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Combinatorics of Classical Unitary Invariant Ensembles and Integrable Systems

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Abstract

The first part of this thesis is devoted to the combinatorics, geometry, and effective computation of correlators of unitary invariant ensembles of random hermitian matrices with classical potentials. The main results are the subject of the publications [92, 93] with my supervisors T. Grava and G. Ruzza, and are summarized as follows.

We provide *generating functions for correlators* of general Hermitian matrix models; formulæ of this sort have already appeared in the literature [23, 78], we rederive them here with different methods which lend themselves to further generalizations. Such formulæ are not recursive in the genus and hence particularly effective. Moreover, these formulae express the correlators of classical unitary ensembles as linear combinations of products of discrete hypergeometric polynomials; this generalizes relations to discrete orthogonal polynomials for the *one-point* correlators $\left\langle \operatorname{tr} M^k \right\rangle$ of the classical ensembles recently discovered by Cunden et al. [52].

Hence, we turn our attention on the combinatorial interpretation of correlators for the Laguerre and Jacobi ensembles. We prove that the coefficients in the topological expansion of Jacobi correlators are *multiparametric single Hurwitz numbers* involving combinations of triple monotone Hurwitz numbers. Via a simple limit, this reproduces formulæ of [51] on the Laguerre ensemble. This completes the combinatorial interpretation of correlators of unitary ensembles with classical potential.

Combining results of Dubrovin et al. [62], and of Norbury [148] connecting integrable systems with enumerative geometry, we obtain *ELSV-like formulæ* linking the multiparametric single Hurwitz numbers of LUE and JUE respectively to cubic Hodge integrals and Θ-GW invariants.

In the second part of the thesis we analyse various integrable dynamical systems from a probabilistic point of view. Specifically, we study the spectrum of their *random Lax Matrix* equipped with the associated Gibbs Measure, in the spirit of [102, 156]. This is the content of the preprint [91], in collaboration with T. Grava, G. Gubbiotti and G. Mazzuca.

We explicitly compute the density of states for the *exponential Toda lattice* and the *Volterra lattice* showing they are connected to the *Laguerre β-ensemble* at high temperatures and the *β-antisymmetric Gaussian ensemble* at high temperatures respectively. For generalizations of these system we derive numerically their density of states and compute their ground states.

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Contents

Introduction

Overview

In the study of dynamical systems, a key concept is that of *integrability*. Many definitions can be given but as Birkhoff writes [29] "let us not forget the dictum of Poincaré, that a system of differential equations is only more or less integrable". In this thesis we both seek and reap the rewards of integrability.

Amongst the first and most studied integrable systems is the *Toda lattice* [159], describing particles on the real line interacting via the Hamiltonian

$$\mathcal{H}(p,q) = \frac{1}{2}\sum_{j=1}^{N} p_j^2 + \sum_{j=1}^{N-1} e^{(q_j - q_{j+1})}. \tag{1}$$

It is a *Liouville integrable system*, in the sense that it admits a maximal set of independent invariants in involution. This can be proved by constructing a pair of matrices (L, A) that reproduces the equations of motion via the commutator relation

$$\dot{L}(t) = [A(t), L(t)]\,. \tag{2}$$

Then, the eigenvalues of L form a complete set of first integrals for the system. The matrix L takes the name of *Lax matrix*, and the pair (L, A) of *Lax pair*. In the case of the Toda lattice, it can be used to explicitly compute the time evolution of the dynamical system [127].

Matrix Models. The very concept of Lax matrix is at the origin of much of the arguments treated in this work. Foremost, the Lax formulation allows to define in a simple way an extension of the Toda lattice to infinitely many time variables. The *Toda lattice-hierarchy* in the time variables $\mathbf{t} = (t_1, t_2, \dots)$ is given by the infinite set of *commuting flows*

$$\frac{\partial}{\partial t_k} L(\mathbf{t}) = \left[\Big(L^k(\mathbf{t})\Big)_+, L(\mathbf{t})\right], \qquad k = 1, 2, \dots, \tag{3}$$

from which the classical Toda lattice is recovered for $k = 1$. Correspondingly, one seeks for a function in infinitely many time variables from which solutions for all equations in the hierarchy can be constructed. In a nutshell, this is the concept of *tau function* $\tau(t_1, t_2, \dots)$, which also has the asset of taking all equations in the hierarchy in bilinear form, see Hirota [113].

Remarkably, *partition functions* of Hermitian matrix models happen to be Toda tau functions. They are defined via the matrix integral

$$Z_N(\mathbf{t}) = \int_{\mathcal{H}_N} e^{\operatorname{tr} V(M;\mathbf{t})} \mathrm{d}M \qquad V(x;\mathbf{t}) = V_0(x) + \sum_{j\geq 1} t_j x^j, \tag{4}$$

for some regular enough background potential $V_0(x)$. Via the spectral theorem for Hermitian matrices, one can give a complete description of the partition functions in terms of the orthogonal polynomials associated to $V_0(x)$. The identification with a specific Toda tau function passes through the remarkable fact that the Toda Lax matrix has essentially the same structure of the *Jacobi operator* describing the three term recurrence of orthogonal polynomials.

For a dynamical system, the existence of commuting flows is [133] “Fundamentally [...] well-definedness of correlation functions”. A matrix model can be called *integrable* if its partition function is the tau function of some integrable hierarchy; in this case, the correlators are defined as

$$\left\langle \operatorname{tr} M^{k_1} \cdots \operatorname{tr} M^{k_\ell} \right\rangle, \tag{5}$$

and are the coefficients of the partition function in the monomial base of symmetric polynomials in the variables $\mathbf{t}$. Well-behavedness of correlators, in this context, also means they can be recovered at all orders starting from the base cases $\ell = 1$ and $\ell = 2$, as described by the theories e.g. of the Topological recursion and Loop equations [79]. More to that, in the case of Hermitian one-cut matrices they can be expanded in Laurent series of N^2, with N the size of the matrix [40, 75], as

$$\left\langle \operatorname{tr} M^{k_1} \cdots \operatorname{tr} M^{k_\ell} \right\rangle \sim \sum_{g\geq 0} \frac{f_g(k_1,\dots,k_\ell)}{N^{2g-2}}, \qquad N \to \infty. \tag{6}$$

In some exceptional cases, one can get the explicit expression of the coefficients $f_g(k_1,\dots,k_\ell)$ in this expansion. When the background potential is quadratic – $V_0(x) = x^2$ – it was shown in the seminal paper of Bessis, Itzykson and Zuber [28] how they are connected to the *counting problem of ribbon graphs*. Ever since there has always been interest in finding, in this sense, combinatorial interpretations of matrix models.

Hurwitz numbers are a recurrent object in this context. They were first introduced by Adolf Hurwitz in [114] and concern the counting of equivalence classes (up to biholomorphism) of ramified coverings over $\mathbb{P}^1$; equivalently, via the *Riemann existence Theorem*, they describe factorization problems in the symmetric group. Amongst the many species of Hurwitz numbers, *Simple Hurwitz numbers* are somewhat the base case and the most studied one. In 2011 Borot, Eynard, Mulase and Safnuk [38] were able to construct generating functions of simple Hurwitz numbers via the *external matrix model*

$$Z \propto \int_{\mathcal{H}_N(\mathcal{C})} \mathrm{d}M \exp\left(-\frac{1}{g_s}\operatorname{tr}\left(V(M) - M\,A\right)\right), \tag{7}$$

we refer to loc. cit. for the precise definitions of the objects; here *external* pertains to the presence of the matrix A in the measure, whose eigenvalues serve as extra parameters for the system. Another important result was the proof by Goulden, Guay-Paquet and Novak [96, 97] that the *Harish-Chandra-Itzykson-Zuber integral* [67, 111, 118]

$$I_N(z) = \int\limits_{U(N)} e^{zN\operatorname{tr}(AUBU^{-1})}\mathrm{d}U \tag{8}$$

is a generating function for *double weakly monotone Hurwitz numbers*; a similar result holds for the *BGW model* which is related to *single weakly monotone Hurwitz numbers* [149]. Recently, external matrix models for generic *multiparametric Hurwitz numbers* have been worked out by Bertola and Harnad [25]. Remarkably, *internal* – in the sense of (4), with no additional matrices in the measure – matrix models for Hurwitz numbers do exist but have been investigated only recently in relation to unitary invariant ensembles with classical weights. The correlators of the GUE have been related to double strictly monotone Hurwitz numbers with a prescribed partition (also called orbifold Hurwitz numbers) by Borot and Garcia-Failde [39]. The *Laguerre Unitary Ensemble (LUE)* is related to multiparametric single Hurwitz numbers involving combinations of double (both weakly and strictly) monotone Hurwitz numbers. Glances of the combinatorial structure of the LUE first appeared in [101]; the idea was later systematized by Collins et al. in [50] and the study completed by Cunden Dahlqvist and O'Connell [51]. The last classical matrix integral is the *Jacobi Unitary Ensemble (JUE)*, and it is related to multiparametric Hurwitz numbers involving combinations of triple weakly monotone Hurwitz numbers; this constitutes one of the original contributions in this thesis and appeared in [93].

The interplay between tau functions, matrix models and Hurwitz numbers is deep and spreads over many branches of Mathematics. One of the most striking ones is their connection through the Kontsevich–Witten theorem [129, 166] with enumerative geometry. This celebrated result states that the generating function of certain *intersection numbers* on the moduli spaces of curves – specifically intersection numbers of *psi-classes* – is a tau function for the *KdV hierarchy*. It was later reproved by Kazarian and Lando [125] essentially inverting the *ELSV formula* [73], named after its discoverers: Ekedahl, Lando, Shapiro, Vainshtein. The ELSV formula gives a close expression for simple Hurwitz numbers in terms of Hodge integrals,

$$h_g(\mu) = \frac{(2g-2+|\mu|+\ell)!}{|\mathrm{Aut}(\mu)|}\prod_{i=1}^{\ell}\frac{\mu_i^{\mu_i}}{\mu_i!}\int_{\overline{\mathcal{M}}_{g,n}}\frac{\sum_{j=0}^{g}(-1)^j\lambda_j}{\prod_{i=1}^{\ell}(1-\mu_i\psi_i)}. \tag{9}$$

The first closed formula for general Hurwitz numbers was given long ago by Burnside [43] but the spark has revived in the early 2000s; along with the ELSV formula, important work on properties of Hurwitz numbers has been done, amongst the others, by Dubrovin, Yang, Zagier [64] and Goulden, Jackson, Vakil [95, 98]. The latter authors also conjectured an ELSV formula for *one-part double Hurwitz numbers* [99] then proved in [59]. Regarding double Hurwitz numbers, very recently an ELSV-like formula has been obtained in [37] by deforming the Johnson–Pandharipande–Tseng [119] formula for *orbifold Hurwitz numbers*. In this work we present two *ELSV-like formulæ* relating the Hurwitz numbers associated to the LUE/JUE with some specific intersection numbers. These formulæ involve weighted sums of the considered objects, and are similar in spirit to those in [39].

Random Lax systems We turn our attention back to Lax matrices. As mentioned, they can serve as a tool both to prove integrability of a dynamical system and to explicitly integrate it when the initial data are known. The latter can be a significantly difficult task, and has been worked out only in a few cases, e.g. [127]. Nonetheless, even if the exact solvability of a system is hardly achievable, it is still possible to study it from a probabilistic point of view.

When the initial data (p,q) are chosen randomly, the Lax matrix itself inherits an entrywise distribution and thus becomes a *random matrix*. In general, Hamiltonian systems have a natural invariant measure with respect to the Hamiltonian flow, defined in terms of the Hamiltonian itself, the so called *Gibbs measure* [126],

$$\mu_H = \frac{1}{Z_H}e^{-\beta H(p,q)}\mathrm{d}p\mathrm{d}q. \tag{10}$$

The eigenvalues of a Lax matrix L are constants of motion for the associated system, so that understanding the behaviour of its spectrum with random initial data remains a sensible question. In order to trigger (global features of) the randomness, these systems are analysed in the regime where the number N of degrees of freedom goes to infinity. This allows to define the *density of* *states* ν_L from the empirical measure on the eigenvalues λ_j of the Lax matrix L,

$$\frac{1}{N}\sum_{j=1}^{N}\delta_{\lambda_j} \xrightarrow{N\to\infty} \nu_L\,. \tag{11}$$

Recently, Spohn [156] connected the spectrum of the random Lax matrix of the Toda lattice with the one of the sparse matrix of the Gaussian β-ensemble [68] at high temperature [11]. The sparse matrix of the Gaussian β-ensemble is the random matrix

$$\begin{pmatrix} \mathcal{N}(0,2) & \chi_{(N-1)\beta} & & & \\ \chi_{(N-1)\beta} & \mathcal{N}(0,2) & \chi_{(N-2)\beta} & & \\ & \ddots & \ddots & \ddots & \\ & & \chi_{2\beta} & \mathcal{N}(0,2) & \chi_{\beta} \\ & & & \chi_{\beta} & \mathcal{N}(0,2) \end{pmatrix}. \tag{12}$$

It is equivalent in distribution to the $G\beta E$ of full matrices – e.g. the potential $V_0(x) = x^2$ in (4) recovers the $\beta = 2$ case – in the sense that their eigenvalues distributions coincide [69]. Roughly, the identification with the Toda lattice takes place since its Lax matrix has the same tridiagonal form of (12) and its entries for $N \to \infty$ converge to the same distributions under the associated Gibbs measure. This observation sparks the interest in linking other classical dynamical systems with known β-ensembles of matrices. In this direction, work has been done also by Mazzuca et al. [102, 136]. In this thesis we provide the link to the *Laguerre β-ensemble* and the *β-antisymmetric Gaussian ensemble* at high temperatures studying two lattice systems, the *exponential Toda lattice* and the *Volterra lattice*.

Structure of the thesis and original contributions

In the first two Chapters of the thesis we recall the main definitions from the theory of random matrix ensembles and orthogonal polynomials, as well as their connection with integrable systems. We also introduce the geometric and combinatorial definitions of Hurwitz numbers and recollect known results linking them to integrable systems.

Chapter 3 deals with **generating functions for correlators of classical unitary ensembles.** These results are based on the work done with T. Grava and G. Ruzza in [92, 93]. General formulæ for generating functions of correlators of Hermitian matrix models appeared in the work of Eynard et al. [77, 78] and Dubrovin and Yang [63] (see also [22]), through the so called matrix resolvent $R(z)$. They read

$$\mathscr{C}_1^{\mathsf{c}}(z) = \left(Y_N^{-1}(z)Y_N'(z)\right)_{1,1}, \tag{13}$$

$$\mathscr{C}_2^{\mathsf{c}}(z_1,z_2) = \frac{\operatorname{tr}\,(R(z_1)R(z_2)) - 1}{(z_1-z_2)^2}, \tag{14}$$

$$\mathscr{C}_\ell^{\mathsf{c}}(z_1,\dots,z_\ell) = -\sum_{(i_1,\dots,i_\ell)\in\mathrm{cyc}((\ell))} \frac{\operatorname{tr}\,(R(z_{i_1})\dots R(z_{i_\ell}))}{(z_{i_1}-z_{i_2})\cdots(z_{i_\ell}-z_{i_1})}, \quad \ell\geq 3, \tag{15}$$

where $\mathscr{C}_\ell^{\mathsf{c}}(z_1,\dots,z_\ell)$ is the ℓ-point cumulant function, $Y_N(z)$ is the *Fokas-Its-Kitaev matrix* [117], and $R(z) := Y_N(z)\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)Y_N^{-1}(z)$; in Section 3.1 we present a new derivation of the above. These formulæ hold for finite, N where N is the matrix dimension, and are different in flavour both from the topological recursion formulæ, [46] that permit to evaluate the coefficients of the large N

expansions of the correlators, and from the Dubrovin-Zhang approach [66] via Dubrovin-Frobenius manifolds. We also point out how a similar approach could be extended to ensembles arising from discrete and multiple orthogonal polynomials, see Section 3.1.1. We apply the matrix resolvent to the Laguerre and Jacobi unitary ensemble; this completes the study on generating functions for correlators of classical ensembles, the Gaussian case having been dealt with by Dubrovin and Yang in [63]. We obtain the solution in the form

$$R(z) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \sum_{\ell \geq 0} \frac{1}{z^{2\ell+2}} \begin{pmatrix} \mathcal{A}_{\ell,N} & -z\mathcal{B}_{\ell,N+1} \\ z\mathcal{B}_{\ell,N} & \mathcal{A}_{\ell,N} \end{pmatrix}, \tag{16}$$

as $z \to \infty$, where the entries $\mathcal{A}_{\ell,N}$ and $\mathcal{B}_{\ell,N}$ are shown to be *discrete orthogonal polynomials* indexed by the variable ℓ. It was first proved by Cunden, Mezzadri, O'Connell and Simm [52] that *one-point* correlators $\langle \operatorname{tr} M^k \rangle$ of all three classical unitary invariant ensembles are discrete orthogonal polynomials. Our results imply that *multi-point* correlators can be expressed as combinations of them as well. Explicit formulæ are presented in Theorems 3.2.3, 3.2.5 and 3.2.12.
In Chapter 4 we obtain the **combinatorial interpretation of the JUE.** We show that correlators of the Jacobi partition function admit a topological expansion in terms of *multiparametric single Hurwitz numbers [104, 112]*

$$\left\langle \prod_{j=1}^{\ell} \operatorname{tr} X^{\lambda_j} \right\rangle_{JUE} = (-1)^{|\lambda|} \frac{z_\lambda}{|\lambda|!} \sum_{g \geq 0} \frac{1}{N^{2g-2-\ell(\lambda)}} \sum_{\mu,\nu \vdash |\lambda|} \frac{c_\alpha^{\ell(\nu)}}{(-c_\alpha - c_\beta)^{\ell(\mu)+\ell(\nu)+\ell(\lambda)+2g-2}} h_g^{\geq}(\lambda,\mu,\nu),$$

which are expressed as a combination of triple weakly monotone Hurwitz numbers, Theorem 4.1.5. The derivation is carried out using the Selberg-Aomoto integral to explicitly compute the coefficients in the Schur expansion of the Jacobi partition function. Identical Hurwitz numbers are related to negative correlators (i.e. expectation of products of traces of *negative* powers of the random matrix). In Section 4.2 we show how the same technique retrieve analogous results of [51] on the Laguerre unitary ensemble.
In Chapter 5 we obtain an **ELSV-like formula for the multipoint correlators.** In [62] the *modified GUE* partition function (mGUE) was introduced and proved to be a generating function for *cubic Hodge integrals*. We point out a symmetry $(N,\alpha) \to (N+\alpha,-\alpha)$ in the Laguerre partition function $Z_N^{(\alpha)}$ which allows us to connect it with the mGUE partition function for the particular value $\alpha = -\frac{1}{2}$. As a consequence we relate the Hurwitz numbers of LUE with Hodge integrals according to

$$\sum_{g \geq 0} \epsilon^{2g-2} \mathscr{H}_{g,\mu} = 2^{\ell} \sum_{\gamma \geq 0} (2\epsilon)^{2\gamma-2} \sum_{\nu \vdash |\mu|} \left(\omega + \frac{\epsilon}{2}\right)^{2-2\gamma+|\mu|-\ell-\ell(\nu)} \left(\omega - \frac{\epsilon}{2}\right)^{\ell(\nu)} h_\gamma^{>}(\mu,\nu), \tag{17}$$

$$\begin{aligned} \mathscr{H}_{g,\mu} := 2^{g-1} \sum_{m \geq 0} \frac{(\omega-1)^m}{m!} \int_{\overline{\mathcal{M}}_{g,\ell+m}} \Lambda^2(-1)\Lambda\left(\frac{1}{2}\right) \exp\left(-\sum_{d \geq 1} \frac{\kappa_d}{d}\right) \prod_{a=1}^{\ell} \frac{\mu_a \binom{2\mu_a}{\mu_a}}{1-\mu_a\psi_a} \\ + \frac{\delta_{g,0}\delta_{\ell,1}}{2}\left(\omega - \frac{\mu_1}{\mu_1+1}\right)\binom{2\mu_1}{\mu_1} + \frac{\delta_{g,0}\delta_{\ell,2}}{2} \frac{\mu_1\mu_2}{\mu_1+\mu_2}\binom{2\mu_1}{\mu_1}\binom{2\mu_2}{\mu_2}, \end{aligned} \tag{18}$$

where $\mathscr{H}_{g,\mu}$ is a finite sum of Hodge integrals, see Theorem 5.1.1. The *Theta classes* on moduli spaces of curves have been introduced by Norbury [47, 147] as the analogue of standard intersection numbers of psi classes (associated to the *Airy topological recursion*) for the *Bessel topological* *recursion*. In the work [148] Norbury relates the generating function of Theta classes coupled to GW invariants of $\mathbb{P}^1$ with the Legendre matrix model. In view of the fact that this is a special case of JUE, we obtain

$$\sum_{\mu,\nu\vdash|\lambda|} \frac{h_g^{\geq}(\lambda,\mu,\nu)}{(-2)^{\ell(\mu)+\ell(\nu)+\ell+2g-2}} = \frac{(-1)^{|\lambda|}|\lambda|!}{2^{|\lambda|}\prod_{i\geq 1} i^{m_i}} \sum_{k_1,\dots,k_\ell\geq 1} \frac{\left\langle \Theta\cdot \prod_{j=1}^{\ell}\binom{\lambda_i}{k_j}\tau_{k_j-1}(\omega)\right\rangle_d^g}{2^{k_1+\cdots+k_\ell}}. \tag{19}$$

This is the content of Section 5.2.
Chapter 6 is partially unrelated to the first part of the thesis and concerns the **spectra of random Lax matrices.** We show that the exponential Toda lattice, and the Volterra lattice are related, respectively, to the Laguerre β-ensemble at high temperature and the antisymmetric Gaussian β-ensemble at high temperature. The results are in Section 6.2 and Section 6.3. We thus fill in rows two and five of the following table, the others appearing in [102, 136, 156]. We explicitly compute

β**-ensemble at high temperature**	**Integrable System**
Gaussian	Toda lattice
Laguerre	Exponential Toda lattice
Jacobi	Defocusing Schur flow
Circular	Defocusing Ablowitz-Ladik lattice
Antisymmetric Gaussian	Volterra lattice

the density of states of the associated random Lax matrices endowed with their generalized Gibbs measures as

$$\nu_{ET}(x) = \beta\partial_\alpha(\alpha\mu_{\alpha,\gamma}(\beta x))\mathrm{d}x, \qquad \nu_{\text{Volt}}(x) = \sqrt{\beta}\partial_\eta\left(\eta\theta_\eta(\sqrt{\beta}x)\right)\mathrm{d}x\,, \tag{20}$$

where $\mu_{\alpha,\gamma}$ and θ_η are related to the Tricomi's confluent hypergeometric and the Whittaker functions respectively.
In Chapter 7 we numerically investigate the eigenvalues distribution of other integrable systems, namely the additive and multiplicative INB lattices, the focusing Ablowitz-Ladik lattice and the focusing Schur flow.

The content of Chapter 6 and Chapter 7 is based on the work done with T. Grava, G. Gubbiotti and G. Mazzuca in [91].

Chapter 1

Random matrix ensembles

A *random matrix ensemble* is the datum of a probability measure $\mathrm{d}\mu(M)$ over a family, an *ensemble*, of matrices $\mathcal{M} \subset Mat\,(\mathbb{C}, N_1 \times N_2)$. It carries an entrywise distribution on the elements M_{ij} of M,

$$M_{ij} \propto \mathrm{d}\mu_{ij}(M). \tag{1.1}$$

Usually, the ensemble $\mathcal{M}$ is taken to be some vector space acted upon by a group of symmetries (e.g. Hermitian, Symmetric matrices,...), thus endowed with a natural Lebesgue measure $\mathrm{d}M$, with respect to which we ask the measure $\mathrm{d}\mu(M)$ to be absolutely continuous.

Random matrices serve as a model for many phenomena coming from the physics world; they first appeared in the fifties in the work of physicist E.P. Wigner, who was investigating properties of the energy levels of highly excited states of heavy nuclei [164]. He was interested in studying the spacings between those energy levels and conjectured they were related to the eigenvalues spacings of a certain random matrix ensemble. The book of Mehta [137] also played a foundational role in the development and spread of the theory.

Dealing with probabilistic quantities, one is often interested in computing expectations of functions with respect to the given measure, write

$$\langle f(M) \rangle = \int_{\mathcal{M}} f(M)\mathrm{d}\mu(M). \tag{1.2}$$

In this thesis we will be particularly interested in the *correlators* of Hermitian matrix ensembles, defined as averages of products of traces of powers of matrices,

$$\left\langle \mathrm{tr}\, M^{k_1} \cdots \mathrm{tr}\, M^{k_\ell} \right\rangle = \int_{\mathcal{H}_N} \mathrm{tr}\, M^{k_1} \cdots \mathrm{tr}\, M^{k_\ell}\, \mathrm{d}\mu(M), \tag{1.3}$$

where $k_1, \dots k_\ell \in \mathbb{Z}$. They can be regarded as the building blocks, coefficients of Taylor expansions, of *partition functions* of matrix models, see Section 1.3. We will extensively investigate these objects in Chapters 3 and 4 for specific measures $\mathrm{d}\mu$.

1.1 Unitary invariant ensembles

In the general scenario, random matrix ensembles can be extremely complicated to analyze. However, most of the models drawn from the real world exhibit an invariance property under the action of specific groups, most notably, the unitary, the orthogonal, and the symplectic group. It is a natural feature, which can be thought of as invariance of a system under change of coordinates [57]. We will almost entirely focus on unitary invariant ensembles.

Definition 1.1.1. *Define* $U(N)$ *the unitary group of size* $N \in \mathbb{N}$ *as*

$$U(N) := \{U \in Mat(\mathbb{C}, N) \,|\, U^{-1} = U^\dagger\}, \tag{1.4}$$

with $U^\dagger$ *denoting the conjugate transpose. A measure* $\mathrm{d}\mu(M)$ *on an ensemble* $\mathcal{M}$ *is said to be unitary invariant if*

$$\mu(M) = \mu(UMU^\dagger), \qquad \forall M \in \mathcal{M}, \forall U \in U(N). \tag{1.5}$$

The natural ensemble of matrices for unitary invariant measures is the N^2-dimensional vector space of *Hermitian matrices* of size N,

$$\mathcal{H}_N := \{M \in Mat(\mathbb{C}, N) \,|\, M = M^\dagger\}. \tag{1.6}$$

Indeed, by the *spectral theorem*, any hermitian matrix can be diagonalized as

$$M = UDU^\dagger, \tag{1.7}$$

with $U \in U(N)$ a unitary matrix and $D = diag(\lambda_1, \ldots, \lambda_N)$ the diagonal matrix with entries the eigenvalues of M. We consider measures of the form

$$\mathrm{d}\mu(M) = \frac{1}{C_N} e^{\operatorname{tr} V(M)} \mathrm{d}M, \tag{1.8}$$

with

$$\mathrm{d}M = \prod_{1 \le i < j \le N} \mathrm{d}\,\mathrm{Re}\, M_{ij}\, \mathrm{d}\,\mathrm{Im}\, M_{ij} \prod_{i=1}^{N} \mathrm{d}M_{ii} \tag{1.9}$$

the Lebesgue measure over the vector space $\mathcal{H}_N$, a regular enough (specific assumptions will be made down the road) scalar function $V(x)$, named *potential*, and C_N the normalization constant,

$$C_N = \int_{\mathcal{H}_N} e^{\operatorname{tr} V(M)} \mathrm{d}M. \tag{1.10}$$

Measures of the form (1.8) are automatically unitary invariant by elementary properties of the trace, $\operatorname{tr}(M) = \operatorname{tr}(UMU^\dagger)$, and by unitary invariance of the Lebesgue measure $\mathrm{d}M$.

In view of (1.7), unitary invariant measures over $\mathcal{H}_N$ of the form (1.8) can be factorized in a constant angular part and a random one depending on its eigenvalues. Indeed consider the map

$$\psi \colon U(N)/\left[U(1)\right]^N \times \mathcal{D}_N \to \mathcal{H}_N, \qquad (U, [D]) \mapsto UDU^\dagger \tag{1.11}$$

where $\mathcal{D}_N$ is the set of diagonal $N \times N$ matrices $diag(\lambda_1, \lambda_2, \ldots, \lambda_N)$ with real ordered eigenvalues $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_N$. Then ψ can be parametrized in a smooth way on a Zariski open set of $\mathcal{H}_N$. It can be proved that the square of the *Vandermonde determinant*

$$\Delta(\underline{\lambda}) := \prod_{1 \le i < j \le N} (\lambda_i - \lambda_j) = \det \begin{pmatrix} 1 & \lambda_1 & \lambda_1^2 & \cdots & \lambda_1^{n-1} \\ 1 & \lambda_2 & \lambda_2^2 & \cdots & \lambda_2^{n-1} \\ \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \vdots \\ 1 & \lambda_n & \lambda_n^2 & \cdots & \lambda_n^{n-1} \end{pmatrix}, \tag{1.12}$$

is the Jacobian $\mathrm{d}M = \Delta^2(\underline{\lambda})\mathrm{d}\underline{\lambda}$ for the change of coordinates induced by ψ. Then we can write

$$\int_{\mathcal{H}_N} e^{\mathrm{tr}\, V(M)} \mathrm{d}M = Vol\left(U(N)/\left[U(1)\right]^N\right) \int_{\mathcal{D}_N} \prod |\lambda_i - \lambda_j|^2 \, e^{\sum_{k=1}^N V(\lambda_k)} d\lambda_1 \dots d\lambda_N, \tag{1.13}$$

and passing from $\mathcal{D}_N$ to $\mathbb{R}^N$, upon adding measure zero terms,

$$\int_{\mathcal{H}_N} e^{\mathrm{tr}\, V(M)} \mathrm{d}M = \frac{1}{N!} \mathrm{Vol}(N) \int_{\mathbb{R}^N} \mathsf{P}_N(\lambda_1, \dots, \lambda_N) \, d\lambda_1 \dots d\lambda_N, \tag{1.14}$$

where we introduced

$$\mathrm{Vol}(N) := \int_{U(N)/[U(1)]^N} \mathrm{d}U = \frac{\pi^{\frac{N(N-1)}{2}}}{\prod_{j=0}^{N-1} j!}, \tag{1.15}$$

and the probability measure over the eigenvalues

$$\mathsf{P}_N(\lambda_1, \dots, \lambda_N) := \frac{1}{C_N} e^{\sum_{k=1}^N V(\lambda_k)} \Delta^2(\underline{\lambda}) \, d\lambda_1 \dots d\lambda_N. \tag{1.16}$$

Notice that in this way we reduced a matrix integral to a standard integral over $\mathbb{R}^N$.

When computing averages of functions of M invariant under the action of the unitary group, one can apply the same argument to dimensionally reduce the matrix integral. An example is given by the correlators (1.3), which are indeed symmetric in the eigenvalues, depending on traces of powers of the matrix only,

$$\left\langle \mathrm{tr}\, M^{k_1} \cdots \mathrm{tr}\, M^{k_\ell} \right\rangle = \frac{\int_{\mathbb{R}^N} \prod_{j=1}^{\ell} \left(\sum_{i=1}^N \lambda_i^{k_j} \right) \Delta^2(\underline{\lambda}) \, e^{\sum_{i=1}^N V(\lambda_i)} d\lambda_1 \dots d\lambda_N}{\int_{\mathbb{R}^N} \Delta^2(\underline{\lambda}) \, e^{\sum_{i=1}^N V(\lambda_i)} d\lambda_1 \dots d\lambda_N}. \tag{1.17}$$

A very efficient tool in computing the so obtained space integrals is supplied by the theory of *orthogonal polynomials*, [56, 57]. In a nutshell, the idea is to further reduce the N-dimensional integrals to lower dimensional ones; this line of thought will be a building block in proofs of Chapter 3. In the next section we recall the basics on orthogonal polynomials.

1.2 Orthogonal polynomials

Orthogonal polynomials are ubiquitous objects in mathematics. We will now state a number of standard facts which can be retrieved e.g. in [48, 115].

Definition 1.2.1. *Let $\{P_n(x)\}_{n\geq 0}$ be a collection of polynomials defined over an open interval $I \subseteq \mathbb{R}$ such that $\deg P_n(x) = n$. They form a family of orthogonal polynomials with respect to the weight $w(x) : I \to \mathbb{R}_+$ provided*

$$\int_I P_n(x) P_m(x) w(x) dx = \delta_{nm} h_n, \qquad \forall m, n \in \mathbb{N}. \tag{1.18}$$

Here, δ_{nm} is the standard Kronecker delta, valued 1 if $m = n$ and 0 otherwise, and $h_n \in \mathbb{R}$ are called norming constants. The weight $w(x) : I \to \mathbb{R}_+$ shall be positive and continuous, and such that all moments

$$m_k(x) := \int_I x^k \, w(x), \qquad k = 0, 1, 2, \dots \tag{1.19}$$

exist finite.

Such a family is unique up to a multiplicative constant, so that we can take them to be *monic*

$$P_n(x) = x^n + c_{n-1}x^{n-1} + \cdots + c_0. \tag{1.20}$$

In particular, once a weight $w(x)$ as in the above definition is given, an associated family of orthogonal polynomials can always be constructed by means of the *Gram-Schmidt procedure*. Namely, consider the inner product on $L^2(w(x)dx)$

$$\langle f, g\rangle = \int_I f(x)g(x)w(x)dx, \tag{1.21}$$

and simply define

$$P_0(x) = 1, \qquad P_1(x) = x - \frac{\langle x, P_0\rangle}{\langle P_0, P_0\rangle}, \tag{1.22}$$

$$P_n(x) = x^n - \frac{\langle x, P_{n-1}\rangle}{\langle P_{n-1}, P_{n-1}\rangle}P_{n-1}(x) - \cdots - \frac{\langle x, P_0\rangle}{\langle P_0, P_0\rangle}P_0(x), \tag{1.23}$$

which are orthogonal by construction.

We can already start to appreciate the usefulness of orthogonal polynomials in the matrix models context looking back at (1.14). Indeed, it is easy to see that the following holds

$$\int_{\mathbb{R}^N} \Delta^2(\underline{x})\, e^{\sum_{i=1}^N V(x_i)}dx_1 \dots dx_N, = N!\, h_0 h_1 \cdots h_{N-1}, \tag{1.24}$$

where the h_i's are the norming constants associated to the weight $e^{V(x)}$. Indeed, by simple row operations we can rewrite the Vandermonde determinant (1.12) as

$$\Delta(\underline{x}) = \det\begin{pmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{N-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{N-1} \\ \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \vdots \\ 1 & x_N & x_N^2 & \cdots & x_N^{N-1} \end{pmatrix} = \det\begin{pmatrix} 1 & P_1(x_1) & P_2(x_1) & \cdots & P_{N-1}(x_1) \\ 1 & P_1(x_2) & P_2(x_2) & \cdots & P_{N-1}(x_2) \\ \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \vdots \\ 1 & P_1(x_N) & P_2(x_N) & \cdots & P_{N-1}(x_N) \end{pmatrix}, \tag{1.25}$$

so that, expanding the squared determinant and then using the orthogonality property (1.18),

$$\int_{\mathbb{R}^N} \Delta^2(\underline{x})\, e^{\sum_{i=1}^N V(x_i)}\mathrm{d}\underline{x} = \int_{\mathbb{R}^N} \left(\sum_{\sigma\in\mathfrak{S}_n} P_0(x_{\sigma(1)})\cdots P_{N-1}(x_{\sigma(n)})\right)^2 e^{\sum_{i=1}^N V(x_i)}\mathrm{d}\underline{x} \tag{1.26}$$

$$= \sum_{\sigma\in\mathfrak{S}_n} \int_{\mathbb{R}^N} P_0^2(x_{\sigma(1)})\cdots P_{N-1}^2(x_{\sigma(n)})\, e^{\sum_{i=1}^N V(x_i)}\mathrm{d}\underline{x} \tag{1.27}$$

$$= N!\, h_0 h_1 \cdots h_{N-1}, \tag{1.28}$$

and finally from (1.14) we get

$$\frac{1}{\mathrm{Vol}(N)}\int_{\mathcal{H}_N} e^{\mathrm{tr}\, V(M)}\mathrm{d}M = h_0 h_1 \cdots h_{N-1}. \tag{1.29}$$

From the Gram-Schmidt procedure is possible to prove the following characterizing feature of orthogonal polynomials.

Theorem 1.2.2. *Any family $\{P_n(x)\}_{n\geq 0}$ of monic orthogonal polynomials satisfies a three term recurrence, which is, there exist sequences of real numbers $\{a_n\}_{n\geq 0}$, $\{b_n\}_{n\geq 0}$, such that*

$$x \cdot P_n(x) = P_{n+1}(x) + a_n P_n(x) + b_n P_{n-1}(x). \tag{1.30}$$

Clearly, the knowledge of the sequences $\{a_n\}_{n\geq 0}$, $\{b_n\}_{n\geq 0}$, together with the monic condition $P_0(x) = 1$ and assuming $P_{-1}(x) = 0$, allows one to completely reconstruct the family of orthogonal polynomials. More to that, the following viceversa to Theorem 1.2.2 holds.

Theorem 1.2.3 (Favard Theorem, [81]). *If there exist sequences $\{a_n\}_{n\geq 0}$, $\{b_n\}_{n\geq 0}$, such that*

$$x \cdot P_n(x) = P_{n+1}(x) + a_n P_n(x) + b_n P_{n-1}(x), \tag{1.31}$$

with $P_{-1}(x) = 0$ and $P_0(x) = 1$, then there exists $w(x)$ of bounded variation such that

$$\int_I P_n(x) P_m(x) w(x) dx = \delta_{nm} h_n, \qquad \forall m, n \in \mathbb{N}. \tag{1.32}$$

The three term recurrence can be rewritten in (semi-infinite) matrix form by means of the *Jacobi operator*, which is the tridiagonal matrix

$$L = \begin{pmatrix} a_0 & 1 & 0 & \cdots \\ b_1 & a_1 & 1 & \cdots \\ 0 & b_2 & a_2 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{1.33}$$

Then, denoting $\mathbf{P}(x) = (P_0(x), P_1(x), \dots)^T$ equation (1.30) is recasted as

$$x \cdot \mathbf{P}(x) = L \cdot \mathbf{P}(x), \tag{1.34}$$

and the vector of monic orthogonal polynomials $\mathbf{P}(x)$ can be thought of as an eigenvector of the operator L. An important feature of orthogonal polynomials is the *Christoffel-Darboux identity.*

Proposition 1.2.4. *Let $\{P_n(x)\}_{n\geq 0}$ be a family of monic orthogonal polynomials with norming constants $\{h_j\}_{j\geq 0}$, and N a positive integer. Then,*

$$\sum_{j=0}^{N-1} \frac{P_j(x) P_j(y)}{h_j} = \frac{1}{h_{N-1}} \frac{P_N(x) P_{N-1}(y) - P_{N-1}(x) P_N(y)}{x - y}, \tag{1.35}$$

In particular, if $\{P_n(x)\}_{n\geq 0}$ are orthogonal with respect to the weight $w(x) = e^{V(x)}$ we name *Christoffel-Darboux kernel* the quantity

$$K_N(x, y) := \frac{e^{\frac{V(x)+V(y)}{2}}}{h_{N-1}} \frac{P_N(x) P_{N-1}(y) - P_{N-1}(x) P_N(y)}{x - y}. \tag{1.36}$$

It consists of the right hand side of (1.35) multiplied by the square root of the weights in the variables x and y. The Christoffel-Darboux kernel satisfies two important properties,

1. normalization $\int_I K_N(x, x) \mathrm{d}x = N$,
2. reproducibility $\int_I K_N(x, y) K_N(y, z) \mathrm{d}y = K_N(x, z)$.

Comparing with (1.35) we realize that $K_N(x,y)$ is the kernel of the orthogonal projection of $L^2(I, w(x)\mathrm{d}x)$ onto the space of polynomials of degree $< N$. Given a unitary invariant matrix ensemble with the same weight $e^{V(x)}$, the Christoffel-Darboux identity also gives an immediate way to reconstruct the probability measure over the eigenvalues and, more generally, all its marginals $\rho_k(x_1,\dots,x_k)$, which here take the name of *k-point correlation functions*.

Proposition 1.2.5. *Let $\mathsf{P}_N(\underline{x})\mathrm{d}\underline{x}$ be the eigenvalue measure (1.16), then we have*

$$\mathsf{P}_N(\underline{x})\,\mathrm{d}\underline{x} := \frac{1}{C_N} e^{\sum_{i=1}^N V(x_i)} \Delta^2(\underline{x}) = \det\left(K_N(x_i,x_j)\right)_{1\le i,j,\le N}. \tag{1.37}$$

$$\rho_k(x_1,\dots,x_k) := \int_{\mathbb{R}^{N-k}} \mathsf{P}_N(x_1,\dots,x_k,x_{k+1},\dots,x_N)\mathrm{d}x_{k+1}\cdots\mathrm{d}x_N = \det\left(K_N(x_i,x_j)\right)_{1\le i,j,\le k}. \tag{1.38}$$

Proof. A standard proof can be found in [56]. □

Finally, family of orthogonal polynomials can be characterized in a third way as the unique solution of the associated *Riemann-Hilbert problem*, first established in the seminal work by Fokas, Its and Kitaev in [117], and which we hereby recall.

Definition 1.2.6. *The Cauchy-Transform of the orthogonal polynomial $P_N(x)$ with respect to the weight $\mathrm{e}^{V(x)}$ is given by*

$$\widehat{P}_N(x) := \frac{1}{2\pi\mathrm{i}} \int_I \frac{P_N(\xi)}{\xi - x} \mathrm{e}^{V(\xi)}\mathrm{d}\xi. \tag{1.39}$$

It is analytic for $x \in \mathbb{C}\backslash\overline{I}$ and continuous up to the boundary of I, where it has a jump.

Introduce the *Fokas-Its-Kitaev matrix*

$$Y_N(x) := \begin{pmatrix} P_N(x) & \widehat{P}_N(x) \\ -\frac{2\pi\mathrm{i}}{h_{N-1}} P_{N-1}(x) & -\frac{2\pi\mathrm{i}}{h_{N-1}} \widehat{P}_{N-1}(x) \end{pmatrix}. \tag{1.40}$$

It is the unique matrix analytic in $z \in \mathbb{C} \setminus I$ satisfying

1. $Y_N(x)$ has a jump on the real axis given by

$$Y_{N,+}(x) = Y_{N,-}(x) \begin{pmatrix} 1 & \mathrm{e}^{V(x)} \\ 0 & 1 \end{pmatrix}, \qquad x \in I, \tag{1.41}$$

where we use the notation

$$Y_{N,\pm}(x) = \lim_{\epsilon\to 0_+} Y_N(x \pm \mathrm{i}\epsilon), \qquad x \in I^\circ, \tag{1.42}$$

and I° is the interior of the interval I,

2. for $x \to \infty$ it has the behaviour

$$Y_N(x) = \left(\mathbf{1} + \mathcal{O}(x^{-1})\right) x^{N\sigma_3}, \tag{1.43}$$

3. at any endpoint x_0 of I, for $x \to x_0$

$$Y_N(x) = \mathcal{O}\left(\log(x - x_0)\right). \tag{1.44}$$

where we denote $\mathbf{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. In Chapter 3 we will exploit such a characterization to investigate probabilistic quantities related to the weight $e^{V(x)}$.

1.2.1 Classical unitary ensembles

Amongst the many, infinite, families of orthogonal polynomials $\{P_n(x)\}_{n\geq 0}$ we distinguish a particular class, the so called *classical orthogonal polynomials*. They are characterized by the following equivalent properties, [30, 128]

1. They are solutions of the system of differential equations
$$A(x)y_n''(x) + B(x)y_n'(x) + \lambda_n y_n(x) = 0, \qquad n = 0, 1, \ldots, \tag{1.45}$$
required $A(x), B(x)$ are independent of n, λ_n independent of x and solutions $y_n(x)$ are polynomials of degree $\deg y_n(x) = n$,
2. Their derivatives $\{P_n'(x)\}_{n\geq 0}$ form a family of orthogonal polynomials as well,
3. *Rodrigues-type formula:* there exist constants k_n and n-independent polynomials $w(x), T(x)$ such that
$$P_n(x) = \frac{k_n}{w(x)} \frac{\mathrm{d}^n}{\mathrm{d}x^n} \left[w(x)\,(T(x))^n\right]. \tag{1.46}$$

It turns out that up to *a linear change of variables*, which amounts to scale and shift of the domain and standardization of the polynomials, there are exactly three measures on $\mathbb{R}$ which satisfy any of the properties above, each coming with its associated family of *classical orthogonal polynomials*. Correspondingly the three associated matrix ensembles go by the name of *classical unitary ensembles*. We recall their definition and main properties in the next subsections.

Hermite polynomials

Hermite polynomials $\{P_n^{\mathsf{H}}(x)\}_{n\geq 0}$ are orthogonal with respect to the Gaussian measure
$$\mu^{\mathsf{H}}(x) = e^{-\frac{x^2}{2}}, \qquad x \in (-\infty, \infty), \tag{1.47}$$
with orthogonality relation
$$\int_{\mathbb{R}} P_n^{\mathsf{H}}(x) P_m^{\mathsf{H}}(x) e^{-\frac{x^2}{2}} \mathrm{d}x = \delta_{nm} h_n, \qquad h_n = n!\sqrt{2\pi} \tag{1.48}$$

They can be computed via the Rodrigues formula
$$P_n^{\mathsf{H}}(x) = (-1)^n e^{\frac{x^2}{2}} \frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(e^{-\frac{x^2}{2}}\right) \tag{1.49}$$
and the three term recurrence reads
$$x \cdot P_n^{\mathsf{H}}(x) = P_{n+1}^{\mathsf{H}}(x) + n\, P_{n-1}^{\mathsf{H}}(x). \tag{1.50}$$

The associated unitary invariant matrix model is known as *Gaussian Unitary Ensemble*, or simply *GUE*. The ensemble of matrix is that of *Hermitian matrices* endowed with the measure
$$\mu_N^{\mathsf{H}}(M) = \frac{1}{C_N} e^{-\mathrm{tr}\,\frac{M^2}{2}} \mathrm{d}M, \tag{1.51}$$
where $\mathrm{d}M$ is the standard Lebesgue measure. In virtue of the discussions in the previous section, the normalization constant is explicitly computed as
$$C_N^{\mathsf{H}} = \mathrm{Vol}(N) \cdot \prod_{j=0}^{N-1} h_j = \frac{\pi^{\frac{N(N-1)}{2}}}{\prod_{j=0}^{N-1} j!} \cdot \prod_{j=0}^{N-1} \left(j!\sqrt{2\pi}\right) = \sqrt{2}^N \sqrt{\pi}^{N^2} \tag{1.52}$$

Laguerre polynomials

Laguerre polynomials $\{P_n^{\mathsf{L}}\}_{n\geq 0}$ are orthogonal with respect to the measure

$$\mu^{\mathsf{L}}(x) = x^{\alpha} e^{-x}, \qquad x \in (0, \infty), \tag{1.53}$$

with $\alpha \in \mathbb{C}$ a complex parameter, and orthogonality relation

$$\int_0^{\infty} P_n^{\mathsf{L}}(x) P_m^{\mathsf{L}}(x) x^{\alpha} e^{-x} \mathrm{d}x = \delta_{nm} h_n^{\mathsf{L}}, \qquad h_n^{\mathsf{L}} = n! \Gamma(\alpha + n + 1) \tag{1.54}$$

They can be computed via the Rodrigues formula

$$P_n^{\mathsf{L}}(x) = (-1)^n x^{-\alpha} e^x \frac{\mathrm{d}^n}{\mathrm{d}x^n} \left(x^{\alpha+n} e^{-x} \right), \tag{1.55}$$

in an explicit way as

$$P_n^{\mathsf{L}}(x) = \sum_{j=0}^{n} \frac{(-1)^{n-j} (n-j+1)_j (j+1+\alpha)_{n-j}}{j!} x^j \tag{1.56}$$

where $(p)_j := p(p+1)\cdots(p+j-1)$ denotes the *rising factorial*; the three term recurrence reads

$$x \cdot P_n^{\mathsf{L}}(x) = P_{n+1}^{\mathsf{L}}(x) + (2n + \alpha + 1) P_n^{\mathsf{L}}(x) + n(n+\alpha) L_{n-1}^{(\alpha)}(x). \tag{1.57}$$

The associated unitary invariant matrix model is known as *Laguerre Unitary Ensemble*, or simply *LUE*. The ensemble of matrix is the cone $\mathcal{H}_N^+$ of *positive definite Hermitian matrices*, which is hermitian matrices with positive eigenvalues only. The measure is given by

$$\mu_N^{\mathsf{L}}(M) \frac{1}{C_N^{\mathsf{L}}} \det^{\alpha}(M) \, e^{-\operatorname{tr} M} \mathrm{d}M, \tag{1.58}$$

the normalization constant being

$$C_N^{\mathsf{L}} = \mathrm{Vol}(N) \cdot \prod_{j=0}^{N-1} h_j^{\mathsf{L}} = \pi^{\frac{N(N-1)}{2}} \prod_{j=0}^{N-1} \Gamma(\alpha + j + 1). \tag{1.59}$$

Remark 1.2.7. *The LUE can also be realized in terms of the product of two full, rectangular, Gaussian Wigner matrices, e.g. [85, 165]. Specifically, let W be an $N \times (N + \alpha)$ matrix with independent identically distributed Gaussian entries. Then, the matrix*

$$M := \frac{1}{N} W W^T \tag{1.60}$$

is positive definite and with entries distributed according to (1.58) and parameter $\alpha \in \mathbb{N}$.

Jacobi polynomials

Jacobi polynomials $\{P_n^{\mathsf{J}}(x)\}_{n\geq 0}$ are orthogonal with respect to the measure

$$\mu^J(x) = x^{\alpha} (1-x)^{\beta}, \qquad x \in (0, 1), \tag{1.61}$$

with $\alpha, \beta \in \mathbb{C}$ complex parameters, and orthogonality relation

$$\int_0^1 P_n^{\mathsf{J}}(x) P_m^{\mathsf{J}}(x)\, x^\alpha (1-x)^\beta, \mathrm{d}x = \delta_{nm} h_n^{\mathsf{J}}, \tag{1.62}$$

$$h_n^{\mathsf{J}} = n! \frac{\Gamma(\alpha+n+1)\Gamma(\beta+n+1)}{\Gamma(\alpha+\beta+2n+2)} \frac{\Gamma(\alpha+\beta+n+1)}{\Gamma(\alpha+\beta+2n+1)}. \tag{1.63}$$

Jacobi polynomials can also be defined over the symmetric interval $(-1,1)$ via the affine transformation $x \mapsto \frac{x+1}{2}$ and renormalizing. However, measure (1.61) will best fit our needs. The Rodrigues formula here reads

$$P_n^{\mathsf{J}}(x) = (-1)^n \frac{\Gamma(\alpha+\beta+n+1)}{\Gamma(\alpha+\beta+2n+1)} x^{-\alpha}(1-x)^{-\beta} \frac{\mathrm{d}^n}{\mathrm{d}x^n} \left(x^{\alpha+n}(1-x)^{\beta+n} \right), \tag{1.64}$$

the explicit form

$$P_n^{\mathsf{J}}(x) = \frac{n!}{(\alpha+\beta+n+1)_n} \sum_{k=0}^n \binom{n+\alpha}{k} \binom{n+\beta}{n-k} (x-1)^k\, x^{n-k}, \tag{1.65}$$

and the three term recurrence

$$x \cdot P_n^{\mathsf{J}}(x) = P_{n+1}^{\mathsf{J}}(x) + a_n P_n^{\mathsf{J}}(x) + b_n P_{n-1}^{\mathsf{J}}(x), \tag{1.66}$$

$$a_n = 1 + \frac{n(\beta+n)}{\alpha+\beta+2n} - \frac{(n+1)(\beta+n+1)}{\alpha+\beta+2n+2}, \tag{1.67}$$

$$b_n = \frac{n(\beta+n)(\alpha+n)(\alpha+\beta+n)}{(\alpha+\beta+2n-1)(\alpha+\beta+2n)^2(\alpha+\beta+2n+1)}. \tag{1.68}$$

The associated unitary invariant matrix model is known as *Jaocobi Unitary Ensemble*, or simply *JUE*. The ensemble of matrix is the space $\mathcal{H}_N(I)$ with $I = (0,1)$ which denotes hermitian matrices with all the eigenvalues lying in I. The measure is given by

$$\mu_N^{\mathsf{J}}(M) = \frac{1}{C_N^J} \det{}^\alpha (M) \det{}^{1-\beta} (M)\, \mathrm{d}M, \tag{1.69}$$

the normalization constant being

$$C_N^J = \mathrm{Vol}(N) \cdot \prod_{j=0}^{N-1} h_j^{\mathsf{J}} = \pi^{\frac{N(N-1)}{2}} \prod_{j=0}^{N-1} \frac{\Gamma(\alpha+j+1)\Gamma(\beta+j+1)}{\Gamma(\alpha+\beta+2N-j)}. \tag{1.70}$$

Remark 1.2.8. *The JUE can also be realized in terms of a rational function of two Gaussian Wigner matrices [61, 85]. Specifically, let $W_\alpha := A^\dagger A$ and $W_\beta := B^\dagger B$ where A and B are, respectively, $N \times (N+\alpha)$ and $N \times (N+\beta)$ matrix with independent identically distributed Gaussian entries. Then, the probability measure (1.69) describes the distribution of the matrix*

$$M := (W_\alpha + W_\beta)^{-1/2} W_\alpha (W_\alpha + W_\beta)^{-1/2}, \tag{1.71}$$

which has spectrum in $(0,1)$ and entries distributed according to (1.69), with parameters $\alpha, \beta \in \mathbb{N}$.

1.3 Partition functions and the Toda lattice hierarchy

Random matrix ensembles are deeply connected with integrable systems. Indeed, exponentially perturbed measure of unitary invariant ensembles are *Toda tau-functions*, with initial conditions given by the coefficients of the three term recurrence of the associated orthogonal polynomials. We make this statement more precise in the following.

Definition 1.3.1. *The Toda lattice is the dynamical system of N particles $x_1(t), \dots, x_N(t)$ on the real line interacting via the Hamiltonian*

$$H = \frac{1}{2}\sum_{j=1}^{N} p_j^2 + \sum_{j=1}^{N-1} e^{(x_j - x_{j+1})}, \tag{1.72}$$

which, with respect to the standard Poisson bracket $\{\cdot, \cdot\}$, yields equations of motion

$$\dot{x}_j = \{x_j, H\} = p_j, \qquad \dot{p}_j = \{p_j, H\} = e^{(x_{j-1}-x_j)} - e^{(x_j - x_{j+1})}, \qquad j = 1, \dots, N \tag{1.73}$$

In particular, as first recognized in the seminal works of Flaschka and Manakov [82, 83, 132] the change of variables $a_j := -\frac{p_j}{2}$ and $b_j := \frac{1}{2} e^{\frac{(x_j - x_{j+1})}{2}}$ allows to rewrite the equations of motion (1.73) in the Lax form

$$\dot{L(t)} = \frac{\partial}{\partial t} L(t) = [A(t), L(t)] \tag{1.74}$$

with

$$L = \begin{pmatrix} b_1 & a_1 & & & \\ a_1 & b_2 & a_2 & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & a_{N-1} \\ & & & a_{N-1} & b_N \end{pmatrix}, \qquad A = \begin{pmatrix} 0 & a_1 & & & \\ -a_1 & 0 & a_2 & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & a_{N-1} \\ & & & -a_{N-1} & 0 \end{pmatrix}. \tag{1.75}$$

Notice that the matrix L has essentially, up to a gauge transformation, the same form of the Jacobi operator (1.33), while A is the difference of the upper and lower triangular parts of L, write $A = L_+ - L_-$. The Lax formulation allows to define in a simple way an extension of the Toda lattice to infinite time variables.

