



SCUOLA INTERNAZIONALE SUPERIORE DI STUDI AVANZATI

SISSA Digital Library

The d-critical structure on the Quot scheme of points of a Calabi-Yau 3-fold

Original

The d-critical structure on the Quot scheme of points of a Calabi-Yau 3-fold / Ricolfi, At; Savvas, M. - In: COMMUNICATIONS IN CONTEMPORARY MATHEMATICS. - ISSN 0219-1997. - (2024), pp. 1-38. [10.1142/S0219199723500384]

Availability:

This version is available at: 20.500.11767/135032 since: 2023-11-28T11:56:22Z

Publisher:

Published

DOI:10.1142/S0219199723500384

Terms of use:

Testo definito dall'ateneo relativo alle clausole di concessione d'uso

Publisher copyright

World Scientific Publishing Co.

This version is available for education and non-commercial purposes.

note finali coverpage

(Article begins on next page)

The d-critical structure on the Quot scheme of points of a Calabi–Yau 3-fold

ANDREA T. RICOLFI AND MICHAEL SAVVAS

ABSTRACT. The Artin stack \mathcal{M}_n of 0-dimensional sheaves of length n on \mathbb{A}^3 carries two natural d-critical structures in the sense of Joyce. One comes from its description as a quotient stack $[\text{crit}(f_n)/\text{GL}_n]$, another comes from derived deformation theory of sheaves. We show that these d-critical structures agree. We use this result to prove the analogous statement for the Quot scheme of points $\text{Quot}_{\mathbb{A}^3}(\mathcal{O}^{\oplus r}, n) = \text{crit}(f_{r,n})$, which is a global critical locus for every $r > 0$, and also carries a derived-in-flavour d-critical structure besides the one induced by the potential $f_{r,n}$. Again, we show these two d-critical structures agree. Moreover, we prove that they locally model the d-critical structure on $\text{Quot}_X(F, n)$, where F is a locally free sheaf of rank r on a projective Calabi–Yau 3-fold X .

Finally, we prove that the perfect obstruction theory on $\text{Hilb}^n \mathbb{A}^3 = \text{crit}(f_{1,n})$ induced by the Atiyah class of the universal ideal agrees with the *critical* obstruction theory induced by the Hessian of the potential $f_{1,n}$.

CONTENTS

0. Introduction	1
1. Background material	5
2. Proof of Theorem A	8
3. The d-critical structure(s) on \mathcal{M}_n	11
4. The d-critical structure(s) on $\text{Quot}_{\mathbb{A}^3}(\mathcal{O}^{\oplus r}, n)$	24
5. The case of a compact Calabi–Yau 3-fold	28
6. The special case of the Hilbert scheme	32
References	37

0. INTRODUCTION

0.1. Overview. In [21], Joyce introduced *d-critical structures* on schemes and Artin stacks, both in algebraic and analytic language. Roughly speaking, a d-critical scheme is a scheme X suitably covered by ‘d-critical charts’, i.e. schemes of the form $\text{crit}(f) \hookrightarrow V$, where $f \in \mathcal{O}(V)$ is a regular function on a smooth scheme V and $\text{crit}(f)$ denotes the scheme-theoretic vanishing locus of $df \in H^0(V, \Omega_V)$. The compatibility between d-critical charts is governed by the behaviour of a section s of a canonically defined sheaf of \mathbb{C} -vector spaces \mathcal{S}_X^0 living over X . Such a well-behaving section is called a ‘d-critical structure’ on X (see Section 1.2 for more details). We use the notation (X, s) to represent a d-critical scheme.

Joyce’s d-critical loci play a central role in categorified Donaldson–Thomas theory: the compatibility between d-critical charts encoded in their structure is crucial to associate to a

moduli space M of sheaves on a Calabi–Yau 3-fold a canonical¹ perverse sheaf Φ_M (the so-called *DT sheaf*), as well as a canonical virtual motive Φ_M^{mot} (the so-called *motivic DT invariant*).

Any d -critical chart $\text{crit}(f) \hookrightarrow V$ has a canonical symmetric perfect obstruction theory in the sense of Behrend–Fantechi [4], induced by the Hessian of f . Since these symmetric obstruction theories do not necessarily glue as two-term complexes (see [21, Example 2.17]), a d -critical scheme (X, s) cannot be equipped with a global symmetric perfect obstruction theory in general. However, the compatibility between d -critical charts is enough to ensure that these locally defined obstruction theories give rise to a slightly weaker structure on X with similar properties, called an *almost perfect obstruction theory*, as was shown in [23].

On the other hand, Pantev–Toën–Vaquié–Vezzosi [26] defined *k -shifted symplectic structures* on derived schemes and derived Artin stacks (we recall their definition in Section 1.1), for every $k \in \mathbb{Z}$. It is explained in [26, Section 3.2] that every -1 -shifted symplectic structure ω on a derived scheme \mathcal{X} induces a symmetric perfect obstruction theory on the underlying classical scheme $X = t_0(\mathcal{X}) \hookrightarrow \mathbf{X}$. Furthermore, as we recall in Theorem 1.3, there is a *truncation functor*

$$(0.1) \quad \{-1\text{-shifted symplectic derived Artin stacks}\} \xrightarrow{\tau} \{d\text{-critical Artin stacks}\}$$

which takes $(\mathcal{X}, \omega) \mapsto (\mathcal{X}, s_\omega)$, where $\mathcal{X} = t_0(\mathcal{X}) \hookrightarrow \mathbf{X}$ in the underlying classical Artin stack of \mathcal{X} and $s_\omega \in H^0(\mathcal{S}_{\mathcal{X}}^0)$ is a natural d -critical structure constructed out of ω . For a -1 -shifted symplectic derived scheme (\mathcal{X}, ω) , the induced symmetric perfect obstruction theory on $X = t_0(\mathcal{X}) \hookrightarrow \mathbf{X}$ is isomorphic to the almost perfect obstruction theory arising from (X, s_ω) .

The structures described so far are reproduced in Figure 1, inspired from a larger picture in [21], where a dotted arrow means that the association only works locally. We will explain the question mark ‘?’ appearing in the diagram in the next subsection.

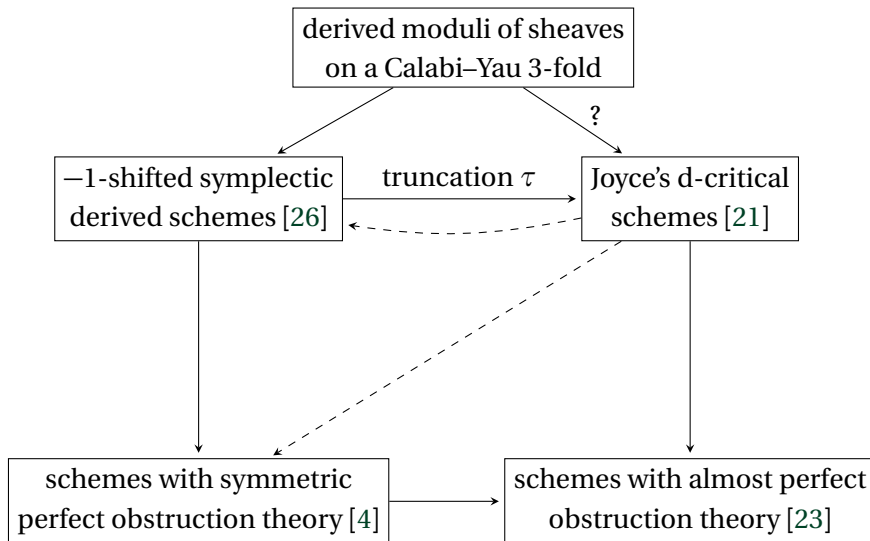


FIGURE 1. Landscape of the structures and their relations appearing in this paper.

0.2. Motivation. A classical example of -1 -shifted symplectic derived scheme is that of a *derived critical locus* $\mathbb{R}\text{crit}(f)$, for $f \in \mathcal{O}(V)$ a regular function on a smooth scheme V [26, Corollary 2.11]. The space $\mathbb{R}\text{crit}(f)$ is defined as the derived fibre product of the zero section

¹Strictly speaking, a choice of orientation data is also needed, see [7] and [10] for the precise statements.

of Ω_V , carrying its canonical symplectic structure, with the section $df \in H^0(V, \Omega_V)$. The -1 -shifted symplectic structure is denoted ω_f in this case. We can view the classical scheme $U = \text{crit}(f) = t_0(\mathbb{R}\text{crit}(f))$ as a d-critical locus with d-critical structure

$$(0.2) \quad s_f = f + (df)^2 \in H^0(\mathcal{S}_U^0)$$

determined by a single d-critical chart, and the diagonal dotted arrow in Figure 1 can in fact, in this special case, be filled in by means of the *critical symmetric obstruction theory*

$$(0.3) \quad \mathbb{E}_f = [T_V|_U \xrightarrow{\text{Hess}(f)} \Omega_V|_U] \rightarrow \mathbb{L}_U,$$

where \mathbb{L}_Y is the truncated cotangent complex of a scheme Y . Moreover, in this case, the functor τ in (0.1) sends $(\mathbb{R}\text{crit}(f), \omega_f)$ to $(\text{crit}(f), s_f)$.

A further example of a -1 -shifted symplectic derived scheme is the derived moduli scheme $\mathbf{M}_X(\text{ch})$ of simple coherent sheaves on a projective Calabi–Yau 3-fold X , with fixed Chern character $\text{ch} \in H^*(X, \mathbb{Q})$, see [26]. In particular, the underlying classical scheme $M_X(\text{ch})$ is naturally a d-critical locus via the truncation functor (0.1). However, as Behrend pointed out in [2], it is “an embarrassment of the theory” that one cannot construct the d-critical structure on $M_X(\text{ch})$ directly, i.e. without passing through derived Algebraic Geometry. This is the meaning of the question mark in Figure 1.

This “embarrassment” was our main motivation for starting this project. This paper dissolves such embarrassment in the following sense. On the Quot scheme of n points on \mathbb{A}^3 , namely the space

$$(0.4) \quad Q_{r,n} = \text{Quot}_{\mathbb{A}^3}(\mathcal{O}^{\oplus r}, n) = \{ [\mathcal{O}^{\oplus r} \rightarrow E] \mid \dim E = 0, \chi(E) = n \},$$

there are in principle *two* d-critical structures:

- (1) one arising from its critical structure $\text{crit}(f_{r,n}) \simeq Q_{r,n} [1]$ (and so a priori independent of derived geometry),
- (2) another arising as follows: the derived moduli stack \mathcal{M}_n of 0-dimensional sheaves of length n on \mathbb{A}^3 is -1 -shifted symplectic [9], so by truncation one has a d-critical structure s_n^{der} on the underlying classical Artin stack \mathcal{M}_n . Its pullback along the (smooth) morphism $Q_{r,n} \rightarrow \mathcal{M}_n$ forgetting the surjection defines yet another d-critical structure on $Q_{r,n}$.

We prove that the d-critical structures described in (1) and (2) agree, which shows that the d-critical structure coming from derived geometry is ‘morally underived’, and is the simplest possible.

We also make some progress in the projective case. More precisely, let F be a locally free sheaf of rank $r > 0$ on a projective Calabi–Yau 3-fold X . There is a natural ‘derived’ d-critical structure on the Quot scheme of points $\text{Quot}_X(F, n)$. We devote Section 5 to showing that such d-critical structure looks étale locally like the d-critical structure in (2) above, which in turn agrees with the one induced by the unique d-critical chart on the local model $\text{Quot}_{\mathbb{A}^3}(\mathcal{O}^{\oplus r}, n)$.

0.3. Main results. We discuss in greater details our main results in the rest of this introduction.

0.3.1. The local case. The Quot scheme of points (0.4) parametrising isomorphism classes of quotients $\mathcal{O}^{\oplus r} \rightarrow E$, where E is a 0-dimensional sheaf of length n on \mathbb{A}^3 , is proven in [1, Theorem 2.6] to be a global critical locus (cf. Theorem 2.3). More precisely, there is a diagram

$$(0.5) \quad \text{ncQuot}_r^n \supset \text{crit}(f_{r,n}) \xrightarrow[\sim]{t_{r,n}} Q_{r,n}$$

where $f_{r,n}$ is a regular function on the *noncommutative Quot scheme* ncQuot_r^n , a smooth $(2n^2 + rn)$ -dimensional variety which can be viewed as the moduli space of (isomorphism classes of) $(1, n)$ -dimensional stable r -framed representations $(A, B, C, v_1, \dots, v_r) \in \text{End}_{\mathbb{C}}(\mathbb{C}^n)^3 \times (\mathbb{C}^n)^r$ of the 3-loop quiver (cf. Figure 3). The function $f_{r,n} \in \mathcal{O}(\text{ncQuot}_r^n)$ is defined to be the trace of the potential $A[B, C]$. It defines a d-critical structure as in (0.2), namely

$$s_{r,n}^{\text{crit}} = s_{f_{r,n}} = f_{r,n} + (df_{r,n})^2 \in H^0(\mathcal{S}_{\text{crit}(f_{r,n})}^0).$$

On the other hand, derived symplectic geometry endows the derived moduli stack \mathcal{M}_n of 0-dimensional sheaves of length n on \mathbb{A}^3 with a -1 -shifted symplectic structure ω_n (see [9]), which can be truncated to produce a d-critical structure $s_n^{\text{der}} = s_{\omega_n}$ on the (underived) moduli stack $\mathcal{M}_n = t_0(\mathcal{M}_n)$. The morphism $q_{r,n}: Q_{r,n} \rightarrow \mathcal{M}_n$ sending $[\mathcal{O}^{\oplus r} \rightarrow E] \mapsto [E]$ is smooth (cf. Section 1.4), so the pullback

$$s_{r,n}^{\text{der}} = q_{r,n}^* s_n^{\text{der}}$$

defines a d-critical structure on $Q_{r,n}$ (cf. Definition 4.3).

The following is our first main result.

Theorem A. *Fix $r \geq 1$ and $n \geq 0$. There is an identity*

$$l_{r,n}^* s_{r,n}^{\text{der}} = s_{r,n}^{\text{crit}} \in H^0(\mathcal{S}_{\text{crit}(f_{r,n})}^0).$$

0.3.2. *The global case.* Consider now the case where \mathbb{A}^3 is replaced by a smooth, *projective* Calabi–Yau 3-fold X and $\mathcal{O}^{\oplus r}$ by a locally free sheaf F of rank r on X . Let $Q_{F,n} = \text{Quot}_X(F, n)$ be the Quot scheme parametrising quotients $[F \rightarrow E]$ with E a 0-dimensional sheaf of length n on X . The derived moduli stack $\mathcal{M}_X(n)$ of 0-dimensional sheaves of length n on X carries a canonical -1 -shifted symplectic structure by [26], which can be truncated to give a d-critical structure $s_{X,n} \in H^0(\mathcal{S}_{\mathcal{M}_X(n)}^0)$. Its pullback along the forgetful morphism $Q_{F,n} \rightarrow \mathcal{M}_X(n)$ is a d-critical structure on the Quot scheme, denoted $s_{F,n}$, see Equation (5.1).

The methods used in the proof of Theorem A also yield (after a bit of work) the following result, which says that the d-critical scheme $(Q_{F,n}, s_{F,n})$ is locally modelled on the d-critical scheme $(Q_{r,n}, s_{r,n}^{\text{der}})$.

Theorem B. *Fix $r \geq 1$ and $n \geq 0$. Let X be a projective Calabi–Yau 3-fold, F a locally free sheaf of rank r on X . There exists an analytic open cover $\{\rho_\lambda: T_\lambda \hookrightarrow Q_{F,n}\}_{\lambda \in \Lambda}$, such that for any index λ we have a diagram*

$$\begin{array}{ccc} & T_\lambda & \\ \pi_\lambda \swarrow & & \searrow \rho_\lambda \\ Q_{r,n} & & Q_{F,n} \end{array}$$

with π_λ étale, satisfying $\pi_\lambda^* s_{r,n}^{\text{der}} = \rho_\lambda^* s_{F,n} \in H^0(\mathcal{S}_{T_\lambda}^0)$.

0.3.3. *The two obstruction theories on $\text{Hilb}^n \mathbb{A}^3$.* Our third main result is a comparison between perfect obstruction theories on the Hilbert scheme of points $\text{Hilb}^n \mathbb{A}^3$. If $V_n = \text{ncQuot}_1^n$ is the noncommutative Hilbert scheme, Diagram (0.5) becomes

$$(0.6) \quad V_n \supset \text{crit}(f_{1,n}) \xrightarrow[\sim]{l_{1,n}} \text{Hilb}^n \mathbb{A}^3,$$

and the Hessian construction (0.3) defines a symmetric obstruction theory

$$(0.7) \quad \mathbb{E}_{f_{1,n}} \xrightarrow{\varphi^{\text{crit}}} \mathbb{L}_{\text{crit}(f_{1,n})}.$$

On the other hand, viewing the Hilbert scheme as a parameter space for ideal sheaves of colength n , one obtains the symmetric obstruction theory (more details are found in [27, 28])

$$\mathbb{E}_{\text{der}} = \mathbf{R}\pi_* \mathbf{R}\mathcal{H}om(\mathcal{I}, \mathcal{I})_0[2] \xrightarrow{\varphi_{\text{der}}} \mathbb{L}_{\text{Hilb}^n \mathbb{A}^3},$$

where $\mathcal{I} \subset \mathcal{O}_{\mathbb{A}^3 \times \text{Hilb}^n \mathbb{A}^3}$ is the universal ideal sheaf, $\pi: \mathbb{A}^3 \times \text{Hilb}^n \mathbb{A}^3 \rightarrow \text{Hilb}^n \mathbb{A}^3$ is the second projection, and $\mathbf{R}\mathcal{H}om(\mathcal{I}, \mathcal{I})_0$ denotes the $[-1]$ -shifted cone of the trace map $\mathbf{R}\mathcal{H}om(\mathcal{I}, \mathcal{I}) \rightarrow \mathcal{O}$.

The following result, established in Section 6, proves Conjecture 9.9 in [15].

Theorem C. *The isomorphism $\iota_{1,n}$ in (0.6) induces an isomorphism of perfect obstruction theories*

$$\begin{array}{ccc} \iota_{1,n}^* \mathbb{E}_{\text{der}} & \xrightarrow{\sim} & \mathbb{E}_{f_{1,n}} \\ \downarrow \iota_{1,n}^* \varphi_{\text{der}} & & \downarrow \varphi_{\text{crit}} \\ & \mathbb{L}_{\text{crit}(f_{1,n})} & \end{array}$$

Conventions. We work over \mathbb{C} throughout. The ‘font’ used in this paper for schemes, derived schemes, stacks and derived stacks will be X , \mathbf{X} , \mathcal{X} and \mathcal{X} respectively. For a quasiprojective variety X , we let $\mathcal{M}_X(n)$ denote the moduli stack of 0-dimensional coherent sheaves of length n on X . If F is a coherent sheaf on X , we also set $Q_{F,n} = \text{Quot}_X(F, n)$, where the right hand side is Grothendieck’s Quot scheme, parametrising quotients $F \rightarrow E$ where E is a 0-dimensional sheaf of length n . When $(X, F) = (\mathbb{A}^3, \mathcal{O}^{\oplus r})$, we set $\mathcal{M}_n = \mathcal{M}_{\mathbb{A}^3}(n)$ and $Q_{r,n} = \text{Quot}_{\mathbb{A}^3}(\mathcal{O}^{\oplus r}, n)$.

Acknowledgments. We wish to thank Pierrick Bousseau, Ben Davison, Dominic Joyce, Dragos Oprea, Sarah Scherotzke and Nicolò Sibilla and Okke van Garderen for helpful discussions. We also thank the anonymous referee for their observations and suggestions.

1. BACKGROUND MATERIAL

1.1. Shifted symplectic structures. Let $\text{cdga}_{\mathbb{C}}^{\leq 0}$ be the category of non-positively graded commutative differential graded \mathbb{C} -algebras. There is a spectrum functor $\mathbf{Spec}: \text{cdga}_{\mathbb{C}}^{\leq 0} \rightarrow \mathbf{dSt}_{\mathbb{C}}$ to the category of derived stacks (see [36, Definition 2.2.2.14] or [34, Definition 4.2]). An object of the form $\mathbf{Spec} A$ is called an *affine derived \mathbb{C} -scheme*. Such objects provide the Zariski local charts for general *derived \mathbb{C} -schemes*, see [34, Section 4.2]. An object \mathcal{X} in $\mathbf{dSt}_{\mathbb{C}}$ is called a *derived Artin stack* if it is m -geometric (cf. [34, Definition 1.3.3.1]) for some m and its ‘classical truncation’ $t_0(\mathcal{X})$ is an Artin stack (and not a higher stack). A derived Artin stack \mathcal{X} admits an *atlas*, i.e. a smooth surjective morphism $\mathbf{U} \rightarrow \mathcal{X}$ from a derived scheme. Every derived Artin stack \mathcal{X} has a *cotangent complex* $\mathbb{L}_{\mathcal{X}}$ of finite cohomological amplitude in $[-m, 1]$ and a dual tangent complex $\mathbb{T}_{\mathcal{X}}$. Both are objects in a suitable stable ∞ -category $L_{\text{qcoh}}(\mathcal{X})$ (see [34] or [36] for its definition).

