A MINIMIZATION APPROACH TO THE WAVE EQUATION ON TIME-DEPENDENT DOMAINS

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Abstract. We prove the existence of weak solutions to the homogeneous wave equation on a suitable class of time-dependent domains. Using the approach suggested by De Giorgi and developed by Serra and Tilli, such solutions are approximated by minimizers of suitable functionals in space-time.

Keywords: wave equation, time-dependent domains, minimization

AMS subject classifications: 35L15, 49J10, 35Q74, 74R10, 35L90

Introduction

Several problems in dynamic fracture mechanics lead to the study of the wave equation in time-dependent domains (see [6, 7, 3]). The main difficulty is that at every time t the solution belongs to a different function space V_t. It is not restrictive to assume that all spaces V_t are embedded in a given Hilbert space H.

In the case of fracture mechanics, a common situation is $V_t = H^1(\Omega \setminus \Gamma_t)$ and $H = L^2(\Omega)$, where Ω is a domain in $\mathbb{R}^d$ and Γ_t is a closed $(d-1)$-dimensional subset of Ω, representing the crack at time t. A natural assumption on Γ_t is that it is monotonically increasing with respect to t, thus encoding the fact that, once created, a crack cannot disappear. As a consequence. the spaces V_t are increasing in time too.

To deal with possibly irregular cracks a more general increasing family of spaces has been considered in [2]: $V_t = GSBV_2^2(\Omega, \Gamma_t)$, defined as the space of functions $u \in GSBV(\Omega)$ such that $u \in L^2(\Omega)$, $\nabla u \in L^2(\Omega; \mathbb{R}^d)$, and $J_u \subset \Gamma_t$ (see [1] for the definition and properties of these spaces and for the definition of the approximate gradient ∇u and of the jump set J_u).

Given $u^0 \in V_0$ and $u^1 \in H$, the Cauchy problem we are interested in is formally written as

$$\begin{cases} u''(t) + Au(t) = 0 & \text{for a.e. } t > 0\,, \\ u(t) \in V_t & \text{for a.e. } t > 0\,, \\ u(0) = u^0\,,\ u'(0) = u^1\,, \end{cases} \tag{0.1}$$

where $'$ denotes the time derivative and A is a continuous and coercive linear operator ($A = -\Delta$ with homogeneous Neumann boundary conditions in the examples considered above).

The existence of a solution for (0.1) has already been proven in [2], through a time-discrete approach, by solving suitable incremental minimum problems and then passing to the limit as the time step tends to zero.

The purpose of this paper is to prove that a solution of (0.1) can be approximated by global minimizers of suitable energy functionals defined as integrals on $[0, \infty)$ with respect to time. On the one hand this shows a link between solutions of the hyperbolic problem (0.1) and solutions of minimum problems for integral functionals on the same time domain. On the other hand this result provides a new proof of the existence of a solution to (0.1).

The seminal idea of this approximation process goes back to a conjecture by De Giorgi [5] on the nonlinear wave equation. Such a conjecture has been proven by Serra and Tilli in [8] and, in a more general setting, in [9].

In our paper we extend their result to the case of time-dependent domains. To illustrate the global minimization approach in our setting, we focus on the model case $V_t = H^1(\Omega \setminus \Gamma_t)$ and $A = -\Delta$. The main idea is to associate to the Cauchy problem (0.1) a functional of the form

$$\mathcal{F}_\varepsilon(u) := \frac{1}{2}\int_0^\infty e^{-t/\varepsilon}\Big(\varepsilon^2\|u''(t)\|^2_{L^2(\Omega)} + \|\nabla u(t)\|^2_{L^2(\Omega;\mathbb{R}^d)}\Big)\,\mathrm{d}t\,, \tag{0.2}$$

This functional is to be minimized, for every fixed $\varepsilon > 0$, among all the functions $t \mapsto u(t)$ satisfying the initial conditions $u(0) = u^0$ and $u'(0) = u^1$ and the time-dependent constraint $u(t) \in V_t$ for a.e. $t > 0$. Once the existence of a minimizer u_ε is proven, the Euler-Lagrange equation of (0.2) formally reads as

$$\varepsilon^2 u_\varepsilon''''(t) - 2\varepsilon u_\varepsilon'''(t) + u_\varepsilon''(t) - \Delta u_\varepsilon(t) = 0 \qquad \text{in } \Omega \setminus \Gamma_t\,,$$

and hence, letting $\varepsilon \to 0$, one *formally* obtains a solution to the wave equation in (0.1).