Definition 1.3.2. *The Toda Lattice-hierarchy in the time variables $\mathbf{t} = (t_1, t_2, \dots)$ is given by the infinite set of commuting flows*

$$\frac{\partial}{\partial t_k} L(\mathbf{t}) = [A_k(\mathbf{t}), L(\mathbf{t})], \qquad A_k(\mathbf{t}) = \left(L^k(\mathbf{t})\right)_+ - \left(L^k(\mathbf{t})\right)_- . \tag{1.76}$$

A τ-function for the Toda lattice is a solution of the Toda equation; essentially it satisfies

$$a = \epsilon \frac{\partial}{\partial t_1} \log \frac{\tau(x+\epsilon)}{\tau(x)}, \qquad b = \log \frac{\tau(x+\epsilon)\tau(x-\epsilon)}{\tau^2(x)}, \tag{1.77}$$

where $a_n =: a(\epsilon n)$ and $b_n =: b(\epsilon n)$, we refer to the literature [160, 167] for a precise definition.

It can be proven that deformations of unitary invariant matrix ensembles are tau functions of the Toda lattice hierarchy, with choice of parameter $\epsilon = 1/N$. Specifically, given a unitary invariant measure $e^{\operatorname{tr} V_0(M)}\mathrm{d}M$ on a certain ensemble of matrices $\mathcal{M}$, we consider the deformed potential

$$V(x;\mathbf{t}) = V_0(x) + \sum_{j\geq 1} t_j M^j \tag{1.78}$$

and name *partition function* the object

$$Z_N(\mathbf{t}) = \int_{\mathcal{H}_N} e^{\operatorname{tr} V(M;\mathbf{t})}\mathrm{d}M. \tag{1.79}$$

Then, we have the following.

Proposition 1.3.3. *$Z_N(\mathbf{t})$ is a tau-function for the Toda lattice hierarchy (1.76). Moreover,*

1. *$a_j(\mathbf{t})$ and $b_j(\mathbf{t})$ are exactly the coefficients of three term recurrence associated to the potential $V(x;\mathbf{t})$, see Theorem 1.2.2.*
2. *the partition function $Z_N(\mathbf{t})$ is explicitly evaluated as*

$$Z_N(\mathbf{t}) = \int_{\mathcal{H}_N} e^{\operatorname{tr} V(M;\mathbf{t})}\mathrm{d}M = \frac{\pi^{\frac{N(N-1)}{2}}}{\prod_{j=0}^{N-1} j!} h_0(\mathbf{t})\cdots h_{N-1}(\mathbf{t}), \tag{1.80}$$

with $h_j(t)$ the norming constants associated to $V(x;\mathbf{t})$, as in (1.29).

Notice that taking derivatives of the partition function (1.79) with respect to the time variables $\mathbf{t} = (t_1, t_2, \dots)$, and evaluating at $\mathbf{t} = \mathbf{0}$, exactly recovers the correlators (1.3) of the matrix model associated to $e^{V_0(\operatorname{tr} M)}$, namely

$$\left.\frac{\partial^\ell Z_N(\mathbf{t})}{\partial t_{k_1}\cdots\partial t_{k_\ell}}\right|_{\mathbf{t}=\mathbf{0}} = \left\langle \operatorname{tr} M^{k_1}\cdots \operatorname{tr} M^{k_\ell}\right\rangle. \tag{1.81}$$

Given a partition $\lambda = (\lambda_1, \dots, \lambda_\ell)$, see also Definition 2.1.1, we denote

$$\left\langle \operatorname{tr} M^{\lambda}\right\rangle := \left\langle \operatorname{tr} M^{\lambda_1}\cdots \operatorname{tr} M^{\lambda_\ell}\right\rangle. \tag{1.82}$$

In view of (1.81), the partition function can be regarded as a (formal) generating function for the correlators in the basis of monomial symmetric polynomials $t_\lambda = t_{\lambda_1}\cdots t_{\lambda_\ell}$,

$$Z_N(\mathbf{t}) := \int_{\mathcal{H}_N} \exp\left(V_0(M) + \sum_{j\geq 1} t_j \operatorname{tr} M^j\right)\mathrm{d}M = \sum_{\lambda\in\mathcal{P}} \left\langle \operatorname{tr} M^{\lambda}\right\rangle \frac{t_\lambda}{\prod_{i\geq 1} m_i!}, \tag{1.83}$$

where m_i are the parts of λ equal to i, see also Definition 2.1.1.

Remark 1.3.4 (Connected correlators)**.** *Closely related are the connected correlators, defined as*

$$\left\langle \prod_{j=1}^{\ell} \operatorname{tr} M^{\lambda_j}\right\rangle^{\mathsf{c}} = \sum_{\mathcal{P}([\ell])} (-1)^{|\mathcal{P}|-1}(|\mathcal{P}|-1)! \prod_{A\in\mathcal{P}} \left\langle \prod_{a\in A} \operatorname{tr} M^{\lambda_a}\right\rangle \tag{1.84}$$

where $\mathcal{P}([\ell])$ denotes the set of partitions of $\{1,\dots,\ell\}$. For example

$$\left\langle \operatorname{tr} M^{k_1} \right\rangle^{\mathsf{c}} := \left\langle \operatorname{tr} M^{k_1} \right\rangle, \qquad \left\langle \operatorname{tr} M^{k_1} \operatorname{tr} M^{k_2} \right\rangle^{\mathsf{c}} := \left\langle \operatorname{tr} M^{k_1} \operatorname{tr} M^{k_2} \right\rangle - \left\langle \operatorname{tr} M^{k_1} \right\rangle \left\langle \operatorname{tr} M^{k_2} \right\rangle. \tag{1.85}$$

By standard combinatorial methods [157] one can argue that connected correlators are the Taylor coefficients of the logarithm of the partition function,

$$\left. \frac{\partial^\ell \log Z_N(\mathbf{t})}{\partial t_{k_1} \cdots \partial t_{k_\ell}} \right|_{\mathbf{t}=\mathbf{0}} = \left\langle \operatorname{tr} M^{k_1} \cdots \operatorname{tr} M^{k_\ell} \right\rangle^{\mathsf{c}}. \tag{1.86}$$

and write, similarly to (1.83),

$$\log Z_N(\mathbf{t}) = \log Z_N(\mathbf{0}) + \sum_{r\geq 1} \sum_{k_1,\dots,k_r \geq 1} \frac{t_{k_1}\cdots t_{k_r}}{r!} \left\langle \operatorname{tr} M^{k_1} \cdots \operatorname{tr} M^{k_r} \right\rangle^{\mathsf{c}}. \tag{1.87}$$

Remark 1.3.5 (Negative correlators)**.** *Assuming the potential $V_0(M^{-1})$ still results in an invariant measure over $\mathcal{H}_N$, deforming it with negative exponents*

$$V(x;\mathbf{t}) = V_0(x) + \sum_{j\geq 1} t_j M^{-j}, \tag{1.88}$$

we get a different Toda tau function. Indeed it is the datum of a different underlying measure, $\mathrm{d}\widetilde{\mu}(M)$ corresponding to the change of variables $\widetilde{M} = M^{-1}$. Writing $\mathrm{d}U$ for the projection of the Lebesgue measure onto the angular variables, the Jacobian is computed as

$$\mathrm{d}M = \mathrm{d}U\, \Delta^2(\underline{x})\, \mathrm{d}\underline{x} \iff \mathrm{d}\widetilde{M} = \mathrm{d}U\, \Delta^2\left(\frac{1}{x_1},\dots,\frac{1}{x_N}\right) \mathrm{d}\left(\frac{1}{x_1}\right) \dots \mathrm{d}\left(\frac{1}{x_1}\right) = \frac{\mathrm{d}M}{\det^{2N} M} \tag{1.89}$$

so that the new measure reads

$$\mathrm{d}\widetilde{\mu}(M)\mathrm{d}\widetilde{M} = \frac{e^{\operatorname{tr} V(M^{-1};\mathbf{t})}}{\det^{2N} M} \mathrm{d}M. \tag{1.90}$$

Similarly, we can perform formal expansions of the associated partition function and compute the correlators as its logarithmic derivatives.

That the above expansions actually converge is not given for granted. More to that, partition functions related to integrable systems admit a *topological expansion*, which is an asymptotic expansion in series of N^2, with N being the size of the matrices, see [40, 75]. Part of the work exhibited in this thesis, is to give an exact characterization of the coefficients which comes with such expansions. Remarkably, in the classical ensembles, they are integer numbers counting specific combinatorial objects, see Chapter 4.

Chapter 2

Hurwitz numbers and symmetric functions

Hurwitz numbers were first introduced by Adolf Hurwitz in [114]. He was interested in the counting problem of *Riemann surfaces* with assigned ramification profiles, encoded by *partitions*, up to biholomorphic equivalence. He also realized the same question could be posed as a factorization problem in the *symmetric group*, and sketched the proof of formulæ which later reappeared in the work of Goulden and Jackson [94]. We recall some concepts and definitions in the following.

2.1 Geometric and combinatoric definition of Hurwitz numbers

Definition 2.1.1. *A partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_\ell)$ of $n \in \mathbb{N}$ is an ordered sequence of positive integers $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell$ such that*

$$\lambda_1 + \lambda_2 + \cdots + \lambda_\ell = n. \tag{2.1}$$

We shall denote by $\lambda \vdash n$ that λ is a partition of n, with $\ell(\lambda) := \ell$ the length of the partition and $|\lambda| = n$ for its weight. As an alternative notation let

$$m_i = \#\{\lambda_j = i,\ j = 1, \dots, \ell\}, \tag{2.2}$$

then the partition λ is equivalently denoted as $\lambda = (1^{m_1}, 2^{m_2}, \dots)$.

In some contexts it can be useful to identify the partition λ with its *diagram*, i.e. the set of $(i, j) \in \mathbb{Z}^2$ satisfying $1 \leq i \leq \ell(\lambda)$, $1 \leq j \leq \lambda_i$. For example, the diagram of the partition $\lambda = (4, 2, 2, 1) \vdash 9$ is depicted below.

$$\begin{array}{c|cccc} & j=1 & j=2 & j=3 & j=4 \\ \hline i=1 & \bullet & \bullet & \bullet & \bullet \\ i=2 & \bullet & \bullet & & \\ i=3 & \bullet & \bullet & & \\ i=4 & \bullet & & & \end{array} \tag{2.3}$$

Let us now give the definitions of Riemann surfaces and maps between them. We refer to the standard books for further references [139].

Definition 2.1.2. *A Riemann surface $\mathcal{S}$ is a 1-dimensional complex manifold.*

– *Two Riemann surfaces* $\mathcal{S}, \mathcal{S}'$ *are biholomorphically equivalent if there exists a map* $f : \mathcal{S} \to \mathcal{S}'$ *holomorphic, bijective and with holomorphic inverse* $f^{-1} : \mathcal{S}' \to \mathcal{S}$

– *Two holomorphic maps between Riemann surfaces,* $f : \mathcal{S} \to \mathcal{R}$ *and* $f' : \mathcal{S}' \to \mathcal{R}$ *are said to be equivalent if and only if there exists an isomorphism* $\varphi : \mathcal{S} \to \mathcal{S}'$ *such that* $f' \circ \varphi = f$

– *The automorphism group of a holomorphic map* $f : \mathcal{S} \to \mathcal{R}$ *is*

$$\mathrm{Aut} f = \{\varphi : \mathcal{S} \to \mathcal{S} \ \textit{isomorphism s.t } \ f \circ \varphi = f\} \tag{2.4}$$

– *The degree of the map* $f : \mathcal{S} \to \mathcal{R}$ *is defined as the cardinality of the fibers* $f^{-1}(r)$, $r \in \mathcal{R}$ *counting multiplicity*

Remark 2.1.3 (Riemann-Hurwitz formula)**.** *Given a non constant degree* d *holomorphic map* $f : \mathcal{S} \to \mathcal{R}$ *between Riemann surfaces of, respectively, genus* $g_{\mathcal{S}}$ *and* $g_{\mathcal{R}}$*, the Riemann-Hurwitz formula states that*

$$2\, g_{\mathcal{S}} - 2 = d\, (2\, g_{\mathcal{R}} - 2) + \sum_{x \in \mathcal{S}} (e_x - 1)\,, \tag{2.5}$$

where e_x *is the ramification index of the map* f *at the point* $x \in \mathcal{S}$*. Notice that if* f *is not ramified at* x *then* $e_x = 1$*, and the sum in* (2.5) *is finite.*

The fundamental tool to give both a geometric definition, in the first place, and hence combinatoric interpretation of Hurwitz numbers is the *Riemann Existence Theorem* [139, 152] (the definition of transitive group can be found below in Definition 2.1.6.)

Theorem 2.1.4 (Riemann Existence Theorem)**.** *Let* $\mathcal{S}$ *be a compact and connected Riemann surface, and* $\Delta \subset \mathcal{S}$ *a finite subset. Let* q *be a base point of* $\mathcal{S} \setminus \Delta$*, then there is a one-to-one correspondence between the following*

$$\left\{ \begin{array}{c} \textit{isomorphism classes of holomorphic maps} \\ \textit{of Riemann surfaces } \varphi : \mathcal{S}' \to \mathcal{S} \textit{ of} \\ \textit{degree } d \textit{ whose branch points lie in } \Delta \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} \textit{up to conjugacy, group} \\ \textit{homomorphisms } \rho : \pi_1(\mathcal{S} \setminus \Delta, q) \to \mathfrak{S}_d \\ \textit{with transitive image} \end{array} \right\} \tag{2.6}$$

with representation given by the monodromy representation.

The Riemann existence theorem tells us that every Riemann surfaces can be realized as a branched covering of the complex projective line $P^1_{\mathbb{C}}$. This allows us to define

Definition 2.1.5 (Geometric definition of Hurwitz numbers)**.** *Let* d *be a positive integer and* $\mu^{(1)}, \ldots, \mu^{(k)} \vdash d$ *partitions of* d*. Define the (connected) Hurwitz number* $h_d(\mu^{(1)}, \ldots, \mu^{(k)})$ *as*

$$h_d(\mu^{(1)}, \ldots, \mu^{(k)}) = \sum_{[f]} \frac{1}{|\mathrm{Aut} f|} \tag{2.7}$$

where the sum ranges over all equivalence classes of connected, genus g *and degree* d *covers of* $P^1_{\mathbb{C}}$ *with* k *branching points with ramification profiles* $\mu^{(1)}, \ldots, \mu^{(k)}$*. The weight of each Hurwitz cover is the inverse of the order of its automorphism group. Notice that* g *and* d *are related according to* (2.5)*.*

On the other hand, Theorem 2.1.4 states that the problem of constructing a Riemann surface with assigned branching points and ramification profiles, is equivalent to a factorization problem in $\mathfrak{S}_d$. Let us recall a few notions on the symmetric group.

Definition 2.1.6. *The symmetric group of order d, write $\mathfrak{S}_d$ is the group of permutations of the set $\{1, 2, \dots, d\}$.*

- *Every permutation $\sigma \in \mathfrak{S}_d$ is the product of disjoint cycles,*

$$\sigma = \sigma_1 \sigma_2 \dots \sigma_\ell, \qquad \sigma_i = (a_1^i \cdots a_{\ell_i}^i), \tag{2.8}$$

  their number ℓ is the length of the permutation σ, write $\ell = \ell(\sigma)$

- *The profile of a permutation $\sigma \in \mathfrak{S}_d$ as in (2.8) is the partition $\mu \vdash d$ consisting of k parts exactly equal to the lengths of the cycles σ_j of σ, namely $\mu = (\ell_1, \ell_2, \dots, \ell_k)$.*

  For example the permutation $\mathfrak{S}_8 \ni \sigma = (146)(23)(57)(8)$ has profile $\mu = (3, 2, 2, 1)$.

- *A subgroup $G \leq \mathfrak{S}_d$ is called transitive if it acts transitively on the set $\{1, 2, \dots, d\}$*

Thanks to Theorem 2.1.4, we can give an equivalent combinatorial definition of Hurwitz numbers.

Definition 2.1.7 (Combinatorial definition of Hurwitz numbers)**.** *Let d be a positive integer and $\mu^{(1)}, \dots, \mu^{(k)} \vdash d$ partitions of d. Define*

$$h_d(\mu^{(1)}, \dots, \mu^{(k)}) = \frac{1}{d!} \# \left\{ \begin{array}{c} \sigma_1, \dots, \sigma_k \in \mathfrak{S}_d \;\; \textit{such that} \;\; \sigma_1 \cdots \sigma_k = id_{\mathfrak{S}_d}, \\ \sigma_i \text{ has profile } \mu^{(i)} \;\; \textit{and} \;\; \langle \sigma_1, \dots, \sigma_k \rangle \text{ is transitive} \end{array} \right\}, \tag{2.9}$$

where $\langle \sigma_1, \dots, \sigma_k \rangle$ denotes the subgroup in $\mathfrak{S}_d$ generated by $\sigma_1, \dots, \sigma_k$.

The factor $(d!)^{-1}$ in (2.9) takes into account overcounting, and corresponds to relabelling of the sheets of the cover. Moreover, in comparing with Theorem 2.1.4, the condition $\sigma_1 \cdots \sigma_k = id_{\mathfrak{S}_d}$ is a consequence of the fact that we take the Riemann surface $\mathcal{S}$ to be compact; similarly, the condition that $\langle \sigma_1, \dots, \sigma_k \rangle$ generates the whole $\mathfrak{S}_d$ comes from the connectedness hypothesis.

Remark 2.1.8. *Disconnected Hurwitz numbers are defined in the same way, respectively dropping the connectedness hypothesis of the cover from Definition 2.1.5, and the transitivity condition on $\langle \sigma_1, \dots, \sigma_k \rangle$ from Definition 2.1.7. We shall denote them with*

$$h_d^\circ(\mu^{(1)}, \dots, \mu^{(k)}) \tag{2.10}$$

Definitions 2.1.5 and 2.1.7 are quite general, and the Hurwitz numbers they define are not trivial to compute in the general case. Some closed formulæ do exist, see e.g. (2.24) below, but they are rarely computational friendly. A majorly studied version of Hurwitz numbers is the following base case, involving a single non trivial partition.

Definition 2.1.9. *Let g, d be non negative integers and $\lambda = (\lambda_1, \dots, \lambda_\ell)$ a partition of d. The simple Hurwitz number $h_g(\lambda)$ is defined as the general Hurwitz number $h_d(\lambda, \mu^{(1)}, \dots, \mu^{(k)})$ where $\mu^{(i)} = (2, 1, 1, \dots, 1)$. Due to (2.5), there holds the relation*

$$2g - 2 = k - \ell - d. \tag{2.11}$$

In regard of Definitions 2.1.5 and 2.1.7, simple Hurwitz numbers can be defined as

- *Geometrically: correspond to covers of $\mathbb{P}^1_{\mathbb{C}}$ where a specific point, e.g. the point $z = 0$, has branching profile λ, and all the others are simple, which means they have $(d - 1)$ preimages,*

– *Combinatorially: all but one permutations are transpositions (we recall that a transposition is a permutation $\tau = (a\,b)$).*

Some solid results have been made throughout the history in the study of Simple Hurwitz numbers. Amongst the many we recall the *cut and join equation* of Goulden and Jackson [94], the *ELSV formula* [73] (see also Theorem 5.0.1), the connections with integrable systems by Okounkov, Harnad and Guay-Paquet [104, 150] as well as to matrix model by Borot et al. [38].

2.2 The symmetric group algebra

Many and various modification and specializations of Hurwitz numbers exist; in this section we introduce the necessary tools to study a vast class of them. This is done giving a *third way* to look at Hurwitz numbers in the setting of the group algebra of the symmetric group.

Definition 2.2.1. *The group algebra of the symmetric group $\mathfrak{S}_d$, denoted $\mathbb{C}[\mathfrak{S}_d]$, is the set of linear combinations with complex coefficients of elements in the symmetric group*

$$\sum_{\sigma\in\mathfrak{S}_d} c_\sigma \sigma, \qquad c_\sigma \in \mathbb{C}, \tag{2.12}$$

equipped with the sum

$$\sum_{\sigma\in\mathfrak{S}_d} c_\sigma\,\sigma + \sum_{\sigma\in\mathfrak{S}_d} d_\sigma\,\sigma = \sum_{\sigma\in\mathfrak{S}_d} (c_\sigma + d_\sigma)\sigma, \tag{2.13}$$

and multiplication induced by the symmetric group,

$$\left(\sum_{\sigma\in\mathfrak{S}_d} c_\sigma\,\sigma\right)\cdot\left(\sum_{\sigma'\in\mathfrak{S}_d} c_{\sigma'}\,\sigma'\right) = \sum_{\sigma''\in\mathfrak{S}_d} c_{\sigma''}\sigma'', \qquad c_{\sigma''} = \sum_{\substack{\sigma,\sigma'\in\mathfrak{S}_d\\ \sigma\circ\sigma'=\sigma''}} c_\sigma\cdot c_{\sigma'} \tag{2.14}$$

The center of the group algebra (i.e. the multiplicative subgroup of the group algebra consisting of elements commuting with every other element) is denoted by $Z(\mathbb{C}[\mathfrak{S}_d])$ and is called the class algebra of $\mathfrak{S}_d$.

There are some distinguished elements in the group algebra. First off, recall the definition of conjugacy class of a permutation $\sigma\in\mathfrak{S}_d$,

$$\mathrm{cyc}(\sigma) = \{\rho\sigma\rho^{-1} \;\; s.t. \;\; \rho\in\mathfrak{S}_d\}. \tag{2.15}$$

It is easy to see that all elements in $\mathrm{cyc}(\sigma)$ must have the same cycle structure of σ. In particular, the conjugacy class depends only the *profile* of σ, see Definition 2.1.6, and if σ has profile $\mu = (\mu_1,\dots,\mu_\ell)$ we will simply write $\mathrm{cyc}(\mu)$ for $\mathrm{cyc}(\sigma)$. The cardinality of $\mathrm{cyc}(\mu)$ with $\mu = (1^{m_1}, 2^{m_2}, \dots)$ is

$$|\mathrm{cyc}(\mu)| = \frac{d!}{z_\mu}, \qquad z_\mu := \prod_{j\geq 1} j^{m_j} m_j! \tag{2.16}$$

We can consider the elements in the group algebra

$$\mathcal{C}_\mu = \sum_{\sigma\in\mathrm{cyc}(\mu)} \sigma, \tag{2.17}$$

labelled by permutations $\mu \vdash d$. In particular notice that $\mathcal{C}_{1^d} = id_{\mathfrak{S}_d}$. It is well known [154] that the elements $\mathcal{C}_\mu$ form a basis for the class algebra $Z(\mathbb{C}[\mathfrak{S}_d])$. Definition (2.10), can then be reformulated as

$$h_d^\circ(\mu^{(1)}, \dots, \mu^{(k)}) = \frac{1}{d!}\,[\mathcal{C}_{1^d}] \prod_{i=1}^{k} \mathcal{C}_{\mu^{(i)}}, \tag{2.18}$$

where the operator $[\mathcal{C}_{1^d}]$ takes the coefficient of the identity element. Here we take *disconnected Hurwitz numbers*, see also Remark 2.1.8.

We would like to obtain a more accessible form of equation (2.18). To this end, the representation theory of symmetric groups is crucial: let us first recall the following basic facts of this theory [154]. Irreducible finite dimensional representations of the symmetric group $\mathfrak{S}_d$ are labelled by partitions $\lambda \vdash d$, namely for every such λ we have

$$\rho_\lambda : \mathfrak{S}_d \to GL(V_\lambda) \tag{2.19}$$

with V_λ some vector space. Define the *irreducible characters* χ^λ *of the symmetric group* as

$$\chi^\lambda = \sum_{\sigma \in \mathfrak{S}_d} \mathrm{tr}\,(\rho_\lambda(\sigma))\sigma, \tag{2.20}$$

which is an element in $\mathbb{C}[\mathfrak{S}_d]$. In particular, write χ_μ^λ for the evaluation of χ^λ at an element $\sigma \in \mathrm{cyc}(\mu)$; we are rightfully doing so since expression (2.20) is invariant by conjugation. The irreducible characters form a basis for the class algebra $Z(\mathbb{C}[\mathfrak{S}_d])$, and the following change of bases formulæ hold,

$$\chi^\lambda = \sum_{\mu \vdash d} \chi_\mu^\lambda \mathcal{C}_\mu, \qquad \Longleftrightarrow \qquad \mathcal{C}_\mu = \sum_{\lambda \vdash d} \frac{\chi_\mu^\lambda}{z_\mu}\chi^\lambda. \tag{2.21}$$

Moreover, they are orthogonal to each and idempotents according to the formulæ

$$\sum_\lambda \chi_\mu^\lambda \chi_{\mu'}^\lambda = z_\mu \delta_{\mu,\mu'} \qquad\qquad \chi^\lambda\,\chi^{\lambda'} = \delta_{\lambda,\lambda'}\frac{d!}{\dim\lambda}\chi^\lambda, \tag{2.22}$$

in particular, the latter allows to define a *basis of idempotents*, denoted $\{\mathcal{E}_\lambda\}_{\lambda\vdash d}$, as

$$\mathcal{E}_\lambda = \frac{\dim\lambda}{d!}\sum_{\mu\vdash d}\chi_\mu^\lambda \mathcal{C}_\mu, \quad \Longleftrightarrow \quad \mathcal{C}_\mu = \sum_{\mu\vdash d}\frac{d!}{\dim\lambda}\frac{\chi_\mu^\lambda}{z_\mu}\mathcal{E}_\lambda, \qquad \mathcal{E}_\lambda\mathcal{E}_{\lambda'} = \delta_{\lambda,\lambda'}\cdot\mathcal{E}_\lambda. \tag{2.23}$$

We can thus recast (2.18) in terms of the characters χ_μ^λ.

Proposition 2.2.2 (Burnside formula)**.** *Let $\mu^{(1)}, \dots, \mu^{(k)} \vdash d$, then the following holds*

$$h_d^\circ(\mu^{(1)}, \dots, \mu^{(k)}) = \frac{1}{d!}\,[\mathcal{C}_{1^d}]\prod_{i=1}^{k}\mathcal{C}_{\mu^{(i)}} = \frac{|\mathrm{cyc}(\mu^{(1)})|\cdots|\mathrm{cyc}(\mu^{(k)})|}{d!}\sum_{\lambda\vdash d}\frac{\chi_{\mu^{(1)}}^\lambda\cdots\chi_{\mu^{(k)}}^\lambda}{(\dim\lambda)^{k-2}}. \tag{2.24}$$

Proof. Rewrite (2.18), explicitly including the $\mathcal{C}_{1^d}$ factor and using (2.21), as

$$h_d^\circ(\mu^{(1)}, \dots, \mu^{(k)}) = \frac{1}{d!}\sum_{\lambda^{(0)},\lambda^{(1)},\dots,\lambda^{(k)}}\frac{\chi_{1^d}^{\lambda^{(0)}}}{d!}\frac{\chi_{\mu^{(1)}}^{\lambda^{(1)}}}{z_{\mu^{(1)}}}\cdots\frac{\chi_{\mu^{(k)}}^{\lambda^{(k)}}}{z_{\mu^{(k)}}}\left(\chi^{\lambda^{(0)}}\chi^{\lambda^{(1)}}\cdots\chi^{\lambda^{(k)}}\right)(id), \tag{2.25}$$

using repeatedly the idempotency property (2.22)

$$h_d^{\circ}(\mu^{(1)}, \dots, \mu^{(k)}) = \frac{1}{d!}\,[\mathcal{C}_{1^d}] \prod_{i=1}^{k} \mathcal{C}_{\mu^{(i)}} = \frac{1}{d!} \sum_{\lambda} \frac{\dim \lambda}{d!} \frac{\chi^{\lambda}_{\mu^{(1)}}}{z_{\mu^{(1)}}} \cdots \frac{\chi^{\lambda}_{\mu^{(k)}}}{z_{\mu^{(k)}}} \left(\frac{d!}{\dim \lambda}\right)^{k-1} \chi^{\lambda}(id), \tag{2.26}$$

and recalling (2.16) the proof is complete. □

Equation (2.24) traces back to Burnside [43] and it allows us, in principle, to compute any Hurwitz number. Indeed, characters of the symmetric group had already been studied for quite a while and their values easily computable for small d, for example via the *Murnaghan–Nakayama rule*, see [143, 157]; however, the computational complexity drastically increases as d grows [20] and it can thus be tricky to apply (2.24) in the generic case.

We introduce one last important set of elements in the group algebra, the *Young–Jucys–Murphy* (YJM) elements [121, 144] $\mathcal{J}_a$, for $a = 1, \dots, d$, defined as

$$\mathcal{J}_1 = 0, \quad \mathcal{J}_a = (1\,a) + (2\,a) + \cdots + (a-1\,a), \qquad 2 \le a \le d, \tag{2.27}$$

denoting $(a\,b)$ (with $a < b$) the transposition of $\{1, \dots, d\}$ switching a, b and fixing everything else. The following relation [121] takes place in $Z(\mathbb{C}[\mathfrak{S}_d])[\epsilon]$,

$$\prod_{a=1}^{d} (1 + \epsilon \mathcal{J}_a) = \sum_{\lambda \vdash d} \epsilon^{d - \ell(\lambda)} \mathcal{C}_{\lambda}. \tag{2.28}$$

Although singularly the YJM elements are not central, they commute amongst themselves, and *symmetric polynomials of d variables evaluated at $\mathcal{J}_1, \dots, \mathcal{J}_d$ generate $Z(\mathbb{C}[\mathfrak{S}_d])$*, in particular for any symmetric polynomial $p(y_1, \dots, y_d)$ in d variables, $p(\mathcal{J}_1, \dots, \mathcal{J}_d)$ belongs to $Z(\mathbb{C}[\mathfrak{S}_d])$. Now, central elements are diagonal on the basis of idempotents, see e.g. [154], and it is proven by Jucys in [121] that

$$p(\mathcal{J}_1, \dots, \mathcal{J}_d)\mathcal{E}_{\lambda} = p\left(\{j - i\}_{(i,j) \in \lambda}\right) \mathcal{E}_{\lambda}, \tag{2.29}$$

where in the right hand side we denote $p\left(\{j - i\}_{(i,j) \in \lambda}\right)$ the evaluation of the symmetric polynomial p at the d values of $j - i$ for $(i, j) \in \mathbb{Z}^2$ in the diagram of $\lambda \vdash d$, see Definition 2.1.1. With respect to the partition $\lambda = (4, 2, 2, 1) \vdash 9$ in (2.3), this denotes the evaluation $p(0, 1, 2, 3, -1, 0, -2, -1, -3)$.

YJM elements allow us to define in a simple way a variation of Hurwitz numbers, in the spirit of (2.18). The construction was pioneered by Guay-Paquet, Harnad and Orlov, see [26, 104, 112].

Definition 2.2.3. *Fix real parameters $\gamma_1, \dots, \gamma_L$ and $\delta_1, \dots, \delta_M$ $(L, M \ge 0)$ and collect them into the rational function*

$$G(z) := \frac{\prod_{i=1}^{L} (1 + \gamma_i z)}{\prod_{j=1}^{M} (1 - \delta_j z)}. \tag{2.30}$$

Then, the (rationally weighted) multiparametric (single) Hurwitz numbers $h_r^G(\mu)$, associated to the function G in (2.30) and labeled by the integer $r \ge 1$ and by the partition $\mu \vdash N$, are defined by

$$h_r^G(\mu) := \frac{1}{z_{\mu}} [\epsilon^r \mathcal{C}_{\mu}] \prod_{a=1}^{N} G\left(\epsilon \mathcal{J}_a\right), \tag{2.31}$$

where $[\epsilon^r \mathcal{C}_{\lambda}]$ denotes the coefficient in front of $\epsilon^r \mathcal{C}_{\lambda}$ in the expansion of $\prod_{a=1}^{N} G\left(\epsilon \mathcal{J}_a\right)$ as an element in $Z(\mathbb{C}[\mathfrak{S}_N])[[\epsilon]]$ in the basis $\{\mathcal{C}_{\lambda}\}$; to compute the expression $G\left(\epsilon \mathcal{J}_a\right) \in Z(\mathbb{C}[\mathfrak{S}_N])[[\epsilon]]$, the denominators in (2.30) are to be understood as $(1 - \delta_j z)^{-1} = \sum_{k \ge 0} \delta_j^k z^k$.

Different Hurwitz number (not necessarily *rationally weighted*) can be defined choosing a different form specifications of the function $G(z)$. For example $G(z) = 1+\sum_{j\geq 1} g_j z^j$ with $g_j = (j!)^{-1}$ returns the exponential function, which is related to simple Hurwitz numbers, see e.g. [26, 104].

Example 2.2.4 (Single weakly monotone Hurwitz numbers)**.** *Take* $G(z) = \frac{1}{1-\delta z}$*, then*

$$h_d^{G=(1+\gamma z)}(\mu) := \frac{1}{z_\mu}[\epsilon^d \mathcal{C}_\mu] \prod_{a=1}^{n} G(\epsilon \mathcal{J}_a) = \frac{1}{z_\mu}[\epsilon^d \mathcal{C}_\mu] \prod_{a=1}^{n} \frac{1}{1-\epsilon\delta\mathcal{J}_a} \tag{2.32}$$

$$= \frac{1}{z_\mu}[\epsilon^d \mathcal{C}_\mu] \left(\sum_{r\geq 0} (\epsilon\delta)^r \sum_{1\leq a_1\leq \cdots \leq a_r \leq n} \mathcal{J}_{a_1} \cdots \mathcal{J}_{a_r} \right). \tag{2.33}$$

Then, it is clear how the above Hurwitz numbers counts factorizations in the symmetric group of permutations in $\sigma \in \mathrm{cyc}(\mu)$ *in* d *weakly monotone transpositions, i.e.*

$$\sigma = \tau_1 \cdots \tau_d, \qquad \tau_i = (a_i\, b_i), \quad a_i < b_i, \quad b_1 \leq b_2 \leq \cdots \leq b_d, \tag{2.34}$$

Similarly $G(z) = (1+\gamma z)$ *generates single strictly monotone Hurwitz numbers, where the inequalities in* (2.34) *are taken to be only strict,* $b_1 < b_2 < \cdots < b_d$*.*

2.3 Hurwitz numbers and integrable systems

In this section we recall results linking Hurwitz numbers with the theory of integrable systems. In particular, we state how generating functions of Hurwitz numbers often happen to be tau-functions of some integrable hierarchies.

The groundbreaking result in this direction was obtained by Okounkov, who proved that generating functions of *double Hurwitz numbers*, defined akin the single ones of Definition 2.1.9, are tau-functions of the 2D-Toda lattice [150]. The result was then generalized to double Multiparametric Hurwitz numbers by Harnad and Guay-Paquet [104], the Theorem is below. To be self contained, we prove a weaker version in Chapter 4, Proposition 4.1.1, we prove a weaker version; relevant definitions can be found there.

Theorem 2.3.1 ([104])**.** *The generating function*

$$\tau^G(\epsilon; \mathbf{t}, \mathbf{s}) = \sum_{d\geq 1} \epsilon^d \sum_{\mu,\nu\in\mathcal{P}} h_d^G(\mu,\nu) p_\mu(\mathbf{t}) p_\nu(\mathbf{s}) \tag{2.35}$$

of multiparametric weighted double Hurwitz numbers associated to the rational function (2.30) *is a 2D-Toda tau-function. Moreover, the series* (2.35) *admits the equivalent expansion*

$$\tau_G(\epsilon; \mathbf{t}, \mathbf{s})) = \sum_{\lambda\in\mathcal{P}} r_\lambda^{(G,\epsilon)} s_\lambda(\mathbf{t}) s_\lambda(\mathbf{s}). \tag{2.36}$$

In the above, $p_\mu(\mathbf{t})$ *denote the power sum symmetric polynomial,* $s_\lambda(\mathbf{t})$ *the Schur polynomials* (4.3) *and the coefficients* $r_\lambda^{(G,\epsilon)}$ *are given explicitly by*

$$r_\lambda^{(G,\epsilon)} = \prod_{(i,j)\in\lambda} G(\epsilon(j-i)), \tag{2.37}$$

A notable question in Random matrix theory, and one of our main purposes in this work, is to find partition functions, in the sense of (1.79), whose correlators admit topological expansions in terms of Hurwitz numbers. Notably, a matrix model for simple Hurwitz number has been found only recently by Borot, Eynard, Mulase and Safnuk [38], which we report for completeness.

$$Z \propto \int_{\mathcal{H}_N(\mathcal{C})} \mathrm{d}M \exp\left(-\frac{1}{g_s}\mathrm{tr}\,(V(M) - M\,A)\right), \tag{2.38}$$

where $\mathcal{H}_N(\mathcal{C})$ is the ensemble of unitarily diagonalizable matrices with eigenvalues in a suitable contour $\mathcal{C}$ in the complex plane and $V(x)$ is the potential

$$V(x) = -\frac{x^2}{2} + g_s(N - \frac{1}{2})x + x\ln(g_s/t) + i\pi x - g_s \ln\big(\Gamma(-x/g_s)\big). \tag{2.39}$$

We refer to loc. cit. for further details. A few years later, Goulden, Guay-Paquet and Novak, [96, 97] were able to conceive the *Harish-Chandra-Itzykson-Zuber integral* [67, 111, 118]

$$I_N(z) = \int\limits_{U(N)} e^{zN\,\mathrm{tr}\,(AUBU^{-1})}\mathrm{d}U, \tag{2.40}$$

as a generating function for double monotone Hurwitz numbers.

Recently Bertola and Harnad [25] were able to construct matrix models for generic multiparametric Hurwitz numbers; however, as in the case of (2.38) and (2.40), they depend on additional parameters given by (the eigenvalues of) the matrices A and B appearing in the measures. Matrix models of this kind take the name of *external* matrix models.

In Chapter 4 we prove how the Laguerre and Jacobi partition functions provide generating functions for specific multiparametric single Hurwitz numbers; the result is different from the ones cited above as we deal with *internal* matrix models, namely we have just a measure on the space of Hermitian matrices (or restriction of it) of the form $V(M)\mathrm{d}M$ where $V(x)$ is a scalar function which may depend on additional complex parameters.

Classical Unitary Invariant Ensembles

Chapter 3

Correlators of unitary invariant ensembles

In this chapter we provide generating functions for the *connected correlators* of unitary invariant ensembles with a regular enough potential and explicitly compute them for the classical unitary invariant ensembles.

The results of the former are contained in Section 3.1, specifically Theorem 3.1.1. Formulæ of this sort for correlators of hermitian matrix models have been recently discussed in the literature, see e.g. [63, 78]. They are directly related to the theory of tau functions (formal [63] and isomonodromic [24, 92]) and to topological recursion theory [18, 19, 46, 79], incidentally, similar formulæ also appear for matrix models with external source [21–23, 27, 129]. Generating functions for correlators in the free probability setting were recently presented in [36]. We provide a direct derivation based on the Riemann–Hilbert characterization of the matrix $Y_N(z)$ outlined in Section 1.2. In Section 3.1.1 we give the ideas of how this approach could be extended to the case of discrete and multiple orthogonal polynomials.

In Section 3.2 we explicitly compute the just found generating functions in the case of the classical potentials introduced in Section 1.2.1, namely the Gaussian, Laguerre and Jacobi unitary ensemble. The GUE case had already been dealt with by Dubrovin and Yang [63], while the LUE and JUE were tackled in [92, 93]. Formulæ of different kind for multipoint correlators of classical ensembles have been recently derived by Jonnadula, Keating, and Mezzadri in [120] via the connection with the theory of multivariate orthogonal polynomials.

It is a nontrivial observation that these generating functions can be expressed via discrete orthogonal polynomials; this has first been proved for one-point correlators $\left\langle \operatorname{tr} M^k \right\rangle$ for all three classical unitary invariant ensembles by Cunden, Mezzadri, O'Connell and Simm [52]. It is worth noting that in loc. cit. similar results have been proved also for one-point correlators of the orthogonal and symplectic ensembles with classical potentials.

3.1 Analytic generating functions for correlators

We consider the general case of a measure on $\mathcal{H}_N(I)$, the space of Hermitian matrices with eigenvalues in the interval I, of the form

$$\mathrm{d}m_N(X) = \frac{1}{C_N} \exp \operatorname{tr} V(X) \mathrm{d}X, \tag{3.1}$$

with normalizing constant $C_N = \int_{\mathcal{H}_N(I)} \exp \operatorname{tr} V(X) \mathrm{d}X$. Here $V(x)$ is a smooth function of $x \in I^\circ$ (the interior of I) and we assume that $\exp V(x) = \mathcal{O}\left(|x - x_0|^{-1+\varepsilon}\right)$ for some $\varepsilon > 0$ as $x \in I^\circ$ approaches a finite endpoint x_0 of I; further, if I extends to $\pm\infty$ we assume that $V(x) \to -\infty$ fast enough as $x \to \pm\infty$ in order for the measure (3.1) to have finite moments of all orders, so that the associated orthogonal polynomials exist, recall Definition 1.2.1.

Introduce the *moment functions*

$$\mathscr{C}_\ell(z_1, \dots, z_\ell) := \int_{\mathcal{H}_N(I)} \prod_{i=1}^{\ell} \operatorname{tr}\left[(z_i - X)^{-1}\right] \mathrm{d}m_N(X), \qquad \ell \geq 1, \tag{3.2}$$

which are analytic functions of $z_1, \dots, z_\ell \in \mathbb{C} \setminus I$, symmetric in the variables $z_1, \dots, z_\ell$. To simplify the analysis it is convenient to introduce their *connected version* which take the name of *cumulant functions*,

$$\mathscr{C}_\ell^{\mathsf{c}}(z_1, \dots, z_\ell) = \sum_{\mathcal{P} \text{ partition of } \{1,\dots,\ell\}} (-1)^{|\mathcal{P}|-1} (|\mathcal{P}| - 1)! \prod_{A \in \mathcal{P}} \mathscr{C}_{|A|}(\{z_a\}_{a \in A}), \tag{3.3}$$

from which the moments can be recovered by

$$\mathscr{C}_\ell(z_1, \dots, z_\ell) = \sum_{\mathcal{P} \text{ partition of } \{1,\dots,\ell\}} \prod_{A \in \mathcal{P}} \mathscr{C}_{|A|}^{\mathsf{c}}(\{z_a\}_{a \in A}). \tag{3.4}$$

For example, $\mathscr{C}_1(z) = \mathscr{C}_1^{\mathsf{c}}(z)$, $\mathscr{C}_2^{\mathsf{c}}(z_1, z_2) = \mathscr{C}_2(z_1, z_2) - \mathscr{C}_1(z_1)\mathscr{C}_1(z_2)$,

$$\begin{aligned}\mathscr{C}_3^{\mathsf{c}}(z_1, z_2, z_3) = \mathscr{C}_3(z_1, z_2, z_3) &- \mathscr{C}_2(z_1, z_2)\mathscr{C}_1(z_3) - \mathscr{C}_2(z_2, z_3)\mathscr{C}_1(z_1) \\ &- \mathscr{C}_2(z_1, z_3)\mathscr{C}_1(z_2) + 2\,\mathscr{C}_1(z_1)\mathscr{C}_1(z_2)\mathscr{C}_1(z_3).\end{aligned} \tag{3.5}$$

We now want to express the cumulant functions in terms of the monic orthogonal polynomials $P_\ell(z) = z^\ell + \dots$ uniquely defined by

$$\int_I P_\ell(x) P_m(x) \mathrm{e}^{V(x)} \mathrm{d}x = h_\ell \delta_{\ell,m}, \tag{3.6}$$

and of the 2×2 matrix $Y_N(z)$ solution to the *Riemann–Hilbert problem* of orthogonal polynomials, see Section 1.2 and equation (1.40) which we hereby recall,

$$Y_N(z) := \begin{pmatrix} P_N(z) & \widehat{P}_N(z) \\ -\frac{2\pi\mathrm{i}}{h_{N-1}} P_{N-1}(z) & -\frac{2\pi\mathrm{i}}{h_{N-1}} \widehat{P}_{N-1}(z) \end{pmatrix}, \tag{3.7}$$

together with its defining properties,

1. $Y_N(z)$ has a jump on the real axis given by

$$Y_{N,+}(z) = Y_{N,-}(z) \begin{pmatrix} 1 & \mathrm{e}^{V(z)} \\ 0 & 1 \end{pmatrix}, \qquad z \in I, \tag{3.8}$$

2. for $z \to \infty$ it has the behaviour

$$Y_N(z) = \left(\mathbf{1} + \mathcal{O}(z^{-1})\right) z^{N\sigma_3}, \tag{3.9}$$

3. at any endpoint z_0 of I, for $z \to z_0$

$$Y_N(x) = \mathcal{O}\left(\log(z - z_0)\right). \tag{3.10}$$

The result is the following.

Theorem 3.1.1. *Let*

$$R(z) := Y_N(z) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} Y_N^{-1}(z), \tag{3.11}$$

with $Y_N(z)$ as in (3.7). Then, the cumulant functions (3.3) are given by

$$\mathscr{C}_1^{\mathsf{c}}(z) = \left(Y_N^{-1}(z) Y_N'(z)\right)_{1,1}, \tag{3.12}$$

$$\mathscr{C}_2^{\mathsf{c}}(z_1, z_2) = \frac{\mathrm{tr}\ \left(R(z_1)R(z_2)\right) - 1}{(z_1 - z_2)^2}, \tag{3.13}$$

$$\mathscr{C}_\ell^{\mathsf{c}}(z_1, \dots, z_\ell) = - \sum_{(i_1, \dots, i_\ell) \in \mathrm{cyc}((\ell))} \frac{\mathrm{tr}\ \left(R(z_{i_1}) \dots R(z_{i_\ell})\right)}{(z_{i_1} - z_{i_2}) \cdots (z_{i_\ell} - z_{i_1})}, \quad \ell \geq 3, \tag{3.14}$$

where prime in the second formula denotes derivative with respect to z and $\mathrm{cyc}((\ell))$ in the last formula is the set of ℓ-cycles in the symmetric group $\mathfrak{S}_\ell$.

We give two remarks: in the first we explain how generating functions for positive and negative correlators are related to formulæ of Theorem 3.1.1; in the second one we point out how, in the same way, is possible to obtain generating functions for *mixed correlators*.

Remark 3.1.2. *When expanded at $z = \infty$, the cumulant functions serve as generating functions for connected correlators, namely*

$$\mathscr{C}_1(z) \overset{z \to \infty}{\sim} \mathscr{F}_{1,\infty}^{\mathsf{c}}(z) - \frac{N}{z}, \qquad \mathscr{C}_\ell^{\mathsf{c}}(z_1, \dots, z_\ell) \overset{z \to \infty}{\sim} \mathscr{F}_{\ell,\infty}^{\mathsf{c}}(z_1, \dots, z_\ell), \tag{3.15}$$

where

$$\mathscr{F}_{\ell,\infty}^{\mathsf{c}}(z_1, \dots z_\ell) := \sum_{k_1, \dots, k_\ell \geq 1} \frac{\left\langle \prod_{j=1}^{\ell} \mathrm{tr}\, X^{k_j} \right\rangle^{\mathsf{c}}}{z_1^{k_1+1} \cdots z_\ell^{k_\ell+1}}. \tag{3.16}$$

Similarly, when expanded at $z = 0$ they yields the negative correlators, see Remark 1.3.5,

$$\mathscr{C}_1(z) \overset{z \to 0}{\sim} \mathscr{F}_{1,0}^{\mathsf{c}}(z), \qquad \mathscr{C}_\ell^{\mathsf{c}}(z_1, \dots, z_\ell) \overset{z \to 0}{\sim} \mathscr{F}_{\ell,0}^{\mathsf{c}}(z_1, \dots, z_\ell), \tag{3.17}$$

where

$$\mathscr{F}_{\ell,0}^{\mathsf{c}}(z_1, \dots, z_\ell) := (-1)^\ell \sum_{k_1, \dots, k_\ell \geq 1} \left\langle \prod_{j=1}^{\ell} \mathrm{tr}\, X^{-k_j} \right\rangle^{\mathsf{c}} z_1^{k_1-1} \cdots z_\ell^{k_\ell-1}. \tag{3.18}$$

The same formulæ hold for the disconnected counterparts.

Remark 3.1.3. *We could consider more general generating functions as follows; take $\ell_1,\dots,\ell_p\geq 0$ with $\sum_{i=1}^p \ell_i>0$ and expand the cumulant function*

$$\mathscr{C}_{\ell_1+\dots+\ell_p}(z_1^{(1)},\dots,z_{\ell_1}^{(1)},\dots,z_1^{(p)},\dots,z_{\ell_p}^{(p)}) \tag{3.19}$$

as $z_j^{(1)}\to x_1,\dots,z_j^{(p)}\to x_p$ where $x_1,\dots,x_p$ are endpoints of the support of $V(x)$. For example, for the Jacobi measure $V(x)=\alpha\log x+\beta\log(1-x)$, see also Section 3.2.3 below, taking $(x_1,x_2,x_3)=(\infty,0,1)$ we obtain generating functions

$$\sum_{\substack{k_1,\dots,k_q\geq 1\\ i_1,\dots,i_r\geq 1\\ j_1,\dots,j_s\geq 1}}\int_{\mathcal{H}_N(0,1)}\mathrm{tr}\,X^{k_1}\cdots\mathrm{tr}\,X^{k_q}\mathrm{tr}\,X^{-i_1}\cdots\mathrm{tr}\,X^{-i_r}\mathrm{tr}\,(\mathbf{1}-X)^{j_1}\cdots\mathrm{tr}\,(\mathbf{1}-X)^{j_s}\mathrm{d}m_N^{\mathsf{J}}(X)$$
$$\times\frac{w_1^{i_1-1}\cdots w_r^{i_r-1}(y_1-1)^{j_1-1}\cdots(y_s-1)^{j_s-1}}{z_1^{k_1+1}\cdots z_q^{k_q+1}}. \tag{3.20}$$

It is then clear that we can compute the coefficients of such series in terms of the asymptotic series $R^{[\infty]},R^{[0]},R^{[1]}$, of the matrix $R(z)$ of JUE at $(\infty,0,1)$.

Proof of Theorem 3.1.1

The strategy for the proof of Theorem 3.1.1 is based on the observation that setting

$$\mathscr{Z}_N(t,z):=\int_{\mathcal{H}_N(I)}\exp\big(\mathrm{tr}\,(V(X)+\sum_{i=1}^{\ell}t_i(z_i-X)^{-1})\big)\mathrm{d}X,\quad t=(t_1,\dots,t_\ell),\ z=(z_1,\dots,z_\ell), \tag{3.21}$$

we have

$$\mathscr{C}_\ell^{\mathsf{c}}(z_1,\dots,z_\ell)=\frac{\partial^\ell}{\partial t_1\cdots\partial t_\ell}\log\mathscr{Z}(t,z)\bigg|_{t_i=0}. \tag{3.22}$$

Here and below it is assumed that $z_i\notin I$. Let us recall here for convenience a couple of formulæ from Chapter 1. In particular, the Christoffel–Darboux identity (1.35)

$$K_N(x,y):=\mathrm{e}^{\frac{V(x)+V(y)}{2}}\sum_{i=0}^{N-1}\frac{P_i(x)P_i(y)}{h_i}=\frac{\mathrm{e}^{\frac{V(x)+V(y)}{2}}}{h_{N-1}}\frac{P_N(x)P_{N-1}(y)-P_{N-1}(x)P_N(y)}{x-y}, \tag{3.23}$$

which can be conveniently rewritten in terms of the matrix $Y_N(z)$ in (3.7) as

$$K_N(x,y)=-\frac{\mathrm{e}^{\frac{V(x)+V(y)}{2}}}{2\pi\mathrm{i}(x-y)}\begin{pmatrix}0&1\end{pmatrix}Y_N^{-1}(x)Y_N(y)\begin{pmatrix}1\\0\end{pmatrix}, \tag{3.24}$$

which is independent of the choice of boundary value of Y_N because of the jump condition (3.8), namely because of the jump matrix being upper triangular,

$$\begin{pmatrix}0&1\end{pmatrix}Y_{N,+}^{-1}(x)Y_{N,+}(y)\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}0&1\end{pmatrix}\begin{pmatrix}1&\mathrm{e}^{V(x)}\\0&1\end{pmatrix}Y_{N,-}^{-1}(x)Y_{N,-}(y)\begin{pmatrix}1&-\mathrm{e}^{V(y)}\\0&1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} \tag{3.25}$$
$$=\begin{pmatrix}0&1\end{pmatrix}Y_{N,-}^{-1}(x)Y_{N,-}(y)\begin{pmatrix}1\\0\end{pmatrix}. \tag{3.26}$$

Next, we recall the expression (1.80) in order to expresses the partition function (3.21) as a product of the norming constants. Here we use a slightly different perturbation of the potential $V(x)$ but the observations made in Proposition 1.3.3 stand still. In particular, introducing

$$V_{t,z}(x) := V(x) + \sum_{i=1}^{\ell} \frac{t_i}{z_i - x}, \qquad t = (t_1, \dots, t_\ell), \quad z = (z_1, \dots, z_\ell), \tag{3.27}$$

and, for convenience, writing explicitly the dependence of $P_\ell = P_\ell^V$ and $h_\ell = h_\ell^V$ on the potential V, we have

$$\mathscr{Z}_N(t, z) = N! h_0^{V_{t,z}} \cdots h_{N-1}^{V_{t,z}}. \tag{3.28}$$

Since our final goal is to be able to give generating functions for the (logarithmic) time derivatives of $\mathscr{Z}_N(t, z)$, it will be convenient to analyse such operation with respect to expression (3.28).

Lemma 3.1.4. *We have*

$$\partial_{t_j} h_i^{V_{t,z}} = \int_I \left(P_i^{V_{t,z}}(x)\right)^2 \mathrm{e}^{V_{t,z}(x)} \frac{\mathrm{d}x}{z_j - x}. \tag{3.29}$$

Proof. We have $h_i^{V_{t,z}} = \int_I \left(P_i^{V_{t,z}}(x)\right)^2 \mathrm{e}^{V_{t,z}(x)} \mathrm{d}x$ hence

$$\partial_{t_j} h_i^{V_{t,z}} = 2 \int_I P_i^{V_{t,z}}(x) \left(\partial_{t_j} P_i^{V_{t,z}}(x)\right) \mathrm{e}^{V_{t,z}(x)} \mathrm{d}x + \int_I \left(P_i^{V_{t,z}}(x)\right)^2 \mathrm{e}^{V_{t,z}(x)} \left(\partial_{t_j} V_{t,z}(x)\right) \mathrm{d}x\,, \tag{3.30}$$

but the first term vanishes by orthogonality because $P_i^{V_{t,z}}(x)$ are normalized to be monic and, therefore, $\partial_{t_j} P_i^{V_{t,z}}(x)$ is a polynomial of degree strictly less than i. □

We now begin the proof of Theorem 3.1.1. We will proceed by induction on ℓ.