Shifted symplectic structures on derived Artin stacks were introduced by Pantev–Toën–Vaquié–Vezzosi in [26]. The definition is given in the affine case first, and then generalised by showing the local notion satisfies smooth descent. We recall the local definition: let us set $\mathbf{X} = \mathbf{Spec} A$, so that $L_{\text{qcoh}}(\mathbf{X}) \cong \mathbf{D}(\text{dg-Mod}_A)$. For all $p \geq 0$ one can define the exterior power complex $(\Lambda^p \mathbb{L}_{\mathbf{X}}, d) \in L_{\text{qcoh}}(\mathbf{X})$, where the differential d is induced by the differential of the algebra A . For a fixed $k \in \mathbb{Z}$, define a k -shifted p -form on \mathbf{X} to be an element $\omega^0 \in (\Lambda^p \mathbb{L}_{\mathbf{X}})^k$ such that $d\omega^0 = 0$. To define the notion of closedness, consider the de Rham differential $d_{\text{dR}}: \Lambda^p \mathbb{L}_{\mathbf{X}} \rightarrow \Lambda^{p+1} \mathbb{L}_{\mathbf{X}}$. A k -shifted closed p -form is a sequence $(\omega^0, \omega^1, \dots)$, with $\omega^i \in (\Lambda^{p+i} \mathbb{L}_{\mathbf{X}})^{k-i}$, such that $d\omega^0 = 0$ and $d_{\text{dR}}\omega^i + d\omega^{i+1} = 0$. When $p = 2$, any k -shifted 2-form $\omega^0 \in (\Lambda^2 \mathbb{L}_{\mathbf{X}})^k$ induces a morphism

$\omega^0: \mathbb{T}_X \rightarrow \mathbb{L}_X[k]$ in $L_{\text{qcoh}}(\mathbf{X})$, and we say that ω^0 is *non-degenerate* if this morphism is an isomorphism in $L_{\text{qcoh}}(\mathbf{X})$.

Definition 1.1 ([26, Definition 1.18]). A k -shifted closed 2-form $\omega = (\omega^0, \omega^1, \dots)$ is called a *k -shifted symplectic structure* if ω^0 is non-degenerate. We say that (\mathbf{X}, ω) is a *k -shifted symplectic (affine) derived scheme*.

1.2. d-critical schemes and Artin stacks. Let X be a scheme over \mathbb{C} . Joyce [21] proved the existence of a canonical sheaf of \mathbb{C} -vector spaces \mathcal{S}_X such that for every triple (R, V, i) , where $R \subset X$ is an open subscheme, V is a smooth scheme and $i: R \hookrightarrow V$ is a closed immersion with ideal \mathcal{I} , one has an exact sequence

$$0 \longrightarrow \mathcal{S}_X|_R \longrightarrow \mathcal{O}_V/\mathcal{I}^2 \xrightarrow{d} \Omega_V/\mathcal{I} \cdot \Omega_V,$$

where the last map is induced by the exterior derivative; see [21, Theorem 2.1] for the full list of properties characterising \mathcal{S}_X . Joyce also proved the existence of a subsheaf $\mathcal{S}_X^0 \subset \mathcal{S}_X$ and a direct sum decomposition

$$\mathcal{S}_X = \mathcal{S}_X^0 \oplus \mathbb{C}_X,$$

where \mathbb{C}_X is the constant sheaf on X and, for any triple (R, V, i) as above, one has

$$\mathcal{S}_X^0|_R = \ker(\mathcal{S}_X|_R \hookrightarrow \mathcal{O}_V/\mathcal{I}^2 \twoheadrightarrow \mathcal{O}_{R_{\text{red}}}) \subset \mathcal{S}_X|_R.$$

If V is a smooth scheme carrying a regular function $f \in \mathcal{O}(V)$ with critical locus $R = \text{crit}(f) \subset V$, and $f|_{R_{\text{red}}} = 0$, then $\mathcal{I} = (df) \subset \mathcal{O}_V$ and a natural element of $H^0(\mathcal{S}_X^0|_R)$ is the section $f + (df)^2$.

Definition 1.2 ([21, Definition 2.5]). A *d -critical scheme* is a pair (X, s) , where X is an ordinary scheme and s is a section of \mathcal{S}_X^0 with the following property: for every point $p \in X$ there is a quadruple (R, V, f, i) , called a *d -critical chart*, where $R \hookrightarrow X$ is an open neighbourhood of p , V is a smooth scheme, $f \in \mathcal{O}(V)$ is a regular function such that $f|_{R_{\text{red}}} = 0$, having critical locus $i: R \hookrightarrow V$, and $s|_R = f + (df)^2 \in H^0(\mathcal{S}_X^0|_R)$. The section s is called a *d -critical structure* on X .

By [21, Proposition 2.8], if $g: X \rightarrow Y$ is a smooth morphism of schemes and $t \in H^0(\mathcal{S}_Y^0)$ is a d -critical structure on Y , then $g^*t \in H^0(\mathcal{S}_X^0)$ is a d -critical structure on X .

For an Artin stack \mathcal{X} , Joyce defined the sheaf $\mathcal{S}_{\mathcal{X}}^0$ by smooth descent [21, Corollary 2.52]. Recall that to give a sheaf \mathcal{F} on \mathcal{X} one has to specify an étale sheaf $\mathcal{F}(U, u)$ for every smooth 1-morphism $u: U \rightarrow \mathcal{X}$ from a scheme, along with a series of natural compatibilities. Similarly, to give a section $s \in H^0(\mathcal{F})$ is to give a collection of compatible sections $s(U, u) \in H^0(\mathcal{F}(U, u))$ for every smooth 1-morphism $u: U \rightarrow \mathcal{X}$ from a scheme.

Joyce defined a *d -critical Artin stack* (see [21, Definition 2.53]) to be a pair (\mathcal{X}, s) , where \mathcal{X} is a classical Artin stack, $s \in H^0(\mathcal{S}_{\mathcal{X}}^0)$ is a section such that $s(U, u)$ defines a d -critical structure on U (being a section of $\mathcal{S}_{\mathcal{X}}^0(U, u) = \mathcal{S}_U^0$) according to Definition 1.2, for every smooth 1-morphism $u: U \rightarrow \mathcal{X}$.

If G is an algebraic group acting on a scheme Y , the sheaf \mathcal{S}_Y^0 is naturally G -equivariant, so there is a well-defined subspace $H^0(\mathcal{S}_Y^0)^G \subset H^0(\mathcal{S}_Y^0)$ of G -invariant sections. If $\mathcal{X} = [Y/G]$, then $H^0(\mathcal{S}_{\mathcal{X}}^0) = H^0(\mathcal{S}_Y^0)^G$, and the d -critical structures on \mathcal{X} are canonically identified with the G -invariant d -critical structures on Y , see [21, Example 2.55]. We will make this identification throughout without further mention.

Theorem 1.3 ([5, Theorem 3.18]). *Let (\mathcal{X}, ω) be a -1 -shifted symplectic derived Artin stack. Then the underlying classical Artin stack $\mathcal{X} = t_0(\mathcal{X})$ extends in a canonical way to a d -critical Artin stack (\mathcal{X}, s_ω) . This defines a ‘truncation functor’ τ , as in (0.1), from the ∞ -category of -1 -shifted symplectic derived Artin stacks to the 2-category of d -critical Artin stacks.*

See also [8, Theorem 6.6] for the analogous result proved for schemes.

1.3. Symmetric obstruction theories. Let M be a \mathbb{C} -scheme with full cotangent complex $L_M^\bullet \in \mathbf{D}^{(-\infty, 0]}(\mathrm{QCoh}_M)$, and let $\mathbb{L}_M \in \mathbf{D}^{[-1, 0]}(\mathrm{QCoh}_M)$ denote its cutoff at -1 . A *perfect obstruction theory* on M , as defined by Behrend–Fantechi [3], is a pair (\mathbb{E}, ϕ) , where \mathbb{E} is a perfect complex of perfect amplitude contained in $[-1, 0]$, and $\phi: \mathbb{E} \rightarrow \mathbb{L}_M$ a morphism in the derived category, such that $h^0(\phi)$ is an isomorphism and $h^{-1}(\phi)$ is onto. A perfect obstruction theory (\mathbb{E}, ϕ) is called *symmetric* if there exists an isomorphism $\vartheta: \mathbb{E} \xrightarrow{\sim} \mathbb{E}^\vee[1]$ such that $\vartheta^\vee[1] = \vartheta$. See [4] for background on symmetric obstruction theories.

If (M, ω) is a -1 -shifted symplectic derived scheme, with underlying classical scheme $i: M \hookrightarrow \mathbf{M}$. Then $\mathbb{E} = i^*\mathbb{L}_M \rightarrow \mathbb{L}_M$ is a perfect obstruction theory, which is furthermore symmetric thanks to the non-degenerate pairing $\vartheta = i^*\omega^0$. This association explains the left vertical arrow in Figure 1.

1.4. Smoothness of the forgetful map. Let X be a quasiprojective variety, $n \geq 0$ an integer and $F \in \mathrm{Coh}(X)$ a coherent sheaf on X . Set $Q_{F,n} = \mathrm{Quot}_X(F, n)$ and let $\pi: Q_{F,n} \times X \rightarrow X$ be the projection. Forgetting the surjection defining the universal quotient

$$[\pi^*F \rightarrow \mathcal{Q}] \mapsto \mathcal{Q}$$

defines a map $q_{F,n}: Q_{F,n} \rightarrow \mathcal{M}_X(n)$ to the moduli stack of length n coherent sheaves on X . Such a map exists since $\mathcal{Q} \in \mathrm{Coh}(Q_{F,n} \times X)$ is flat over $Q_{F,n}$ by definition of the Quot scheme. We call $q_{F,n}$ the *forgetful morphism* throughout.

We will need the following result.

Proposition 1.4. *Let X be a reduced projective variety, F a locally free sheaf of rank $r \geq 1$ on X . For every $n \geq 0$, the forgetful morphism*

$$q_{F,n}: Q_{F,n} \rightarrow \mathcal{M}_X(n)$$

is smooth of relative dimension rn .

Before proving the result, we recall a classical result by Grothendieck. According to [17, Théorème 7.7.6] (but see also [25, Theorem 5.7]), if $f: Y \rightarrow B$ is a proper morphism to a locally noetherian scheme B , and E is a coherent B -flat sheaf on Y , there exists a coherent sheaf \mathcal{Q}_E on B along with functorial isomorphisms

$$\eta: f_*(E \otimes_{\mathcal{O}_B} \mathcal{M}) \xrightarrow{\sim} \mathcal{H}om_{\mathcal{O}_B}(\mathcal{Q}_E, \mathcal{M})$$

for all quasicohherent sheaves \mathcal{M} on B . The sheaf \mathcal{Q}_E is unique up to a unique isomorphism, it behaves well with respect to pullback, and moreover it is locally free exactly when f is cohomologically flat in dimension 0 [17, Proposition 7.8.4]. For instance, any proper flat morphism with geometrically reduced fibres is cohomologically flat in dimension 0 [17, Proposition 7.8.6].

Proof of Proposition 1.4. Let $\mathcal{X}_{F,n}$ be the stack of pairs (E, α) , where E is a 0-dimensional sheaf of length n and α is an \mathcal{O}_X -linear homomorphism $\alpha: F \rightarrow E$. Then we have an open immersion

$Q_{F,n} \hookrightarrow \mathcal{X}_{F,n}$ and the morphism $q_{F,n}$ extends to a morphism

$$\pi_{F,n}: \mathcal{X}_{F,n} \rightarrow \mathcal{M}_X(n).$$

Let B be a scheme. By standard methods, we may reduce to the case where B is locally noetherian. Given a map $B \rightarrow \mathcal{M}_X(n)$, let us consider the fibre products

$$\begin{array}{ccc} P & \longrightarrow & Q_{F,n} \\ \downarrow & \square & \downarrow q_{F,n} \\ B & \longrightarrow & \mathcal{M}_X(n) \end{array} \quad \text{and} \quad \begin{array}{ccc} V & \longrightarrow & \mathcal{X}_{F,n} \\ \downarrow & \square & \downarrow \pi_{F,n} \\ B & \longrightarrow & \mathcal{M}_X(n) \end{array}$$

where $B \rightarrow \mathcal{M}_X(n)$ corresponds to a B -flat family of 0-dimensional sheaves $E \in \text{Coh}(X \times B)$. The associated sheaf $\mathcal{Q}_E \in \text{Coh}(B)$ is locally free of rank rn , since the projection $f: X \times B \rightarrow B$ is cohomologically flat in dimension 0 since it is proper with reduced fibres. Moreover, the projection $V \rightarrow B$ is canonically isomorphic to the structure morphism

$$\text{Spec Sym}_{\mathcal{O}_B} \mathcal{Q}_E \rightarrow B,$$

because of Grothendieck's theorem [17, Cor. 7.7.8, Rem. 7.7.9], which says the following: Let $f: Y \rightarrow B$ be a projective morphism, \mathcal{F} and \mathcal{E} two coherent sheaves on Y . Consider the functor $\text{Sch}_B^{\text{op}} \rightarrow \text{Sets}$ sending a B -scheme $T \rightarrow B$ to the set of morphism $\text{Hom}_{Y_T}(\mathcal{F}_T, \mathcal{E}_T)$, where \mathcal{F}_T and \mathcal{E}_T are the pullbacks of \mathcal{F} and \mathcal{E} along the projection $Y_T = Y \times_B T \rightarrow Y$. Then, if \mathcal{E} is flat over B , the above functor is represented by a linear scheme $\text{Spec Sym}_{\mathcal{O}_B} \mathcal{H} \rightarrow B$, where \mathcal{H} is a coherent sheaf on B . However, in our case we have $Y = X \times B$, $f = p_2$ the second projection and $\mathcal{F} = p_1^* F$, and the functor described above is precisely the functor of points of V , thus we must have $\mathcal{H} = \mathcal{Q}_E$. More details can be found in the proof of Grothendieck's result found in [25, Theorem 5.8].

Therefore, since $P \hookrightarrow V$ is open and $V \rightarrow B$ is an affine bundle of rank rn , the projection $P \rightarrow B$ is smooth of relative dimension rn . \square

Remark 1.5. Let $X^\circ \hookrightarrow X$ be an open subscheme, where X is a reduced projective variety as in Proposition 1.4, and let $F^\circ = F|_{X^\circ}$ be the restriction of a locally free sheaf F over X . Then

$$q_{F^\circ, n}: \text{Quot}_{X^\circ}(F^\circ, n) \rightarrow \mathcal{M}_{X^\circ}(n)$$

is again smooth, being the pullback of the smooth morphism $q_{F,n}: Q_{F,n} \rightarrow \mathcal{M}_X(n)$ along the open immersion $\mathcal{M}_{X^\circ}(n) \hookrightarrow \mathcal{M}_X(n)$. This applies for instance to $(X, F, X^\circ) = (\mathbb{P}^3, \mathcal{O}^{\oplus r}, \mathbb{A}^3)$. Thus we get the next corollary.

Corollary 1.6. *The forgetful map $q_{r,n}: \text{Quot}_{\mathbb{A}^3}(\mathcal{O}^{\oplus r}, n) \rightarrow \mathcal{M}_n$ is smooth of relative dimension rn .*

Note that, for fixed $[E] \in \mathcal{M}_n$, the choice of a surjection $\mathcal{O}^{\oplus r} \rightarrow E$ is nothing but the datum of r (general enough) sections $\sigma_1, \dots, \sigma_r \in \text{Hom}(\mathcal{O}, E) = H^0(E) = \mathbb{C}^n$. So the fibre of $q_{r,n}$ over $[E]$ is an open subset of \mathbb{C}^{rn} .

2. PROOF OF THEOREM A

In this section we prove Theorem A (see Theorem 2.6) granting the (fundamental) auxiliary result Theorem 2.2, which will be proved in Theorem 3.11.

The moduli stack \mathcal{M}_n of 0-dimensional coherent sheaves on length n on \mathbb{A}^3 can be seen as a stack of representations of the Jacobi algebra of a quiver with potential. Indeed, consider the 3-loop quiver L_3 (Figure 2), equipped with the potential $W = A[B, C]$.

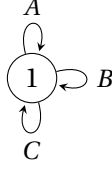


FIGURE 2. The 3-loop quiver L_3 .

Notation 2.1. Fix a quiver $Q = (Q_0, Q_1, s, t)$, where Q_0 is the vertex set, Q_1 is the edge set, s and t are the source and target maps $Q_1 \rightarrow Q_0$ respectively; for a dimension vector $\mathbf{d} = (d_i)_i \in \mathbb{N}^{Q_0}$, we let $\text{Rep}_{\mathbf{d}}(Q)$ denote the space of \mathbf{d} -dimensional representations of Q , namely the affine space $\prod_{a \in Q_1} \text{Hom}_{\mathbb{C}}(\mathbb{C}^{d_{s(a)}}, \mathbb{C}^{d_{t(a)}})$. The group $\text{GL}_{\mathbf{d}} = \prod_{i \in Q_0} \text{GL}_{d_i}$ acts on $\text{Rep}_{\mathbf{d}}(Q)$ by conjugation; the moduli stack of \mathbf{d} -dimensional representations is the quotient stack $\mathfrak{M}_{\mathbf{d}}(Q) = [\text{Rep}_{\mathbf{d}}(Q)/\text{GL}_{\mathbf{d}}]$.

The critical locus of the GL_n -invariant regular function

$$(2.1) \quad f_n = (\text{Tr } W)_n : \text{Rep}_n(L_3) \rightarrow \mathbb{C}, \quad (A, B, C) \mapsto \text{Tr } A[B, C],$$

defines a closed GL_n -invariant subscheme U_n of the $3n^2$ -dimensional affine space $\text{Rep}_n(L_3) \cong \text{End}_{\mathbb{C}}(\mathbb{C}^n)^3$ parametrising triples of *pairwise commuting* endomorphisms. There is an isomorphism of Artin stacks

$$(2.2) \quad \iota_n : [U_n/\text{GL}_n] \xrightarrow{\sim} \mathcal{M}_n$$

defined on closed points by sending the GL_n -orbit $[A, B, C]$ of a triple $(A, B, C) \in U_n$ to the point $[E]$ represented by the direct sum $E = \mathcal{O}_{p_1} \oplus \cdots \oplus \mathcal{O}_{p_n}$, where the points p_i are not necessarily distinct, and are determined by the diagonal entries of the matrices A (for the x -coordinate), B (for the y -coordinate) and C (for the z -coordinate). Indeed, the matrices can be simultaneously put in upper triangular form precisely because they pairwise commute, and we are working modulo GL_n . The source of the isomorphism ι_n is equipped with the algebraic d -critical structure

$$s_n^{\text{crit}} = f_n + (df_n)^2 \in H^0(\mathcal{S}_{[U_n/\text{GL}_n]}^0) = H^0(\mathcal{S}_{U_n}^0)^{\text{GL}_n}.$$

On the other hand, Brav and Dyckerhoff [9] proved that there is a -1 -shifted symplectic structure ω_n on the derived Artin stack \mathcal{M}_n of 0-dimensional coherent sheaves of length n on \mathbb{A}^3 (see Subsection 3.2 for details). We denote by

$$(2.3) \quad (\mathcal{M}_n, s_n^{\text{der}}) = \tau(\mathcal{M}_n, \omega_n)$$

its truncation, defined through the ‘truncation functor’ τ recalled in Theorem 1.3.

The following result, granted for now, will be proved in Section 3.4.

Theorem 2.2. *The isomorphism (2.2) induces an identity*

$$\iota_n^* s_n^{\text{der}} = s_n^{\text{crit}}$$

of algebraic d -critical structures.

Now, for a fixed integer $r \geq 1$, let us consider the Quot scheme $Q_{r,n} = \text{Quot}_{\mathbb{A}^3}(\mathcal{O}^{\oplus r}, n)$. As we now recall, this Quot scheme is a global critical locus, or, in other words, it admits a d -critical structure consisting of a single d -critical chart. This result in the case $r = 1$ is due to Szendrői [31, Theorem 1.3.1]. The ' r -framed 3-loop quiver' \tilde{L}_3 (Figure 3) plays a crucial role.

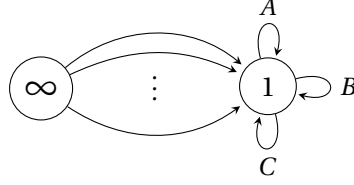


FIGURE 3. The r -framed 3-loop quiver \tilde{L}_3 .

Theorem 2.3 ([1, Theorem 2.6]). *Fix a vector $\theta = (\theta_1, \theta_2) \in \mathbb{R}^2$, with $\theta_1 > \theta_2$. Let*

$$\text{ncQuot}_r^n = \text{Rep}_{(1,n)}^{\theta\text{-st}}(\tilde{L}_3) / \text{GL}_n$$

be the moduli space of θ -stable representations of the r -framed 3-loop quiver \tilde{L}_3 , and consider the regular function $f_{r,n}: \text{ncQuot}_r^n \rightarrow \mathbb{A}^1$ induced by the potential $W = A[B, C]$. Then there is an isomorphism

$$\iota_{r,n}: \text{crit}(f_{r,n}) \xrightarrow{\sim} Q_{r,n}.$$

The main result of this paper (Theorem A, proved in Theorem 2.6 below) states that the d -critical structure

$$s_{r,n}^{\text{crit}} = f_{r,n} + (df_{r,n})^2 \in H^0(\mathcal{S}_{\text{crit}(f_{r,n})}^0)$$

determined by the function $f_{r,n}$ in Theorem 2.3 agrees, up to the isomorphism $\iota_{r,n}$, with the pullback of s_n^{der} along the smooth forgetful morphism $q_{r,n}$.