As mentioned above, a quite general scheme to pass to the limit rigorously has been introduced by Serra and Tilli in [9] when time-dependent constraint $u(t) \in V_t$ is not present. The proof consists in finding suitable estimates on the minimizers u_ε of the functionals $\mathcal{F}_\varepsilon$ and to exploit these estimates in order to obtain, by compactness, the convergence of u_ε to a weak solution u to the wave equation.

In this paper we implement this scheme in the case of time-dependent domains. This requires some changes in the proof, since all competitors of the minimum problem for (0.2) must satisfy the constraint $u(t) \in V_t$ for a.e. $t > 0$.

The main change is in the proof of the key estimate for $u_\varepsilon(t)$, which is obtained in [9] by using an inner variation $u_\varepsilon(\varphi_\delta(t))$ for a suitable function $\varphi_\delta\colon [0,\infty) \to [0,\infty)$. Since in our case we have to require that $u_\varepsilon(\varphi_\delta(t)) \in V_t$ for a.e. $t > 0$, this variation is admissible only if $\varphi_\delta(t) \le t$ for a.e. $t > 0$. By the technical definition of φ_δ, this leads to the constraint $\delta > 0$. Therefore the standard comparison between the functional on $u_\varepsilon(\varphi_\delta(t))$ and on the minimizer $u_\varepsilon(t)$, in the limit as $\delta \to 0+$, gives only an inequality, instead of the equality proven in [9, formula (4.7)]. This inequality, however, turns out to be enough to obtain the other estimates of [9] with minor changes.

A further difficulty appears when proving that the limit u of u_ε is a weak solution of (0.1), since also the test functions η must satisfy the constraint $\eta(t) \in V_t$ for a.e. $t > 0$. Therefore, to adapt the proof of [9], we have to approximate an arbitrary test function η satisfying the constraint $\eta(t) \in V_t$ for a.e. $t > 0$ by sums of functions of the form $\varphi(t)v$ with $v \in V_s$ and $\varphi \in C^2(\mathbb{R})$ with $\mathrm{supp}(\varphi) \subset [s,\infty)$, which still satisfy the constraint.

1. Description of the problem

1.1. **Setting.** To study the wave equation in time-dependent domains we adopt the functional setting introduced in [4]. Let H be a separable Hilbert space and let $(V_t)_{t\in[0,\infty)}$ be a family of separable Hilbert spaces with the following properties

- (H1) for every $t \in [0,\infty)$ the space V_t is contained and dense in H with continuous embedding;
- (H2) for every $s,t \in [0,\infty)$, with $s < t$, V_s is a closed subspace of V_t with the induced scalar product.

The scalar product in H is denoted by $(\cdot,\cdot)$ and the corresponding norm by $\|\cdot\|$. The norm in V_t is denoted by $\|\cdot\|_t$. By (H2) for every $0\le s<t$ we have $\|v\|_s=\|v\|_t$ for every $v\in V_s$.

The dual of H is identified with H, while for every $t\in[0,T]$ the dual of V_t is denoted by V_t^*. Note that the adjoint of the continuous embedding of V_t into H provides a continuous embedding of H into V_t^* and that H is dense in V_t^*. Let $\langle\cdot,\cdot\rangle_t$ be the duality product between V_t^* and V_t and let $\|\cdot\|_t^*$ be the corresponding dual norm. Note that $\langle\cdot,\cdot\rangle_t$ is the unique continuous bilinear map on $V_t^*\times V_t$ satisfying

$$\langle h,v\rangle_t=(h,v)\qquad \text{for every } h\in H \text{ and } v\in V_t\,. \tag{1.1}$$

Let $V_\infty:=\bigcup_{t\ge0}V_t$ and let $a\colon V_\infty\times V_\infty\to\mathbb{R}$ be a bilinear symmetric form satisfying the following conditions:

(H3) continuity: there exists $M_0>0$ such that

$$|a(u,v)|\le M_0\|u\|_t\|v\|_t\quad \text{for every } t\ge0 \text{ and every } u,v\in V_t\,; \tag{1.2}$$

(H4) coercivity: there exist $\lambda_0\ge0$ and $\nu_0>0$ such that

$$a(u,u)+\lambda_0\|u\|^2\ge\nu_0\|u\|_t^2\quad \text{for every } t\ge0 \text{ and every } u\in V_t\,; \tag{1.3}$$

(H5) positive semidefiniteness:

$$a(u,u)\ge0\quad \text{for every } u\in V_\infty\,. \tag{1.4}$$

For every $\tau,t\in[0,\infty)$ let $A_\tau^t:V_t\to V_\tau^*$ be the continuous linear operator defined by

$$\langle A_\tau^t u,v\rangle_\tau:=a(u,v)\quad \text{for every } u\in V_t \text{ and } v\in V_\tau\,. \tag{1.5}$$

Note that

$$\|A_\tau^t u\|_\tau^*\le M_0\|u\|_t\quad \text{for every } u\in V_t\,. \tag{1.6}$$

Finally, we set $Q(u):=a(u,u)$ for every $u\in V_\infty$.

