what we mean precisely by the algebra of quantum fields in general spacetimes. In section 5.2.1, the elements of the algebra in consideration ($\tilde{\mathcal{A}}_{L/R}$) was given in terms of the dual CFT subtracted single trace operators in the large N limit, which form an algebra of generalized free fields. These operators are well defined even at non perturbative level, though they do not close to form an algebra without the addition of operators not available in the large N limit, for instance product of N^2 single trace operators. Even though we do not aspire to define operators that make sense non perturbtatively for general spacetime (including non holographic) at the moment, we would like to describe what the elements of the algebra of observables, $\mathcal{A}$, is when gravitational interactions are added in some limited fashion.

We take this algebra to be a slight modification to the algebra of observables of a quantum field theory on a general curved spacetime [Wald, 1995]. This modification can be a perturbative interaction between the matter field and the graviton or an addition of modes that correspond to large diffieomorphisms. So, they would not be present for a quantum field theory, without gravity.

We start with the classical field theory describing matter fields and metric fluctuations on top of some fixed globally hyperbolic geometry. If we assume that the matter fields satisfy equations of motion with a well defined initial value problem, then we can describe the system in terms of a phase space, $\mathcal{M}$, of a pair of smooth functions (Π, Φ) defined on a Cauchy hypersurface Σ. This phase space also has a symplectic structure; that means there is a non degenerate, closed two form $\Omega(.,.)$ on $\mathcal{M}$. The states of the classical system correspond to points on $\mathcal{M}$, while functions (or more precisely functionals), $f : \mathcal{M} \to \mathbb{R}$, are the observables (we take linear functions of (Π, Φ) as the basis for this set of observables). On the other hand, a quantum theory is described by a Hilbert space, $\mathcal{H}$, and self adjoint bounded operators acting on the Hilbert space. Quantization of the classical system is thus the problem of finding a map from the classical phase space $\mathcal{M}$ and functions on $\mathcal{M}$ to a Hilbert space and self adjoint operators acting on the Hilbert space. But since Hilbert spaces with infinite dimensions (we expect this since the phase space is infinite dimensional) are all isomorphic, the physical content of the problem concerns with the map from the linear functions (classical observables) on $\mathcal{M}$ to operators acting on a certain Hilbert space. Canonical quantization provides such a map by requiring Poisson brackets satisfied by functions on $\mathcal{M}$ be mapped to commutators satisfied by the operators.

In the case where the geometry has a killing vector that is everywhere timelike, there is a simple way to implement this, that also ensures the correct short distance behaviour of the matrix elements of the quantum observables. For definiteness, let's consider a free scalar field with the equation of motion,

$$\frac{1}{\sqrt{g}}\partial_\mu(\sqrt{g}g^{\mu\nu}\partial_\nu\Phi - m^2\Phi) = 0 \tag{5.5}$$

where $g_{\mu\nu}$ is the metric.

The metric can be written in a time independent way and we can define a self adjoint Hamiltonian that is bounded from below. A solution of (5.5) can be decomposed into a positive and negative frequency modes of complex functions,

$$\Phi(x,t) = \int d\omega dk\; a_{\omega k}f_{\omega k}(x)e^{i\omega t} + a^\dagger_{\omega k}f^*_{\omega k}(x)e^{-i\omega t} \tag{5.6}$$

where $\omega > 0$. The phase space of the classical system is the space of functions (Π, Φ), with $\Pi = n^\mu D_\mu\Phi$ where D_μ is the covariant derivative and n^μ is a normal vector to Σ. Then, mapping Poisson brackets to commutators leads to the canonical commutation condition, $[a_{\omega k}, a^\dagger_{\omega' k'}] = \delta(\omega - \omega')\delta(k - k')$, while the rest of the commutators vanish.

In addition, we can also define an inner product to be the real part of the Klein-Gordon inner product of the positive frequency parts of the solutions corresponding to the initial data[2]. Thus, in this case, we can consider the algebra of operators generated by the a finite but arbitrary product of a and $a^\dagger$'s and take our algebra of observables for the quantum field theory to be the completion of this algebra with respect to the inner product just described. Similarly, the algebra can be extended to include other matter fields and free gravitons.

But the above analysis is only for a stationary spacetimes. In particular, we would not have the decomposition of the fields into positive and negative frequency modes (5.6) in a general curved spacetime.[3] Thus to proceed, we go back to the symplectic structure of the phase space and note that the function $f = \Omega(q, .)$, for $q \in \mathcal{M}$ is a linear function on $\mathcal{M}$ since $f(p) = \Omega(q, p) \in \mathbb{R}$ and the symplectic form is non degenerate. Therefore, we can take $\Omega(q, .)$ as the basis for our set of classical observables. In most cases the phase space is defined as the cotangent bundle of the configuration space of the system and thus the symplectic form is taken to be the usual anti-symmetric bilinear map on a contangent bundle. For example, in the case of the free scalar field mentioned above,

$$\Omega[(\Pi_1, \Phi_1), (\Pi_2, \Phi_2)] = \int_\Sigma (\Pi_1 \Phi_2 - \Pi_2 \Phi_1). \tag{5.7}$$

Following this, we can summarize the Poisson bracket conditions on (Π, Φ) in terms of the symplectic form as

$$\{\Omega(q_1, .), \Omega(q_2, .)\} = -\Omega(q_1, q_2) I \tag{5.8}$$

where $q_i = (\Pi_i, \Phi_i)$, $i = 1, 2$. For instance if we take $q_1 = (\delta_\varepsilon(x), 0)$ and $q_2 = (0, -\delta_\varepsilon(y))$ where δ_ε is a smooth approximation of the delta function such that $\lim_{\varepsilon \to 0} \delta_\varepsilon(x) = \delta(x)$,

[2] Check section 4.3 of [Wald, 1995] for a precise construction.

[3] And with it, we lose the nice particle interpretation of the Hilbert space.

then in the limit $\varepsilon \to 0$, $\Omega(q_1, (\Pi, \Phi)) = \Phi(x)$ and $\Omega(q_2, (\Pi, \Phi)) = \Pi(y)$ and we find,

$$\{\Phi(x), \Pi(y)\} = \delta(x - y). \tag{5.9}$$

Since $\Omega(q, .)$ is an observable, we can promote it into self adjoint operator so that

$$[\bar{\Omega}(q_1, .), \bar{\Omega}(q_2, .)] = -i\Omega(q_1, q_2)I \tag{5.10}$$

is satisfied. The bounded functions[4] of these operators form an algebra[5] and we can also define an inner product from the symplectic form so that [Wald, 1995],

$$\langle q_1, q_1 \rangle = \frac{1}{2}\mathrm{sup}_{q_2 \neq 0} \frac{|\Omega(q_1, q_2)|^2}{\langle q_2, q_2 \rangle}. \tag{5.11}$$

Taking the completion of the algebra with respect to this inner product, we get the algebra that corresponds to a quantum field theory on the curved spacetime.[6] It should be noted that there is a wide class of inner products that satisfy the above equation and that there is arbitrariness in the definition of the inner product, but this will not create a problem since the resulting algebras are all isomorphic as abstract algebras [Wald, 1995]. However, this becomes a problem when we construct a Hilbert space.

To describe the algebra of observables in the presence of perturbative gravitational in-

[4] A basis for the bounded functions of the operators can be taken to be the unitary operators generated by $\bar{\Omega}(q_1, .)$. The self adjointness and the commutation condition on the operators translates to the unitary operators satisfying thw Weyl relation.

[5] This algebra is called a $^*-$algebra [Haag, 1992b]

[6] If there is a sequence of operators $\{a_n\}$ and if the completion of the algebra is taken such that the operator a is in included in the algebra when, for any ψ in the Hilbert space, $a\psi = \lim_{n\to\infty} a_n\psi$, then the algebra is what is called a C^* algebra. If we rather take the completion of the $^*-$algebra by including the limit points of the expectation values of the operators, we will get a von Neumann algebra.

teractions, we will consider the same algebra but now we can also take linear combination of the operators with powers of G_N as coefficients. In the case where we just include a large diffeomorphism mode, we form the algebra associated with it from the unitaries constructed from the generator of the diffeomorphism. The algebra includes bounded functions of the generator and any bounded function can be expressed in terms of the unitaries. We will reformulate the latter case in terms of the covariant representation of the algebra of the paragraph above, in the sections that follow.

Coming back to the construction of the Hilbert space, we will first take the Hilbert space completion of $\mathcal{M}$ with respect to a given inner product, (5.11). We then complexify the manifold using the antisymmetric two form Ω. This complex manifold, $\bar{\mathcal{M}}$, can be used to create the Fock space of states, which is given by the direct sum of the vacuum ($\mathbb{C}$), one particle space, two particle space and so on; where the n-particle space is given by the symmetric (bosonic) or antisymmetric (fermionic) tensor product of n copy of $\bar{\mathcal{M}}$. But as mentioned before, the condition (5.11) does not uniquely determine an inner product and the different inner products in general lead to different Hilbert spaces. Naturally, it is reasonable to take the direct sum of all the Hilbert spaces and consider a bigger Hilbert space (this Hilbert space is the one studied by von Neumann et. al. [von Neumann, 1949, von Neumann, 1940] as infinite tensor product Hilbert spaces, where in the same papers shown that they are equivalent to the infinite direct sum Hilbert spaces). But there are a couple of issues with this bigger Hilbert space [Streater & Wightman, 1989, Bahiru, 2022], first it is a Hilbert space with unaccountably infinite dimensions. Second, the individual Hilbert spaces are what are called unitarily inequivalent and they correspond to different superselection sectors, in the sense that states in different sectors will not form a coherent superposition and a superposition only results in a mixed state and any dynamics, implemented by a unitary evolution will not evolve states in one sector to a different sector. Hoping to avoid this non-separable Hilbert space and because of the fact that each sector can be treated completely independently, studying the individual sectors only is generally expected to be enough to describe a physical system [Streater & Wightman, 1989]. But again a question arises as to which of the sectors we should choose, to correspond to the quantization of the classical system.

In the case where the geometry has a timelike Killing vector that is globally defined, there is a canonical choice of inner product, in particular a generalization of the inner product mentioned for the free scalar field. This is also directly related with the fact that there is a canonical choice of the vacuum state. In addition to stationary spacetimes, it was also shown that [Witten, 2021c] for asymptotically AdS (with boundary conditions as that of AdS/CFT), asymptotically flat spacetimes (with theories with a mass gap) and compact spacetimes, there is a canonical choice of vacuum and a natural choice of Hilbert space. For a more general open spacetimes though, it is expected that there is indeed not a canonical inner product (or choice of 'vacuum' state) and an algebraic treatment of the theory is necessary as it includes the content of all the unitarily inequivalent Hilbert spaces in a systematic manner, as was discussed in the introduction[7].

This algebra is the same for any spacetime in the sense that it is generated by the (5.10). But the elements differ for the specific choice of geometry since the symplectic form of the classical theory will in general be different. For instance, for flat spacetimes, the operators

[7] Additional condition on the states is making sure that they reproduce the correct short distance behaviour for the operators. This is analyzed by checking that the UV behaviour of the two point functions has the correct singularity. This condition goes by the name, the Hadamard condition. Check [Wald, 1995] for more careful discussion.

(5.10) can be shown to be the sum of the annihilation and creation operator [Wald, 1995]. On the other hand, the fact that even for the same classical theory, there are several unitraily inequivalent Hilbert space will lead to several representations (π) of the algebra on the Hilbert spaces. These representations can be constructed following the GNS construction (see 5.2.1) on a certain cyclic state in the Hilbert space. In the following sections, we discuss the covariant representation of this algebra.

5.3.2 Covariant representation of the algebra

One of the central principles of algebraic approach towards understanding of quantum field theory (and statistical physics) has been that physical content is actually algebraic and does not depend on the representation. Still, there are certain physically sensible and useful representations. In particular, if $\mathcal{A}$ is a von Neumann algebra with an automorphism group G, then the most interesting representation (and probably the only relevant representation [Borchers, 1969, Borchers, 1966]) for physics is one where the automorphism/symmetry group is implemented by an action of a unitary operator on the Hilbert space (the reason for considering unitary operators is that the symmetry group is expected to preserve transition probabilities and this is true if they are represented by a unitary operator). This is to say that one concentrates on a set of special states which are related to each other by a unitary action of the automorphism group. The automorphism group can be one associated with space-time translations or some internal symmetry of the system in consideration. Such representation is called a covariant representation of the algebra $\mathcal{A}$ [Doplicher et al., 1966, Borchers, 1966, Masamichi Takesaki, 1967, Borchers, 1969].