Remark 2.4. According to our conventions, in a dimension vector (or a stability condition) on a framed quiver $\infty \rightarrow Q$ the first entry always refers to the framing vertex ∞ . As explained in [1, 13], stability of a representation $(A, B, C, v_1, \dots, v_r) \in \text{Rep}_{(1,n)}(\tilde{L}_3)$ with respect to θ translates into the condition that the framing vectors $v_1, \dots, v_r \in \text{Hom}_{\mathbb{C}}(V_\infty, V_1) = \mathbb{C}^n$ generate the underlying (unframed) representation $(A, B, C) \in \text{Rep}_n(L_3)$. Since GL_n acts freely on $\text{Rep}_{(1,n)}^{\theta\text{-st}}(\tilde{L}_3)$, this exhibits ncQuot_r^n as a *smooth* quasiprojective variety of dimension $2n^2 + rn$.

We have a cartesian diagram

$$(2.4) \quad \begin{array}{ccc} \text{crit}(f_{r,n}) & \xrightarrow[\sim]{\iota_{r,n}} & Q_{r,n} \\ \tilde{q}_{r,n} \downarrow & \square & \downarrow q_{r,n} \\ [U_n / \text{GL}_n] & \xrightarrow[\sim]{\iota_n} & \mathcal{M}_n \end{array}$$

where $\iota_{r,n}$ is the isomorphism of Theorem 2.3 and ι_n is the isomorphism (2.2).

Proposition 2.5. *There is an identity of d -critical structures*

$$\tilde{q}_{r,n}^* s_n^{\text{crit}} = s_{r,n}^{\text{crit}} \in H^0(\mathcal{S}_{\text{crit}(f_{r,n})}^0),$$

where $s_{r,n}^{\text{crit}} = f_{r,n} + (df_{r,n})^2$ is the d -critical structure defined by the function $f_{r,n}$.

Proof. Forgetting the framing data yields a smooth morphism

$$p_{r,n}: \text{ncQuot}_r^n \rightarrow \mathfrak{M}_n(L_3) = [\text{Rep}_n(L_3)/\text{GL}_n]$$

fitting in a cartesian diagram

$$\begin{array}{ccc} \text{crit}(f_{r,n}) & \hookrightarrow & \text{ncQuot}_r^n \\ \tilde{q}_{r,n} \downarrow & \square & \downarrow p_{r,n} \\ [U_n/\text{GL}_n] & \hookrightarrow & \mathfrak{M}_n(L_3) \end{array}$$

where the horizontal arrows are closed immersions. The function f_n introduced in (2.1) descends to a function $\mathfrak{M}_n(L_3) \rightarrow \mathbb{A}^1$, still denoted f_n , with critical locus $[U_n/\text{GL}_n]$. The conclusion then follows by observing that $p_{r,n}^* f_n = f_{r,n}$, which is true because $f_{r,n}$ does not interact with the framing data. \square

We can now complete the proof of Theorem A.

Theorem 2.6. *Let $s_{r,n}^{\text{der}} \in H^0(\mathcal{S}_{Q_{r,n}}^0)$ be the pullback of the d-critical structure s_n^{der} (defined in (2.3) by truncating ω_n) along $q_{r,n}$. Then the isomorphism $\iota_{r,n}$ of Theorem 2.3 induces an identity*

$$\iota_{r,n}^* s_{r,n}^{\text{der}} = s_{r,n}^{\text{crit}} \in H^0(\mathcal{S}_{\text{crit}(f_{r,n})}^0).$$

Proof. We have

$$\begin{aligned} \iota_{r,n}^* s_{r,n}^{\text{der}} &= \iota_{r,n}^* q_{r,n}^* s_n^{\text{der}} && \text{by definition} \\ &= \tilde{q}_{r,n}^* \iota_n^* s_n^{\text{der}} && \text{by Diagram (2.4)} \\ &= \tilde{q}_{r,n}^* s_n^{\text{crit}} && \text{by Theorem 2.2} \\ &= s_{r,n}^{\text{crit}} && \text{by Proposition 2.5,} \end{aligned}$$

which concludes the proof. \square

3. THE D-CRITICAL STRUCTURE(S) ON \mathcal{M}_n

In this section we compare the two d-critical structures

$$s_n^{\text{crit}} \in H^0(\mathcal{S}_{[U_n/\text{GL}_n]}^0), \quad s_n^{\text{der}} \in H^0(\mathcal{S}_{\mathcal{M}_n}^0)$$

on the isomorphic spaces $[U_n/\text{GL}_n]$ and \mathcal{M}_n , introduced in Section 2. These d-critical structures come from quiver representations and symplectic derived algebraic geometry respectively. Our goal (achieved in Theorem 3.11) is to prove Theorem 2.2.

To do so, we first analyse and give explicit local d-critical charts for the two d-critical structures. For s_n^{crit} , we use the geometry of the quotient stack $[U_n/\text{GL}_n]$ and Luna’s étale slice theorem. For s_n^{der} , we take advantage of the derived deformation theory of the -1 -shifted symplectic stack \mathcal{M}_n to obtain explicit formal charts in Darboux form (cf. [5]). Finally, we argue that one can pass from formal neighbourhoods to honest smooth neighbourhoods to obtain the equality of the d-critical structures.

Notation 3.1. Throughout we shall use the notation $\text{Mat}_{a,b}(R)$ to denote the space of matrices with a rows and b columns with entries in a ring R . If $R = \mathbb{C}$, we simply write $\text{Mat}_{a,b}$.

3.1. The \mathbf{d} -critical structure s_n^{crit} coming from quiver representations. Let $[E] \in \mathcal{M}_n$ be a closed point. Then the corresponding sheaf E must be polystable, i.e. of the form

$$(3.1) \quad E = \bigoplus_{i=1}^k \mathbb{C}^{a_i} \otimes \mathcal{O}_{p_i}$$

where, for $i = 1, \dots, k$, the points $p_i = (\alpha_i, \beta_i, \gamma_i) \in \mathbb{A}^3$ are pairwise distinct and a_i are positive integers such that $a_1 + \dots + a_k = n$.

We fix some notation for convenience. Let $Y_n = \text{Rep}_n(L_3) = \text{End}_{\mathbb{C}}(\mathbb{C}^n)^3$ be the vector space of triples of $n \times n$ matrices, and let $f_n: Y_n \rightarrow \mathbb{C}$ be the regular function (2.1). Denote by $\mathbf{a} = (a_1, \dots, a_k)$ the k -tuple of positive integers determined by (3.1). Form the product

$$Y_{\mathbf{a}} = \prod_{i=1}^k Y_{a_i}.$$

Then there is a closed embedding $\Phi_{\mathbf{a}}: Y_{\mathbf{a}} \hookrightarrow Y_n$ by block diagonal matrices with square diagonal blocks of sizes a_1, \dots, a_k . The reductive algebraic group $\text{GL}_{\mathbf{a}} = \prod_{i=1}^k \text{GL}_{a_i}$ acts on $Y_{\mathbf{a}}$ by componentwise conjugation and $\Phi_{\mathbf{a}}$ is equivariant with respect to the inclusion $\text{GL}_{\mathbf{a}} \subset \text{GL}_n$ by block diagonal matrices of the same kind.

On the space $Y_{\mathbf{a}}$ we have the $\text{GL}_{\mathbf{a}}$ -invariant potential

$$(3.2) \quad g_{\mathbf{a}} = f_{a_1} \oplus \dots \oplus f_{a_k}: Y_{\mathbf{a}} \rightarrow \mathbb{A}^1, \quad (A_i, B_i, C_i)_i \mapsto \sum_{i=1}^k \text{Tr } A_i[B_i, C_i],$$

where $(A_i, B_i, C_i) \in Y_{a_i}$ for $i = 1, \dots, k$. We have the obvious relation $f_n \circ \Phi_{\mathbf{a}} = g_{\mathbf{a}}$. If we set

$$U_n = \text{crit}(f_n) \subset Y_n, \quad U_{\mathbf{a}} = \text{crit}(g_{\mathbf{a}}) \subset Y_{\mathbf{a}},$$

the restriction $\Phi_{\mathbf{a}}: U_{\mathbf{a}} \hookrightarrow U_n$ is still equivariant with respect to the inclusion $\text{GL}_{\mathbf{a}} \subset \text{GL}_n$, and so it induces a morphism of algebraic stacks

$$(3.3) \quad \psi_{\mathbf{a}}: [U_{\mathbf{a}}/\text{GL}_{\mathbf{a}}] \rightarrow [U_n/\text{GL}_n].$$

Note that the identification $\text{crit}(g_{\mathbf{a}}) = \prod_i \text{crit}(f_{a_i}) = \prod_i U_{a_i}$ combined with the product of the isomorphisms ι_{a_i} as in (2.2) induces a canonical identification

$$[U_{\mathbf{a}}/\text{GL}_{\mathbf{a}}] = \prod_{i=1}^k [U_{a_i}/\text{GL}_{a_i}] \xrightarrow{\sim} \prod_{i=1}^k \mathcal{M}_{a_i} = \mathcal{M}_{\mathbf{a}}.$$

We can identify the map $\psi_{\mathbf{a}}$ with the direct sum map

$$\psi_{\mathbf{a}}: \mathcal{M}_{\mathbf{a}} \rightarrow \mathcal{M}_n,$$

denoted the same way, taking a B -valued point of $\mathcal{M}_{\mathbf{a}}$, i.e. a k -tuple $(\mathcal{E}_1, \dots, \mathcal{E}_k)$ of B -flat families of 0-dimensional sheaves $\mathcal{E}_i \in \text{Coh}(B \times \mathbb{A}^3)$, to their direct sum $\bigoplus_{1 \leq i \leq k} \mathcal{E}_i$. (The direct sum of flat sheaves is flat by [30, Tag 05NC]).

Lemma 3.2. *Let $\mathcal{X}_{\mathbf{a}} \subset \mathcal{M}_{\mathbf{a}}$ be the open substack parametrising k -tuples of sheaves with pairwise disjoint support. Then the morphism*

$$\psi_{\mathbf{a}}|_{\mathcal{X}_{\mathbf{a}}}: \mathcal{X}_{\mathbf{a}} \rightarrow \mathcal{M}_n$$

is étale.

Proof. According to [30, Tag 0CIK], to show that $\psi := \psi_{\mathbf{a}}|_{\mathcal{X}_{\mathbf{a}}}$ is étale it is enough to find an algebraic space W , a faithfully flat morphism $\rho: W \rightarrow \mathcal{M}_n$ locally of finite presentation and an étale morphism $T \rightarrow W \times_{\rho, \mathcal{M}_n, \psi} \mathcal{X}_{\mathbf{a}}$ such that $T \rightarrow W$ is étale. We start by picking $W = \text{Quot}_{\mathbb{A}^3}(\mathcal{O}^{\oplus n}, n)$ along with the smooth (cf. Corollary 1.6) morphism

$$\rho = q_{n,n}: \text{Quot}_{\mathbb{A}^3}(\mathcal{O}^{\oplus n}, n) \rightarrow \mathcal{M}_n$$

sending, for any test scheme B , a B -flat quotient $\mathcal{O}_{B \times \mathbb{A}^3}^{\oplus n} \rightarrow \mathcal{T}$ to the object $\mathcal{T} \in \mathcal{M}_n(B)$. Note that ρ is surjective in the sense of [30, Tag 04ZR]. Indeed, for any closed point $[E]$, we have a direct sum decomposition as in (3.1), thus the sheaf E receives a surjection from $\mathcal{O}^{\oplus n}$, which is nothing but the direct sum of the surjections $\mathcal{O}^{\oplus a_i} \rightarrow \mathbb{C}^{a_i} \otimes \mathcal{O}_{p_i}$ — their direct sum is again surjective because $p_i \neq p_j$ for all $i \neq j$.

Next, we form the cartesian diagram

$$\begin{array}{ccc} T & \longrightarrow & \mathcal{X}_{\mathbf{a}} \\ \mu \downarrow & \square & \downarrow \psi \\ \text{Quot}_{\mathbb{A}^3}(\mathcal{O}^{\oplus n}, n) & \xrightarrow{\rho} & \mathcal{M}_n \end{array}$$

and we pick the identity $T = \text{Quot}_{\mathbb{A}^3}(\mathcal{O}^{\oplus n}, n) \times_{\rho, \mathcal{M}_n, \psi} \mathcal{X}_{\mathbf{a}}$ as an étale map. We need to show that μ is étale. But this follows from [1, Proposition A.3], after observing that T is the open subscheme of

$$\prod_{i=1}^k \text{Quot}_{\mathbb{A}^3}(\mathcal{O}^{\oplus n}, a_i)$$

parametrising k -tuples of quotients with pairwise disjoint support, and μ is nothing but the map taking a tuple of surjections to their direct sum. \square

Let E be a polystable sheaf supported on points $p_i = (\alpha_i, \beta_i, \gamma_i)$ for $1 \leq i \leq k$, as in (3.1). The point $[E] \in \mathcal{M}_n$ corresponds under ι_n to the orbit of the triple of matrices $y_E = \Phi_{\mathbf{a}}(v_E) \in U_n \subset Y_n$, where $v_E = (\underline{\alpha}, \underline{\beta}, \underline{\gamma}) \in U_{\mathbf{a}} \subset Y_{\mathbf{a}}$ is given by

$$(3.4) \quad \underline{\alpha} = (\alpha_1 \text{Id}_{a_1}, \dots, \alpha_k \text{Id}_{a_k}), \quad \underline{\beta} = (\beta_1 \text{Id}_{a_1}, \dots, \beta_k \text{Id}_{a_k}), \quad \underline{\gamma} = (\gamma_1 \text{Id}_{a_1}, \dots, \gamma_k \text{Id}_{a_k}).$$

We let \bar{v}_E denote the image of v_E along the smooth atlas $U_{\mathbf{a}} \rightarrow [U_{\mathbf{a}}/\text{GL}_{\mathbf{a}}]$, and similarly for $\bar{y}_E \in [U_n/\text{GL}_n]$. The morphism $\psi_{\mathbf{a}}$ maps $\bar{v}_E \in [U_{\mathbf{a}}/\text{GL}_{\mathbf{a}}]$ to $\bar{y}_E \in [U_n/\text{GL}_n]$. For convenience, we will identify $\underline{\alpha}, \underline{\beta}, \underline{\gamma}$ with their images in Y_n under $\Phi_{\mathbf{a}}$, considering them as $n \times n$ block diagonal matrices and more generally we identify elements of $Y_{\mathbf{a}}$ with their image in Y_n under $\Phi_{\mathbf{a}}$.

Lemma 3.3. *There is an open $\text{GL}_{\mathbf{a}}$ -invariant neighbourhood $v_E \in V \subset Y_{\mathbf{a}}$ such that, if $U = \text{crit}(g_{\mathbf{a}}|_V)$, the morphism $\psi_E := \psi_{\mathbf{a}}|_{[U/\text{GL}_{\mathbf{a}}]}$ is étale. In particular $\psi_{\mathbf{a}}$ is étale at \bar{v}_E . Moreover, if ϕ_E denotes the composition*

$$\phi_E: U \longrightarrow [U/\text{GL}_{\mathbf{a}}] \longleftarrow [U_{\mathbf{a}}/\text{GL}_{\mathbf{a}}] \xrightarrow{\psi_{\mathbf{a}}} [U_n/\text{GL}_n],$$

then we have an identity of d-critical structures

$$\phi_E^* s_n^{\text{crit}} = g_{\mathbf{a}}|_V + (\text{dg}_{\mathbf{a}}|_V)^2 \in H^0(\mathcal{S}_U^0)^{\text{GL}_{\mathbf{a}}} \subset H^0(\mathcal{S}_U^0).$$

Proof. The first statement follows from Lemma 3.2, up to replacing the open substack $[U/\text{GL}_{\mathbf{a}}] \hookrightarrow [U_{\mathbf{a}}/\text{GL}_{\mathbf{a}}]$ with its intersection with $\mathcal{X}_{\mathbf{a}}$. We now compare f_n and $g_{\mathbf{a}}$.

For a complex matrix $A \in \text{Mat}_{n,n}$ (cf. Notation 3.1), we consider it as a block matrix with diagonal blocks A_{11}, \dots, A_{kk} of sizes $a_1 \times a_1, \dots, a_k \times a_k$ and off-diagonal blocks A_{ij} of sizes $a_i \times a_j$.

Let $Y_n = Y_a \oplus Y_n^\perp$ be the direct sum decomposition where Y_n^\perp is the subspace of triples of matrices (A, B, C) whose diagonal blocks are zero. This decomposition is GL_a -invariant, since the conjugation action of GL_a is given on blocks by the formula

$$(h \cdot (A, B, C))_{ij} = (h_{ii} A_{ij} h_{jj}^{-1}, h_{ii} B_{ij} h_{jj}^{-1}, h_{ii} C_{ij} h_{jj}^{-1}).$$

The derivative $\mathfrak{gl}_n \rightarrow T_{y_E} Y_n$ of the GL_n -action on Y_n at the point $y_E = (\underline{\alpha}, \underline{\beta}, \underline{\gamma})$ is given by the map

$$\begin{aligned} \sigma: \mathfrak{gl}_n = \text{Mat}_{n,n} &\longrightarrow T_{y_E} Y_n \simeq Y_n \\ X &\mapsto ([X, \underline{\alpha}], [X, \underline{\beta}], [X, \underline{\gamma}]) \end{aligned}$$

Notice that the ij -th blocks of the triple of matrices $\sigma(X)$ are given by the triple of $a_i \times a_j$ matrices

$$(3.5) \quad \sigma(X)_{ij} = -((\alpha_i - \alpha_j)X_{ij}, (\beta_i - \beta_j)X_{ij}, (\gamma_i - \gamma_j)X_{ij}).$$

In particular, $\text{im}(\sigma)$ is a GL_a -invariant subspace of Y_n^\perp .

Define a subspace $\text{im}(\sigma)^\perp \subset Y_n$ by the condition that $(X, Y, Z) \in \text{im}(\sigma)^\perp$ if and only if for all indices $i \neq j$ we have

$$(3.6) \quad \overline{(\alpha_i - \alpha_j)X_{ij}} + \overline{(\beta_i - \beta_j)Y_{ij}} + \overline{(\gamma_i - \gamma_j)Z_{ij}} = 0.$$

We thus have a GL_a -invariant direct sum decomposition

$$(3.7) \quad T_{y_E} Y_n \simeq Y_n = Y_a \oplus \text{im}(\sigma) \oplus Y_n^\sigma$$

where $Y_a \oplus Y_n^\sigma = \text{im}(\sigma)^\perp$ and Y_n^σ is the GL_a -invariant complement of Y_a in $\text{im}(\sigma)^\perp$ consisting of the matrices that have zero diagonal blocks. Since $\text{im}(\sigma) = T_{y_E}(\text{GL}_n \cdot y_E)$, by Luna's étale slice theorem there exists a sufficiently small GL_a -invariant open neighbourhood S of $y_E \in Y_a \oplus Y_n^\sigma$ which is an étale slice for the GL_n -action on Y_n at y_E . Therefore, $T = U_n \times_{Y_n} S$ is an étale slice for the GL_n -action on U_n at y_E . In particular, T is cut out by the equations $df_n|_S = 0$ in S and thus $T = \text{crit}(f_n|_S) \subset S$.

Since $f_n|_S = \text{Tr} A[B, C]|_S$, it is easy to check that the derivatives of f_n with respect to the coordinates of Y_n^σ vanish on Y_a . Moreover, the Hessian H of f_n at the point $y_E \in Y_n$ is given by the quadratic form whose value at $(X, Y, Z) \in T_{y_E} Y_n \simeq Y_n$ is

$$(3.8) \quad H(X, Y, Z) = \text{Tr} \left(X[\underline{\beta}, Z] + X[\underline{\gamma}, \underline{\alpha}] + \underline{\alpha}[Y, Z] \right).$$

By direct computation, the form H is non-degenerate on Y_n^σ . To see this, working in terms of blocks, we have

$$H(X, Y, Z) = \sum_{i \neq j} \text{Tr} \left((\beta_i - \beta_j) Z_{ij} X_{ji} - (\gamma_i - \gamma_j) Y_{ij} X_{ji} - (\alpha_i - \alpha_j) Z_{ij} Y_{ji} \right).$$

For simplicity, we write each pair of summands corresponding to the indices $i \neq j$ suggestively in the form

$$(3.9) \quad \beta \text{Tr}(Z_1 X_2) - \gamma \text{Tr}(Y_1 X_2) - \alpha \text{Tr}(Z_1 Y_2) - \beta \text{Tr}(Z_2 X_1) + \gamma \text{Tr}(Y_2 X_1) + \alpha \text{Tr}(Z_2 Y_1)$$

where X_1, Y_1, Z_1 (resp. X_2, Y_2, Z_2) stand for X_{ij}, Y_{ij}, Z_{ij} (resp. X_{ji}, Y_{ji}, Z_{ji}) and α, β, γ stand for $\alpha_i - \alpha_j, \beta_i - \beta_j, \gamma_i - \gamma_j$ respectively.