Definition 1.1. Given $T>0$, we define $\mathcal{W}_T^{0,1}:=L^2((0,T);V_T)\cap H^1((0,T);H)$, with the Hilbert space structure induced by the scalar product

$$(u,v)_{\mathcal{W}_T^{0,1}}=(u,v)_{L^2((0,T);V_T)}+(u',v')_{L^2((0,T);H)}\,,$$

where u' and v' denote the distributional derivatives. The norm induced by the scalar product $(\cdot,\cdot)_{\mathcal{W}_T^{0,1}}$ is denoted by $\|\cdot\|_{\mathcal{W}_T^{0,1}}$. Moreover, we define

$$\mathcal{V}_T^{0,1}:=\{u\in\mathcal{W}_T^{0,1}\,:\,u(t)\in V_t \text{ for a.e. } t\in(0,T)\}\,,$$

and note that it is a closed subspace of $\mathcal{W}_T^{0,1}$.

Analogously, we define $\mathcal{W}_T^{0,2}:=L^2((0,T);V_T)\cap H^2((0,T);H)$, with the Hilbert space structure induced by the scalar product

$$(u,v)_{\mathcal{W}_T^{0,2}}=(u,v)_{L^2((0,T);V_T)}+(u',v')_{L^2((0,T);H)}+(u'',v'')_{L^2((0,T);H)}\,,$$

and the space

$$\mathcal{V}_T^{0,2}:=\{u\in\mathcal{W}_T^{0,2}\,:\,u(t)\in V_t \text{ for a.e. } t\in(0,T)\}\,,$$

which is a closed subspace of $\mathcal{W}_T^{0,2}$.

Finally, $\mathcal{V}^{0,1}$ (resp. $\mathcal{V}^{0,2}$) is defined as the space of functions $u\colon(0,+\infty)\to H$ whose restrictions to $(0,T)$ belong to $\mathcal{V}_T^{0,1}$ (resp. $\mathcal{V}_T^{0,2}$) for every $T>0$.

Remark 1.2. It is well known that every function $u \in H^1((0,T);H)$ (resp. $u \in H^2((0,T);H)$) admits a representative, still denoted by u, which belongs to the space $C^0([0,T];H)$ (resp. $C^1([0,T];H)$). With this convention we have $\mathcal{V}^{0,1}_T \subset C^0([0,T];H)$ (resp. $\mathcal{V}^{0,2}_T \subset C^1([0,T];H)$) for every $T>0$.

Definition 1.3. We say that u is a weak solution of the equation

$$u''(t) + A^t_t u(t) = 0, \quad u(t) \in V_t \qquad \text{for } t \in [0,\infty) \tag{1.7}$$

if $u \in \mathcal{V}^{0,1}$ and for every $T>0$

$$\int_0^T (u'(t), \psi'(t))\,\mathrm{d}t = \int_0^T a(u(t),\psi(t))\,\mathrm{d}t \tag{1.8}$$

for every $\psi \in \mathcal{V}^{0,1}_T$ with $\psi(0)=\psi(T)=0$.

For every Banach space X let $C_w([0,T];X)$ be the space of functions $u\colon [0,T]\to X$ that are continuous for the weak topology of X.

Remark 1.4. If u is a weak soltution of (1.7) with $u \in L^\infty((0,T);V_T)$ and $u' \in L^\infty((0,T);H)$ for every $T>0$, then [4, Theorem 2.17 and Proposition 2.18] imply that, after a modification on a set of measure zero, $u \in C_w([0,T];V_T)$ and $u' \in C_w([0,T];H)$ for every $T>0$.