More precisely, a covariant representation (π, U) of $\mathcal{A}$ is pair of a non-degenerate representation of $\mathcal{A}$, $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ and a unitary representation of the automorphism group G, $U : G \to \mathcal{U}(\mathcal{H})$, where $\mathcal{B}(\mathcal{H})$ is a set of bounded operators acting on $\mathcal{H}$ and $\mathcal{U}(\mathcal{H})$ is a set of unitary operators acting on $\mathcal{H}$, such that;

$$U(g)^\dagger \pi(a) U(g) = \pi(\alpha_g(a)) \,, \qquad \text{for all } a \in \mathcal{A} \tag{5.12}$$

where $\alpha : G \to \text{Auto}\mathcal{A}$ and $\text{Auto}\mathcal{A}$ is the automorphism group of $\mathcal{A}$. We will write the unitary representations of the automorphism group as $U(g)$.

Thus, given an algebra of operators of a quantum system, gravitational or not, *we should ask what is the Hilbert space where there is a non degenerate representation of the algebra. In addition, there should also be a unitary representation of the automorphism group of the algebra acting on the same Hilbert space such that 5.12 is satisfied.*

For instance, in the context of section 5.2.1, once we are given an algebra of operators $\tilde{A}_R$ it is then physically necessary to ask what the covariant representation of the algebra is. The first guess at the relevant Hilbert space may be $\mathcal{H}_{HH}$, as one is interested in a quantum field theory on the black hole or deSitter background. But, since this Hilbert space does not carry a unitary representation of the automorphism group of $\tilde{A}_R$, one should consider an extension of it, say $\mathcal{H} = \mathcal{H}_{HH} \otimes L^2(\mathbf{g})$. Then we take the representation, π to be the representation furnished on the GNS Hilbert space, π_β (see section 5.2.1) and identity on the $L^2(\mathbf{g})$ factor. The unitary representation of the automorphism group on the other hand acts as identity on $\mathcal{H}_{HH}$ but acts by mulitplication on the $L^2(\mathbf{g})$ factor. We will discuss in section 5.4 another representation of $\tilde{A}_R$ that is also covariant.

Even though, as stated before, this statement is to be applied to either gravitational or non gravitational systems, there are quite interesting features that arise in gravitational setting and also in non gravitational systems when we think of them as limits of gravitational systems. By a gravitational system we mean that a system where gravitational modes are present and interact with the rest of matter fields perturbatively (AdS/CFT will be the special case where 'we know what we mean' non-perturbatively).

Such a system in general will be described by an action of the form,

$$S = \frac{1}{G_N} \int dx^d \sqrt{-g}(R + \ldots) \tag{5.13}$$

where the ... represents the matter part of the action with minimal coupling to gravity and a boundary Gibbons-Hawking-Yorks term. A non gravitational system will be described by the strict $G_N \to 0$ limit of the above action. We assume the full geometry to be asymptotically flat or AdS but we have not specified the background geometry with respect to which we take this limit, $g_{\mu\nu} = g^0_{\mu\nu} + \sqrt{G_N} h_{\mu\nu}$. We can consider the matter fluctuations (and free graviton) acting on the full spacetime or a subregion and take the algebra of these operators as the starting point. The physically relevant question will be what is the covariant representation of this algebra, i.e, the appropriate Hilbert space and representation for the algebra, where the automorphisms of act unitarily?

To answer this question, 1) we must know what the symmetries (automorphisms) of the algebra of operators are; 2) we must find the Hilbert space where they act unitarily. As we consider the $G_N \to 0$ limit, we note the following; the associated charges for the automorphisms can be obtained from the action (5.13). Thus the generators for the autmorphisms (rather for the unitary representation of the automorphisms) will in general also have the following form[8],

$$Q = \frac{1}{G_N} \int dx^d f(x) \,, \qquad \text{and } [Q, \pi(a)] = O(1) \tag{5.14}$$

where $f(x)$ is a function of the background metric, the metric fluctuations, the matter fields and their derivatives.

As it is the case for the action, we have to subtract the contribution from the background space-time and redefine the charge Q to get a possibly finite result. But, as we take the limit $G_N \to 0$, this only gives finite expectation value for Q except in the cases where the background spacetime is flat or pure AdS. (There are also other exceptions, specifically states with $O(1)$ variance for the charges. We will defer the discussion of these states to section 5.4.) In the other most interesting examples, black holes or semiclassical geometries of coherent states [Skenderis & van Rees, 2009, Botta-Cantcheff et al., 2016, Marolf et al., 2018, Belin et al., 2019], (5.14) diverges specifically as a result of the $G_N \to 0$ limit. In particular such semiclassical states will have $O(1/G_N)$ variance in the charges [Bahiru et al., 2023].[9]

Thus in the strict $G_N \to 0$ limit in particular, the charge that is well defined is,

$$q = \sqrt{G_N} Q. \tag{5.15}$$

and together with (5.14), we have $[q, \pi(a)] = 0$. This implies that in most interesting cases (except in the cases where the background metric is flat or pure AdS) where the renomar-

[8] In general there are more than one generators for the representation of G

[9] For each Hilbert space we assume the existence of a Hadamrad vacuum with respect to which the n-point functions will satisfy the Wick contraction and factorize to 2 point functions.

lization (5.15) is necessary, the automorphism generators will not produce $O(1)$ transformation, in fact they will be central to the representation of the operator algebra $\mathcal{A}$. The consequence of this fact is that the condition (5.12) for the covariant representation will be trivial in the strict large N limit.

Thus the covariant representation of the algebra $\mathcal{A}$ with a symmetry group G ($\mathbf{g}$ being the corresponding lie algebra) will be (π, U) acting on a Hilbert space $\mathcal{H} = \mathcal{H}_{bulk} \otimes L^2(\mathbf{g})$ where $\pi(a)$ will act on $\mathcal{H}_{bulk}$ creating bulk fluctuations and acts as identity on $L^2(\mathbf{g})$ while $U(g) = e^{iq^{\alpha}\nu_{\alpha}}$ acts as identity on $\mathcal{H}_{bulk}$ and by multiplication on $L^2(\mathbf{g})$. In addition, $U(g)$ will commute with the bulk fluctuations $\pi(a)$. But note that when the background geometry is flat or pure AdS, $U(g)$ will have non-trivial action on $\pi(a)$ given by (5.12).

This property, that for general gravitational systems in the strict G_N going to zero limit the automorphism generators do not generate transformations, may look a little unattractive. Let's divide the symmetry groups into the compact and non compact components. Even though the non compact group be could more general and even non abelian, we specifically treat time translations as the non compact group in part because time translation is usually one of the symmetry generators for $\mathcal{A}$ in part because we want to make contact with the cross product construction of [Witten, 2021b, Chandrasekaran et al., 2022a] (we defer the discussion on the compact subgroup to section 5.4). The above argument implies that time development generator commutes with operators in the algebra. *How would we then implement time translations for our operators?*

Keeping the above question in mind, let's consider the modification of the covariant representation (π, U) when gravitational interactions are added perturbatively. Looking at representation π, we can construct operators with coefficients in the ring of power series in $\sqrt{G_N}$. On the other hand, the unitary representations of the symmetry group will transform as follows; concentrating on time translations, as we back away from the strict G_N going to zero limit, since $\sqrt{G_N}Q$ generates transformations of $O(G_N)$, the most natural modification to (5.15) would be

$$\sqrt{G_N}Q_t = q_t - \frac{\sqrt{G_N}}{\beta}\log\Delta. \tag{5.16}$$

For asymptotically AdS eternal black hole spacetime, the above modifications can be showed to be enough to all orders in perturbation theory [Witten, 2021b], "otherwise it is valid to first order in perturbation theory." The unitary representation for time translation symmetry acting on the Hilbert space $\mathcal{H}_{bulk} \otimes L^2(\mathbf{g})$ will thus be $U(g_t) = e^{i(q_t - \sqrt{G_N}/\beta \log\Delta)\nu}$.

This looks quite similar to the cross product construction but our starting point is different and the crossed product construction is in fact distinct from it. The crossed product algebra is a special kind of what is called a covariance algebra (see appendix 5.5). It was shown in [Doplicher et al., 1966] that the covariant representation of an algebra $\mathcal{A}$ and the representations of the covariance algebra associated with $\mathcal{A}$ are in one to one correspondence.

Coming back to the question we asked above, the bulk fluctuations now transform non trivially under the action of $U(g)$, following the equation (5.12). Still, since we are working in perturbation theory, the transformation is infinitesimally small. To produce an $O(1)$ time translations, we propose (inspired by [Leutheusser & Liu, 2021a]) $U(s)$, the half sided modular translation [Leutheusser & Liu, 2021b, Leutheusser & Liu, 2021a, Bahiru,], as the appropriate operator whenever it is possible to define it, particularly because its generator has bounded spectrum. Note that for the other candidate for the generator of a translation operator, $-\log\Delta$, the spectrum is not bounded from below, in general.

5.4 Gravitational interaction: continued

5.4.1 Proto-QG Hilbert space

In the previous section, we introduced the covariaint representation of an algebra. The algebra of our interest was the one discussed in subsection 5.3.1. Up till now we have focused mainly on the algebra where perturbative corrections are added, controlled by G_N, to the algebra of operators for a quantum field theory on a curved spacetime. Now, we discuss the case where no perturbative correction is added to the QFT operators, rather we consider a 'background' state that has $O(1)$ variance in the charges. Thus, we consider a Hilbert space built on top of such a state with bulk fluctuations and where the symmetries of the spacetime are implemented unitarily. As we will see, such a representation corresponds to a system where gravitational modes that are associated with large diffeomorphism is added to the description. Since this representation is not the naively considered Hilbert space of a QFT on a fixed background and yet it arises from the strict $G_N \to 0$ limit of perturbative quantum gravity, we call it proto-quantum gravity (proto-QG) Hilbert space.

Let's consider the covariant representation of the algebra $\mathcal{A}$ on flat or pure AdS spacetimes. Then $\pi(a)$ will be operators that create bulk fluctuations on the flat or AdS spacetime. The unitary representations of the symmetries will be constructed as discussed in subsection 5.3.2. The only difference being the additional normalization of the charges, multiplication by $\sqrt{G_N}$, is no longer necessary. That is, Q is already well defined in the limit $G_N \to 0$, and $[Q, \pi(a)]$ is nonzero $O(1)$ number. Thus, the covariant representation (π, U) where $U = e^{iQv}$ will act on $\mathcal{H}_{bulk} \otimes L^2(\mathbf{g})$. The operators $\pi(a)$ will act on

$\mathcal{H}_{bulk}$ creating bulk fluctuations while acting as identity on $L^2(\mathbf{g})$ and $U(\mathbf{g})$ acts on $\mathcal{H}_{bulk}$ as identity while acting on $L^2(\mathbf{g})$ by multiplication. The fact that Q, for gravitational theories in general, is a boundary quantity together with $[Q, \pi(a)]$ shows that the operators $\pi(a)$ are dressed with respect to the boundary of the spacetime and alludes to the gravitational nature of the Hilbert space even in this $G_N = 0$ limit. Note that the one to one correspondence with the cross product algebra implies that this algebra is a type II_∞ algebra. The cross product algebra in Minkowski and pure AdS was discussed in [Ali Ahmad & Jefferson, 2023].