Case $\ell = 1$

It follows from (3.24) that

$$K_N(x, x) = \lim_{y \to x} K_N(x, y) = \frac{\mathrm{e}^{V(x)}}{2\pi\mathrm{i}} \begin{pmatrix} 0 & 1 \end{pmatrix} Y_N^{-1}(x) Y_N'(x) \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{3.31}$$

In the following we shall use the notation

$$\Delta f(x) := f_+(x) - f_-(x), \qquad x \in I^\circ, \tag{3.32}$$

for the jump of a function f across I, namely $f_\pm(x) := \lim_{\epsilon \to 0_+} f(x \pm \mathrm{i}\epsilon)$. The next preliminary lemma is well known, see e.g. [49], and it is proven here for the reader's convenience.

Lemma 3.1.5. *We have*

$$K_N(x, x) = -\frac{1}{2\pi\mathrm{i}} \Delta \left[\mathrm{tr}\left(Y_N^{-1}(x) \frac{\partial Y_N(x)}{\partial x} \mathrm{E}_{1,1}\right)\right], \qquad \mathrm{E}_{1,1} := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{3.33}$$

Proof. Let us denote $' := \partial_x$. It follows from the jump condition (3.8) for Y_N that

$$Y_{N,+}'(x) = Y_{N,-}'(x) \begin{pmatrix} 1 & \mathrm{e}^{V(x)} \\ 0 & 1 \end{pmatrix} + Y_{N,-}(x) \begin{pmatrix} 0 & V'(x)\mathrm{e}^{V(x)} \\ 0 & 0 \end{pmatrix}, \qquad x \in I^\circ. \tag{3.34}$$

Therefore we compute

$$\begin{aligned}\Delta\left[\mathrm{tr}\,\left(Y_N^{-1}(x)Y_N'(x)\mathrm{E}_{1,1}\right)\right] &= \mathrm{tr}\,\left(Y_{N,+}^{-1}(x)Y_{N,+}'(x)\mathrm{E}_{1,1}\right) - \mathrm{tr}\,\left(Y_{N,-}^{-1}(x)Y_{N,-}'(x)\mathrm{E}_{1,1}\right)\\ &= \mathrm{tr}\,\left(\begin{pmatrix}1 & -\mathrm{e}^{V(x)}\\ 0 & 1\end{pmatrix}Y_{N,-}^{-1}(x)Y_{N,-}'(x)\begin{pmatrix}1 & \mathrm{e}^{V(x)}\\ 0 & 1\end{pmatrix}\mathrm{E}_{1,1}\right) - \mathrm{tr}\,\left(Y_{N,-}^{-1}(x)Y_{N,-}'(x)\mathrm{E}_{1,1}\right)\\ &\qquad + \mathrm{tr}\,\left(\begin{pmatrix}1 & -\mathrm{e}^{V(x)}\\ 0 & 1\end{pmatrix}\begin{pmatrix}0 & V'(x)\mathrm{e}^{V(x)}\\ 0 & 0\end{pmatrix}\mathrm{E}_{1,1}\right). \end{aligned} \tag{3.35}$$

The last term vanishes and so, by the cyclic property of the trace, we have

$$\Delta\left[\mathrm{tr}\,\left(Y_N^{-1}(x)Y_N'(x)\mathrm{E}_{1,1}\right)\right] = \mathrm{tr}\,\left[Y_{N,-}^{-1}(x)Y_{N,-}'(x)\left(\begin{pmatrix}1 & \mathrm{e}^{V(x)}\\ 0 & 1\end{pmatrix}\mathrm{E}_{1,1}\begin{pmatrix}1 & -\mathrm{e}^{V(x)}\\ 0 & 1\end{pmatrix} - \mathrm{E}_{1,1}\right)\right] \tag{3.36}$$

which is easily seen to be equivalent, up to multiplying by $-1/(2\pi\mathrm{i})$, to (3.31). □

We are ready for the proof of the case $\ell = 1$. In such case, $t = t_1, z = z_1$ and $V_{t,z}(x) = V(x) + t/(z-x)$. By (3.28), Lemma 3.1.4, and (3.23), we have

$$\partial_t \log \mathscr{Z}_N(t,z) = \sum_{i=0}^{N-1}\frac{1}{h_i^{V_{t,z}}}\partial_t h_i^{V_{t,z}} = \sum_{i=0}^{N-1}\frac{1}{h_i^{V_{t,z}}}\int_I \left(P_i^{V_{t,z}}(x)\right)^2 \mathrm{e}^{V_{t,z}(x)}\frac{\mathrm{d}x}{z-x} = \int_I K_N^{V_{t,z}}(x,x)\frac{\mathrm{d}x}{z-x}, \tag{3.37}$$

where we denote explicitly the dependence of the Christoffel–Darboux kernel on the potential. Let Γ be an oriented contour in the complex plane which surrounds I in counterclockwise sense (i.e. I lies on the left of Γ) and leaves z outside (i.e. z lies to the right of Γ). Then, in virtue of Lemma 3.1.5 we get

$$\begin{aligned}\partial_t \log \mathscr{Z}_N(t,z) &= -\int_I \Delta\left[\mathrm{tr}\,\left(Y_N^{-1}(x;t,z)\frac{\partial Y_N(x;t,z)}{\partial x}\mathrm{E}_{1,1}\right)\right]\frac{\mathrm{d}x}{2\pi\mathrm{i}(z-x)}\\ &= \int_\Gamma \mathrm{tr}\,\left(Y_N^{-1}(x;t,z)\frac{\partial Y_N(x;t,z)}{\partial x}\mathrm{E}_{1,1}\right)\frac{\mathrm{d}x}{2\pi\mathrm{i}(z-x)}, \end{aligned} \tag{3.38}$$

where $Y_N(\cdot;t,z)$ is the matrix (3.7) for the potential $V_{t,z}$. The last contour integral can be evaluated by a residue computation as

$$\partial_t \log \mathscr{Z}_N(t,z) = \left(-\operatorname*{res}_{x=z} - \operatorname*{res}_{x=\infty}\right)\mathrm{tr}\,\left(Y_N^{-1}(x;t,z)\frac{\partial Y_N(x;t,z)}{\partial x}\mathrm{E}_{1,1}\right)\frac{\mathrm{d}x}{z-x}. \tag{3.39}$$

It can be checked from the normalization condition at $x = \infty$ for $Y_N(x;t,z)$, equation (3.9), that the residue at $x = \infty$ vanishes. Therefore

$$\partial_t \log \mathscr{Z}_N(t,z) = \mathrm{tr}\,\left(Y_N^{-1}(z;t,z)\left.\frac{\partial Y_N(x;t,z)}{\partial x}\right|_{x=z}\mathrm{E}_{1,1}\right). \tag{3.40}$$

Evaluating this identity at $t = 0$, taking into account (3.22), we obtain exactly (3.12), namely

$$\mathscr{C}_1^{\mathsf{c}}(z) = \left(Y_N^{-1}(z)Y_N'(z)\right)_{1,1}. \tag{3.41}$$

Case $\ell = 2$

Let us first formulate a result that will be needed for all $\ell \geq 2$.

Lemma 3.1.6. *Let*

$$R(x; z, t) := Y_N(x; t, z)\mathrm{E}_{1,1}Y_N^{-1}(x; t, z), \tag{3.42}$$

and

$$V_{t,z}(x) = V(x) + \sum_{i=1}^{\ell} \frac{t_i}{z_i - x}, \tag{3.43}$$

where $t = (t_1, \ldots, t_\ell)$*, and* $z = (z_1, \ldots, z_\ell)$*. For all* $1 \leq j \leq \ell$*, we have*

$$\frac{1}{z_j - x}R(x; t, z) + \left(\frac{\partial Y_N}{\partial t_j}(x; t, z)\right) Y_N^{-1}(x; t, z) = \frac{1}{z_j - x}R(z_j; t, z). \tag{3.44}$$

Proof. Let us denote by $\Omega_j(x; t, z)$ the left-hand side of (3.44). Using (3.8) we get the identities

$$Y_N(x_+; t, z) = Y_N(x_-; t, z)\begin{pmatrix} 1 & \mathrm{e}^{V_{t,z}(x)} \\ 0 & 1 \end{pmatrix}, \tag{3.45}$$

$$\frac{\partial Y_N}{\partial t_j}(x_+; t, z) = \frac{\partial Y_N}{\partial t_j}(x_-; t, z)\begin{pmatrix} 1 & \mathrm{e}^{V_{t,z}(x)} \\ 0 & 1 \end{pmatrix} + Y_N(x_-; t, z)\begin{pmatrix} 0 & \frac{1}{z_j - x}\mathrm{e}^{V_{t,z}(x)} \\ 0 & 0 \end{pmatrix}, \tag{3.46}$$

from which we readily ascertain that $\Delta\Omega_j(x; t, z) = 0$ for all $x \in \mathbb{R}$. Hence, $\Omega_j(x; t, z)$ is a meromorphic function of x with a single simple pole at $x = z_j$ and which vanishes at $x = \infty$, because of (3.9), and so the statement follows. $\square$

Let us consider the case $\ell = 2$, in which $t = (t_1, t_2)$, $z = (z_1, z_2)$, and $V_{t,z}(x) = V(x) + \frac{t_1}{z_1 - x} + \frac{t_2}{z_2 - x}$. By the argument used for $\ell = 1$, cf. (3.40), we obtain

$$\partial_{t_1} \log \mathscr{Z}_N(t, z) = \mathrm{tr}\left(Y_N^{-1}(z_1; t, z)\left.\frac{\partial Y_N(x; t, z)}{\partial x}\right|_{x=z_1}\mathrm{E}_{1,1}\right). \tag{3.47}$$

Next we have to take a derivative in t_2: omitting the explicit dependence on t, z, we have

$$\partial_{t_2}\partial_{t_1} \log \mathscr{Z}_N(t, z) = \mathrm{tr}\left(-Y_N^{-1}(z_1)\frac{\partial Y_N}{\partial t_2}(z_1)Y_N^{-1}(z_1)\left.\frac{\partial Y_N(x)}{\partial x}\right|_{x=z_1}\mathrm{E}_{1,1} + Y_N^{-1}(z_1)\left.\frac{\partial^2 Y_N(x)}{\partial t_2 \partial x}\right|_{x=z_1}\mathrm{E}_{1,1}\right). \tag{3.48}$$

We use (3.44) to rewrite the first term inside the trace in the right-hand side as

$$-Y_N^{-1}(z_1)\frac{R(z_2) - R(z_1)}{z_2 - z_1}\left.\frac{\partial Y_N(x)}{\partial x}\right|_{x=z_1}\mathrm{E}_{1,1} \tag{3.49}$$

and the second term as

$$\begin{aligned}
Y_N^{-1}(z_1)\left.\frac{\partial^2 Y_N(x)}{\partial x \partial t_2}\right|_{x=z_1}\mathrm{E}_{1,1} &= Y_N^{-1}(z_1)\frac{\partial}{\partial x}\left.\left(\frac{R(z_2) - R(x)}{z_2 - x}Y_N(x)\right)\right|_{x=z_1}\mathrm{E}_{1,1} \\
&= Y_N^{-1}(z_1)\Bigg(\frac{R(z_2) - R(z_1)}{(z_2 - z_1)^2}Y_N(z_1) - \frac{\left[\left.\frac{\partial Y_N(x)}{\partial x}\right|_{x=z_1}Y_N^{-1}(z_1), R(z_1)\right]}{z_2 - z_1}Y_N(z_1) \\
&\qquad + \frac{R(z_2) - R(z_1)}{z_2 - z_1}\left.\frac{\partial Y_N(x)}{\partial x}\right|_{x=z_1}\Bigg)\mathrm{E}_{1,1},
\end{aligned}$$

where $[A,B] := AB - BA$ is the commutator. The term in the last row exactly cancels with (3.49), and so, rearranging terms

$$\partial_{t_2}\partial_{t_1}\log \mathscr{Z}_N(t,z) = \frac{\mathrm{tr}\,\big(R(z_1)R(z_2)\big) - 1}{(z_2-z_1)^2} + \frac{\mathrm{tr}\left(\left[Y_N^{-1}(z_1)\frac{\partial Y_N(x)}{\partial x}\Big|_{x=z_1}, \mathrm{E}_{1,1}\right]\mathrm{E}_{1,1}\right)}{z_2-z_1} \tag{3.50}$$

and, since $\mathrm{tr}\,([A,B]B) = \mathrm{tr}\,([AB,B]) = 0$, the proof of the case $\ell = 2$ is completed by setting $t_1 = t_2 = 0$. This yields formula (3.13).

Case $\ell \geq 3$

Let us denote

$$S_\ell(z_1,\dots,z_\ell;t) := -\sum_{(i_1,\dots,i_\ell)\in\mathrm{cyc}((\ell))} \frac{\mathrm{tr}\,\big(R(z_{i_1};t,z)\cdots R(z_{i_\ell};t,z)\big)}{(z_{i_1}-z_{i_2})\cdots(z_{i_{\ell-1}}-z_{i_\ell})(z_{i_\ell}-z_{i_1})} - \frac{\delta_{\ell,2}}{(z_1-z_2)^2}, \tag{3.51}$$

where the sum extends over cyclic permutations of $\{1,\dots,\ell\}$. We aim at proving that

$$\frac{\partial^\ell \log \mathscr{Z}_N(t,z)}{\partial t_\ell\cdots\partial t_1} = S_\ell(z_1,\dots,z_\ell;t). \tag{3.52}$$

where $Y_N(x;t,z)$, and so $R(x;t,z)$, are computed for the potential $V_{t,z}(x) = V(x) + \sum_{i=1}^{\ell}\frac{t_i}{z_i-x}$. Then, the seeked formula (3.14) follows by taking $t_i = 0$. The proof of (3.52) is by induction on $\ell \geq 2$ and it is similar in spirit to that in [22, 23, 92].

Let us assume (3.52) for ℓ and let us prove it for $\ell+1$, the base case having been established in the previous section. Since the potential V is arbitrary, we can assume (3.52) holds true for $V_{t,z}(x) = V(x) + \sum_{j=1}^{\ell+1}\frac{t_j}{z_j-x}$, and so we just have to show that

$$\partial_{t_{\ell+1}} S_\ell(z_1,\dots,z_\ell;t) = S_{\ell+1}(z_1,\dots,z_\ell,z_{\ell+1};t). \tag{3.53}$$

To this end we first observe that by (3.44) we have

$$\frac{\partial R(x;t,z)}{\partial t_j} = \left[\frac{R(z_j;t,z) - R(x;t,z)}{z_j - x}, R(x;t,z)\right] = \frac{[R(z_j;t,z), R(x;t,z)]}{z_j-x}, \tag{3.54}$$

and therefore

$$\frac{\partial S_\ell(z_1,\dots,z_\ell;t)}{\partial t_{\ell+1}} = -\sum_{(i_1,\dots,i_\ell)\in\mathrm{cyc}((\ell))}\sum_{j=1}^{\ell} \frac{\mathrm{tr}\,\big(R(z_{i_1};t,z)\cdots[R(z_{\ell+1};t,z), R(z_{i_j};t,z)]\cdots R(z_{i_\ell};t,z)\big)}{(z_{i_1}-z_{i_2})\cdots(z_{i_\ell}-z_{i_1})(z_{\ell+1}-z_{i_j})}. \tag{3.55}$$

Expanding the commutator

$$[R(z_{\ell+1};t,z), R(z_{i_j};t,z)] = R(z_{\ell+1};t,z)R(z_{i_j};t,z) - R(z_{i_j};t,z)R(z_{\ell+1};t,z),$$

we note that, in the previous sum, each term involving the expression

$$\mathrm{tr}\,\big(R(z_{i_1};t,z)\cdots R(z_{\ell+1};t,z)R(z_{i_j};t,z)\cdots R(z_{i_\ell};t,z)\big) \tag{3.56}$$

appears exactly twice, but with different denominators. Collecting such terms yields

$$\begin{aligned}
&\sum_{(i_1,\dots,i_\ell)\in\mathrm{cyc}((\ell))}\sum_{j=1}^{\ell}\frac{\mathrm{tr}\,\left(R(z_{i_1};t,z)\cdots R(z_{\ell+1};t,z)R(z_{i_j};t,z)\cdots R(z_{i_\ell};t,z)\right)}{(z_{i_1}-z_{i_2})\cdots(z_{i_\ell}-z_{i_1})}\left(\frac{1}{z_{i_j}-z_{\ell+1}}-\frac{1}{z_{i_{j-1}}-z_{\ell+1}}\right)\\
&\quad=-\sum_{(i_1,\dots,i_\ell)\in\mathrm{cyc}((\ell))}\sum_{j=1}^{\ell}\frac{\mathrm{tr}\,\left(R(z_{i_1};t,z)\cdots R(z_{\ell+1};t,z)R(z_{i_j};t,z)\cdots R(z_{i_\ell};t,z)\right)}{(z_{i_1}-z_{i_2})\cdots(z_{i_{j-1}}-z_{\ell+1})(z_{\ell+1}-z_{i_j})\cdots(z_{i_\ell}-z_{i_1})}\\
&\quad=S_{\ell+1}(z_1,\dots,z_\ell,z_{\ell+1}),
\end{aligned}\tag{3.57}$$

where we set $i_0 := i_\ell$ in the internal summation. The proof is complete.

Notice that the functions $S_\ell(z_1,\dots,z_\ell)$ are regular along the diagonals $z_i = z_j$. In the case $\ell = 2$ this can be seen from the fact that

$$\mathrm{tr}\,(R^2(z))\equiv 1 \tag{3.58}$$

hence the function $\mathrm{tr}\,(R(z_1)R(z_2))-1$ is symmetric in z_1 and z_2 and vanishes for $z_1 = z_2$. Therefore the zero on the diagonal $z_1 = z_2$ is of order at least 2 and so $S_2(z_1, z_2)$ is regular for $z_1 = z_2$. For $\ell \geq 3$ instead we can reason as follows; since S_ℓ is manifestly symmetric, we can focus on the case $z_{\ell-1} = z_\ell$ and keeping only the summands in S_ℓ which are singular for $z_{\ell-1} = z_\ell$ gives

$$\sum_{(i_1,\dots,i_{\ell-2})\in\mathrm{cyc}((\ell-2))}\frac{\mathrm{tr}\,\left(R(z_{\ell-1})R(z_\ell)R(z_{i_1})\cdots R(z_{i_{\ell-2}})\right)}{(z_{\ell-1}-z_\ell)(z_\ell-z_{i_1})\cdots(z_{i_{\ell-2}}-z_{\ell-1})}+\frac{\mathrm{tr}\,\left(R(z_{\ell-1})R(z_{i_1})\cdots R(z_{i_{\ell-2}})R(z_\ell)\right)}{(z_{\ell-1}-z_{i_1})\cdots(z_{i_{\ell-2}}-z_\ell)(z_\ell-z_{\ell-1})}, \tag{3.59}$$

but this sum is regular for $z_{\ell-1} = z_\ell$ by the cyclic property of the trace, as terms cancel out pairwise.

We end this section with a couple of remarks.

Remark 3.1.7. *We note here that since $R(z)$ is a rank one matrix, the formulae of Theorem 3.1.1 for $\mathscr{C}^{\mathsf{c}}_\ell$, $\ell \geq 2$, can be expressed in terms of the scalar quantities*

$$w(x,y) := \frac{2\pi \mathrm{i}}{h_{N-1}}\frac{P_N(x)\widehat{P}_{N-1}(y)-P_{N-1}(x)\widehat{P}_N(y)}{x-y} \tag{3.60}$$

as

$$\mathscr{C}^{\mathsf{c}}_\ell(z_1,\dots,z_\ell) = -\sum_{(i_1,\dots,i_\ell)\in\mathrm{cyc}((\ell))} w(z_{i_1},z_{i_2})\cdots w(z_{i_{\ell-1}},z_{i_\ell})w(z_{i_\ell},z_{i_1})-\frac{\delta_{\ell,2}}{(z_1-z_2)^2},\qquad \ell\geq 2, \tag{3.61}$$

compare for instance with [65, 167].

3.1.1 The multiple and discrete orthogonal polynomials case

Formulæ of Theorem 3.1.1 lend themselves to be generalized in various directions, namely to *discrete orthogonal polynomials* and *multiple orthogonal polynomials*. Here we only give a blueprint of the strategy to be adopted in the two cases, which will be subject of future work. Essentially, one just has to write out the corresponding Riemann-Hilbert problem and use the solution – akin the Fokas-Its-Kitaev matrix $Y_N(z)$ – to construct the matrix $R(z)$ as depicted in Section 3.1; the cumulant functions are expressed via $R(z)$ in the same fashion.

Discrete orthogonal polynomials

Let $\mathfrak{X} \subset \mathbb{R}$ be a discrete set and $w : \mathfrak{X} \to \mathbb{R}_{>0}$ a probability mass function on $\mathfrak{X}$ with moments of all orders. *Discrete orthogonal polynomial ensemble* are defined as the probability distribution on $\mathfrak{X}^N$

$$Q_N(x_1, \dots, x_N) = \frac{1}{Z_N} w(x_1) \cdots w(x_N) \prod_{i<j} (x_j - x_i)^2, \tag{3.62}$$

where Z_N is the normalizing constant

$$Z_N = \sum_{x_1,\dots,x_N \in \mathfrak{X}} w(x_1) \cdots w(x_N)(x_j - x_i)^2. \tag{3.63}$$

The probability distribution Q_N defines a *determinantal point process* on $\mathfrak{X}$: all marginals of (3.62) can be expressed as determinants of the *correlation kernel* $K_N(x, y)$,

$$\sum_{x_{\ell+1},\dots,x_N \in \mathfrak{X}} Q_N(x_1, \dots, x_N) = \frac{(N-\ell)!}{N!} \left(\det_{1 \le i,j \le \ell} K_N(x_i, x_j) \right),$$

where as in the continuous case, see (1.35), it holds

$$K_N(x, y) = \sqrt{w(x)w(y)} \sum_{n=0}^{N-1} h_n^{-1} P_n(x) P_n(y) = \frac{\sqrt{w(x)w(y)}}{h_{N-1}} \frac{P_N(x)P_{N-1}(y) - P_{N-1}(x)P_N(y)}{x - y}. \tag{3.64}$$

Here we denote by $P_n(x)$ the (unique) monic orthogonal polynomial with respect to the weight w, and with $\widehat{P}_n(z)$ its (discrete) *Hilbert transform*,

$$\sum_{x \in \mathfrak{X}} P_n(x) P_m(x) w(x) = h_n \delta_{mn}, \quad h_n > 0, \qquad \widehat{P}_n(z) := \sum_{x \in \mathfrak{X}} P_n(x) \frac{w(x)}{z - x}, \quad z \in \mathbb{C} \setminus \mathfrak{X}. \tag{3.65}$$

The matrix

$$Y_N(z) := \begin{pmatrix} P_N(z) & \widehat{P}_N(z) \\ \frac{1}{h_{N-1}} P_{N-1}(z) & \frac{1}{h_{N-1}} \widehat{P}_{N-1}(x) \end{pmatrix} \tag{3.66}$$

is uniquely characterised by the following properties.

1. $Y_N(z)$ is meromorphic in z, with simple poles at $z \in \mathfrak{X}$.
2. $\operatorname*{res}_{z=x} Y_N(z) \mathrm{d}z = \lim_{z \to x} Y_N(z) \begin{pmatrix} 0 & w(x) \\ 0 & 0 \end{pmatrix}$.
3. $Y_N(z) = (I + \mathcal{O}(z^{-1})) z^{N\sigma_3}$ as $z \to \infty$ away from $\mathfrak{X}$.

The three conditions above constitute the *discrete Riemann-Hilbert* characterization of discrete orthogonal polynomials; their proof from the definition (3.66) is straightforward, and it turns out that they uniquely define the matrix $Y_N(z)$, [17] (see also [35]). The crucial observation is that the Christoffel-Darboux identity (3.64) allows us to write the correlation kernel via the matrix $Y_N(z)$ akin to (3.24) (see also Remark 3.1.7), namely

$$K_N(x, y) = -\frac{\sqrt{w(x)w(y)}}{x - y} \begin{pmatrix} 0 & 1 \end{pmatrix} Y_N^{-1}(x) Y_N(y) \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{3.67}$$

Then, analogously to (3.31) the confluent version of (3.67) is recovered as

$$K_N(x,x) = \operatorname*{res}_{z=x} \operatorname{tr}\left(Y_N^{-1}(z)Y_N'(z)\mathrm{E}_{1,1}\right)\mathrm{d}z, \tag{3.68}$$

from which the one-point function is derived as in the continuous case, Lemma 3.1.5. For the two point functions, again define $R(z) := Y_N(z)\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)Y_N^{-1}(z)$; in the discrete setting, $R(z)$ is a meromorphic function of z with simple poles at $\mathfrak{X}$. The analogue of Lemma 3.1.6 is obtained proving Ω has no poles on the support $\mathfrak{X}$ (rather than no jumps on the interval I). Multi point functions are a byproduct of this Lemma and the definition of $R(z)$.

Multiple orthogonal polynomials

Multiple orthogonal polynomials are polynomials of one variable which satisfy orthogonality conditions with respect to several measures [134]. Consider r weight functions $w_1,\dots,w_r$ on the real line with support on intervals I_j. Given a multi index $\mathbf{n}=(n_1,\dots,n_r)$ of positive integers with $|\mathbf{n}| := n_1+\cdots+n_r$, there are two, somewhat dual, families of orthogonal polynomials one can construct

1. *Type I multiple orthogonal polynomial:* the vector of polynomials $(A_{\mathbf{n},1},\dots,A_{\mathbf{n},r})$ such that $A_{\mathbf{n},j}$ has degree at most (n_j-1), satisfying the orthogonality condition

$$\sum_{j=1}^{r}\int x^k A_{\mathbf{n},j}(x)w_j(x)\mathrm{d}x = 0, \qquad 0\le k\le |\mathbf{n}|-2. \tag{3.69}$$

2. *Type II multiple orthogonal polynomials:* the monic polynomial $P_{\mathbf{n}}$ of degree $|\mathbf{n}|$ satisfying the r orthogonality conditions

$$\int x^k P_{\mathbf{n}}(x)w_j(x)\mathrm{d}x = 0, \qquad 0\le k\le n_j-1, \qquad 1\le j\le r. \tag{3.70}$$

The uniqueness of both types of multiple orthogonal polynomials is guaranteed under additional assumptions on the weights (e.g. *Angelescu* and *AT systems*, see [161]).

Let us focus on type II multiple orthogonal polynomials, the type I case being analogous. In [14] the following characterization via a Riemann-Hilbert problem is given.

Theorem 3.1.8. *The unique $(r+1)\times(r+1)$ matrix function $Y_N(z)$ satisfying*

1. *$Y_N(z)$ is analytic in $\mathbb{C}\setminus\mathbb{R}$,*
2. *on the real line it satisfies the jump condition*

$$Y_{N,+}(z) = Y_{N,-}(z)\begin{pmatrix} 1 & -2\pi\mathrm{i}w_1(z) & -2\pi\mathrm{i}w_2(z) & \cdots & -2\pi\mathrm{i}w_r(z) \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix} \tag{3.71}$$

3. *as $z\to\infty$ we have*

$$Y_N(z) \sim \left(\mathbf{1}+\mathcal{O}(z^{-1})\right)\begin{pmatrix} z^{|\mathbf{n}|} & & & \\ & z^{-n_1} & & \\ & & \ddots & \\ & & & z^{-n_r}\end{pmatrix} \tag{3.72}$$

is given by

$$Y_N(z) = \begin{pmatrix} P_{\mathbf{n}}(z) & R_{\mathbf{n},1}(z) & \cdots & R_{\mathbf{n},r}(z) \\ c_1 P_{\mathbf{n}-e_1}(z) & c_1 R_{\mathbf{n}-e_1,1}(z) & & c_1 R_{\mathbf{n}-e_1,r}(z) \\ \vdots & & \ddots & \\ c_r P_{\mathbf{n}-e_r}(z) & c_r R_{\mathbf{n}-e_r,1}(z) & c_r & R_{\mathbf{n}-e_r,r}(z) \end{pmatrix}. \tag{3.73}$$

Here, $e_k = (0, \dots, 1, \dots, 0)$ is the r-dimensional vector with $k-th$ entry equal to 1, $R_{\mathbf{n},j}(z)$ is the Hilbert transform of $P_{\mathbf{n}}(z)$ with respect to the weight $w_j(x)$ and d_j are constants,

$$R_{\mathbf{n},j}(z) = \int \frac{P_{\mathbf{n}}(x)}{x-z} w_j(x)\mathrm{d}x, \qquad d_j^{-1} = \int x^{n_j - 1} P_{\mathbf{n}-e_j}(x) w_j(x)\mathrm{d}x. \tag{3.74}$$

The related ensemble is a *determinantal point process*, as first pointed out by Borodin in the *biorthogonal* case, [34]. Remarkably, it is proven in [55] that the correlation kernel $K_N(x,y)$ can be expressed as

$$K_N(x,y) = \frac{1}{2\pi \mathrm{i}} \begin{pmatrix} 0 & w_1(y) & \cdots & w_r(y) \end{pmatrix} Y_N^{-1}(y) Y_N(x) \begin{pmatrix} 1 & 0 & \cdots & 0 \end{pmatrix}^T, \tag{3.75}$$

from which its confluent version is readily recovered as

$$K_N(x,x) = \frac{1}{2\pi \mathrm{i}} \begin{pmatrix} 0 & w_1(x) & \cdots & w_r(x) \end{pmatrix} Y_N^{-1}(x) Y_N'(x) \begin{pmatrix} 1 & 0 & \cdots & 0 \end{pmatrix}^T. \tag{3.76}$$

Then again, defining the matrix $R(z) := Y_N(z) \mathrm{E}_{1,1} Y_N^{-1}(z)$, where now $\mathrm{E}_{1,1}$ is the $(r+1)\times(r+1)$ matrix with 1 in the upper left entry and 0 elsewhere, the same strategy outlined in Section 3.1 yields generating functions for the correlators of the associated process.

3.2 Generating functions for correlators of classical unitary invariant ensembles

In this section we employ Theorem 3.1.1 to obtain explicit expressions for generating functions of correlators related to the classical potentials introduced in Section 1.2.1, namely the Gaussian, Laguerre and Jacobi ones. We derive for each of these case an expression for the functions $\mathscr{F}^{\mathsf{c}}_{\ell,\infty}(z_1, \dots z_\ell)$ and $\mathscr{F}^{\mathsf{c}}_{\ell,0}(z_1, \dots z_\ell)$ of Remark 3.1.2 in terms of asymptotic expansions at $z = 0, \infty$ of the matrix $R(z)$. Let us first see how said matrix can be written in terms of the associated orthogonal polynomials and their Cauchy transforms.

Remark 3.2.1. *The jump matrix in (3.8) has unit determinant and $\det Y_N(z) \sim 1$ when $z \to \infty$ by (3.9). We conclude by Liouville theorem that $\det Y_N(z) \equiv 1$ identically, this implies*

$$Y_N^{-1}(z) = \begin{pmatrix} -\frac{2\pi \mathrm{i}}{h_{N-1}} \widehat{P}_{N-1}(z) & -\widehat{P}_N(z) \\ \frac{2\pi \mathrm{i}}{h_{N-1}} P_{N-1}(z) & P_N(z) \end{pmatrix}. \tag{3.77}$$

In turn, using again $\det Y_N(z) \equiv 1$ and the definition $R(z) := Y_N(z) \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) Y_N^{-1}(z)$, we get

$$R(z) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} -\frac{2\pi \mathrm{i}}{h_{N-1}} P_{N-1}(z) \widehat{P}_N(z) & -P_N(z) \widehat{P}_N(z) \\ \left(\frac{2\pi \mathrm{i}}{h_{N-1}}\right)^2 P_{N-1}(z) \widehat{P}_{N-1}(z) & \frac{2\pi \mathrm{i}}{h_{N-1}} P_{N-1}(z) \widehat{P}_N(z) \end{pmatrix}. \tag{3.78}$$

Thus, what we really need in computing asymptotic expansions of $R(z)$ is the expansion for the products of the orthogonal polynomials and their Cauchy transforms.

As we shall see in the next section, asymptotic series for Cauchy transforms of polynomials from the classical ensembles are relatively easy to carry out, due to the existence of a Rodrigues formula of type (1.46). Moreover, it turns out their product satisfies a three term recurrence which we are able to identify with specific families of discrete orthogonal polynomials, see Remarks 3.2.4, 3.2.11 and 3.2.16. Let us make final observation which will be useful in the writing of more aesthetically pleasing formulæ.

Remark 3.2.2. *A z-independent (but possibly N-dependent) gauge transformation of the matrix $R(z)$, i.e. $R(z) \mapsto \tilde{R}(z) := T_N R(z) T_N^{-1}$ or equivalently $Y_N(z) \mapsto \tilde{Y}_N(z) := T_N Y_N(z)$, does not affect formulæ of Theorem 3.1.1, as*

$$\left(\tilde{Y}_N^{-1}(z)\tilde{Y}_N'(z)\right)_{1,1} = \left(Y_N^{-1}(z)T_N^{-1}T_N \cdot Y_N'(z)\right)_{1,1} = \left(Y_N^{-1}(z)Y_N'(z)\right)_{1,1}, \tag{3.79}$$

$$\frac{\operatorname{tr}\left(\tilde{R}(z_{i_1})\dots\tilde{R}(z_{i_\ell})\right)}{(z_{i_1}-z_{i_2})\cdots(z_{i_\ell}-z_{i_1})} = \frac{\operatorname{tr}\left(T_N R(z_{i_1})T_N^{-1}\dots T_N R(z_{i_\ell})T_N^{-1}\right)}{(z_{i_1}-z_{i_2})\cdots(z_{i_\ell}-z_{i_1})} = \frac{\operatorname{tr}\left(R(z_{i_1})\dots R(z_{i_\ell})\right)}{(z_{i_1}-z_{i_2})\cdots(z_{i_\ell}-z_{i_1})}. \tag{3.80}$$

The latter follows from the basic conjugation-invariant property of the trace.

3.2.1 Correlators generating functions for GUE

The correlators generating functions for the GUE were first computed by Dubrovin and Yang in [63] and were part of the motivation for our subsequent work on the Laguerre and Jacobi ensemble. There, the authors compute the matrix $R(z)$ from its difference equation. A proof on the line of this thesis has already been written by Ruzza in his PhD thesis [153](Theorem 3.5.3). We report the relevant Theorem for the sake of completeness in our analysis on the classical unitary ensembles.

Theorem 3.2.3 ([63])**.** *Introduce the formal series*

$$R(z) \sim \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \sum_{\ell\geq 0} \frac{1}{z^{2\ell+2}} \begin{pmatrix} N\mathcal{A}_{\ell,N} & -zN\mathcal{B}_{\ell,N+1} \\ z\mathcal{B}_{\ell,N} & -N\mathcal{A}_{\ell,N} \end{pmatrix} \tag{3.81}$$

where

$$\mathcal{A}_{\ell,N} := (2\ell+1)!! \sum_{j=0}^{\ell} 2^j \binom{\ell}{j}\binom{N}{j+1} = N(2\ell+1)!!\, {}_2F_1\left(\begin{smallmatrix} -\ell,1-N \\ 2 \end{smallmatrix} \middle| 2\right) \tag{3.82}$$

$$\mathcal{B}_{\ell,N} := N(2\ell-1)!! \sum_{j=0}^{\ell} 2^j \binom{\ell}{j}\binom{N-1}{j} = N(2\ell-1)!!\, {}_2F_1\left(\begin{smallmatrix} -\ell,1-N \\ 1 \end{smallmatrix} \middle| 2\right). \tag{3.83}$$

as $z \to \infty$ within any of the two sectors in $\mathbb{C}\setminus(-\infty,+\infty)$. Then, the correlator generating functions introduced in Remark 3.1.2 are

$$\mathscr{F}_{1,\infty}(z) = \int_z^{\infty} (R_{11}(y)-1)\,\mathrm{d}y \tag{3.84}$$

$$\mathscr{F}^{\mathsf{c}}_{2,\infty}(z_1,z_2) = \frac{\operatorname{tr}\left(R(z_1)R(z_2)\right)-1}{(z_1-z_2)^2} \tag{3.85}$$

$$\mathscr{F}^{\mathsf{c}}_{\ell,\infty}(z_1,\dots,z_\ell) = -\sum_{(i_1,\dots,i_\ell)\in\operatorname{cyc}((\ell))} \frac{\operatorname{tr}\left(R(z_{i_1})\dots R(z_{i_\ell})\right)}{(z_{i_1}-z_{i_2})\cdots(z_{i_\ell}-z_{i_1})}, \quad \ell\geq 3, \tag{3.86}$$

Equivalent formulæ for the one and two point functions were given by Harer and Zagier and by Morozov, [110, 140]. Interestingly, they have recently been reproved by Giacchetto, Lewański and Norbury by the sole use of the theory on moduli spaces of curves in [90].

Remark 3.2.4. *As first pointed out in [52], the normalized moments of GUE are Meixner polynomials, defined via their hypergeometric representation*

$$M_n(x;\gamma,c) = {}_2F_1\left(\begin{matrix} -n, & -x \\ & \gamma \end{matrix} \middle| 1-\frac{1}{c} \right). \tag{3.87}$$

As it turns out, both the $\mathcal{A}_{\ell,N}$ and $\mathcal{B}_{\ell,N}$ can be put in the form (3.87) as

$$\frac{\mathcal{A}_{\ell,N}}{N(2\ell+1)!!} = M_\ell(N-1;2,-1), \qquad \frac{\mathcal{B}_{\ell,N}}{N(2\ell-1)!!} = M_\ell(N-1;1,-1). \tag{3.88}$$

As a consequence, we deduce that also the GUE multipoint correlators are linear combinations of products of Meixner polynomials.

3.2.2 Correlators generating functions for LUE

Theorem 3.2.5 ([92])**.** *Introduce the matrix-valued formal series*

$$R^{[\infty]}(z) := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \sum_{\ell\geq 0} \frac{1}{z^{\ell+1}} \begin{pmatrix} \ell\mathcal{A}_\ell(N,N+\alpha) & \mathcal{B}_\ell(N+1,N+\alpha+1) \\ -N(N+\alpha)\mathcal{B}_\ell(N,N+\alpha) & -\ell\mathcal{A}_\ell(N,N+\alpha) \end{pmatrix} \tag{3.89}$$

$$R^{[0]}(z) := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \sum_{\ell\geq 0} \frac{z^\ell}{(\alpha-\ell)_{2\ell+1}} \begin{pmatrix} (\ell+1)\mathcal{A}_\ell(N,N+\alpha) & -\mathcal{B}_\ell(N+1,N+\alpha+1) \\ N(N+\alpha)\mathcal{B}_\ell(N,N+\alpha) & -(\ell+1)\mathcal{A}_\ell(N,N+\alpha) \end{pmatrix} \tag{3.90}$$

where we denote by $(p)_j := p(p+1)\cdots(p+j-1)$ the rising factorial and

$$\mathcal{A}_\ell(N,M) := \begin{cases} N, & \ell = 0, \\ \frac{1}{\ell}\sum_{j=0}^{\ell-1}(-1)^j \frac{(N-j)_\ell (M-j)_\ell}{j!(\ell-1-j)!}, & \ell \geq 1, \end{cases} \qquad \mathcal{B}_\ell(N,M) := \sum_{j=0}^{\ell}(-1)^j \frac{(N-j)_\ell(M-j)_\ell}{j!(\ell-j)!}. \tag{3.91}$$

Then, the one-point correlators generating functions introduced in Remark 3.1.2 are

$$\mathscr{F}_{1,\infty}(z) = \frac{N}{z} + \frac{1}{z}\int_y^\infty \left[\left(R^{[\infty]}(y)\right)_{11} - 1\right] \mathrm{d}y, \qquad \mathscr{F}_{1,0}(z) = \frac{1}{z}\int_0^y \left[\left(R^{[0]}(y)\right)_{11} - 1\right] \mathrm{d}y, \tag{3.92}$$

and similarly, the multipoint generating functions admit the expression

$$\mathscr{F}^{\mathsf{c}}_{2,p}(z_1,z_2) = \frac{\operatorname{tr}\left(R^{[p]}(z_1)R^{[p]}(z_2)\right) - 1}{(z_1-z_2)^2}, \tag{3.93}$$

$$\mathscr{F}^{\mathsf{c}}_{\ell,p}(z_1,\dots,z_\ell) = -\sum_{(i_1,\dots,i_\ell)\in\operatorname{cyc}((\ell))} \frac{\operatorname{tr}\left(R^{[p]}(z_{i_1})\dots R^{[p]}(z_{i_\ell})\right)}{(z_{i_1}-z_{i_2})\cdots(z_{i_\ell}-z_{i_1})}, \quad \ell\geq 3, \qquad p = 0,\infty. \tag{3.94}$$

Remark 3.2.6. *The one-point generating series boil down to the following identities*

$$\left\langle \mathrm{tr}\, X^k \right\rangle = A_k(N, N+\alpha), \qquad \left\langle \mathrm{tr}\, X^{-k-1} \right\rangle = \frac{A_k(N,N+\alpha)}{(\alpha-k)_{2k+1}}, \qquad k \geq 0 \tag{3.95}$$

which were already derived in the literature [52, 109]. From from Theorem 3.2.5 for example one can deduce compact expressions for correlators of the form

$$\left\langle \mathrm{tr}\, X^k \mathrm{tr}\, X \right\rangle_{\mathsf{c}} = k A_k(N, N+\alpha), \qquad \left\langle \mathrm{tr}\, X^{-k} \mathrm{tr}\, X^{-1} \right\rangle_{\mathsf{c}} = \frac{k A_k(N,N+\alpha)}{\alpha(\alpha-k)_{2k+1}}.$$

We begin the proof of Theorem 3.2.5 computing $R^{[\infty]}(z)$, the expansion of $R(z)$ as $z \to \infty$.

Proposition 3.2.7. *The matrix $R(z)$ admits the asymptotic expansion*

$$T R(z) T^{-1} \sim R^{[\infty]}(z), \qquad z \to \infty \tag{3.96}$$

uniformly within the sector $0 < \arg z < 2\pi$. Here $R^{[\infty]}$ is the formal series introduced in Theorem 3.2.5, see (3.89), and T is defined as

$$T := \begin{pmatrix} 1 & 0 \\ 0 & \frac{h_N}{2\pi \mathrm{i}} \end{pmatrix} \tag{3.97}$$

where $h_N = N!\Gamma(N+\alpha+1)$ as in (1.54).

Proof. As outlined at the beginning of Section 3.1, the first step is to get our hands on the asymptotic expansion at $z = \infty$ of the Laguerre polynomials and their Cauchy transforms. For convenience we report formula (1.56) here,

$$\pi_N^{(\alpha)}(x) = \sum_{j=0}^{N} \frac{(-1)^{N-j}(N-j+1)_j (j+1+\alpha)_{N-j}}{j!} x^j, \qquad N \geq 0. \tag{3.98}$$

We can expand $\widehat{\pi}_N^{(\alpha)}$ when $z \to \infty$ as

$$\begin{aligned}
\widehat{\pi}_N^{(\alpha)}(z) &= \frac{1}{2\pi \mathrm{i}} \int_0^{+\infty} \pi_N^{(\alpha)}(\xi) \xi^\alpha \mathrm{e}^{-\xi} \frac{\mathrm{d}\xi}{\xi - z} \\
&\sim -\frac{1}{2\pi \mathrm{i}} \sum_{j \geq 0} \frac{1}{z^{j+1}} \int_0^{+\infty} \pi_N^{(\alpha)}(\xi) \xi^{\alpha+j} \mathrm{e}^{-\xi} \mathrm{d}\xi \\
&= -\frac{1}{2\pi \mathrm{i}} \sum_{j \geq 0} \frac{1}{z^{j+N+1}} \int_0^{+\infty} \pi_N^{(\alpha)}(\xi) \xi^{\alpha+j+N} \mathrm{e}^{-\xi} \mathrm{d}\xi \\
&= -\frac{1}{2\pi \mathrm{i}} \sum_{j \geq 0} \frac{1}{z^{j+N+1}} \int_0^{+\infty} (-1)^N \left(\frac{\mathrm{d}^N}{\mathrm{d}\xi^N} (\mathrm{e}^{-\xi} \xi^{\alpha+N}) \right) \xi^{j+N} \mathrm{d}\xi \\
&= -\frac{1}{2\pi \mathrm{i}} \sum_{j \geq 0} \frac{1}{z^{j+N+1}} \int_0^{+\infty} \left(\frac{\mathrm{d}^N}{\mathrm{d}\xi^N} \xi^{j+N} \right) \xi^{\alpha+N} \mathrm{e}^{-\xi} \mathrm{d}\xi \\
&= -\frac{1}{2\pi \mathrm{i}} \sum_{j \geq 0} \frac{(j+1)_N \Gamma(j+N+1+\alpha)}{z^{N+j+1}} &(3.99) \\
&= -\frac{1}{2\pi \mathrm{i}} \sum_{j \geq 0} \frac{h_{N-1}}{j!} \frac{(N)_{j+1}(N+\alpha)_{j+1}}{z^{N+j+1}}, &(3.100)
\end{aligned}$$

where we have used the orthogonality property to shift the sum in the first place, then Rodrigues formula (1.55), integration by parts and the trivial formula (A). The expansion (3.100) is formal; however, it has an analytic meaning of asymptotic expansion within $0 < \arg z < 2\pi$. Indeed, we note that for any $J \geq 0$ the difference between the Cauchy transform and its truncated formal expansion is

$$\widehat{\pi}_N^{(\alpha)}(z) + \frac{1}{2\pi\mathrm{i}}\sum_{j=0}^{J-1}\frac{1}{z^{j+1}}\int_0^{+\infty}\pi_N^{(\alpha)}(\xi)\xi^{\alpha+j}\mathrm{e}^{-\xi}\mathrm{d}\xi = \frac{1}{2\pi\mathrm{i}z^J}\int_0^{+\infty}\pi_N^{(\alpha)}(\xi)\xi^{\alpha+J}\mathrm{e}^{-\xi}\frac{\mathrm{d}\xi}{\xi - z} = \mathcal{O}\left(\frac{1}{z^{J+1}}\right) \tag{3.101}$$

where the last step holds as $z \to \infty$, uniformly within any closed subsector $0 < \arg z < 2\pi$. Rotating the contour of integration we see that the expansion actually holds uniformly in the full sector $0 < \arg z < 2\pi$. Hence from (3.7), together with (3.98) and (3.100), we have

$$Y_N(z) \sim \sum_{j\geq 0}\frac{1}{j!z^j}\begin{pmatrix} (-1)^j(N-j+1+\alpha)_j(N-j+1)_j & -\frac{h_{N-1}}{2\pi\mathrm{i}z}(N+\alpha)_{j+1}(N)_{j+1} \\ -\frac{2\pi\mathrm{i}}{h_{N-1}z}(-1)^j(N-j+\alpha)_j(N-j)_j & (N+\alpha)_j(N)_j \end{pmatrix} z^{N\sigma_3} \tag{3.102}$$

as $z \to \infty$ within the sector $0 < \arg z < 2\pi$, and in turn we can compute $R(z)$ via (3.78). For example for the $(1,1)$-entry we have

$$R_{11}(z) = 1 - \frac{2\pi\mathrm{i}}{h_{N-1}}\pi_{N-1}^{(\alpha)}(z)\widehat{\pi}_N^{(\alpha)}(z) \sim 1 + \sum_{\ell\geq 0}\frac{1}{z^{\ell+2}}\sum_{j=0}^{\ell}\frac{(-1)^j(N)_{\ell-j+1}(N-j+\alpha)_j(N-j)_j(N+\alpha)_{\ell-j+1}}{j!(\ell-j)!} \tag{3.103}$$

and noting a trivial simplification of rising factorials

$$(N+\alpha)_{\ell-j+1}(N-j+\alpha)_j = (N-j+\alpha)_{\ell+1}, \qquad (N)_{\ell-j+1}(N-j)_j = (N-j)_{\ell+1} \tag{3.104}$$

it follows that as $z \to \infty$, and since diagonal conjugation only affects off-diagonal entries,

$$(TR(z)T^{-1})_{11} \sim 1 + \sum_{\ell\geq 0}\frac{1}{z^{\ell+2}}(\ell+1)\mathcal{A}_{\ell+1}(N, N+\alpha) = 1 + \sum_{\ell\geq 0}\frac{1}{z^{\ell+1}}\ell\mathcal{A}_\ell(N, N+\alpha) = (R^{[\infty]})_{11}(z), \tag{3.105}$$

with $\mathcal{A}_\ell(N, M)$ as in (3.91), notice that the first term in the sum is identically zero. In a similar way, now recalling (3.97) as well, we compute the $(1,2)$-entry as

$$\begin{aligned}(TR(z)T^{-1})_{12} &= -\frac{2\pi\mathrm{i}}{h_N}\pi_N^{(\alpha)}(z)\widehat{\pi}_N^{(\alpha)}(z) \\ &\sim \frac{1}{N(N+\alpha)}\sum_{\ell\geq 0}\frac{1}{z^{\ell+1}}\sum_{j=0}^{\ell}(-1)^j\frac{(N+\alpha)_{\ell-j+1}(N)_{\ell-j+1}(N-j+1+\alpha)_j(N-j+1)_j}{j!(\ell-j)!} \\ &= \sum_{\ell\geq 0}\frac{1}{z^{\ell+1}}\sum_{j=0}^{\ell}(-1)^j\frac{(N-j+1+\alpha)_\ell(N-j+1)_\ell}{j!(\ell-j)!}, \\ &= \sum_{\ell\geq 0}\frac{1}{z^{\ell+1}}\mathcal{B}_\ell(N+1, N+1+\alpha) = (R^{[\infty]})_{12}(z)\end{aligned}$$

where we used a similar version of (3.104) and with $\mathcal{B}_\ell(N, M)$ as in (3.91). Finally, the $(2,1)$-entry of the expansion of $TR(z)T^{-1}$ is computed in a similar way as

$$
\begin{aligned}
(TR(z)T^{-1})_{21} &= \frac{h_N}{2\pi\mathrm{i}}\pi^{(\alpha)}_{N-1}(z)\widehat{\pi}^{(\alpha)}_{N-1}(z) \\
&\sim -N(N+\alpha)\sum_{\ell\geq 0}\frac{1}{z^{\ell+1}}\sum_{j=0}^{\ell}(-1)^j\frac{(N)_{\ell-j}}{j!(\ell-j)!(N-j+\alpha)_j(N-j)_j(N+\alpha)_{\ell-j}} \\
&= -N(N+\alpha)\sum_{\ell\geq 0}\frac{1}{z^{\ell+1}}\sum_{j=0}^{\ell}(-1)^j\frac{(N-j+\alpha)_\ell(N-j)_\ell}{j!(\ell-j)!} \\
&= -N(N+\alpha)\sum_{\ell\geq 0}\frac{1}{z^{\ell+1}}\mathcal{B}_\ell(N,N+\alpha)
\end{aligned}
$$

and the proof of the Proposition is complete. □

Before considering the asymptotic expansion as $z\to 0$, let us dwell on the properties of the coefficients $\mathcal{A}_\ell(N,M)$ and $\mathcal{B}_\ell(N,M)$ entering the expansion $R^{[\infty]}(z)$. We claim they satisfy a *three term recurrence* and hence are *orthogonal polynomials* themselves, this is the content of Lemma 3.2.9 and Remark 3.2.11.

In preparation let us recall the matrix differential equation satisfied by the Laguerre polynomials and their Cauchy transform via the matrix $Y(z)$.

Proposition 3.2.8. *Let us introduce the following dressing transformation of the matrix $Y_N(z)$ of the Laguerre polynomials*[1]*,*

$$\Psi_N(z) := Y_N(z)z^{\alpha\frac{\sigma_3}{2}}e^{-z\frac{\sigma_3}{2}}, \tag{3.106}$$

then, $\Psi(z)$ has a constant jump

$$\Psi_{N,+}(z) = \Psi_{N,-}(z)\begin{pmatrix} \mathrm{e}^{-\mathrm{i}\pi\alpha} & \mathrm{e}^{-\mathrm{i}\pi\alpha} \\ 0 & \mathrm{e}^{\mathrm{i}\pi\alpha}\end{pmatrix}. \tag{3.107}$$

Moreover it satisfies the matrix differential equation

$$\frac{\partial\Psi_N(z)}{\partial z} = A(z)\Psi_N(z), \qquad A(z) := -\frac{1}{2}\sigma_3 + \frac{1}{z}\begin{pmatrix} N+\frac{\alpha}{2} & -\frac{h_N}{2\pi\mathrm{i}} \\ \frac{2\pi\mathrm{i}}{h_{N-1}} & -N-\frac{\alpha}{2}\end{pmatrix} \tag{3.108}$$

with $h_N = N!\Gamma(\alpha+N+1)$ as in (1.54). Notice that (3.108) has a Fuchsian singularity at $x=0$ and an irregular singularity of Poincaré rank 1 at $x=\infty$.

Proof. It is a classical result, see e.g. [115, 128] □

Notice that $R(z)$ can equivalently be defined via the matrix $\Psi(z)$ in (3.106) as

$$\Psi_N(z)\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\Psi_N^{-1}(z) = Y_N(z)\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}Y_N^{-1}(z) = R(z), \tag{3.109}$$

in particular from the above and (3.108) we infer

$$\frac{\partial\Psi_N(z)}{\partial z} = A(z)\Psi_N(z) \qquad \Longrightarrow \qquad \frac{\partial}{\partial z}R(z) = [A(z),R(z)]. \tag{3.110}$$

We are now ready to prove the following Lemma.

[1] Here we choose the branch of the logarithm for the potential $V_\alpha(x;\mathbf{t}) := \alpha\log x - x + \sum_{k>0}t_x x^k$ analytic for $x\in\mathbb{C}\setminus[0,\infty)$ satisfying $\lim_{\epsilon\to 0_+}\log(x+\mathrm{i}\epsilon)\in\mathbb{R}$; to be consistent we shall identify $V_\alpha(x;\mathbf{t})$, without further mention, with $V_{\alpha,+}(x;\mathbf{t}) = \lim_{\epsilon\to 0_+}V_\alpha(x+\mathrm{i}\epsilon;\mathbf{t})$ whenever $x>0$.