Then (3.6) takes the equivalent form

$$\begin{aligned}\bar{\alpha}X_1 + \bar{\beta}Y_1 + \bar{\gamma}Z_1 &= 0 \\ \bar{\alpha}X_2 + \bar{\beta}Y_2 + \bar{\gamma}Z_2 &= 0.\end{aligned}$$

At least one of α, β, γ is non-zero, so without loss of generality we assume $\alpha \neq 0$, as the computation is analogous in the other two cases. Solving for X_1 and X_2 , substituting into (3.9) and simplifying, gives the expression

$$\frac{1}{\bar{\alpha}}(|\alpha|^2 + |\beta|^2 + |\gamma|^2)\text{Tr}(Y_1Z_2 - Y_2Z_1).$$

This is non-degenerate and therefore H must be non-degenerate.

Since $f_n|_{Y_a} = g_a$, the non-degeneracy of H on Y_n^σ implies that, after possibly replacing S by a GL_a -invariant open neighbourhood of y_E in S , we may assume that the ideal $I = (df_n|_S)$ coincides with the ideal (dg_a, K) , where K is the ideal of Y_a in S generated by the coordinates of Y_n^σ .

Thus, letting $U = U_a \times_S Y_a$, we see that U is an open neighbourhood of v_E in U_a and the étale map

$$\kappa = \psi_a|_{[U/\text{GL}_a]}: [U/\text{GL}_a] \longrightarrow [U_a/\text{GL}_a] \longrightarrow [U_n/\text{GL}_n]$$

satisfies by construction

$$(\kappa^* s_n^{\text{crit}})|_U = f_n|_S + I^2 \in H^0(S_U^0)^{\text{GL}_a}.$$

Let $V = Y_a \times_S Y_n^\sigma$. By definition, V is a GL_a -invariant open neighbourhood of y_E in Y_a . We have the embedding $\Psi: V \rightarrow S$ such that $\Psi^* f_n|_S = g_a|_V$ and $U = \text{crit}(g_a|_V)$. Write J for the ideal $(dg_a|_V)$. Then, by the definition of the sheaf S_U^0 , the commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & S_U^0 & \longrightarrow & \mathcal{O}_S/I^2 & \xrightarrow{d} & \Omega_S/I \cdot \Omega_S \\ & & \parallel & & \downarrow \Psi^* & & \downarrow \Psi^* \\ 0 & \longrightarrow & S_U^0 & \longrightarrow & \mathcal{O}_V/J^2 & \xrightarrow{d} & \Omega_V/J \cdot \Omega_V \end{array}$$

implies that $f_n|_S + I^2 = g_a|_V + J^2 \in H^0(S_U^0)^{\text{GL}_a}$, which is exactly what we want. \square

3.2. The derived stack \mathcal{M}_n of 0-dimensional coherent sheaves of length n on \mathbb{A}^3 . Let \mathcal{M}_n be the derived moduli stack of 0-dimensional coherent sheaves of length n on \mathbb{A}^3 . Its classical truncation is the stack \mathcal{M}_n . We now recall the definition of \mathcal{M}_n and give an explicit description as a derived quotient stack for the convenience of the reader.

Let $A \in \text{cdga}_{\mathbb{C}}^{\leq 0}$. By [35] or [9, Example 2.7], \mathcal{M}_n assigns to A the mapping space

$$(3.10) \quad \text{Map}_{\text{dgCat}}(\mathbb{C}[x, y, z], \text{Proj}_n(A)).$$

Here $\mathbb{C}[x, y, z]$ is viewed as the dg-category with one object and morphisms concentrated in degree 0 given by $\mathbb{C}[x, y, z]$, whereas $\text{Proj}_n(A)$ is the dg-category whose objects are rank n projective A -modules and the morphism complexes are given by $\text{Hom}^\bullet(E, F)$ for two objects E, F .

Since projective modules are locally trivial and, as differential graded algebras, we have $\text{Hom}^\bullet(A^n, A^n) = \mathfrak{gl}_n \otimes_{\mathbb{C}} A$, we can further simplify (3.10) (up to possible shrinking) into

$$(3.11) \quad \text{Map}_{\text{dga}_{\mathbb{C}}}(\mathbb{C}[x, y, z], \mathfrak{gl}_n \otimes_{\mathbb{C}} A),$$

up to conjugation by GL_n .

To compute this mapping space with respect to the model structure of the category $\text{dga}_{\mathbb{C}}$, we need a resolution of the algebra $\mathbb{C}[x, y, z]$ by a semi-free dg-algebra. This is provided by the following proposition.

Proposition 3.4. *Let Q_3 be the semi-free dg-algebra with generators x, y, z in degree 0, x^*, y^*, z^* in degree -1 and w in degree -2 and differential δ satisfying*

$$(3.12) \quad \begin{aligned} \delta x &= \delta y = \delta z = 0 \\ \delta x^* &= [y, z], \quad \delta y^* = [z, x], \quad \delta z^* = [x, y] \\ \delta w &= [x, x^*] + [y, y^*] + [z, z^*]. \end{aligned}$$

Then the natural morphism $Q_3 \rightarrow \mathbb{C}[x, y, z]$ is a quasi-isomorphism.

Proof. The claim follows from the fact that the \mathbb{C} -algebra $\mathbb{C}\langle x, y, z \rangle$ generated over \mathbb{C} freely by three variables x, y, z is Calabi–Yau of dimension 3 and [16, Theorem 5.3.1]. \square

It follows from the proposition that the mapping space (3.11) is computed by

$$\text{Hom}_{\text{dga}_{\mathbb{C}}}(Q_3, \mathfrak{gl}_n \otimes_{\mathbb{C}} A)$$

which, using the results of [6], is naturally isomorphic to

$$\text{Hom}_{\text{cdga}_{\mathbb{C}}^{\leq 0}}((Q_3)_n, A).$$

Here $(Q_3)_n$ is the commutative dg-algebra with generators corresponding to the matrix entries of the $n \times n$ matrices X, Y, Z in degree 0, X^*, Y^*, Z^* in degree -1 and W in degree -2 with differential determined by (3.12). In particular we have sets of generators given by the matrix entries of

$$\text{Rep}_n(L_3) \simeq \mathfrak{gl}_n^{\oplus 3}, \quad \text{Rep}_n(L_3) \simeq \mathfrak{gl}_n^{\oplus 3}, \quad \mathfrak{gl}_n$$

in degrees 0, -1 and -2 respectively.

Quotienting by the conjugation action of GL_n , we obtain the following proposition.

Proposition 3.5. *There exists an isomorphism of derived Artin stacks*

$$(3.13) \quad \iota_n: [\mathbf{Spec}(Q_3)_n / \text{GL}_n] \xrightarrow{\sim} \mathcal{M}_n.$$

3.3. The d-critical structure s_n^{der} coming from derived symplectic geometry. Recall that Equation (2.3) defines s_n^{der} to be the truncation of the -1 -shifted symplectic structure ω_n on \mathcal{M}_n constructed by Brav and Dyckerhoff [9].

Let $[E] \in \mathcal{M}_n$ be a closed point, so that the sheaf E is polystable as in (3.1). We now proceed to find formally smooth d-critical charts for $(\mathcal{M}_n, s_n^{\text{der}})$ around such points $[E]$ using the derived deformation theory of \mathcal{M}_n , following [33]. In order to do this, we first need some preparations.

Let $p = (a_0, b_0, c_0) \in \mathbb{A}^3$ be a point in coordinates x_0, y_0, z_0 for \mathbb{A}^3 . We denote the functions $x_0 - a_0, y_0 - b_0, z_0 - c_0$ by x, y, z respectively. For convenience, we also write \mathcal{O} in place of the ring $\mathcal{O}_{\mathbb{A}^3} = \mathbb{C}[x_0, y_0, z_0]$ in what follows.

We have the Koszul resolution $Q_p^\bullet \rightarrow \mathcal{O}_p$, given by the complex

$$(3.14) \quad Q_p^\bullet = [Q^{-3} \rightarrow Q^{-2} \rightarrow Q^{-1} \rightarrow Q^0] = [\mathcal{O} \xrightarrow{A} \mathcal{O}^{\oplus 3} \xrightarrow{B} \mathcal{O}^{\oplus 3} \xrightarrow{C} \mathcal{O}],$$

where the morphisms A, B, C are multiplications by the matrices (we are very slightly abusing notation here)

$$A = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad B = \begin{pmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{pmatrix}, \quad C = (x \ y \ z).$$

Let

$$(3.15) \quad \mathfrak{g}_p = \mathrm{Hom}(Q_p^\bullet, Q_p^\bullet) = \mathbf{R}\mathrm{Hom}(\mathcal{O}_p, \mathcal{O}_p).$$

By composing morphisms in each degree, \mathfrak{g}_p has the structure of an (infinite-dimensional) dg-algebra $(\mathfrak{g}_p, \cdot, \delta)$ over \mathbb{C} . Moreover, let

$$(3.16) \quad \mathfrak{g}_p^{\min} = \bigoplus_{i=0}^3 \mathrm{H}^i(\mathfrak{g}_p) = \bigoplus_{i=0}^3 \mathrm{Ext}^i(\mathcal{O}_p, \mathcal{O}_p).$$

Using the Yoneda product

$$m_2: \mathrm{Ext}^i(\mathcal{O}_p, \mathcal{O}_p) \otimes \mathrm{Ext}^j(\mathcal{O}_p, \mathcal{O}_p) \longrightarrow \mathrm{Ext}^{i+j}(\mathcal{O}_p, \mathcal{O}_p),$$

one can endow \mathfrak{g}_p^{\min} with the structure of a graded commutative algebra over \mathbb{C} and thus of a dg-algebra $(\mathfrak{g}_p^{\min}, m_2, 0)$ with zero differential. As a graded algebra, it is clear that \mathfrak{g}_p^{\min} is isomorphic to the exterior algebra $\Lambda^\bullet \mathbb{C}^3$.

Lemma 3.6. *There exists a morphism $I_p: (\mathfrak{g}_p^{\min}, m_2, 0) \rightarrow (\mathfrak{g}_p, \cdot, \delta)$ of dg-algebras, which induces an isomorphism on homology. In particular, the dg-algebra \mathfrak{g}_p is formal as an A_∞ -algebra.*

Proof. We construct the morphism I_p explicitly. For brevity, we identify $\mathrm{Hom}_\mathcal{O}(\mathcal{O}^{\oplus s}, \mathcal{O}^{\oplus t})$ with $t \times s$ matrices $\mathrm{Mat}_{t,s}(\mathcal{O})$ with entries in \mathcal{O} .

The graded pieces of \mathfrak{g}_p in degrees $0, \dots, 4$ are

$$\begin{aligned} \mathfrak{g}_p^0 &= \bigoplus_i \mathrm{Hom}(Q^i, Q^i) = \mathcal{O} \oplus \mathrm{Mat}_{3,3}(\mathcal{O}) \oplus \mathrm{Mat}_{3,3}(\mathcal{O}) \oplus \mathcal{O} \\ \mathfrak{g}_p^1 &= \bigoplus_i \mathrm{Hom}(Q^i, Q^{i+1}) = \mathrm{Mat}_{3,1}(\mathcal{O}) \oplus \mathrm{Mat}_{3,3}(\mathcal{O}) \oplus \mathrm{Mat}_{1,3}(\mathcal{O}) \\ \mathfrak{g}_p^2 &= \bigoplus_i \mathrm{Hom}(Q^i, Q^{i+2}) = \mathrm{Mat}_{3,1}(\mathcal{O}) \oplus \mathrm{Mat}_{1,3}(\mathcal{O}) \\ \mathfrak{g}_p^3 &= \bigoplus_i \mathrm{Hom}(Q^i, Q^{i+3}) = \mathcal{O} \\ \mathfrak{g}_p^4 &= \bigoplus_i \mathrm{Hom}(Q^i, Q^{i+4}) = 0. \end{aligned}$$

Define elements $\widehat{1} \in \mathfrak{g}_p^0$ and $\widehat{x}, \widehat{y}, \widehat{z} \in \mathfrak{g}_p^1$ by

$$\begin{aligned}\widehat{1} &= (1, \text{Id}_3, \text{Id}_3, 1) \\ \widehat{x} &= \left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, (1 \ 0 \ 0) \right) \\ \widehat{y} &= \left(\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, (0 \ 1 \ 0) \right) \\ \widehat{z} &= \left(\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, (0 \ 0 \ 1) \right).\end{aligned}$$

We may verify directly that these satisfy the relations

$$\begin{aligned}\widehat{1} \cdot \widehat{x} &= \widehat{x}, \quad \widehat{1} \cdot \widehat{y} = \widehat{y}, \quad \widehat{1} \cdot \widehat{z} = \widehat{z} \\ \widehat{x} \cdot \widehat{x} &= 0, \quad \widehat{y} \cdot \widehat{y} = 0, \quad \widehat{z} \cdot \widehat{z} = 0,\end{aligned}$$

as well as

$$\begin{aligned}\widehat{x} \cdot \widehat{y} &= -\widehat{y} \cdot \widehat{x} = \left(\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, (1 \ 0 \ 0) \right) \\ \widehat{y} \cdot \widehat{z} &= -\widehat{z} \cdot \widehat{y} = \left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, (1 \ 0 \ 0) \right) \\ \widehat{z} \cdot \widehat{x} &= -\widehat{x} \cdot \widehat{z} = \left(\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, (0 \ 1 \ 0) \right)\end{aligned}$$

$$\widehat{x} \cdot \widehat{y} \cdot \widehat{z} = 2.$$

So the associative algebra R generated by $\widehat{1}, \widehat{x}, \widehat{y}, \widehat{z}$ is a sub-algebra of \mathfrak{g}_p isomorphic to $\Lambda^\bullet \mathbb{C}^3$. Moreover, all of the elements of R are cocycles for the differential δ of \mathfrak{g}_p and their images in homology give bases for the homology groups $H^i(\mathfrak{g}_p) = \text{Ext}^i(\mathcal{O}_p, \mathcal{O}_p)$. By the definition of the Yoneda product m_2 , it is immediate that R is isomorphic to \mathfrak{g}_p^{\min} , giving a morphism of dg-algebras $I_p: \mathfrak{g}_p^{\min} \rightarrow \mathfrak{g}_p$ which, by construction, induces the identity on homology, as desired. \square

Remark 3.7. The element $\widehat{x} \in \mathfrak{g}_p^1$ is the evaluation of the triple (A, B, C) at the point $(1, 0, 0) \in \mathbb{C}^3$. Analogous statements hold for \widehat{y}, \widehat{z} and all products of $\widehat{x}, \widehat{y}, \widehat{z}$. This motivates their definition.

Let now E be a polystable sheaf as in (3.1) and write $Q_E^\bullet = \bigoplus_{i=1}^k \mathbb{C}^{a_i} \otimes Q_{p_i}^\bullet$. We then define

$$\mathfrak{g}_E = \text{Hom}(Q_E^\bullet, Q_E^\bullet) = \bigoplus_{i,j=1}^k \text{Hom}(Q_{p_i}^\bullet, Q_{p_j}^\bullet) \otimes \text{Mat}_{a_j, a_i} = \mathbf{R}\text{Hom}(E, E)$$

and

$$(3.17) \quad \begin{aligned} \mathfrak{g}_E^{\text{ss}} &= \bigoplus_{i=1}^k \mathfrak{g}_{p_i} \otimes \text{Mat}_{a_i, a_i} \\ \mathfrak{g}_E^{\text{min}} &= \bigoplus_{i=1}^k \mathfrak{g}_{p_i}^{\text{min}} \otimes \text{Mat}_{a_i, a_i}. \end{aligned}$$

Then \mathfrak{g}_E is a dg-algebra with differential δ induced from that of each summand \mathfrak{g}_{p_i} and $\mathfrak{g}_E^{\text{ss}} \subset \mathfrak{g}_E$ is a dg-subalgebra. In the same fashion $\mathfrak{g}_E^{\text{min}}$ is a commutative graded algebra. Since for any $i \neq j$ and any ℓ we have $\text{Ext}^\ell(\mathcal{O}_{p_i}, \mathcal{O}_{p_j}) = 0$, the inclusion $\mathfrak{g}_E^{\text{ss}} \subset \mathfrak{g}_E$ is a quasi-isomorphism and $\mathfrak{g}_E^{\text{min}}$ is isomorphic as a dg-algebra to $(\text{Ext}^*(E, E), m_2, 0)$.

Corollary 3.8. *Let E be a polystable sheaf as in (3.1). Then there exists a morphism*

$$I_E: (\mathfrak{g}_E^{\text{min}}, m_2, 0) \rightarrow (\mathfrak{g}_E, \cdot, \delta)$$

of dg-algebras, which induces an isomorphism on homology. Thus, the dg-algebra \mathfrak{g}_E is formal as an A_∞ -algebra.

Proof. The map I_E is given by the composition of the morphism $\bigoplus_{i=1}^k I_{p_i} \otimes \text{id}_{\text{Mat}_{a_i, a_i}}$ and the inclusion $\mathfrak{g}_E^{\text{ss}} \subset \mathfrak{g}_E$. \square

We fix some notation and useful facts before proceeding.

Let $S = \widehat{S}(\mathfrak{g}_E^{\text{min}, >0}[1])^\vee$ be the dual of the abelianisation of the Bar construction of the dg-algebra $\mathfrak{g}_E^{\text{min}, >0}$, which is the Chevalley–Eilenberg dg-algebra of the dg-Lie algebra associated to the commutative dg-algebra $\mathfrak{g}_E^{\text{min}}$, with Lie bracket on each $\mathfrak{g}_{p_i}^{\text{min}} \otimes \text{Mat}_{a_i, a_i}$ determined by the equations

$$(3.18) \quad \begin{aligned} [(\widehat{x}_i^\vee \otimes A_i), (\widehat{y}_i^\vee \otimes B_i)] &= \widehat{z}_i^\vee \otimes [A_i, B_i] \\ [(\widehat{y}_i^\vee \otimes B_i), (\widehat{z}_i^\vee \otimes C_i)] &= \widehat{x}_i^\vee \otimes [B_i, C_i] \\ [(\widehat{z}_i^\vee \otimes C_i), (\widehat{x}_i^\vee \otimes A_i)] &= \widehat{y}_i^\vee \otimes [C_i, A_i], \end{aligned}$$

where each $\mathfrak{g}_{p_i}^{\text{min}} \cong \Lambda^\bullet \mathbb{C}^3$ using the given basis $\{\widehat{x}_i, \widehat{y}_i, \widehat{z}_i\}$. Here we have used the fact that the Lie bracket in the dg-Lie algebra structure of the derived tangent complex $\mathfrak{g}_E[1]$ of \mathcal{M}_n at $[E]$ is given by commutator of matrices (see [9, Proposition 3.3] or [19, Proposition 2.4.4]).

Thus, the definition (3.17) implies that S is generated by subalgebras S_i , where each S_i is a negatively graded commutative dg-algebra and the differential ϵ in degree -1 is determined by the morphisms

$$(3.19) \quad \Lambda^2 \left((\mathbb{C}^3)^\vee \otimes \text{Mat}_{a_i, a_i}^\vee \right) \xrightarrow{\epsilon_i} \text{Sym}^\bullet \left((\mathbb{C}^3)^\vee \otimes \text{Mat}_{a_i, a_i}^\vee \right)$$

satisfying, due to the Lie bracket relations (3.18),

$$(3.20) \quad \begin{aligned} (\widehat{x}_i^\vee \otimes A_i) \wedge (\widehat{y}_i^\vee \otimes B_i) &\mapsto \widehat{z}_i^\vee \otimes [A_i, B_i] \\ (\widehat{y}_i^\vee \otimes B_i) \wedge (\widehat{z}_i^\vee \otimes C_i) &\mapsto \widehat{x}_i^\vee \otimes [B_i, C_i] \\ (\widehat{z}_i^\vee \otimes C_i) \wedge (\widehat{x}_i^\vee \otimes A_i) &\mapsto \widehat{y}_i^\vee \otimes [C_i, A_i]. \end{aligned}$$

Let \widehat{Y}_{a_i} be the formal completion of $Y_{a_i} = \text{End}_{\mathbb{C}}(\mathbb{C}^{a_i})^3$ at the origin and let \widehat{f}_{a_i} be the formal function

$$\widehat{f}_{a_i}: \widehat{Y}_{a_i} \rightarrow \mathbb{C}, \quad \widehat{f}_{a_i}(A_i, B_i, C_i) = \text{Tr } A_i[B_i, C_i].$$

Similarly, let

$$(3.21) \quad \widehat{g}_a = \widehat{f}_{a_1} \oplus \cdots \oplus \widehat{f}_{a_k} : \widehat{Y}_a = \prod_{i=1}^k \widehat{Y}_{a_i} \rightarrow \mathbb{C}$$

be the function $\widehat{g}_a(v_1, \dots, v_k) = \widehat{f}_{a_1}(v_1) + \cdots + \widehat{f}_{a_k}(v_k)$.