The gravitational nature of the Hilbert space is even more elaborated in the example of the GNS Hilbert space of the microcanonical thermofield double state [Chandrasekaran et al., 2022b]. We consider a new covariant represenation of the same algebra of operators, $\tilde{\mathcal{A}}_R$ introduced in section 5.2.1. Rather than using the GNS Hilbert space of the thermofield double, we consider the state,

$$|\Phi_\beta\rangle = \frac{1}{e^{S(E_0)}} \sum_i e^{-\beta(E_i - E_0)/2} f(E_i - E_0) \, |E_i\rangle_R \, |E_i\rangle_L \tag{5.17}$$

where, $\int dx \, |f(x)|^2 = 1$ and define an inner product as in (5.1),

$$\langle a|b\rangle = \lim_{N\to\infty} \langle \Phi_\beta| \, ab \, |\Phi_\beta\rangle \ , \qquad \forall a, b \in \tilde{\mathcal{A}}_R \, . \tag{5.18}$$

If Y is the ideal of the algebra with respect to this inner product, we can define the equivalence class, $\tilde{\mathcal{A}}_R/Y$ and complete it to a Hilbert space, H_Φ. Thus, we take the non degenerate representation π_Φ acting on H_Φ as,

$$\pi_\beta(c)\,|a\rangle = |ca\rangle\ , \qquad \forall c, a \in \tilde{\mathcal{A}}_R \tag{5.19}$$

But we notice that the symmetry generator of the algebra $\tilde{\mathcal{A}}_R$ (we only consider the non-compact subgroup for the moment) is also present in this Hilbert space and is well defined in the large N limit. In particular, it the Hamiltonian of the right system. This is in contrast to the Hartle-Hawking state where this observable is not present and we had to consider an extension of the Hilbert space built on $|\psi_{HH}\rangle$ (by the action of the bulk fluctuations) to get a covariant representation. Even the canonical thermofield double Hilbert space includes a covariant representation, i.e, no extension is needed. The only issue with the canonical thermofield double is that the generator has to be normalized twice to be well defined in the large N limit. The obvious difference between the canonical or microcanonical thermofield double and the Hartle Hawking state is that, the former are well defined even in non-perturbtive theory of quantum gravity. This again points to the intrinsically quantum gravitational nature the the covariant representation of an algebra.

Since the subtracted generator, Q of the symmetry is already included in the Hilbert space and well defined large N limit, we take $U(g_t) = e^{iQv}$ as the unitary representation of the symmetry group, where it acts as.

$$Q = q - \frac{1}{\beta}\log\Delta_\Psi. \tag{5.20}$$

This is because it translates the operators $\pi_\Phi(a)$ but differs from the modular Hamiltonian by a central operator, q. Thus the representation (π_Φ, U) acting on the Hilbert space

H_Φ will be a covariant representation of $\tilde{\mathcal{A}}_R$ and it is an example where the variance of the generator is $O(1)$.

The states in the Hilbert space correspond to different semiclassical states in general even though we have set G_N to zero. The reason is that the solution $|\Phi_\beta\rangle$ breaks the symmetry of the theory. In particular, while the symmetry (corresponding to time translations) of the full theory is $\mathbb{R}_R \otimes \mathbb{R}_L$ generated by the left and right Hamiltonians, the solution is only symmetric under the diagonal sub group $\mathbb{R}$, generated by the difference of the two Hamiltonians. This implies that there is a moduli space of classical solutions given again by $\mathbb{R}$. Since U only acts on the right boundary, states related by the action of $U(g_t)$ are associated with different classical solutions on the moduli space. The action of $\pi_\Phi(a)$ on any of the classical solutions would correspond to a given 'Hilbert space' of QFT on a curved spacetime. That is to say that, excluding U, one of these states acted upon by the bulk fluctuations $\pi_\Phi(a)$ is what we expect QFT on a curved spacetime describes. Note that these Hilbert spaces are not the unitraily inequivalent Hilbert spaces described in section 5.3.1, rather they more similar to the sectors corresponding to different charge sectors of a gauge theory. This sectors in a gauge theory are related by a large gauge transformation that does not vanish at infinity. Similarly, the transformation by Q is a diffeomorphism that actually does not vanish in the boundary. This mode parameterizes the timeshift between the left and the right boundary. Thus, we can think of the algebra of operators (π_Φ, U) as an addition of a gravitational mode to the algebra of operators describing bulk fluctuations at $G_N = 0$.

A general state will be a superposition of states with different classical background, thus does not have a semiclassical bulk dual geometry. But, following [Chandrasekaran et al., 2022b], we call a semiclassical state a state with a very large variance in Q (very small variance in the dual mode). The ideal case would be if the variance of Q is $O(1/\sqrt{G_N})$, where we get the canonical thermofield double and so a fixed background geometry.

This feature of the spontaneous breaking of the symmetries can be generalized to any state. Starting with an algebra of operator $\mathcal{A}$ with a symmetry group G, we can choose a classical solution in the theory that breaks the symmetry to G_s. This state can be a coherent state like a collection of galaxies, a black hole or a collapsing geometry. Taking this state as the background geometry we can construct the GNS Hilbert space and describe the quantum field theory on this fixed background (with all the subtleties described in section 5.3.1). Because the 'vacuum' breaks the symmetry G, there is a moduli space of classical solutions parameterized by the quotient G/G_s. But if we look at this state in the limit $G_N \to 0$ of perturbative quantum gravity, charges of the symmetry group G/G_s must have variance of $O(1/G_N)$, since the strict $G_N \to 0$ limit is a quantum field theory on a fixed spacetime while the action of the the generators of G/G_s map a classical solution of the theory to a different one (see also section 5.3.2). Thus these generators will have to be normalized (multiplied by $\sqrt{G_N}$) to be well defined. On the other hand, if we rather take an $O(1)$ superposition of states that correspond to different classical solutions on the moduli space as a starting point for the covariant representation of $\mathcal{A}$, we have mitigated the diverging variance of the charges, at the cost of not having a well defined semiclassical spacetime.

As mentioned before there will be a Hilbert space for each element of the moduli group G/G_s that corresponds to the classical solution. Thus we have a fiber of Hilbert spaces $\mathcal{F}$ on the base space G/G_s. The Hilbert space the covariant representation (π, U) will act on will be L^2 sections of the fiber $\mathcal{F} \to G/G_s$. This is the proto-QG Hilbert space for general states and general symmetry group G that can also be compact. For a symmetry group that is non compact, if the algebra generated by $\pi(a)$ for $a \in \mathcal{A}$ is a type III_1 algebra, then the one to one correspondence with the crossed product algebra implies that the algebra generated by $\pi(a)$ and U will be a type II_∞ algebra, with a well defined trace and entropy.

5.4.2 Subregions and the observer

The last comment we would like to add is concerning sub regions. The naive algebra of operators we associate to subregions are not so well defined in the perturbative theory of quantum gravity since the spacetime is fluctuating, we can not define a fixed subregion with respect to which we define the algebra, all the while the spactime fluctuation itself is expected to be included in the algebra of observables (check [Witten, 2023b, Jensen et al., 2023, Aguilar-Gutierrez et al., 2023] for discussions of algebra of operators associated with subregions in perturbative quantum gravity). But we have a well defined notion of subregions in the QFT on a curved spacetime limit and we can define a covariant representation for this algebra associated with a subregion. The symmetry of a sub region has to preserve the causal diamond of the subregion and one such symmetry is the one that looks like the Lorentz boost symmetry close to the horizon, that acts both on the subregion and its complement. But note that the boost generator that only acts on the subregion is not a well defined operator. In fact we have to do a brick-wall regularization to the operator by demanding a Dirichlet boundary conditions on the fields at some proper distance ε from the horizon. In the limit $\varepsilon \to 0$, we will have a divergence similar to the $G_N \to 0$ divergence of the symmetry generators discussed in 5.3.2. This brick-wall regularization is clear and unambiguous (up to the Weyl anomaly present in even dimensions) in the cases where the theory has a holographic dual, where the renomlization corresponds to a renomalization of Euclidean hyperbolic manifolds in the bulk dual theory, [Bahiru, 2022] (assuming large rank for the gauge group of the boundary theory). Then the algebra generated $\pi(a)$ and U will be a type II_∞ algebra.

Recently, the role of an observer in connection to the algebra of observables in subregions and compact spacetimes like deSitter has been given some attention [Witten, 2023b, Jensen et al., 2023]. The connection between the two is given by the timelike tube theorem ([Witten, 2023b, Strohmaier & Witten, 2023b]) which states that for the so called 'complete' theories, where all the possible electrically and magnetically charged objects (particles, strings, ...) that can couple to the gauge fields (and higher form gauge fields) in the theory are also present, that the algebra of quantum fields smeared along a timelike world line is same as the algebra of quantum fields that are causally accessible to the world line. In other words, if we pick two points, p and q, one to the future of the other along a timelike world line, the algebra of operators that we can construct by smearing the fields in this segment of the worldline is the same as the algebra of operators that are defined in the causal diamond defined by the points p and q. The theorem is proved for quantum field theories in flat and curved spacetimes, [Strohmaier & Witten, 2023a].

The fact that the static patch of deSitter can be understood as the region that is causally accessible to an observer and that algebra of operators along the worldline of the observer is the same as that of the static patch (in QFT on a curved spacetime limit) motivated including an observer in the study the algebra of operators in the static patch in perturbative quantum gravity, where since the spacetime has compact spacelike slices the symmetries of the spacetime are to be treated as gravitational constraints. The algebra of operators is defined in the presence of an observer at one of the boundaries of the Penrose diagram of deSitter, which would otherwise be trivial (just c-numbers).

In our situations, even though it is not shown that the timelike tube theorem is still present in perturbative quantum gravity (as far as the author is aware), the fact that it holds in the QFT limit is enough motivation to study the covariant representation of the algebra of operators along a timelike worldline. Thus we consider an observer in a certain spacetime and the algebra of observables, $\mathcal{A}_{ob}$ constructed from the quantum fields smeared along its worldline.

$$\Phi = \int d\tau f(\tau)\varphi(\tau, x) \tag{5.21}$$

These operators are well defined operators that act on the code subspace of states, that map normalized states to normalizable states.[10] Since in quantum filed theory the algebra of this operators is the same as the algebra of operators in the subregion that is causally accessible to the worldline, the symmetries of this subregion are also the symmetries of the algebra along the world line. Thus the covariant representation for the algebra of operators along a timelike world line includes a representation of the algebra of operators $\pi(\Phi)$[11] and unitaries, U, generated by the symmetry generators of the subregion (let's call them generically Q) associated to the worldline by timelike tube theorem. Note that this symmetry generators will preserve the worldline but since we assumed the existence of an observer, we will need to add operators that are associated with the observer itself. Following a minimal model for the observer as a clock with a given rest mass, m, the only operator that needs to

[10] A general smearing of a quantum field would not map normalized states to normalized states because of the OPE singularities of the quantum fields.

[11] This representation is a representation of operators of the theory on the specific code subspace that includes the observer.

be modified is the time translation operator. If Q_t is one of the symmetry generators that generate time translation along the worldline of the observer, the actual operator that acts on the observer, generating time translations, has to be modified to $Q_t + q_{ob}$ so that its spectrum is bounded from below by m. With the appropriate renormalization of the symmetry generators, as discussed at the beginning of this subsection, the covariant representation of the algebra of operators for an observer will be $\pi(\Phi)$ and U, generated by the symmetry generators where the time translation generator is modified as $Q_t + q_{ob}$.

Following the timelike tube theorem, the operators $\pi(\Phi)$ associated with a time segment on the worldline (or for an observer living at the boundary of one of the static patches) is actually a type III$_1$ von Neumann algebra. On the other, the algebra of operators generated by $\pi(\Phi)$ and U will be type II. Simply renormalizing the generators would result in a type II$_\infty$ algebra but with the presence of an observer, the algebra will be type II$_1$, because of the bounded spectrum of the time translation generator.

Notice that the reason for the algebra associated, for example with the deSitter static patch, is a type II$_1$ algebra is different from the above argument. In [Chandrasekaran et al., 2022a], with the addition of the observer to the deSitter patch the algebra is modified from $\mathcal{A}_0$ (algebra of quantum fluctuations in the patch) to $\mathcal{A}_0 \otimes B(L^2(\mathbb{R}))$, where $B(L^2(\mathbb{R}))$ is the set of bounded functions of the observer's Hamiltonian, $h_{ob} \geq m$. The algebra of operators associated with perturbative QG is the subalgebra that is invariant under the action of the full symmetry generator $h + h_{ob}$, where h is the symmetry generator of deSitter that preserve the static patches[12]. This subalgebra will be composed of h_{ob} and

$$e^{iph} a_0 e^{-iph}, \qquad \forall a_0 \in \mathcal{A}_0, \tag{5.22}$$

if p is the conjugate operator to h_{ob}. This set of operators is a type II_1 algebra by Takesaki duality.