Lemma 3.2.9. *The entries $\mathcal{A}_\ell(N,M), \mathcal{B}_\ell(N,M)$ ($\ell \geq 0$) defined in (3.91) satisfy the following three term recursions*

$$(\ell+2)\mathcal{A}_{\ell+1}(N,M) = (2\ell+1)(N+M)\mathcal{A}_\ell(N,M) + (\ell-1)(\ell^2-(M-N)^2)\mathcal{A}_{\ell-1}(N,M), \quad (3.111)$$
$$(\ell+1)\mathcal{B}_{\ell+1}(N,M) = (2\ell+1)(N+M-1)\mathcal{B}_\ell(N,M) + \ell(\ell^2-(M-N)^2)\mathcal{B}_{\ell-1}(N,M), \quad (3.112)$$

for $\ell \geq 1$, with initial data given as

$$\mathcal{A}_0(N,M) = N, \quad \mathcal{A}_1(N,M) = NM, \qquad \mathcal{B}_0(N,M) = 1, \quad \mathcal{B}_1(N,M) = N+M-1. \quad (3.113)$$

Proof. Introduce the matrices

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad (3.114)$$

and write

$$TR(z)T^{-1} = \frac{1}{2}\mathbf{1} + r_3\sigma_3 + r_+\sigma_+ + r_-\sigma_-, \quad (3.115)$$

where we used that $\operatorname{tr} R \equiv 1$; hereafter we omit the dependence on z for brevity. Since the gauge T is independent of z, from (3.110) we also have

$$\frac{\partial}{\partial z}R(z) = [A(z), R(z)] \Rightarrow \frac{\partial}{\partial z}\left(TR(z)T^{-1}\right) = [TA(z)T^{-1}, TR(z)T^{-1}] \quad (3.116)$$

and from (3.108) we set

$$TA(z)T^{-1} = -\frac{1}{2}\sigma_3 + \frac{1}{z}\begin{pmatrix} N+\frac{\alpha}{2} & -1 \\ N(N+\alpha) & -N-\frac{\alpha}{2} \end{pmatrix} = a_3\sigma_3 + a_+\sigma_+ + a_-\sigma_-, \quad (3.117)$$

which together with (3.116) yields the system of linear ODEs

$$\partial_z r_3 = a_+r_- - a_-r_+, \qquad \partial_z r_+ = 2(a_3r_+ - a_+r_3), \qquad \partial_z r_- = 2(a_-r_3 - a_3r_-). \quad (3.118)$$

In turn, these imply the following *decoupled* third order equations for $\partial_z r_3, r_+, r_-$

$$3(2N+\alpha-z)\partial_z r_3 + (4-\alpha^2+2(2N+\alpha)z-z^2)\partial_z^2 r_3 + 5z\partial_z^3 r_3 + z^2\partial_z^4 r_3 = 0, \quad (3.119)$$
$$(2N+\alpha\pm1-z)r_\pm + (1-\alpha^2+2(2N+\alpha\pm1)z-z^2)\partial_z r_\pm + 3z\partial_z^2 r_\pm + z^2\partial_z^3 r_\pm = 0. \quad (3.120)$$

Finally, using the Wishart parameter $M = N+\alpha$, we substitute the expansion at $z=\infty$ given by (3.89) into the ODEs (3.119) and (3.120) to obtain the claimed recursion relations. □

We will now make use of Lemma 3.2.9 to get the asymptotic expansion of the matrix $R(z)$ at $z=0$.

Proposition 3.2.10. *The matrix $R(z)$ admits the asymptotic expansion*

$$TR(z)T^{-1} \sim R^{[0]}(z), \qquad z \to 0 \quad (3.121)$$

uniformly within the sector $0 < \arg z < 2\pi$. Here $R^{[0]}$ is the formal series claimed in Theorem 3.2.5, see (3.90), and T is defined in (3.97).

Proof. First we observe that by arguments which are entirely analogous to those employed in the proof of Proposition 3.2.7, the matrices $Y(z)$ and (consequently) $R(z)$ possesses asymptotic expansions in integer powers of z as $z \to 0$, which are uniform in a sector properly containing $0 < \arg z < 2\pi$. The first coefficients of these expansions at $z = 0$ can be computed from

$$\pi_N^{(\alpha)} = (-1)^N \left((\alpha+1)_N - N(\alpha+2)_{N-1}z + \mathcal{O}\left(z^2\right)\right) \tag{3.122}$$

$$\widehat{\pi}_N^{(\alpha)} \sim \frac{(-1)^N}{2\pi\mathrm{i}} \left(N!\Gamma(\alpha) + (N+1)!\Gamma(\alpha-1)z + \mathcal{O}\left(z^2\right)\right) \tag{3.123}$$

where the former is found directly from (3.98) and the latter by a computation analogous to (3.100); hence recalling the definition (3.7) we have

$$\begin{aligned} Y(x) \sim & (-1)^N \begin{pmatrix} (\alpha+1)_N & \frac{N!\Gamma(\alpha)}{2\pi\mathrm{i}} \\ \frac{2\pi\mathrm{i}}{h_{N-1}}(\alpha+1)_{N-1} & \frac{(N-1)!\Gamma(\alpha)}{h_{N-1}} \end{pmatrix} \\ & + (-1)^N \begin{pmatrix} -N(\alpha+2)_{N-1} & \frac{(N+1)!\Gamma(\alpha-1)}{2\pi\mathrm{i}} \\ -\frac{2\pi\mathrm{i}}{h_{N-1}}(N-1)(\alpha+2)_{N-2} & \frac{N!}{h_{N-1}}\Gamma(\alpha-1) \end{pmatrix} x + \mathcal{O}\left(x^2\right) \end{aligned} \tag{3.124}$$

as $x \to 0$ within $0 < \arg x < 2\pi$; this implies that in the same regime we have

$$\begin{aligned} TR(x)T^{-1} \sim & \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \frac{1}{\alpha}\begin{pmatrix} N & -1 \\ N(N+\alpha) & -N \end{pmatrix} \\ & + \begin{pmatrix} 2N(N+\alpha) & -2N-\alpha-1 \\ N(N+\alpha)(2N+\alpha-1) & -2N(N+\alpha) \end{pmatrix} \frac{x}{(\alpha-1)\alpha(\alpha+1)} + \mathcal{O}\left(x^2\right). \end{aligned} \tag{3.125}$$

Therefore, our goal is just to show that the coefficients of the latter expansion are related to those of the expansion at $z = \infty$ as stated in the formulæ (3.89) and (3.90). To this end let us write, in terms of the decomposition (3.115),

$$r_3(z) \sim \frac{1}{2} + \sum_{\ell \geq 0} (\ell+1)\widetilde{\mathcal{A}}_\ell(N, N+\alpha)\frac{z^\ell}{(\alpha-\ell)_{2\ell+1}}, \qquad r_\pm(z) \sim \sum_{\ell \geq 0} \widetilde{\mathcal{B}}_\ell^\pm(N, N+\alpha)\frac{z^\ell}{(\alpha-\ell)_{2\ell+1}} \tag{3.126}$$

for some, yet undetermined coefficients $\widetilde{\mathcal{A}}_\ell(N,M), \widetilde{\mathcal{B}}_\ell^\pm(N,M)$. From (3.125) we read the first coefficients $\widetilde{\mathcal{A}}_\ell(N,M)$, $\widetilde{\mathcal{B}}_\ell^\pm(N,M)$ in (3.126) as

$$\begin{aligned} &\widetilde{\mathcal{A}}_0(N,M) = N = \mathcal{A}_0(N,M), && \widetilde{\mathcal{A}}_1(N,M) = NM = \mathcal{A}_1(N,M), \\ &\widetilde{\mathcal{B}}_0^+(N,M) = -1 = -\mathcal{B}_0(N+1,M+1), && \widetilde{\mathcal{B}}_1^+(N,M) = -N-M-1 = -\mathcal{B}_1(N+1,M+1), \\ &\widetilde{\mathcal{B}}_0^-(N,M) = NM = NM\mathcal{B}_0(N,M), && \widetilde{\mathcal{B}}_1^-(N,M) = NM(N+M-1) = NM\mathcal{B}_1(N,M) \end{aligned} \tag{3.127}$$

Finally, it can be checked that inserting (3.126) in (3.119) and (3.120) we obtain, again using $M = N + \alpha$, the recursions

$$\begin{aligned} &(\ell+2)\widetilde{\mathcal{A}}_{\ell+1}(N,M) = (2\ell+1)(N+M)\widetilde{\mathcal{A}}_\ell(N,M) + (\ell-1)(\ell^2-(M-N)^2)\widetilde{\mathcal{A}}_{\ell-1}(N,M), \\ &(\ell+1)\widetilde{\mathcal{B}}_{\ell+1}^+(N,M) = (2\ell+1)(N+M+1)\widetilde{\mathcal{B}}_\ell^+(N,M) + \ell(\ell^2-(M-N)^2)\widetilde{\mathcal{B}}_{\ell-1}^+(N,M), \\ &(\ell+1)\widetilde{\mathcal{B}}_{\ell+1}^-(N,M) = (2\ell+1)(N+M-1)\widetilde{\mathcal{B}}_\ell^-(N,M) + \ell(\ell^2-(M-N)^2)\widetilde{\mathcal{B}}_{\ell-1}^-(N,M) \end{aligned} \tag{3.128}$$

for $\ell \geq 1$. In view of Lemma 3.2.9, the linear recursions (3.128) with initial data (3.127) are uniquely solved as

$$\widetilde{\mathcal{A}}_\ell(N,M) = \mathcal{A}_\ell(N,M), \qquad \widetilde{\mathcal{B}}_\ell^+(N,M) = -\mathcal{B}_\ell(N+1,M+1), \qquad \widetilde{\mathcal{B}}_\ell^-(N,M) = NM\mathcal{B}_\ell(N,M). \tag{3.129}$$

Therefore from (3.115), (3.126) and the above relation we obtain

$$TR(z)T^{-1} \sim \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \sum_{\ell\geq 0} \frac{z^\ell}{(\alpha-\ell)_{2\ell+1}} \begin{pmatrix} (\ell+1)\mathcal{A}_\ell(N,N+\alpha) & -\mathcal{B}_\ell(N+1,N+1+\alpha) \\ N(N+\alpha)\mathcal{B}_\ell(N,N+\alpha) & -(\ell+1)\mathcal{A}_\ell(N,N+\alpha) \end{pmatrix} \tag{3.130}$$

with $\alpha = M-N$ and $\mathcal{A}_\ell(N,M)$ and $\mathcal{B}_\ell(N,M)$ as in (3.91). The proof is complete □

We now complete the proof of Theorem 3.2.5 deriving formulæ (3.92) and (3.94).

Proof of Theorem 3.2.5. The equations (3.94) for the multipoint generating functions follow immediately from the general Theorem 3.1.1.

As far as it concerns the one-point formulæ , we claim the following equation holds,

$$\partial_z\left(z\mathscr{C}_1(z)\right) = 1 - R_{11}(z). \tag{3.131}$$

First off, recall that from the general Theorem 3.1.1,

$$\mathscr{C}_1^{\mathsf{c}}(z) = \left(Y_N^{-1}(z)Y_N'(z)\right)_{1,1} = \mathrm{tr}\,\left(Y_N^{-1}(z)Y_N'(z)\frac{\sigma_3}{2}\right). \tag{3.132}$$

Plugging in the above the definition (3.109) for $R(z)$, the one (3.106) for $\Psi_N(z)$ and the differential equation (3.108) yields

$$\mathscr{C}_1^{\mathsf{c}}(z) = \mathrm{tr}\,\left(A(z)R(z)\right) - \frac{1}{2}\left(\frac{\alpha}{z}-1\right). \tag{3.133}$$

Following the claim (3.131) we then compute

$$\partial_z\left(z\mathscr{C}_1(z)\right) = \mathrm{tr}\,\left(A(z)R(z)\right) - \frac{1}{2}\left(\frac{\alpha}{z}-1\right) + z\left(\mathrm{tr}\,\left((\partial_z A z)R(z)\right) + \mathrm{tr}\,\left(A(z)\partial_z R(z)\right) + \frac{\alpha^2}{2z}\right). \tag{3.134}$$

Notice that the terms in z^{-1} cancel out, as does

$$\mathrm{tr}\,\left(A(z)\partial_z R(z)\right) = \mathrm{tr}\,\left(A(z)\left[A(z)R(z)\right]\right) = \mathrm{tr}\,\left(\left[A^2(z)R(z)\right]\right), \tag{3.135}$$

being the trace of a commutator. On the other side, from (3.108)

$$z\,\partial_z A(z) = -A(z) - \frac{\sigma_3}{2}. \tag{3.136}$$

so that (3.134) simplifies to

$$\begin{aligned} \partial_z\left(z\mathscr{C}_1(z)\right) &= \mathrm{tr}\,\left(A(z)R(z)\right) + \frac{1}{2} - \mathrm{tr}\,\left(\left(A(z)+\frac{\sigma_3}{2}\right)R(z)\right) && (3.137)\\ &= \frac{1}{2} - \mathrm{tr}\,\left(\frac{\sigma_3}{2}R(z)\right) = 1 - R_{11}(z). && (3.138)\end{aligned}$$

Upon integrating and using formulæ of Remark 3.1.2 to pass from the $\mathscr{C}$ to the $\mathscr{F}$ generating functions, we retrieve exactly (3.92). □

Remark 3.2.11. *Let us remark that the recursion for $\mathcal{A}_\ell(N,M)$ in Lemma 3.2.9 is also deduced, by different means, in [107]. In [52] it is pointed out that such three term recursion is a manifestation of the fact that $\mathcal{A}_\ell(N,M)$ is expressible in terms of hypergeometric orthogonal polynomials; this property extends to the entries $\mathcal{B}_\ell(N,M)$, as we now show. Introducing the generalized hypergeometric function ${}_3F_2$*

$$ {}_3F_2\left(\begin{matrix} p_1,\ p_2,\ p_3 \\ q_1,\ q_2 \end{matrix}\middle| \zeta\right) := \sum_{j\geq 0} \frac{(p_1)_j (p_2)_j (p_3)_j}{(q_1)_j (q_2)_j} \frac{\zeta^j}{j!} \tag{3.139} $$

we can rewrite the coefficients $\mathcal{A}_\ell(N,M)$ and $\mathcal{B}_\ell(N,M)$ in the form

$$ \mathcal{A}_\ell(N,M) := \frac{(N)_\ell (M)_\ell}{\ell!} {}_3F_2\left(\begin{matrix} 1-N,\ 1-M,\ 1-\ell \\ 1-N-\ell,\ 1-M-\ell \end{matrix}\middle| 1\right), \tag{3.140} $$

$$ \mathcal{B}_\ell(N,M) := \frac{(N)_\ell (M)_\ell}{\ell!} {}_3F_2\left(\begin{matrix} 1-N,\ 1-M,\ -\ell \\ 1-N-\ell,\ 1-M-\ell \end{matrix}\middle| 1\right). \tag{3.141} $$

Alternatively, introducing the Hahn and dual Hahn polynomials [52, 128]

$$ Q_j(x;\mu,\nu,k) := {}_3F_2\left(\begin{matrix} -x,\ j+\mu+\nu+1,\ -j \\ -k,\ \mu+1 \end{matrix}\middle| 1\right), \tag{3.142} $$

$$ R_j(\lambda(x);\gamma,\delta,k) := {}_3F_2\left(\begin{matrix} -j,\ x+\gamma+\delta+1,\ -x \\ -k,\ \gamma+1 \end{matrix}\middle| 1\right), \qquad \lambda(x) = x(x+\gamma+\delta+1) \tag{3.143} $$

the coefficients $\mathcal{A}_\ell(N,M)$ and $\mathcal{B}_\ell(N,M)$ can be rewritten in the form

$$ \frac{\ell!}{(N)_\ell (M)_\ell} \mathcal{A}_\ell(N,M) = Q_{\ell-1}(N-1;-M-\ell,1,N+\ell-1) = R_{N-1}(\ell-1;-M-\ell,1,N+\ell-1), \tag{3.144} $$

$$ \frac{\ell!}{(N)_\ell (M)_\ell} \mathcal{B}_\ell(N,M) = Q_{\ell}(N-1;-M-\ell,0,N+\ell-1) = R_{N-1}(\ell;-M-\ell,0,N+\ell-1). \tag{3.145} $$

3.2.3 Correlators generating functions for JUE

Theorem 3.2.12 ([93])**.** *Introduce the matrix-valued formal series*

$$ R^{[\infty]}(z) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \sum_{\ell\geq 0} \frac{1}{z^{\ell+1}} \frac{1}{\alpha+\beta+2N} \begin{pmatrix} \ell\mathcal{A}_\ell(N) & N(\alpha+N)(\beta+N)(\alpha+\beta+N)\mathcal{B}_\ell(N+1) \\ -\mathcal{B}_\ell(N) & -\ell\mathcal{A}_\ell(N) \end{pmatrix} \tag{3.146} $$

$$ R^{[0]}(z) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \sum_{\ell\geq 0} \frac{z^\ell}{\alpha+\beta+2N} \begin{pmatrix} (\ell+1)\widetilde{\mathcal{A}}_\ell(N) & -N(\alpha+N)(\beta+N)(\alpha+\beta+N)\widetilde{\mathcal{B}}_\ell(N+1) \\ \widetilde{\mathcal{B}}_\ell(N) & -(\ell+1)\widetilde{\mathcal{A}}_\ell(N) \end{pmatrix} \tag{3.147} $$

where

$$\mathcal{A}_0(N) = \frac{N(\beta+N)}{\alpha+\beta+2N}, \tag{3.148}$$

$$\mathcal{A}_\ell(N) = \frac{N(\alpha+N)(\beta+N)(\alpha+\beta+N)(\alpha+2)_{\ell-1}}{(\alpha+\beta+2N-1)_{\ell+2}}\,{}_4F_3\left(\left.\begin{matrix}1-\ell,\ \ell+2,\ 1-\beta-N,\ 1-N\\ 2,\ \alpha+2,\ 2-\alpha-\beta-2N\end{matrix}\right|1\right), \quad \ell \geq 1, \tag{3.149}$$

$$\mathcal{B}_\ell(N) = \frac{(\alpha+1)_\ell}{(\alpha+\beta+2N-1)_{\ell+1}}\,{}_4F_3\left(\left.\begin{matrix}-\ell,\ \ell+1,\ 1-\beta-N,\ 1-N\\ 1,\ \alpha+1,\ 2-\alpha-\beta-2N\end{matrix}\right|1\right), \quad \ell \geq 0, \tag{3.150}$$

and

$$\widetilde{\mathcal{A}}_\ell(N) = \frac{(\alpha+\beta+2N-\ell)_{2\ell+1}}{(\alpha-\ell)_{2\ell+1}}\mathcal{A}_\ell(N), \quad \widetilde{\mathcal{B}}_\ell(N) = \frac{(\alpha+\beta+2N-1-\ell)_{2\ell+1}}{(\alpha-\ell)_{2\ell+1}}\mathcal{B}_\ell(N), \quad \ell \geq 0. \tag{3.151}$$

Then, the one-point correlators generating functions introduced in Remark 3.1.2 are

$$\mathscr{F}_{1,\infty}(z) = \frac{\alpha+\beta+2N}{z(1-z)}\int_\infty^z \left(1 - R_{1,1}^{[\infty]}(w)\right)\mathrm{d}w - \frac{N(\alpha+N)}{z(1-z)(\alpha+\beta+2N)}, \tag{3.152}$$

$$\mathscr{F}_{1,0}(z) = \frac{\alpha+\beta+2N}{z(1-z)}\int_0^z \left(1 - R_{1,1}^{[0]}(w)\right)\mathrm{d}w - \frac{N}{1-z}, \tag{3.153}$$

and similarly, the multipoint generating functions admit the expression

$$\mathscr{F}_{2,p}^{\mathsf{c}}(z_1,z_2) = \frac{\mathrm{tr}\,\left(R^{[p]}(z_1)R^{[p]}(z_2)\right) - 1}{(z_1-z_2)^2}, \tag{3.154}$$

$$\mathscr{F}_{\ell,p}^{\mathsf{c}}(z_1,\dots,z_\ell) = -\sum_{(i_1,\dots,i_\ell)\in\mathrm{cyc}((\ell))} \frac{\mathrm{tr}\,\left(R^{[p]}(z_{i_1})\dots R^{[p]}(z_{i_\ell})\right)}{(z_{i_1}-z_{i_2})\cdots(z_{i_\ell}-z_{i_1})}, \quad \ell \geq 3, \qquad p = 0,\infty. \tag{3.155}$$

The proof is similar to that of the previous section for the LUE. We start by recalling the explicit expression for the monic Jacobi polynomials (1.65) defined in Section 1.2.1

$$P_\ell^{\mathsf{J}}(z) = \frac{\ell!}{(\alpha+\beta+\ell+1)_\ell}\sum_{k=0}^{\ell}\binom{\ell+\alpha}{k}\binom{\ell+\beta}{\ell-k}(z-1)^k z^{\ell-k}, \tag{3.156}$$

and then by computing their Cauchy transforms.

Lemma 3.2.13. *The following relations hold true;*

$$\widehat{P}_\ell^{\mathsf{J}}(z) = -\frac{1}{2\pi\mathrm{i}(\alpha+\beta+\ell+1)_\ell}\sum_{j\geq 0}\frac{1}{z^{j+\ell+1}}(j+1)_\ell\frac{\Gamma(\alpha+\ell+j+1)\Gamma(\beta+\ell+1)}{\Gamma(\alpha+\beta+2\ell+j+1)}, \quad |z| > 1, \tag{3.157}$$

$$\widehat{P}_\ell^{\mathsf{J}}(z) \overset{z\to 0}{\sim} (-1)^\ell\frac{1}{2\pi\mathrm{i}(\alpha+\beta+\ell+1)_\ell}\sum_{j\geq 0} z^j (j+1)_\ell\frac{\Gamma(\alpha-j)\Gamma(\beta+\ell+1)}{\Gamma(\alpha+\beta+\ell-j+1)}, \tag{3.158}$$

where the first relation is a genuine Taylor expansion at $z=\infty$ whilst the second one is a Poincaré asymptotic expansion at $z=0$ uniform in the sector $0<\arg z<2\pi$.

Proof. We start with the expansion (3.157) at $z=\infty$, which is computed as follows;

$$
\begin{aligned}
\widehat{P^{\mathsf{J}}_\ell}(z) &= \frac{1}{2\pi\mathrm{i}}\int_0^1 P^{\mathsf{J}}_\ell(x)x^\alpha(1-x)^\beta\frac{\mathrm{d}x}{x-z}\\
&\overset{(i)}{=} -\frac{1}{2\pi\mathrm{i}}\sum_{j\geq 0}\frac{1}{z^{j+1}}\int_0^1 P^{\mathsf{J}}_\ell(x)x^{\alpha+j}(1-x)^\beta\mathrm{d}x\\
&\overset{(ii)}{=} -\frac{1}{2\pi\mathrm{i}}\sum_{j\geq 0}\frac{1}{z^{j+\ell+1}}\int_0^1 P^{\mathsf{J}}_\ell(x)x^{\alpha+j+\ell}(1-x)^\beta\mathrm{d}x\\
&\overset{(iii)}{=} -\frac{1}{2\pi\mathrm{i}}\frac{(-1)^\ell}{(\alpha+\beta+\ell+1)_\ell}\sum_{j\geq 0}\frac{1}{z^{j+\ell+1}}\int_0^1\left(\frac{\mathrm{d}^\ell}{\mathrm{d}x^\ell}x^{\alpha+\ell}(1-x)^{\beta+\ell}\right)x^{j+\ell}\mathrm{d}x\\
&\overset{(iv)}{=} -\frac{1}{2\pi\mathrm{i}}\frac{1}{(\alpha+\beta+\ell+1)_\ell}\sum_{j\geq 0}\frac{1}{z^{j+\ell+1}}\int_0^1 x^{\alpha+\ell}(1-x)^{\beta+\ell}\frac{\mathrm{d}^\ell}{\mathrm{d}x^\ell}(x^{j+\ell})\mathrm{d}x\\
&\overset{(v)}{=} -\frac{1}{2\pi\mathrm{i}}\frac{1}{(\alpha+\beta+\ell+1)_\ell}\sum_{j\geq 0}\frac{(j+1)_\ell}{z^{j+\ell+1}}\int_0^1 x^{\alpha+\ell+j}(1-x)^{\beta+\ell}\mathrm{d}x\\
&\overset{(vi)}{=} -\frac{1}{2\pi\mathrm{i}}\frac{1}{(\alpha+\beta+\ell+1)_\ell}\sum_{j\geq 0}\frac{1}{z^{j+\ell+1}}(j+1)_\ell\frac{\Gamma(\alpha+\ell+j+1)\Gamma(\beta+\ell+1)}{\Gamma(\alpha+\beta+2\ell+j+1)}.
\end{aligned}
\tag{3.159}
$$

In (i) we have expanded the geometric series and exchanged sum and integral by Fubini theorem, in (ii) we use that $P^{\mathsf{J}}_\ell(z)$ is orthogonal to z^j for $j<\ell$, in (iii) we use the Rodrigues' formula (1.64), in (iv) we integrate by parts, in (v) we compute the derivative, and finally in (vi) we use the Euler beta integral. The computation at $z=0$ is completely analogous, with the only difference that in (i) it is not legitimate to exchange sum and integral so this step holds only in the sense of a Poincaré asymptotic series. □

Unlike the Laguerre case, in the Jacobi one is not trivial to get a succinct expression for the product of polynomials with their Cauchy transforms. However, we bypass this problem exploiting directly the associated matrix differential equation and the recurrence it induces on the entries of $R(z)$. Indeed, we have the following Proposition analogue to Proposition 3.2.8.

Proposition 3.2.14. *Consider the following matrix obtained dressing the matrix $Y_N(z)$ of Jacobi polynomials,*

$$\Psi_N(z) := Y_N(z)z^{\alpha\sigma_3/2}(1-z)^{\beta\sigma_3/2}. \tag{3.160}$$

Then, $\Psi_N(z)$ satisfies the following linear differential equation

$$\partial_z\Psi_N(z) = U(z)\Psi_N(z) \tag{3.161}$$

and the matrix $R(z)$ satisfies the following Lax differential equation,

$$\partial_z R(z) = [U(z), R(z)]. \tag{3.162}$$

Here the matrix $U(z)$ is explicitly given as

$$U(z) = \frac{U_0}{z} + \frac{U_1}{1-z}, \tag{3.163}$$

with

$$U_0 = \begin{pmatrix} \frac{2N(\alpha+\beta+N)+\alpha(\alpha+\beta)}{2(\alpha+\beta+2N)} & -\frac{h^{\mathsf{J}}_N}{2\pi\mathrm{i}}(\alpha+\beta+2N+1) \\ \frac{2\pi\mathrm{i}}{h^{\mathsf{J}}_{N-1}}(\alpha+\beta+2N-1) & -\frac{2N(\alpha+\beta+N)+\alpha(\alpha+\beta)}{2(\alpha+\beta+2N)} \end{pmatrix}, \tag{3.164}$$

$$U_1 = \begin{pmatrix} -\frac{2N(\alpha+\beta+N)+\beta(\alpha+\beta)}{2(\alpha+\beta+2N)} & -\frac{h^{\mathsf{J}}_N}{2\pi\mathrm{i}}(\alpha+\beta+2N+1) \\ \frac{2\pi\mathrm{i}}{h^{\mathsf{J}}_{N-1}}(\alpha+\beta+2N-1) & \frac{2N(\alpha+\beta+N)+\beta(\alpha+\beta)}{2(\alpha+\beta+2N)} \end{pmatrix}. \tag{3.165}$$

Proof. Equation (3.162) is derived as in the Laguerre case, see (3.110). Equation (3.161) is a classical property of Jacobi orthogonal polynomials, see [115]. □

We can now prove the claimed asymptotic expansions.

Proposition 3.2.15. *We have the Taylor expansion at $z=\infty$*

$$TR(z)T^{-1} \sim R^{[\infty]}(z), \qquad z\to\infty, \tag{3.166}$$

where T is the constant matrix

$$T = \begin{pmatrix} 1 & 0 \\ 0 & \frac{h^{\mathsf{J}}_{N-1}}{2\pi i}\frac{1}{(\alpha+\beta+2N)(\alpha+\beta+2N-1)} \end{pmatrix}, \tag{3.167}$$

the h^{J}_{N-1}'s have been defined in (1.63) and $R^{[\infty]}(z)$ is the matrix-valued power series in z^{-1} in (3.146).

Proof. As in Lemma 3.2.9, in proving Proposition 3.2.15 we can perform our computations via the matrix $TR(z)T^{-1}$ since, again, generating functions for the correlators are invariant under gauge transformations of $R(z)$. It follows from Proposition 3.2.14 that

$$\frac{\partial}{\partial z}TR(z)T^{-1} = [TU(z)T^{-1}, TR(z)T^{-1}], \qquad U(z) = \frac{U_0}{z} + \frac{U_1}{1-z}, \tag{3.168}$$

and we can explicitly compute

$$TU_0T^{-1} = \frac{1}{\alpha+\beta+2N}\begin{pmatrix} \frac{2N(\alpha+\beta+N)+\alpha(\alpha+\beta)}{2} & -N(\alpha+N)(\beta+N)(\alpha+\beta+N) \\ 1 & -\frac{2N(\alpha+\beta+N)+\alpha(\alpha+\beta)}{2} \end{pmatrix},$$

$$TU_1T^{-1} = \frac{1}{\alpha+\beta+2N}\begin{pmatrix} -\frac{2N(\alpha+\beta+N)+\beta(\alpha+\beta)}{2} & -N(\alpha+N)(\beta+N)(\alpha+\beta+N) \\ 1 & \frac{2N(\alpha+\beta+N)+\beta(\alpha+\beta)}{2} \end{pmatrix}. \tag{3.169}$$

Introduce the matrices

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \tag{3.170}$$

and write

$$TR(z)T^{-1} = \frac{1}{2}\mathbf{1} + r_3\sigma_3 + r_+\sigma_+ + r_-\sigma_-, \qquad TU(z)T^{-1} = u_3\sigma_3 + u_+\sigma_+ + u_-\sigma_-, \tag{3.171}$$

where we used that $\operatorname{tr} R(z) = 1, \operatorname{tr} U(z) = 0$. For the sake of brevity we omit the dependence on z in the $\mathfrak{sl}_2$ components. The Lax equation (3.168) yields the coupled first order linear ODEs

$$\partial_z r_3 = u_+r_- - u_-r_+, \qquad \partial_z r_+ = 2(u_3r_+ - u_+r_3), \qquad \partial_z r_- = 2(u_-r_3 - u_3r_-), \tag{3.172}$$

which is equivalent to three decoupled third order linear ODEs, one for $\partial_z r_3$

$$\begin{aligned}3\left[2N(\alpha+\beta+N)+\alpha(\alpha+\beta)-2-z\left((\alpha+\beta+2N)^2-4\right)\right]\partial_z r_3\\ -\left[\alpha^2-4-2z\left(2N(\alpha+\beta+N)+\alpha(\alpha+\beta)-12\right)+z^2\left((\alpha+\beta+2N)^2-24\right)\right]\partial_z^2 r_3\\ -5z(z-1)(1-2z)\partial_z^3 r_3+z^2(z-1)^2\partial_z^4 r_3=0,\end{aligned} \tag{3.173}$$

and for $r_\pm$

$$\begin{aligned}\left[2N(\alpha+\beta+N\pm1)+(\alpha\pm1)(\alpha+\beta)-z(\alpha+\beta+2N\pm2)(\alpha+\beta+2N)\right]r_\pm\\ -\left[\alpha^2-1-z\left(-4N(\alpha+\beta+N\pm1)-2(\alpha+\beta)(\alpha\pm1)+6\right)\right.\\ \left.+z^2\left(4N(\alpha+\beta+N\pm1)+(\alpha+\beta)(\alpha+\beta\pm2)-6\right)\right]\partial_z r_\pm+3z(z-1)(2z-1)\partial_z^2 r_\pm+z^2(z-1)^2\partial_z^3 r_\pm=0.\end{aligned} \tag{3.174}$$

The following ansatz is quite natural in view of the Laguerre case scenario, see Lemma 3.2.9namely we write the expansions of the entries of $R(z)$ at $z=\infty$ as

$$r_3(z)\sim\frac{1}{2}+\frac{1}{\alpha+\beta+2N}\sum_{\ell\geq0}\frac{1}{z^{\ell+1}}\ell\mathcal{A}_\ell(N), \tag{3.175}$$

$$r_+(z)\sim\frac{1}{\alpha+\beta+2N}\sum_{\ell\geq0}\frac{N(\alpha+N)(\beta+N)(\alpha+\beta+N)}{z^{\ell+1}}\mathcal{B}_\ell(N+1), \tag{3.176}$$

$$r_-(z)\sim-\frac{1}{\alpha+\beta+2N}\sum_{\ell\geq0}\frac{1}{z^{\ell+1}}\mathcal{B}_\ell(N), \tag{3.177}$$

for some coefficients $\mathcal{A}_\ell(N)=\mathcal{A}_\ell(N,\alpha,\beta)$ and $\mathcal{B}_\ell(N)=\mathcal{B}_\ell(N,\alpha,\beta)$. By substitution in (3.173) and (3.174) we see that the ansatz is consistent with them; in particular we get the following *three term recurrence relations* for $\mathcal{A}_\ell(N),\mathcal{B}_\ell(N)$,

$$\begin{aligned}(2\ell+1)\left(\alpha(\alpha+\beta)-\ell(\ell+1)+2N(\alpha+\beta+N)\right)\mathcal{A}_\ell(N)\\ +(\ell-1)(\ell^2-\alpha^2)\mathcal{A}_{\ell-1}(N)+(\ell+2)\left((\ell+1)^2-(\alpha+\beta+2N)\right)\mathcal{A}_{\ell+1}(N)=0,\end{aligned} \tag{3.178}$$

$$\begin{aligned}(2\ell+1)\left((\alpha-1)(\alpha+\beta)-\ell(\ell+1)+2N(\alpha+\beta+N-1)\right)\mathcal{B}_\ell(N)\\ +\ell(\ell^2-\alpha^2)\mathcal{B}_{\ell-1}(N)+(\ell+1)\left((\ell+1)^2-(\alpha+\beta+2N-1)\right)\mathcal{B}_{\ell+1}(N)=0,\end{aligned} \tag{3.179}$$

for $\ell\geq1$, together with the initial conditions

$$\mathcal{A}_0(N,\alpha,\beta)=\frac{N(\beta+N)}{\alpha+\beta+2N},\qquad \mathcal{A}_1(N,\alpha,\beta)=\frac{N(\alpha+N)(\beta+N)(\alpha+\beta+N)}{(\alpha+\beta+2N-1)(\alpha+\beta+2N)(\alpha+\beta+2N+1)}, \tag{3.180}$$

$$\mathcal{B}_0(N,\alpha,\beta)=\frac{1}{(\alpha+\beta+2N-1)},\qquad \mathcal{B}_1(N,\alpha,\beta)=\frac{(\alpha-1)(\alpha+\beta)+2N(\alpha+\beta+N-1)}{(\alpha+\beta+2N-2)(\alpha+\beta+2N-1)(\alpha+\beta+2N)}. \tag{3.181}$$

The initial conditions are obtained from (3.156) and (3.157). It can be checked that the recurrence relation for the coefficients of $r_+(z)$ are actually those of $r_-(z)$, modulo a shift in N, as claimed in (3.177). □

Analogously to the Laguerre case, the three term recurrence relations (3.178) and (3.179) can be solved in terms of *hypergeometric orthogonal polynomials*, specifically *Wilson polynomials*.

Remark 3.2.16. *In this case as well, the coefficients $\mathcal{A}_\ell(N,\alpha,\beta)$ and $\mathcal{B}_\ell(N,\alpha,\beta)$ can be expressed in terms of discrete orthogonal polynomials, namely Wilson Polynomials. They are defined as*

$$\frac{W_n(k^2;a,b,c,d)}{(a+b)_n(a+c)_n(a+d)_n} := {}_4F_3\left(\left.\begin{matrix} -n,\ n+a+b+c+d-1,\ a+\mathrm{i}k,\ a-\mathrm{i}k \\ a+b,\ a+c,\ a+d \end{matrix}\right| 1\right), \tag{3.182}$$

and the identification is the following

$$\mathcal{A}_\ell(N,\alpha,\beta) = \frac{(-1)^{N-1}(\alpha+\ell)!(\alpha+\beta+N)!(\beta+N)}{(N-1)!(\alpha+N-1)!(\alpha+\beta+2N+\ell)!} W_{N-1}\left(-\left(\ell+\frac{1}{2}\right)^2; \frac{3}{2},\frac{1}{2},\alpha+\frac{1}{2},\frac{1}{2}-\alpha-\beta-2N\right),$$
$$\mathcal{B}_\ell(N,\alpha,\beta) = \frac{(-1)^{N-1}(\alpha+\ell)!(\alpha+\beta+N-1)!}{(N-1)!(\alpha+N-1)!(\alpha+\beta+2N+\ell-1)!} W_{N-1}\left(-\left(\ell+\frac{1}{2}\right)^2; \frac{1}{2},\frac{1}{2},\alpha+\frac{1}{2},\frac{3}{2}-\alpha-\beta-2N\right),$$

or equivalently in hypergeometric notation

$$\mathcal{A}_\ell(N,\alpha,\beta) = N(\alpha+N)(\beta+N)(\alpha+\beta+N)\frac{(\alpha+2)_{\ell-1}}{(\alpha+\beta+2N-1)_{\ell+2}}\,{}_4F_3\left(\left.\begin{matrix} 1-\ell,\ \ell+2,\ 1-\beta-N,\ 1-N \\ 2,\ \alpha+2,\ 2-\alpha-\beta-2N \end{matrix}\right| 1\right),$$
$$\mathcal{B}_\ell(N,\alpha,\beta) = \frac{(\alpha+1)_\ell}{(\alpha+\beta+2N-1)_{\ell+1}}\,{}_4F_3\left(\left.\begin{matrix} -\ell,\ \ell+1,\ 1-\beta-N,\ 1-N \\ 1,\ \alpha+1,\ 2-\alpha-\beta-2N \end{matrix}\right| 1\right).$$

In this case, the identification with is obtained by comparing the recurrence relations (3.178) and (3.179) with the difference equation for this family of orthogonal polynomials, which reads

$$n(n+a+b+c+d-1)w(k) = C(k)w(k+\mathrm{i}) - [C(k)+D(k)]\,w(k) + D(k)w(k-\mathrm{i}), \tag{3.183}$$

where $w(k) = W_n(k^2;a,b,c,d)$ and

$$C(k) = \frac{(a-\mathrm{i}k)(b-\mathrm{i}k)(c-\mathrm{i}k)(d-\mathrm{i}k)}{2\mathrm{i}k(2\mathrm{i}k-1)}, \quad D(k) = \frac{(a+\mathrm{i}k)(b+\mathrm{i}k)(c+\mathrm{i}k)(d+\mathrm{i}k)}{2\mathrm{i}k(2\mathrm{i}k+1)}. \tag{3.184}$$

The hypergeometric representation of $\mathcal{A}_\ell, \mathcal{B}_\ell$ then directly follows from that of the Wilson polynomials in (3.182).

The asymptotic expansion of $R(z)$ at $z=0$ is obtained in a similar way.

Proposition 3.2.17. *We have the Poincaré asymptotic expansion at $z=0$ uniformly within the sector $0<\arg z<2\pi$*

$$TR(z)T^{-1} \sim R^{[0]}(z), \tag{3.185}$$

where T is the constant matrix (3.167) and $R^{[0]}(z)$ is the matrix-valued (formal) power series in z in (3.147).

Proof. We claim that the expansion at $z=0$ of the entries of $\widehat{R}(z)$ reads as

$$r_3(z) \sim \frac{1}{2} + \frac{1}{\alpha+\beta+2N}\sum_{\ell\geq 0}\frac{(\alpha+\beta+2N-\ell)_{2\ell+1}}{(\alpha-\ell)_{2\ell+1}}(\ell+1)\mathcal{A}_\ell(N,\alpha,\beta)z^\ell,$$
$$r_+(z) \sim -\frac{N(\beta+N)(\alpha+N)(\alpha+\beta+N)}{\alpha+\beta+2N}\sum_{\ell\geq 0}\frac{(\alpha+\beta+2N+1-\ell)_{2\ell+1}}{(\alpha-\ell)_{2\ell+1}}\mathcal{B}_\ell(N+1,\alpha,\beta)z^\ell,$$
$$r_-(z) \sim \frac{1}{\alpha+\beta+2N}\sum_{\ell\geq 0}\frac{(\alpha+\beta+2N-1-\ell)_{2\ell+1}}{(\alpha-\ell)_{2\ell+1}}\mathcal{B}_\ell(N,\alpha,\beta)z^\ell. \tag{3.186}$$

This can be proven by checking that plugging the formulæ (3.186) in the equations (3.173), (3.174), one obtains the *same* recurrence relations (3.178) and (3.179). The associated initial conditions can again be computed from (3.156) and (3.158). This concludes the proof of Proposition 3.2.15. ◻

We now complete the proof of Theorem 3.2.12 deriving formulæ(3.152), (3.153) and (3.155).

Proof of Theorem 3.2.12. From the general Theorem 3.1.1, the definition of $R(z)$ and the differential equation (3.161) we have

$$\mathscr{C}_1(z) = \mathrm{tr}\,\left(Y_N^{-1}(z)Y_N'(z)\mathrm{E}_{1,1}\right) = \mathrm{tr}\,(U(z)R(z)) - \frac{1}{2}\left(\frac{\alpha}{z} - \frac{\beta}{1-z}\right). \tag{3.187}$$

On the other side, we notice that the following holds

$$\partial_z\left[z(1-z)\mathrm{tr}\,(U(z)R(z))\right] = (1-2z)\mathrm{tr}\,(U(z)R(z)) + z(1-z)\mathrm{tr}\,(U'(z)R(z)) \tag{3.188}$$
$$+ z(1-z)\mathrm{tr}\,(U(z)R'(z)) \tag{3.189}$$
$$= -(\alpha+\beta+2N)\mathrm{tr}\,\left(R(z)\frac{\sigma_3}{2}\right). \tag{3.190}$$

where we used that $\mathrm{tr}\,(U(z)[U(z),R(z)]) = 0$ and the identity

$$(1-2z)U(z) + z(1-z)U'(z) = -\frac{\alpha+\beta+2N}{2}\sigma_3, \tag{3.191}$$

which can be checked directly from (3.163). Equation (3.187) and (3.190) imply

$$\partial_z[z(1-z)\mathscr{C}_1(z)] = -(\alpha+\beta+2N)\left(R_{11}(z) - \frac{1}{2}\right) + \frac{\alpha+\beta}{2} = -(\alpha+\beta+2N)\left(R_{11}(z)-1\right) - N, \tag{3.192}$$

which, upon integrating, yields for any $p \in \mathbb{C}\setminus[0,1]$

$$z(1-z)\mathscr{C}_1(z) - p(1-p)\mathscr{C}_1(p) = (\alpha+\beta+2N)\int_p^z \left(1 - R_{1,1}(w)\right)\mathrm{d}w + N(p-z). \tag{3.193}$$

Letting $p \to \infty$ we have $p(1-p)\mathscr{C}_1(p) \sim (1-p)N - \langle \mathrm{tr}\, X\rangle + \mathcal{O}(1/p)$ and therefore from (3.193) we have (noting that $R_{11}(w) = 1 + \mathcal{O}(w^{-2})$ so the integral is well defined)

$$z(1-z)\mathscr{C}_1(z) = (\alpha+\beta+2N)\int_\infty^z \left(1 - R_{11}(w)\right)\mathrm{d}w + (1-z)N - \langle \mathrm{tr}\, X\rangle\,. \tag{3.194}$$

We can compute

$$\langle \mathrm{tr}\, X\rangle = \frac{N(\alpha+N)}{\alpha+\beta+2N} \tag{3.195}$$

by expanding the general formula $\mathscr{C}_1(z) = \mathrm{tr}\,\left(Y_N^{-1}(z)Y_N'(z)\sigma_3/2\right)$ at $z=\infty$, using (3.156) and the first few terms in (3.157). We finally obtain

$$\mathscr{C}_1(z) = \frac{\alpha+\beta+2N}{z(1-z)}\int_\infty^z \left(1 - R_{1,1}(w)\right)\mathrm{d}w + \frac{N}{z} - \frac{N(\alpha+N)}{z(1-z)(\alpha+\beta+2N)}. \tag{3.196}$$

In view of Remark 3.1.2 and Proposition 3.2.15 (note that $(TR(z)T)_{11} = R_{11}(z)$ because T is diagonal) the formula for $\mathscr{F}_{1,\infty}(z)$ is proved.

Letting instead $p \to 0$ in (3.193) we have $p(1-p)\mathscr{C}_1(p) \to 0$ and so

$$\mathscr{C}_1(z) = \frac{(\alpha+\beta+2N)}{z(1-z)}\int_0^z \left(1 - R_{11}(w)\right)\mathrm{d}w - \frac{N}{1-z}. \tag{3.197}$$

Expanding this identity at $z=0$ we get at the left hand side

$$\mathscr{C}_1(z) \sim -\sum_{k\geq 0}\left\langle \mathrm{tr}\, X^{-k-1}\right\rangle z^k = \mathscr{F}_{1,0}(z), \tag{3.198}$$

and again by Remark 3.1.2 and Proposition 3.2.17 the formula for $\mathscr{F}_{1,0}(z)$ is proved. $\square$

Chapter 4

Combinatorics of classical unitary invariant ensembles

In this chapter we provide a combinatorial interpretation to the partition functions of classical unitary Invariant Ensembles. Specifically we show how their (positive and negative) correlators admit a *topological expansion* in even powers of N of the type

$$\left\langle \operatorname{tr} X^{k_1} \cdots \operatorname{tr} X^{k_\ell} \right\rangle \sim \sum_{g\geq 0} \frac{f_g^{(k_1,\dots,k_\ell)}}{N^{2g-2}}, \qquad N \to \infty, \tag{4.1}$$

where the coefficients $f_g^{(k_1,\dots,k_\ell)}$ have meaningful combinatorial quantities. For the GUE it has been know since the seminal work of Bessis, Itzykson and Zuber [28, 158], that these coefficients count the number of *ribbon graphs* whose features (number of vertices, valencies and genus) are all encoded by the set of parameters $(g, k_1, \dots, k_\ell)$ appearing in (4.1). This case has been extensively studied in the literature [42, 63] and we will not dwell on it further here.

We analyze the LUE in Section 4.2. The combinatorial interpretation for positive correlators has been folklore in the literature, remarkably first appearing in a paper in *The Annals of Statistics* [101], and later systematized by Collins et al. in [50] via the theory of Weingarten functions and later (also extended to negative correlators) by Cunden et al. [51, 53]. The quantities of interest are, in this case, combinations of double monotone Hurwitz numbers where, for one of the two fixed partitions, only its length is distinguishable (and not its parts). Positive correlators are related to *strictly* double monotone Hurwitz numbers, while the negative ones involve their *weakly monotone* version.

In Section 4.1 we supply the combinatorial interpretation for correlators of the JUE; it is one of the original results in this thesis. They are *weighted multiparametric single Hurwitz numbers*, involving combinations of *triple weakly monotone Hurwitz numbers* where, for two of the three fixed partitions, only the length is distinguishable. Interestingly, positive and negative correlators carry the same combinatorial coefficients. The proof is inspired by theory of hypergeometric tau-functions [104, 112], see also Section 2.3, but is self contained and relies on an accurate use of the Selberg-Aomoto integral. This technique also allows us to reprove the results for the LUE, and as a consequence we also identify the Hurwitz numbers here involved as *multiparametric* Hurwitz numbers.

4.1 Combinatorics of the JUE correlators

The idea of the proof is to expand both the generating function of multiparametric Hurwitz numbers and the JUE partition function in the basis of symmetric functions given by the Schur polynomials and, then, identify the coefficients of these two formal power series. Remarkably, in both settings the coefficient can be explicitly written out. In the former, this happens thanks to the properties of the YJM elements, see (2.29) and Proposition 4.1.1 below. In the latter, this is due to the Selberg-Aomoto integral, see (4.29).

4.1.1 Preliminaries: expansions in Schur basis

We will need some notation. Denote the standard Schur polynomials in n variables $x_1, \dots, x_n$

$$\chi_\lambda(\underline{x}) := \frac{\det \left[x_j^{N-i+\lambda_i}\right]_{i,j=1}^N}{\Delta(\underline{x})}. \tag{4.2}$$

The definition of Schur polynomials can be extended to allow infinitely many variables by an inductive limit (see Chapter 1 in [131]). In this context, they are regarded as the basis $\{s_\lambda(\mathbf{t})\}$, with λ running in the set of all partitions, of the space of weighted homogeneous polynomials in $\mathbf{t} = (t_1, t_2, \dots)$, with $\deg t_k = k$,

$$s_\lambda(\mathbf{t}) = \det \left[h_{\lambda_i - i + j}(\mathbf{t})\right]_{i,j=1}^{\ell(\lambda)}, \tag{4.3}$$

where the complete homogeneous symmetric polynomials $h_k(\mathbf{t})$ are defined by the generating series

$$\sum_{k\geq 0} w^k h_k(\mathbf{t}) = \exp\left(\sum_{k\geq 1} \frac{t_k}{k} w^k\right). \tag{4.4}$$

We now prove the following general Proposition, see also [112].

Proposition 4.1.1 ([112])**.** *The generating function*

$$\tau_G(\epsilon; \mathbf{t}) = \sum_{d\geq 1} \epsilon^d \sum_{\lambda \in \mathcal{P}} h_d^G(\lambda) \prod_{i=1}^{\ell(\lambda)} t_{\lambda_i} \tag{4.5}$$

of multiparametric weighted Hurwitz numbers (2.31) *associated to the rational function* (2.30) *is equivalently expressed as*

$$\tau_G(\epsilon; \mathbf{t}) = \sum_{\lambda \in \mathcal{P}} \frac{\dim \lambda}{|\lambda|!} r_\lambda^{(G,\epsilon)} s_\lambda(\mathbf{t}), \tag{4.6}$$

where $s_\lambda(\mathbf{t})$ *are the Schur polynomials* (4.3) *and the coefficients are given explicitly by*

$$r_\lambda^{(G,\epsilon)} = \prod_{(i,j)\in\lambda} G(\epsilon(j-i)), \tag{4.7}$$

$\dim \lambda = \chi^\lambda_{1^{|\lambda|}}$ *being the dimension of the irreducible representation of* $\mathfrak{S}_{|\lambda|}$ *associated with* λ.

Proof. First off, let us recall formula (2.23)

$$\mathcal{E}_\lambda = \frac{\chi^\lambda_{1^d}}{d!} \sum_{\mu \vdash d} \chi^\mu_\lambda \mathcal{C}_\mu, \qquad \mathcal{E}_\lambda \mathcal{E}_{\lambda'} = \delta_{\lambda,\lambda'} \cdot \mathcal{E}_\lambda, \tag{4.8}$$

which gives a basis of idempotents for the class algebra $Z(\mathbb{C}[\mathfrak{S}_n])$, as well as equation (2.29) expressing the action of symmetric polynomials on the YJM elements on said basis,

$$p(\mathcal{J}_1, \dots, \mathcal{J}_n)\mathcal{E}_\lambda = p\left(\{j-i\}_{(i,j)\in\lambda}\right)\mathcal{E}_\lambda. \tag{4.9}$$

In particular this immediately implies

$$\left[\prod_{a=1}^{n} G(\epsilon \mathcal{J}_a)\right] \mathcal{E}_\lambda = r_\lambda^{(\epsilon,G)} \mathcal{E}_\lambda, \tag{4.10}$$

with the $r_\lambda^{(\epsilon,G)}$ defined in (4.7). Indeed, this is just (4.9) in the case the function $G(z)$ is a polynomials, but it remains valid if we expand via geometric series possible denominators, as the resulting series will still be symmetric. Using the orthogonality of the $\{\mathcal{E}_\lambda\}_{\lambda\vdash n}$ amongst themselves, equation (4.10) then yields

$$\prod_{a=1}^{n} G(\epsilon \mathcal{J}_a) = \sum_{\lambda \vdash n} r_\lambda^{(\epsilon,G)} \mathcal{E}_\lambda. \tag{4.11}$$

Secondly, recall the definition (2.31) for $h^d_G(\mu)$,

$$h^G_d(\mu) := \frac{1}{z_\mu}[\epsilon^d \mathcal{C}_\lambda] \prod_{a=1}^{n} G(\epsilon \mathcal{J}_a), \tag{4.12}$$

together with (4.11) allows to obtain the first equality in the following

$$\sum_{\mu\vdash n}\sum_{d\geq 1} \epsilon^d z_\mu h^G_d(\mu)\mathcal{C}_\mu = \sum_{\lambda\vdash n} r_\lambda^{(\epsilon,G)}\mathcal{E}_\lambda = \sum_{\lambda,\mu\vdash n} \frac{\dim\lambda}{|\lambda|!} r_\lambda^{(\epsilon,G)} \chi^\mu_\lambda \mathcal{C}_\mu, \tag{4.13}$$

the second being obtained by the definition of $\mathcal{E}_\lambda$. Since $\mathcal{C}_\mu$ form a basis of $Z(\mathbb{C}[\mathfrak{S}_n])$, comparing the left and right most sides of (4.13) we get that for any partition μ

$$\sum_{d\geq 1} \epsilon^d h^G_d(\mu) = \sum_{\lambda\vdash|\mu|} \frac{\dim\lambda}{|\lambda|!} r_\lambda^{(\epsilon,G)} \frac{\chi^\mu_\lambda}{z_\mu}. \tag{4.14}$$

Multiplying this identity by $\prod_{i=1}^{\ell(\mu)} t_{\mu_i}$ and summing over all partitions μ, on the left we obtain (4.5). On the right hand side, the well-known identity [131]

$$s_\lambda(\mathbf{t}) = \sum_{\mu\vdash|\lambda|} \frac{\chi^\mu_\lambda}{z_\mu} \prod_{i=1}^{\ell(\mu)} t_{\mu_i}, \tag{4.15}$$

yields

$$\sum_{d\geq 1} \epsilon^d \sum_{\mu\in\mathcal{P}} h^G_d(\mu) \prod_{i=1}^{\ell(\mu)} t_{\mu_i} = \sum_{\mu\in\mathcal{P}}\sum_{\lambda\vdash|\mu|} \frac{\dim\lambda}{|\lambda|!} r_\lambda^{(\epsilon,G)} \frac{\chi^\mu_\lambda}{z_\mu} \prod_{i=1}^{\ell(\mu)} t_{\mu_i} = \sum_{\lambda\in\mathcal{P}} \frac{\dim\lambda}{|\lambda|!} r_\lambda^{(G,\epsilon)} s_\lambda(\mathbf{t}). \tag{4.16}$$

The proof is complete. $\square$

As far as it concerns the JUE partition function, it is a general fact that Hermitian partition functions possess a Schur expansion which can be carried out by a use of the Cauchy Identity, see (4.20). The coefficients are identified with a N-dimensional integral; remarkably in the Jacobi and Laguerre cases these can be explicitly evaluated by means of the Selberg-Aomoto integral, see (4.29) below. The general Lemma is the following.

Lemma 4.1.2. *For any potential $V(x)$ ($x \in I$) we have*

$$\frac{\int_{\mathcal{H}_N(I)} \exp \operatorname{tr} \left(V(X) + \sum_{k\geq 1} \frac{u_k}{k} X^{\pm k} \right) \mathrm{d}X}{\int_{\mathcal{H}_N(I)} \exp \operatorname{tr} \left(V(X) \right) \mathrm{d}X} = \sum_{\lambda \in \mathcal{P}:\, \ell(\lambda) \leq N} c^{\pm}_{\lambda,N} s_\lambda(\mathbf{u}), \tag{4.17}$$

where the coefficients are

$$c^{\pm}_{\lambda,N} = \frac{\int_{I^N} \chi_\lambda(\underline{x}^{\pm 1}) \Delta^2(\underline{x}) \prod_{a=1}^N \exp\left[V(x_a)\right] \mathrm{d}^N \underline{x}}{\int_{I^N} \Delta^2(\underline{x}) \prod_{a=1}^N \exp\left[V(x_a)\right] \mathrm{d}^N \underline{x}}. \tag{4.18}$$

and we denote $\underline{x} = (x_1, \dots, x_N)$ and $\underline{x}^{-1} = (x_1^{-1}, \dots, x_N^{-1})$.