It follows immediately by (3.19) that

$$(3.22) \quad \text{Spec} H^0(S) = \prod_{i=1}^k \text{crit}(\widehat{f}_{a_i}) = \text{crit}\left(\bigoplus_{i=1}^k \widehat{f}_{a_i}\right) = \text{crit}(\widehat{g}_a) \subset \widehat{Y}_a.$$

We also note that by [9], the pairing induced from the -1 -shifted symplectic structure ω_n on the shifted tangent complex \mathfrak{g}_E^{\min} and its components $\mathfrak{g}_{p_i}^{\min} \otimes \text{Mat}_{a_i, a_i}$ coincides with the Serre duality pairing. We denote this pairing by ζ_E .

We may now prove the following key lemma.

Lemma 3.9. *Let $[E] \in \mathcal{M}_n$ be a polystable sheaf of the form (3.1), and set $\widehat{U} = \text{crit}(\widehat{g}_a)$, where $\widehat{g}_a : \widehat{Y}_a \rightarrow \mathbb{C}$ is given by (3.21). Then there exists a formally smooth morphism*

$$(3.23) \quad \widehat{\phi}_E : \widehat{U} \rightarrow \mathcal{M}_n,$$

mapping $0 \in \widehat{U}$ to $[E] \in \mathcal{M}_n$, inducing an isomorphism at the level of tangent spaces and an identity

$$(\widehat{\phi}_E)^* s_n^{\text{der}} = \widehat{g}_a + (d\widehat{g}_a)^2 \in H^0(\mathcal{S}_{\widehat{U}}^0).$$

Proof. We continue to work formally locally at the point $[E] \in \mathcal{M}_n$. Write $\{\widehat{y}_\ell\}$ for the set of natural coordinates on \widehat{Y}_a induced by the equalities $Y_{a_i} = \text{End}_{\mathbb{C}}(\mathbb{C}^{a_i})^3$. Since S^{-1} is isomorphic to $T_{\widehat{Y}_a}$ as an S^0 -module by definition, we have a natural basis $\{\check{y}_\ell = \frac{\partial}{\partial \widehat{y}_\ell}\}$ of S^{-1} , which is dual to the basis $\{d_{\text{dR}} \widehat{y}_\ell\}$.

By the definition of S and properties of formal derived stacks and formal completions (see, for example, [11] and [19]), there is a formally smooth morphism $\mathbf{Spec} S \rightarrow \mathcal{M}_n$. Observe that, by construction, the morphism maps the closed point $0 \in \mathbf{Spec} S$ to $[E] \in \mathcal{M}_n$ and is minimal at 0 , i.e. it induces an equivalence between the cotangent complexes of $\mathbf{Spec} S$ at 0 and of \mathcal{M}_n at $[E]$ in non-positive degrees, and the restriction of the differential of the cotangent complex of $\mathbf{Spec} S$ is zero.

Starting with this morphism as input, by [8, Proposition 5.7], it follows that the pullback of the closed 2-form ω_n to $\mathbf{Spec} S$ can be brought to the form $(\omega_n^0, 0, 0, \dots)$, where ω_n^0 is a non-degenerate -1 -shifted 2-form that satisfies $d_{\text{dR}} \omega_n^0 = \delta \omega_n^0 = 0$. More precisely, we can put this formally smooth chart in Darboux form, meaning that there exist $\Phi \in S^0$ and $\phi \in (\Omega_S^1)^{-1}$, which satisfy the equations $\omega_n^0 = d_{\text{dR}} \phi$, $d_{\text{dR}} \Phi = \delta \phi$, and by [8, Theorem 5.18], we may take ϕ to be of the form $\phi = \sum_{\ell} \alpha_{\ell} d_{\text{dR}} \widehat{y}_{\ell}$, where $\alpha_{\ell} \in S^{-1}$. The lemma will follow if we can show that $\Phi = \widehat{g}_a$ and $\phi = \sum_{\ell} \check{y}_{\ell} d_{\text{dR}} \widehat{y}_{\ell}$, possibly after an automorphism of the cdga S .

Now, $\omega_n^0|_0 = \zeta_E = \sum_{\ell} d_{\text{dR}} \check{y}_{\ell}|_0 d_{\text{dR}} \widehat{y}_{\ell}|_0$, so, by a graded application of [24, Proposition 10.2, Lemma 10.3], there exists an affine change of coordinates Ψ for $\mathbf{Spec} S^0$ such that with respect to these new coordinates, which we denote by \widehat{y}'_{ℓ} , we have $\omega_n^0 \otimes_S S^0 = \sum_{\ell} d_{\text{dR}} \check{y}_{\ell} d_{\text{dR}} \widehat{y}'_{\ell}$.

Hence, still writing $\phi = \sum_{\ell} \alpha_{\ell} d_{\text{dR}} \widehat{y}'_{\ell}$ with respect to the new coordinates, we obtain $\omega_n^0 \otimes_S S^0 = \sum_{\ell} d_{\text{dR}} \check{y}_{\ell} d_{\text{dR}} \widehat{y}'_{\ell} = d_{\text{dR}} \phi \otimes_S S^0 = \sum_{\ell} d_{\text{dR}} \alpha_{\ell} d_{\text{dR}} \widehat{y}'_{\ell}$, which shows that $\alpha_{\ell} = \check{y}_{\ell}$ for all values of the index ℓ .

We now have a commutative diagram

$$\begin{array}{ccccc} S^{-1} & \xrightarrow{\sim} & T_{S^0} & \xrightarrow{d\widehat{g}_a^\vee} & S^0 \\ & & \downarrow (d\Psi)^{-1} & & \downarrow \Psi \\ S^{-1} & \xrightarrow{\sim} & T_{S^0} & \xrightarrow{d(\widehat{g}_a \circ \Psi)^\vee} & S^0 \end{array}$$

which extends to an automorphism of the cdga S . By construction, the pullback of ϕ under this automorphism is equal to $\sum_\ell \check{y}'_\ell d_{\text{dR}} \widehat{y}'_\ell$, where \check{y}'_ℓ is the basis element of T_{S^0} corresponding to the partial derivative $\frac{\partial}{\partial \widehat{y}'_\ell}$.

In particular, after pulling back, we have equalities $d_{\text{dR}}(\Phi \circ \Psi) = \delta \phi = d_{\text{dR}}(\widehat{g}_a \circ \Psi)$, so that $\Phi \circ \Psi = \widehat{g}_a \circ \Psi$ and $\Phi = \widehat{g}_a$, as desired. \square

Remark 3.10. In a previous version of this paper, the claim that the natural morphism

$$\text{Spec } S \rightarrow \mathcal{M}_n$$

gives a formal atlas for \mathcal{M}_n at the point $[E]$ which is *already* in Darboux form with respect to the -1 -shifted symplectic structure ω_n was based on an appendix in the paper [32]. This has since been removed, so we have added an independent proof of the lemma to reflect this change.

The basic idea behind the proof is that a shifted symplectic form on a formal scheme can be taken to have constant coefficients after reparametrization. In particular, its value at the unique closed point should determine a Darboux form in a unique way. Hence, if the formal scheme is already in Darboux form and the restriction of the naturally associated symplectic structure at the point matches another given one, then the Darboux forms and symplectic structures must match as well. In this spirit, the claim is a simple, particular case of a general principle, whose proof we expect will appear in full form in the literature sooner rather than later.

3.4. The two d-critical structures are equal. We now prove Theorem 2.2, showing that the two d-critical structures studied in this section actually coincide.

Theorem 3.11. *The isomorphism $\iota_n: [U_n/\text{GL}_n] \xrightarrow{\sim} \mathcal{M}_n$ from (2.2) induces an identity*

$$\iota_n^* s_n^{\text{der}} = s_n^{\text{crit}}$$

of algebraic d-critical structures.

Proof. Let $[E] \in \mathcal{M}_n$ be a polystable sheaf, given by the expression (3.1). Applying Lemma 3.3, we have an étale morphism $\psi_E: [U/\text{GL}_a] \rightarrow [U_n/\text{GL}_n]$ such that the pullback under the induced morphism $\phi_E: U \rightarrow [U/\text{GL}_a] \rightarrow [U_n/\text{GL}_n]$ satisfies

$$(3.24) \quad \phi_E^* s_n^{\text{crit}} = g_a|_V + (dg_a|_V)^2 \in H^0(\mathcal{S}_U^0)^{\text{GL}_a} \subset H^0(\mathcal{S}_U^0),$$

where $V \subset Y_a$ is an open neighbourhood of the point $\nu_E \in Y_a$ corresponding to $[E]$ and g_a is given by (3.2). By Lemma 3.9, we have a formal atlas

$$\widehat{\phi}_E: \widehat{U} \rightarrow \mathcal{M}_n, \quad 0 \mapsto [E],$$

satisfying

$$(3.25) \quad \widehat{\phi}_E^*(s_n^{\text{der}}) = \widehat{g}_a + (d\widehat{g}_a)^2 \in H^0(\mathcal{S}_{\widehat{U}}^0).$$

Let $j_m: U_{E,m} \rightarrow U$ be the m -th order thickening of $v_E \in U$. We get induced morphisms $\iota_n \circ \phi_E \circ j_m: U_{E,m} \rightarrow \mathcal{M}_n$. Since $\widehat{\phi}_E: \widehat{U} \rightarrow \mathcal{M}_n$ is formally smooth, this compatible system lifts via $\widehat{\phi}_E$ to give rise to a morphism $\gamma: \widehat{U} \rightarrow \widehat{U}$ such that $\widehat{\phi}_E \circ \gamma = \iota_n \circ \phi_E \circ \eta$, where $\eta: \widehat{U} \rightarrow U$ is the formal completion morphism of U at v_E . That is, we have a commutative diagram

$$\begin{array}{ccccc} \widehat{U} & \xrightarrow{\gamma} & \widehat{U} & & \\ & \searrow \eta & & \searrow \widehat{\phi}_E & \\ & & U & \xrightarrow{\phi_E} & [U_n/\mathrm{GL}_n] \xrightarrow{\iota_n} \mathcal{M}_n \end{array}$$

where, since $\iota_n \circ \phi_E$ and η are isomorphisms to first order at v_E (i.e. at the level of tangent spaces), the same is true for γ , so γ is an isomorphism.

The inclusion $\widehat{U} \subset \widehat{V}$ is cut out by quadrics (the derivatives of the cubic function g_a) and thus γ extends to an isomorphism $\widehat{\gamma}: \widehat{V} \rightarrow \widehat{V}$. Let $h = \widehat{g}_a \circ \widehat{\gamma}$ so that obviously $(dh) = (d\widehat{g}_a)$. The commutativity of the diagram, along with Equation (3.25), implies that

$$\eta^* \phi_E^* \iota_n^* (s_n^{\mathrm{der}}) = \gamma^* \widehat{\phi}_E^* (s_n^{\mathrm{der}}) = h + (dh)^2 \in H^0(\mathcal{S}_{\widehat{U}}^0).$$

After possible further shrinking of V around v_E , we have

$$(3.26) \quad \phi_E^* \iota_n^* (s_n^{\mathrm{der}}) = k + (dk)^2$$

for some function $k: V \rightarrow \mathbb{C}$ satisfying $(dk) = (dg_a|_V)$. Therefore, taking completions, we find

$$\eta^* \phi_E^* \iota_n^* (s_n^{\mathrm{der}}) = \widehat{k} + (d\widehat{k})^2 \in H^0(\mathcal{S}_{\widehat{U}}^0).$$

Since $(dh) = (d\widehat{k}) = (d\widehat{g}_a)$ it follows that \widehat{k} has no quadratic terms and we must have $\widehat{k} - h \in (d\widehat{g}_a)^2$. The ideal $(d\widehat{g}_a)^2$ is generated by quartics so by comparing cubic terms we deduce that

$$\widehat{k}_3 = \widehat{g}_a \circ d\gamma = h_3$$

where \widehat{k}_3, h_3 are the cubic terms of \widehat{k}, h respectively. As k is a polynomial, we get for the higher order terms

$$\begin{aligned} \widehat{k} - h &= \widehat{k}_{\geq 4} - h_{\geq 4} \in (d\widehat{g}_a)^2 \\ h_{\geq \mathrm{deg} k} &\in (d\widehat{g}_a)^2. \end{aligned}$$

Since by definition $h = \widehat{g}_a \circ \gamma$, where \widehat{g}_a is a cubic, we may truncate the power series defining γ at sufficiently high degree to get a polynomial isomorphism $\widehat{\rho}: \widehat{V} \rightarrow \widehat{V}$ such that $h' = \widehat{g}_a \circ \widehat{\rho}$ still satisfies $(dh') = (dh) = (d\widehat{g}_a)$ and $h' - h$ consists of summands of h of degree at least $\mathrm{deg} k$. By the above, $h - h' \in (d\widehat{g}_a)^2$ and therefore

$$(3.27) \quad \eta^* \phi_E^* \iota_n^* (s_n^{\mathrm{der}}) = \widehat{k} + (d\widehat{k})^2 = h + (dh)^2 = h' + (dh')^2 \in H^0(\mathcal{S}_{\widehat{U}}^0).$$

But V is an open subscheme of an affine space and hence $\widehat{\rho}$ is the completion of an isomorphism $\rho: V \rightarrow V$. In particular, h' is the completion of the function $g'_a = g_a|_V \circ \rho: V \rightarrow \mathbb{C}$,

$$(3.28) \quad h' = \widehat{g'_a}: \widehat{V} \rightarrow \mathbb{C}.$$

Writing $I = (dg_a|_V) = (dg'_a)$, the commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{S}_U & \longrightarrow & \mathcal{O}_V/I^2 & \xrightarrow{d} & \Omega_V/I \cdot \Omega_V \\ & & \parallel & & \downarrow \rho^* & & \downarrow \rho^* \\ 0 & \longrightarrow & \mathcal{S}_U & \longrightarrow & \mathcal{O}_V/I^2 & \xrightarrow{d} & \Omega_V/I \cdot \Omega_V \end{array}$$

and (3.24) show that

$$(3.29) \quad \phi_E^*(s_n^{\text{crit}}) = g'_a + (dg'_a)^2 \in H^0(\mathcal{S}_U^0).$$

Using equations (3.26), (3.27), (3.28) and (3.29), the following proposition applied to the functions $k, g'_a: V \rightarrow \mathbb{C}$ shows that there exists a Zariski open neighbourhood $v_E \in U' \subset U$ and a smooth morphism

$$\phi'_E: U' \rightarrow U \xrightarrow{\phi_E} [U_n/\text{GL}_n]$$

such that

$$(\phi'_E)^* s_n^{\text{crit}} = (\phi'_E)^* \iota_n^* s_n^{\text{der}} \in H^0(\mathcal{S}_{U'}^0).$$

As $[E]$ varies in \mathcal{M}_n , the morphisms ϕ'_E give a smooth surjective cover of \mathcal{M}_n , on which the d-critical structures s_n^{crit} and $\iota_n^* s_n^{\text{der}}$ agree. Hence we must have $s_n^{\text{crit}} = \iota_n^* s_n^{\text{der}}$, as desired. \square

Proposition 3.12. *Let V be a smooth scheme and let $f_1, f_2: V \rightarrow \mathbb{C}$ be two functions with $(df_1) = (df_2) = I$ such that $U = \text{crit}(f_1) = \text{crit}(f_2) \subset V$. Fix a point $u \in U$. Let $\eta: \widehat{U} \rightarrow U$ be the formal completion of U at the point u and $\widehat{I} = I \otimes_{\mathcal{O}_V} \mathcal{O}_{\widehat{V}}$ the ideal of \widehat{U} in \widehat{V} .*

Suppose that

$$(3.30) \quad f_1 + \widehat{I}^2 = f_2 + \widehat{I}^2 \in H^0(\mathcal{S}_{\widehat{U}}^0).$$

Then there exists a Zariski open neighbourhood $V' \subset V$ of u in V such that for $U' = U \times_V V'$ we have

$$f_1|_{V'} + I|_{V'}^2 = f_2|_{V'} + I|_{V'}^2 \in H^0(\mathcal{S}_{U'}^0).$$

Proof. Since the problem is local in V , we may assume that V is affine. Write \mathfrak{m} for the maximal ideal of the closed point $0 \in \widehat{V}$ and \mathfrak{m}_u for the ideal of $u \in V$. We use $\eta: \widehat{V} \rightarrow V$ to denote the formal completion map as well. Let V_{loc} denote the localisation of V at u . The map η clearly factors through the localisation morphism $V_{\text{loc}} \rightarrow V$ by a local, faithfully flat morphism $\mu: \widehat{V} \rightarrow V_{\text{loc}}$.

Let $\delta = f_1 - f_2: \mathcal{O}_{V,u} \rightarrow \mathcal{O}_{V,u}/I^2$. The assumption (3.30) implies that $\mu^* \delta = 0$. Therefore, since μ is faithfully flat, we must have $\delta = 0$.

Hence there exist functions $h \in I^2, g \in \mathcal{O}_{V,u} \setminus \mathfrak{m}_u$ and a positive integer N such that

$$g^N (f_1 - f_2) = h \in I^2.$$

Letting V' be the non-vanishing locus of g completes the proof. \square

3.5. Some remarks on the proof of Theorem 2.2. We finally collect some observations to conclude the section and suggest an alternative, stronger approach to prove Theorem 2.2.

One can check that the classical truncation of the isomorphism

$$\iota_n: [\mathbf{Spec}(Q_3)_n/\text{GL}_n] \xrightarrow{\sim} \mathcal{M}_n$$

given in (3.13) is the isomorphism

$$\iota_n: [U_n/\text{GL}_n] \xrightarrow{\sim} \mathcal{M}_n.$$

The target \mathcal{M}_n of ι_n admits the -1 -shifted symplectic structure ω_n .

On the other hand, using the model for shifted symplectic forms for (affine) derived quotient stacks described in [26, Section 1], we can construct an explicit -1 -shifted symplectic form $\omega = (\omega^0, 0, 0, \dots)$ on $[\mathbf{Spec}(Q_3)_n/\text{GL}_n]$, combining the -1 -shifted symplectic forms for $\mathbb{R}\text{crit}(f_n)$ and BGL_n , such that the induced d-critical structure on the truncation $[U_n/\text{GL}_n]$ is s_n^{crit} .

Theorem 2.2 will then follow from the stronger statement that $\iota_n^* \omega_n = \omega$ as -1 -shifted symplectic forms. Since the appearance of the present paper, the equality $\iota_n^* \omega_n = \omega$ has been established by Katz–Shi [22].

We have chosen to pursue a local argument to obtain the equality of d-critical structures using the derived deformation theory of \mathcal{M}_n . This gives us more control over the derived tangent complex and explicit Darboux charts, which we are comfortable with, and also avoids possible technicalities on shifted symplectic structures on derived quotient stacks. Our local approach should be applicable any time one can get a concrete handle on the formal completion of a derived stack at a point.

4. THE D-CRITICAL STRUCTURE(S) ON $\text{Quot}_{\mathbb{A}^3}(\mathcal{O}^{\oplus r}, n)$

Fix integers $n \geq 0$ and $r \geq 1$. Recall from Theorem 2.3 the critical structure

$$\text{ncQuot}_r^n \supset \text{crit}(f_{r,n}) \xrightarrow{\sim} Q_{r,n} = \text{Quot}_{\mathbb{A}^3}(\mathcal{O}^{\oplus r}, n)$$

on the Quot scheme $Q_{r,n}$ of length n quotients $\mathcal{O}^{\oplus r} \rightarrow E$ of the trivial rank r sheaf on \mathbb{A}^3 . In this section, we compare two d-critical structures

$$s_{r,n}^{\text{crit}} \in H^0(\mathcal{S}_{\text{crit}(f_{r,n})}^0), \quad s_{r,n}^{\text{der}} \in H^0(\mathcal{S}_{Q_{r,n}}^0),$$

by bootstrapping the arguments in our discussion of the d-critical structure of \mathcal{M}_n in the preceding section.

To motivate, let $\mathbf{Q}_{r,n}$ be the derived Quot scheme [14]. Then we have a commutative diagram

$$(4.1) \quad \begin{array}{ccc} Q_{r,n} & \xleftarrow{j_{r,n}} & \mathbf{Q}_{r,n} \\ q_{r,n} \downarrow & & \downarrow \mathbf{q}_{r,n} \\ \mathcal{M}_n & \xleftarrow{j_n} & \mathcal{M}_n \end{array}$$

where $j_{r,n}$ and j_n are the inclusions of the underlying classical spaces and $q_{r,n}$ is the forgetful morphism taking $[\mathcal{O}^{\oplus r} \rightarrow E] \in Q_{r,n}$ to $[E] \in \mathcal{M}_n$, which can be viewed as the truncation of its derived enhancement $\mathbf{q}_{r,n}$.