1.2. **Main results.** Throughout the paper we fix $u^0 \in V_0$, $u^1 \in H$, and a sequence $\{u^1_\varepsilon\} \subset V_0$ such that

$$\|u^1_\varepsilon - u^1\|_H \to 0 \text{ as } \varepsilon \to 0+ \quad \text{and} \quad \varepsilon\|u^1_\varepsilon\|_0 \le C_1, \tag{1.9}$$

for some constant $C_1>0$. For every $\varepsilon>0$ we consider the functional

$$\mathcal{F}_\varepsilon(u) := \frac{1}{2}\int_0^\infty e^{-t/\varepsilon}\Big(\varepsilon^2\|u''(t)\|^2 + Q(u(t))\Big)\,\mathrm{d}t, \tag{1.10}$$

defined on the set

$$\mathcal{V}^{0,2}(u^0,u^1_\varepsilon) := \{u \in \mathcal{V}^{0,2} \ : u(0)=u^0,\ u'(0)=u^1_\varepsilon\}, \tag{1.11}$$

which is well-defined in view of Remark 1.2.

We now state our main results, which are proven in Sections 2, 3, and 4.

Theorem 1.5. *For every $\varepsilon \in (0,1)$ the functional $\mathcal{F}_\varepsilon$ admits a unique global minimizer u_ε in the set $\mathcal{V}^{0,2}(u^0,u^1_\varepsilon)$. Moreover,*

$$\mathcal{F}_\varepsilon(u_\varepsilon) \le \bar{C}\varepsilon, \tag{1.12}$$

for some constant $\bar{C}>0$ depending only on $\|u^0\|_0$ and C_1.

In particular, if $\varepsilon\|u^1_\varepsilon\|_0 \to 0$ as $\varepsilon \to 0+$, then

$$\mathcal{F}_\varepsilon(u_\varepsilon) \le \varepsilon\Big(\frac{1}{2}Q(u^0) + r_\varepsilon\Big), \tag{1.13}$$

where $r_\varepsilon \to 0$ as $\varepsilon \to 0+$.

Theorem 1.6. *There exists a constant $C>0$ such that for every $\varepsilon \in (0,1)$ the minimizer u_ε of $\mathcal{F}_\varepsilon$ in $\mathcal{V}^{0,2}(u^0,u^1_\varepsilon)$ satisfies the estimates:*

$$\int_t^{t+\tau} Q(u_\varepsilon(s))\,\mathrm{d}s \le C\,\tau \qquad \text{for every } t\ge 0,\ \tau \ge \varepsilon, \tag{1.14}$$

$$\|u_\varepsilon(t)\|^2 \le C(1+t^2) \qquad \text{for every } t \ge 0, \tag{1.15}$$

$$\|u'_\varepsilon(t)\| \le C \qquad \text{for every } t\ge 0. \tag{1.16}$$

Theorem 1.7. *For every* $\varepsilon \in (0,1)$ *let* u_ε *be the minimizer of* $\mathcal{F}_\varepsilon$ *in* $\mathcal{V}^{0,2}(u^0, u^1_\varepsilon)$ *. Then for every sequence* $\{\varepsilon_n\} \subset (0,1)$ *, with* $\varepsilon_n \to 0$ *as* $n \to \infty$ *, there exist a subsequence, not relabeled, and a weak solution* u *of* (1.7) *such that* $u_{\varepsilon_n} \rightharpoonup u$ *weakly in* $\mathcal{W}^{0,1}_T$ *for every* $T > 0$ *. Moreover the following properties hold:*

(a) *weak continuity:* $u \in C_w([0,T]; V_T)$ *and* $u' \in C_w([0,T]; H)$ *for every* $T > 0$ *;*
(b) *initial conditions:* $u(0) = u^0$ *and* $u'(0) = u^1$ *.*

If, in addition, $\varepsilon \|u^1_\varepsilon\|_0 \to 0$ *as* $\varepsilon \to 0+$ *, then the following energy inequality holds:*

$$\|u'(t)\|^2 + Q(u(t)) \le \|u^1\|^2 + Q(u^0) \quad \text{for every } t > 0\,. \tag{1.17}$$

2. Proof of Theorem 1.5

Before proving our results we introduce a change of variables that will be useful throughout the paper.