Proof. We have

$$\frac{\int_{\mathcal{H}_N(I)} \exp \operatorname{tr} \left(V(X) + \sum_{k\geq 1} \frac{u_k}{k} X^{\pm k} \right) \mathrm{d}X}{\int_{H_N(I)} \exp \operatorname{tr} \left(V(X) \right) \mathrm{d}X} = \frac{\int_{I^N} \Delta^2(\underline{x}) \prod_{a=1}^N \exp\left[V(x_a) + \sum_{k\geq 1} \frac{u_k}{k} x_a^{\pm k}\right] \mathrm{d}^N \underline{x}}{\int_{I^N} \Delta^2(\underline{x}) \prod_{a=1}^N \exp\left[V(x_a)\right] \mathrm{d}^N \underline{x}}, \tag{4.19}$$

where we use the standard decomposition $\mathrm{d}X = \Delta^2(\underline{x}) \mathrm{d}^N \underline{x} \mathrm{d}U$ of the Lebesgue measure into eigenvalues $\underline{x} = (x_1, \dots, x_N)$ and eigenvectors $U \in \mathrm{U}_N$ of the hermitian matrix $X = UXU^\dagger$, with $\mathrm{d}U$ a Haar measure on U_N (whose normalization is irrelevant as it cancels in (4.19) between numerator and denominator). The proof follows by an application of the identity

$$\exp\left[\sum_{k\geq 1} \frac{u_k}{k} (x_1^{\pm 1} + \dots + x_N^{\pm 1})^k\right] = \sum_{\lambda \in \mathcal{P}:\, \ell(\lambda) \leq N} \chi_\lambda(\underline{x}^{\pm 1}) s_\lambda(\mathbf{u}), \tag{4.20}$$

which is nothing but a form of Cauchy identity, see e.g. [157]. $\square$

Notice that the Schur polynomials evaluated at the reciprocal variables $\underline{x}^{-1}$ can be written in terms of standard ones $\underline{x}$ as follows.

Lemma 4.1.3. *For any partition $\lambda = (\lambda_1, \dots, \lambda_\ell)$ of length $\ell \leq N$ we have*

$$\chi_\lambda(\underline{x}^{-1}) = \left(\prod_{a=1}^N x_a^{-\lambda_1} \right) \chi_{\widehat{\lambda}}(\underline{x}), \tag{4.21}$$

where $\widehat{\lambda}$ is the partition of length $< N$ whose parts are $\widehat{\lambda}_j = \lambda_1 - \lambda_{N-j+1}$.

Proof. The proof follows from the following chain of equalities;

$$\begin{aligned} \chi_\lambda(\underline{x}^{-1}) &= \frac{\det\left[x_i^{-N+j-\lambda_j}\right]_{i,j=1}^N}{\det\left[x_i^{-N+j}\right]_{i,j=1}^N} = \frac{\det\left[x_i^{1-j-\lambda_{N-j+1}}\right]_{i,j=1}^N}{\det\left[x_i^{1-j}\right]_{i,j=1}^N} \\ &= \left(\prod_{a=1}^N x_a^{-\lambda_1} \right) \frac{\det\left[x_i^{N-j+\lambda_1-\lambda_{N-j+1}}\right]_{i,j=1}^N}{\det\left[x_i^{N-j}\right]_{i,j=1}^N} = \left(\prod_{a=1}^N x_a^{-\lambda_1} \right) \chi_{\widehat{\lambda}}(\underline{x}). \end{aligned} \tag{4.22}$$

In the first step we have shuffled the columns as $j \mapsto N-j+1$, then we have multiplied both numerator and denominator by $(x_1 \cdots x_N)^{N+\lambda_1}$, and finally we have applied the definition (4.2). $\square$

4.1.2 Topological expansion in triple Hurwitz numbers

Let us introduce the formal generating functions

$$Z_N^{\pm}(\mathbf{u}) := \int_{\mathcal{H}_N(0,1)} \exp\left(\sum_{k\geq 1} \frac{u_k}{k} \operatorname{tr} X^{\pm k}\right) \mathrm{d}m_N^{\mathsf{J}}(X) = \sum_{\lambda\in\mathcal{P}} \frac{\left\langle \prod_{j=1}^{\ell} \operatorname{tr} X^{\pm\lambda_j}\right\rangle}{z_\lambda} \prod_{i=1}^{\ell(\lambda)} u_{\lambda_i}, \tag{4.23}$$

of JUE correlators, where we recall that

$$\mathrm{d}m_N^{\mathsf{J}}(X) = \frac{1}{C_N^{\mathsf{J}}} \det{}^{\alpha}(X) \det{}^{\beta}(\mathbf{1}-X)\mathrm{d}X, \tag{4.24}$$

with parameters α, β satisfying $\operatorname{Re}\alpha, \operatorname{Re}\beta > -1$, C_N^{J} defined in (1.70) and $\mathcal{H}_N(0,1)$ the set of Hermitian matrices with eigenvalues lying in the interval $(0,1)$. We call $Z_N^{+}(\mathbf{u})$ (resp. $Z_N^{-}(\mathbf{u})$) the *positive* (resp. *negative*) *JUE partition function*. Our description of the JUE correlators involves (weighted sums of) *triple weakly monotone Hurwitz numbers*, which we promptly define.

Definition 4.1.4. *Given $n \geq 0$, three partitions $\lambda, \mu, \nu \vdash n$ and an integer $g \geq 0$, we define $h_g^{\geq}(\lambda, \mu, \nu)$ to be the number of tuples $(\pi_1, \pi_2, \tau_1, \ldots, \tau_r)$ of permutations in $\mathfrak{S}_n$ such that*

1. *$r = 2g-2-n+\ell(\mu)+\ell(\nu)+\ell(\lambda)$,*
2. *$\pi_1 \in \operatorname{cyc}(\mu)$, $\pi_2 \in \operatorname{cyc}(\nu)$,*
3. *$\tau_i = (a_i, b_i)$ are transpositions, with $a_i < b_i$ and $b_1 \leq \cdots \leq b_r$, and*
4. *$\pi_1\pi_2\tau_1 \cdots \tau_r \in \operatorname{cyc}(\lambda)$.*

The relation of these Hurwitz numbers to the JUE is expressed by the following result.

Theorem 4.1.5. *Under the re-scaling $\alpha = (c_\alpha - 1)N$, $\beta = (c_\beta - 1)N$, for any partition λ we have the following Laurent expansions as $N \to \infty$;*

$$(-1)^{|\lambda|} N^{\ell(\lambda)} \frac{|\lambda|!}{z_\lambda} \left\langle \prod_{j=1}^{\ell} \operatorname{tr} X^{\lambda_j} \right\rangle = \sum_{g\geq 0} \frac{1}{N^{2g-2}} \sum_{\mu,\nu\vdash|\lambda|} \frac{c_\alpha^{\ell(\nu)}}{(-c_\alpha - c_\beta)^{\ell(\mu)+\ell(\nu)+\ell(\lambda)+2g-2}} h_g^{\geq}(\lambda, \mu, \nu), \tag{4.25}$$

$$(-1)^{|\lambda|} N^{\ell(\lambda)} \frac{|\lambda|!}{z_\lambda} \left\langle \prod_{j=1}^{\ell} \operatorname{tr} X^{-\lambda_j} \right\rangle = \sum_{g\geq 0} \frac{1}{N^{2g-2}} \sum_{\mu,\nu\vdash|\lambda|} \frac{(1-c_\alpha-c_\beta)^{\ell(\nu)}}{(c_\alpha - 1)^{\ell(\mu)+\ell(\nu)+\ell(\lambda)+2g-2}} h_g^{\geq}(\lambda, \mu, \nu), \tag{4.26}$$

where z_λ is given in (2.16) and $h_g^{\geq}(\lambda, \mu, \nu)$ are the weakly monotone triple Hurwitz numbers of Definition 4.1.4.

As anticipated, we want to explicitly compute the coefficients c_λ in (4.28) for the Jacobi measure and identify them with the $r_\lambda^{(G,\epsilon)}$ in (4.7) for a suitable function $G(z)$ and parameter ϵ. To this end, we apply Lemma 4.1.2 to $I = [0,1]$ and $V(x) = \alpha \log x + \beta \log(1-x)$ and expand the positive and negative JUE partition functions (4.23) in the Schur basis as

$$Z_N^{\pm}(\mathbf{u}) = \sum_{\lambda \in \mathcal{P}\colon \ell(\lambda) \le N} c_{\lambda,N}^{\pm} s_\lambda(\mathbf{u}), \tag{4.27}$$

where

$$c_{\lambda,N}^{\pm} = \frac{\int_{(0,1)^N} \chi_\lambda(\underline{x}^{\pm 1}) \Delta^2(\underline{x}) \prod_{a=1}^N x_a^\alpha (1-x_a)^\beta \mathrm{d}^N \underline{x}}{\int_{(0,1)^N} \Delta^2(\underline{x}) \prod_{a=1}^N x_a^\alpha (1-x_a)^\beta \mathrm{d}^N \underline{x}}. \tag{4.28}$$

These can be explicitly evaluated via the following *Selberg-Aomoto integral*

$$\int_{(0,1)^N} \chi_\lambda(\underline{x}) \Delta^2(\underline{x}) \prod_{a=1}^n x_a^\alpha (1-x_a)^\beta \mathrm{d}^N \underline{x} = N! \prod_{1 \le i < j \le N} (\lambda_i - \lambda_j + j - i) \prod_{k=1}^N \frac{\Gamma(\beta+k)\Gamma(\alpha+N+\lambda_k-k+1)}{\Gamma(\alpha+\beta+2N+\lambda_k-k+1)}, \tag{4.29}$$

for which we refer e.g. to [131, page 385]. Specifically, we write the coefficients $c_{\lambda,N}^{\pm}$ as functions of the contents of λ only.

Proposition 4.1.6. *We have*

$$c_{\lambda,N}^{+} = \frac{\dim \lambda}{|\lambda|!} \prod_{(i,j)\in\lambda} \frac{(N-i+j)(\alpha+N-i+j)}{(\alpha+\beta+2N-i+j)}, \quad c_{\lambda,N}^{-} = \frac{\dim \lambda}{|\lambda|!} \prod_{(i,j)\in\lambda} \frac{(N-i+j)(\alpha+\beta+N+i-j)}{(\alpha+i-j)}, \tag{4.30}$$

where

$$\frac{\dim \lambda}{|\lambda|!} = \frac{\prod_{1\le i<j\le N} (\lambda_i - \lambda_j + j - i)}{\prod_{k=1}^N (\lambda_k - k + N)!} \tag{4.31}$$

is the dimension of the irreducible representation associated to the partition λ.

Proof. We start with $c_{\lambda,N}^+$; using (4.28), (4.29), and (4.31) we compute

$$\begin{aligned} c_{\lambda,N}^{+} &= \frac{\prod_{1\le i<j\le N}(\lambda_i - \lambda_j + j - i)}{\prod_{1\le i<j\le N}(j-i)} \prod_{k=1}^N \frac{\Gamma(\alpha+N+\lambda_k-k+1)\Gamma(\alpha+\beta+2N-k+1)}{\Gamma(\alpha+\beta+2N+\lambda_k-k+1)\Gamma(\alpha+N-k+1)} \\ &= \frac{\dim\lambda}{|\lambda|!} \prod_{k=1}^{N-1} \frac{(N-k+1)_{\lambda_k}(\alpha+N-k+1)_{\lambda_k}}{(\alpha+\beta+2N-k+1)_{\lambda_k}} \\ &= \frac{\dim\lambda}{|\lambda|!} \prod_{(i,j)\in\lambda} \frac{(N-i+j)(\alpha+N-i+j)}{(\alpha+\beta+2N-i+j)}. \end{aligned} \tag{4.32}$$

We remind that $(r)_j := r(r+1)\cdots(r+j-1)$ denotes the rising factorial. For $c_{\lambda,N}^-$ we first note that, thanks to Lemma 4.1.3 and (4.29), we have

$$\int_{(0,1)^N} \chi_\lambda(\underline{x}^{-1}) \Delta^2(\underline{x}) \prod_{a=1}^N x_a^\alpha (1-x_a)^\beta \mathrm{d}^N \underline{x} = N! \prod_{1\le i<j\le N} (\lambda_i - \lambda_j + j - i) \prod_{k=1}^N \frac{\Gamma(\beta+k)\Gamma(\alpha-\lambda_k+k)}{\Gamma(\alpha+\beta+N-\lambda_k+k)}, \tag{4.33}$$

then with similar computations as above we obtain

$$
\begin{aligned}
c^-_{\lambda,N} &= \frac{\prod_{1\le i<j\le N}(\lambda_i-\lambda_j+j-i)}{\prod_{1\le i<j\le N}(j-i)} \prod_{k=1}^{N} \frac{\Gamma(\alpha-\lambda_k+k)\Gamma(\alpha+\beta+N+k)}{\Gamma(\alpha+\beta+N-\lambda_k+k)\Gamma(\alpha+k)} \\
&= \frac{\dim\lambda}{|\lambda|!} \prod_{k=1}^{N-1} \frac{(N-k+1)_{\lambda_k}(\alpha+\beta+N-\lambda_k+k)_{\lambda_k}}{(\alpha-\lambda_k+k)_{\lambda_k}} \\
&= \frac{\dim\lambda}{|\lambda|!} \prod_{(i,j)\in\lambda} \frac{(N-i+j)(\alpha+\beta+N+i-j)}{(\alpha+i-j)}. \qquad (4.34)
\end{aligned}
$$

□

This proposition enables us to identify the Jacobi generating function (4.23) with the generating function of multiparametric weighted Hurwitz numbers in (4.5).

Corollary 4.1.7. *Let $c_\alpha := 1+\alpha/N$ and $c_\beta := 1+\beta/N$; then the Jacobi formal partition functions in (4.23) take the form*

$$
Z^+_N(\mathbf{u}) = \tau_{G^+}\left(\epsilon=\frac{1}{N},\mathbf{t}\right), \quad G^+(z) = \frac{(1+z)\left(1+\frac{z}{c_\alpha}\right)}{1+\frac{z}{c_\alpha+c_\beta}}, \qquad t_k = \left(\frac{c_\alpha N}{c_\alpha+c_\beta}\right)^k u_k, \qquad (4.35)
$$

$$
Z^-_N(\mathbf{u}) = \tau_{G^-}\left(\epsilon=\frac{1}{N},\mathbf{t}\right), \quad G^-(z) = \frac{(1+z)\left(1-\frac{z}{c_\alpha+c_\beta-1}\right)}{1-\frac{z}{c_\alpha-1}}, \quad t_k = \left(\frac{(c_\alpha+c_\beta-1)N}{c_\alpha-1}\right)^k u_k, \qquad (4.36)
$$

where τ_G is introduced in Theorem 4.1.1. i.e. it serves as a generating function for the multiparametric Hurwitz numbers $h^{\frac{1}{N}}_{G^\pm}(\mu)$.

Proof. We first note that we can rewrite the expansion (4.27) as

$$
Z^\pm_N(\mathbf{u}) = \sum_{\lambda\in\mathcal{P}} c^\pm_{\lambda,N} s_\lambda(\mathbf{u}), \qquad (4.37)
$$

with the sum over all partitions $\mathcal{P}$ and no longer restricted to $\ell(\lambda)\le N$; this is clear as $c^\pm_{N,\lambda}=0$ whenever $N=0,1,2,\dots$ and $\ell(\lambda)>N$. Then the proof is immediate by the formula (4.7) for the coefficients $r^{(G,\epsilon)}_\lambda$, since (4.30) can be rewritten as

$$
c^+_{\lambda,N} = \frac{\dim\lambda}{|\lambda|!}\left(\frac{c_\alpha N}{c_\alpha+c_\beta}\right)^{|\lambda|} \prod_{(i,j)\in\lambda} \frac{\left(1+\frac{1}{N}(j-i)\right)\left(1+\frac{1}{c_\alpha N}(j-i)\right)}{1+\frac{1}{(c_\alpha+c_\beta)N}(j-i)}, \qquad (4.38)
$$

$$
c^-_{\lambda,N} = \frac{\dim\lambda}{|\lambda|!}\left(\frac{(c_\alpha+c_\beta-1)N}{c_\alpha-1}\right)^{|\lambda|} \prod_{(i,j)\in\lambda} \frac{\left(1+\frac{1}{N}(j-i)\right)\left(1-\frac{1}{(c_\alpha+c_\beta-1)N}(j-i)\right)}{1-\frac{1}{(c_\alpha-1)N}(j-i)}. \qquad (4.39)
$$

□

We now connect the multiparametric Hurwitz numbers (2.2.3) for the functions $G^{\pm}(z)$, appearing in Corollary 4.1.7, with the counting problem in Definition 4.1.4.

Proposition 4.1.8. *If $G(z) = \frac{(1+z)(1+\gamma z)}{1-\delta z}$, with γ and δ parameters, then for all partitions $\lambda \vdash n$ and all integers $g \geq 0$ we have*

$$h^G_{2g-2+n+\ell(\lambda)}(\lambda) = \frac{1}{n!} \sum_{\mu,\nu \vdash n} \gamma^{n-\ell(\nu)} \delta^{\ell(\mu)+\ell(\nu)+\ell(\lambda)+2g-2-n} h_g(\lambda,\mu,\nu), \tag{4.40}$$

where the triple monotone Hurwitz number $h_g(\lambda,\mu,\nu)$ has been introduced in Definition 4.1.4.

Proof. We apply the relation between YJM elements and the conjugacy classes (2.28) to the first two factors of the following to get

$$\begin{aligned} \prod_{a=1}^{n} G(\epsilon J_a) &= \prod_{a=1}^{n} (1+\epsilon \mathcal{J}_a)(1+\epsilon\gamma\mathcal{J}_a) \frac{1}{1-\epsilon\delta\mathcal{J}_a} \\ &= \left(\sum_{\mu \vdash n} \epsilon^{n-\ell(\mu)} \mathcal{C}_\mu \right) \left(\sum_{\nu \vdash n} (\epsilon\gamma)^{n-\ell(\nu)} \mathcal{C}_\nu \right) \left(\sum_{r\geq 0} (\epsilon\delta)^r \sum_{1\leq a_1 \leq \cdots \leq a_r \leq n} \mathcal{J}_{a_1} \cdots \mathcal{J}_{a_r} \right). \end{aligned} \tag{4.41}$$

By definition (2.31), extracting the coefficient of $\epsilon^d \mathcal{C}_\lambda$ and dividing by z_λ we obtain $h^G_d(\lambda)$; therefore

$$h^G_d(\lambda) = \frac{1}{z_\lambda |\mathrm{cyc}(\lambda)|} \sum_{\mu,\nu \vdash n} \gamma^{n-\ell(\nu)} \delta^r h_g(\lambda,\mu,\nu), \tag{4.42}$$

where d, r, g in this identity are related via

$$r = \ell(\lambda) + \ell(\mu) + \ell(\nu) + 2g - 2 - n, \qquad d = 2n - \ell(\mu) - \ell(\nu) + r. \tag{4.43}$$

The proof is complete by the identity $z_\lambda |\mathrm{cyc}(\lambda)| = n!$, see (2.16). ☐

We finally have all the elements to complete the proof of Theorem 4.1.5.

Proof of Theorem 4.1.5. From Corollary 4.1.7 we have, with the scaling $\alpha = (c_\alpha - 1)N$, $\beta = (c_\beta - 1)N$,

$$Z^+_N(\mathbf{u}) = \sum_{d\geq 1} \frac{1}{N^d} \sum_{\lambda \in \mathcal{P}} \left(\frac{c_\alpha N}{c_\alpha + c_\beta} \right)^{|\lambda|} h^{G^+}_d(\lambda) \prod_{i=1}^{\ell(\lambda)} u_{\lambda_i}, \tag{4.44}$$

$$Z^-_N(\mathbf{u}) = \sum_{d\geq 1} \frac{1}{N^d} \sum_{\lambda \in \mathcal{P}} \left(\frac{(c_\alpha + c_\beta - 1)N}{c_\alpha - 1} \right)^{|\lambda|} h^{G^-}_d(\lambda) \prod_{i=1}^{\ell(\lambda)} u_{\lambda_i}, \tag{4.45}$$

where we have used Proposition 4.1.1. It follows from (4.23) that

$$\frac{\left\langle \prod_{j=1}^{\ell} \operatorname{tr} X^{\lambda_j} \right\rangle}{z_\lambda} = \sum_{d\geq 1} N^{|\lambda|-d} \left(\frac{c_\alpha}{c_\alpha + c_\beta} \right)^{|\lambda|} h^{G^+}_d(\lambda), \tag{4.46}$$

$$\frac{\left\langle \prod_{j=1}^{\ell} \operatorname{tr} X^{-\lambda_j} \right\rangle}{z_\lambda} = \sum_{d\geq 1} N^{|\lambda|-d} \left(\frac{c_\alpha + c_\beta - 1}{c_\alpha - 1} \right)^{|\lambda|} h^{G^-}_d(\lambda), \tag{4.47}$$

and using finally Proposition 4.1.8 we have

$$\frac{\left\langle \prod_{j=1}^{\ell} \operatorname{tr} X^{\lambda_j} \right\rangle}{z_\lambda} = \frac{1}{|\lambda|!} \sum_{g\geq 0} N^{2-2g-\ell(\lambda)} \sum_{\mu,\nu\vdash|\lambda|} (-1)^{|\lambda|} \frac{c_\alpha^{\ell(\nu)}}{(-c_\alpha - c_\beta)^{\ell(\mu)+\ell(\nu)+\ell(\lambda)+2g-2}} h_g^{\geq}(\lambda,\mu,\nu), \tag{4.48}$$

$$\frac{\left\langle \prod_{j=1}^{\ell} \operatorname{tr} X^{-\lambda_j} \right\rangle}{z_\lambda} = \frac{1}{|\lambda|!} \sum_{g\geq 0} N^{2-2g-\ell(\lambda)} \sum_{\mu,\nu\vdash n} (-1)^{|\lambda|} \frac{(1 - c_\alpha - c_\beta)^{\ell(\nu)}}{(c_\alpha - 1)^{\ell(\mu)+\ell(\nu)+\ell(\lambda)+2g-2}} h_g^{\geq}(\lambda,\mu,\nu). \tag{4.49}$$

□

Example 4.1.9. *From Theorem 4.1.5, together with the explicit formulæ of Theorem 3.2.12, we can compute*

$$\left\langle (\operatorname{tr} X)^3 \right\rangle^{\mathsf{c}} = \frac{2N(\alpha+\beta)(\beta-\alpha)(\alpha+N)(\beta+N)(\alpha+\beta+N)}{(\alpha+\beta+2N-2)(\alpha+\beta+2N-1)(\alpha+\beta+2N)^3(\alpha+\beta+2N+1)(\alpha+\beta+2N+2)}. \tag{4.50}$$

With the substitution $\alpha = (c_\alpha - 1)N$ and $\beta = (c_\beta - 1)N$ we have the large N expansion

$$\begin{aligned}\left\langle (\operatorname{tr} X)^3 \right\rangle^{\mathsf{c}} = \frac{1}{N}\Bigg[& c_\alpha \left(\frac{2}{(c_\alpha+c_\beta)^3} - \frac{6}{(c_\alpha+c_\beta)^4} + \frac{4}{(c_\alpha+c_\beta)^5} \right) \\ & + c_\alpha^2 \left(-\frac{6}{(c_\alpha+c_\beta)^4} + \frac{18}{(c_\alpha+c_\beta)^5} - \frac{12}{(c_\alpha+c_\beta)^6} \right) \\ & + c_\alpha^3 \left(\frac{4}{(c_\alpha+c_\beta)^5} - \frac{12}{(c_\alpha+c_\beta)^6} + \frac{8}{(c_\alpha+c_\beta)^7} \right) \Bigg] + \mathcal{O}\left(\frac{1}{N^3}\right).\end{aligned} \tag{4.51}$$

Matching the coefficients as in Theorem 4.1.5 we get the values for $h^{\mathsf{c}}_{g=0}(\lambda = (1,1,1),\mu,\nu)$ (the connected Hurwitz numbers defined in Remark 1.3.4) reported in the following table;

	$\nu = (3)$	$\nu = (2,1)$	$\nu = (1,1,1)$
$\mu = (3)$	2	6	4
$\mu = (2,1)$	6	18	12
$\mu = (1,1,1)$	4	12	8

(4.52)

For example, the numbers in the first row ($\mu = (3)$) can be read from the following factorizations in $\mathfrak{S}_3$. To list them let us first note that we have $\operatorname{cyc}(\lambda) = \{\mathrm{Id}\}$ and $\operatorname{cyc}(\mu) = \{(123),(132)\}$; therefore for $\nu = (3)$ we have 2 factorizations (r = number of transpositions = 0)

$$(123)(132) = \mathrm{Id}, \qquad (132)(123) = \mathrm{Id}, \tag{4.53}$$

for $\nu = (2,1)$ ($\operatorname{cyc}(\nu) = \{(12),(23),(13)\}$) we have 6 factorizations ($r = 1$)

$$(123)(12)(13) = \mathrm{Id}, \qquad (123)(13)(23) = \mathrm{Id}, \qquad (123)(23)(12) = \mathrm{Id}, \tag{4.54}$$

$$(132)(13)(12) = \mathrm{Id}, \qquad (132)(12)(23) = \mathrm{Id}, \qquad (132)(23)(13) = \mathrm{Id}, \tag{4.55}$$

and for $\nu = (1,1,1)$ we have the 4 factorizations ($r = 2$, here the monotone condition plays a role)

$$(123)\mathrm{Id}(12)(13) = \mathrm{Id}, \qquad (123)\mathrm{Id}(13)(23) = \mathrm{Id}, \tag{4.56}$$

$$(132)\mathrm{Id}(12)(23) = \mathrm{Id}, \qquad (123)\mathrm{Id}(23)(13) = \mathrm{Id}. \tag{4.57}$$

Similarly we can compute

$$\begin{aligned}\left\langle \left(\operatorname{tr} X^{-1}\right)^3 \right\rangle^{\mathsf{c}} &= \frac{2N(\alpha+N)(\alpha+2N)(\beta+N)(\alpha+\beta+N)(\alpha+2\beta+2N)}{(\alpha-2)(\alpha-1)\alpha^3(\alpha+1)(\alpha+2)} \\ &= \frac{1}{N}\left[\left(\frac{2}{(c_\alpha-1)^3}+\frac{6}{(c_\alpha-1)^4}+\frac{4}{(c_\alpha-1)^5}\right)(c_\alpha+c_\beta-1)\right. \\ &\quad -\left(\frac{6}{(c_\alpha-1)^4}+\frac{18}{(c_\alpha-1)^5}+\frac{12}{(c_\alpha-1)^6}\right)(c_\alpha+c_\beta-1)^2 \\ &\quad \left.+\left(\frac{4}{(c_\alpha-1)^5}+\frac{12}{(c_\alpha-1)^6}+\frac{8}{(c_\alpha-1)^7}\right)(c_\alpha+c_\beta-1)^3\right]+\mathcal{O}\left(\frac{1}{N^3}\right) \end{aligned} \tag{4.58}$$

and from Theorem 4.1.5 we recognize the connected Hurwitz numbers tabulated above.

4.1.3 Generalization to the Jacobi beta ensemble

The same strategy used in deriving the connection of the Jacobi unitary ensemble to multiparametric Hurwitz numbers can be employed to link the Jacobi *beta* ensemble (JβE) [142] to the so called b-Hurwitz numbers [45]. Here we give an idea of how to prove such a statement.

The joint eigenvalues probability distribution function of the N-dimensional JβE with parameters α_1 and α_2 is given by

$$f_\beta(\underline{x}) = c_J^{\beta,\alpha_1,\alpha_2}\prod_{i<j}\Delta^\beta(\underline{x})\prod_{j=1}^{N} x^{\frac{\beta}{2}\alpha_1-1}(1-x)^{\frac{\beta}{2}\alpha_2-1}\mathrm{d}\underline{x}. \tag{4.59}$$

For $\beta = 1,2,4$ it corresponds to the eigenvalue distribution associated to a *full* matrix model, respectively the Jacobi orthogonal/unitary/symplectic ensembles. For generic β a *tridiagonal* matrix model has been given in the foundational paper of Dumitriu-Edelman [69].

The b-Hurwitz numbers have been introduced by Chapuy and Dołega in [45]. They are associated to the counting problem of generalized branched coverings of the Riemann sphere by (not necessarily orientable) surfaces with appropriate b-weighting and generalizes the weighted multiparametric Hurwitz numbers of Definition 2.2.3; see loc.cit. for the complete picture. The b and β parameters here and in the following are related as

$$b = \frac{2-\beta}{\beta}. \tag{4.60}$$

In [45] the authors study the generating functions of these objects, which can be in turn considered a b-deformed version of the tau-functions of multiparametric weighted Hurwitz numbers of Theorem 2.3.1. They are expressed as an expansion in *Jack symmetric polynomials* $J_\lambda^{(b)}$, see [131],

$$\tau_G^b(u,\epsilon;\mathbf{t},\mathbf{s})) = \sum_{n\geq 0}u^n\sum_{\lambda\vdash n}\frac{J_\lambda^{(b)}(\mathbf{t})J_\lambda^{(b)}(\mathbf{s})}{||J_\lambda^{(b)}||^2}\prod_{(i,j)\in\lambda}G(\epsilon\cdot c_b(j-i)), \tag{4.61}$$

where $G(z)$ can be taken to be a rational function as in (2.30) and c_b is the weighted content of partitions,

$$c_b(i,j) = b(i-1)+i-j. \tag{4.62}$$

The familiar expansion in Schur symmetric polynomials (2.36) is recovered from $b=0$; it is proved in [33] that for $b=1$ the function (4.61) obeys the *BKP-integrable hierarchy*.

Recall that the identification between the JUE and $G(z) = \dfrac{(1+z)(1+\gamma z)}{1-\delta z}$ multiparametric Hurwitz numbers consists of three steps:

1. Lemma 4.1.2: consider the deformation of the Jacobi potential $V(x;\mathbf{t}) = \alpha_1 \log x + \alpha_2 \log(1-x) + \sum_{k>0} t_k x^k$ and use the Cauchy identity (4.20) to expand it in the basis of Schur polynomials $s_\lambda(\mathbf{t})$ with coefficients $c_{\lambda,N}$

2. Proposition 4.1.6: use the Selberg-Aomoto integral (4.29) to explicitly compute the coefficients $c_{\lambda,N}$ as a function of the contents of λ for some rational $G(z)$

3. Corollary 4.1.7: use previous results and Proposition 4.1.1 to connect the JUE with the Hurwitz numbers associated to the specific $G(z)$

The tentative relation between the JβE and generating functions of type (4.61) follows similar lines, replacing Schur polynomials with Jack ones:

1. A Cauchy identity of the form (4.20) holds for Jack polynomials as well, namely
$$\exp\left[\sum_{k\geq 1} \frac{t_k}{k(1+b)}(x_1^k + \cdots + x_N^k)\right] = \sum_{\lambda\in\mathcal{P}} \mathcal{J}_\lambda^{(1+b)}(x_1,\ldots,x_N)\frac{J_\lambda(\mathbf{t})}{j_\lambda^{(b)}}, \tag{4.63}$$
where $j_\lambda^{(b)}$ is the analogue of $\dfrac{\dim\lambda}{|\lambda|!}$ in the $b=0$ case.

2. The Selberg-Aomoto integral can be generalized to arbitrary β to compute averages of Jack polynomials $J_\lambda^{(b)}$ with respect to the Jacobi measure. It can be found e.g. in [84],
$$\int_{(0,1)^N} J_\lambda^{(1/k)}(\underline{x})\Delta^{2k}(\underline{x})\prod_{i=1}^N x_i^{r-1}(1-x_i)^{s-1}\mathrm{d}\underline{x} = \nu_\lambda(k)\cdot\prod_{i=1}^N \frac{\Gamma(\lambda_i + r + k(N-i))\Gamma(s+k(N-i))}{\Gamma(\lambda_i + r + s + k(2N-i-1))}, \tag{4.64}$$
$$\nu_\lambda(k) = \prod_{1\leq i<j\leq N} \frac{\Gamma(\lambda_i - \lambda_j + k(j-i+1))}{\Gamma(\lambda_i-\lambda_j+k(j-i))}, \tag{4.65}$$
and allows to express the coefficients in the Jack expansion in terms of b-contents

3. The connection with the b-Hurwitz numbers now passes through the results of Chapuy et al. and formula (4.61).

The detail will be sorted out in a future work.

4.2 LUE and double Hurwitz numbers

Let us introduce the formal generating functions
$$Z_N^{\pm}(\mathbf{u}) := \int_{\mathcal{H}_N(0,1)} \exp\left(\sum_{k\geq 1}\frac{u_k}{k}\mathrm{tr}\,X^{\pm k}\right)\mathrm{d}m_N^{\mathsf{L}}(X) = \sum_{\lambda\in\mathcal{P}} \frac{\left\langle \prod_{j=1}^{\ell}\mathrm{tr}\,X^{\pm\lambda_j}\right\rangle}{z_\lambda}\prod_{i=1}^{\ell(\lambda)} u_{\lambda_i}, \tag{4.66}$$
of LUE correlators, where we recall that
$$\mathrm{d}m_N^{\mathsf{L}}(X) = \frac{1}{C_N^{\mathsf{L}}}\det{}^{\alpha}\exp\left(-\mathrm{tr}\,X\right)\mathrm{d}X, \tag{4.67}$$
with parameter α satisfying $\mathrm{Re}\,\alpha, \mathrm{Re}\,\beta > -1$, C_N^{L} defined in (1.59) and $\mathcal{H}_N^+$ the set of positive definite Hermitian matrices, i.e. with eigenvalues lying in $(0,\infty)$. We call $Z_N^+(\mathbf{u})$ (resp. $Z_N^-(\mathbf{u})$) the

positive (resp. *negative*) *LUE partition function.* Our description of the LUE correlators involves (weighted sums of) *double* (both strictly and weakly) *monotone Hurwitz numbers*, a reduction of the triple Hurwitz numbers introduced in Definition 4.1.4.

Definition 4.2.1. *Given $n \geq 0$, partitions $\lambda, \mu \vdash n$ and an integer $g \geq 0$, we define $h_g^{>}(\lambda;\mu)$ (resp. $h_g^{\geq}(\lambda;\mu)$) as the number of tuples $(\pi_1, \pi_2, \tau_1, \ldots, \tau_r)$ of permutations in $\mathfrak{S}_n$ such that*

1. $r = 2g - 2 + \ell(\lambda) + \ell(\mu)$,
2. $\pi_1 \in \mathrm{cyc}(\lambda)$, $\pi_2 \in \mathrm{cyc}(\mu)$,
3. $\tau_i = (a_i, b_i)$ *are transpositions, with* $a_i < b_i$ *and* $b_1 < \cdots < b_r$, *(resp.* $b_1 \leq \cdots \leq b_r$*)*
4. $\pi_1 \tau_1 \cdots \tau_r \in \mathrm{cyc}(\mu)$.

The relation of these Hurwitz numbers to the LUE is expressed by the following result.

Theorem 4.2.2 ([51])**.** *Under the re-scaling $\alpha = (c-1)N$, for any partition λ we have the following Laurent expansions as $N \to \infty$;*

$$N^{\ell(\lambda)-|\lambda|} \frac{|\lambda|!}{z_\lambda} \left\langle \prod_{j=1}^{\ell} \mathrm{tr}\, X^{\lambda_j} \right\rangle = \sum_{g \geq 0} \frac{1}{N^{2g-2}} \sum_{\mu \vdash |\lambda|} c^{\ell(\mu)}\, h_g^{>}(\lambda;\mu), \qquad c > 1 - \frac{1}{N}, \tag{4.68}$$

$$N^{\ell(\lambda)+|\lambda|} \frac{|\lambda|!}{z_\lambda} \left\langle \prod_{j=1}^{\ell} \mathrm{tr}\, X^{-\lambda_j} \right\rangle = \sum_{g \geq 0} \frac{1}{N^{2g-2}} \sum_{\mu \vdash |\lambda|} \frac{h_g^{\geq}(\lambda;\mu)}{(c-1)^{2g-2+|\lambda|+\ell(\lambda)+\ell(\mu)}}, \qquad c > 1 + \frac{|\lambda|}{N}. \tag{4.69}$$

Remark 4.2.3. *The proof of Theorem 4.2.2 could be carried out following the same strategy adopted in the previous section in the proof of Theorem 4.1.5. Indeed a Selberg-Aomoto integral formula holds for the Laguerre measure as well,*

$$\int_{(0,\infty)^N} \chi_\lambda(\underline{x}) \Delta^2(\underline{x}) \prod_{a=1}^{n} x_a^{\alpha} e^{-x_a} \mathrm{d}^N \underline{x} = N! \prod_{1 \leq i < j \leq N} (\lambda_i - \lambda_j + j - i) \prod_{k=1}^{N} \Gamma(\alpha + N + \lambda_k - k + 1), \tag{4.70}$$

from which we find a suitable function G and parameter ϵ to match the coefficients c_λ with $r^{(G,\epsilon)}$.

However, a much more direct way is to just pass from JUE to LUE via the following limit,

$$\lim_{\beta \to +\infty} \beta^{k_1 + \cdots + k_\ell} \left\langle \prod_{j=1}^{\ell} \mathrm{tr}\, X^{k_j} \right\rangle_{JUE} = \frac{\int_{\mathcal{H}_N(0,+\infty)} \left(\prod_{j=1}^{\ell} \mathrm{tr}\, X^{k_j} \right) \det^{\alpha}(X) \exp(-\mathrm{tr}\, X) \mathrm{d}X}{\int_{\mathcal{H}_N(0,+\infty)} \det^{\alpha}(X) \exp(-\mathrm{tr}\, X) \mathrm{d}X}, \tag{4.71}$$

valid for $k_1, \cdots, k_\ell$ arbitrary integers. Indeed, let us prove the expansion (4.68) starting from (4.46), which we rewrite here as

$$\frac{\left\langle \prod_{j=1}^{\ell} \mathrm{tr}\, X^{\lambda_j} \right\rangle_{JUE}}{z_\lambda} = \sum_{d \geq 1} N^{|\lambda| - d} \left(\frac{c_\alpha}{c_\alpha + c_\beta} \right)^{|\lambda|} H_d^{G^+}(\lambda). \tag{4.72}$$

Then, applying (4.71) to (4.72) and substituting $\beta = N(c_\beta - 1)$ yields

$$\frac{\left\langle \prod_{j=1}^{\ell} \operatorname{tr} X^{\lambda_j} \right\rangle_{LUE}}{z_\lambda} = \lim_{c_\beta \to +\infty} N^{|\lambda|} c_\beta^{|\lambda|} \sum_{d \geq 1} N^{|\lambda|-d} \left(\frac{c_\alpha}{c_\alpha + c_\beta} \right)^{|\lambda|} h^d_{G^+}(\lambda) \tag{4.73}$$

$$= \lim_{c_\beta \to +\infty} \frac{1}{z_\mu} [\epsilon^d \mathcal{C}_\lambda] \prod_{a=1}^{n} \left(1 + \frac{1}{N} \mathcal{J}_a\right) \left(1 + \frac{1}{N\, c_\alpha} \mathcal{J}_a\right) \frac{1}{1 - \frac{1}{N(c_\alpha + c_\beta)} \mathcal{J}_a} \tag{4.74}$$

$$= \sum_{d \geq 1} N^{|\lambda|-d} c_\beta^{|\lambda|} \left(\frac{c_\alpha}{c_\alpha + c_\beta} \right)^{|\lambda|} h^d_{\widetilde{G}}(\lambda) \tag{4.75}$$

where

$$\widetilde{G}^{(z)} = \lim_{c_\beta \to +\infty} G^+(z; c_\alpha, c_\beta) = \lim_{c_\beta \to +\infty} \frac{(1+z)\left(1 + \frac{z}{c_\alpha}\right)}{1 + \frac{z}{c_\alpha + c_\beta}} = (1+z)\left(1 + \frac{z}{c_\alpha}\right). \tag{4.76}$$

The associated multiparametric Hurwitz numbers are found as in Proposition 4.1.8,

$$\prod_{a=1}^{|\lambda|} \widetilde{G}\,(\epsilon J_a) = \prod_{a=1}^{|\lambda|} (1 + \frac{\epsilon}{c_\alpha} \mathcal{J}_a)(1 + \epsilon \mathcal{J}_a)$$

$$= \left(\sum_{\mu \vdash |\lambda|} \left(\frac{\epsilon}{c_\alpha} \right)^{|\lambda| - \ell(\mu)} \mathcal{C}_\mu \right) \left(\sum_{r \geq 0} \epsilon^r \sum_{1 < a_1 < \cdots < a_r < |\lambda|} \mathcal{J}_{a_1} \cdots \mathcal{J}_{a_r} \right), \tag{4.77}$$

so that

$$h^d_{\widetilde{G}}(\lambda) = \frac{1}{|\lambda|} \sum_{\mu \vdash |\lambda|} \left(\frac{1}{c_\alpha} \right)^{|\lambda| - \ell(\mu)} h^>_g(\lambda, \mu), \tag{4.78}$$

where d, r, g in this identity are related via

$$r = 2g - 2 + \ell(\lambda) + \ell(\mu), \qquad d = |\lambda| - \ell(\mu) + r. \tag{4.79}$$

Finally, putting together equations (4.75), (4.78) and (4.79) we have

$$N^{-|\lambda|} \frac{|\lambda|!}{z_\lambda} \left\langle \prod_{j=1}^{\ell} \operatorname{tr} X^{\lambda_j} \right\rangle_{LUE} = \sum_{g \geq 0} N^{2-2g-\ell(\lambda)} \sum_{\mu \vdash |\lambda|} c_\alpha^{\ell(\mu)} h^>_g(\lambda, \mu), \tag{4.80}$$

which is exactly (4.68). Formula (4.69) is proved likewise starting from (4.47).

Remark 4.2.4. *Notice that, in contrast to the JUE case, positive and negative correlators of the LUE are expressed via different objects, respectively strictly and weakly monotone Hurwitz numbers. In the Jacobi case, it is somewhat a generalization of the reflection formula for the Gamma function; specifically, when $N = 1$ moments are expressed via the beta function as*

$$\frac{\int_0^1 x^k \cdot x^\alpha (1-x)^\beta \mathrm{d}x}{\int_0^1 x^\alpha (1-x)^\beta \mathrm{d}x} = \frac{\Gamma(\alpha + k + 1)\Gamma(\alpha + \beta + 2)}{\Gamma(\alpha + 1)\Gamma(\alpha + \beta + k + 2)} =: m(k, \alpha, \beta) \tag{4.81}$$

and using Euler's reflection formula

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}, \tag{4.82}$$

the following relation is immediate for $k \in \mathbb{Z}$*,*

$$m(k, \alpha, \beta) = m(-k, -2 - \alpha - \beta, \beta). \tag{4.83}$$

For general N*, a similar relation holds between negative and positive correlators, namely writing out the dependence on* α, β *explicitly*

$$\left\langle \prod_{j=1}^{\ell} \operatorname{tr} X^{\lambda_j} \right\rangle_{JUE} (\alpha, \beta) = \left\langle \prod_{j=1}^{\ell} \operatorname{tr} X^{-\lambda_j} \right\rangle_{JUE} (-2N - \alpha - \beta, \beta). \tag{4.84}$$

This can already be seen from formulæ (4.25) *and* (4.26) *of Theorem 4.1.5. Indeed, recall we defined* $c_\alpha = 1 + \alpha/N$ *and* $c_\beta = 1 + \beta/N$*, so that the parameters relation* (4.84) *is equivalent to*

$$(\alpha, \beta) \leftrightarrow (-2N - \alpha - \beta, \beta) \iff (c_\alpha, c_\alpha + c_\beta) \leftrightarrow ((1 - c_\alpha - c_\beta)N, -(c_\alpha - 1)N), \tag{4.85}$$

which automatically brings (4.25) *in* (4.26)*. Such a symmetry is not preserved in the Laguerre case via the limit* (4.71) *since it is obtained via the parameter limit* $\beta \to \infty$*.*

Example 4.2.5. *Here we employ the formulæ of Theorems 3.2.5 and 4.2.2 to obtain the genus zero limit for one- and two-point correlators. In these cases, formulæ of the same kind have already appeared in the literature [54, 87, 169]. In the regime* $\alpha = N(c-1)$ *with* $N \to \infty$ *we have*

$$\lim_{N\to\infty} \frac{A_\ell(N, cN)}{N^{\ell+1}} = \frac{1}{\ell} \sum_{b=0}^{\ell-1} \binom{\ell}{b+1} \binom{\ell}{b} c^{b+1}, \tag{4.86}$$

$$\lim_{N\to\infty} \frac{B_\ell(N, cN)}{N^{\ell}} = \sum_{b=0}^{\ell} \binom{\ell}{b}^2 c^b. \tag{4.87}$$

In particular, $\langle \operatorname{tr} X^\ell \rangle = A_\ell(N, M)$*, in the regime* $N \to \infty$ *with* $\alpha = N(c-1)$ *we have*

$$\lim_{N\to\infty} \frac{\langle \operatorname{tr} X^\ell \rangle}{N^{\ell+1}} = \sum_{s=1}^{\ell} \mathcal{N}_{\ell,s} c^s \tag{4.88}$$

where

$$\mathcal{N}_{\ell,s} := \frac{1}{\ell} \binom{\ell}{s} \binom{\ell}{s-1}, \qquad \ell \geq 1, \ s = 1, \dots, \ell \tag{4.89}$$

are the Narayana numbers. Formula (4.88) *agrees with Wigner's computation of positive moments of the Laguerre equilibrium measure* $\rho(x) = \frac{\sqrt{(x_+ - x)(x - x_-)}}{2\pi c x} 1_{x \in (x_-, x_+)}$ *with* $x_\pm := (1 \pm \sqrt{c})^2$*, see [85].*

From the one-point function, matching the coefficients in N *(i.e. the genus* g*) and in* c *(i.e. the length* s *of the summand partition) we obtain, for the weighted strictly monotone and weakly monotone double Hurwitz numbers of genus zero, for a partition* $\mu = (k)$*,*

$$\frac{z_{(k)}}{k!} \sum_{\nu \text{ of length } s} h^{>}_{g=0}((k); \nu) = \frac{1}{(k-1)!} \sum_{\nu \text{ of length } s} h^{>}_{g=0}((k); \nu) = \mathcal{N}_{k,s} = \frac{1}{k} \binom{k}{s-1} \binom{k}{s}, \tag{4.90}$$

$$\frac{z_{(k)}}{k!} \sum_{\nu \text{ of length } s} h^{\geq}_{g=0}((k); \nu) = \frac{1}{(k-1)!} \sum_{\nu \text{ of length } s} h^{\geq}_{g=0}((k); \nu) = \binom{k-1}{k-s} \frac{(s+1)_{k-2}}{(k-1)!}. \tag{4.91}$$

Similarly, for all two-point generating functions, we obtain the planar limit $g = 0$ as

$$\lim_{N\to\infty} \mathscr{F}^{\mathsf{c}}_{2,\infty}(Nz_1, Nz_2) = \lim_{N\to\infty} \mathscr{F}^{\mathsf{c}}_{2,0}(Nz_1, Nz_2) = -\frac{\phi(z_1, z_2) + \sqrt{\phi(z_1, z_1)\phi(z_2, z_2)}}{2\sqrt{\phi(z_1, z_1)\phi(z_2, z_2)}(z_1 - z_2)^2} \tag{4.92}$$

where

$$\phi(z_1, z_2) := c^2 - c(2 + z_1 + z_2) + (z_1 - 1)(z_2 - 1). \tag{4.93}$$

The two-point planar limit is strictly related [79] to the so called canonical symmetric bi-differential (called also Bergman kernel) associated to the spectral curve $x^2y^2 = (x - x_+)(x - x_-) = c^2 - 2c(x + 1) + (x - 1)^2$.

Chapter 5

Connections with intersection theory on moduli spaces of curves

In recent years, Hurwitz numbers have found their way to connect with the *intersection theory on moduli spaces of curves*. In this respect, a fundamental result is the classical ELSV formula, which takes its name from its discoverers, Ekedahl, Lando, Shapiro, Vainshtein, [73], and expresses simple Hurwitz numbers in terms of intersection numbers on moduli spaces of curves.

Theorem 5.0.1 (ELSV-Formula, [73])**.** *Let $\mu = (\mu_1, \ldots, \mu_\ell)$ be a partition of n, then*

$$h_g(\mu) = \frac{(2g-2+|\mu|+\ell)!}{|\mathrm{Aut}(\mu)|} \prod_{i=1}^{\ell} \frac{\mu_i^{\mu_i}}{\mu_i!} \int_{\overline{\mathcal{M}}_{g,n}} \frac{\sum_{j=0}^{g} (-1)^j \lambda_j}{\prod_{i=1}^{\ell}(1-\mu_i\psi_i)}. \tag{5.1}$$

In the above $h_g(\mu)$ denotes indeed the *simple Hurwitz numbers* of genus g and partition μ, and $\mathrm{Aut}(\mu)$ the permutation group of symmetries of the parts of μ. We denote by $\overline{\mathcal{M}}_{g,n}$ the Deligne-Mumford moduli space of stable nodal Riemann surfaces, $\psi_i \in H^2\left(\overline{\mathcal{M}}_{g,n}, \mathbb{Q}\right)$ the first Chern class of the cotangent line bundle to the i-th marked point and with $\lambda_i \in H^{2i}\left(\overline{\mathcal{M}}_{g,n}, \mathbb{Q}\right)$ the Chern classes of the Hodge bundle $\mathbb{E}$ so that $\lambda_i = c_i(\mathbb{E}) \in H^{2i}\left(\overline{\mathcal{M}}_{g,n}, \mathbb{Q}\right)$. For the precise definition of these objects we refer to the literature, see e.g. [171] and references therein.

The ELSV formula is a cornerstone of the modern enumerative geometry. Even though it was obtained by purely geometrical means [80, 100] it can be connected to the Witten-Kontsevich Theorem and hence to the KdV hierarchy, a direction pioneered by Kazarian and Lando [124, 125]. On the other side *ELSV-like formulæ* have been discovered recently, in the aim of counting general Hurwitz numbers; we find such examples in [37] for double Hurwitz numbers generalizing a previous result of Johnson-Pandharipande-Tseng [119], in [10] for single monotone Hurwitz numbers, in [39] for *2-orbifold* strictly monotone Hurwitz numbers, in [41] and [71] for spin Hurwitz numbers following conjectural formulæ of Zvonkine [170].

The purpose of this chapter is exactly to give a contribution to this new plethora of ELSV-like formulæ, combining our results on the combinatorial interpretation of classical unitary ensemble with recent work connecting integrable systems and enumerative geometry. Specifically, from Dubrovin et al. [62] we are able link the multiparametric Hurwitz numbers of LUE with *cubic Hodge integrals*; work on the connection between cubic Hodge integrals and integrable hierarchies, namely the *KP-hierarchy*, was also carried out by Alexandrov in [8, 9]. In a similar fashion, we connect the explicit (Legendre) matrix model presented by Norbury in [148] as a generating function of Θ-GW invariants with the multiparametric Hurwitz numbers of JUE. The resulting ELSV-like formulæ are reported in (5.2) and (5.62).

We thank Di Yang for pointing out the connection between our results on JUE and Norbury's one on intersection numbers.

5.1 Hodge integrals and LUE

The main Theorem of this section is the following *ELSV-like formula*.

Theorem 5.1.1. *For any partition $\mu = (\mu_1, \dots, \mu_\ell)$ of length ℓ we have*

$$\sum_{g\geq 0} \epsilon^{2g-2} \mathscr{H}_{g,\mu} = 2^\ell \sum_{\gamma\geq 0} (2\epsilon)^{2\gamma-2} \sum_{\nu\vdash|\mu|} \left(\omega + \frac{\epsilon}{2}\right)^{2-2\gamma+|\mu|-\ell-\ell(\nu)} \left(\omega - \frac{\epsilon}{2}\right)^{\ell(\nu)} h^{>}_\gamma(\mu,\nu) \tag{5.2}$$

where

$$\mathscr{H}_{g,\mu} := 2^{g-1} \sum_{m\geq 0} \frac{(\omega-1)^m}{m!} \int_{\overline{\mathcal{M}}_{g,\ell+m}} \Lambda^2(-1)\Lambda\left(\frac{1}{2}\right) \exp\left(-\sum_{d\geq 1}\frac{\kappa_d}{d}\right) \prod_{a=1}^{\ell} \frac{\mu_a\binom{2\mu_a}{\mu_a}}{1-\mu_a\psi_a}$$
$$+ \frac{\delta_{g,0}\delta_{\ell,1}}{2}\left(\omega - \frac{\mu_1}{\mu_1+1}\right)\binom{2\mu_1}{\mu_1} + \frac{\delta_{g,0}\delta_{\ell,2}}{2}\frac{\mu_1\mu_2}{\mu_1+\mu_2}\binom{2\mu_1}{\mu_1}\binom{2\mu_2}{\mu_2}. \tag{5.3}$$

Here $\kappa_j \in H^{2j}\left(\overline{\mathcal{M}}_{g,n}, \mathbb{Q}\right)$ $(j = 1, 2, \dots)$ are the Mumford-Morita-Miller classes, and $\Lambda(\xi) := 1+\lambda_1\xi+\dots+\lambda_g\xi^g$ is the Chern polynomial of the Hodge bundle, $\lambda_i \in H^{2i}\left(\overline{\mathcal{M}}_{g,n}, \mathbb{Q}\right)$, the remaining objects have been defined below Theorem 5.0.1; again we refer to the literature for a more accurate definition. Finally $h^{>}_\gamma(\mu;\nu)$ are the Hurwitz numbers of Definition 4.2.1. Note that $\mathscr{H}_{g,\mu}$ in (5.3) is a well defined formal power series in $\mathbb{C}[[\omega-1]]$, as for dimensional reasons each coefficient of $(\omega-1)^m$ in (5.3) is a finite sum of intersection numbers of Mumford-Morita-Miller and Hodge classes on the moduli spaces of curves.

As anticipated, this will be proven by identifying the LUE partition function with the generating function of the intersection numbers depicted above. The latter is connected to the *modified GUE partition function*, termed mGUE and denoted $\widetilde{Z}_N(\mathbf{s})$, introduced in [62] and which we now define. Consider the classical GUE partition function with couplings to odd powers set to zero, namely

$$Z_N^{\mathsf{even}}(\mathbf{s}) := \int_{\mathcal{H}_N} \exp\operatorname{tr}\left(-\frac{1}{2}X^2 + \sum_{k\geq 1} s_k X^{2k}\right) \mathrm{d}X, \qquad \mathbf{s} = (s_1, s_2, \dots). \tag{5.4}$$

It is then argued in [62] that the identity

$$\frac{Z_N^{\mathsf{even}}(\mathbf{s})}{(2\pi)^N \mathrm{Vol}(N)} = \widetilde{Z}_{N-\frac{1}{2}}(\mathbf{s})\widetilde{Z}_{N+\frac{1}{2}}(\mathbf{s}), \qquad \mathrm{Vol}(N) := \frac{\pi^{\frac{N(N-1)}{2}}}{\mathrm{G}(N+1)}, \tag{5.5}$$

uniquely defines $\widetilde{Z}_N(\mathbf{s})$; above, $\mathrm{Vol}(N)$ is the usual volume (1.15), while $\mathrm{G}(z)$ is the Barnes G-function, with the particular evaluation

$$\mathrm{G}(N+1) = 1!2!\cdots(N-1)! \tag{5.6}$$

for any integer $N > 0$. With respect to the normalizations in [62] we are setting $\epsilon \equiv 1$ for simplicity; the dependence on ϵ can be restored by the scaling $N = x\epsilon$. The *Hodge-GUE correspondence* works as follows; introduce the generating function

$$\mathcal{H}(\mathbf{p};\epsilon) = \sum_{g\geq 0} \epsilon^{2g-2} \sum_{n\geq 0} \sum_{k_1,\dots,k_n\geq 0} \frac{p_{k_1}\cdots p_{k_n}}{n!} \int_{\overline{\mathcal{M}}_{g,n}} \Lambda^2(-1)\Lambda\left(\frac{1}{2}\right) \prod_{i=1}^n \psi_i^{k_i}, \tag{5.7}$$

for *special cubic Hodge integrals*, here $\mathbf{p} = (p_0, p_1, \dots)$. Then the following is proved in [62].