4.1. The d-critical structure $s_{r,n}^{\text{crit}}$ coming from quiver representations. The d-critical structure

$$s_{r,n}^{\text{crit}} = f_{r,n} + (df_{r,n})^2$$

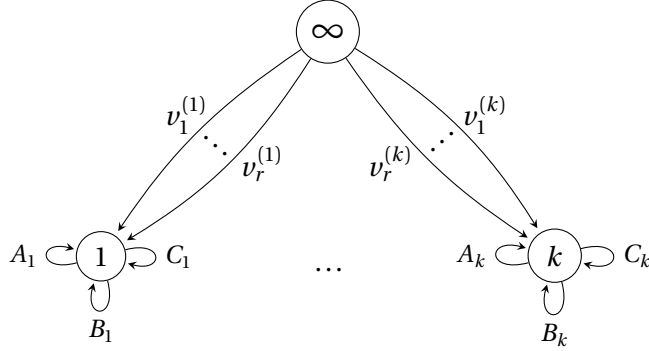
on $\text{crit}(f_{r,n}) \hookrightarrow \text{ncQuot}_r^n$ admits a local description similar to that of s_n^{crit} . We fix some notation for convenience, extending the corresponding notation used for \mathcal{M}_n in Section 3.1.

Let $Y_{r,n} = \text{Rep}_{(1,n)}(\tilde{L}_3) = \text{End}_{\mathbb{C}}(\mathbb{C}^n)^3 \oplus \text{Hom}_{\mathbb{C}}(\mathbb{C}, \mathbb{C}^n)^r = Y_n \oplus Z_{r,n}$, i.e. we denote by $Z_{r,n}$ the space of r -tuples (v_1, \dots, v_r) of vectors $v_i \in \mathbb{C}^n$. Let $f_{r,n}: Y_{r,n} \rightarrow \mathbb{A}^1$ be the regular (and GL_n -invariant) function induced by the potential $W = A[B, C]$. It is crucial to observe that this function does not interact with the component $Z_{r,n}$. According to Theorem 2.3, there is a stability condition θ on \tilde{L}_3 along with a commutative diagram

$$\begin{array}{ccc} Y_{r,n}^{\theta\text{-st}} & \xrightarrow{\text{open}} & Y_{r,n} \\ \downarrow / \text{GL}_n & & \downarrow f_{r,n} \\ Y_{r,n} //_{\theta} \text{GL}_n & \xlongequal{\quad} & \text{ncQuot}_r^n \xrightarrow{f_{r,n}} \mathbb{A}^1 \end{array}$$

defining the noncommutative Quot scheme, on which the function $f_{r,n}$ descends, and there is an isomorphism $\iota_{r,n}: \text{crit}(f_{r,n}) \xrightarrow{\sim} Q_{r,n}$. For instance, one could take $\theta = (n, -1)$ for any fixed $n > 0$.

Let $\mathbf{a} = (a_1, \dots, a_k)$ denote a k -tuple of positive integers such that $\sum_{1 \leq i \leq k} a_i = n$. Denote by Q the quiver



so that

$$Y_{r,\mathbf{a}} = \text{Rep}_{(1,a_1,\dots,a_k)}(Q) = \prod_{i=1}^k Y_{r,a_i} = \prod_{i=1}^k \text{Rep}_{(1,a_i)}(\tilde{L}_3).$$

Consider the stability condition $\theta = (n, -1, \dots, -1)$ on Q , so that a Q -representation with dimension vector $(1, a_1, \dots, a_k)$ has slope 0. We then have the following.

Lemma 4.1. *Set $\theta_i = (a_i, -1)$ and $\theta = (n, -1, \dots, -1)$. Then there are open immersions*

$$Y_{r,\mathbf{a}}^{\theta\text{-st}} \hookrightarrow \prod_{i=1}^k Y_{r,a_i}^{\theta_i\text{-st}} \hookrightarrow Y_{r,\mathbf{a}}.$$

Proof. By the very definition of $Y_{r,\mathbf{a}}$, only the first inclusion needs a proof. Set $\mathbf{d} = (1, a_1, \dots, a_k)$ and note that $\mathbf{d} \cdot \theta = 0$. Any \mathbf{d} -dimensional Q -representation M can be written in the form $M = (M_1, \dots, M_k)$, where $M_i \in \text{Rep}_{(1,a_i)}(\tilde{L}_3)$. We want to show that if such an M is θ -stable, then each M_i is θ_i -stable. So let us assume by contradiction that there exists $0 \neq N_i \subsetneq M_i$ such that $(\underline{\dim} N_i) \cdot (a_i, -1) \geq 0$ for some $1 \leq i \leq k$. Now, write $\underline{\dim} N_i = (d_\infty, d_i)$ where $0 \leq d_\infty \leq 1$ and $0 \leq d_i \leq a_i$, so the inequality we are assuming is $d_\infty a_i - d_i \geq 0$, from which it follows that $d_\infty = 1$ (otherwise $N_i = 0$), and hence $d_i < a_i$ (otherwise $N_i = M_i$).

Consider now the proper nonzero subrepresentation $N = (M_1, \dots, M_{i-1}, N_i, M_{i+1}, \dots, M_k) \subset M$, whose dimension vector is $(1, a_1, \dots, a_{i-1}, d_i, a_{i+1}, \dots, a_k)$. We have

$$\begin{aligned} (\underline{\dim} N) \cdot \theta &= (1, a_1, \dots, a_{i-1}, d_i, a_{i+1}, \dots, a_k) \cdot (n, -1, \dots, -1) \\ &= n - \left(\sum_{j \neq i} a_j \right) - d_i \\ &= \sum_j a_j - \left(\sum_{j \neq i} a_j \right) - d_i \\ &= a_i - d_i > 0, \end{aligned}$$

contradicting stability of M . The result follows. \square

It is clear from the definitions and Lemma 4.1 that if a representation

$$\left(A_i, B_i, C_i, v_1^{(i)}, \dots, v_r^{(i)} \right)_{1 \leq i \leq k} \in Y_{r,\mathbf{a}}$$

is θ -stable then the n -dimensional (unframed) representation

$$(A_1 \oplus \dots \oplus A_k, B_1 \oplus \dots \oplus B_k, C_1 \oplus \dots \oplus C_k) \in \text{Rep}_n(L_3)$$

is spanned by the vectors

$$v_j = \begin{pmatrix} v_j^{(1)} \\ v_j^{(2)} \\ \vdots \\ v_j^{(k)} \end{pmatrix} \in \mathbb{C}^n, \quad 1 \leq j \leq r.$$

There is a closed embedding $\Phi_{r,\mathbf{a}}: Y_{r,\mathbf{a}} \hookrightarrow Y_{r,n}$, which on Y_n restricts to the embedding $\Phi_{\mathbf{a}}: Y_{\mathbf{a}} \hookrightarrow Y_n$ (considered in Section 3.1) and concatenates the elements of $\prod_{i=1}^k Z_{r,a_i}$ to produce an element of $Z_{r,n}$.

The reductive algebraic group $\text{GL}_{\mathbf{a}} = \prod_{i=1}^k \text{GL}_{a_i}$ acts on $Y_{r,\mathbf{a}}$ by componentwise conjugation on the summand $Y_{\mathbf{a}}$ and $\Phi_{r,\mathbf{a}}$ is equivariant with respect to the inclusion $\text{GL}_{\mathbf{a}} \subset \text{GL}_n$ by block diagonal matrices of the same kind.

On the space $Y_{r,\mathbf{a}}$ we have the $\text{GL}_{\mathbf{a}}$ -invariant potential

$$(4.2) \quad g_{r,\mathbf{a}} = f_{r,a_1} \oplus \dots \oplus f_{r,a_k}: Y_{r,\mathbf{a}} \rightarrow \mathbb{A}^1, \quad (A_i, B_i, C_i, Z_i)_i \mapsto \sum_{i=1}^k \text{Tr } A_i[B_i, C_i],$$

where $(A_i, B_i, C_i) \in Y_{a_i}$ and $Z_i = (v_1^{(i)}, \dots, v_r^{(i)}) \in Z_{r,a_i} = \text{Hom}_{\mathbb{C}}(\mathbb{C}, \mathbb{C}^{a_i})^r$ for $i = 1, \dots, k$. We have the obvious relation $f_{r,n} \circ \Phi_{r,\mathbf{a}} = g_{r,\mathbf{a}}$, and we observed above that the θ -stable locus embeds in the θ -stable locus, which yields a commutative diagram

$$\begin{array}{ccc} Y_{r,\mathbf{a}}^{\theta\text{-st}} & \hookrightarrow & Y_{r,n}^{\theta\text{-st}} \\ \downarrow & & \downarrow \\ Y_{r,\mathbf{a}} & \xrightarrow{\Phi_{r,\mathbf{a}}} & Y_{r,n} \xrightarrow{f_{r,n}} \mathbb{A}^1 \\ & \searrow g_{r,\mathbf{a}} & \nearrow \end{array}$$

where the vertical arrows are open immersions and $\theta = (n, -1)$.

If we set

$$U_{r,n} = \text{crit}(f_{r,n}) \subset Y_{r,n}, \quad U_{r,\mathbf{a}} = \text{crit}(g_{r,\mathbf{a}}) \subset Y_{r,\mathbf{a}},$$

the restriction $\Phi_{r,\mathbf{a}}: U_{r,\mathbf{a}} \hookrightarrow U_{r,n}$ is still equivariant with respect to the inclusion $\text{GL}_{\mathbf{a}} \subset \text{GL}_n$, and so it induces a morphism of schemes

$$(4.3) \quad \psi_{r,\mathbf{a}}: U_{r,\mathbf{a}} //_{\theta} \text{GL}_{\mathbf{a}} \rightarrow U_{r,n} //_{\theta} \text{GL}_n,$$

which fits in a diagram

$$\begin{array}{ccc} U_{r,\mathbf{a}} //_{\theta} \text{GL}_{\mathbf{a}} & \xrightarrow{\text{open}} & \prod_{i=1}^k U_{r,a_i} //_{\theta_i} \text{GL}_{a_i} \xrightarrow{\sim} \prod_{i=1}^k Q_{r,a_i} \\ \psi_{r,\mathbf{a}} \downarrow & & \swarrow \text{---} \\ U_{r,n} //_{\theta} \text{GL}_n & \xrightarrow{\sim} & Q_{r,n} \end{array}$$

Let $Q_{r,\mathbf{a}}^{\circ} \hookrightarrow Q_{r,\mathbf{a}} = \prod_{i=1}^k Q_{r,a_i}$ be the open subscheme of k -tuples of quotients

$$K_{\mathbf{a}} = ([\mathcal{O}^{\oplus r} \twoheadrightarrow E_1], \dots, [\mathcal{O}^{\oplus r} \twoheadrightarrow E_k]) \in Q_{r,\mathbf{a}}$$

such that $\text{Supp}(E_i) \cap \text{Supp}(E_j) = \emptyset$ for all $i \neq j$. We can use the above diagram to identify the map $\psi_{r,a}$ with the ‘union of points’ map (denoted the same way)

$$\psi_{r,a}: \mathcal{Q}_{r,a}^\circ \rightarrow \mathcal{Q}_{r,n},$$

which takes a point $K_a \in \mathcal{Q}_{r,a}^\circ$ as above to the joint surjection $[\mathcal{O}^{\oplus r} \rightarrow E_1 \oplus \cdots \oplus E_k] \in \mathcal{Q}_{r,n}$. This morphism is étale by [1, Proposition A.3].

Let $K = [\mathcal{O}^{\oplus r} \rightarrow E] \in \mathcal{Q}_{r,n}$ be in the image of the map $\psi_{r,a}$, i.e. assume we can write $K = \psi_{r,a}(K_a)$. The local structure of the d-critical locus $(\text{crit}(f_{r,n}), s_{r,n}^{\text{crit}})$ around $\iota_{r,n}^{-1}(K)$ is then described as follows.

Lemma 4.2. *If $\phi_{r,a}$ denotes the composition*

$$\phi_{r,a}: U_{r,a}^{\theta\text{-st}} \longrightarrow U_{r,a} //_{\theta} \text{GL}_a \xrightarrow{\psi_{r,a}} U_{r,n} //_{\theta} \text{GL}_n,$$

then we have an identity of d-critical structures

$$\phi_{r,a}^* s_{r,n}^{\text{crit}} = g_{r,a} + (\text{d}g_{r,a})^2 \in H^0(\mathcal{S}_{U_{r,a}^{\theta\text{-st}}}^0)^{\text{GL}_a}.$$

Proof. The commutative diagram

$$\begin{array}{ccc} U_{r,a} //_{\theta} \text{GL}_a & \hookrightarrow & Y_{r,a} //_{\theta} \text{GL}_a \\ \psi_{r,a} \downarrow & & \Phi_{r,a} \downarrow \searrow^{g_{r,a}} \\ U_{r,n} //_{\theta} \text{GL}_n & \hookrightarrow & Y_{r,n} //_{\theta} \text{GL}_n \xrightarrow{f_{r,n}} \mathbb{A}^1 \end{array}$$

and the fact that $\psi_{r,a}$ is étale immediately imply the claim, using the defining properties of the sheaf $\mathcal{S}_{U_{r,a}^{\theta\text{-st}}}^0$. \square

4.2. The d-critical structure $s_{r,n}^{\text{der}}$ coming from derived symplectic geometry. We start right away with the following definition.

Definition 4.3. The *derived d-critical structure* on $\mathcal{Q}_{r,n}$ is defined as

$$s_{r,n}^{\text{der}} = q_{r,n}^* s_n^{\text{der}} \in H^0(\mathcal{S}_{\mathcal{Q}_{r,n}}^0).$$

Remark 4.4. We observed in Theorem 2.6 how Theorem 3.11, combined with Proposition 2.5, proves the relation

$$\iota_{r,n}^* s_{r,n}^{\text{der}} = s_{r,n}^{\text{crit}} \in H^0(\mathcal{S}_{\text{crit}(f_{r,n})}^0)$$

that is the content of Theorem A.

In the rest of this subsection, we motivate and justify the above definition by explaining how the d-critical structure $s_{r,n}^{\text{der}}$ arises naturally using the derived Quot scheme in Diagram (4.1).

We proceed to describe the derived Quot scheme $\mathbf{Q}_{r,n}$.

Recall that as in the case of classical Quot schemes [25], the derived Quot scheme $\mathbf{Q}_{r,n}$ is defined as follows: Let $\mathbb{A}^3 \subset \mathbb{P}$ be any compactification of \mathbb{A}^3 , for example $\mathbb{P} = \mathbb{P}^3$. Then $\mathbf{Q}_{r,n}$ is the open derived subscheme of the derived Quot scheme $\mathbf{Q}_{\mathbb{P},r,n}$ [14] parametrising quotients $[\mathcal{O}_{\mathbb{P}}^{\oplus r} \rightarrow E]$ where E is a 0-dimensional length n sheaf on \mathbb{P} whose support is contained in \mathbb{A}^3 .

By direct computation, we have that $\mathbf{Q}_{r,n}$ is a quasi-affine dg-scheme (or more generally derived scheme) $[\mathbf{Spec} R/\text{GL}_n]$ where R is a sheaf of commutative dg-algebras generated in

degrees $0, -1, -2$ respectively by

$$\mathcal{O}_{Y_r, n}^{\theta\text{-st}}, \quad \text{Rep}_n(L_3)^{\oplus 3} \simeq \mathfrak{gl}_n^{\oplus 3}, \quad \text{Rep}_n(L_3) \simeq \mathfrak{gl}_n$$

and the differentials are exactly the same as for the algebra $(Q_3)_n$. Notice that we are slightly abusing the notation **Spec** R here.

It is now clear that there is an inclusion $(Q_3)_n \subset R$ of dg-algebras, where $(Q_3)_n$ is as in Equation (3.13). This induces precisely the forgetful map

$$\mathbf{q}_{r,n}: \mathbf{Q}_{r,n} \longrightarrow \mathcal{M}_n.$$

Since the map of dg-algebras is evidently smooth, it follows that $\mathbf{q}_{r,n}$ is smooth.

Given these descriptions of $\mathbf{Q}_{r,n}$, \mathcal{M}_n and the simple ‘‘product’’ structure of the map $\mathbf{q}_{r,n}$, we see that while the derived Quot scheme is not -1 -shifted symplectic, any Darboux chart of \mathcal{M}_n will give rise to a smooth chart of $\mathbf{Q}_{r,n}$ which is a smooth d-critical chart for the classical truncation $Q_{r,n}$. These d-critical charts induce precisely the d-critical structure $s_{r,n}^{\text{der}}$ of Definition 4.3. The (closed, degenerate) 2-form $\mathbf{q}_{r,n}^* \omega_n$ is compatible with this d-critical structure in the appropriate sense.

5. THE CASE OF A COMPACT CALABI–YAU 3-FOLD

Let F be a locally free sheaf of rank r on a smooth, projective Calabi–Yau 3-fold X , where we have fixed a trivialisation $\zeta: \Lambda^3 \Omega_X \xrightarrow{\sim} \mathcal{O}_X$. Form the Quot scheme

$$Q_{F,n} = \text{Quot}_X(F, n).$$

In this section we prove Theorem B, showing that the derived d-critical structure on $Q_{F,n}$, defined in (5.1) below, is locally modelled on the derived critical structure $s_{r,n}^{\text{der}}$ of Definition 4.3.

We will need the following algebraic result.

Lemma 5.1. *Let $x \in X$ be a point on a smooth projective Calabi–Yau 3-fold X , and let $0 \in \mathbb{A}^3$ be the origin. There is an equivalence of dg-algebras*

$$\text{Ext}^*(\mathcal{O}_x, \mathcal{O}_x) \cong \text{Ext}^*(\mathcal{O}_0, \mathcal{O}_0).$$

Moreover the higher Massey products m_n vanish for $n \geq 3$.

Proof. Let X be any smooth 3-fold, $x \in X$ a point. The dg-algebra $\text{Ext}^*(\mathcal{O}_x, \mathcal{O}_x)$ can be computed in either the algebraic or analytic category, the result and its A_∞ -structure being the same, capturing the deformation theory of a point inside a smooth 3-fold. The first statement then follows after identifying suitable analytic neighbourhoods of $x \in X$ and $0 \in \mathbb{A}^3$. The vanishing of the Massey products $m_n: \text{Ext}^1(\mathcal{O}_x, \mathcal{O}_x)^{\otimes n} \rightarrow \text{Ext}^2(\mathcal{O}_x, \mathcal{O}_x)$ for $n \geq 3$ is a consequence of Lemma 3.6. \square

Remark 5.2. By the results of [37], it is actually the case that there is an equivalence $\text{Ext}^*(\mathcal{O}_x, \mathcal{O}_x) \cong \text{Ext}^*(\mathcal{O}_0, \mathcal{O}_0)$ as cyclic dg-algebras.

We have a commutative diagram

$$\begin{array}{ccc} Q_{F,n} & \xrightarrow{q} & \mathcal{M}_X(n) \\ & \searrow h & \downarrow p \\ & & \text{Sym}^n X \end{array}$$

where $\mathcal{M}_X(n)$ is the moduli stack of 0-dimensional sheaves of length n over X , the morphism p is the map to the coarse moduli space (the n -th symmetric product $\text{Sym}^n X = X^n/\mathfrak{S}_n$) and $q = q_{E,n}$ is the (smooth) forgetful morphism sending a surjection $[F \rightarrow E]$ to the point $[E]$. The composition $h = p \circ q$ agrees with the Quot-to-Chow map [18, Section 6]. The trivialisation $\zeta: \Lambda^3 \Omega_X \xrightarrow{\sim} \mathcal{O}_X$ induces a canonical -1 -shifted symplectic structure $\omega_{X,n}$ on $\mathcal{M}_X(n)$, whose truncation $s_{X,n} = \tau(\omega_{X,n}) \in H^0(\mathcal{S}_{\mathcal{M}_X(n)}^0)$ induces a d-critical structure

$$(5.1) \quad s_{F,n} = q^* s_{X,n}$$

on the Quot scheme $Q_{F,n}$.

Fix a 0-cycle $[E] \in \text{Sym}^n X$ represented, as ever, by a polystable sheaf

$$(5.2) \quad E = \bigoplus_{i=1}^k \mathbb{C}^{a_i} \otimes \mathcal{O}_{x_i}, \quad x_i \in X, \quad x_i \neq x_j \text{ for } i \neq j.$$

In this particular case, the *Ext quiver* Q_{E_\bullet} associated to E (see Toda's paper [33, Section 3.3] for a more general definition) is the quiver with vertex set $V(Q_{E_\bullet}) = \{1, 2, \dots, k\}$ and edge set

$$E(Q_{E_\bullet}) = \coprod_{1 \leq i, j \leq k} E_{i,j},$$

where $E_{i,j} \subset \text{Ext}^1(\mathcal{O}_{x_i}, \mathcal{O}_{x_j})^\vee$ is a \mathbb{C} -linear basis. It follows that $E_{i,j} = \emptyset$ for $i \neq j$, and

$$(5.3) \quad E_{i,i} = \{e_{i,1}, e_{i,2}, e_{i,3}\} \subset \text{Ext}^1(\mathcal{O}_{x_i}, \mathcal{O}_{x_i})^\vee$$

contains 3 elements. The source and target maps s and t from $E(Q_{E_\bullet})$ to the vertex set $V(Q_{E_\bullet})$ both send $E_{i,i}$ to the vertex i . In other words, the quiver Q_{E_\bullet} is a disjoint union of k copies of the 3-loop quiver (see Figure 4).

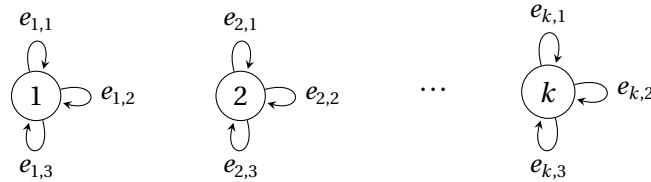


FIGURE 4. The Ext quiver of a 0-dimensional polystable sheaf (5.2).