To relate this algebra to the covariant representation given above, we use the timelike tube theorem and claim that $\mathcal{A}_0$ is the same as the algebra generated by $\pi(\Phi)$'s and conjugating the set of operators $\{e^{iph} a_0 e^{-iph}, h_{ob}\}$ by e^{-iph} would give the covariant representation of $\mathcal{A}_{ob}$ and the condition $h_{ob} \geq m$ transforms to $h + h_{ob} \geq m$, which is the condition satisfied by $Q_t + q_{ob}$ for the covariant representation.

Thus we find that the covariant representation of $\mathcal{A}_{ob}$ is the same as the algebra of operators for the deSitter static patch in perturbative quantum gravity. The same can also be said for a subregion, which by the timelike tube theorem is associated to a proper time segment of the observer's worldline. But there is a subtlety here in that the subregion is not well defined in perturbative QG because the spacetime fluctuates with fluctuations of $O(\sqrt{G_N})$, in the same sense there is uncertainty of $O(\sqrt{G_N})$ in specifying a point in time along the worldline, and so there is uncertainty in defining the segment.

5.5 Appendix: Covariance algebra

We will now introduce the definition of the covariance algebra and only state some of its properties. We refer the reader to [Doplicher et al., 1966] for complete discussion.

[12] the operator h is in fact the modular Hamiltonian of $\mathcal{A}_0$.

We consider a C^* algebra $\mathcal{A}$ and a locally compact group G, which to simplify the writing we assume to be Abelian, acting on $\mathcal{A}$ such that for every $g \in G$, there is a linear map $\bar{g}$: $a \in \mathcal{A} \to a(g) \in \mathcal{A}$ with the properties,

$$ab(g) = a(g)b(g) \qquad \text{and } a^\dagger(g) = a(g)^\dagger \qquad \text{for } a, b \in \mathcal{A} \tag{5.23}$$

that preserves the norm of a in $\mathcal{A}$, $|a| = |a(g)|$. In other words the map preserves the group structure of $\mathcal{A}$. It is also a linear map over complex numbers such that,

$$(\alpha a + \beta b)(g) = \alpha a(g) + \beta b(g) \qquad \text{for } \alpha, \beta \text{ complex numbers.} \tag{5.24}$$

Note that this furnishes a representation of the group G in the automorphism group of $\mathcal{A}$, that is,

$$[a(g_1)](g_2) = a(g_1 + g_2) \tag{5.25}$$

$$a(0) = a. \tag{5.26}$$

In addition, the function $a(g)$ is a continuous function of $g \in G$ in the norm topology of $\mathcal{A}$.

We define the covariance algebra $(\mathcal{A}, G)$ as a set of all measurable functions, A, from G to $\mathcal{A}$,

$$A : g \in G \to A_g \in \mathcal{A} \tag{5.27}$$

defined up to a measure zero and absolutely integrable set as long as,

$$|A|_1 = \int |A_g| dg < \infty, \tag{5.28}$$

This means the function A is not necessarily exactly everywhere defined and dg is the Haar measure on G. Note that the element A is not an operator in $\mathcal{A}$, rather a function which takes values in $\mathcal{A}$. The product of A and B, elements of $(\mathcal{A}, G)$, is defined as follows,

$$(A.B)_g = \int A_u B_{g-u}(u) du \tag{5.29}$$

where $B_{g-u}(u)$ is the image under the action of the automorphism group by u, of the element B_{g-u}; while the element B_{g-u} is just the image of B, an element of the covariance algebra at $g - u \in G$. We define the adjoint of A as,

$$A_g^\dagger = A_{-g}(g)^\dagger \tag{5.30}$$

so that $|A^\dagger| = \int |A_{-g}(g)^\dagger| dg = \int |A_{-g}| dg = |A|$.

With these definitions of the adjoint and products of the elements of the covariance algebra, it can be shown that it forms a Banach $^*-$ algebra. In section 3 of [Doplicher et al., 1966] the representations of this algebra are shown to be in one to one correspondence with the covariant representation of $\mathcal{A}$. This rather unfamiliar form for the covariance algebra is discussed in the familiar form of crossed product algebra [Witten, 2021b, Chandrasekaran et al., 2022b] when $\mathcal{A}$ is a von Neumann algebra, in work by Takesaki [Masamichi Takesaki, 1973].

6

Conclusion

In summary, in chapter 3 we have shown that the proposed operation to reconstruct the entanglment wedge, the Petz map, can be applied and explicitly computed in the case of the AdS-Rindler wedge. We have demonstrated that the Petz map is equivalent to the the AdS-Rindler reconstruction of the bulk operators. On the other hand, we have found that for general entanglement wedges, the Petz map reconstruction is equivalent to the the bulk reconstruction proposed by [Jafferis et al., 2016b, Faulkner & Lewkowycz, 2017b] using the modular Hamiltonian. An important future direction would be to continue the explicit Petz map reconstruction for more subtle regions as bulk duals of big disconnected boundary subregions.

In addition, chapter 2 was mostly concerned with defining localized operators in the CFT that commute with the algebra of operators in a small time band. The main obstacle to defining these operators was addressed. In particular, explicit operators that commute with the Hamiltonian constructed and an existence proof is given for operators that commute with all operators in the time band. A small time band in boundary is dual to an annular subregion of the bulk that is connected to the boundary. Therefore, these proposed operators are dual to the bulk operators that are localized on the compliment subregion of the annular region. These operators are natural candidates for local operators associated with a bulk subregions in the presence of gravity. Therefore, an important future direction would be to study their algebraic properties of and check if it is possible to define a von Neumann entropy for the subregions. An ambitious goal can also be to study a net of such local algebras and study what that implies for perturbative quantum gravity in general, in the sense of algebraic quantum field theory. On the other hand, this chapter seems to give evidence for the existence of split states in perturbative quantum gravity and thus strictly localized states. This implies a big steps towards answering whether localization of information is possible in perturbative quantum gravity.

In chapter 4, we have studied the algebra of operators in the AdS-Rindler wedge, which is dual to a spherically symmteric ball in the boundary. The algebra of of operators in the subregions are generally type III_1, due to the diverging entanglment between the subregion and its compliment. The origin this divergence can also be isolated from the bulk dual perspective. We will consider the theory in the large N limit, where there a well defined bulk geometry dual to the boundary subregion. In this limit, there are two sources of divergences discussed in this chapter. These two divergences were regularized and renormalized to define a well behaved operator. This renormaliztion will improve the algebra of operators to a more manageable one called type II_∞, using what is called the cross product construction. An interesting future direction is to study what this renormalized algebra implies for this subregion in holographic CFTs, as in general this is expected to be a type III_1 von Neumann algebra. Even though, we have used the cross product construction is used in this chapter, its physical significance was not quite clear. In chapter 5, we have elaborated that this construction equivalent to a covariant representation of an algebra. Therefore, it is equivalent to extending the Hilbert space and implementing the symmetries of the algebra by a unitary operators. It would be interesting to study what properties we can learn by implementing more abstract symmetries like higher form and non-invertible symmetries on an extended Hilbert space.

Bibliography

[Abt & Erdmenger, 2018] Abt, R. & Erdmenger, J. (2018). Properties of modular hamiltonians on entanglement plateaux. *Journal of High Energy Physics*, 2018(11), 1–23.

[Aguilar-Gutierrez et al., 2023] Aguilar-Gutierrez, S. E., Bahiru, E., & Espíndola, R. (2023). The centaur-algebra of observables.

[Aharony et al., 2008] Aharony, O., Bergman, O., Jafferis, D. L., & Maldacena, J. (2008). N=6 superconformal Chern-Simons-matter theories, M2-branes and their gravity duals. *JHEP*, 10, 091.

[Aharony et al., 2006] Aharony, O., Clark, A. B., & Karch, A. (2006). The CFT/AdS correspondence, massive gravitons and a connectivity index conjecture. *Phys. Rev. D*, 74, 086006.

[Aharony et al., 2007] Aharony, O., David, J. R., Gopakumar, R., Komargodski, Z., & Razamat, S. S. (2007). Comments on worldsheet theories dual to free large N gauge theories. *Phys. Rev. D*, 75, 106006.

[Aharony et al., 2000] Aharony, O., Gubser, S. S., Maldacena, J. M., Ooguri, H., & Oz, Y. (2000). Large N field theories, string theory and gravity. *Phys. Rept.*, 323, 183–386.

[Aharony & Komargodski, 2008] Aharony, O. & Komargodski, Z. (2008). The Space-time operator product expansion in string theory duals of field theories. *JHEP*, 01, 064.

[Aharony et al., 2004] Aharony, O., Marsano, J., Minwalla, S., Papadodimas, K., & Van Raamsdonk, M. (2004). The Hagedorn - deconfinement phase transition in weakly coupled large N gauge theories. *Adv. Theor. Math. Phys.*, 8, 603–696.

[Akers & Penington, 2021] Akers, C. & Penington, G. (2021). Leading order corrections to the quantum extremal surface prescription. *JHEP*, 04, 062.

[Ali Ahmad & Jefferson, 2023] Ali Ahmad, S. & Jefferson, R. (2023). Crossed product algebras and generalized entropy for subregions.

[Almheiri et al., 2015a] Almheiri, A., Dong, X., & Harlow, D. (2015a). Bulk Locality and Quantum Error Correction in AdS/CFT. *JHEP*, 04, 163.

[Almheiri et al., 2015b] Almheiri, A., Dong, X., & Harlow, D. (2015b). Bulk locality and quantum error correction in ads/cft. *Journal of High Energy Physics*, 2015(4), 1–34.

[Almheiri et al., 2019] Almheiri, A., Engelhardt, N., Marolf, D., & Maxfield, H. (2019). The entropy of bulk quantum fields and the entanglement wedge of an evaporating black hole. *JHEP*, 12, 063.

[Almheiri et al., 2020] Almheiri, A., Mahajan, R., Maldacena, J., & Zhao, Y. (2020). The Page curve of Hawking radiation from semiclassical geometry. *JHEP*, 03, 149.

[Almheiri et al., 2018a] Almheiri, A., Mousatov, A., & Shyani, M. (2018a). Escaping the Interiors of Pure Boundary-State Black Holes.

[Almheiri et al., 2018b] Almheiri, A., Mousatov, A., & Shyani, M. (2018b). Escaping the interiors of pure boundary-state black holes. *arXiv preprint arXiv:1803.04434*.

[Anand et al., 2018] Anand, N., Chen, H., Fitzpatrick, A. L., Kaplan, J., & Li, D. (2018). An Exact Operator That Knows Its Location. *JHEP*, 02, 012.

[Arias et al., 2020] Arias, R., Botta-Cantcheff, M., Martinez, P. J., & Zarate, J. F. (2020). Modular hamiltonian for holographic excited states. *Physical Review D*, 102(2), 026021.

[Bahiru,] Bahiru, E. The infalling observer and quantum gravity.

[Bahiru, 2023] Bahiru, E. (2023). Algebras and their Covariant representations in quantum gravity.

[Bahiru et al., 2022] Bahiru, E., Belin, A., Papadodimas, K., Sarosi, G., & Vardian, N. (2022). State-dressed local operators in AdS/CFT.

[Bahiru et al., 2023] Bahiru, E., Belin, A., Papadodimas, K., Sarosi, G., & Vardian, N. (2023). Holography and Localization of Information in Quantum Gravity.

[Bahiru & Vardian, 2022] Bahiru, E. & Vardian, N. (2022). Explicit reconstruction of the entanglement wedge via the Petz map.

[Bahiru, 2022] Bahiru, E. D. (2022). Algebra of Operators in AdS-Rindler.

[Balakrishnan et al., 2019] Balakrishnan, S., Faulkner, T., Khandker, Z. U., & Wang, H. (2019). A general proof of the quantum null energy condition. *Journal of High Energy Physics*, 2019(9), 1–86.

[Balasubramanian et al., 2014] Balasubramanian, V., Chowdhury, B. D., Czech, B., de Boer, J., & Heller, M. P. (2014). Bulk curves from boundary data in holography. *Phys. Rev. D*, 89(8), 086004.