Theorem 5.1.2 (Hodge-GUE correspondence [62])**.** *Introduce the formal series*

$$A(\omega,\mathbf{s}) := \frac{1}{4} \sum_{j_1,j_2\geq 1} \frac{j_1 j_2}{j_1+j_2} \binom{2j_1}{j_1}\binom{2j_2}{j_2} s_{j_1}s_{j_2} + \frac{1}{2}\sum_{j\geq 1}\left(\omega - \frac{j}{j+1}\right)\binom{2j}{j} s_j, \tag{5.8}$$

and a transformation of an infinite vector of times $\mathbf{s} = (s_1, s_2, \dots) \mapsto \mathbf{p} = (p_0, p_1, \dots)$ *depending on a parameter* ω *as*

$$p_k(\omega,\mathbf{s}) := \sum_{j\geq 1} j^{k+1}\binom{2j}{j} s_j + \delta_{k,1} + \omega\delta_{k,0} - 1, \qquad k\geq 0. \tag{5.9}$$

Then we have

$$\mathcal{H}\left(\mathbf{p}\left(\omega,\mathbf{s}\right);\sqrt{2}\epsilon\right) + \epsilon^{-2}A(\omega,\mathbf{s}) = \log \widetilde{Z}_{\frac{\omega}{\epsilon}}\left((s_1, \epsilon s_2, \epsilon^2 s_3, \dots)\right) + B(\omega,\epsilon) \tag{5.10}$$

where $B(\omega,\epsilon)$ *is a constant depending on* ω *and* ϵ *only and* $\widetilde{Z}_{\frac{\omega}{\epsilon}}$ *the mGUE partition function* (5.5).

On the other side, recall the positive LUE partition function (4.66),

$$Z_N(\alpha;\mathbf{t}) = \frac{1}{C_N^{\mathsf{L}}} \int_{\mathcal{H}_N^+} \det{}^{\alpha} X \exp\operatorname{tr}\left(-X + \sum_{k>0} t_k X^k\right) \mathrm{d}X, \tag{5.11}$$

where for convenience we rescaled the time variables as $u_k \mapsto k\,t_k$ and explicitly wrote out the dependence on the complex parameter α. We have the following relation with the mGUE partition function.

Theorem 5.1.3. *The modified GUE partition function* $\widetilde{Z}_N(\mathbf{s})$ *in* (5.5) *is identified with the Laguerre partition function* $Z_N(\alpha;\mathbf{t})$ *in* (5.11) *by the relation*

$$\widetilde{Z}_{2N-\frac{1}{2}}(\mathbf{s}) = C_N Z_N\left(\alpha = -\frac{1}{2};\mathbf{t}\right) \tag{5.12}$$

where $\mathbf{t},\mathbf{s}$ *are related by*

$$t_k = 2^k s_k \tag{5.13}$$

and C_N *is an explicit constant depending on* N *only;*

$$C_N = \frac{2^{N^2-\frac{3}{2}N+\frac{1}{4}}}{\pi^{\frac{N(N+1)}{2}}}\mathrm{G}(N+1). \tag{5.14}$$

Identity (5.12) can be recast as the following explicit relation,

$$\widetilde{Z}_{2N-\frac{1}{2}}(\mathbf{s}) = \frac{2^{-N+\frac{1}{4}}}{\pi^{\frac{N(N+1)}{2}}} \mathrm{G}(N+1) \int_{H_N^+} \exp \mathrm{tr} \left(-\frac{X}{2} + \sum_{k\geq 1} s_k X^k \right) \frac{\mathrm{d}X}{\sqrt{\det X}}\,, \tag{5.15}$$

which is obtained from (5.12) by a change of variable $X \mapsto \frac{X}{2}$ in the LUE partition function.

Theorem 5.1.1 is proven combining Theorems 5.1.2 and 5.1.3.

5.1.1 Relation between LUE and mGUE partition functions

Recall that the mGUE partition function is *uniquely defined* by the factorization (5.5) of the even GUE partition function; we will now seek for an identical factorization in terms of two LUE partition functions.

To this end, let us introduce two sequences of monic orthogonal polynomials; $p_n^{\mathsf{even}}(x) = x^n + \dots$ satisfying

$$\int_{-\infty}^{+\infty} p_n^{\mathsf{even}}(x) p_m^{\mathsf{even}}(x) \mathrm{e}^{V(x^2)} \mathrm{d}x = h_n^{\mathsf{even}} \delta_{n,m} \tag{5.16}$$

and, for $\mathrm{Re}\,\alpha > -1$, $p_n^{(\alpha)}(x) = x^n + \dots$ satisfying

$$\int_0^{+\infty} p_n^{(\alpha)}(x) p_m^{(\alpha)}(x) x^\alpha \mathrm{e}^{V(x)} \mathrm{d}x = h_n^{(\alpha)} \delta_{n,m}, \tag{5.17}$$

where $V(x)$ is an arbitrary potential for which the polynomials are well defined. The following lemma is elementary and can be found e.g. in [49].

Lemma 5.1.4. *For all $n \geq 0$ we have*

$$p_{2n}^{\mathsf{even}}(x) = p_n^{(-\frac{1}{2})}(x^2), \qquad p_{2n+1}^{\mathsf{even}}(x) = x p_n^{(\frac{1}{2})}(x^2) \tag{5.18}$$

and

$$h_{2n}^{\mathsf{even}} = h_n^{(-\frac{1}{2})}, \quad h_{2n+1}^{\mathsf{even}} = h_n^{(\frac{1}{2})}. \tag{5.19}$$

Proof. We prove the first formulæ in (5.18) and (5.19), the second ones follow likewise. Rewrite (5.17) for $\alpha = -\frac{1}{2}$ and make the change of variables $x = y^2$, then

$$h_n^{(-\frac{1}{2})} \delta_{n,m} = \int_0^{+\infty} p_n^{(-\frac{1}{2})}(x) p_m^{(-\frac{1}{2})}(x) x^{-\frac{1}{2}} \mathrm{e}^{V(x)} \mathrm{d}x = \int_0^{+\infty} p_n^{(-\frac{1}{2})}(y^2) p_m^{(-\frac{1}{2})}(y^2) y^{-1} \mathrm{e}^{V(y^2)} 2y \mathrm{d}y \tag{5.20}$$

$$= \int_{-\infty}^{+\infty} p_n^{(-\frac{1}{2})}(y^2) p_m^{(-\frac{1}{2})}(y^2) \mathrm{e}^{V(y^2)} \mathrm{d}y, \tag{5.21}$$

where we also used the parity of the last integrand. The claim follows comparing with the orthogonality property (5.16). □

Next, recall the relation between matrix integrals and the norming constants (1.29) so that with respect to the above orthogonal polynomials

$$\frac{1}{\mathrm{Vol}(N)} \int_{\mathcal{H}_N} \exp \mathrm{tr} \left(-V(X^2) \right) \mathrm{d}X = h_0^{\mathsf{even}} h_1^{\mathsf{even}} \cdots h_{N-1}^{\mathsf{even}}, \tag{5.22}$$

$$\frac{1}{\mathrm{Vol}(N)} \int_{\mathcal{H}_N^+} \det{}^\alpha X \exp \mathrm{tr} \left(-V(X) \right) \mathrm{d}X = h_0^{(\alpha)} h_1^{(\alpha)} \cdots h_{N-1}^{(\alpha)}, \tag{5.23}$$

where $\mathrm{Vol}(N)$ is defined in (5.5). Using the above relations and (5.19) in the case $V(x) = \frac{x}{2} - \sum_{k\geq 1} s_k x^k$, we obtain the following identity between the GUE partition function $Z_{2N}^{\mathsf{even}}(\mathbf{s})$ in (5.4) and the Laguerre partition function $Z_N\left(\pm\frac{1}{2};\mathbf{t}\right)$ in (5.11),

$$\frac{Z_{2N}^{\mathsf{even}}(\mathbf{s})}{Z_{2N}^{\mathsf{even}}(\mathbf{0})} = \frac{Z_N\left(-\frac{1}{2};\mathbf{t}\right)}{Z_N\left(-\frac{1}{2};\mathbf{0}\right)}\frac{Z_N\left(\frac{1}{2};\mathbf{t}\right)}{Z_N\left(\frac{1}{2};\mathbf{0}\right)}, \qquad t_k := 2^k s_k, \tag{5.24}$$

where $Z_N^{\mathsf{even}}(\mathbf{0}) = \sqrt{2^N \pi^{N^2}}$ is given as in (1.52) and $Z_N\left(\pm\frac{1}{2};\mathbf{0}\right)$ in (1.59). There is a similar, slightly more involved, factorization for the matrix model Z_{2N+1}^{even}, but we do not need its formulation for our present purposes. The following symmetry property of the LUE partition function allows us to match the parameters $\alpha = \pm\frac{1}{2}$ in (5.24).

Lemma 5.1.5. *The LUE connected correlator $\langle \mathrm{tr}\, X^{k_1} \cdots \mathrm{tr}\, X^{k_r}\rangle_{\mathsf{c}}$ with $k_1, \ldots, k_r > 0$ is a polynomial in N, α, and it is invariant under the involution $(N,\alpha) \mapsto (N+\alpha, -\alpha)$.*

Proof. It follows directly from Theorem 3.2.5, as the coefficients of $R^{[\infty]}$, defined in (3.89), are polynomials in N, α which are manifestly symmetric under the aforementioned transformation. Indeed

$$(N,\alpha) \mapsto (N+\alpha, -\alpha) \quad \Longleftrightarrow \quad (N, M-N) \mapsto (M, N-M) \quad \Longleftrightarrow \quad (N,M) \mapsto (M,N), \tag{5.25}$$

and from (3.91) we see that all the coefficients $A_\ell(N,M), B_\ell(N,M)$ but $A_0(N,M) = N$ are symmetric in $N, M := N+\alpha$; however $R^{[\infty]}$ only contains the combination $\ell A_\ell(N,M)$, which is always symmetric in N, M. ☐

Let us restate Lemma 5.1.5, in view of the formal expansion (5.41), as the following identity

$$\frac{Z_N(\alpha;\mathbf{t})}{Z_N(\alpha;\mathbf{0})} = \frac{Z_{N+\alpha}(-\alpha;\mathbf{t})}{Z_{N+\alpha}(-\alpha;\mathbf{0})}. \tag{5.26}$$

In particular, for $\alpha = \frac{1}{2}$ it reads

$$Z_{N+\frac{1}{2}}\left(-\frac{1}{2};\mathbf{t}\right) = \frac{\pi^{\frac{3}{8}+\frac{N}{2}}\mathrm{G}(N+1)}{\mathrm{G}\left(N+\frac{3}{2}\right)} Z_N\left(\frac{1}{2};\mathbf{t}\right) \tag{5.27}$$

which can be applied to (5.24) to prove Theorem 5.1.3.

Proof of Theorem 5.1.3. We use the uniqueness of the decomposition (5.5) which defines the mGUE partition function; rewriting it under the substitution $N \mapsto 2N$ we have

$$\frac{Z_{2N}^{\mathsf{even}}(\mathbf{s})}{(2\pi)^{2N}\mathrm{Vol}(2N)} = \widetilde{Z}_{2N-\frac{1}{2}}(\mathbf{s})\widetilde{Z}_{2N+\frac{1}{2}}(\mathbf{s}). \tag{5.28}$$

On the other hand, from (5.24) we have

$$Z_{2N}^{\mathsf{even}}(\mathbf{s}) = D_N Z_N\left(-\frac{1}{2};\mathbf{t}\right) Z_N\left(\frac{1}{2};\mathbf{t}\right) \tag{5.29}$$

where here and below we are identifying $t_k = 2^k s_k$. The proportionality constant D_N is explicitly evaluated from the normalization constant of LUE (1.59) and (1.52) as

$$D_N = \frac{Z_{2N}^{\mathsf{even}}(\mathbf{0})}{Z_N\left(-\frac{1}{2};\mathbf{0}\right) Z_N\left(\frac{1}{2};\mathbf{0}\right)} = \frac{2^N \pi^{N^2+N+\frac{1}{2}}\mathrm{G}(\frac{1}{2})^2}{\mathrm{G}(N+\frac{1}{2})\mathrm{G}(N+\frac{3}{2})}. \tag{5.30}$$

It is then enough to show that the two factorizations (5.28) and (5.29) are consistent once we identify $\widetilde{Z}_{2N-\frac{1}{2}}(\mathbf{s}) = C_N Z_N\left(-\frac{1}{2};\mathbf{t}\right)$ with C_N a constant depending on N only. Such consistency follows from the chain of equalities

$$
\begin{aligned}
\frac{Z_{2N}^{\mathsf{even}}(\mathbf{s})}{(2\pi)^{2N}\mathrm{Vol}(2N)} &= \widetilde{Z}_{2N-\frac{1}{2}}(\mathbf{s})\widetilde{Z}_{2N+\frac{1}{2}}(\mathbf{s}) \\
&= \widetilde{Z}_{2N-\frac{1}{2}}(\mathbf{s})\widetilde{Z}_{2(N+\frac{1}{2})-\frac{1}{2}}(\mathbf{s}) \\
&= C_N Z_N\left(-\frac{1}{2};\mathbf{t}\right) C_{N+\frac{1}{2}} Z_{N+\frac{1}{2}}\left(-\frac{1}{2};\mathbf{t}\right) \\
&= C_N C_{N+\frac{1}{2}} \frac{\pi^{\frac{3}{8}+\frac{N}{2}}\mathrm{G}(N+1)}{\mathrm{G}\left(N+\frac{3}{2}\right)} Z_N\left(-\frac{1}{2};\mathbf{t}\right) Z_N\left(\frac{1}{2};\mathbf{t}\right)
\end{aligned}
\tag{5.31}
$$

where we have used the symmetry property (5.27). This shows that the two factorizations (5.28) and (5.29) are consistent, so that equation (5.12), i.e.

$$
\widetilde{Z}_{2N-\frac{1}{2}}(\mathbf{s}) = C_N Z_N\left(\alpha = -\frac{1}{2};\mathbf{t}\right), \qquad t_k = 2^k s_k, \tag{5.32}
$$

holds, provided we also identify the proportionality constants (5.30) and (5.31),

$$
C_N C_{N+\frac{1}{2}} \frac{\pi^{\frac{3}{8}+\frac{N}{2}}\mathrm{G}(N+1)}{\mathrm{G}\left(N+\frac{3}{2}\right)} = \frac{D_N}{(2\pi)^{2N}\mathrm{Vol}(2N)} = 4^{N(N-1)}\pi^{-N(N+1)}\mathrm{G}(N+1)^2, \tag{5.33}
$$

where in the last step we use the duplication formula for the Barnes G-function in the form

$$
\mathrm{G}(2N+1) = \frac{2^{N(2N-1)}\pi^{-N-\frac{1}{2}}}{\mathrm{G}\left(\frac{1}{2}\right)^2}\mathrm{G}\left(N+\frac{1}{2}\right)\mathrm{G}(N+1)^2\mathrm{G}\left(N+\frac{3}{2}\right). \tag{5.34}
$$

Equation (5.33) fixes the constant to be

$$
C_N = 2^{N^2-\frac{3}{2}N+\frac{1}{4}}\pi^{-\frac{N(N+1)}{2}}\mathrm{G}(N+1), \tag{5.35}
$$

as stated in (5.14). □

Remark 5.1.6. *The identification of the mGUE and LUE partition functions is manifest also from the Virasoro constraints of the two models. Indeed, Virasoro constraints for the modified GUE partition function have been derived in [62], directly from those of the GUE partition function, and they assume the form* $\widetilde{\mathcal{L}}_n\widetilde{Z}_N(\mathbf{s}) = 0$*, for* $n \geq 0$*, where*

$$
\widetilde{\mathcal{L}}_n := \begin{cases} \sum_{k\geq 1} k\left(s_k - \frac{1}{2}\delta_{k,1}\right)\frac{\partial}{\partial s_k} + \frac{N^2}{4} - \frac{1}{16}, & n = 0, \\ \sum_{k=1}^{n-1}\frac{\partial^2}{\partial s_k \partial s_{n-k}} + \sum_{k\geq 1} k\left(s_k - \frac{1}{2}\delta_{k,1}\right)\frac{\partial}{\partial s_{k+n}} + N\frac{\partial}{\partial s_n}, & n \geq 1. \end{cases} \tag{5.36}
$$

On the other hand, it is well known [7, 108] that the LUE partition function with only positive couplings $\mathbf{t}_+$ *satisfies the Virasoro constraints* $\mathcal{L}_n^{(\alpha)}Z_N(\alpha;\mathbf{t}) = 0$*, for* $n \geq 0$*, where*

$$
\mathcal{L}_n^{(\alpha)} := \begin{cases} \sum_{k\geq 1} k\left(t_k - \delta_{k,1}\right)\frac{\partial}{\partial t_k} + N\left(N+\alpha\right), & n = 0, \\ \sum_{k=1}^{n-1}\frac{\partial^2}{\partial t_k \partial t_{n-k}} + \sum_{k\geq 1} k\left(t_k - \delta_{k,1}\right)\frac{\partial}{\partial t_{k+n}} + (2N+\alpha)\frac{\partial}{\partial t_n}, & n \geq 1. \end{cases} \tag{5.37}
$$

The Virasoro constraints $\widetilde{\mathcal{L}}_n = \widetilde{\mathcal{L}}_n(N,\mathbf{s})$ *in (5.36) and* $= \mathcal{L}_n^{(\alpha)} = \mathcal{L}_n^{(\alpha)}(N,\mathbf{t})$ *in (5.37) satisfy*

$$
2^n\widetilde{\mathcal{L}}_n\left(2N-\frac{1}{2},\mathbf{s}\right) = \mathcal{L}_n^{(-\frac{1}{2})}(N,\mathbf{t}) \tag{5.38}
$$

under the identification $t_k = 2^k s_k$*, in agreement with Theorem 5.1.3.*

Formal matrix models

In this section we justify the (formal) identifications derived above. We start from the definition of mGUE partition function. First, the logarithm of the even GUE partition function can be considered as a formal Taylor expansion for small s_k as

$$\log Z_N^{\mathsf{even}}(\mathbf{s}) := \log Z_N^{\mathsf{even}}(\mathbf{0}) + \sum_{r\geq 1}\sum_{k_1,\dots,k_r\geq 1} \frac{s_{k_1}\cdots s_{k_r}}{r!}\left\langle \operatorname{tr} X^{k_1}\cdots \operatorname{tr} X^{k_r}\right\rangle_{\mathsf{even}}^{\mathsf{c}} \tag{5.39}$$

where the connected even GUE correlators are introduced as usual,

$$\left\langle \operatorname{tr} X^{k_1}\cdots \operatorname{tr} X^{k_r}\right\rangle_{\mathsf{even}}^{\mathsf{c}} := \left.\frac{\partial^r \log Z_N^{\mathsf{even}}(\mathbf{s})}{\partial s_{k_1}\cdots\partial s_{k_r}}\right|_{\mathbf{s}=\mathbf{0}} \tag{5.40}$$

and the normalizing constant $Z_N^{\mathsf{even}}(\mathbf{0}) = \sqrt{2^N\pi^{N^2}}$, see (1.52). The infinite sum in (5.39) can be given a rigorous formal meaning in the algebra $\mathbb{C}[N][[\mathbf{s}]]$; introducing the grading $\deg s_k := k$, the latter algebra is obtained taking the inductive limit $K\to\infty$ from the algebras of polynomials in $\mathbf{s}$ of degree $< K$, with coefficients in $\mathbb{C}[N]$. Equivalently, this grading can be encoded, up to an inessential shift, by a (small) variable ϵ via the transformation $s_k \mapsto \epsilon^{k-1}s_k$, which is the same as considering the matrix model $\int_{\mathcal{H}_N}\exp\left[-\frac{1}{\epsilon}\left(\frac{X^2}{2}-\sum_{k\geq 1}s_kX^{2k}\right)\right]\mathrm{d}X$. For simplicity we have preferred to avoid the explicit ϵ-dependence, even though we shall restore it for the statement of the Hodge-GUE and Hodge-LUE correspondence, respectively Theorem 5.1.2 and Corollary 5.1.7. It must be stressed that (5.39) makes sense for any complex N, and not just for positive integers as it would be required by the genuine matrix integral interpretation; indeed the correlators are polynomials in N.

The same arguments apply to the Laguerre partition function (5.11), which can similarly be identified via the formal series

$$\log Z_N(\alpha;\mathbf{t}) = \log Z_N(\alpha;\mathbf{0}) + \sum_{r\geq 1}\sum_{k_1,\dots,k_r\geq 1}\frac{t_{k_1}\cdots t_{k_r}}{r!}\left\langle \operatorname{tr} X^{k_1}\cdots\operatorname{tr} X^{k_r}\right\rangle^{\mathsf{c}}. \tag{5.41}$$

Since the correlators are polynomials in N,α, see Theorem 3.2.5, the expression (5.41) can be viewed as an element of $\mathbb{C}[N,\alpha][[\mathbf{t}]]$, in particular it makes sense also for N complex.

This remark is crucial for a correct understanding of formulæ (5.26) and (5.27), as well as to formally justify the factorization identity (5.5).

5.1.2 Hodge-LUE correspondence

The identification Theorem 5.1.3 and the Hodge-GUE correspondence Theorem 5.1.2, allow us to deduce the following.

Corollary 5.1.7 (Hodge-LUE correspondence)**.** *Let* $\mathcal{H}\left(\mathbf{p}\left(\omega,\mathbf{s}\right);\sqrt{2}\epsilon\right)$ *as in* (5.10) *and* $Z_N\left(-\frac{1}{2};\mathbf{t}\right)$ *the Laguerre partition function* (5.11) *with parameter* $\alpha=-\frac{1}{2}$. *We have*

$$\mathcal{H}\left(\mathbf{p}(\omega,\mathbf{s});\sqrt{2}\epsilon\right) + \epsilon^{-2}A\left(\omega,\mathbf{s}\right) = \log Z_N\left(-\frac{1}{2};\mathbf{t}\right) + C(N,\epsilon), \tag{5.42}$$

where we identify

$$\omega = \epsilon\left(2N-\frac{1}{2}\right),\qquad t_k = 2^k\epsilon^{k-1}s_k, \tag{5.43}$$

$C(N,\epsilon)$ is a constant depending on N and ϵ only, and we recall the definitions of $A(\omega,\mathbf{s})$ and $\mathbf{p}(\omega,\mathbf{s})$, respectively (5.8) and (5.9), namely

$$A(\lambda,\mathbf{s}) := \frac{1}{4}\sum_{j_1,j_2\geq 1}\frac{j_1 j_2}{j_1+j_2}\binom{2j_1}{j_1}\binom{2j_2}{j_2}s_{j_1}s_{j_2} + \frac{1}{2}\sum_{j\geq 1}\left(\lambda - \frac{j}{j+1}\right)\binom{2j}{j}s_j, \tag{5.44}$$

$$p_k(\lambda,\mathbf{s}) := \sum_{j\geq 1} j^{k+1}\binom{2j}{j}s_j + \delta_{k,1} + \lambda\delta_{k,0} - 1, \qquad k\geq 0. \tag{5.45}$$

Proof. It follows from (5.10) upon the substitution $\omega \mapsto \epsilon\left(2N - \frac{1}{2}\right)$ and applying Theorem 5.1.3 for the set of times $\epsilon^{k-1}s_k$, $k\geq 1$. ☐

Finally, Theorem 5.1.1 is obtained matching the coefficients in (5.42) and the topological expansion for the positive correlators of the LUE in terms of Hurwitz numbers equation (4.68).

Proof of Corollary 5.1.1. We apply $\frac{\partial^\ell}{\partial s_{\mu_1}\cdots\partial s_{\mu_\ell}}\Big|_{\mathbf{s}=\mathbf{0}}$, for $\ell>0$, on both sides of (5.10). On the right side, in view of Theorem 5.1.3 and the Hodge-GUE correspondence (5.10) we get

$$\frac{\partial^\ell}{\partial s_{\mu_1}\cdots\partial s_{\mu_\ell}}\bigg|_{\mathbf{s}=\mathbf{0}} \log\widetilde{Z}_{\frac{\omega}{\epsilon}}\left((s_1,\epsilon s_2,\epsilon^2 s_3,\dots)\right) = \epsilon^{|\mu|-\ell}2^{|\mu|}\left\langle \operatorname{tr}X^{\mu_1}\cdots\operatorname{tr}X^{\mu_\ell}\right\rangle^{\mathsf{c}}_{LUE}\Big|_{N=\frac{\omega}{2\epsilon}+\frac{1}{4},\,\alpha=-\frac{1}{2}},$$

which in view of the topological expansion (4.68) for the LUE correlators equals

$$= \epsilon^{|\mu|-\ell}2^{|\mu|}\sum_{\gamma\geq 0}\sum_{\nu\vdash|\mu|}\left(\frac{\omega+\frac{\epsilon}{2}}{2\epsilon}\right)^{2-2\gamma+|\mu|-\ell}\left(\frac{\omega-\frac{\epsilon}{2}}{\omega+\frac{\epsilon}{2}}\right)^{s} h^{>}_{\gamma}(\mu,\nu). \tag{5.46}$$

Note that the substitutions $2N-\frac{1}{2}=\frac{\omega}{\epsilon}$, $\alpha=-\frac{1}{2}$, from Theorem 5.1.3, yield $N=\frac{\omega+\frac{\epsilon}{2}}{2\epsilon}$, $c=\frac{\omega-\frac{\epsilon}{2}}{\omega+\frac{\epsilon}{2}}$. On the other side we get

$$\frac{\partial^\ell}{\partial s_{\mu_1}\cdots\partial s_{\mu_\ell}}\bigg|_{\mathbf{s}=\mathbf{0}}\mathcal{H}(\mathbf{p}(\omega,\mathbf{s});\sqrt{2}\epsilon) + \epsilon^{-2}\frac{\partial^\ell}{\partial s_{\mu_1}\cdots\partial s_{\mu_\ell}}\bigg|_{\mathbf{s}=\mathbf{0}}A(\omega,\mathbf{s}). \tag{5.47}$$

The contributions from the last term is directly evaluated from (5.8) and give the second line of (5.3). For the first term we recall the *affine* change of variable (5.9) and compute

$$\frac{\partial^\ell}{\partial s_{\mu_1}\cdots\partial s_{\mu_\ell}}\mathcal{H}\left(\mathbf{p}(\omega,\mathbf{s});\sqrt{2}\epsilon\right) = \sum_{i_1,\dots,i_\ell\geq 0}\prod_{b=1}^{\ell}\mu_b^{i_b+1}\binom{2\mu_b}{\mu_b}\frac{\partial^\ell}{\partial p_{i_1}\cdots\partial p_{i_\ell}}\mathcal{H}\left(\mathbf{p}(\omega,\mathbf{s});\sqrt{2}\epsilon\right) \tag{5.48}$$

$$= \sum_{g,n\geq 0}\left(\sqrt{2}\epsilon\right)^{2g-2}\sum_{\substack{k_1,\dots,k_n\geq 0\\ i_1,\dots,i_\ell\geq 0}}\frac{p_{k_1}(\omega,\mathbf{s})\cdots p_{k_n}(\omega;\mathbf{s})}{n!}\int_{\overline{\mathcal{M}}_{g,n+\ell}}\Lambda^2(-1)\Lambda\left(\frac{1}{2}\right)\prod_{a=1}^{n}\psi_a^{k_a}\prod_{b=1}^{\ell}\mu_b^{i_b+1}\binom{2\mu_b}{\mu_b}\psi_{n+b}^{i_b}. \tag{5.49}$$

Evaluation at $\mathbf{s}=\mathbf{0}$ corresponds to $p_k=\delta_{k,1}+\omega\delta_{k,0}-1$; thus, in the previous expression, we set $n=m+r$, where m is the number of k_a's equal to zero, and the remaining $k_1,\dots,k_r$'s are all ≥ 2

(we are evaluating at $p_1 = 0$), and so the evaluation of the (5.48) at $p_k = \delta_{k,1} + \omega\delta_{k,0} - 1$ reads

$$\sum_{g,m,r\geq 0} \left(\sqrt{2}\epsilon\right)^{2g-2} \sum_{\substack{k_1,\dots,k_r\geq 2\\ i_1,\dots,i_\ell\geq 0}} \frac{(\omega-1)^m(-1)^r}{m!r!} \int_{\overline{\mathcal{M}}_{g,\ell+m+r}} \Lambda^2(-1)\Lambda\left(\frac{1}{2}\right) \prod_{a=1}^{r} \psi_a^{k_a} \prod_{b=1}^{\ell} \mu_b^{i_b+1}\binom{2\mu_b}{\mu_b}\psi_{m+r+b}^{i_b}$$
$$= \sum_{g,m,r\geq 0} \left(\sqrt{2}\epsilon\right)^{2g-2} \sum_{d_1,\dots,d_r\geq 1} \frac{(\omega-1)^m(-1)^r}{m!r!} \int_{\overline{\mathcal{M}}_{g,\ell+m+r}} \Lambda^2(-1)\Lambda\left(\frac{1}{2}\right) \prod_{a=1}^{r} \psi_a^{d_a+1} \prod_{b=1}^{\ell} \frac{\mu_b\binom{2\mu_b}{\mu_b}}{1-\mu_b\psi_{m+r+b}}, \tag{5.50}$$

where in the last step we rename $k_a = d_a + 1$, $d_a \geq 1$.

We can trade the $\psi_1, \dots, \psi_r$ classes in (5.50) for a suitable combination of Mumford κ classes, following ideas from [39]. Let $\pi : \overline{\mathcal{M}}_{g,\ell+m+r} \to \overline{\mathcal{M}}_{g,\ell+m}$ be the map forgetting the first r marked points (and contracting the resulting unstable components), then we have the following iterated version of the *dilaton equation*

$$\pi_*\left((\pi^*\mathcal{X})\prod_{a=1}^{r}\psi_a^{d_a+1}\right) = \mathcal{X}\sum_{\sigma\in\mathfrak{S}_r}\prod_{\gamma\in\mathrm{Cycles}(\sigma)}\kappa_{\sum_{a\in\gamma}d_a}, \qquad d_a,\dots,d_r\geq 1, \tag{5.51}$$

for any $\mathcal{X} \in H^\bullet\left(\overline{\mathcal{M}}_{g,\ell+m},\mathbb{Q}\right)$. Here and below, $\mathfrak{S}_r$ is the group of permutations of $\{1,\dots,r\}$ and $\mathrm{Cycles}(\sigma)$ is the set of disjoint cycles in the permutation σ, $\sigma = \prod_{\gamma\in\mathrm{Cycles}(\sigma)}\gamma$. In our case it is convenient to set

$$\mathcal{X} = \Lambda^2(-1)\Lambda\left(\frac{1}{2}\right)\prod_{b=1}^{\ell}\frac{\mu_b\binom{2\mu_b}{\mu_b}}{1-\mu_b\psi_{m+b}}, \qquad \pi^*\mathcal{X} = \Lambda^2(-1)\Lambda\left(\frac{1}{2}\right)\prod_{b=1}^{\ell}\frac{\mu_b\binom{2\mu_b}{\mu_b}}{1-\mu_b\psi_{m+r+b}}, \tag{5.52}$$

so that the sum over $r \geq 0$ and $d_1,\dots,d_r \geq 1$ in (5.50) can be expressed as

$$\sum_{r\geq 0}\frac{(-1)^r}{r!}\sum_{d_1,\dots,d_r\geq 1}\int_{\overline{\mathcal{M}}_{g,\ell+m+r}}(\pi^*X)\prod_{a=1}^{r}\psi_a^{d_a+1} = \sum_{r\geq 0}\frac{(-1)^r}{r!}\sum_{d_1,\dots,d_r\geq 1}\int_{\overline{\mathcal{M}}_{g,\ell+m}} X\sum_{\sigma\in\mathfrak{S}_r}\prod_{\gamma\in\mathrm{Cycles}(\sigma)}\kappa_{\sum_{a\in\gamma}d_a}. \tag{5.53}$$

Let us now recall that for any set of variables $F_1, F_2, \dots,$ we have the identity of symmetric functions

$$\exp\left(\sum_{r\geq 1}\frac{\xi^r}{r}F_r\right) = \sum_{\nu}\frac{\xi^{|\nu|}}{z_\nu}F_{\nu_1}\cdots F_{\nu_{\ell(\nu)}} \tag{5.54}$$

where the sum on the right side extends over the set of all partitions $\nu = (\nu_1,\dots,\nu_{\ell(\nu)})$, $|\nu| = \nu_1+\dots+\nu_{\ell(\nu)}$, and z_ν has the same definition as above, namely $z_\nu := \prod_{i\geq 1}(i^{m_i})\,m_i!$, m_i being the multiplicity of i in the partition ν. Applying this relation to

$$F_r = \sum_{d_1,\dots,d_r\geq 1}\kappa_{\sum_{a=1}^{t}d_a} = \sum_{d\geq r}\binom{d-1}{r-1}\kappa_d, \qquad \xi = -1, \tag{5.55}$$

since for any partition ν of r the quantity $r!/z_\nu$ is the cardinality of the conjugacy class labeled by ν in $\mathfrak{S}_r$, we deduce that

$$\sum_{r\geq 0}\frac{(-1)^r}{r!}\sum_{d_1,\dots,d_r\geq 1}\int_{\overline{\mathcal{M}}_{g,\ell+m}}\mathcal{X}\sum_{\sigma\in\mathfrak{S}_r}\prod_{\gamma\in Cycles(\sigma)}\kappa_{\sum_{a\in\gamma}d_a} = \int_{\overline{\mathcal{M}}_{g,\ell+m}}\mathcal{X}\exp\left(-\sum_{d\geq 1}\frac{\kappa_d}{d}\right) \tag{5.56}$$

where we also use the identity $\sum_{r\geq 1}\frac{(-1)^r}{r}\binom{d-1}{r-1} = -\frac{1}{d}$. The proof is complete. □

Example 5.1.8. *Comparing the coefficients of ϵ^{-2} on both sides of (5.2) we obtain the following relation in genus zero*

$$\mathscr{H}_{0,\mu} = 2^{\ell-2}\omega^{|\mu|+2-\ell}\frac{z_\mu}{|\mu|!}\sum_{\nu\vdash|\mu|} h^{>}_{g=0}(\mu,\nu) \tag{5.57}$$

valid for any partition μ of length ℓ. One can check that (5.57) is consistent with the computations of Hurwitz numbers in genus zero performed in Example 4.2.5. E.g. for $\ell=1$ we compute the first terms in the $(\omega-1)$-expansion of the left side of (5.57), directly from (5.2),

$$\mathscr{H}_{0,\mu=(\mu_1)} = \frac{1}{2}\frac{1}{\mu_1+1}\binom{2\mu_1}{\mu_1} + \frac{(\omega-1)}{2}\binom{2\mu_1}{\mu_1} + \frac{(\omega-1)^2}{4}\mu_1\binom{2\mu_1}{\mu_1} + \mathcal{O}\left((\omega-1)^3\right). \tag{5.58}$$

On the other hand, the right side of (5.57) is computed as

$$\begin{aligned}\frac{1}{2}\omega^{\mu_1+1}\sum_{s=1}^{\mu_1}\frac{1}{(\mu_1-1)!}\sum_{\nu\vdash|\mu|}h^{>}_{g=0}(\mu,\nu) &= \frac{1}{2\mu_1}\omega^{\mu_1+1}\sum_{s=1}^{\mu_1}\binom{\mu_1}{s}\binom{\mu_1}{s-1}\\ &= \frac{1}{2\mu_1}\omega^{\mu_1+1}\binom{2\mu_1}{\mu_1-1}\\ &= \frac{1}{2(\mu_1+1)}\binom{2\mu_1}{\mu_1}\sum_{b=0}^{\mu_1+1}\binom{\mu_1+1}{b}(\omega-1)^b\end{aligned} \tag{5.59}$$

where we use (4.90) and the identity

$$\sum_{s=1}^{\mu_1}\binom{\mu_1}{s-1}\binom{\mu_1}{s} = \sum_{s=0}^{\mu_1-1}\binom{\mu_1}{s}\binom{\mu_1}{\mu_1-1-s} = \binom{2\mu_1}{\mu_1-1}, \tag{5.60}$$

which follows from the Chu-Vandermonde identity $\sum_{s=0}^{k-1}\binom{a}{s}\binom{b}{k-1-s} = \binom{a+b}{k-1}$ for $a=b=k=\mu_1$. Expressions (5.58) and (5.59) match.

5.2 Θ-GW invariants of $\mathbb{P}^1$ and the Legendre unitary ensemble

In [147] Norbury defines a new collection of cohomological class $\Theta_{g,k} \in H^{2(2g-2+k)}\left(\overline{\mathcal{M}}_{g,k}\right)$ on the moduli space $\overline{\mathcal{M}}_{g,k}$ of stable algebraic curves. The *k-point, genus g and degree d stationary Θ-Gromov-Witten (Θ-GW) invariants of* $\mathbb{P}^1$ are defined as the integrals

$$\left\langle \Theta\cdot\prod_{j=1}^{k} b_j!\,\tau_{b_j}(\omega)\right\rangle^{\mathbb{P}^1}_{g,k,d} := \int_{\left[\mathbb{P}^1_{g,k,d}\right]}\left(\prod_{j=1}^{k}\psi_j^{b_j}\mathrm{ev}_j^*(\omega)\right)\cdot\Theta^{\mathbb{P}^1}_{g,k}, \tag{5.61}$$

where $\omega\in H^*(\mathbb{P}^1)$ is the Kähler class of $\mathbb{P}^1$, the b_i are non-negative integers, $\left[\mathbb{P}^1_{g,k,d}\right]$ is the virtual fundamental class of the moduli space of stable maps from curves of genus g with k distinct marked points to the target $\mathbb{P}^1$ of degree d, the ev_i are the evaluation maps $ev_i : \mathbb{P}^1_{g,k,d}\to\mathbb{P}^1$ and $\Theta^{\mathbb{P}^1}_{g,k} = p^*\Theta_{g,k}$ with $p:\overline{\mathcal{M}}_{g,k}\to\mathbb{P}^1_{g,k,d}$ the forgetful map. We refer to loc. cit. and references therein for a more accurate description of these objects. It is proved in [148], see also [47], that the generating function of the Θ-GW invariants (5.61) is a *tau-function* of the *KdV hierarchy*, this is the content of Theorem 5.2.2. More to that, it is expressible as the partition function of a matrix model closely related to the JUE, which allows to derive the following *ELSV-like formula*.

Theorem 5.2.1. *The triple monotone Hurwitz numbers of Definition 4.1.4 and the stationary Θ-GW invariants of $\mathbb{P}^1$ are related by*

$$\sum_{\mu,\nu\vdash|\lambda|} \frac{h_g^{\geq}(\lambda,\mu,\nu)}{(-2)^{\ell(\mu)+\ell(\nu)+\ell+2g-2}} = \frac{(-1)^{|\lambda|}|\lambda|!}{2^{|\lambda|}\prod_{i\geq 1} i^{m_i}} \sum_{k_1,\dots,k_\ell\geq 1} \frac{\left\langle \Theta\cdot\prod_{j=1}^{\ell}\binom{\lambda_i}{k_j}\tau_{k_j-1}(\omega)\right\rangle_d^g}{2^{k_1+\cdots+k_\ell}}, \tag{5.62}$$

where $\lambda = (\lambda_1,\dots,\lambda_\ell) = (1^{m_1},2^{m_2},\dots)$ is an arbitrary partition, and the right hand side is subject to the dimensional constraint

$$2d = \sum_{j=1}^{\ell}(k_j-1)+\ell = \sum_{j=1}^{\ell}k_j. \tag{5.63}$$

Conversely,

$$\left\langle \Theta\cdot\prod_{j=1}^{\ell}\tau_{\lambda_j-1}(\omega)\right\rangle_d^g = \frac{2^{|\lambda|}}{\prod_{i\geq 1} m_i!}\sum_{k_1,\dots,k_\ell\geq 1} 2^{|\kappa|}\frac{z_\kappa}{|\kappa|!}\prod_{j=1}^{\ell}\binom{\lambda_i}{k_j}\sum_{\mu,\nu\vdash|\kappa|}\frac{h_g^{\geq}(\kappa,\mu,\nu)}{(-2)^{\ell(\mu)+\ell(\nu)+\ell+2g-2}}, \tag{5.64}$$

where we used the multi-index notation $|\kappa| := (k_1,\dots,k_\ell)$ and $|\kappa| = \sum k_1$. Entering $h_g^{\geq}(\kappa,\mu,\nu)$, we regard κ as the unique partition of $|\kappa|$ with parts the (ordered) entries of κ.

Proof of Theorem 5.2.1

The main ingredient in the proof is the following Theorem.

Theorem 5.2.2 ([148])**.** *Consider the generating function for the Θ-GW invariants of $\mathbb{P}^1$*

$$F_{\mathbb{P}^1}^{\Theta}(\epsilon,\mathbf{s}=\{s_k\}_{k\geq 0}) = \sum_{g\geq 0}\epsilon^{2g-2}F_g^{\Theta} = \sum_{g,d}\epsilon^{2g-2}\left\langle\Theta\cdot\exp\left\{\sum_{k\geq 0}^{\infty}\tau_k(\omega)s_k\right\}\right\rangle_d^g + \frac{1}{4}\log\epsilon \tag{5.65}$$

and define the partition function $Z_{\mathbb{P}^1}^{\Theta}(\epsilon,\mathbf{s}) = \exp F_{\mathbb{P}^1}^{\Theta}(\epsilon,\mathbf{s})$. Then, setting $\epsilon = N^{-1}$ we have

$$Z_{\mathbb{P}^1}^{\Theta}\left(\epsilon=N^{-1},\mathbf{s}\right) = \frac{c}{N!}\int_{-2}^{2}\int_{-2}^{2}\dots\int_{-2}^{2}dx_1\dots dx_N\prod_{i<j}(x_i-x_j)^2\exp\left(N\sum_{k\geq 0}s_k\sum_{i=1}^{N}x_i^{k+1}\right). \tag{5.66}$$

where $c\in\mathbb{C}$ is a constant. More precisely, $Z_{\mathbb{P}^1}^{\Theta}$ coincides with an asymptotic expansion of the integral (5.66) as $N\to\infty$.

Consider the following partition functions where, respectively, the first one is the Jacobi partition function (4.23) with parameters $\alpha=\beta=0$, and the second one is essentially (5.66) ,

$$Z_N^{(0,0)}(\mathbf{u}) := \frac{1}{C_N^{(0,0)}}\int_{\mathcal{H}_N(0,1)}\exp\left(\sum_{k\geq 1}\frac{u_k}{k}\operatorname{tr}X^k\right)\mathrm{d}X = \sum_{\lambda\in\mathcal{P}}\frac{\left\langle\prod_{j=1}^{\ell}\operatorname{tr}X^{\pm\lambda_j}\right\rangle_{(0,0)}}{z_\lambda}\prod_{i=1}^{\ell(\lambda)}u_{\lambda_i}, \tag{5.67}$$

$$Z_N^{\Theta}(\mathbf{v}) := Z_{\mathbb{P}^1}^{\Theta}\left(\epsilon=N^{-1},\mathbf{v}=\{v_k=Nk\,s_{k-1}\}_{k\geq 1}\right) = \frac{1}{C_N}\int_{\mathcal{H}_N(-2,2)}\exp\left(\sum_{k\geq 1}\frac{v_k}{k}\operatorname{tr}Y^k\right)\mathrm{d}Y. \tag{5.68}$$

The two models are the – respectively antisymmetric and symmetric – *Legendre unitary ensemble*, and are equivalent up to an affine transformation. At the level of the disconnected correlators the relation reads

$$\left\langle \prod_{j=1}^{\ell} \operatorname{tr} X^{\lambda_j} \right\rangle_{(0,0)} = \sum_{j_1,\dots,j_\ell \geq 0} 2^{-|\lambda|-j_1-\dots-j_\ell} \left\langle \prod_{i=1}^{\ell} \binom{\lambda_i}{j_i} \operatorname{tr} Y^{j_i} \right\rangle_{\Theta} \tag{5.69}$$

$$\left\langle \prod_{j=1}^{\ell} \operatorname{tr} Y^{\lambda_j} \right\rangle_{\Theta} = \sum_{j_1,\dots,j_\ell \geq 0} (-1)^{|\lambda|} (-2)^{|\lambda|+j_1+\dots+j_\ell} \left\langle \prod_{i=1}^{\ell} \binom{\lambda_i}{j_i} \operatorname{tr} X^{j_i} \right\rangle_{(0,0)} \tag{5.70}$$

notice that the sums are bounded due to the presence of the binomial factors. The connected correlators admit the same relation, but the indices are taken to be different from zero,

$$\left\langle \prod_{j=1}^{\ell} \operatorname{tr} X^{\lambda_j} \right\rangle^{\mathsf{c}}_{(0,0)} = \sum_{j_1,\dots,j_\ell \geq 1} 2^{-|\lambda|-j_1-\dots-j_\ell} \left\langle \prod_{i=1}^{\ell} \binom{\lambda_i}{j_i} \operatorname{tr} Y^{j_i} \right\rangle^{\mathsf{c}}_{\Theta} \tag{5.71}$$

$$\left\langle \prod_{j=1}^{\ell} \operatorname{tr} Y^{\lambda_j} \right\rangle^{\mathsf{c}}_{\Theta} = \sum_{j_1,\dots,j_\ell \geq 1} (-1)^{|\lambda|} (-2)^{|\lambda|+j_1+\dots+j_\ell} \left\langle \prod_{i=1}^{\ell} \binom{\lambda_i}{j_i} \operatorname{tr} X^{j_i} \right\rangle^{\mathsf{c}}_{(0,0)} \tag{5.72}$$

This happens because, in general, if $j_i = 0$ for some $i \in \{1, \dots, \ell\}$, then the whole connected correlator vanishes; for example if $\ell = 2$,

$$\left\langle \operatorname{tr} X^j \operatorname{tr} X^0 \right\rangle^{\mathsf{c}} = \left\langle \operatorname{tr} X^j \operatorname{tr} X^0 \right\rangle - \left\langle \operatorname{tr} X^j \right\rangle \left\langle \operatorname{tr} X^0 \right\rangle = \left\langle \operatorname{tr} X^j \cdot N \right\rangle - \left\langle \operatorname{tr} X^j \right\rangle \langle N \rangle = 0. \tag{5.73}$$

The next step is to rewrite relations (5.71) and (5.72) in terms of the topological expansion of the correlators, given in Theorems 4.1.5 and 5.2.2. With respect to the former, we just substitute the parameters $\alpha = \beta = 0 \iff c_\alpha = c_\beta = 1$ to get

$$\left\langle \prod_{j=1}^{\ell} \operatorname{tr} X^{\lambda_j} \right\rangle^{\mathsf{c}}_{(0,0)} = (-1)^{|\lambda|} N^{-\ell} \frac{z_\lambda}{|\lambda|!} \sum_{g \geq 0} \frac{1}{N^{2g-2}} \sum_{\mu,\nu \vdash |\lambda|} \frac{h_g^{\geq}(\lambda, \mu, \nu)}{(-2)^{\ell(\mu)+\ell(\nu)+\ell+2g-2}}. \tag{5.74}$$

For the latter, first notice that by Theorem 5.2.2 we have

$$\left. \frac{\partial^\ell}{\partial s_{k_1} \cdots \partial s_{k_\ell}} \right|_{\mathbf{s}=\mathbf{0}} F^{\Theta}_{\mathbb{P}^1}(\epsilon, \mathbf{s}) = \sum_{g \geq 0} \epsilon^{2g-2} \left\langle \Theta \cdot \prod_{j=1}^{\ell} \tau_{k_j}(\omega) \right\rangle^g_d \tag{5.75}$$

so that, recalling the change of times $\mathbf{s} = \left\{ s_k = \frac{1}{N} \frac{v_{k+1}}{k+1} \right\}_{k \geq 0}$, for $\lambda = (\lambda_1, \dots, \lambda_\ell) = (1^{m_1}, 2^{m_2}, \dots)$,

$$\left\langle \prod_{j=1}^{\ell} \operatorname{tr} Y^{\lambda_j} \right\rangle^{\mathsf{c}}_{\Theta} = z_\lambda \left. \frac{\partial^\ell}{\partial v_{\lambda_1} \cdots \partial v_{\lambda_\ell}} \right|_{\mathbf{v}=\mathbf{0}} \log Z_N^{\Theta}(\mathbf{v}) \tag{5.76}$$

$$= z_\lambda \sum_{k_1,\dots,k_\ell \geq 0} \frac{\partial s_{k_1}}{\partial v_{\lambda_1}} \cdots \frac{\partial s_{k_\ell}}{\partial v_{\lambda_\ell}} \left. \frac{\partial^\ell}{\partial s_{k_1} \cdots \partial s_{k_\ell}} \right|_{\mathbf{s}=\mathbf{0}} \log Z^{\Theta}_{\mathbb{P}^1}\left(\epsilon = N^{-1}, \mathbf{s}\right) \tag{5.77}$$

$$= z_\lambda \frac{\delta_{k_1}^{\lambda_1 - 1}}{\lambda_1} \cdots \frac{\delta_{k_\ell}^{\lambda_\ell - 1}}{\lambda_\ell} \left. \frac{\partial^\ell}{\partial s_{k_1} \cdots \partial s_{k_\ell}} \right|_{\mathbf{s}=\mathbf{0}} F^{\Theta}_{\mathbb{P}^1}\left(\epsilon = N^{-1}, \mathbf{s}\right) \tag{5.78}$$

$$= N^{-\ell} m_1! \cdots m_\ell! \sum_{g \geq 0} \frac{1}{N^{2g-2}} \left\langle \Theta \cdot \prod_{j=1}^{\ell} \tau_{\lambda_j - 1}(\omega) \right\rangle^g_d. \tag{5.79}$$

Plugging formulæ (5.74) and (5.79) in (5.71) and matching the coefficients in N yields

$$\sum_{\mu,\nu\vdash|\lambda|} \frac{h_g^{\geq}(\lambda,\mu,\nu)}{(-2)^{\ell(\mu)+\ell(\nu)+\ell+2g-2}} = \frac{(-1)^{|\lambda|}|\lambda|!}{2^{|\lambda|}\prod_{i\geq 1} i^{m_i}} \sum_{k_1,\dots,k_\ell\geq 1} \frac{\left\langle \Theta\cdot\prod_{j=1}^{\ell}\binom{\lambda_i}{k_j}\tau_{k_j-1}(\omega)\right\rangle_d^g}{2^{k_1+\cdots+k_\ell}}, \tag{5.80}$$

which is exactly (5.62). Similarly, from (5.72) we get (5.64) as

$$\left\langle \Theta\cdot\prod_{j=1}^{\ell}\tau_{\lambda_j-1}(\omega)\right\rangle_d^g = \frac{2^{|\lambda|}}{\prod_{i\geq 1} m_i!} \sum_{k_1,\dots,k_\ell\geq 1} 2^{|\kappa|}\frac{z_\kappa}{|\kappa|!}\prod_{j=1}^{\ell}\binom{\lambda_i}{k_j} \sum_{\mu,\nu\vdash|\kappa|} \frac{h_g^{\geq}(\kappa,\mu,\nu)}{(-2)^{\ell(\mu)+\ell(\nu)+\ell+2g-2}}, \tag{5.81}$$

where we denoted $|\kappa| := (k_1,\dots,k_\ell)$ and used the multi-index notations, $|\kappa| = \sum k_1$, while z_κ is still defined as usual, see (2.16).

Discrete Integrable Systems and Random Lax Matrices

Chapter 6

Hermitian Lax systems

In this chapter, we consider two specific integrable systems: the *exponential Toda lattice* in Section 6.2 and the *Volterra lattice* in Section 6.3. The goal is to explore the behaviour of the spectrum of their random Lax matrices when the number of degrees of freedom $N \to \infty$, and the initial data is sampled according to a properly chosen Gibbs measure. Analyzing the Lax matrix we connect them respectively to the Laguerre β-ensemble at high temperature and the antisymmetric Gaussian β-ensemble at high temperature. This allows us to explicitly compute their density of states, see Corollary 6.45 and 6.3.3. The fact that both these systems admit a Lax representation with an Hermitian Lax matrix plays a crucial role in the derivations, as rank one perturbations arguments can be applied, see [16].

We briefly recall some useful formulæ and definitions from the classical theory of integrable systems in Section 6.1.

6.1 Background material

In this section we recall some standard tools to study Hamiltonian integrable systems that we need throughout the paper. For further details, we refer to various textbooks and monographs [12, 13, 15, 151].

Definition 6.1.1. *A Poisson manifold is a pair $(P, \{.,.\})$ where P is a n-dimensional differentiable manifold and $\{.,.\}$ is an antisymmetric bilinear operation on the space $\mathcal{C}^\infty(P)$ of smooth functions over P,*

$$\begin{aligned} \mathcal{C}^\infty(P)\times\mathcal{C}^\infty(P) &\to \mathcal{C}^\infty(P) \\ (f,g) &\longrightarrow \{f,g\} \end{aligned} \tag{6.1}$$

such that for all functions $f, g, h \in \mathcal{C}^\infty(P)$, it satisfies:

1. *the Jacobi identity*
$$\{\{f,g\},h\} + \{\{h,f\},g\} + \{\{g,h\},f\} = 0, \tag{6.2}$$
2. *the Leibniz rule*
$$\{hf,g\} = h\{f,g\} + \{h,g\}f. \tag{6.3}$$

The operator $\{.,.\}$ is called a Poisson bracket. When there is no risk of confusion, we simply denote a Poisson manifold by P, where the Poisson bracket is assumed to be fixed and given.

In local coordinates $\mathbf{a} = (a_1, \ldots, a_n)$ the Poisson bracket is specified by an antisymmetric $(2,0)$ tensor $\pi^{ij}(\boldsymbol{a})$, the *Poisson tensor*, acting on the coordinates as

$$\{a_i, a_j\} = \pi^{ij}(\boldsymbol{a}), \qquad i, j = 1, \ldots n. \tag{6.4}$$

The Jacobi identity on the coordinates is equivalent to the relation

$$\frac{\partial \pi^{ij}(\boldsymbol{a})}{\partial a_s}\pi^{sk}(\boldsymbol{a}) + \frac{\partial \pi^{ki}(\boldsymbol{a})}{\partial a_s}\pi^{sj}(\boldsymbol{a}) + \frac{\partial \pi^{jk}(\boldsymbol{a})}{\partial a_s}\pi^{si}(\boldsymbol{a}) = 0, \quad 1 \le i < j < k \le n, \tag{6.5}$$

where we are summing over repeated indices. In an open subset of P the Poisson tensor has a fixed even rank $2r \le n$. By antisymmetry, it follows that the Poisson tensor can be non-degenerate, meaning that $\det \pi(\boldsymbol{a}) \neq 0$, if and only if the dimension n of the base space is even, namely $n = 2N$.