Remark 2.1. For every $\varepsilon > 0$ and every $T > 0$ we set

$$\mathcal{W}^{0,2}_{\varepsilon,T} := L^2((0,T); V_{\varepsilon T}) \cap H^2((0,T); H)\,,$$
$$\mathcal{V}^{0,2}_{\varepsilon,T} := \{v \in \mathcal{W}^{0,2}_{\varepsilon,T} \,:\, v(t) \in V_{\varepsilon t} \text{ for a.e. } t \in (0,T)\}\,.$$

Note that $\mathcal{W}^{0,2}_{\varepsilon,T}$ is a Hilbert space with the scalar product

$$(u,v)_{\mathcal{W}^{0,2}_{\varepsilon,T}} = (u,v)_{L^2((0,T);V_{\varepsilon T})} + (u',v')_{L^2((0,T);H)} + (u'',v'')_{L^2((0,T);H)}\,,$$

and $\mathcal{V}^{0,2}_{\varepsilon,T}$ is a closed subspace of $\mathcal{W}^{0,2}_{\varepsilon,T}$. Furthermore, $\mathcal{V}^{0,2}_{\varepsilon}$ denotes the space of functions $u\colon [0,\infty) \to H$ whose restrictions to $(0,T)$ belong to $\mathcal{V}^{0,2}_{\varepsilon,T}$ for every $T > 0$. By Remark 1.2 every $u \in \mathcal{W}^{0,2}_{\varepsilon,T}$ admits a representative, still denoted by u, which belongs to $C^1([0,T]; H)$. With this convention we have $\mathcal{V}^{0,2}_{\varepsilon,T} \subset C^1([0,T]; H)$ for every $T > 0$. Finally, we define

$$\mathcal{V}^{0,2}_{\varepsilon}(u^0, \varepsilon u^1_\varepsilon) := \{v \in \mathcal{V}^{0,2}_{\varepsilon} \,:\, v(0) = 0, v'(0) = \varepsilon u^1_\varepsilon\}\,.$$

It is easy to see that if $u \in \mathcal{V}^{0,2}(u^0, u^1_\varepsilon)$, then the function v defined by

$$v(t) := u(\varepsilon t) \tag{2.1}$$

belongs to $\mathcal{V}^{0,2}_{\varepsilon}(u^0, \varepsilon u^1_\varepsilon)$ and

$$\mathcal{F}_\varepsilon(u) = \varepsilon \mathcal{G}_\varepsilon(v)\,, \tag{2.2}$$

where

$$\mathcal{G}_\varepsilon(v) := \frac{1}{2}\int_0^\infty e^{-t}\Big(\frac{\|v''(t)\|^2}{\varepsilon^2} + Q(v(t))\Big)\,\mathrm{d}t\,.$$

In view of Remark 2.1, Theorem 1.5 is a consequence of the following result for the functional $\mathcal{G}_\varepsilon$.

Theorem 2.2. *For every* $\varepsilon \in (0,1)$ *the functional* $\mathcal{G}_\varepsilon$ *admits a unique global minimizer* v_ε *in the class* $\mathcal{V}^{0,2}_{\varepsilon}(u^0, \varepsilon u^1_\varepsilon)$ *. Moreover,*

$$\mathcal{G}_\varepsilon(v_\varepsilon) \le \bar{C}, \tag{2.3}$$

for some constant $\bar{C} < \infty$ *depending only on* $\|u^0\|_0$ *and* C_1 *.*

Furthermore $u_\varepsilon(t) := v_\varepsilon(\frac{t}{\varepsilon})$ *is the unique global minimizer of* $\mathcal{F}_\varepsilon$ *in* $\mathcal{V}^{0,2}(u^0, u^1_\varepsilon)$ *and satisfies* (1.12) *.*

Finally, if $\varepsilon\|u_1^\varepsilon\|_0 \to 0$ *as* $\varepsilon \to 0+$ *, then*

$$\mathcal{G}_\varepsilon(v_\varepsilon) \le \frac{1}{2}Q(u^0) + r_\varepsilon\,, \tag{2.4}$$

where $r_\varepsilon \to 0$ *as* $\varepsilon \to 0$ *and* u_ε *satisfies* (1.13) *.*

Proof. Fix $\varepsilon > 0$ and set $v(t) := u^0 + \varepsilon t u_\varepsilon^1$ for every $t \ge 0$. Note that $v \in \mathcal{V}_\varepsilon^{0,2}(u^0, \varepsilon u_\varepsilon^1)$, since $u^0,\, u_\varepsilon^1 \in V_0 \subset V_t$ for every $t \ge 0$. By (H3) and by (1.9), we have

$$\mathcal{G}_\varepsilon(v) = \frac{1}{2}\int_0^\infty e^{-t} Q(v(t))\,\mathrm{d}t \le \frac{1}{2}Q(u^0) + M_0\,\varepsilon\|u_\varepsilon^1\|_0(\varepsilon\|u_\varepsilon^1\|_0 + \|u^0\|_0) \le \bar{C}\,, \tag{2.5}$$

where $\bar{C}$ is a constant depending only on C_1 and $\|u_0\|_0$. Note that, if $\varepsilon\|u_\varepsilon^1\|_0 \to 0$ as $\varepsilon \to 0+$, then by (2.3) it follows that

$$\mathcal{G}_\varepsilon(v) \le \frac{1}{2}Q(u^0) + r_\varepsilon\,, \tag{2.6}$$

where $r_\varepsilon \to 0$ as $\varepsilon \to 0$.