[Balasubramanian et al., 2013] Balasubramanian, V., Czech, B., Chowdhury, B. D., & de Boer, J. (2013). The entropy of a hole in spacetime. *JHEP*, 10, 220.

[Balasubramanian & Kraus, 1999] Balasubramanian, V. & Kraus, P. (1999). A Stress tensor for Anti-de Sitter gravity. *Commun. Math. Phys.*, 208, 413–428.

[Banerjee et al., 2016] Banerjee, S., Bryan, J.-W., Papadodimas, K., & Raju, S. (2016). A toy model of black hole complementarity. *JHEP*, 05, 004.

[Banks et al., 1998] Banks, T., Douglas, M. R., Horowitz, G. T., & Martinec, E. J. (1998). AdS dynamics from conformal field theory.

[Barnum & Knill, 2002] Barnum, H. & Knill, E. (2002). Reversing quantum dynamics with near-optimal quantum and classical fidelity. *Journal of Mathematical Physics*, 43(5), 2097–2106.

[Belin et al., 2022] Belin, A., Biswas, S., & Sully, J. (2022). The spectrum of boundary states in symmetric orbifolds. *JHEP*, 01, 123.

[Belin et al., 2018] Belin, A., Iqbal, N., & Lokhande, S. F. (2018). Bulk entanglement entropy in perturbative excited states. *SciPost Physics*, 5(3), 024.

[Belin et al., 2019] Belin, A., Lewkowycz, A., & Sárosi, G. (2019). The boundary dual of the bulk symplectic form. *Phys. Lett. B*, 789, 71–75.

[Belin & Withers, 2020] Belin, A. & Withers, B. (2020). From sources to initial data and back again: on bulk singularities in Euclidean AdS/CFT. *JHEP*, 12, 185.

[Bena, 2000] Bena, I. (2000). On the construction of local fields in the bulk of AdS(5) and other spaces. *Phys. Rev. D*, 62, 066007.

[Bény et al., 2009] Bény, C., Kempf, A., & Kribs, D. W. (2009). Quantum error correction on infinite-dimensional hilbert spaces. *Journal of mathematical physics*, 50(6), 062108.

[Berenstein, 2004] Berenstein, D. (2004). A Toy model for the AdS / CFT correspondence. *JHEP*, 07, 018.

[Berenstein & Miller, 2017] Berenstein, D. & Miller, A. (2017). Superposition induced topology changes in quantum gravity. *JHEP*, 11, 121.

[Berenstein & Miller, 2018] Berenstein, D. & Miller, A. (2018). Code subspaces for LLM geometries. *Class. Quant. Grav.*, 35(6), 065003.

[Bergmann & Komar, 1960] Bergmann, P. G. & Komar, A. B. (1960). Poisson brackets between locally defined observables in general relativity. *Phys. Rev. Lett.*, 4, 432–433.

[Blanco et al., 2018] Blanco, D., Casini, H., Leston, M., & Rosso, F. (2018). Modular energy inequalities from relative entropy. *Journal of High Energy Physics*, 2018(1), 1–21.

[Blanco & Casini, 2013] Blanco, D. D. & Casini, H. (2013). Localization of negative energy and the bekenstein bound. *Physical review letters*, 111(22), 221601.

[Borchers, 1966] Borchers, H. J. (1966). Energy and momentum as observables in quantum field theory. *Communications in Mathematical Physics*, 2, 49–54.

[Borchers, 1969] Borchers, H. J. (1969). On the implementability of automorphism groups. *Communications in Mathematical Physics*, 14, 305–314.

[Borchers, 2000] Borchers, H.-J. (2000). On revolutionizing quantum field theory with tomita's modular theory. *Journal of mathematical Physics*, 41(6), 3604–3673.

[Botta-Cantcheff et al., 2016] Botta-Cantcheff, M., Martínez, P., & Silva, G. A. (2016). On excited states in real-time AdS/CFT. *JHEP*, 02, 171.

[Bousso, 2013] Bousso, R. (2013). Complementarity Is Not Enough. *Phys. Rev. D*, 87(12), 124023.

[Bousso et al., 2018] Bousso, R., Chandrasekaran, V., Halpern, I. F., & Wall, A. (2018). Asymptotic Charges Cannot Be Measured in Finite Time. *Phys. Rev. D*, 97(4), 046014.

[Bratteli & Robinson, 2012] Bratteli, O. & Robinson, D. W. (2012). *Operator algebras and quantum statistical mechanics: Volume 1: C*-and W*-Algebras. Symmetry Groups. Decomposition of States*. Springer Science & Business Media.

[Brézin et al., 1978] Brézin, E., Itzykson, C., Parisi, G., & Zuber, J.-B. (1978). Planar diagrams. *Communications in Mathematical Physics*, 59(1), 35–51.

[Brown & Henneaux, 1986] Brown, J. D. & Henneaux, M. (1986). Central Charges in the Canonical Realization of Asymptotic Symmetries: An Example from Three-Dimensional Gravity. *Commun. Math. Phys.*, 104, 207–226.

[Buchholz, 1974] Buchholz, D. (1974). PRODUCT STATES FOR LOCAL ALGEBRAS. *Commun. Math. Phys.*, 36, 287–304.

[Buchholz & Wichmann, 1986] Buchholz, D. & Wichmann, E. H. (1986). Causal Independence and the Energy Level Density of States in Local Quantum Field Theory. *Commun. Math. Phys.*, 106, 321.

[Cardy & Tonni, 2016] Cardy, J. & Tonni, E. (2016). Entanglement hamiltonians in two-dimensional conformal field theory. *Journal of Statistical Mechanics: Theory and Experiment*, 2016(12), 123103.

[Cardy, 2004] Cardy, J. L. (2004). Boundary conformal field theory.

[Casini, 2008] Casini, H. (2008). Relative entropy and the bekenstein bound. *Classical and Quantum Gravity*, 25(20), 205021.

[Casini et al., 2011] Casini, H., Huerta, M., & Myers, R. C. (2011). Towards a derivation of holographic entanglement entropy. *Journal of High Energy Physics*, 2011(5), 1–41.

[Castro et al., 2018] Castro, A., Iqbal, N., & Llabrés, E. (2018). Wilson lines and Ishibashi states in $\mathrm{AdS}_3/\mathrm{CFT}_2$. *JHEP*, 09, 066.

[Ceyhan & Faulkner, 2020] Ceyhan, F. & Faulkner, T. (2020). Recovering the qnec from the anec. *Communications in Mathematical Physics*, 377(2), 999–1045.

[Chakravarty, 2021] Chakravarty, J. (2021). Overcounting of interior excitations: A resolution to the bags of gold paradox in AdS. *JHEP*, 02, 027.

[Chandrasekaran et al., 2022a] Chandrasekaran, V., Longo, R., Penington, G., & Witten, E. (2022a). An Algebra of Observables for de Sitter Space.

[Chandrasekaran et al., 2022b] Chandrasekaran, V., Penington, G., & Witten, E. (2022b). Large N algebras and generalized entropy.

[Chen et al., 2020a] Chen, C.-F., Penington, G., & Salton, G. (2020a). Entanglement Wedge Reconstruction using the Petz Map. *JHEP*, 01, 168.

[Chen et al., 2020b] Chen, C.-F., Penington, G., & Salton, G. (2020b). Entanglement wedge reconstruction using the petz map. *Journal of High Energy Physics*, 2020(1), 1–14.

[Chen et al., 2019] Chen, H., Kaplan, J., & Sharma, U. (2019). AdS_3 reconstruction with general gravitational dressings. *JHEP*, 07, 141.

[Chen et al., 2020c] Chen, H. Z., Myers, R. C., Neuenfeld, D., Reyes, I. A., & Sandor, J. (2020c). Quantum Extremal Islands Made Easy, Part II: Black Holes on the Brane. *JHEP*, 12, 025.

[Chen, 2022] Chen, Y. (2022). Spectral form factor for free large N gauge theory and strings. *JHEP*, 06, 137.

[Chen et al., 2018] Chen, Y., Dong, X., Lewkowycz, A., & Qi, X.-L. (2018). Modular flow as a disentangler. *Journal of High Energy Physics*, 2018(12), 1–27.

[Choi et al., 2022] Choi, S., Kim, S., & Song, J. (2022). Supersymmetric Spectral Form Factor and Euclidean Black Holes.

[Chowdhury et al., 2022] Chowdhury, C., Godet, V., Papadoulaki, O., & Raju, S. (2022). Holography from the Wheeler-DeWitt equation. *JHEP*, 03, 019.

[Chowdhury et al., 2021] Chowdhury, C., Papadoulaki, O., & Raju, S. (2021). A physical protocol for observers near the boundary to obtain bulk information in quantum gravity. *SciPost Phys.*, 10(5), 106.

[Cooper et al., 2019a] Cooper, S., Rozali, M., Swingle, B., Van Raamsdonk, M., Waddell, C., & Wakeham, D. (2019a). Black hole microstate cosmology. *JHEP*, 07, 065.

[Cooper et al., 2019b] Cooper, S., Rozali, M., Swingle, B., Van Raamsdonk, M., Waddell, C., & Wakeham, D. (2019b). Black hole microstate cosmology. *Journal of High Energy Physics*, 2019(7), 1–70.

[Corley et al., 2002] Corley, S., Jevicki, A., & Ramgoolam, S. (2002). Exact correlators of giant gravitons from dual N=4 SYM theory. *Adv. Theor. Math. Phys.*, 5, 809–839.

[Corvino & Schoen, 2006] Corvino, J. & Schoen, R. M. (2006). On the asymptotics for the vacuum Einstein constraint equations. *J. Diff. Geom.*, 73(2), 185–217.

[Cotler et al., 2019a] Cotler, J., Hayden, P., Penington, G., Salton, G., Swingle, B., & Walter, M. (2019a). Entanglement Wedge Reconstruction via Universal Recovery Channels. *Phys. Rev. X*, 9(3), 031011.

[Cotler et al., 2019b] Cotler, J., Hayden, P., Penington, G., Salton, G., Swingle, B., & Walter, M. (2019b). Entanglement wedge reconstruction via universal recovery channels. *Physical Review X*, 9(3), 031011.

[Cotler et al., 2017] Cotler, J. S., Gur-Ari, G., Hanada, M., Polchinski, J., Saad, P., Shenker, S. H., Stanford, D., Streicher, A., & Tezuka, M. (2017). Black Holes and Random Matrices. *JHEP*, 05, 118. [Erratum: JHEP 09, 002 (2018)].

[Czech et al., 2019] Czech, B., De Boer, J., Ge, D., & Lamprou, L. (2019). A modular sewing kit for entanglement wedges. *Journal of High Energy Physics*, 2019(11), 1–30.

[Czech et al., 2012] Czech, B., Karczmarek, J. L., Nogueira, F., & Van Raamsdonk, M. (2012). The Gravity Dual of a Density Matrix. *Class. Quant. Grav.*, 29, 155009.

[Czech et al., 2017] Czech, B., Lamprou, L., McCandlish, S., & Sully, J. (2017). Modular berry connection. *arXiv preprint arXiv:1712.07123*.

[Das, 1992] Das, S. R. (1992). The one-dimensional matrix model and string theory. In *Spring School on Superstrings*.

[Das, 2004] Das, S. R. (2004). D-branes in 2-d string theory and classical limits. In *3rd International Symposium on Quantum Theory and Symmetries* (pp. 218–233).

[Das & Jevicki, 2003] Das, S. R. & Jevicki, A. (2003). Large N collective fields and holography. *Phys. Rev. D*, 68, 044011.

[de Boer et al., 2022] de Boer, J., Jafferis, D. L., & Lamprou, L. (2022). On black hole interior reconstruction, singularities and the emergence of time.

[De Boer & Lamprou, 2020] De Boer, J. & Lamprou, L. (2020). Holographic order from modular chaos. *Journal of High Energy Physics*, 2020(6), 1–24.

[de Boer et al., 2019] de Boer, J., Van Breukelen, R., Lokhande, S. F., Papadodimas, K., & Verlinde, E. (2019). On the interior geometry of a typical black hole microstate. *JHEP*, 05, 010.

[De Boer et al., 2020] De Boer, J., Van Breukelen, R., Lokhande, S. F., Papadodimas, K., & Verlinde, E. (2020). Probing typical black hole microstates. *JHEP*, 01, 062.