Given a function $H(\mathbf{a}) \in \mathcal{C}^\infty(P)$, it generates a set of so-called *Hamilton's equations* through the relation

$$\dot{a}_j = \{a_j, H\} = \sum_{j=1}^{n} \pi^{ij}(\mathbf{a})\frac{\partial H}{\partial a_j}, \quad j = 1, \ldots, n. \tag{6.6}$$

The function H itself is called a *Hamiltonian*. The previous set of equations defines a continuum time flow from an initial condition $\mathbf{a}(0) \in P$ to its time evolution $t > 0$, namely $\Phi_t \colon \mathbf{a}(0) \to \mathbf{a}(t)$. A function $K = K(\boldsymbol{a})$ is constant under evolution Φ_t if and only if

$$\dot{K} = \{K, H\} = 0. \tag{6.7}$$

In this case the quantity K is called a *first integral* or a *constant of motion*. The notion of Liouville integrability is strictly related to the number of first integrals and the rank of the associated Poisson tensor.

Definition 6.1.2 (Liouville integrability)**.** *A Hamiltonian system* (6.6) *on a Poisson manifold P of rank $2r \le n$ is Liouville integrable if there are $k = n - r$ first integrals $H_1, \ldots, H_k$ in involution*

$$\{H_i, H_j\} = 0, \quad i, j = 1, \ldots, k\,, \tag{6.8}$$

and functionally independent, namely

$$\operatorname{rank}\left(\frac{\partial H_i}{\partial a_j}\right)_{\substack{i=1,\ldots,k\\ j=1,\ldots,n}} = k\,, \tag{6.9}$$

in a dense subset of P.

Finding first integrals is often a complicated task, and during the past decades several algorithms to construct them have been developed. One of the most effective methods to produce first integrals of a given mechanical system is the so-called *Lax pair representation*[1]. The concept of Lax pair originates from the work of P. D. Lax on the theory of PDEs [130], where it was used to produce exact solutions through the so-called inverse scattering method [1, 4]. We give the definition in the finite dimensional setting, [12, 15].

Definition 6.1.3. *Let $L = L(\mathbf{a})$ and $A = A(\mathbf{a})$ be $N \times N$ matrices, with $N = N(n)$ and such that the equation*

$$\dot{L} = [A, L], \quad [A, L] = AL - LA. \tag{6.10}$$

is equivalent to the Hamiltonian flow (6.6)*. Then the matrices L and A are a Lax pair for the Hamiltonian system and the matrix L is called Lax matrix.*

[1] Often L-A pair in the Russian literature.

The main consequence of the Lax equation (6.10) is that the eigenvalues of L are first integrals of the Hamiltonian flow (6.6). So, provided that we can prove these eigenvalues give enough functionally independent quantities in involution, we can infer the Liouville integrability of Hamilton's equations (6.6) through the Lax pair.

Remark 6.1.4. *The fact that in many cases an integrable system can be equivalent to a matrix relation gives the connection with random matrix theory. Indeed, when the initial data* $\mathbf{a}(0)$ *is chosen randomly, the Lax matrix* $L = L(\mathbf{a}(0))$ *becomes a random matrix.*

To define a random initial data, we consider *invariant measures* with respect to the Hamiltonian flow. In general, such objects have the form

$$\mu = m\left(\mathbf{a}\right) \mathrm{d}a_1 \wedge \cdots \wedge \mathrm{d}a_n, \tag{6.11}$$

where the density $m(\mathbf{a})$ is such that the measure $\mu \in L^1(\mathcal{M})$, with $\mathcal{M}$ being a sub-manifold of the manifold P.

Definition 6.1.5. *Given an Hamiltonian system equipped with an invariant measure* μ*, define the Gibbs measure [126] associated to the Hamiltonian as*

$$\mu_H = \frac{1}{Z_H} e^{-\beta H(\mathbf{a})} \mu, \tag{6.12}$$

where we assume the normalization constant Z_H *to be finite,*

$$Z_H = \int_{\mathcal{M}} e^{-\beta H(\mathbf{a})} \mu < \infty. \tag{6.13}$$

Similarly, given $H_1, \ldots, H_k$ *first integrals and* $\beta_1, \ldots, \beta_k$ *constants, define the generalized Gibbs measure as*

$$\mu_G = \frac{1}{Z_G} e^{-\sum_{j=1}^{k} \beta_j H_j(\mathbf{a})} \mu. \tag{6.14}$$

As above, we assume that the normalization constant Z_G *is finite,*

$$Z_G = \int_{\mathcal{M}} e^{-\sum_{j=1}^{N} \beta_j H_j(\mathbf{a})} \mu < \infty. \tag{6.15}$$

As discussed above, random initial data are obtained from an invariant measure μ of the form (6.11). More precisely, this means that the measure of every subset $S \subset \mathcal{M}$ with respect to μ is preserved under the time-evolution Φ_t,

$$\int_{\Phi_t(S)} \mu = \int_S \mu. \tag{6.16}$$

Interpreting the evolution as a coordinate transformation, we have

$$\int_{\Phi_t(S)} \mu = \int_S \Phi_t^*(\mu), \tag{6.17}$$

where $\Phi_t^*(\mu)$ is the pull-back of μ through Φ_t. This shows that the condition (6.16) is satisfied if $\Phi_t^*(\mu) = \mu$. In the following, we will only work in Euclidean coordinates, for a measure written in the form (6.11), so that the above condition can be rephrased as

$$\operatorname{div}\left(m(\mathbf{a})\mathbf{f}_H(\boldsymbol{a})\right) := \sum_{i=1}^{m} \frac{\partial}{\partial a_i}(m(\mathbf{a})(\mathbf{f}_H(\mathbf{a}))_i) = 0, \tag{6.18}$$

where div is the usual euclidean divergence, see e.g. [106, Chapter 1]. The vector field $\mathbf{f}_H$ is specified by the Hamiltonian H via the relation $(\mathbf{f}_H)_i = \{a_i, H\}$. The condition (6.18) can be written in the form

$$\{m, H\} + m\,\mathrm{div}\,(\mathbf{f}_H) = 0. \tag{6.19}$$

Remark 6.1.6. *The condition (6.18) depends just on $m(\mathbf{a})$ and the vector field $\mathbf{f}_H$, thus it is independent of the Hamiltonian nature of the dynamical system at hand. In particular, we conclude that a measure μ as in (6.11) is invariant for the dynamical system*

$$\dot{a}_i = f_i(\mathbf{a}), \quad i = 1, \dots, N, \tag{6.20}$$

if and only if (6.18) holds.

From formula (6.19) we immediately have two important consequences.

– If the Hamiltonian vector field is divergence free, like in the case of a canonical Poisson bracket, it follows that the standard Euclidean measure

$$\mu_0 = \mathrm{d}a_1 \wedge \mathrm{d}a_2 \wedge \cdots \wedge \mathrm{d}a_n \tag{6.21}$$

is an invariant measure.

– If K is a first integral and m is the density of an invariant measure, then from the Leibniz rule (6.3) it follows that

$$\widetilde{m} := f(K)m \tag{6.22}$$

is the density of another invariant measure for every scalar function $f \in \mathcal{C}^\infty(\mathcal{M})$.

In all the examples we analyse in this thesis, *all the Hamiltonian vector fields are divergence free*, so we will be allowed to consider the Generalized Gibbs ensemble with $\mu = \mu_0$ the Euclidean measure in (6.21).

6.2 Laguerre β-ensemble and the exponential Toda lattice

In this section, we introduce an integrable model that we call *exponential Toda lattice*, since it resembles the well-know Toda lattice. We construct the Lax pair for this system, and we define its Generalized Gibbs measure. Finally, we compute the mean density of states of the Lax matrix.

The exponential Toda lattice is the Hamiltonian system on $M = \mathbb{R}^{2N}$ with canonical Poisson bracket described by the Hamiltonian

$$H_E(\mathbf{p}, \mathbf{q}) = \sum_{j=1}^N e^{-p_j} + \sum_{j=1}^N e^{q_j - q_{j+1}}, \quad p_j, q_j \in \mathbb{R}. \tag{6.23}$$

We consider periodic boundary conditions

$$q_{j+N} = q_j + \Delta, \quad p_{j+N} = p_j, \qquad \forall\, j \in \mathbb{Z}, \tag{6.24}$$

and $\Delta \geq 0$ is an arbitrary constant. The equations of motion are given in Hamiltonian form as

$$\begin{aligned} \dot{q}_j &= \frac{\partial H_E}{\partial p_j} = -e^{-p_j}, \\ \dot{p}_j &= -\frac{\partial H_E}{\partial q_j} = e^{q_{j-1} - q_j} - e^{q_j - q_{j+1}}. \end{aligned} \tag{6.25}$$

The system possesses two trivial constants of motion,

$$H_0(\mathbf{p},\mathbf{q}) = \sum_{j=1}^{N}(q_j - q_{j+1}), \qquad H_1(\mathbf{p},\mathbf{q}) = \sum_{j=1}^{N} p_j, \tag{6.26}$$

the first one due to periodicity, the second one due to the translational invariance of the Hamiltonian (6.23). In order to obtain a Lax pair for this system we introduce, in the spirit of Flaschka and Manakov [82, 83, 132], the variables

$$x_j = e^{-\frac{p_j}{2}}, \qquad y_j = e^{\frac{q_j - q_{j+1}}{2}} = e^{-\frac{r_j}{2}}, \;\; r_j = q_{j+1} - q_j, \quad j = 1, \dots, N, \tag{6.27}$$

where we notice that $\prod_{j=1}^{N} y_j = e^{-\frac{\Delta}{2}}$. In these variables, the Hamiltonian (6.23) and the constants of motion (6.26) transform into

$$H_E(\mathbf{x},\mathbf{y}) = \sum_{j=1}^{N}(x_j^2 + y_j^2), \qquad H_0(\mathbf{x},\mathbf{y}) = 2\sum_{j=1}^{N}\log y_j, \qquad H_1(\mathbf{x},\mathbf{y}) = -2\sum_{j=1}^{N}\log x_j. \tag{6.28}$$

The Hamilton's equations (6.25) become

$$\dot{x_j} = \frac{x_j}{2}\left(y_j^2 - y_{j-1}^2\right), \qquad \dot{y_j} = \frac{y_j}{2}\left(x_{j+1}^2 - x_j^2\right), \quad j = 1, \dots, N, \tag{6.29}$$

where $x_{N+1} = x_1$, $y_0 = y_N$.

One can explicitly construct a Lax pair for this system. Let us introduce the matrix $E_{r,s}$, defined as $(E_{r,s})_{ij} = \delta_r^i \delta_s^j$. Set

$$L = \sum_{j=1}^{N}(x_j^2 + y_{j-1}^2)E_{j,j} + \sum_{j=1}^{N} x_j y_j (E_{j,j+1} + E_{j+1,j}), \tag{6.30}$$

$$A = \sum_{j=1}^{N} \frac{x_j y_j}{2}(E_{j,j+1} - E_{j+1,j}), \tag{6.31}$$

where, accounting for periodic boundary conditions, indices are taken modulo N, so that $E_{i,j+N} = E_{i+N,j} = E_{i,j}$ for all $i, j \in \mathbb{Z}$. For example, the matrix L in (6.30) has the explicit form

$$L = \begin{pmatrix} x_1^2 + y_N^2 & x_1 y_1 & & & x_N y_N \\ x_1 y_1 & x_2^2 + y_1^2 & x_2 y_2 & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & x_{N-1} y_{N-1} \\ x_N y_N & & & x_{N-1} y_{N-1} & x_N^2 + y_{N-1}^2 \end{pmatrix}. \tag{6.32}$$

The system of equations (6.29) then admits the Lax representation

$$\dot{L} = [A, L]. \tag{6.33}$$

Hence, the quantities $H_m = \operatorname{Tr}\left(L^{m-1}\right)$, $m = 2, \dots, N+1$ are constants of motion as well as the eigenvalues of L. For the exponential Toda lattice, we define the generalized Gibbs ensemble as

$$\mu_{ET} = \frac{1}{Z_N^E(\beta,\eta,\theta)} \exp\left(-\beta H_E + \theta H_0 - \eta H_1\right) \mathrm{d}\mathbf{r}\mathrm{d}\mathbf{p}\,, \tag{6.34}$$

where $\beta, \eta, \theta > 0$, the Hamiltonians H_E, H_0 and H_1 are defined in (6.23) and (6.26) respectively, Z_N^E is the normalization constant, $\mathrm{d}\mathbf{r} = \mathrm{d}r_1 \dots \mathrm{d}r_N$ and analogously for $\mathrm{d}\mathbf{p}$. We notice that according to this measure, all the variables are independent, moreover all p_j are identically distributed, and so are the r_j. After introducing the variables $(\mathbf{r}, \mathbf{p}) \to (\mathbf{x}, \mathbf{y})$, the previous measure turns into

$$\mu_{ET} = \frac{1}{Z_N^{H_E}(\beta, \eta, \theta)} \prod_{j=1}^{N} x_j^{2\eta-1} e^{-\beta x_j^2} \mathrm{d}x_j \prod_{j=1}^{N} y_j^{2\theta-1} e^{-\beta y_j^2} \mathrm{d}y_j \,. \tag{6.35}$$

Let $\chi_{2\alpha}$ be the *chi-distribution*, defined by its density

$$f_{2\alpha}(r) = \frac{r^{2\alpha-1} e^{-\frac{r^2}{2}}}{2^{\alpha-1}\Gamma(\alpha)}, \qquad r \in \mathbb{R}^+, \tag{6.36}$$

where $\alpha > 0$. The variables x_j and y_j in the Gibbs measure (6.35) are independent random variables with scaled *chi*-distribution, respectively $f_{2\eta}(\sqrt{2\beta}x_j)\sqrt{2\beta}\mathrm{d}x_j$ and $f_{2\theta}(\sqrt{2\beta}y_j)\sqrt{2\beta}\mathrm{d}y_j$.

The Lax matrix L in (6.32) becomes a random matrix when the entries are sampled according to (6.35). Such random matrix can be linked to the so-called *Laguerre α-ensemble* [135]. The connection is obtained noticing that the matrix L admits the following decomposition

$$L = BB^\intercal, \qquad B = \sum_{j=1}^{N} x_j E_{j,j} + \sum_{j=1}^{N} y_j E_{j+1,j}, \tag{6.37}$$

where $B^\intercal$ is the matrix transpose. On the other hand, the Laguerre α-ensemble is given by the set of matrices $L_{\alpha,\gamma} = B_{\alpha,\gamma}(B_{\alpha,\gamma})^\intercal$, where $B_{\alpha,\gamma} \in \mathrm{Mat}(N \times M)$, $M \geq N$, and

$$B_{\alpha,\gamma} = \frac{1}{\sqrt{2}} \begin{pmatrix} x_1 & & & & \\ y_1 & x_2 & & & \\ & \ddots & \ddots & & \mathbf{0}_{N\times(M-N)} \\ & & y_{N-1} & x_N & \end{pmatrix}, \tag{6.38}$$

here $\mathbf{0}_{N\times(M-N)}$ is the zero matrix of dimension $N \times (M-N)$. The variables x_n, y_n are distributed according to chi-distribution

$$x_n \sim \chi_{\frac{2\alpha}{\gamma}} \quad n = 1, \dots, N, \tag{6.39}$$

$$y_n \sim \chi_{2\alpha} \quad n = 1, \dots, N-1. \tag{6.40}$$

Thus, the following entry wise measure on the matrices $B_{\alpha,\gamma}$ can be defined,

$$\mu_{B_{\alpha,\gamma}} = \frac{1}{\left(2^{\frac{\alpha}{\gamma}-1}\Gamma(\frac{\alpha}{\gamma})\right)^N \left(2^{\alpha-1}\Gamma(\alpha)\right)^{N-1}} \prod_{j=1}^{N} x_j^{\frac{2\alpha}{\gamma}-1} e^{-\frac{x_j^2}{2}} \mathrm{d}x_j \prod_{j=1}^{N-1} y_j^{2\alpha-1} e^{-\frac{y_j^2}{2}} \mathrm{d}y_j \,. \tag{6.41}$$

We observe that the matrix B in (6.37) has the same form of $B_{\alpha,\gamma}$ in (6.39), with the addition of the corner element $y_N E_{1,N}$. Furthermore, the rescaling of the variables $(x_j, y_j) \mapsto \frac{1}{\sqrt{2\beta}}(x_j, y_j)$ in (6.35), amounts to the matrix rescaling $B \mapsto \frac{1}{\sqrt{2\beta}} B$, and comparing with (6.41) we see that $\frac{1}{\sqrt{2\beta}} B$ is a rank one perturbation of the matrix $B_{\theta, \frac{\theta}{\eta}}$.

We are interested in studying the density of states ν_{ET} for the Lax matrix L when the entries are distributed according to the Gibbs measure μ_{ET} in (6.35). The density of states ν_{ET} is obtained from the weak convergence in $L^1(\mathbb{R})$ of the empirical measure of the Lax matrix L, namely

$$\frac{1}{N}\sum_{j=1}^{N}\delta_{\lambda_j} \xrightharpoonup{N\to\infty} \nu_{ET}\,, \tag{6.42}$$

where λ_j are the eigenvalues of L and δ_x is the Dirac delta function centred at x.

In order to study the density of states of the Exponential Toda lattice, we recall the following result proved in [135].

Theorem 6.2.1 (cf. [135], Theorem 1.1)**.** *Consider the matrix $L_{\alpha,\gamma} = B_{\alpha,\gamma}B_{\alpha,\gamma}^{\intercal}$ distributed according to $\mu_{B_{\alpha,\gamma}}$ in (6.41). Then, its mean density of states $\nu_{L_{\alpha,\gamma}}$ takes the form*

$$\nu_{L_{\alpha,\gamma}} = \partial_\alpha\left(\alpha\mu_{\alpha,\gamma}(x)\right)\mathrm{d}x\,,\quad x\geq 0, \tag{6.43}$$

where

$$\mu_{\alpha,\gamma}(x) := \frac{1}{\Gamma(\alpha+1)\Gamma\left(1+\frac{\alpha}{\gamma}+\alpha\right)}\frac{x^{\frac{\alpha}{\gamma}}e^{-x}}{\left|\psi\left(\alpha,-\frac{\alpha}{\gamma};xe^{-i\pi}\right)\right|^2},\qquad x\geq 0\,, \tag{6.44}$$

here $\psi(v,w;z)$ is the Tricomi's confluent hypergeometric function [5].

In view of Theorem 6.2.1 and the previous discussion we deduce.

Corollary 6.2.2. *Consider the Lax matrix $L = BB^{\intercal}$ in (6.37) of the exponential Toda lattice with Hamiltonian (6.23) and endow the entries of the matrix B in (6.37) with the Gibbs measure μ_{ET} (6.35). Then, the density of states ν_{ET} of the Lax matrix $L = BB^{\intercal}$ takes the form*

$$\nu_{ET} = \beta\partial_\alpha(\alpha\mu_{\alpha,\gamma}(\beta x))_{|_{\alpha=\theta,\gamma=\frac{\theta}{\eta}}}\mathrm{d}x,\quad x\geq 0\,, \tag{6.45}$$

where the measure $\mu_{\alpha,\gamma}$ is defined in (6.44).

Proof. First, we notice that by virtue of general theory of Hermitian matrices, see [16, Theorem A.43], we can restrict to the case $y_N = 0$ in (6.35). As observed above, performing the change of variables $(x_j,y_j)\mapsto\frac{1}{\sqrt{2\beta}}(x_j,y_j)$, which amounts to rescale $B\mapsto\frac{1}{\sqrt{2\beta}}B$, one has that the matrix entries of $\frac{1}{\sqrt{2\beta}}B$ are distributed as the matrix entries of $B_{\theta,\frac{\theta}{\eta}}$. Applying Theorem 6.2.1 we obtain the claim. $\square$

Parameter Limit

In this section, we examine the low-temperature limit of the Hamiltonian system (6.23). Namely, we want to compute the eigenvalues of the Lax matrix L in (6.30) in the limit $\beta,\theta,\eta\to\infty$, in such a way that

$$\eta = \widetilde{\eta}\beta,\quad \theta = \widetilde{\theta}\beta,$$

where $\widetilde{\eta}$ and $\widetilde{\theta}$ are in compact sets of $\mathbb{R}_+$.

Since all x_j and y_j are independent random variables, we just have to consider the weak limit of the rescaled chi-distributions, respectively

$$f_{2\tilde{\eta}\beta}(\sqrt{2\beta}x)\sqrt{2\beta}\mathrm{d}x, \quad f_{2\tilde{\theta}\beta}(\sqrt{2\beta}y)\sqrt{2\beta}\mathrm{d}y\,.$$

We explicitly work out one of the cases above.

We consider a continuous and bounded function $h : \mathbb{R}_+ \to \mathbb{R}$ and evaluate the limit

$$\lim_{\beta\to\infty}\int_0^\infty h(x)f_{2\eta\beta}(\sqrt{2\beta}x)\sqrt{2\beta}\mathrm{d}x = \lim_{\beta\to\infty}\frac{\int_0^\infty h(x)e^{\beta(2\widetilde{\eta}\log x - x^2)}\mathrm{d}x}{\int_0^\infty x^{2\widetilde{\eta}\beta}e^{-\beta x^2}\mathrm{d}x} = h\left(\sqrt{\widetilde{\eta}}\right)\,. \tag{6.46}$$

The last identity has been obtained by applying the Laplace method (see [138]) and observing that the minimizer of the term $2\widetilde{\eta}\log(x) - x^2$ in the exponent of the integral is $x_0 = \sqrt{\widetilde{\eta}}$.

As a consequence, we conclude that $x_j \rightharpoonup \sqrt{\widetilde{\eta}}$ and $y_j \rightharpoonup \sqrt{\widetilde{\theta}}$, $j = 1, \ldots, N$ as $\beta \to \infty$, where with $\rightharpoonup$ we denote weak convergence. The previous limit implies that the measure ν_{ET} in (6.45) converges, in the low temperature limit, to the density of states of the matrix L_∞

$$L_\infty = \begin{pmatrix} \widetilde{\eta}+\widetilde{\theta} & \sqrt{\widetilde{\eta}\widetilde{\theta}} & & & \sqrt{\widetilde{\eta}\widetilde{\theta}} \\ \sqrt{\widetilde{\eta}\widetilde{\theta}} & \widetilde{\eta}+\widetilde{\theta} & \sqrt{\widetilde{\eta}\widetilde{\theta}} & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & \sqrt{\widetilde{\eta}\widetilde{\theta}} \\ \sqrt{\widetilde{\eta}\widetilde{\theta}} & & & \sqrt{\widetilde{\eta}\widetilde{\theta}} & \widetilde{\eta}+\widetilde{\theta} \end{pmatrix}. \tag{6.47}$$

Indeed, the fact that L is tridiagonal with iid entries along the diagonals, implies its k-th moment depends on a multiple of k number of variables only; specifically looking back at the Lax matrix L in (6.32)

$$\left\langle \mathrm{tr}\left(L^k\right)\right\rangle = \left\langle \sum_{j=1}^N \left(L^k\right)_{jj}\right\rangle = N\cdot\left\langle \left(L^k\right)_{11}\right\rangle =: N\cdot\langle f\left(x_{N-k},\ldots,x_{N+k};y_{N-k},\ldots,y_{N+k}\right)\rangle\,, \tag{6.48}$$

for some function $f(\cdot)$ of its entries. Then, passing to the density of states (6.42) and renaming the iid variables, the scaling factor N identically cancels out and moments converge,

$$\lim_{N\to\infty}\frac{1}{N}\left\langle \mathrm{tr}\left(L^k\right)\right\rangle = \langle f\left(x_1,\ldots,x_{2k};y_1,\ldots,y_{2k}\right)\rangle\,. \tag{6.49}$$

The eigenvalues being functions of moments, density of states converges as well. In particular, this also shows that the two limit commute in taking the density of states at low temperature,

$$\lim_{N\to\infty}\lim_{\beta\to\infty}\frac{1}{N}\sum_{j=1}^N\delta_{\lambda_j} = \lim_{\beta\to\infty}\lim_{N\to\infty}\frac{1}{N}\sum_{j=1}^N\delta_{\lambda_j} = \nu_\infty\,, \tag{6.50}$$

since the limits can be passed directly to the variables x_i, y_i.

The matrix L_∞ is a circulant matrix, so its eigenvalues can be computed explicitly [103] as

$$\lambda_j = \widetilde{\eta}+\widetilde{\theta}+2\sqrt{\widetilde{\eta}\widetilde{\theta}}\cos\left(2\pi\frac{j}{N}\right)\,, \quad j = 1,\ldots,N\,. \tag{6.51}$$

Then, from the explicit expression (6.51), it follows that the density of states of L_∞ is

$$\nu_{L_\infty} = \frac{1}{2\pi}\frac{dx}{\sqrt{(c_+ - x)(x - c_-)}}\mathbb{1}_{(c_-,c_+)}, \quad c_\pm = (\sqrt{\widetilde{\eta}} \pm \sqrt{\widetilde{\theta}})^2. \tag{6.52}$$

here $\mathbb{1}_{(c_-,c_+)}$ is the indicator function of the set (c_-, c_+). Thus, we proved the following result.

Proposition 6.2.3. *Consider the random Lax matrix L in (6.30) sampled from the Gibbs ensemble μ_{ET} (6.35) of the Exponential Toda lattice (6.29). The density of states of the matrix L in the low-temperature limit, i.e. when $\beta, \theta, \eta \to \infty$ in such a way that $\eta = \widetilde{\eta}\beta$, $\theta = \widetilde{\theta}\beta$, with $\widetilde{\eta}, \widetilde{\theta}$ in compact subsets of $\mathbb{R}_+$, is the hard edge distribution given by (6.52), namely an arcsine distribution.*

6.3 Volterra lattice

The *Volterra lattice*, also known as the *discrete KdV equation*, describes the motion of N particles on the line with equations

$$\dot{a_j} = a_j\left(a_{j+1} - a_{j-1}\right), \qquad j = 1, \dots, N. \tag{6.53}$$

It was originally introduced by Kac and Van Moerbeke in [122] to study population evolution in a hierarchical system of competing species. It was first solved by Kac and van Moerbeke in [10], using a discrete version of inverse scattering due to Flaschka [83]. Equations (6.53) can be considered as a finite-dimensional approximation of the Korteweg–de Vries equation.

The phase space is $\mathbb{R}_+^N$ and we consider periodic boundary conditions $a_j = a_{j+N}$ for all $j \in \mathbb{Z}$. The Volterra lattice is a reduction of the *second flow of the Toda lattice* [123]. Indeed, the latter is described by the dynamical system

$$\dot{a_j} = a_j\left(b_{j+1}^2 - b_j^2 + a_{j+1} - a_{j-1}\right), \qquad j = 1, \dots, N, \tag{6.54}$$

$$\dot{b_j} = a_j(b_{j+1} + b_j) - a_{j-1}(b_j + b_{j-1}), \qquad j = 1, \dots, N, \tag{6.55}$$

and equations (6.53) are recovered just by setting $b_j \equiv 0$. The Hamiltonian structure of the equations follows from the one of the Toda lattice. On the phase space $\mathbb{R}_+^N$ we introduce the Poisson bracket

$$\{a_j, a_i\}_{\text{Volt}} = a_j a_i(\delta_{i,j+1} - \delta_{i,j-1}) \tag{6.56}$$

and the Hamiltonian $H_1 = \sum_{j=1}^N a_j$ so that the equations of motion (6.53) can be written in the Hamiltonian form

$$\dot{a}_j = \{a_j, H_1\}_{\text{Volt}}\,. \tag{6.57}$$

An elementary constant of motion for the system is $H_0 = \prod_{j=1}^N a_j$ that is independent of H_1.

The Volterra lattice is a completely integrable system, and it admits several equivalent *Lax representations*, see e.g. [123, 141]. The classical one reads

$$\dot{L}_1 = [A_1, L_1]\,, \tag{6.58}$$

where

$$\begin{aligned} L_1 &= \sum_{j=1}^N a_j E_{j+1,j} + E_{j,j+1}, \\ A_1 &= \sum_{j=1}^N (a_j + a_{j+1})E_{j,j} + E_{j,j+2}\,, \end{aligned} \tag{6.59}$$

where we recall that the matrix $E_{r,s}$ is defined as $(E_{r,s})_{ij} = \delta_r^i \delta_s^j$ and $E_{j+N,i} = E_{j,i+N} = E_{j,i}$. There exists also a *symmetric* formulation due to Moser [141],

$$
\begin{aligned}
&\dot{L}_2 = [A_2, L_2] \\
&L_2 = \sum_{j=1}^{N} \sqrt{a}_j (E_{j,j+1} + E_{j+1,j}) \,, \\
&A_2 = \sum_{j=1}^{N} \sqrt{a_j a_{j+1}} (E_{j,j+2} - E_{j+2,j}) \,,
\end{aligned}
\tag{6.60}
$$

which assumes that all $a_j > 0$.

Furthermore, we point out that there exists also an *antisymmetric* formulation for this Lax pair, indeed a straightforward computation yields

Proposition 6.3.1. *Let $a_j > 0$ for all $j = 1, \dots, N$. Then, the dynamical system (6.53) admits an antisymmetric Lax matrix L_3 with companion matrix A_3, namely the equations of motion are equivalent to $\dot{L}_3 = [A_3, L_3]$ with*

$$
L_3 = \sum_{j=1}^{N} \sqrt{a_j} (E_{j,j+1} - E_{j+1,j}), \tag{6.61}
$$

$$
A_3 = \sum_{j=1}^{N} \sqrt{a_j a_{j+1}} (E_{j+2,j} - E_{j,j+2}). \tag{6.62}
$$

Gibbs Ensemble

We introduce a Gibbs ensemble for the Volterra lattice (6.53) by observing that its vector field $f_j = a_j \left(a_{j+1} - a_{j-1}\right)$ is divergence free, due to the periodic boundary conditions. Therefore, an invariant measure can be obtained from (6.22). We use $H_0 = \prod_{j=1}^N a_j$, and $H_1 = \sum_{j=1}^N a_j$ as constants of motion to construct the invariant measure

$$
\mu_{\mathrm{Volt}}(\mathbf{a}) = \frac{1}{Z_N^{\mathrm{Volt}}(\beta,\eta)} e^{-\beta H_1 + (\eta-1)\log H_0} \mathrm{d}\mathbf{a}, \quad \beta, \eta > 0, \tag{6.63}
$$

where

$$
Z_N^{\mathrm{Volt}}(\beta,\eta) = \left(\frac{\Gamma(\eta)}{\beta^\eta} \right)^N < \infty, \tag{6.64}
$$

and $\Gamma(\eta)$ is the Euler Gamma function [58, §5]. We notice that according to this measure, all the variables are independent and identically distributed (i.i.d.).

Next we want to characterize the density of states of the antisymmetric Lax L_3 of the Volterra lattice given in Proposition 6.3.1. Among the three Lax matrices of the Volterra lattice, the matrix L_3 is particularly useful since it allows to connect the Volterra Lattice with a specific α-ensemble, namely the antisymmetric Gaussian α-ensemble. The antisymmetric Gaussian α-ensemble, see [86], is the family of random antisymmetric tridiagonal matrices

$$
L_\alpha = \begin{bmatrix}
0 & y_1 & & & \\
-y_1 & 0 & y_2 & & \\
 & \ddots & \ddots & \ddots & \\
 & & -y_{N-2} & 0 & y_{N-1} \\
 & & & -y_{N-1} & 0
\end{bmatrix}, \tag{6.65}
$$

where y_i are i.i.d. random variables with chi-distribution with density

$$f_{2\alpha}(y) = \frac{y^{2\alpha-1}e^{-y^2}}{\Gamma(\alpha)}, \quad y \in \mathbb{R}^+. \tag{6.66}$$

Even though we use a different normalization of the chi-distribution with respect to Section 6.2, we keep the same notation $f_{2\alpha}(y)$ for the density. This distribution induces a measure on the entries of the matrix L_α, namely

$$\mu_{L_\alpha} = \frac{\prod_{i=1}^{N-1} y_i^{2\alpha-1}e^{-y_i^2}\,\mathbb{1}_{\mathbb{R}_+}(y_i)\mathrm{d}\mathbf{y}}{\Gamma(\alpha)^{N-1}}. \tag{6.67}$$

In [86] the authors studied this matrix ensemble in connection with the Antisymmetric Gaussian β-ensemble introduced by Dumitriu and Forrester [70] in the high temperature regime, and computed explicitly its density of states $\nu_{L_\alpha}(x)$, defined as

$$\frac{1}{N}\sum_{j=1}^{N} \delta_{\mathrm{Im}\,(\lambda_j)} \xrightarrow{N\to\infty} \nu_{L_\alpha}(x)\,, \tag{6.68}$$

where λ_j are the eigenvalues of L_α. Since the matrix L_α is antisymmetric with real entries, its eigenvalues are purely imaginary numbers.

Theorem 6.3.2. *[86] The density of states of the random matrix L_α in (6.65), is explicitly given by*

$$\nu_{L_\alpha}(x) = \partial_\alpha(\alpha\theta_\alpha(x))\mathrm{d}x\,, \tag{6.69}$$

where

$$\theta_\alpha(x) = \left|\Gamma(\alpha)W_{-\alpha+1/2,0}(-y)\right|^{-2}\,, \tag{6.70}$$

here $W_{k,\mu}(z)$ is the Whittaker function [5].

We notice that performing the change of coordinates $a_j = x_j^2$, the Gibbs ensemble (6.63) reads:

$$\mu_{\mathrm{Volt}}(\mathbf{x}) = \frac{\prod_{j=1}^{N} x_j^{2\eta-1}e^{-\beta\sum_{j=1}^{N} x_j^2}\,\mathbb{1}_{\mathbb{R}_+}(x_j)\mathrm{d}\mathbf{x}}{Z_N^{\mathrm{Volt}}(\beta,\eta)}\,, \tag{6.71}$$

which, up to a rescaling $x_j \to x_j/\sqrt{\beta}$ and for the extra term x_N in the probability distribution, is exactly the distribution (6.67) of the matrix L_α. Furthermore, the matrix L_3 is a 2 rank perturbation of the matrix L_α. Therefore, by a corollary of [16, Theorem A.41] and Theorem 6.3.2, we obtain the following.

Corollary 6.3.3. *Consider the matrix L_3 in (6.62) endowed with the Gibbs measure μ_{Volt} (6.63). Then, the density of states of the matrix L_3 is explicitly given by*

$$\nu_{Volt}(x) = \sqrt{\beta}\partial_\eta\left(\eta\theta_\eta(\sqrt{\beta}x)\right)\mathrm{d}x\,, \tag{6.72}$$

where $\theta_\alpha(x)$ is given in (6.70).

Parameter Limit

As for the case of Exponential Toda (6.29), in this section we consider the low-temperature regime of the Volterra lattice, namely the limit $\eta, \beta \to \infty$, in such a way that $\eta = \beta\widetilde{\eta}$, with $\widetilde{\eta}$ in a compact set of $\mathbb{R}_+$, and we compute the density of states of the matrix L_3 (6.61) in this regime.

Applying the same techniques of Section 6.2, we conclude that the density of states of the matrix L_3 in the low-temperature limit coincides with the one of the matrix L_∞, where

$$L_\infty = \begin{pmatrix} 0 & \sqrt{\widetilde{\eta}} & & & -\sqrt{\widetilde{\eta}} \\ -\sqrt{\widetilde{\eta}} & 0 & \sqrt{\widetilde{\eta}} & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & \sqrt{\widetilde{\eta}} \\ \sqrt{\widetilde{\eta}} & & & -\sqrt{\widetilde{\eta}} & 0 \end{pmatrix} . \tag{6.73}$$

Since the matrix L_∞ is circulant, we can readily compute its eigenvalues as

$$\lambda_j = 2i\sqrt{\widetilde{\eta}}\sin\left(2\pi\frac{j}{N}\right), \quad j = 1, \ldots, N\,. \tag{6.74}$$

From this explicit formula, it follows that the density of states of the matrix L_∞ reads

$$\nu_{L_\infty} = \frac{1}{2\pi}\frac{1}{\sqrt{4\widetilde{\eta} - x^2}}\mathbb{1}_{(-2\sqrt{\widetilde{\eta}}, 2\sqrt{\widetilde{\eta}})}(x)\mathrm{d}x\,. \tag{6.75}$$

Such measure coincides with the measure ν_{L^α} in the low-temperature limit. Thus, we just proved the following.

Proposition 6.3.4. *Consider the Gibbs ensemble μ_{Volt} of the Volterra lattice (6.63), in the low-temperature limit, i.e. $\beta, \eta \to \infty$, in such a way that $\eta = \widetilde{\eta}\beta$, where $\widetilde{\eta}$ is in a compact subset of $\mathbb{R}_+$. Then, the density of states ν_{Volt} of the Lax matrix L_3 in (6.61) converges, in this regime, to an arcsine distribution, namely*

$$\nu_{Volt} = \frac{1}{2\pi}\frac{1}{\sqrt{4\widetilde{\eta} - x^2}}\mathbb{1}_{(-2\sqrt{\widetilde{\eta}}, 2\sqrt{\widetilde{\eta}})}(x)\mathrm{d}x\,. \tag{6.76}$$

Chapter 7

Non-Hermitian Lax systems

In this chapter we continue our pursuit in the description of Hamiltonian integrable systems with random initial data sampled according to the associated generalized Gibbs measure. We consider: generalizations of the Volterra lattice to short range interactions[32] , Section 7.1, the focusing Ablowitz–Ladik lattice[2, 3], Section 7.2, and the focusing Schur flow, Section 7.3. In these cases the corresponding random Lax matrices are not symmetric nor self-adjoint and we derive numerically their density of states that has support in the complex plane. Interesting patterns of the density of states emerge as we vary the parameters of the system. For all the systems under analysis we are still able to compute the density of states in the low-temperature limit, namely in the ground state. The background material is in Section 6.1.

7.1 Generalization of the Volterra lattice: the INB k-lattices

The Volterra lattice (6.53) can be generalized in a variety of ways. The most natural ones are two families of lattices described in [32] (see also [31, 116, 145]) which include short range interactions,

$$\dot{a}_i = a_i \left(\sum_{j=1}^{k} a_{i+j} - \sum_{j=1}^{k} a_{i-j} \right), \quad i = 1, \dots, N, \tag{7.1}$$

$$\dot{a}_i = a_i \left(\prod_{j=1}^{k} a_{i+j} - \prod_{j=1}^{k} a_{i-j} \right), \quad i = 1, \dots, N, \tag{7.2}$$

where $k \in \mathbb{N}$, $N \geq k$, and the periodicity condition $a_{j+N} = a_j$ holds.
These two families are called the *additive Itoh–Narita–Bogoyavleskii (INB) k-lattice* and the *multiplicative Itoh–Narita–Bogoyavleskii (INB) k-lattice* respectively. Setting $k = 1$, we recover from both lattices the Volterra one (6.53). Further generalizations of the INB lattice were recently considered in [76].

A crucial difference in the two models is that in the additive lattice (7.1) the interaction is on arbitrary number of points, but the non-linearity is still quadratic like the original Volterra lattice (6.53); on the other hand , the multiplicative lattice (7.2) admits non-linearity of *arbitrary order*. Moreover, both families admit the KdV equation as continuum limits, see [32].

As mentioned earlier, the additive INB k-lattice is an integrable system for all $k \in \mathbb{N}$ and $i \in \mathbb{Z}$, since they all admit a Lax pair formulation (6.10). For the additive INB lattice (7.1), it reads

$$L^{(+,k)} = \sum_{i=1}^{N} \left(a_{i+k} E_{i+k,i} + E_{i,i+1} \right), \tag{7.3}$$

$$A^{(+,k)} = \sum_{i=1}^{N} \left(\sum_{j=0}^{k} a_{i+j} \right) E_{i,i} + E_{i,i+k+1}, \tag{7.4}$$

we recall that we are always considering periodic boundary conditions, so for all $j \in \mathbb{Z}$, $a_{j+N} = a_j$ and $E_{i,j+N} = E_{i+N,j} = E_{i,j}$. The constants of motion obtained through this Lax pair are in involution with respect to the Poisson bracket

$$\{a_j, a_i\}_{(+,k)} = a_j a_i \left(\sum_{s=1}^{k} \delta_{j+s,i} - \sum_{s=1}^{k} \delta_{j-s,i} \right). \tag{7.5}$$

Then, the additive INB k-lattice (7.1) can be written as

$$\dot{a}_i = \{a_i, H_1\}_{(+,k)}, \tag{7.6}$$

where the Hamiltonian function $H_1 = \sum_{j=1}^{N} a_j$ is the same as in equation (6.57). In the same way, it is possible to prove that the function $H_0 = \prod_{j=1}^{N} a_j$ is a first integral for the additive INB k-lattice (7.1) as well.

Similarly, the multiplicative INB k-lattices can be endowed with a Lax Pair for all $k \in \mathbb{N}$, therefore it is another example of integrable systems. Specifically, for the periodic case we presented in equation (7.2), the Lax pair reads

$$L^{(\times,k)} = \sum_{i=1}^{N} \left(a_i E_{i,i+1} + E_{i+k,i} \right), \tag{7.7}$$

$$A^{(\times,k)} = \sum_{i=1}^{N} \left(\prod_{j=0}^{k} a_{i+j} \right) E_{i,i+k+1}. \tag{7.8}$$

We notice that both $H_1 = \sum_{j=1}^{N} a_j$, and $H_0 = \prod_{j=1}^{N} a_j$ are constants of motion for these systems, for all $k \in \mathbb{N}$.

Remark 7.1.1. *For fixed k, there exists a transformation that maps the multiplicative INB k-lattice to the additive one. Namely, consider the system (7.2) and define the new set of variables*

$$b_i := a_i \cdot \dots \cdot a_{i+k-1}, \quad i = 1, \dots, N, \tag{7.9}$$

where the indices are taken modulo N. Then, it is immediate to see that

$$\dot{a}_i = a_i \left(b_{i+1} - b_{i+k-1} \right), \quad i = 1, \dots, N, \tag{7.10}$$

which in turn, due to telescopic summations, implies

$$\dot{b}_i = b_i \left(\sum_{j=1}^{k} b_{i+j} - \sum_{j=1}^{k} b_{i-j} \right), \quad i = 1, \dots, N,$$

which is (7.1). The transformation (7.9) is invertible only when k and N are co-prime, for a more detailed discussion see [32].

Gibbs Ensemble

We want to introduce an invariant measure for the INB lattices ((7.1)) and (7.2). Since $H_0 = \prod_{j=1}^{N} a_j$ and $H_1 = \sum_{j=1}^{N} a_j$ are constants of motion for all the INB lattices, and both systems are divergence free, in view of (6.22) we can consider as invariant measure the same one that we used for the Volterra lattice, namely

$$\mu_{\mathrm{INB}}\left(\mathbf{a};\eta,\beta\right) = \frac{e^{-\beta H_1 + (\eta-1)\log H_0}\mathrm{d}\mathbf{a}}{\int_{\mathbb{R}_+^N} e^{-\beta H_1 + (\eta-1)\log H_0}\mathrm{d}\mathbf{a}} = \frac{\prod_{j=1}^{N} a_j^{\eta-1} e^{-\beta\sum_{j=1}^{N} a_j}\mathrm{d}\mathbf{a}}{Z_N^{\mathrm{INB}}(\beta,\eta)}\,, \quad \beta,\eta>0, \tag{7.11}$$

where the normalization constant $Z_N^{\mathrm{INB}}(\beta,\eta)$ has the value given in equation (6.64).

Unlike the Lax matrix of the Volterra lattice, the Lax matrices of these generalizations lack of a known random matrix model to compare with. For this reason, we present numerical investigations of the density of states for these random Lax matrices for several values of the parameters k,η and β, see Figures 7.2-7.3. We notice that, for both the additive lattice and the multiplicative one, the density of states seems to possess a discrete rotational symmetry. In this spirit, we prove the following

Lemma 7.1.2. *Fix $\ell\in\mathbb{N}$. Then for N large enough*

$$Tr\left((L^{(+,k)})^\ell\right) = Tr\left((L^{(\times,k)})^\ell\right) = 0\,, \tag{7.12}$$

if ℓ is not an integer multiple of $k+1$.

Proof. We prove the statement for the additive case, the proof in the multiplicative one is analogous.

The main idea is to relate each addendum appearing in $\mathrm{Tr}\left((L^{(+,k)})^\ell\right)$ to a specific path in the $\mathbb{Z}^2$ plane, and prove that such a path exists if and only if $\ell = m(k+1)$ for some $m\in\mathbb{N}$. In particular, we can focus on the first element of the diagonal of $(L^{(+,k)})^\ell$, write $(L^{(+,k)})^\ell(1,1)$, since all the other ones can be recovered shifting the indices. First, we write $(L^{(+,k)})^\ell(1,1)$ as

$$(L^{(+,k)})^\ell(1,1) = \sum_{i_1,\dots,i_{\ell-1}=1}^{N} L^{(+,k)}(1,i_1)L^{(+,k)}(i_1,i_2)\cdots L^{(+,k)}(i_{\ell-1},1)\,. \tag{7.13}$$

We notice that, due to the structure of $L^{(+,k)}$, if $L^{(+,k)}(1,i_1)\cdots L^{(+,k)}(i_{\ell-1},1)$ is not zero, then either $i_{s+1} = i_s+1$ or $i_{s+1} = i_s - k$ modulo N. Now, consider paths in the $\mathbb{Z}^2$ plane from the point $(0,0)$ to $(\ell,0)$, such that the only permitted steps are the up step $(1,1)$ and the down step $(1,-k)$. Since these paths resemble the classical Dyck paths, we call them $(1,k)$-Dyck paths of length ℓ. Given a non-zero element of the product in (7.13), we can construct the corresponding path in the following way. We start at $(0,0)$, then if $|i_1 - 1| = 1$ we make an up step of height 1, otherwise we make a down step of height k, and so on.

For each path, let n be the number of up steps and m the number of down steps, then

$$m+n=\ell\,,\quad n-mk=0\,, \tag{7.14}$$

since there is a total of ℓ step, and the path has to go back to height 0. Thus, we deduce that

$$m(k+1)=\ell\,, \tag{7.15}$$

and the claim is proven. $\square$

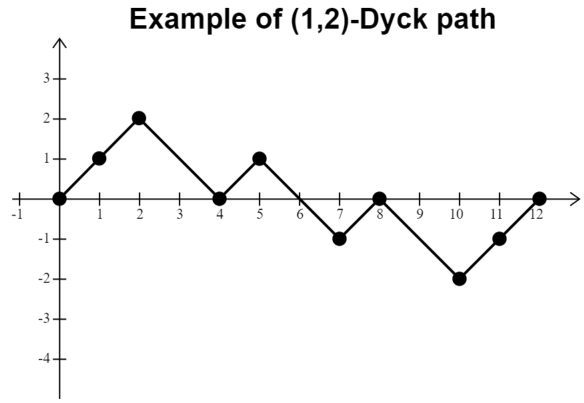

Figure 7.1: Example of $(1, 2)$-Dick path of length 12

Remark 7.1.3. *The previous result implies that the only non-zero moments of the densities of states $\nu_{INB,+,k}, \nu_{INB,\times,k}$, provided they exist, are the ones which are an integer multiple of $k+1$.*

Another interesting feature of these measures is that their supports seem to be exponentially localized to one dimensional contours. Specifically, it appears that the supports are the two hypotrochoids $\gamma_{+,k}$, $\gamma_{\times,k}$, respectively

$$\gamma_{+,k}(t, \eta, \beta) = e^{-it} + \frac{\eta}{\beta} e^{ikt}\,, \qquad \gamma_{\times,k}(t, \eta, \beta) = \frac{\eta}{\beta} e^{-it} + e^{ikt}\,, \quad t \in [0; 2\pi)\,. \tag{7.16}$$

This feature is highlighted in Figures 7.2-7.3, where we plot the empirical density of states and the corresponding hypotrochoid. This characteristic is important since this type of curves are also related to the density of some cyclic digraph, see [6], and may serve as a link between these two topics.

All these observations lead us to formulate the following conjecture

Conjecture 1. *Consider the two matrices $L^{(+,k)}$, $L^{(\times,k)}$ as in (7.3), (7.7) both endowed with the probability distribution μ_{INB} (7.11). Then, the densities of states $\nu^{\gamma,\beta}_{INB,+,k}$ and $\nu^{\gamma,\beta}_{INB,\times,k}$ exist, and have a discrete rotational symmetry, namely*

$$\nu^{\gamma,\beta}_{INB+,k}(\mathrm{d}z) = \nu^{\gamma,\beta}_{INB+,k}\left(e^{\frac{2\pi i}{k+1}}\mathrm{d}z\right), \qquad \nu^{\gamma,\beta}_{INB\times,k}(\mathrm{d}z) = \nu^{\gamma,\beta}_{INB\times,k}\left(e^{\frac{2\pi i}{k+1}}\mathrm{d}z\right)\,. \tag{7.17}$$

Moreover, the densities are exponentially localized in a neighbourhood of the two hypotrochoids $\gamma_{+,k}(t, \eta, \beta)$ and $\gamma_{\times,k}(t, \eta, \beta)$ in (7.16) respectively.

Parameter limit

As in the previous cases, although we are not able to give an explicit formula for the density of states of the INB lattices for general β, η, we can characterize this measure in the low-temperature

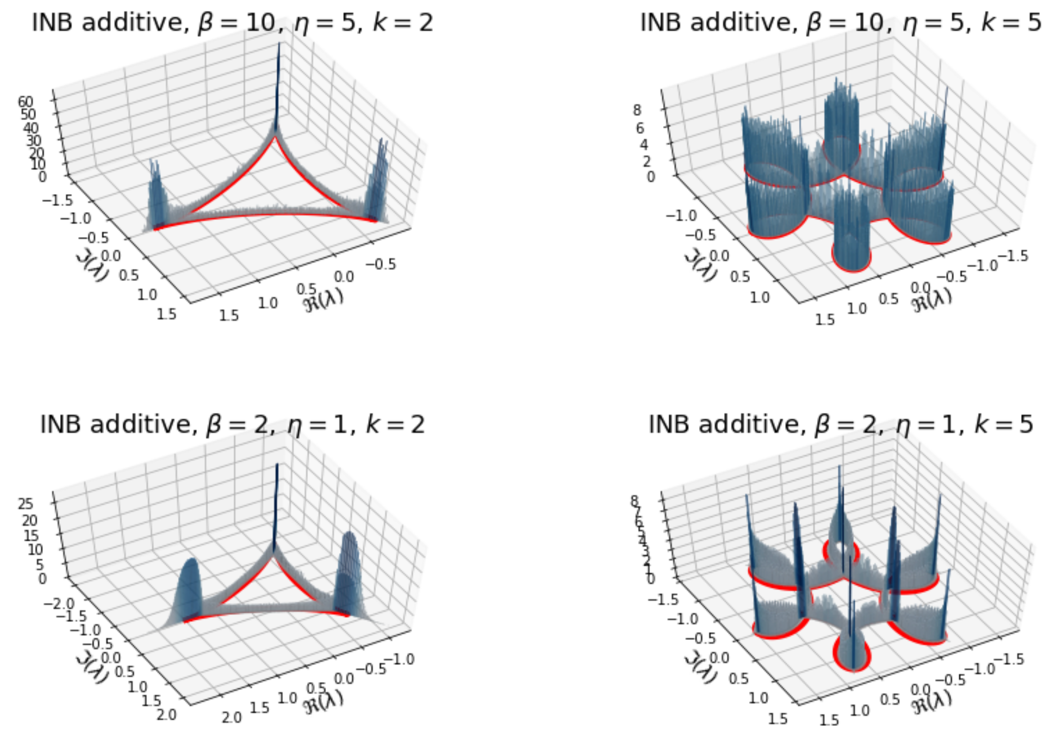

Figure 7.2: Eigenvalues of INB additive lattice for $k = 2$ (left) and $k = 5$ (right). $N = 1000$ and 6000 trials performed, in red the corresponding hypotrochoid $\gamma_{+,k}$ defined in equation (7.16). We observe that the examples on the left panel correspond to the case $\widetilde{\eta} = \frac{\eta}{\beta} = \frac{1}{k}$ that in the limiting case $\eta, \beta \to \infty$ gives the hard edge density of states in (7.21) where the hard edges are the cusps of the hypotrochoid. This observation explains the very high peaks located at the cusps.

limit. Specifically, we consider the limit as $\beta, \eta \to \infty$ in such a way that $\eta = \widetilde{\eta}\beta$, with $\widetilde{\eta}$ in a compact set of $\mathbb{R}_+$, and we compute the density of states of the matrices $L^{(+,k)}, L^{(\times,k)}$, endowed with the probability inherited from μ_{INB} (7.11), in this limit.