There is an associated convergent superpotential (see [33, Sections 2.2, 2.6] for more details)

$$W_{E_\bullet} \in \mathbb{C}\{Q_{E_\bullet}\} \subset \mathbb{C}[[Q_{E_\bullet}]],$$

whose general definition reads as follows. First, for each $\mathcal{O}_{x_i} \in \text{Coh}(X)$, consider the Massey products on the dg-algebra $\text{Ext}^*(\mathcal{O}_{x_i}, \mathcal{O}_{x_i})$, defined by the maps

$$m_n: \text{Ext}^1(\mathcal{O}_{x_i}, \mathcal{O}_{x_i})^{\otimes n} \rightarrow \text{Ext}^2(\mathcal{O}_{x_i}, \mathcal{O}_{x_i}).$$

Denote by

$$(-, -)_{x_i}: \text{Ext}^2(\mathcal{O}_{x_i}, \mathcal{O}_{x_i}) \times \text{Ext}^1(\mathcal{O}_{x_i}, \mathcal{O}_{x_i}) \rightarrow \text{Ext}^3(\mathcal{O}_{x_i}, \mathcal{O}_{x_i}) \xrightarrow{\text{tr}} \mathbb{C}$$

the Serre duality pairing. Then, by the Calabi–Yau condition, for any given elements $a_1, \dots, a_n \in \text{Ext}^1(\mathcal{O}_{x_i}, \mathcal{O}_{x_i})$, one has the cyclicity relation

$$(m_{n-1}(a_1, \dots, a_{n-1}), a_n)_{x_i} = (m_{n-1}(a_2, \dots, a_n), a_1)_{x_i}.$$

Let $E_{i,i}^\vee = \{e_{i,1}^\vee, e_{i,2}^\vee, e_{i,3}^\vee\} \subset \text{Ext}^1(\mathcal{O}_{x_i}, \mathcal{O}_{x_i})$ be the dual basis of (5.3). Then, Toda defines in [33, Section 5.5] the superpotential

$$W_E = \sum_{n \geq 3} \sum_{\psi} \sum_{e_i \in E_{\psi(i), \psi(i+1)}} a_{\psi, e_\bullet} \cdot e_1 \cdots e_n,$$

where ψ runs over the set of maps $\{1, 2, \dots, n+1\} \rightarrow \{1, \dots, k\}$ such that $\psi(1) = \psi(n+1)$, and the coefficients $a_{\psi, e_\bullet} \in \mathbb{C}$ are defined by

$$(5.4) \quad a_{\psi, e_\bullet} = \frac{1}{n} (m_{n-1}(e_1^\vee, \dots, e_{n-1}^\vee), e_n^\vee).$$

We now determine the (trace of the) superpotential W_E explicitly.

Lemma 5.3. *Given a polystable sheaf E as in (5.2), one has, up to a scalar,*

$$\text{Tr } W_E = \sum_{i=1}^k \text{Tr } A_i[B_i, C_i],$$

where we have set $A_i = e_{i,1}$, $B_i = e_{i,2}$ and $C_i = e_{i,3}$. In particular, $\text{Tr } W_E$ defines a regular (everywhere convergent) function on $\text{Rep}_{\mathbf{a}}(Q_E)$, where $\mathbf{a} = (a_1, \dots, a_k)$ is determined by (5.2).

Proof. By Lemma 5.1, $m_n = 0$ for $n \geq 3$, so by (5.4) only $n = 3$ contributes to the sum. This already proves the last statement, about convergence. Since the only nonvanishing Ext groups $\text{Ext}^1(\mathcal{O}_{x_i}, \mathcal{O}_{x_j})$ are those where $i = j$, every ψ in the sum satisfies $\psi(i) = \psi(i+1)$, hence the sum over ψ is actually a sum over integers from 1 up to k . Now, we have relations

$$\begin{aligned} j_i &= (m_2(A_i^\vee, B_i^\vee), C_i^\vee)_{x_i} = (m_2(B_i^\vee, C_i^\vee), A_i^\vee)_{x_i} = (m_2(C_i^\vee, A_i^\vee), B_i^\vee)_{x_i} \\ l_i &= (m_2(A_i^\vee, C_i^\vee), B_i^\vee)_{x_i} = (m_2(C_i^\vee, B_i^\vee), A_i^\vee)_{x_i} = (m_2(B_i^\vee, A_i^\vee), C_i^\vee)_{x_i}. \end{aligned}$$

Thus

$$W_E = \sum_{i=1}^k \frac{j_i}{3} (A_i B_i C_i + B_i C_i A_i + C_i A_i B_i) + \frac{l_i}{3} (A_i C_i B_i + B_i A_i C_i + C_i B_i A_i).$$

But since m_2 agrees with the Yoneda pairing, we have

$$j_i + l_i = (m_2(A_i^\vee, B_i^\vee), C_i^\vee)_{x_i} + (m_2(B_i^\vee, A_i^\vee), C_i^\vee)_{x_i} = (m_2(A_i^\vee, B_i^\vee) + m_2(B_i^\vee, A_i^\vee), C_i^\vee)_{x_i} = 0.$$

Since j_i and l_i do not depend on i , we can set $j = j_i$ and $l = l_i$, so that $l = -j$, thus

$$W_E = \frac{j}{3} \sum_{i=1}^k A_i[B_i, C_i] + B_i[C_i, A_i] + C_i[A_i, B_i].$$

It follows that $\text{Tr } W_E = j \cdot \sum_{1 \leq i \leq k} \text{Tr } A_i[B_i, C_i]$, as required. \square

Let $\mathfrak{M}_{\mathbf{a}}(Q_E)$ denote the moduli stack of \mathbf{a} -dimensional representations of the Ext quiver Q_E . In the diagram

$$\begin{array}{ccc} \mathbb{A}^1 & \xleftarrow{\text{Tr } W_E} & \text{Rep}_{\mathbf{a}}(Q_E) \longrightarrow \mathfrak{M}_{\mathbf{a}}(Q_E) \equiv \prod_{1 \leq i \leq k} \mathfrak{M}_{a_i}(L_3) \\ & & \searrow \pi \quad \downarrow \\ & & M_{\mathbf{a}}(Q_E) \equiv \prod_{1 \leq i \leq k} \text{Sym}^{a_i} \mathbb{A}^3. \end{array}$$

we thus have a canonical identification $(\text{Rep}_{\mathbf{a}}(Q_E), \text{Tr } W_E) = (Y_{\mathbf{a}}, \mathbf{g}_{\mathbf{a}})$, where $\mathbf{g}_{\mathbf{a}}$ was defined in Equation (3.2) and it equals $\text{Tr } W_E$ by Lemma 5.3.

By [32, Theorem 5.3], we can find open *analytic* neighbourhoods

$$0 \in V \subset M_{\mathbf{a}}(Q_E), \quad [E] \in T \subset \text{Sym}^n X$$

and an analytic isomorphism i fitting in a diagram

$$\mathcal{Z}_{\mathbf{a}} = [\text{crit}(g_{\mathbf{a}}|_V)/\text{GL}_{\mathbf{a}}] \xrightarrow{\sim} p^{-1}(T) \xrightarrow{\text{open}} \mathcal{M}_X(n),$$

Lemma 5.4. *Up to potential shrinking of $\pi^{-1}(V)$ around $\pi^{-1}(0)$, we have*

$$(5.5) \quad s_{X,n}|_{\mathcal{Z}_{\mathbf{a}}} = g_{\mathbf{a}}|_V + (\text{d}g_{\mathbf{a}}|_V)^2 \in H^0(\mathcal{S}_{\mathcal{Z}_{\mathbf{a}}}^0),$$

where we are abusing notation and writing $g_{\mathbf{a}}|_V$ for the restriction of $g_{\mathbf{a}}$ to $\pi^{-1}(V) \subset \text{Rep}_{\mathbf{a}}(Q_{E_{\bullet}})$.

Proof. The arguments in the proof of Theorem 2.2 and Proposition 3.12 go through verbatim, as long as we can verify that the statement of Lemma 3.9 applies to $\mathcal{M}_X(n)$. But this is indeed the case, since Lemma 3.9 and its proof only depend on the minimal model of the dg-algebra \mathfrak{g}_E , where E was defined in (5.2). But, by Lemma 5.1, we may assume that the points $x_i \in X$ are distinct points of \mathbb{A}^3 , so we get a formal d-critical chart of the required form at $[E]$. \square

Now form the open subscheme

$$Q_{F,\mathbf{a}} = \mathcal{Z}_{\mathbf{a}} \times_{\mathcal{M}_X(n)} Q_{F,n} \hookrightarrow Q_{F,n},$$

and consider the cartesian diagram

$$(5.6) \quad \begin{array}{ccccc} & & Q_{F,\mathbf{a}} & \hookrightarrow & Q_{F,n} \\ & & \downarrow q & \square & \downarrow q \\ \mathcal{Z}'_{\mathbf{a}} & \longrightarrow & \mathcal{Z}_{\mathbf{a}} & \longrightarrow & \mathcal{M}_X(n) \\ \text{étale} \downarrow & & \downarrow \psi_{\mathbf{a}} & & \\ Q_{r,n} & \xrightarrow{q_{r,n}} & \mathcal{M}_n & & \end{array}$$

defining the scheme $\mathcal{Z}'_{\mathbf{a}}$. Note that, possibly after shrinking V , thanks to Lemma 3.2 and Lemma 3.3 we may assume $\psi_{\mathbf{a}}: \mathcal{Z}_{\mathbf{a}} \rightarrow \mathcal{M}_n$ is étale and satisfies

$$(5.7) \quad \psi_{\mathbf{a}}^* s_n^{\text{der}} = g_{\mathbf{a}}|_V + (\text{d}g_{\mathbf{a}}|_V)^2.$$

Next, we show that the morphisms

$$(5.8) \quad Q_{F,\mathbf{a}} \rightarrow \mathcal{Z}_{\mathbf{a}}, \quad \mathcal{Z}'_{\mathbf{a}} \rightarrow \mathcal{Z}_{\mathbf{a}}$$

look the same étale locally. We work locally analytically around a point $[E] \in \mathcal{Z}_{\mathbf{a}} \subset \mathcal{M}_X(n)$. We let $[E'] \in \mathcal{M}_n$ be the image of $[E]$ under the étale map $\psi_{\mathbf{a}}$. Let

$$B \subset \mathbb{C}^{rn}$$

be the analytic open subset corresponding to surjective maps in

$$\text{Hom}_X(F, E) = \text{Hom}_{\mathbb{A}^3}(\mathcal{O}^{\oplus r}, E') = \mathbb{C}^{rn}.$$

We identify B with the fibre of q (resp. $q_{r,n}$) over the point $[E]$ (resp. $[E']$). Since the morphisms (5.8) are smooth, they are both analytically locally trivial (on the source), so any point in $q^{-1}([E]) \subset Q_{F,\mathbf{a}}$ admits an analytic open neighbourhood of the form $B \times W_{\mathbf{a}}$, where $W_{\mathbf{a}} \subset \mathcal{Z}_{\mathbf{a}}$ is a suitable analytic open neighbourhood of $[E]$. Repeating the same reasoning with the map

$\mathcal{Z}'_a \rightarrow \mathcal{Z}_a$ and shrinking W_a further if necessary, we see that in Diagram (5.6) we can make the replacement

$$\begin{array}{ccc} & \mathcal{Q}_{F,a} & \\ & \downarrow q & \rightsquigarrow \\ \mathcal{Z}'_a & \longrightarrow & \mathcal{Z}_a \end{array} \quad \begin{array}{ccc} & B \times W_a & \\ & \downarrow \text{pr}_2 & \\ B \times W_a & \xrightarrow{\text{pr}_2} & W_a \end{array}$$

where $B \times W_a$ has an étale map (the same ψ_a as above) down to $\mathcal{Q}_{r,n}$. Now we compute

$$\begin{aligned} s_{F,n}|_{B \times W_a} &= \text{pr}_2^*(s_{X,n}|_{W_a}) \\ &= \text{pr}_2^*(g_a|_V + (dg_a|_V)^2) && \text{by (5.5)} \\ &= \text{pr}_2^*\psi_a^*s_n^{\text{der}} && \text{by (5.7)} \\ &= s_n^{\text{der}}|_{B \times W_a}. \end{aligned}$$

This completes the proof of Theorem B.

6. THE SPECIAL CASE OF THE HILBERT SCHEME

This section is devoted to the proof of Theorem C.

Let $H = \text{Hilb}^n \mathbb{A}^3$ be the Hilbert scheme parametrising 0-dimensional subschemes of \mathbb{A}^3 of length n . Denote by

$$0 \rightarrow \mathcal{I}_{\mathcal{Z}} \rightarrow \mathcal{O}_{\mathbb{A}^3 \times H} \rightarrow \mathcal{O}_{\mathcal{Z}} \rightarrow 0$$

the universal short exact sequence living over $\mathbb{A}^3 \times H$, where $\mathcal{Z} \subset \mathbb{A}^3 \times H$ denotes the universal subscheme. Let $\pi: \mathbb{A}^3 \times H \rightarrow H$ be the projection; we use the notation $\mathbf{R}\mathcal{H}om_{\pi}(-, -) = \mathbf{R}\pi_*\mathbf{R}\mathcal{H}om(-, -)$ throughout.

Consider the derived critical locus $\mathbf{Q} = \mathbf{R}\text{crit}(f)$ where $f = f_{1,n}: \text{ncQuot}_1^n \rightarrow \mathbb{A}^1$ is the potential in Theorem 2.3. More precisely, $\mathbf{Q} = \mathbf{Spec} B$ where B is the sheaf of dg-algebras which is generated by $B^0 = \mathcal{O}_{\text{ncQuot}_1^n}$ in degree 0 and $B^{-1} = T_{\text{ncQuot}_1^n}$ in degree -1 with differential given by the dual of the section $df \in H^0(\Omega_{\text{ncQuot}_1^n})$. Recall that $\text{ncQuot}_1^n = Y_{1,n}^{\theta\text{-st}}/\text{GL}_n$.

Then we have the following commutative diagram

$$(6.1) \quad \begin{array}{ccc} & \eta & \\ & \curvearrowright & \\ H & \xrightarrow[\sim]{\epsilon} \text{crit}(f) \xrightarrow{j} & \mathbf{Q} \\ \downarrow q & & \downarrow q \\ \mathcal{M}_n & \xrightarrow{j_n} & \mathcal{M}_n \end{array}$$

where we have set $\epsilon = \iota_{1,n}^{-1}$, and $q = q_{1,n}$ is the forgetful map, whereas j and j_n are the inclusions of the classical spaces into their derived enhancements.

The morphism q is obtained as follows: Recall that by (3.13) \mathcal{M}_n is the quotient stack $[\mathbf{Spec}(Q_3)_n/\text{GL}_n]$. For brevity, let us write $A = (Q_3)_n$ and $A^0 = (Q_3)_n^0 = \mathcal{O}_{Y_n}$, $A^{-1} = (Q_3)_n^{-1} = T_{Y_n}$ be the degree 0 and degree -1 summands of the dg-algebra $(Q_3)_n$ respectively. There are natural morphisms $A^0 \rightarrow B^0$ and $A^{-1} \rightarrow B^{-1}$ which are GL_n -invariant and, together with the trivial map $A^{-2} \rightarrow 0$, induce a morphism $\mathbf{Spec} B \rightarrow \mathbf{Spec} A \rightarrow [\mathbf{Spec} A/\text{GL}_n]$, which is the morphism q .

Note that, by definition,

$$\mathbb{E}_{\text{crit}} = j^*\mathbb{L}_{\mathbf{Q}} \xrightarrow{\varphi_{\text{crit}}} \mathbb{L}_{\text{crit}(f)}$$

is the critical obstruction theory on $\text{crit}(f)$, denoted \mathbb{E}_f in the introduction, see (0.7). The maps η and q induce a commutative diagram

$$\begin{array}{ccc} & & \eta^* q^* \mathbb{L}\mathcal{M}_n \\ & \swarrow \psi & \downarrow \\ \epsilon^* \mathbb{E}_{\text{crit}} & \xlongequal{\quad} & \eta^* \mathbb{L}\mathbf{Q} \\ & \searrow \epsilon^* \varphi_{\text{crit}} & \downarrow \\ & & \mathbb{L}_H \end{array}$$

which after applying the truncation functor $\tau_{[-1,0]}$ becomes

$$(6.2) \quad \begin{array}{ccc} & & \eta^* \tau_{[-1,0]} q^* \mathbb{L}\mathcal{M}_n \\ & \swarrow \bar{\psi} & \downarrow \varphi \\ \epsilon^* \mathbb{E}_{\text{crit}} & & \mathbb{L}_H \\ & \searrow \epsilon^* \varphi_{\text{crit}} & \end{array}$$

where \mathbb{E}_{crit} is unchanged since it already has cohomological amplitude in degrees $[-1, 0]$. The morphism $\bar{\psi}$ is an isomorphism because it is the pullback along η of the second of the isomorphisms (6.3) established in the following lemma.

Lemma 6.1. *The derivative of q induces isomorphisms*

$$(6.3) \quad \begin{array}{ccc} \mathbb{T}_{\mathbf{Q}} & \xrightarrow{\sim} & \tau_{[0,1]} q^* \mathbb{T}\mathcal{M}_n \\ \tau_{[-1,0]} q^* \mathbb{L}\mathcal{M}_n & \xrightarrow{\sim} & \mathbb{L}\mathbf{Q}. \end{array}$$

Moreover, one has an isomorphism

$$(6.4) \quad \gamma: \tau_{[0,1]} \mathbf{R}\mathcal{H}om_{\pi}(\mathcal{O}_{\mathbb{Z}}[-1], \mathcal{O}_{\mathbb{Z}}) \xrightarrow{\sim} \tau_{[0,1]} \eta^* q^* \mathbb{T}\mathcal{M}_n.$$

Proof. The derived tangent complex of \mathbf{Q} is given by

$$\mathbb{T}_{\mathbf{Q}} = [T_{\text{ncQuot}_1^n} \xrightarrow{\text{Hess}(f)} \Omega_{\text{ncQuot}_1^n}]$$

where $\text{Hess}(f)$ denotes the Hessian of f .

Since $\text{ncQuot}_1^n = Y_{1,n}^{\theta\text{-st}} / \text{GL}_n$ and the action of GL_n on $Y_{1,n}^{\theta\text{-st}}$ is free, we may write this as the quasi-isomorphic GL_n -equivariant three-term complex on $Y_{1,n}^{\theta\text{-st}}$ in degrees -1 to 1 (where we are slightly abusing notation by omitting certain pullbacks)

$$\mathbb{T}_{\mathbf{Q}} = [\mathfrak{gl}_n \longrightarrow T_{Y_{1,n}^{\theta\text{-st}}} \xrightarrow{\text{Hess}(f)} \Omega_{\text{ncQuot}_1^n].$$

By the definition of q , we have (again slightly abusing notation)

$$q^* \mathbb{T}\mathcal{M}_n = [\mathfrak{gl}_n \longrightarrow T_{Y_n} \longrightarrow \Omega_{Y_n} \longrightarrow \mathfrak{gl}_n]$$

and the derivative $d\mathbf{q}$ is the morphism of complexes

$$\begin{array}{ccccc} \mathfrak{gl}_n & \longrightarrow & T_{Y_{1,n}^{\theta\text{-st}}} & \xrightarrow{\text{Hess}(f)} & \Omega_{\text{ncQuot}_1^n} \\ \parallel & & \downarrow & & \downarrow \\ \mathfrak{gl}_n & \longrightarrow & T_{Y_n} & \longrightarrow & \Omega_{Y_n} \longrightarrow \mathfrak{gl}_n \end{array}$$

We may identify $T_{Y_{1,n}^{\theta\text{-st}}}$ with $(\mathfrak{gl}_n^{\oplus 3} \oplus \mathbb{C}^n) \otimes \mathcal{O}_B$ and T_{Y_n} with $\mathfrak{gl}_n^{\oplus 3} \otimes \mathcal{O}_B$ so that the middle vertical arrow is the natural projection map.

It is clear that $H^0(d\mathbf{q})$ is surjective.

To show that it is injective, notice that the leftmost arrow $\mathfrak{gl}_n \rightarrow T_{Y_{1,n}^{\theta\text{-st}}}$ maps $X \in \mathfrak{gl}_n$ to

$$([X, A], [X, B], [X, C], Xv) \in \mathfrak{gl}_n^{\oplus 3} \oplus \mathbb{C}^n$$

at the point $(A, B, C, v) \in Y_{1,n}^{\theta\text{-st}}$. Since, by stability, v is a cyclic vector with respect to the action of the matrices A, B, C , for any $w \in \mathbb{C}^n$ there exists a polynomial $f(A, B, C)$ such that $w = f(A, B, C)v$. Letting $X = f(A, B, C)$, the image of X is then $(0, 0, 0, w) \in \mathfrak{gl}_n^{\oplus 3} \oplus \mathbb{C}^n$. Therefore, the composition $\mathfrak{gl}_n \rightarrow T_{Y_{1,n}^{\theta\text{-st}}} = \mathfrak{gl}_n^{\oplus 3} \oplus \mathbb{C}^n \rightarrow \mathbb{C}^n$ is fibrewise surjective and hence is a surjective morphism of locally free sheaves and the injectivity of $H^0(d\mathbf{q})$ follows readily.