[de Mello Koch et al., 2011] de Mello Koch, R., Jevicki, A., Jin, K., & Rodrigues, J. P. (2011). AdS_4/CFT_3 Construction from Collective Fields. *Phys. Rev. D*, 83, 025006.

[DeWitt, 1962] DeWitt, B. (1962). *The Quantization of geometry, Gravitation: An Introduction to Current Research (Edited by L. Witten).* Wiley.

[Dewitt, 1999] Dewitt, B. (1999). The Peierls Bracket. *NATO Sci. Ser. C*, 530, 111–136.

[Dhar et al., 1992] Dhar, A., Mandal, G., & Wadia, S. R. (1992). Nonrelativistic fermions, coadjoint orbits of W(infinity) and string field theory at c = 1. *Mod. Phys. Lett. A*, 7, 3129–3146.

[Dhar et al., 1993a] Dhar, A., Mandal, G., & Wadia, S. R. (1993a). Classical Fermi fluid and geometric action for c=1. *Int. J. Mod. Phys. A*, 8, 325–350.

[Dhar et al., 1993b] Dhar, A., Mandal, G., & Wadia, S. R. (1993b). W(infinity) coherent states and path integral derivation of bosonization of nonrelativistic fermions in one-dimension. *Mod. Phys. Lett. A*, 8, 3557–3568.

[Dong et al., 2016a] Dong, X., Harlow, D., & Wall, A. C. (2016a). Reconstruction of Bulk Operators within the Entanglement Wedge in Gauge-Gravity Duality. *Phys. Rev. Lett.*, 117(2), 021601.

[Dong et al., 2016b] Dong, X., Harlow, D., & Wall, A. C. (2016b). Reconstruction of bulk operators within the entanglement wedge in gauge-gravity duality. *Physical review letters*, 117(2), 021601.

[Dong & Lewkowycz, 2018] Dong, X. & Lewkowycz, A. (2018). Entropy, Extremality, Euclidean Variations, and the Equations of Motion. *JHEP*, 01, 081.

[Donnelly & Giddings, 2016] Donnelly, W. & Giddings, S. B. (2016). Observables, gravitational dressing, and obstructions to locality and subsystems. *Phys. Rev. D*, 94(10), 104038.

[Donnelly & Giddings, 2017] Donnelly, W. & Giddings, S. B. (2017). How is quantum information localized in gravity? *Phys. Rev. D*, 96(8), 086013.

[Donnelly & Giddings, 2018] Donnelly, W. & Giddings, S. B. (2018). Gravitational splitting at first order: Quantum information localization in gravity. *Phys. Rev. D*, 98(8), 086006.

[Doplicher et al., 1966] Doplicher, S., Kastler, D., & Robinson, D. W. (1966). Covariance algebras in field theory and statistical mechanics. *Communications in Mathematical Physics*, 3(1), 1–28.

[Doplicher & Longo, 1984] Doplicher, S. & Longo, R. (1984). Standard and split inclusions of von neumann algebras. *Inventiones mathematicae*, 75(3), 493–536.

[Duetsch & Rehren, 2003] Duetsch, M. & Rehren, K.-H. (2003). Generalized free fields and the AdS - CFT correspondence. *Annales Henri Poincare*, 4, 613–635.

[Eberhardt et al., 2019] Eberhardt, L., Gaberdiel, M. R., & Gopakumar, R. (2019). The Worldsheet Dual of the Symmetric Product CFT. *JHEP*, 04, 103.

[Eberhardt et al., 2020] Eberhardt, L., Gaberdiel, M. R., & Gopakumar, R. (2020). Deriving the AdS_3/CFT_2 correspondence. *JHEP*, 02, 136.

[El-Showk & Papadodimas, 2012a] El-Showk, S. & Papadodimas, K. (2012a). Emergent Spacetime and Holographic CFTs. *JHEP*, 10, 106.

[El-Showk & Papadodimas, 2012b] El-Showk, S. & Papadodimas, K. (2012b). Emergent spacetime and holographic cfts. *Journal of High Energy Physics*, 2012(10), 1–72.

[Emparan et al., 1999] Emparan, R., Johnson, C. V., & Myers, R. C. (1999). Surface terms as counterterms in the AdS / CFT correspondence. *Phys. Rev. D*, 60, 104001.

[Engelhardt & Wall, 2015] Engelhardt, N. & Wall, A. C. (2015). Quantum Extremal Surfaces: Holographic Entanglement Entropy beyond the Classical Regime. *JHEP*, 01, 073.

[Erdmenger et al., 2020] Erdmenger, J., Fries, P., Reyes, I. A., & Simon, C. P. (2020). Resolving modular flow: a toolkit for free fermions. *Journal of High Energy Physics*, 2020(12), 1–32.

[Faulkner et al., 2016] Faulkner, T., Leigh, R. G., Parrikar, O., & Wang, H. (2016). Modular hamiltonians for deformed half-spaces and the averaged null energy condition. *Journal of High Energy Physics*, 2016(9), 1–35.

[Faulkner & Lewkowycz, 2017a] Faulkner, T. & Lewkowycz, A. (2017a). Bulk locality from modular flow. *JHEP*, 07, 151.

[Faulkner & Lewkowycz, 2017b] Faulkner, T. & Lewkowycz, A. (2017b). Bulk locality from modular flow. *Journal of High Energy Physics*, 2017(7), 1–31.

[Faulkner et al., 2013] Faulkner, T., Lewkowycz, A., & Maldacena, J. (2013). Quantum corrections to holographic entanglement entropy. *JHEP*, 11, 074.

[Faulkner et al., 2019] Faulkner, T., Li, M., & Wang, H. (2019). A modular toolkit for bulk reconstruction. *Journal of High Energy Physics*, 2019(4), 1–35.

[Festuccia & Liu, 2007] Festuccia, G. & Liu, H. (2007). The Arrow of time, black holes, and quantum mixing of large N Yang-Mills theories. *JHEP*, 12, 027.

[Fitzpatrick et al., 2011] Fitzpatrick, A. L., Katz, E., Poland, D., & Simmons-Duffin, D. (2011). Effective Conformal Theory and the Flat-Space Limit of AdS. *JHEP*, 07, 023.

[Freivogel et al., 2012] Freivogel, B., McGreevy, J., & Suh, S. J. (2012). Exactly Stable Collective Oscillations in Conformal Field Theory. *Phys. Rev. D*, 85, 105002.

[Fujita et al., 2011] Fujita, M., Takayanagi, T., & Tonni, E. (2011). Aspects of AdS/BCFT. *JHEP*, 11, 043.

[Gaberdiel & Gopakumar, 2011] Gaberdiel, M. R. & Gopakumar, R. (2011). An AdS_3 Dual for Minimal Model CFTs. *Phys. Rev. D*, 83, 066007.

[Geng et al., 2022] Geng, H., Karch, A., Perez-Pardavila, C., Raju, S., Randall, L., Riojas, M., & Shashi, S. (2022). Inconsistency of islands in theories with long-range gravity. *JHEP*, 01, 182.

[Giddings, 2013] Giddings, S. B. (2013). Nonviolent nonlocality. *Phys. Rev. D*, 88, 064023.

[Giddings, 2019] Giddings, S. B. (2019). Gravitational dressing, soft charges, and perturbative gravitational splitting. *Phys. Rev. D*, 100(12), 126001.

[Giddings, 2020] Giddings, S. B. (2020). Holography and unitarity. *JHEP*, 11, 056.

[Giddings, 2022] Giddings, S. B. (2022). On the questions of asymptotic recoverability of information and subsystems in quantum gravity. *JHEP*, 08, 227.

[Giddings & Kinsella, 2018] Giddings, S. B. & Kinsella, A. (2018). Gauge-invariant observables, gravitational dressings, and holography in AdS. *JHEP*, 11, 074.

[Giddings et al., 2006] Giddings, S. B., Marolf, D., & Hartle, J. B. (2006). Observables in effective gravity. *Phys. Rev. D*, 74, 064018.

[Ginsparg & Moore, 1993] Ginsparg, P. H. & Moore, G. W. (1993). Lectures on 2-D gravity and 2-D string theory. In *Theoretical Advanced Study Institute (TASI 92): From Black Holes and Strings to Particles* (pp. 277–469).

[Gomez, 2022] Gomez, C. (2022). Cosmology as a Crossed Product.

[Gomez, 2023] Gomez, C. (2023). Entanglement, Observers and Cosmology: a view from von Neumann Algebras.

[Gopakumar, 2004a] Gopakumar, R. (2004a). Free field theory as a string theory? *Comptes Rendus Physique*, 5, 1111–1119.

[Gopakumar, 2004b] Gopakumar, R. (2004b). From free fields to AdS. *Phys. Rev. D*, 70, 025009.

[Gopakumar, 2004c] Gopakumar, R. (2004c). From free fields to AdS. 2. *Phys. Rev. D*, 70, 025010.

[Gopakumar, 2005] Gopakumar, R. (2005). From free fields to AdS: III. *Phys. Rev. D*, 72, 066008.

[Gopakumar & Mazenc, 2022] Gopakumar, R. & Mazenc, E. A. (2022). Deriving the Simplest Gauge-String Duality – I: Open-Closed-Open Triality.

[Greenberg, 1961] Greenberg, O. (1961). Generalized free fields and models of local field theory. *Annals of Physics*, 16(2), 158–176.

[Guica & Ross, 2015] Guica, M. & Ross, S. F. (2015). Behind the geon horizon. *Class. Quant. Grav.*, 32(5), 055014.

[Guo, 2018] Guo, W.-z. (2018). Entanglement Properties of Boundary State and Thermalization. *JHEP*, 06, 044.

[Haag, 1992a] Haag, R. (1992a). Local quantum mechanics; fields, particles, algebras.

[Haag, 1992b] Haag, R. (1992b). *Local quantum physics: Fields, particles, algebras*.

[Haag & Kastler, 1964] Haag, R. & Kastler, D. (1964). An Algebraic approach to quantum field theory. *J. Math. Phys.*, 5, 848–861.

[Haag & Schroer, 1962a] Haag, R. & Schroer, B. (1962a). Postulates of quantum field theory. *Journal of Mathematical Physics*, 3(2), 248–256.

[Haag & Schroer, 1962b] Haag, R. & Schroer, B. (1962b). The postulates of quantum field theory.

[Haag & Swieca, 1965] Haag, R. & Swieca, J. A. (1965). When does a quantum field theory describe particles? *Communications in Mathematical Physics*, 1(4), 308 – 320.

[Hamilton et al., 2006a] Hamilton, A., Kabat, D., Lifschytz, G., & Lowe, D. A. (2006a). Holographic representation of local bulk operators. *Physical Review D*, 74(6), 066009.

[Hamilton et al., 2006b] Hamilton, A., Kabat, D., Lifschytz, G., & Lowe, D. A. (2006b). Local bulk operators in ads/cft correspondence: A boundary view of horizons and locality. *Physical Review D*, 73(8), 086003.

[Hamilton et al., 2007a] Hamilton, A., Kabat, D., Lifschytz, G., & Lowe, D. A. (2007a). Local bulk operators in ads/cft correspondence: A holographic description of the black hole interior. *Physical Review D*, 75(10), 106001.

[Hamilton et al., 2008a] Hamilton, A., Kabat, D., Lifschytz, G., & Lowe, D. A. (2008a). Local bulk operators in ads/cft and the fate of the btz singularity. *AMS/IP Stud. Adv. Math*, 44, 85.

[Hamilton et al., 2006c] Hamilton, A., Kabat, D. N., Lifschytz, G., & Lowe, D. A. (2006c). Holographic representation of local bulk operators. *Phys. Rev. D*, 74, 066009.

[Hamilton et al., 2006d] Hamilton, A., Kabat, D. N., Lifschytz, G., & Lowe, D. A. (2006d). Local bulk operators in AdS/CFT: A Boundary view of horizons and locality. *Phys. Rev. D*, 73, 086003.

[Hamilton et al., 2007b] Hamilton, A., Kabat, D. N., Lifschytz, G., & Lowe, D. A. (2007b). Local bulk operators in AdS/CFT: A Holographic description of the black hole interior. *Phys. Rev. D*, 75, 106001. [Erratum: Phys.Rev.D 75, 129902 (2007)].