The procedure is the same as in the case of Volterra (see Section 6.3). Indeed, following the same line, we can conclude that the densities of states $\nu^{\infty}_{\mathrm{INB},+,k}$ and $\nu^{\infty}_{\mathrm{INB},\times,k}$ coincide with the densities of $\mathfrak{L}^{(+)}$ and $\mathfrak{L}^{(\times)}$ respectively, where

$$\mathfrak{L}^{(+,k)} = \sum_{i=1}^{N} \left(\widetilde{\eta} E_{i+k,i} + E_{i,i+1}\right), \quad \mathfrak{L}^{(\times,k)} = \sum_{i=1}^{N} \left(E_{i+k,i} + \widetilde{\eta} E_{i,i+1}\right). \tag{7.18}$$

We notice that both matrices are circulant, thus we can compute their eigenvalues explicitly as

$$\lambda_j^{(+,k)} = e^{-2\pi i \frac{j}{N}} + \widetilde{\eta} e^{2\pi i \frac{jk}{N}}, \quad \lambda_j^{(\times,k)} = \widetilde{\eta} e^{-2\pi i \frac{j}{N}} + e^{2\pi i \frac{jk}{N}}, \tag{7.19}$$

here $j = 1, \ldots, N$. Thus, in the large N limit, we deduce that the support of the measures $\nu^{\infty}_{\mathrm{INB}+,k}$ and $\nu^{\infty}_{\mathrm{INB}\times,k}$ are the hypotrochoids

$$\gamma_{+,k}(t, \widetilde{\eta}, 1) = e^{-it} + \widetilde{\eta} e^{ikt}, \quad \gamma_{\times,k}(t, \widetilde{\eta}, 1) = \widetilde{\eta} e^{-it} + e^{ikt}, \quad t \in [0; 2\pi), \tag{7.20}$$

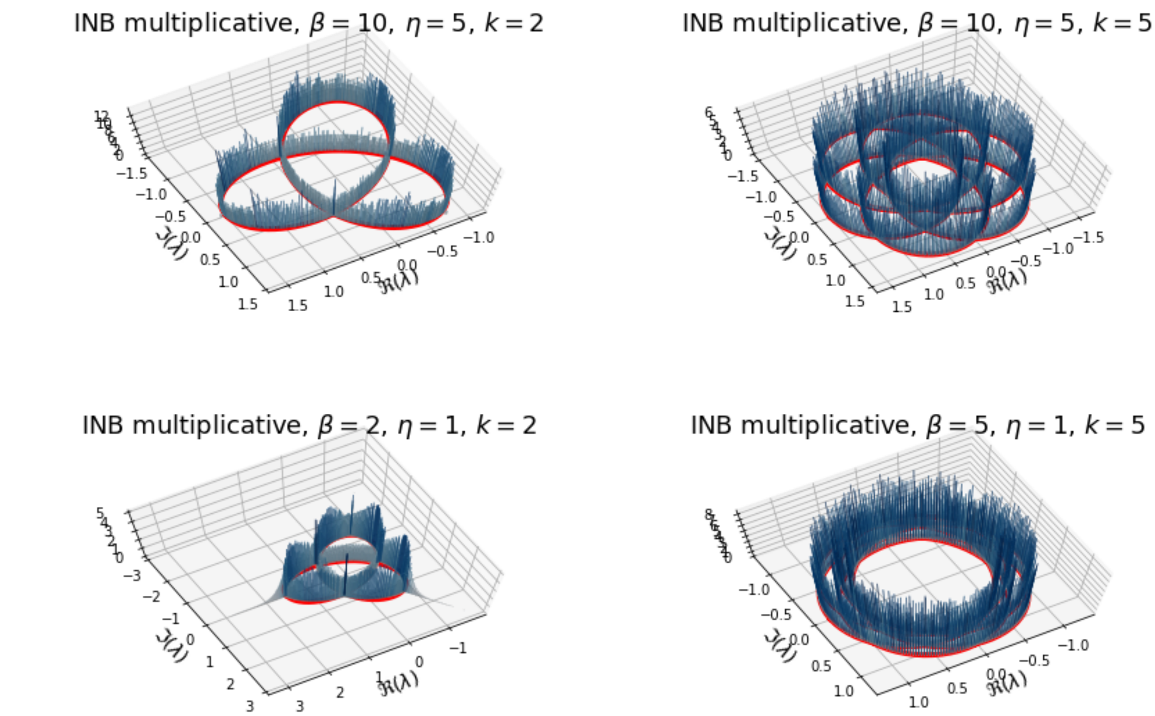

Figure 7.3: Eigenvalues of INB multiplicative lattice for $k = 2$ (left) and $k = 5$ (right). $N = 1000$ and 6000 trials performed, in red the corresponding hypotrochoid $\gamma_{\times,k}$

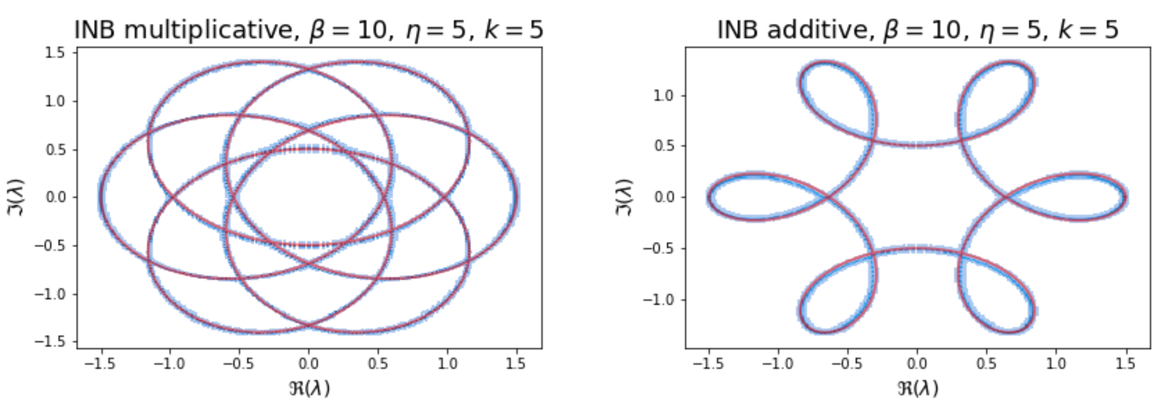

Figure 7.4: Eigenvalues of INB multiplicative and additive lattice for $k = 5$, $N = 1000$ and 6000 trials performed, in red the corresponding hypotrochoid $\gamma_{\times,k}, \gamma_{+,k}$

and the limiting eigenvalue densities are

$$\begin{aligned}\nu^{\infty}_{\mathrm{INB}+,k} &= \frac{|dz|}{2\pi\sqrt{1+\widetilde{\eta}^2k^2-k(|z|^2-1-\widetilde{\eta}^2)}}, \quad z\in\gamma_{+,k}\,,\\ \nu^{\infty}_{\mathrm{INB}\times,k} &= \frac{|dz|}{2\pi\sqrt{\widetilde{\eta}^2+k^2-k(|z|^2-1-\widetilde{\eta}^2)}}, \quad z\in\gamma_{\times,k}.\end{aligned} \tag{7.21}$$

We summarize these results in the following Proposition.

Proposition 7.1.4. *The densities of states of the Lax matrices $L^{(+,k)}$ and $L^{(\times,k)}$ in (7.3) and (7.7) endowed with the Gibbs measure μ_{INB} in (7.11), in the low temperature limit, i.e. when η, $\beta\to\infty$, in such a way that $\eta=\widetilde{\eta}\beta$, with $\widetilde{\eta}$ in a compact set of $\mathbb{R}_+$, are given respectively by $\nu^{\infty}_{INB+,k}$ and $\nu^{\infty}_{INB\times,k}$ in (7.21).*

Remark 7.1.5. *When the parameters satisfy the relation $\widetilde{\eta}k<1$, the curve $\gamma_{+,k}(t,\widetilde{\eta},1)$ is not self-intersecting, while for $\widetilde{\eta}k>1$ the curve is self-intersecting. For $\widetilde{\eta}k=1$ it has cusp singularities [60]. The limiting shape of the support as $\widetilde{\eta}\to 0$ is a circle.*
The same considerations are true for the curve $\gamma_{\times,k}(t,\widetilde{\eta})$ upon substitution $\widetilde{\eta}\mapsto 1/\widetilde{\eta}$. We also observe that the density of states $\nu^{\infty}_{INB+,k}$ ($\nu^{\infty}_{INB\times,k}$) in equation (7.21) is a hard-edge distribution for $\widetilde{\eta}=\frac{1}{k}$ ($\widetilde{\eta}=k$) and the hard edges correspond to the cusps of the curve $\gamma_{+,k}(t,\widetilde{\eta})$ ($\gamma_{\times,k}(t,\widetilde{\eta})$).

7.2 The focusing Ablowitz-Ladik lattice

The *focusing Ablowitz-Ladik lattice* is the following system of spatial discrete differential equations

$$i\dot{a}_j + a_{j+1} + a_{j-1} - 2a_j + |a_j|^2(a_{j-1}+a_{j+1}) = 0\,, \tag{7.22}$$

where $a_j\in\mathbb{C}$, $j=1,\dots,N$, $N\geq 3$, and we consider periodic boundary conditions $a_{j+N}=a_j$ for all $j\in\mathbb{Z}$. This equation was introduced by Ablowitz and Ladik [2, 3], by searching integrable spatial discretization of the cubic non-linear Schrödinger Equation (NLS) for the complex function $\psi(x,t)$, $x\in\mathbb{R}$, $t\in\mathbb{R}^+$

$$i\partial_t\psi(x,t) + \partial_x^2\psi(x,t) + 2|\psi(x,t)|^2\psi(x,t) = 0\,, \tag{7.23}$$

In contrast with what happens in the defocusing case, the particles $(a_1,\dots,a_N)$ are free to explore the whole $\mathbb{C}^N$, which is the phase space of the system.

On the space $\mathcal{C}^\infty(\mathbb{C}^N)$ we consider the Poisson bracket [74, 88]

$$\{f,g\} = i\sum_{j=1}^N \rho_j^2\left(\frac{\partial f}{\partial\overline{a}_j}\frac{\partial g}{\partial a_j} - \frac{\partial f}{\partial a_j}\frac{\partial g}{\partial\overline{a}_j}\right)\,, \quad f,g\in\mathcal{C}^\infty(\mathbb{C}^N). \tag{7.24}$$

We notice that the phase shift $a_j(t)\to e^{-2it}a_j(t)$ transforms the AL lattice (7.22) into the equation

$$\dot{a}_j = i\,\rho_j^2(a_{j+1}+a_{j-1}), \quad \rho_j = \sqrt{1+|a_j|^2}, \tag{7.25}$$

which we call the *reduced AL equation*. We remark that the quantity $H_0 = 2\ln\left(\prod_{j=1}^N\rho_j^2\right)$ is the generator of the shift $a_j(t)\to e^{-2it}a_j(t)$, while $H_1 = -K^{(1)} - \overline{K^{(1)}}$ with

$$K^{(1)} := \sum_{j=1}^N a_j\overline{a}_{j+1}, \tag{7.26}$$

generates the flow (7.25). Therefore, we can rewrite the AL equation as

$$\dot{a}_j = \{a_j, H_{AL}\}\,, \quad H_{AL} = H_0 + H_1\,. \tag{7.27}$$

Moreover, it is straightforward to verify that $\{H_0, H_1\} = 0$. The Poisson bracket induces the symplectic form

$$\omega = i\sum_{j=1}^{N} \frac{1}{\rho_j^2}\mathrm{d}a_j \wedge \mathrm{d}\overline{a}_j\,, \quad \rho_j = \sqrt{1+|a_j|^2}\,, \tag{7.28}$$

that is invariant under the evolution generated by the Hamiltonians H_0 and H_1. Therefore, the volume form

$$\omega^N = \omega \wedge \cdots \wedge \omega,$$

is also invariant. In view of these properties, we can define the Gibbs ensemble for the focusing Ablowitz-Ladik lattice on the phase space $\mathbb{C}^N$ as

$$\mu_{AL} = \frac{1}{Z_N^{AL}(\beta)} e^{\frac{\beta}{2}H_0}\omega^N = \frac{1}{Z_N^{AL}(\beta)} \prod_{j=1}^{N} \left(1+|a_j|^2\right)^{-\beta-1} \mathrm{d}^2\mathbf{a}\,, \quad \beta > 0\,, \tag{7.29}$$

where $\mathbf{a} = (a_1, \ldots, a_N)$, $\mathrm{d}^2\mathbf{a} = \prod_{j=1}^{N}(i\mathrm{d}a_j \wedge \mathrm{d}\overline{a}_j)$ and $Z_N^{AL}(\beta)$ is the normalization constant of the system. We notice that according to this measure, all the variables are i.i.d.

Remark 7.2.1. *The measure with density* $\exp(-\beta H_{AL})$ *and* $\beta > 0$ *is not bounded nor normalizable on the whole phase space. For this reason, we have defined the Gibbs ensemble as in (7.29). Furthemore, we observe that the measure (7.29) has a finite number of moments, which implies that the corresponding density of states of the Lax matrix (see below), if it exists, would have a finite number of moments.*

The focusing AL lattice is a complete integrable system. Indeed it admits a Lax representation, first obtained by Ablowitz and Ladik from the discretization of the Zakharov-Shabat Lax pair for the focusing non-linear Schrödinger equation [168]. Gesztesy, Holden, Michor, and Teschl [89] found a different Lax pair for the infinite case of focusing AL lattice, and for its general hierarchy. To adapt their construction, we double the size of the lattice according to the periodic boundary conditions, thus we consider a chain of $2N$ particles $a_1, \ldots, a_{2N}$ such that $a_j = a_{j+N}$ for $j = 1, \ldots, N$. Define the 2×2 matrix Ξ_j

$$\Xi_j = \begin{pmatrix} -a_j & \rho_j \\ \rho_j & -\overline{a_j} \end{pmatrix}\,, \quad j = 1, \ldots, 2N\,, \tag{7.30}$$

and the $2N \times 2N$ matrices

$$\mathcal{M} = \begin{pmatrix} -\overline{a_{2N}} & & & & & \rho_{2N} \\ & \Xi_2 & & & & \\ & & \Xi_4 & & & \\ & & & \ddots & & \\ & & & & \Xi_{2N-2} & \\ \rho_{2N} & & & & & -a_{2N} \end{pmatrix}\,, \qquad \mathcal{L} = \begin{pmatrix} \Xi_1 & & & \\ & \Xi_3 & & \\ & & \ddots & \\ & & & \Xi_{2N-1} \end{pmatrix}\,. \tag{7.31}$$

Now let us define the Lax matrix

$$\mathcal{E} = \mathcal{L}\mathcal{M}\,, \tag{7.32}$$

that has the structure of a 5-band diagonal matrix

$$
\begin{pmatrix}
* & * & * & & & & & & & * \\
* & * & * & & & & & & & * \\
 & * & * & * & * & & & & & \\
 & * & * & * & * & & & & & \\
 & & & & \ddots & \ddots & & & & \\
 & & & & & * & * & * & * \\
 & & & & & * & * & * & * \\
* & & & & & & * & * & * \\
* & & & & & & * & * & *
\end{pmatrix} .
$$

The N-periodic equation (7.25) is equivalent to the following Lax equation for the matrix $\mathcal{E}$,

$$
\dot{\mathcal{E}} = [\mathcal{A}, \mathcal{E}] \ , \tag{7.33}
$$

where

$$
\mathcal{A} = \frac{i}{2}(\mathcal{E}_+ - \mathcal{E}_- - \mathcal{E}_+^{-1} + \mathcal{E}_-^{-1}) \, , \tag{7.34}
$$

where the two projections M_+, M_- are defined for a $2N \times 2N$ matrix as

$$
M_+ = \begin{cases} M_{\ell,j} \, , & \ell < j \leq \ell + N \\ M_{\ell,j} \, , & \ell > j + N \\ 0 & \text{otherwise} \end{cases} , \quad M_- = \begin{cases} M_{\ell,j} \, , & j < \ell \leq j + N \\ M_{\ell,j} \, , & j > \ell + N \\ 0 & \text{otherwise} \end{cases} . \tag{7.35}
$$

We notice that the Lax matrix $\mathcal{E}$ has a similar structure to the one of the defocusing AL lattice obtained by Nenciu, and Simon [146, 155]. The crucial difference is that while for the defocusing AL lattice the blocks Ξ_j are unitary matrices, for the focusing lattice this is not the case since $\Xi_j \Xi_j^\dagger \neq I_2$ where $I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and † stands for hermitian conjugate.

The measure μ_{AL} induces a probability distribution on the entries of the matrix $\mathcal{E}$, thus it becomes a random matrix. As in the previous cases, one would like to connect the density of states for this random matrix to the density of states of some β-ensemble in the high temperature regime, but, as in the case of the INB lattices, we lack of a matrix representation of some β-ensemble with eigenvalues supported on the plane.

We make the following observations. The matrix Ξ_j (7.30) is complex symmetric, and it can be factorized in the form

$$
\Xi_j = U_j \begin{pmatrix} \frac{\overline{a_j}}{|a_j|\left(|a_j| + \sqrt{1+|a_j|^2}\right)} & 0 \\ 0 & -\frac{a_j}{|a_j|}(|a_j| + \sqrt{1+|a_j|^2}) \end{pmatrix} U_j \tag{7.36}
$$

where the matrices $U_j = \begin{pmatrix} \frac{a_j}{\sqrt{2}|a_j|} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{\overline{a_j}}{\sqrt{2}|a_j|} \end{pmatrix}$ are unitary, $U_j^{-1} = U_j^\dagger$. Thus defining the matrices

$$\widetilde{\mathcal{M}} = \begin{pmatrix} -\frac{\overline{a_{2N}}}{\sqrt{2}|a_{2N}|} & & & & & \frac{1}{\sqrt{2}} \\ & U_2 & & & & \\ & & U_4 & & & \\ & & & \ddots & & \\ & & & & U_{2N-2} & \\ \frac{1}{\sqrt{2}} & & & & & \frac{a_{2N}}{\sqrt{2}|a_{2N}|} \end{pmatrix}, \qquad \widetilde{\mathcal{L}} = \begin{pmatrix} U_1 & & & \\ & U_3 & & \\ & & \ddots & \\ & & & U_{2N-1} \end{pmatrix}, \tag{7.37}$$

we can rewrite the Lax matrix $\mathcal{E}$ (7.32) as

$$\mathcal{E} = \widetilde{\mathcal{L}} \Lambda_{\text{odd}} \widetilde{\mathcal{L}} \widetilde{\mathcal{M}} \Lambda_{\text{even}} \widetilde{\mathcal{M}}\,, \tag{7.38}$$

where, defining $c_j := \frac{\overline{a_j}}{|a_j|\left(|a_j|+\sqrt{1+|a_j|^2}\right)}$, the matrices Λ_{odd} and Λ_{even} are given by

$$\begin{aligned} \Lambda_{\text{odd}} &= \operatorname{diag}\left(c_1, -\frac{1}{c_1}, c_3, -\frac{1}{c_3}, \dots, c_{2N-1}, -\frac{1}{c_{2N-1}}\right) \\ \Lambda_{\text{even}} &= \operatorname{diag}\left(-\frac{1}{c_{2N}}, c_2, -\frac{1}{c_2}, \dots, c_{2N}\right)\,. \end{aligned} \tag{7.39}$$

Since we are interested in the distribution of the eigenvalues of $\mathcal{E}$, it follows from the factorization (7.38) that we can also consider the matrix $\Lambda_{\text{odd}} \widetilde{\mathcal{E}} \Lambda_{\text{even}} \widetilde{\mathcal{E}}^{\intercal}$, where $\widetilde{\mathcal{E}} = \widetilde{\mathcal{L}}\widetilde{\mathcal{M}}$. The eigenvalues of $\Lambda_{\text{even}}, \Lambda_{\text{odd}}$ come in pairs, such that if λ is an eigenvalue, then also $-\lambda^{-1}$ is an eigenvalue. The matrix $\widetilde{\mathcal{E}}$ is a periodic CMV matrix [44], thus its eigenvalues are on the unit circle.

Thus, we are in a similar setting considered in [105, 162, 163]. Indeed in [162] the authors derived the eigenvalues distribution of $U\sqrt{D}$ where U is a Haar distributed unitary matrix and D is a fixed diagonal matrix with positive eigenvalues. They show that the density of states is rotational invariant and it is supported on a single ring whose radii $r_1 < r_2$ satisfy the constraint $d_{min} < r_1 < r_2 < d_{max}$, where d_{min} and d_{max} are the minimum and maximum eigenvalues of D. In [105], the authors considered a similar problem, namely the characterization of the density of states for a matrix of the form UTV, where U, V are independent unitary matrices Haar distributed, and T is a real diagonal matrix independent of U, V. They proved, under some mild conditions, that the density of states of the matrix UTV is radially symmetric and it is supported on a ring.

It is therefore reasonable to expect that the density of states of the random Lax matrix of the Ablowitz-Ladik lattice is rotational invariant, but with unbounded support, indeed the eigenvalues of $\Lambda_{\text{even}}, \Lambda_{\text{odd}}$ could grow to infinity.

To confirm our expectations, we perform several numerical investigations of the eigenvalues of the random Lax matrix of the Ablowitz-Ladik lattice for various values of β (see Figures 7.5-7.6). In Figure 7.5 the eigenvalue density is shown. As expected, the density seems to be rotational invariant, and concentrated on a ring, exactly as in [105, 162, 163]. For this reason, we investigate the behaviour of the modulus of the eigenvalues, see Figure 7.6. They seem to be concentrated in a small region, but, in view of Remark 7.2.1, we expect that the tails should decay just polynomially fast.

Parameter Limit

Despite not being able to explicitly compute the density of states for general values of β, we can perform such an analysis in the low-temperature limit, namely when $\beta \to \infty$.

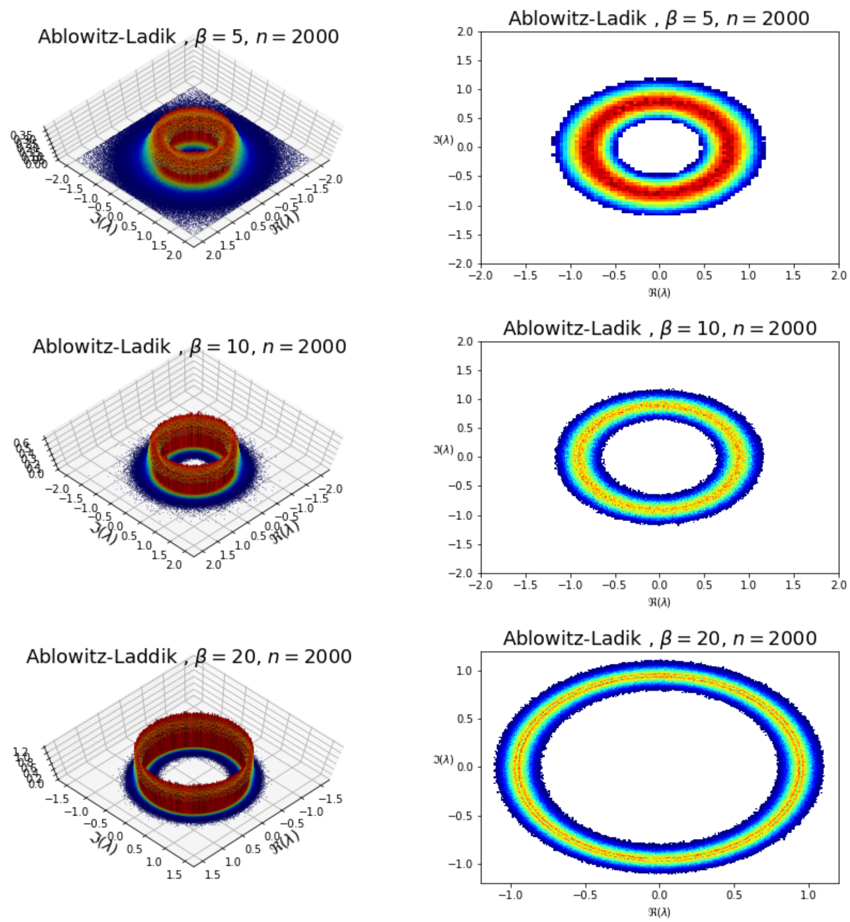

Figure 7.5: Empirical densities for the focusing Ablowitz-Ladik lattice for $\beta = 5, 10, 20$, 10000 trials per picture .

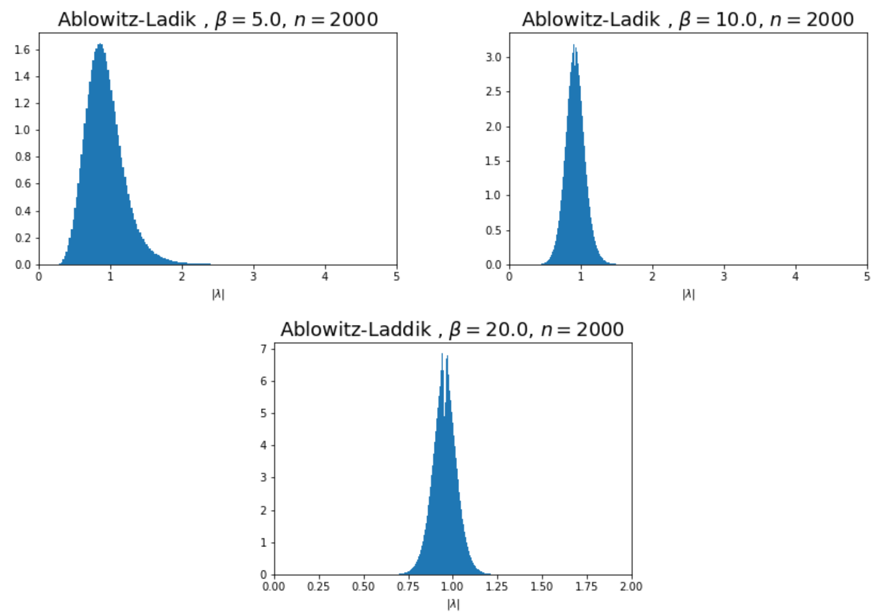

Figure 7.6: Empirical density for the eigenvalues' modulus for the focusing Ablowitz-Ladik lattice for $\beta = 5, 10, 20$, 10000 trials per picture.

We notice that, according to (7.29), all the a_j are independent. Hence, in order to obtain the density of states in the low-temperature limit, we have to compute the weak limit of the density

$$\mu_\beta = \frac{(1+|z|^2)^{-\beta-1}\mathrm{d}z}{\int_{\mathbb{D}}(1+|z|^2)^{-\beta-1}\mathrm{d}z}\,. \tag{7.40}$$

Proceeding as in the previous cases, it follows that the following limit holds for all bounded and continuous $f\;:\overline{\mathbb{D}} \to \mathbb{R}$:

$$\lim_{\beta\to\infty}\int_{\mathbb{D}} f(z)\mu_\beta = f(0) \tag{7.41}$$

The previous limit implies that the density of states of the Ablowitz-Ladik lattice in the low temperature limit is equal to the one of $\widehat{\mathcal{E}} = \widehat{\mathcal{L}}\widehat{\mathcal{M}}$, where $\widehat{\mathcal{L}}$, $\widehat{\mathcal{M}}$ are $2N \times 2N$ matrices

$$\widehat{\mathcal{M}} = \begin{pmatrix} 0 & & & & 1 \\ & \widetilde{\Xi} & & & \\ & & \widetilde{\Xi} & & \\ & & & \ddots & \\ & & & & \widetilde{\Xi} \\ 1 & & & & 0 \end{pmatrix}, \qquad \widehat{\mathcal{L}} = \begin{pmatrix} \widetilde{\Xi} & & & \\ & \widetilde{\Xi} & & \\ & & \ddots & \\ & & & \widetilde{\Xi} \end{pmatrix}, \tag{7.42}$$

and $\widehat{\Xi}$ is defined as the unitary matrix

$$\widehat{\Xi} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{7.43}$$

To compute the density of states for the matrix $\mathcal{E}$, we notice that both $\widehat{\mathcal{M}}$, and $\widehat{\mathcal{L}}$ are permutation matrices. Specifically, we identify them with the following permutations

$$\begin{aligned}\widehat{\mathcal{M}} &\longleftrightarrow (2N,1)(2,3)(4,5)\dots(2N-2,2N-1)\,,\\ \widehat{\mathcal{L}} &\longleftrightarrow (1,2)(3,4)(5,6)\dots(2N-1,2N)\,.\end{aligned} \tag{7.44}$$

As a consequence, the matrix $\widehat{\mathcal{E}}$ itself corresponds to the permutation

$$\widehat{\mathcal{E}} \longleftrightarrow (1,3,5,\dots,2N-1)(2,4,6,\dots,2N)\,. \tag{7.45}$$

This implies that the eigenvalues of $\widehat{\mathcal{E}}$ are all double, and given by

$$\lambda_j = e^{2\pi i \frac{j}{N}}\,, \quad j = 1,\dots,2N\,.$$

From this explicit expression of the eigenvalues, it is straightforward to prove that

$$\nu_{AL} = \frac{\mathrm{d}\theta}{2\pi}\,, \quad \theta \in [0,2\pi)\,. \tag{7.46}$$

Thus, we have proved the following

Proposition 7.2.2. *Consider the Gibbs ensemble μ_{AL} (7.29) of the focusing Ablowitz-Ladik lattice in the low-temperature limit, i.e. $\beta \to \infty$. Then, the density of states ν_{AL} of the Lax matrix $\mathcal{E}$ (7.32) is given by*

$$\nu_{AL} = \frac{\mathrm{d}\theta}{2\pi}\,, \quad \theta \in [0,2\pi)\,. \tag{7.47}$$

7.3 Schur flow

The *focusing Schur flow*, also known as *discrete mKdV*, is an integrable system deeply related to the Ablowitz-Ladik lattice. Its equations of motion are

$$\dot a_j = \rho_j^2(a_{j+1} - a_{j-1})\,, \quad \rho_j = \sqrt{1+|a_j|^2}\,, \tag{7.48}$$

here we consider periodic boundary conditions, $a_j = a_{j+N}$ for all $j \in \mathbb{Z}$. Notice that if $a_j(0) \in \mathbb{R}$ for all $j = 1,\dots,N$, then $a_j(t) \in \mathbb{R}$ for all times, implying that $\mathbb{R}^N$ is an invariant subspace for the dynamics.

Recalling the definition (7.26) for $K^{(1)} = \sum_{j=1}^N a_j \bar a_{j+1}$ and introducing the Poisson bracket

$$\{f,g\} = \sum_{j=1}^N \rho_j^2 \left(\frac{\partial f}{\partial \bar a_j}\frac{\partial g}{\partial a_j} - \frac{\partial f}{\partial a_j}\frac{\partial g}{\partial \bar a_j} \right)\,, \quad f,g \in \mathcal{C}^\infty(\mathbb{C}^N), \tag{7.49}$$

we can rewrite the equations of motion (7.48) as

$$\dot a_j = \{a_j, H_S\}\,, \quad H_S = K^{(1)} - \overline{K^{(1)}}\,. \tag{7.50}$$

Notice that the quantity $H_0 = -2\ln\left(\prod_{j=1}^N \rho_j^2\right)$ is a global first integral for the system. Moreover, one can deduce immediately from the equations of motion that $\mathbb{R}^N$ is invariant for the dynamics. Thus, in view of the Hamiltonian representation and this invariance, we define the Gibbs measure for the Schur flow as

$$\mu_S = \frac{1}{Z_N^S(\beta)} \prod_{j=1}^N \left(1 + a_j^2\right)^{-\beta-1} \mathrm{d}\mathbf{a}\,, \quad a_j \in \mathbb{R}\,, \tag{7.51}$$

where $Z_N^S(\beta)$ is the normalization constant of the system,

$$Z_N^S(\beta) = \int_{\mathbb{R}^N} \prod_{j=1}^N \left(1 + a_j^2\right)^{-\beta-1} \mathrm{d}\mathbf{a}\,. \tag{7.52}$$

Remark 7.3.1. *Similarly to the focusing AL case, the classical Gibbs ensemble is not well-defined on the whole phase space. Indeed, the measure with density $e^{-\beta H_S}$, $\beta > 0$ cannot be normalized on $\mathbb{R}^N$.*

The Schur flow is a completely integrable system since it admits a Lax formulation. Namely, define the 2×2 matrix Ξ_j

$$\Xi_j = \begin{pmatrix} -a_j & \rho_j \\ \rho_j & -\overline{a_j} \end{pmatrix}, \quad j = 1, \dots, 2N\,, \tag{7.53}$$

and, similarly to the Ablowitz-Ladik case, the $2N \times 2N$ matrices

$$\mathcal{M} = \begin{pmatrix} -\overline{a_{2N}} & & & & & \rho_{2N} \\ & \Xi_2 & & & & \\ & & \Xi_4 & & & \\ & & & \ddots & & \\ & & & & \Xi_{2N-2} & \\ \rho_{2N} & & & & & -a_{2N} \end{pmatrix}, \qquad \mathcal{L} = \begin{pmatrix} \Xi_1 & & & \\ & \Xi_3 & & \\ & & \ddots & \\ & & & \Xi_{2N-1} \end{pmatrix}, \tag{7.54}$$

and as in (7.32) the Lax matrix

$$\mathcal{E} = \mathcal{L}\mathcal{M}\,. \tag{7.55}$$

Then, the N-periodic equation (7.48) is equivalent to the following Lax equation for the matrix $\mathcal{E}$:

$$\dot{\mathcal{E}} = [\mathcal{A}, \mathcal{E}]\,, \tag{7.56}$$

where

$$\mathcal{A} = \frac{1}{2}\left(\mathcal{E}_+ + \mathcal{E}_+^{-1} - \mathcal{E}_- - \mathcal{E}_-^{-1}\right), \tag{7.57}$$

where the two projection $+, -$ are defined in (7.35).

Carrying on with the approach of this article, we study the density of states ν_S for the matrix $\mathcal{E}$ when the entries are distributed according to the measure (7.51). First, we notice that Remark 7.2.1 is valid also in the case of the focusing Schur flow. Moreover, since the variables a_j are real, we can factorize the matrices Ξ_j in the following way:

$$\Xi_j = U_0 \,\mathrm{diag}\left(\sqrt{1 + a_j^2} - a_j, -\frac{1}{\sqrt{1 + a_j^2} - a_j}\right) U_0, \quad U_0 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \tag{7.58}$$

where we note that $U_0^{-1} = U_0$, so that the above is a similarity transformation. Thus, defining

$$\widetilde{\mathcal{M}} = \begin{pmatrix} -\frac{1}{\sqrt{2}} & & & & & \frac{1}{\sqrt{2}} \\ & U_0 & & & & \\ & & U_0 & & & \\ & & & \ddots & & \\ & & & & U_0 & \\ \frac{1}{\sqrt{2}} & & & & & \frac{1}{\sqrt{2}} \end{pmatrix}, \qquad \widetilde{\mathcal{L}} = \begin{pmatrix} U_0 & & & \\ & U_0 & & \\ & & \ddots & \\ & & & U_0 \end{pmatrix}, \tag{7.59}$$

we can rewrite the Lax matrix of the Schur flow $\mathcal{E}$ as

$$\mathcal{E} = \widetilde{\mathcal{L}}\Lambda_{\mathrm{odd}}\widetilde{\mathcal{L}}\widetilde{\mathcal{M}}\Lambda_{\mathrm{even}}\widetilde{\mathcal{M}}\,, \tag{7.60}$$

where

$$\begin{aligned} \Lambda_{\mathrm{odd}} &= \mathrm{diag}\left(\sqrt{1+a_1^2}-a_1, -\frac{1}{\sqrt{1+a_1^2}-a_1}, \sqrt{1+a_3^2}-a_3, \dots\right), \\ \Lambda_{\mathrm{even}} &= \mathrm{diag}\left(-\frac{1}{\sqrt{1+a_{2N}^2}-a_{2N}}, \sqrt{1+a_2^2}-a_2, \dots, \sqrt{1+a_{2N}^2}-a_{2N}\right). \end{aligned} \tag{7.61}$$

As in the case of the Ablowitz-Ladik lattice, since we are interested in just the distribution of the eigenvalues of $\mathcal{E}$, we can consider the matrix $\Lambda_{\mathrm{odd}}\widetilde{\mathcal{E}}\Lambda_{\mathrm{even}}\widetilde{\mathcal{E}}^{\intercal}$, where

$$\widetilde{\mathcal{E}} = \widetilde{\mathcal{L}}\widetilde{\mathcal{M}}. \tag{7.62}$$

As in the AL case, the eigenvalues of $\Lambda_{\mathrm{even}}, \Lambda_{\mathrm{odd}}$ come in pairs, such that if λ is an eigenvalue, then also $-\lambda^{-1}$ is an eigenvalue. The main difference with the case of the focusing AL lattice is that in this case the matrix $\widetilde{\mathcal{E}}$ is *deterministic*. Thus, one can be led to think that the eigenvalue distribution of the Schur flow would be similar to the one of the AL lattice, but it is not the case. Indeed, we perform several numerical investigations, reported in Figure 7.7, which shows that the behaviour of the eigenvalues is different in the two situations.

We notice that a consistent part of the eigenvalues tend to stay close to the real axis, see Figure 7.7. This behaviour is also typical of the orthogonal Ginibre ensemble [72]. The main reason is that the eigenvalues of $\widetilde{\mathcal{E}}$ are not evenly spaced on the unit circle, but they are constrained to the left semicircle, and are more dense nearby $\pm i$ (see Figure 7.8). Indeed we can give an accurate description of the spectrum of this matrix.

More precisely, we have the following.

Proposition 7.3.2. *Let $\widetilde{\mathcal{E}}$ be the $2N\times 2N$ matrix defined in (7.62). Its eigenvalues are the solutions of the quadratic equations*

$$\lambda + \frac{1}{\lambda} + 1 = \cos\left(\frac{2\pi j}{N}\right), \qquad j = 0, 1, \dots, N-1, \tag{7.63}$$

counting the multiplicity.

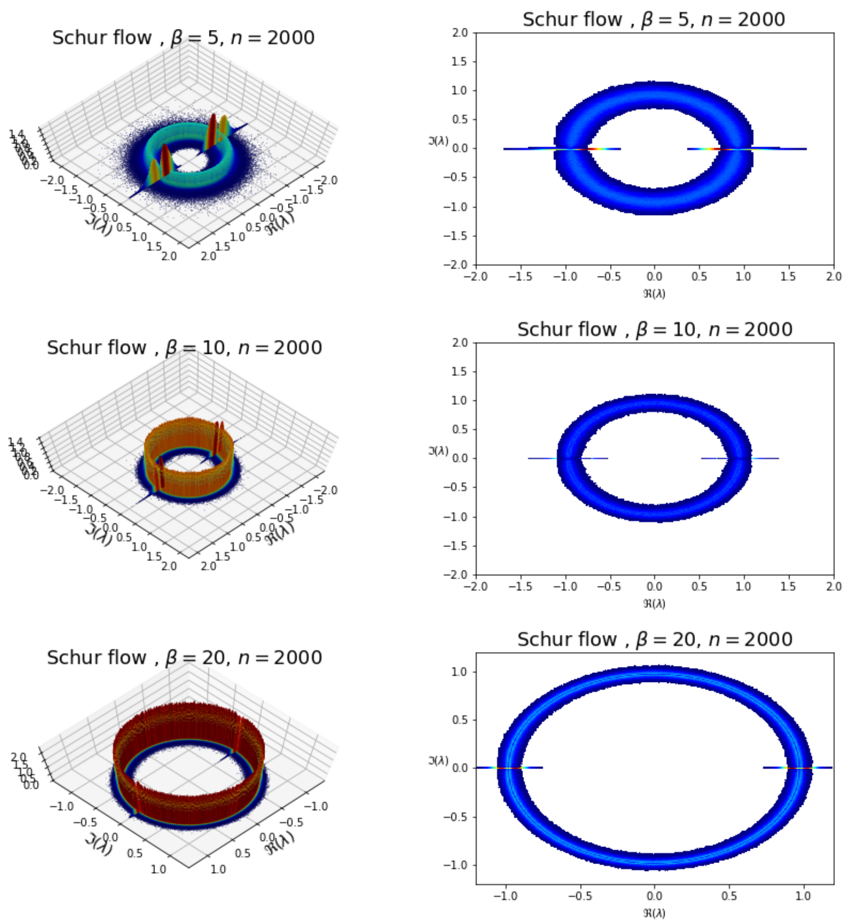

Figure 7.7: Schur flow eigenvalue density for $\beta = 5, 10, 20$, 10000 trials.

Proof. Recall that the matrix $\widetilde{\mathcal{E}}$ is defined as $\widetilde{\mathcal{E}} = \widetilde{\mathcal{L}}\widetilde{\mathcal{M}}$ where $\widetilde{\mathcal{L}}$ and $\widetilde{\mathcal{M}}$ are as in (7.59). It is a block circulant matrix, indeed we can write

$$\widetilde{\mathcal{E}} = \frac{1}{2}\begin{pmatrix} E_0 & E_1 & & & E_{-1} \\ E_{-1} & E_0 & E_1 & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & E_1 \\ E_1 & & & E_{-1} & E_0 \end{pmatrix}, \tag{7.64}$$

with

$$E_0 = \begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix}, \qquad E_{-1} = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \qquad E_1 = \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}. \tag{7.65}$$

One can immediately check that $\lambda = \pm i$ are eigenvalues for $\widetilde{\mathcal{E}}$ with eigenvectors

$$v_{\pm i} = (\mp i, 1, \mp i, 1, \dots, \mp i, 1)^\intercal. \tag{7.66}$$

We now claim that, for fixed N, the remaining eigenvalues have multiplicity 2 and are the $(N-1)$ solutions to $\Omega(\lambda)^N = I_2$, where we defined

$$\Omega(\lambda) = \begin{pmatrix} \lambda & -\lambda - 1 \\ -\lambda - 1 & \frac{\lambda^2 + 2\lambda + 2}{\lambda} \end{pmatrix}. \tag{7.67}$$

Such solutions are obtained by solving the equation

$$\lambda + \frac{1}{\lambda} + 1 = \cos\left(\frac{2\pi j}{N}\right) \qquad for \;\; j = 0, \dots, \left\lfloor \frac{N}{2} \right\rfloor . \tag{7.68}$$

For $j = 0$ the solutions to (7.68) are $\pm i$ which we already treated separately. Indeed $\Omega(\pm i)$ is not diagonalizable and $\Omega(\pm i)^N \neq I_2$ for every N greater than 0. For any other $j \in \{1, \dots, \lfloor \frac{N}{2} \rfloor\}$, the solutions to (7.68) are

$$\lambda_{1,2} = \frac{\cos\left(\frac{2\pi j}{N}\right) - 1}{2} \pm i \frac{\sqrt{3 + 2\cos\left(\frac{2\pi j}{N}\right) - \cos^2\left(\frac{2\pi j}{N}\right)}}{2}. \tag{7.69}$$

Since both the real and imaginary part are monotone functions of j, different $j's$ will correspond to different solutions. Hence, if N is odd, we will have a total of $N-1$ solutions coming from (7.69); if N is even one has $N-2$ distinct solutions coming from the equation in (7.68) plus the double solution $\lambda = -1$ obtained for $j = N/2$.

For a given eigenvalue λ, the corresponding independent eigenvectors are

$$v_1 = \Big((1,0)\Omega(\lambda), \dots, (1,0)\Omega(\lambda)^{N-1}, 1, 0\Big)^{\intercal}, \tag{7.70}$$
$$v_2 = \Big((0,1)\Omega(\lambda), \dots, (0,1)\Omega(\lambda)^{N-1}, 0, 1\Big)^{\intercal}. \tag{7.71}$$

Let us check the correctness of the claim. Write $\widetilde{\mathcal{E}} v_1 := (w_1, \dots, w_N)^{\intercal}$, where w_j are two-dimensional row vectors, then using the fact that $\Omega(\lambda)^N = I_2$, one can compute for any $k = 1, \dots, N$,

$$w_k^{\intercal} = \frac{1}{2}\Big(E_{-1}\,\Omega(\lambda)^{-1} + E_0 + E_1\,\Omega(\lambda)\Big)\,\Omega(\lambda)^k \begin{pmatrix}1\\0\end{pmatrix} \tag{7.72}$$
$$= \frac{1}{2}\left(\begin{pmatrix}\lambda+1 & \lambda\\ \lambda+1 & \lambda\end{pmatrix} + \begin{pmatrix}-1 & 1\\ -1 & -1\end{pmatrix} + \begin{pmatrix}\lambda & -\lambda-1\\ -\lambda & \lambda+1\end{pmatrix}\right)\Omega(\lambda)^k \begin{pmatrix}1\\0\end{pmatrix} \tag{7.73}$$
$$= \lambda \cdot \Omega(\lambda)^k \begin{pmatrix}1\\0\end{pmatrix}, \tag{7.74}$$

which shows that v_1 is an eigenvector with eigenvalue λ. The same proof clearly applies to the other eigenvector v_2. ☐

Remark 7.3.3. *From equation (7.63) we can infer the limiting distributions of the eigenvalues of $\widetilde{\mathcal{E}}$. We already know all of its eigenvalues lie in the unit circle, hence we can write $\lambda = e^{i\varphi}$ for some $\varphi \in [-\pi, \pi)$. Equation (7.63) thus becomes*

$$e^{i\varphi} + e^{-i\varphi} + 1 = \cos\left(\frac{2\pi j}{N}\right) \qquad \Longleftrightarrow \qquad \varphi = \arccos\left(\frac{1}{2}\cos\left(\frac{2\pi j}{N}\right) - \frac{1}{2}\right). \tag{7.75}$$

Passing to the limit $N \to \infty$, by standard methods, we can compute the limiting density of the argument φ of the eigenvalues as

$$\mu(\varphi)\mathrm{d}\varphi = \Big(\mathbb{1}_{[\frac{\pi}{2},\pi]}(\varphi) - \mathbb{1}_{[-\pi,-\frac{\pi}{2}]}(\varphi)\Big)\,\frac{\sin\varphi\,\mathrm{d}\varphi}{\pi\sqrt{1-(1+2\cos\varphi)^2}} \tag{7.76}$$

This behaviour is shown in Figure 7.8.

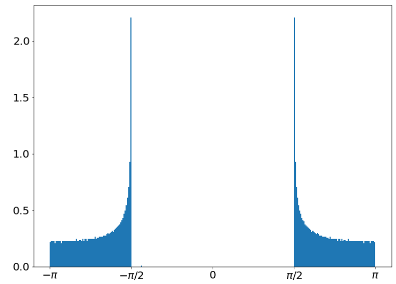

Figure 7.8: Distribution of the eigenvalues arguments for the $\widetilde{\mathcal{E}}$ (7.62), $N = 5000$.

Parameter Limit

Since the Lax matrix $\mathcal{E}$ for the Schur flow coincides with the one of the AL lattice, cf. equations (7.32) and (7.55), the parameter limit analysis coincides with the one performed at the end of Section 7.2. In particular, as in the case of the AL lattice, for large β the eigenvalues tend to accumulate on the unit circle, see Figure 7.7. In a completely similar way as done for the focusing AL lattice we conclude that the density of states of the random Lax matrix $\mathcal{E}$ with probability distribution entries given by the Gibbs measure μ_S in (7.51), converges in the limit $\beta \to \infty$ to the uniform measure on the unit circle, analogously to (7.46).

Appendix A

Numerical Tables

A.1 Tables of some weighted strictly monotone double Hurwitz numbers

We display below tables for the multiparametric Hurwitz numbers

$$H_g^{>}(\mu; s) := \frac{z_\mu}{|\mu|!} \sum_{\nu \text{ of length } s} h_g^{>}(\mu; \nu). \tag{A.1}$$

To obtain them, we first use Theorem 3.2.5 to compute the correlator $\langle \operatorname{tr} X^\mu \rangle$, hence we expand in series and following Theorem 4.2.2 we extract the corresponding coefficients in the variable N, keeping track of the genus, and in c, keeping track of $s = \ell(nu)$.

$\mu = (3,1)$	$g=0$	$g=1$
$s=1$	3	3
$s=2$	9	0
$s=3$	3	0

$\mu = (3,2)$	$g=0$	$g=1$
$s=1$	6	18
$s=2$	30	18
$s=3$	30	0
$s=4$	6	0

$\mu = (3,3)$	$g=0$	$g=1$	$g=2$
$s=1$	9	75	36
$s=2$	72	198	0
$s=3$	138	75	0
$s=4$	72	0	0
$s=5$	9	0	0

$\mu = (4,4)$	$g=0$	$g=1$	$g=2$	$g=3$
$s=1$	16	616	3304	1104
$s=2$	264	4636	8132	0
$s=3$	1200	8496	3304	0
$s=4$	1940	4636	0	0
$s=5$	1200	616	0	0
$s=6$	264	0	0	0
$s=7$	16	0	0	0

$\mu = (6,3)$	$g=0$	$g=1$	$g=2$	$g=3$
$s=1$	18	1428	16002	22872
$s=2$	414	15120	70938	22872
$s=3$	2598	43680	70938	0
$s=4$	6210	43680	16002	0
$s=5$	6210	15120	0	0
$s=6$	2598	1428	0	0
$s=7$	414	0	0	0
$s=8$	18	0	0	0

$\mu = (2,1,1)$	$g=0$
$s=1$	6
$s=2$	6

$\mu = (2,2,1)$	$g=0$	$g=1$
$s=1$	16	8
$s=2$	40	0
$s=3$	16	0

$\mu = (2,2,2)$	$g=0$	$g=1$
$s=1$	40	80
$s=2$	176	80
$s=3$	176	0
$s=4$	40	0

$\mu = (4,3,2,1)$	$g=0$	$g=1$	$g=2$	$g=3$
$s=1$	1728	54432	235872	70848
$s=2$	26136	379512	570672	0
$s=3$	111024	680832	235872	0
$s=4$	175824	379512	0	0
$s=5$	111024	54432	0	0
$s=6$	26136	0	0	0
$s=7$	1728	0	0	0

$\mu = (2,2,2,2)$	$g=0$	$g=1$	$g=2$
$s=1$	672	3360	1008
$s=2$	4464	8016	0
$s=3$	7872	3360	0
$s=4$	4464	0	0
$s=5$	672	0	0

$\mu=(4,4,4)$	$g=0$	$g=1$	$g=2$	$g=3$	$g=4$
$s=1$	704	89760	2631552	18161440	19033344
$s=2$	21312	1568640	24587904	75241920	19033344
$s=3$	204480	8507520	66562944	75241920	0
$s=4$	843648	18934080	66562944	18161440	0
$s=5$	1673856	18934080	24587904	0	0
$s=6$	1673856	8507520	2631552	0	0
$s=7$	843648	1568640	0	0	0
$s=8$	204480	89760	0	0	0
$s=9$	21312	0	0	0	0
$s=10$	704	0	0	0	0

$\mu=(5,4,4,2)$	$g=0$	$g=1$	$g=2$	$g=3$	$g=4$	$g=5$
$s=1$	29120	7047040	444924480	8434666240	42317475200	35974149120
$s=2$	1212800	180513600	6829912320	71893480000	168041817600	35974149120
$s=3$	16616960	1529449920	33913376640	186374568640	168041817600	0
$s=4$	103248000	5796138240	72317482560	186374568640	42317475200	0
$s=5$	331189440	11030467200	72317482560	71893480000	0	0
$s=6$	584935680	11030467200	33913376640	8434666240	0	0
$s=7$	584935680	5796138240	6829912320	0	0	0
$s=8$	331189440	1529449920	444924480	0	0	0
$s=9$	103248000	180513600	0	0	0	0
$s=10$	16616960	7047040	0	0	0	0
$s=11$	1212800	0	0	0	0	0
$s=12$	29120	0	0	0	0	0

$\mu=(2,2,2,1,1)$	$g=0$	$g=1$
$s=1$	1680	3360
$s=2$	7392	3360
$s=3$	7392	0
$s=4$	1680	0

$\mu=(3,3,2,2,2)$	$g=0$	$g=1$	$g=2$	$g=3$
$s=1$	71280	2661120	18461520	18722880
$s=2$	1206144	23973840	75182256	18722880
$s=3$	6314976	63697968	75182256	0
$s=4$	13791600	63697968	18461520	0
$s=5$	13791600	23973840	0	0
$s=6$	6314976	2661120	0	0
$s=7$	1206144	0	0	0
$s=8$	71280	0	0	0

A.2 Tables of some weighted weakly monotone double Hurwitz numbers

We display below tables for the multiparametric Hurwitz numbers

$$H_g^{\geq}(\mu;s) := \frac{z_\mu}{|\mu|!} \sum_{\nu \text{ of length } s} h_g^{\geq}(\mu;\nu). \tag{A.2}$$

The computation are performed as explained in the previous section. Notice that, in general, $H_g^{\geq}(\mu;s) \neq 0$ for every $s \leq |\mu|$ and $g \geq 0$. We calculate $H_g^{\geq}(\mu;s)$ for the first few values of g.

$\mu=(3,1)$	$g=0$	$g=1$	$g=2$
$s=1$	3	45	483
$s=2$	18	255	2688
$s=3$	30	420	4410
$s=4$	15	210	2205

$\mu=(3,2)$	$g=0$	$g=1$	$g=2$
$s=1$	6	168	3402
$s=2$	54	1464	29058
$s=3$	156	4176	82212
$s=4$	180	4800	94260
$s=5$	72	1920	37704

$\mu=(3,3)$	$g=0$	$g=1$	$g=2$
$s=1$	9	462	16443
$s=2$	117	5742	197559
$s=3$	516	24660	833472
$s=4$	1008	47580	1594836
$s=5$	900	42300	1413720
$s=6$	300	14100	471240

$\mu=(1,1,1)$	$g=0$	$g=1$	$g=2$	$g=3$	$g=4$
$s=1$	4	20	84	340	1364
$s=2$	12	60	252	1020	4092
$s=3$	8	40	168	680	2728

$\mu=(3,2,1)$	$g=0$	$g=1$	$g=2$	$g=3$
$s=1$	42	2268	81774	2498496
$s=2$	558	28248	982326	29405736
$s=3$	2472	121320	4143024	122714160
$s=4$	4836	234060	7926312	233606280
$s=5$	4320	208080	7025760	206699040
$s=6$	1440	69360	2341920	68899680

$\mu=(5,3,2)$	$g=0$	$g=1$	$g=2$	$g=3$	$g=4$	$g=5$
$s=1$	330	98670	17117100	2288397540	262779844470	27370788935490
$s=2$	11790	3139530	508126980	64989626220	7244914364850	739256601861510
$s=3$	151140	37555800	5814501240	722008428240	78865374260700	7932095991173640
$s=4$	973200	231506100	34809669720	4236585517200	456285210221400	45429895491347220
$s=5$	3600180	832748640	122812524600	14745786668160	1572851081541420	155505293985110400
$s=6$	8126700	1846504080	268910866680	31999520486160	3391243294051140	333707416656660000
$s=7$	11380320	2557716000	369587047200	43733298023520	4615886297332800	452853891923025600
$s=8$	9649080	2155587000	310123401000	36581098895880	3852087017209200	377274782175656400
$s=9$	4536000	1010772000	145151092800	17098516260000	1798743628584000	176040872796600000
$s=10$	907200	202154400	29030218560	3419703252000	359748725716800	35208174559320000

$\mu=(1,1,1,1)$	$g=0$	$g=1$	$g=2$	$g=3$	$g=4$	$g=5$
$s=1$	30	420	4410	42240	390390	3554460
$s=2$	174	2364	24498	233328	2151222	19565892
$s=3$	288	3888	40176	382176	3521664	32022864
$s=4$	144	1944	20088	191088	1760832	16011432

$\mu=(2,2,1,1)$	$g=0$	$g=1$	$g=2$	$g=3$	$g=4$	$g=5$
$s=1$	224	11760	417648	12652640	353825472	9465041040
$s=2$	2936	145560	5001792	148676240	4111488168	109250057640
$s=3$	12912	623088	21061152	619916064	17042443920	451231651728
$s=4$	25176	1200264	40262736	1179630192	32339018280	854769872184
$s=5$	22464	1066464	35678592	1043606592	28581355584	754984855584
$s=6$	7488	355488	11892864	347868864	9527118528	251661618528

$\mu=(3,2,2,1)$	$g=0$	$g=1$	$g=2$	$g=3$	$g=4$	$g=5$
$s=1$	1080	142560	11891880	808030080	49030839000	2777130588960
$s=2$	24408	2975688	236613384	15604156944	928759785048	51934912866648
$s=3$	195696	22833936	1764985248	114273524448	6718979907216	372620872120176
$s=4$	764208	86946408	6607836864	423012867984	24682857466608	1361716707058488
$s=5$	1622160	181944000	13692581280	870735528000	50576815946160	2781487931040000
$s=6$	1911600	212829120	15934474080	1009718844480	58506896866320	3212163320083200
$s=7$	1175040	130440960	9745954560	616691715840	35698249900800	1958572008345600
$s=8$	293760	32610240	2436488640	154172928960	8924562475200	489643002086400

$\mu=(3,3,3,3)$	$g=0$	$g=1$	$g=2$	$g=3$	$g=4$
$s=1$	14742	6781320	1863064476	397980044280	73027276324002
$s=2$	684774	286543656	73938326364	15124478632344	2690423275640562
$s=3$	11927088	4700315952	1162209509712	230530176869328	40089332784598560
$s=4$	108506304	41049414576	9847619855856	1910059732782864	326635075616752080
$s=5$	591049872	217264375440	50997568912848	9730568084094000	1643434518194147520
$s=6$	2065978224	744104821680	171941934622896	32417467690208400	5425295582074933440
$s=7$	4798180800	1703613513600	389301061256640	72772493528332800	12099023079466665600
$s=8$	7485955200	2632114958400	596891523260160	110918372096491200	18356651181359395200
$s=9$	7754940000	2709582840000	611410862412000	113177279163888000	18674140608815688000
$s=10$	5114988000	1780691688000	400648862930400	73995902520393600	12187705122917006400
$s=11$	1944000000	675695520000	151836376608000	28014789102336000	4610660182447564800
$s=12$	324000000	112615920000	25306062768000	4669131517056000	768443363741260800

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