A similar argument involving duals shows that $H^1(d\mathbf{q})$ is an isomorphism as well.

Thus the truncation $\tau_{[0,1]}d\mathbf{q}$ induces the first isomorphism in (6.3). The second isomorphism is obtained analogously.

For (6.4), since the universal complex of \mathcal{M}_n over the image of \mathbf{q} restricts to $\mathcal{O}_{\mathcal{Z}}$ on \mathcal{M}_n , we have that $\mathbf{R}\mathcal{H}om_{\pi}(\mathcal{O}_{\mathcal{Z}}[-1], \mathcal{O}_{\mathcal{Z}})$ is isomorphic to $q^*j_n^*\mathbb{T}\mathcal{M}_n = \eta^*q^*\mathbb{T}\mathcal{M}_n$, so applying the truncation $\tau_{[0,1]}$ gives the desired isomorphism. \square

We note that by the above explicit description of $\mathbb{T}\mathcal{M}_n$ and the morphism \mathbf{q} we have an isomorphism between the complexes $\eta^*\tau_{[0,1]}q^*\mathbb{T}\mathcal{M}_n$ and $\tau_{[0,1]}\eta^*q^*\mathbb{T}\mathcal{M}_n$, which we use to identify the two complexes from now on.

Our final goal is to produce an isomorphism

$$\rho: \eta^*\tau_{[-1,0]}q^*\mathbb{L}\mathcal{M}_n \xrightarrow{\sim} \mathbb{E}_{\text{der}} = \mathbf{R}\mathcal{H}om_{\pi}(\mathcal{I}_{\mathcal{Z}}, \mathcal{I}_{\mathcal{Z}})_0[2]$$

such that

$$(6.5) \quad \begin{array}{ccc} \eta^*\tau_{[-1,0]}q^*\mathbb{L}\mathcal{M}_n & \xrightarrow{\rho} & \mathbb{E}_{\text{der}} \\ \varphi \downarrow & \swarrow \varphi_{\text{der}} & \\ \mathbb{L}_{\mathcal{H}} & & \end{array}$$

commutes, where φ_{der} is obtained from the Atiyah class of $\mathcal{I}_{\mathcal{Z}}$ via a classical construction (see [20], [27] or [28] for full details).

First of all, we have a diagram

$$\begin{array}{ccccc}
0 & \dashrightarrow & \mathbf{R}\pi_*\mathcal{O}_{\mathbb{A}^3 \times \mathbb{H}}[1] & \xlongequal{\quad} & \mathbf{R}\pi_*\mathcal{O}_{\mathbb{A}^3 \times \mathbb{H}}[1] \\
\downarrow & & \downarrow & & \downarrow \\
\mathbf{R}\mathcal{H}om_\pi(\mathcal{I}_{\mathcal{Z}}, \mathcal{O}_{\mathcal{Z}}) & \longrightarrow & \mathbf{R}\mathcal{H}om_\pi(\mathcal{I}_{\mathcal{Z}}, \mathcal{I}_{\mathcal{Z}})[1] & \longrightarrow & \mathbf{R}\mathcal{H}om_\pi(\mathcal{I}_{\mathcal{Z}}, \mathcal{O}_{\mathbb{A}^3 \times \mathbb{H}})[1] \\
\downarrow \text{id} & & \downarrow & & \downarrow \\
\mathbf{R}\mathcal{H}om_\pi(\mathcal{I}_{\mathcal{Z}}, \mathcal{O}_{\mathcal{Z}}) & \longrightarrow & \mathbf{R}\mathcal{H}om_\pi(\mathcal{I}_{\mathcal{Z}}, \mathcal{I}_{\mathcal{Z}})_0[1] & \longrightarrow & \mathbf{R}\mathcal{H}om_\pi(\mathcal{O}_{\mathcal{Z}}, \mathcal{O}_{\mathbb{A}^3 \times \mathbb{H}})[2]
\end{array}$$

in the derived category of \mathbb{H} , where rows and columns are exact triangles; more precisely:

- (1) the middle column is obtained by applying $\mathbf{R}\pi_*$ to the shifted *dual* of the exact triangle $\mathbf{R}\mathcal{H}om(\mathcal{I}_{\mathcal{Z}}, \mathcal{I}_{\mathcal{Z}})_0 \rightarrow \mathbf{R}\mathcal{H}om(\mathcal{I}_{\mathcal{Z}}, \mathcal{I}_{\mathcal{Z}}) \rightarrow \mathcal{O}_{\mathbb{A}^3 \times \mathbb{H}}$, exploiting the fact that all three objects are self-dual;
- (2) the right column is obtained by applying $\mathbf{R}\mathcal{H}om_\pi(-, \mathcal{O}_{\mathbb{A}^3 \times \mathbb{H}})$ to the exact triangle $\mathcal{O}_{\mathcal{Z}}[-2] \rightarrow \mathcal{I}_{\mathcal{Z}}[-1] \rightarrow \mathcal{O}_{\mathbb{A}^3 \times \mathbb{H}}[-1]$,
- (3) the middle row is obtained by applying $\mathbf{R}\mathcal{H}om_\pi(\mathcal{I}_{\mathcal{Z}}, -)$ to the exact triangle $\mathcal{O}_{\mathcal{Z}} \rightarrow \mathcal{I}_{\mathcal{Z}}[1] \rightarrow \mathcal{O}_{\mathbb{A}^3 \times \mathbb{H}}[1]$.

The last row induces an isomorphism

$$\alpha: \tau_{[0,1]}\mathbf{R}\mathcal{H}om_\pi(\mathcal{I}_{\mathcal{Z}}, \mathcal{O}_{\mathcal{Z}}) \xrightarrow{\sim} \mathbf{R}\mathcal{H}om_\pi(\mathcal{I}_{\mathcal{Z}}, \mathcal{I}_{\mathcal{Z}})_0[1].$$

On the other hand, the exact triangle

$$\mathcal{O}_{\mathbb{A}^3 \times \mathbb{H}}[-1] \longrightarrow \mathcal{O}_{\mathcal{Z}}[-1] \longrightarrow \mathcal{I}_{\mathcal{Z}}$$

induces, via $\mathbf{R}\mathcal{H}om_\pi(-, \mathcal{O}_{\mathcal{Z}})$, an exact triangle

$$\mathbf{R}\mathcal{H}om_\pi(\mathcal{I}_{\mathcal{Z}}, \mathcal{O}_{\mathcal{Z}}) \longrightarrow \mathbf{R}\mathcal{H}om_\pi(\mathcal{O}_{\mathcal{Z}}, \mathcal{O}_{\mathcal{Z}})[1] \longrightarrow \mathbf{R}\mathcal{H}om_\pi(\mathcal{O}_{\mathbb{A}^3 \times \mathbb{H}}, \mathcal{O}_{\mathcal{Z}})[1]$$

such that the truncation of the first arrow

$$\beta: \tau_{[0,1]}\mathbf{R}\mathcal{H}om_\pi(\mathcal{I}_{\mathcal{Z}}, \mathcal{O}_{\mathcal{Z}}) \xrightarrow{\sim} \tau_{[0,1]}\mathbf{R}\mathcal{H}om_\pi(\mathcal{O}_{\mathcal{Z}}, \mathcal{O}_{\mathcal{Z}})[1]$$

is an isomorphism. Summing up, as proved in [12, Proposition 2.2], we have an isomorphism

$$\beta \circ \alpha^{-1}: \mathbf{R}\mathcal{H}om_\pi(\mathcal{I}_{\mathcal{Z}}, \mathcal{I}_{\mathcal{Z}})_0[1] \xrightarrow{\sim} \tau_{[0,1]}\mathbf{R}\mathcal{H}om_\pi(\mathcal{O}_{\mathcal{Z}}[-1], \mathcal{O}_{\mathcal{Z}})$$

along with the identifications

$$\begin{aligned}
\eta^* \tau_{[-1,0]}\mathbf{q}^* \mathbb{L}_{\mathcal{M}_n} &= (\tau_{[0,1]}\eta^* \mathbf{q}^* \mathbb{T}_{\mathcal{M}_n})^\vee \\
&= (\tau_{[0,1]}\mathbf{R}\mathcal{H}om_\pi(\mathcal{O}_{\mathcal{Z}}[-1], \mathcal{O}_{\mathcal{Z}}))^\vee && \text{via } \gamma^\vee \text{ from (6.4)} \\
&= (\mathbf{R}\mathcal{H}om_\pi(\mathcal{I}_{\mathcal{Z}}, \mathcal{I}_{\mathcal{Z}})_0[1])^\vee && \text{via } (\beta \circ \alpha^{-1})^\vee \\
&= \mathbb{E}_{\text{der}} && \text{by Serre duality.}
\end{aligned}$$

We have thus obtained

$$\rho = (\beta \circ \alpha^{-1})^\vee \circ \gamma^\vee: \eta^* \tau_{[-1,0]}\mathbf{q}^* \mathbb{L}_{\mathcal{M}_n} \xrightarrow{\sim} \mathbb{E}_{\text{der}}$$

Thus all that remains to check is the commutativity of Diagram (6.5); that is, we need to check that “up to ρ ” the vertical map

$$\varphi: \eta^* \tau_{[-1,0]}\mathbf{q}^* \mathbb{L}_{\mathcal{M}_n} \longrightarrow \mathbb{L}_{\mathbb{H}}$$

in Diagram (6.2) agrees with the symmetric obstruction theory $\varphi_{\text{der}}: \mathbb{E}_{\text{der}} \rightarrow \mathbb{L}_H$ on the Hilbert scheme, viewed as a moduli space of ideal sheaves.

Recall that the *Atiyah class* of a perfect complex P on a scheme Y is an element

$$\text{At}_P \in \text{Ext}^1(P, P \otimes \mathbb{L}_Y[1]) = \text{Hom}(P[-1], P \otimes \mathbb{L}_Y).$$

Working over $Y = \mathbb{A}^3 \times H$, by projecting along π we will view our Atiyah classes as elements in the group

$$\text{At}_P \in \text{Hom}(P[-1], P \otimes \pi^* \mathbb{L}_H).$$

By the functoriality of Atiyah classes, we have a commutative diagram

$$\begin{array}{ccccc} \mathcal{O}_{\mathbb{A}^3 \times H}[-1] & \longrightarrow & \mathcal{O}_Z[-1] & \longrightarrow & \mathcal{I}_Z \\ \downarrow 0 = \text{At}_\theta & & \downarrow \text{At}_{\theta_Z} & \swarrow u & \downarrow \text{At}_{\mathcal{I}_Z} \\ \mathcal{O}_{\mathbb{A}^3 \times H} \otimes \pi^* \mathbb{L}_H & \longrightarrow & \mathcal{O}_Z \otimes \pi^* \mathbb{L}_H & \longrightarrow & \mathcal{I}_Z[1] \otimes \pi^* \mathbb{L}_H \end{array}$$

where we observed that the Atiyah class of the trivial line bundle vanishes, to give the diagonal arrow

$$u: \mathcal{I}_Z \rightarrow \mathcal{O}_Z \otimes \pi^* \mathbb{L}_H.$$

The commutativity of the two triangles forming the right square translates into a commutative diagram

$$\begin{array}{ccc} \mathbf{R}\mathcal{H}om(\mathcal{I}_Z, \mathcal{I}_Z[1])^\vee & & \\ \downarrow \theta_Z \rightarrow \mathcal{I}_Z[1] & \searrow \text{At}_{\mathcal{I}_Z} & \\ \mathbf{R}\mathcal{H}om(\mathcal{I}_Z, \mathcal{O}_Z)^\vee & \xrightarrow{u} & \pi^* \mathbb{L}_H \\ \uparrow \theta_Z[-1] \rightarrow \mathcal{I}_Z & \nearrow \text{At}_{\theta_Z} & \\ \mathbf{R}\mathcal{H}om(\mathcal{O}_Z[-1], \mathcal{O}_Z)^\vee & & \end{array}$$

After dualising and applying $\tau_{[0,1]} \circ \mathbf{R}\pi_*$, we obtain a commutative diagram

$$\begin{array}{ccccc} \mathbf{R}\mathcal{H}om_\pi(\mathcal{I}_Z, \mathcal{I}_Z)_0[1] & \longleftarrow & \varphi_{\text{der}}^\vee & & \\ \uparrow a & & & & \\ \tau_{[0,1]} \mathbf{R}\mathcal{H}om_\pi(\mathcal{I}_Z, \mathcal{O}_Z) & \longleftarrow & \mathbf{R}\pi_* \pi^* \mathbb{T}_H & \longleftarrow & \mathbb{T}_H \\ \downarrow \beta & & & & \downarrow \varphi^\vee \\ \tau_{[0,1]} \mathbf{R}\mathcal{H}om_\pi(\mathcal{O}_Z[-1], \mathcal{O}_Z) & \longleftarrow & \pi_* \text{At}_{\theta_Z}^\vee & \longleftarrow & \\ & \searrow \gamma & & & \downarrow \\ & & \tau_{[0,1]} \eta^* \mathbf{q}^* \mathbb{T}_{\mathcal{M}_n} & & \end{array}$$

where γ is the isomorphism (6.4), and the lower right part of the diagram, stating that the composition $\gamma \circ \pi_* \text{At}_{\theta_Z}^\vee$ agrees with the map φ^\vee dual to the map φ appearing in Diagram (6.2), is an immediate consequence of [29, Appendix A] or [19, Proposition 2.4.7].

Dualising back, we conclude the proof of Theorem C. Therefore we have proved Conjecture 9.9 in [15].

REFERENCES

1. Sjoerd Beentjes and Andrea T. Ricolfi, *Virtual counts on Quot schemes and the higher rank local DT/PT correspondence*, Math. Res. Lett. **28** (2021), no. 4, 967–1032.
2. Kai Behrend, *Introduction to Donaldson–Thomas theory, II*, MSRI Talk, 2018.
3. Kai Behrend and Barbara Fantechi, *The intrinsic normal cone*, Invent. Math. **128** (1997), no. 1, 45–88.
4. Kai Behrend and Barbara Fantechi, *Symmetric obstruction theories and Hilbert schemes of points on threefolds*, Algebra Number Theory **2** (2008), 313–345.
5. Oren Ben-Bassat, Christopher Brav, Vittoria Bussi, and Dominic Joyce, *A ‘Darboux theorem’ for shifted symplectic structures on derived Artin stacks, with applications*, Geom. Topol. **19** (2015), no. 3, 1287–1359.
6. Yuri Berest, George Khachatryan, and Ajay Ramadoss, *A Simple Construction of Derived Representation Schemes*, [ArXiv:1010.4901](https://arxiv.org/abs/1010.4901), 2010.
7. Christopher Brav, Vittoria Bussi, Delphine Dupont, Dominic Joyce, and Balázs Szendrői, *Symmetries and stabilization for sheaves of vanishing cycles*, J. Singul. **11** (2015), 85–151, With an appendix by Jörg Schürmann.
8. Christopher Brav, Vittoria Bussi, and Dominic Joyce, *A Darboux theorem for derived schemes with shifted symplectic structure*, J. Amer. Math. Soc. **32** (2019), no. 2, 399–443.
9. Christopher Brav and Tobias Dyckerhoff, *Relative Calabi–Yau structures II: Shifted Lagrangians in the moduli of objects*, Selecta Math. **27** (2021), no. 63, 1–45.
10. Vittoria Bussi, Dominic Joyce, and Sven Meinhardt, *On motivic vanishing cycles of critical loci*, J. Algebraic Geom. **28** (2019), no. 3, 405–438.
11. Damien Calaque, Tony Pantev, Bertrand Toën, Michel Vaquié, and Gabriele Vezzosi, *Shifted Poisson structures and deformation quantization*, J. Topol. **10** (2017), no. 2, 483–584.
12. Yalong Cao and Martijn Kool, *Counting zero-dimensional subschemes in higher dimensions*, J. Geom. Phys. **136** (2019), 119–137.
13. Alberto Cazzaniga, Dimbinaina Ralaivaosaona, and Andrea T. Ricolfi, *Higher rank motivic Donaldson–Thomas invariants of \mathbb{A}^3 via wall-crossing, and asymptotics*, Math. Proc. Cambridge Philos. Soc. **174** (2023), no. 1, 97–122.
14. Ionuț Ciocan-Fontanine and Mikhail Kapranov, *Derived Quot schemes*, Ann. Sci. École Norm. Sup. (4) **34** (2001), no. 3, 403–440.
15. Nadir Fasola, Sergej Monavari, and Andrea T. Ricolfi, *Higher rank K -theoretic Donaldson–Thomas theory of points*, Forum Math. Sigma **9** (2021), no. E15, 1–51.
16. Victor Ginzburg, *Calabi–Yau algebras*, [ArXiv:0612139](https://arxiv.org/abs/0612139), 2006.
17. Alexander Grothendieck, *Éléments de Géométrie Algébrique (rédigés avec la collaboration de Jean Dieudonné): III. Étude cohomologique des faisceaux cohérents, Seconde partie*, Inst. Hautes Études Sci. Publ. Math. (1963), no. 32, 5–91.
18. Alexander Grothendieck, *Techniques de construction et théorèmes d’existence en géométrie algébrique. IV. Les schémas de Hilbert*, Séminaire Bourbaki, Vol. 6, Soc. Math. France, Paris, 1995, pp. Exp. No. 221, 249–276.
19. Benjamin Hennion, *Tangent Lie algebra of derived Artin stacks*, J. Reine Angew. Math. **741** (2018), 1–45.
20. Daniel Huybrechts and Richard P. Thomas, *Deformation-obstruction theory for complexes via Atiyah and Kodaira–Spencer classes*, Math. Ann. **346** (2010), no. 3, 545–569.
21. Dominic Joyce, *A classical model for derived critical loci*, J. Differ. Geom. **101** (2015), no. 2, 289–367.
22. Sheldon Katz and Yun Shi, *D -critical loci for length n sheaves on local toric Calabi–Yau 3-folds*, Bull. Lond. Math. Soc. **54** (2022), no. 6, 2101–2116.
23. Young-Hoon Kiem and Michail Savvas, *K -Theoretic Generalized Donaldson–Thomas Invariants*, Int. Math. Res. Not. IMRN **2022** (2022), no. 3, 2123–2158.
24. Maxim Kontsevich and Yan Soibelman, *Notes on A_∞ -algebras, A_∞ -categories and non-commutative geometry, Homological mirror symmetry*, Lecture Notes in Phys., vol. 757, Springer, Berlin, 2009, pp. 153–219.
25. Nitin Nitsure, *Construction of Hilbert and Quot schemes*, Fundamental Algebraic Geometry, Math. Surveys Monogr., vol. 123, Amer. Math. Soc., Providence, RI, 2005, pp. 105–137.
26. Tony Pantev, Bertrand Toën, Michel Vaquié, and Gabriele Vezzosi, *Shifted symplectic structures*, Publ. Math. Inst. Hautes Études Sci. **117** (2013), 271–328.
27. Andrea T. Ricolfi, *Virtual classes and virtual motives of Quot schemes on threefolds*, Adv. Math. **369** (2020), 107182.
28. Andrea T. Ricolfi, *The equivariant Atiyah class*, C. R. Math. Acad. Sci. Paris **359** (2021), no. 3, 257–282.

29. Timo Schürg, Bertrand Toën, and Gabriele Vezzosi, *Derived algebraic geometry, determinants of perfect complexes, and applications to obstruction theories for maps and complexes*, J. Reine Angew. Math. **702** (2015), 1–40.
30. The Stacks Project Authors, *Stacks Project*, [stacks-project](https://stacks-project.org), 2021.
31. Balázs Szendrői, *Non-commutative Donaldson–Thomas invariants and the conifold*, Geom. Topol. **12** (2008), no. 2, 1171–1202.
32. Yukinobu Toda, *Gopakumar–Vafa invariants and wall-crossing*, [ArXiv:1710.01843](https://arxiv.org/abs/1710.01843), 2017.
33. Yukinobu Toda, *Moduli stacks of semistable sheaves and representations of Ext-quivers*, Geom. Topol. **22** (2018), no. 5, 3083–3144.
34. Bertrand Toën, *Higher and derived stacks: a global overview*, Algebraic geometry—Seattle 2005. Part 1, Proc. Sympos. Pure Math., vol. 80, Amer. Math. Soc., Providence, RI, 2009, pp. 435–487.
35. Bertrand Toën and Michel Vaquié, *Moduli of objects in dg-categories*, Ann. Sci. École Norm. Sup. (4) **40** (2007), no. 3, 387–444.
36. Bertrand Toën and Gabriele Vezzosi, *From HAG to DAG: derived moduli stacks*, Axiomatic, enriched and motivic homotopy theory, NATO Sci. Ser. II Math. Phys. Chem., vol. 131, Kluwer Acad. Publ., Dordrecht, 2004, pp. 173–216.
37. Okke van Garderen, *Cyclic A-infinity Algebras and Calabi–Yau Structures in the Analytic Setting*, [ArXiv:2306.00771](https://arxiv.org/abs/2306.00771), 2023.