[Hamilton et al., 2008b] Hamilton, A., Kabat, D. N., Lifschytz, G., & Lowe, D. A. (2008b). Local bulk operators in AdS/CFT and the fate of the BTZ singularity. *AMS/IP Stud. Adv. Math.*, 44, 85–100.

[Harlow, 2014] Harlow, D. (2014). Aspects of the Papadodimas-Raju Proposal for the Black Hole Interior. *JHEP*, 11, 055.

[Harlow & Stanford, 2011] Harlow, D. & Stanford, D. (2011). Operator Dictionaries and Wave Functions in AdS/CFT and dS/CFT.

[Hawking & Ellis, 2011] Hawking, S. W. & Ellis, G. F. R. (2011). *The Large Scale Structure of Space-Time*. Cambridge Monographs on Mathematical Physics. Cambridge University Press.

[Hayden et al., 2004] Hayden, P., Jozsa, R., Petz, D., & Winter, A. (2004). Structure of states which satisfy strong subadditivity of quantum entropy with equality. *Communications in mathematical physics*, 246(2), 359–374.

[Hayden & Penington, 2019] Hayden, P. & Penington, G. (2019). Learning the Alpha-bits of Black Holes. *JHEP*, 12, 007.

[Headrick, 2010] Headrick, M. (2010). Entanglement rényi entropies in holographic theories. *Phys. Rev. D*, 82, 126010.

[Headrick et al., 2014a] Headrick, M., Hubeny, V. E., Lawrence, A., & Rangamani, M. (2014a). Causality & holographic entanglement entropy. *JHEP*, 12, 162.

[Headrick et al., 2014b] Headrick, M., Myers, R. C., & Wien, J. (2014b). Holographic Holes and Differential Entropy. *JHEP*, 10, 149.

[Heemskerk et al., 2012] Heemskerk, I., Marolf, D., Polchinski, J., & Sully, J. (2012). Bulk and Transhorizon Measurements in AdS/CFT. *JHEP*, 10, 165.

[Heemskerk et al., 2009] Heemskerk, I., Penedones, J., Polchinski, J., & Sully, J. (2009). Holography from Conformal Field Theory. *JHEP*, 10, 079.

[Heemskerk & Sully, 2010] Heemskerk, I. & Sully, J. (2010). More Holography from Conformal Field Theory. *JHEP*, 09, 099.

[Hellwig & Kraus, 1970] Hellwig, K.-E. & Kraus, K. (1970). Operations and measurements. ii. *Communications in mathematical physics*, 16(2), 142–147.

[Henningson & Skenderis, 1998] Henningson, M. & Skenderis, K. (1998). The Holographic Weyl anomaly. *JHEP*, 07, 023.

[Hubeny et al., 2007] Hubeny, V. E., Rangamani, M., & Takayanagi, T. (2007). A Covariant holographic entanglement entropy proposal. *JHEP*, 07, 062.

[Isham, 1993] Isham, C. J. (1993). Canonical quantum gravity and the problem of time. *NATO Sci. Ser. C*, 409, 157–287.

[Jacobson & Nguyen, 2019] Jacobson, T. & Nguyen, P. (2019). Diffeomorphism invariance and the black hole information paradox. *Phys. Rev. D*, 100(4), 046002.

[Jafferis et al., 2016a] Jafferis, D. L., Lewkowycz, A., Maldacena, J., & Suh, S. J. (2016a). Relative entropy equals bulk relative entropy. *JHEP*, 06, 004.

[Jafferis et al., 2016b] Jafferis, D. L., Lewkowycz, A., Maldacena, J., & Suh, S. J. (2016b). Relative entropy equals bulk relative entropy. *Journal of High Energy Physics*, 2016(6), 1–20.

[Jafferis & Suh, 2016] Jafferis, D. L. & Suh, S. J. (2016). The gravity duals of modular hamiltonians. *Journal of High Energy Physics*, 2016(9), 1–30.

[Jenčová & Petz, 2006] Jenčová, A. & Petz, D. (2006). Sufficiency in quantum statistical inference. *Communications in mathematical physics*, 263(1), 259–276.

[Jensen et al., 2023] Jensen, K., Sorce, J., & Speranza, A. (2023). Generalized entropy for general subregions in quantum gravity.

[Jevicki & Sakita, 1980] Jevicki, A. & Sakita, B. (1980). The Quantum Collective Field Method and Its Application to the Planar Limit. *Nucl. Phys. B*, 165, 511.

[Junge et al., 2018] Junge, M., Renner, R., Sutter, D., Wilde, M. M., & Winter, A. (2018). Universal recovery maps and approximate sufficiency of quantum relative entropy. In *Annales Henri Poincaré*, volume 19 (pp. 2955–2978).: Springer.

[Kabat & Lifschytz, 2013] Kabat, D. & Lifschytz, G. (2013). CFT representation of interacting bulk gauge fields in AdS. *Phys. Rev. D*, 87(8), 086004.

[Kabat & Lifschytz, 2014] Kabat, D. & Lifschytz, G. (2014). Decoding the hologram: Scalar fields interacting with gravity. *Phys. Rev. D*, 89(6), 066010.

[Kabat et al., 2011] Kabat, D., Lifschytz, G., & Lowe, D. A. (2011). Constructing local bulk observables in interacting AdS/CFT. *Phys. Rev. D*, 83, 106009.

[Kaplan, 2016] Kaplan, J. (2016). Lectures on ads/cft from the bottom up.

[Karch & Randall, 2001] Karch, A. & Randall, L. (2001). Open and closed string interpretation of SUSY CFT's on branes with boundaries. *JHEP*, 06, 063.

[Khavkine, 2015] Khavkine, I. (2015). Local and gauge invariant observables in gravity. *Class. Quant. Grav.*, 32(18), 185019.

[Kim, 2017] Kim, J.-W. (2017). Explicit reconstruction of the entanglement wedge. *Journal of High Energy Physics*, 2017(1), 1–31.

[Kiritsis, 2006] Kiritsis, E. (2006). Product CFTs, gravitational cloning, massive gravitons and the space of gravitational duals. *JHEP*, 11, 049.

[Kiritsis & Niarchos, 2008] Kiritsis, E. & Niarchos, V. (2008). (Multi)Matrix Models and Interacting Clones of Liouville Gravity. *JHEP*, 08, 044.

[Kiritsis & Niarchos, 2011] Kiritsis, E. & Niarchos, V. (2011). Large-N limits of 2d CFTs, Quivers and AdS_3 duals. *JHEP*, 04, 113.

[Klebanov & Polyakov, 2002] Klebanov, I. R. & Polyakov, A. M. (2002). AdS dual of the critical O(N) vector model. *Phys. Lett. B*, 550, 213–219.

[Koeller et al., 2018] Koeller, J., Leichenauer, S., Levine, A., & Shahbazi-Moghaddam, A. (2018). Local modular hamiltonians from the quantum null energy condition. *Physical Review D*, 97(6), 065011.

[Komar, 1958] Komar, A. (1958). Construction of a complete set of independent observables in the general theory of relativity. *Phys. Rev.*, 111, 1182–1187.

[Kontsevich & Segal, 2021] Kontsevich, M. & Segal, G. (2021). Wick Rotation and the Positivity of Energy in Quantum Field Theory. *Quart. J. Math. Oxford Ser.*, 72(1-2), 673–699.

[Kourkoulou & Maldacena, 2017] Kourkoulou, I. & Maldacena, J. (2017). Pure states in the SYK model and nearly-AdS_2 gravity.

[Krasnov & Schlenker, 2008] Krasnov, K. & Schlenker, J.-M. (2008). On the renormalized volume of hyperbolic 3-manifolds. *Commun. Math. Phys.*, 279, 637–668.

[Kraus et al., 1983] Kraus, K., Böhm, A., Dollard, J. D., & Wootters, W. H. (1983). *States, Effects, and Operations Fundamental Notions of Quantum Theory*, volume 190.

[Kuchar, 2011] Kuchar, K. V. (2011). Time and interpretations of quantum gravity. *Int. J. Mod. Phys. D*, 20, 3–86.

[Laddha et al., 2021] Laddha, A., Prabhu, S. G., Raju, S., & Shrivastava, P. (2021). The Holographic Nature of Null Infinity. *SciPost Phys.*, 10(2), 041.

[Lashkari, 2016] Lashkari, N. (2016). Modular hamiltonian for excited states in conformal field theory. *Physical review letters*, 117(4), 041601.

[Lashkari, 2019] Lashkari, N. (2019). Constraining quantum fields using modular theory. *Journal of High Energy Physics*, 2019(1), 1–31.

[Lashkari, 2021] Lashkari, N. (2021). Modular zero modes and sewing the states of qft. *Journal of High Energy Physics*, 2021(4), 1–25.

[Lashkari et al., 2021] Lashkari, N., Liu, H., & Rajagopal, S. (2021). Modular flow of excited states. *Journal of High Energy Physics*, 2021(9), 1–50.

[Leichenauer & Rosenhaus, 2013] Leichenauer, S. & Rosenhaus, V. (2013). Ads black holes, the bulk-boundary dictionary, and smearing functions. *Physical Review D*, 88(2), 026003.

[Leigh & Petkou, 2003] Leigh, R. G. & Petkou, A. C. (2003). Holography of the N=1 higher spin theory on AdS(4). *JHEP*, 06, 011.

[Leutheusser & Liu, 2021a] Leutheusser, S. & Liu, H. (2021a). Causal connectability between quantum systems and the black hole interior in holographic duality.

[Leutheusser & Liu, 2021b] Leutheusser, S. & Liu, H. (2021b). Emergent times in holographic duality.

[Leutheusser & Liu, 2022] Leutheusser, S. & Liu, H. (2022). Subalgebra-subregion duality: emergence of space and time in holography.

[Lin et al., 2004] Lin, H., Lunin, O., & Maldacena, J. M. (2004). Bubbling AdS space and 1/2 BPS geometries. *JHEP*, 10, 025.

[Lindblad, 1975] Lindblad, G. (1975). Completely positive maps and entropy inequalities. *Communications in Mathematical Physics*, 40(2), 147–151.

[Louko & Marolf, 1999] Louko, J. & Marolf, D. (1999). Single exterior black holes and the AdS / CFT conjecture. *Phys. Rev. D*, 59, 066002.

[Maldacena & Stanford, 2016] Maldacena, J. & Stanford, D. (2016). Remarks on the Sachdev-Ye-Kitaev model. *Phys. Rev. D*, 94(10), 106002.

[Maldacena & Susskind, 2013] Maldacena, J. & Susskind, L. (2013). Cool horizons for entangled black holes. *Fortsch. Phys.*, 61, 781–811.

[Maldacena, 1998] Maldacena, J. M. (1998). The Large N limit of superconformal field theories and supergravity. *Adv. Theor. Math. Phys.*, 2, 231–252.

[Maldacena, 2003] Maldacena, J. M. (2003). Eternal black holes in anti-de Sitter. *JHEP*, 04, 021.

[Maldacena & Strominger, 1998] Maldacena, J. M. & Strominger, A. (1998). AdS(3) black holes and a stringy exclusion principle. *JHEP*, 12, 005.

[Maldacena et al., 1997] Maldacena, J. M., Strominger, A., & Witten, E. (1997). Black hole entropy in M theory. *JHEP*, 12, 002.

[Marolf, 2009] Marolf, D. (2009). Unitarity and Holography in Gravitational Physics. *Phys. Rev. D*, 79, 044010.

[Marolf, 2015] Marolf, D. (2015). Comments on Microcausality, Chaos, and Gravitational Observables. *Class. Quant. Grav.*, 32(24), 245003.

[Marolf et al., 2018] Marolf, D., Parrikar, O., Rabideau, C., Izadi Rad, A., & Van Raamsdonk, M. (2018). From Euclidean Sources to Lorentzian Spacetimes in Holographic Conformal Field Theories. *JHEP*, 06, 077.

[Marolf & Wien, 2018] Marolf, D. & Wien, J. (2018). The Torus Operator in Holography. *JHEP*, 01, 105.

[Masamichi Takesaki, 1967] Masamichi Takesaki (1967). Covariant representations ofC*-algebras and their locally compact automorphism groups. *Acta Mathematica*, 119, 273–303.

[Masamichi Takesaki, 1973] Masamichi Takesaki (1973). Duality for crossed products and the structure of von Neumann algebras of type III. *Acta Mathematica*, 131, 249–310.

[Maxfield et al., 2016] Maxfield, H., Ross, S., & Way, B. (2016). Holographic partition functions and phases for higher genus Riemann surfaces. *Class. Quant. Grav.*, 33(12), 125018.

[Minasian et al., 2000] Minasian, R., Moore, G. W., & Tsimpis, D. (2000). Calabi-Yau black holes and (0,4) sigma models. *Commun. Math. Phys.*, 209, 325–352.

[Mintun et al., 2015] Mintun, E., Polchinski, J., & Rosenhaus, V. (2015). Bulk-Boundary Duality, Gauge Invariance, and Quantum Error Corrections. *Phys. Rev. Lett.*, 115(15), 151601.

[Miyaji et al., 2015] Miyaji, M., Ryu, S., Takayanagi, T., & Wen, X. (2015). Boundary States as Holographic Duals of Trivial Spacetimes. *JHEP*, 05, 152.

[Miyaji et al., 2021] Miyaji, M., Takayanagi, T., & Ugajin, T. (2021). Spectrum of End of the World Branes in Holographic BCFTs. *JHEP*, 06, 023.

[Morrison, 2014] Morrison, I. A. (2014). Boundary-to-bulk maps for ads causal wedges and the reeh-schlieder property in holography. *Journal of High Energy Physics*, 2014(5), 1–29.

[Mosonyi & Petz, 2004] Mosonyi, M. & Petz, D. (2004). Structure of sufficient quantum coarse-grainings. *Letters in Mathematical Physics*, 68(1), 19–30.

[Myers et al., 2014] Myers, R. C., Rao, J., & Sugishita, S. (2014). Holographic Holes in Higher Dimensions. *JHEP*, 06, 044.

[Page & Wootters, 1983] Page, D. N. & Wootters, W. K. (1983). EVOLUTION WITHOUT EVOLUTION: DYNAMICS DESCRIBED BY STATIONARY OBSERVABLES. *Phys. Rev. D*, 27, 2885.

[Papadodimas & Raju, 2013] Papadodimas, K. & Raju, S. (2013). An Infalling Observer in AdS/CFT. *JHEP*, 10, 212.

[Papadodimas & Raju, 2014a] Papadodimas, K. & Raju, S. (2014a). Black Hole Interior in the Holographic Correspondence and the Information Paradox. *Phys. Rev. Lett.*, 112(5), 051301.

[Papadodimas & Raju, 2014b] Papadodimas, K. & Raju, S. (2014b). State-Dependent Bulk-Boundary Maps and Black Hole Complementarity. *Phys. Rev. D*, 89(8), 086010.

[Papadodimas & Raju, 2015] Papadodimas, K. & Raju, S. (2015). Local Operators in the Eternal Black Hole. *Phys. Rev. Lett.*, 115(21), 211601.

[Papadodimas & Raju, 2016] Papadodimas, K. & Raju, S. (2016). Remarks on the necessity and implications of state-dependence in the black hole interior. *Phys. Rev. D*, 93(8), 084049.

[Peierls, 1952] Peierls, R. E. (1952). The Commutation laws of relativistic field theory. *Proc. Roy. Soc. Lond. A*, 214, 143–157.

[Penedones, 2011] Penedones, J. (2011). Writing CFT correlation functions as AdS scattering amplitudes. *JHEP*, 03, 025.

[Penington, 2020a] Penington, G. (2020a). Entanglement wedge reconstruction and the information paradox. *Journal of High Energy Physics*, 2020(9), 1–84.

[Penington, 2020b] Penington, G. (2020b). Entanglement Wedge Reconstruction and the Information Paradox. *JHEP*, 09, 002.

[Penington et al., 2019] Penington, G., Shenker, S. H., Stanford, D., & Yang, Z. (2019). Replica wormholes and the black hole interior. *arXiv preprint arXiv:1911.11977*.

[Penington et al., 2022] Penington, G., Shenker, S. H., Stanford, D., & Yang, Z. (2022). Replica wormholes and the black hole interior. *JHEP*, 03, 205.

[Penington & Witten, 2023] Penington, G. & Witten, E. (2023). Algebras and States in JT Gravity.

[Petkou, 2003] Petkou, A. C. (2003). Evaluating the AdS dual of the critical O(N) vector model. *JHEP*, 03, 049.

[Petz, 1986] Petz, D. (1986). Sufficient subalgebras and the relative entropy of states of a von neumann algebra. *Communications in mathematical physics*, 105(1), 123–131.

[Petz, 1988] Petz, D. (1988). Sufficiency of channels over von neumann algebras. *The Quarterly Journal of Mathematics*, 39(1), 97–108.

[Petz, 2003] Petz, D. (2003). Monotonicity of quantum relative entropy revisited. *Reviews in Mathematical Physics*, 15(01), 79–91.

[Polchinski, 1991] Polchinski, J. (1991). Classical limit of (1+1)-dimensional string theory. *Nucl. Phys. B*, 362, 125–140.

[Preskill, 1998] Preskill, J. (1998). Lecture notes for physics 229: Quantum information and computation. *California Institute of Technology*, 16(1), 100–104.

[Reeves et al., 2021] Reeves, W., Rozali, M., Simidzija, P., Sully, J., Waddell, C., & Wakeham, D. (2021). Looking for (and not finding) a bulk brane. *JHEP*, 12, 002.

[Rey & Rosenhaus, 2014] Rey, S.-J. & Rosenhaus, V. (2014). Scanning tunneling macroscopy, black holes and ads/cft bulk locality. *Journal of High Energy Physics*, 2014(7), 1–28.

[Roos, 1970] Roos, H. (1970). Independence of local algebras in quantum field theory. *Commun. Math. Phys.*, 16, 238–246.

[Ryu & Takayanagi, 2006] Ryu, S. & Takayanagi, T. (2006). Holographic derivation of entanglement entropy from AdS/CFT. *Phys. Rev. Lett.*, 96, 181602.

[Saad et al., 2018] Saad, P., Shenker, S. H., & Stanford, D. (2018). A semiclassical ramp in SYK and in gravity.

[Sárosi & Ugajin, 2018] Sárosi, G. & Ugajin, T. (2018). Modular hamiltonians of excited states, ope blocks and emergent bulk fields. *Journal of High Energy Physics*, 2018(1), 1–36.

[Schlenker & Witten, 2022] Schlenker, J.-M. & Witten, E. (2022). No ensemble averaging below the black hole threshold. *JHEP*, 07, 143.

[Sezgin & Sundell, 2002] Sezgin, E. & Sundell, P. (2002). Massless higher spins and holography. *Nucl. Phys. B*, 644, 303–370. [Erratum: Nucl.Phys.B 660, 403–403 (2003)].

[Shapiro, 1981] Shapiro, J. A. (1981). A Test of the Collective Field Method for the $N \to$ Infinity Limit. *Nucl. Phys. B*, 184, 218–224.

[Shenker & Stanford, 2014] Shenker, S. H. & Stanford, D. (2014). Black holes and the butterfly effect. *JHEP*, 03, 067.

[Skenderis & van Rees, 2008] Skenderis, K. & van Rees, B. C. (2008). Real-time gauge/gravity duality. *Phys. Rev. Lett.*, 101, 081601.

[Skenderis & van Rees, 2009] Skenderis, K. & van Rees, B. C. (2009). Real-time gauge/gravity duality: Prescription, Renormalization and Examples. *JHEP*, 05, 085.

[Streater & Wightman, 1989] Streater, R. F. & Wightman, A. S. (1989). *PCT, spin and statistics, and all that*.

[Strohmaier & Witten, 2023a] Strohmaier, A. & Witten, E. (2023a). Analytic states in quantum field theory on curved spacetimes.

[Strohmaier & Witten, 2023b] Strohmaier, A. & Witten, E. (2023b). The Timelike Tube Theorem in Curved Spacetime.

[Sundborg, 2000] Sundborg, B. (2000). The Hagedorn transition, deconfinement and N=4 SYM theory. *Nucl. Phys. B*, 573, 349–363.

[Susskind et al., 1993] Susskind, L., Thorlacius, L., & Uglum, J. (1993). The Stretched horizon and black hole complementarity. *Phys. Rev. D*, 48, 3743–3761.

[Sutter et al., 2016] Sutter, D., Tomamichel, M., & Harrow, A. W. (2016). Strengthened monotonicity of relative entropy via pinched petz recovery map. *IEEE Transactions on Information Theory*, 62(5), 2907–2913.

['t Hooft, 1985] 't Hooft, G. (1985). On the Quantum Structure of a Black Hole. *Nucl. Phys. B*, 256, 727–745.

[Takayanagi, 2011] Takayanagi, T. (2011). Holographic Dual of BCFT. *Phys. Rev. Lett.*, 107, 101602.

[Takesaki, 1970] Takesaki, M. (1970). Tensor product and direct sum of modular hilbert algebras. *Tomita's Theory of Modular Hilbert Algebras and its Applications*, (pp. 57–60).

[Takesaki, 1973] Takesaki, M. (1973). Duality for crossed products and the structure of von neumann algebras of type iii. *Acta Mathematica*, 131, 249–310.

[Takesaki et al., 2003] Takesaki, M. et al. (2003). *Theory of operator algebras II*, volume 125. Springer.

[Uhlmann, 1977] Uhlmann, A. (1977). Relative entropy and the wigner-yanase-dyson-lieb concavity in an interpolation theory. *Communications in Mathematical Physics*, 54(1), 21–32.

[Verlinde & Verlinde, 2013] Verlinde, E. & Verlinde, H. (2013). Black Hole Entanglement and Quantum Error Correction. *JHEP*, 10, 107.

[Verlinde & Zurek, 2020] Verlinde, E. & Zurek, K. M. (2020). Spacetime Fluctuations in AdS/CFT. *JHEP*, 04, 209.

[von Neumann, 1940] von Neumann, J. (1940). On rings of operators. iii. *Annals of Mathematics*, 41(1), 94–161.

[von Neumann, 1949] von Neumann, J. (1949). On rings of operators. reduction theory. *Annals of Mathematics*, 50(2), 401–485.

[Wald, 1984] Wald, R. M. (1984). *General Relativity*. Chicago, USA: Chicago Univ. Pr.

[Wald, 1995] Wald, R. M. (1995). *Quantum Field Theory in Curved Space-Time and Black Hole Thermodynamics*. Chicago Lectures in Physics. Chicago, IL: University of Chicago Press.

[Wall, 2014] Wall, A. C. (2014). Maximin Surfaces, and the Strong Subadditivity of the Covariant Holographic Entanglement Entropy. *Class. Quant. Grav.*, 31(22), 225007.

[Wigner, 1959] Wigner, E. P. . (1959). *Proceedings of the fourth Canadian Mathematical Congress, Banff, 1957*. Toronto: University of Toronto Press.

[Wilde, 2015] Wilde, M. M. (2015). Recoverability in quantum information theory. *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences*, 471(2182), 20150338.

[Witten, 1998] Witten, E. (1998). Anti-de Sitter space and holography. *Adv. Theor. Math. Phys.*, 2, 253–291.

[Witten, 2018] Witten, E. (2018). APS Medal for Exceptional Achievement in Research: Invited article on entanglement properties of quantum field theory. *Rev. Mod. Phys.*, 90(4), 045003.

[Witten, 2021a] Witten, E. (2021a). A Note On Complex Spacetime Metrics.

[Witten, 2021b] Witten, E. (2021b). Gravity and the Crossed Product.

[Witten, 2021c] Witten, E. (2021c). Why Does Quantum Field Theory In Curved Spacetime Make Sense? And What Happens To The Algebra of Observables In The Thermodynamic Limit?

[Witten, 2023a] Witten, E. (2023a). A Background Independent Algebra in Quantum Gravity.

[Witten, 2023b] Witten, E. (2023b). Algebras, Regions, and Observers.

[Yaffe, 1982] Yaffe, L. G. (1982). Large n limits as classical mechanics. *Reviews of Modern Physics*, 54(2), 407.