In particular, $\mathcal{G}_\varepsilon$ has a finite infimum and (2.3) (as well as (2.4)) follows as soon as $\mathcal{G}_\varepsilon$ has an absolute minimizer v_ε. To show this, consider a minimizing sequence $\{v_{\varepsilon,n}\} \subset \mathcal{V}_\varepsilon^{0,2}(u^0, \varepsilon u_\varepsilon^1)$ and fix $T > 0$. By the very definition of $\mathcal{G}_\varepsilon$ and by (2.5),

$$\int_0^T \|v''_{\varepsilon,n}(t)\|^2\,\mathrm{d}t \le e^T \int_0^T e^{-t}\|v''_{\varepsilon,n}(t)\|^2\,\mathrm{d}t \le 2\varepsilon^2 e^T \mathcal{G}_\varepsilon(v_{\varepsilon,n}) \le \varepsilon^2 C_T\,, \tag{2.7}$$

for some constant $C_T > 0$. The bound (2.7), together with the boundary conditions

$$v_{\varepsilon,n}(0) = u^0 \quad \text{and} \quad v'_{\varepsilon,n}(0) = \varepsilon u_\varepsilon^1\,, \tag{2.8}$$

implies

$$\|v_{\varepsilon,n}\|_{H^2((0,T);H)} \le C_{T,\varepsilon} \tag{2.9}$$

for some constant $C_{T,\varepsilon} > 0$ independent of n. Moreover, by (H2) and (H4), for $t \in [0,T]$ we have

$$\nu_0\|v_{\varepsilon,n}(t)\|_T^2 = \nu_0\|v_{\varepsilon,n}(t)\|_t^2 \le \lambda_0\|v_{\varepsilon,n}(t)\|^2 + Q(v_{\varepsilon,n}(t))$$

from which, using (2.5) and (2.9), we get

$$\nu_0\|v_{\varepsilon,n}\|^2_{L^2((0,T);V_T)} \le \lambda_0\|v_{\varepsilon,n}\|^2_{L^2((0,T);H)} + \int_0^T Q(v_{\varepsilon,n}(t))\,\mathrm{d}t \le \widehat{C}_{T,\varepsilon}$$

for some constant $\widehat{C}_{T,\varepsilon} > 0$ independent of n. It follows that $\|v_{\varepsilon,n}\|_{\mathcal{W}^{0,2}_{\varepsilon,T}}$ is uniformly bounded and hence, up to a subsequence, $v_{\varepsilon,n} \rightharpoonup v_\varepsilon$ in $\mathcal{W}^{0,2}_{\varepsilon,T}$ as $n \to \infty$, for some $v_\varepsilon \in \mathcal{W}^{0,2}_{\varepsilon,T}$. Moreover, since $\mathcal{V}^{0,2}_{\varepsilon,T}$ is closed, $v_\varepsilon \in \mathcal{V}^{0,2}_{\varepsilon,T}$. By the arbitrariness of T we have $v_\varepsilon \in \mathcal{V}^{0,2}_\varepsilon$ and by (2.8) we get $v_\varepsilon \in \mathcal{V}_\varepsilon^{0,2}(u^0, \varepsilon u_\varepsilon^1)$. Finally, since $\mathcal{G}_\varepsilon$ is lower semi-continuous and strictly convex by (H5), v_ε is the unique minimizer of $\mathcal{G}_\varepsilon$ in $\mathcal{V}_\varepsilon^{0,2}(u^0, \varepsilon u_\varepsilon^1)$. The statements about $u_\varepsilon(t)$ follow from Remark 2.1. □

3. Proof of Theorem 1.6

We first introduce some notations. Let v_ε be the minimizer of $\mathcal{G}_\varepsilon$ in $\mathcal{V}^{0,2}_\varepsilon(u^0, \varepsilon u^1_\varepsilon)$ and let L_ε be the corresponding Lagrangian defined as

$$L_\varepsilon(t) := D_\varepsilon(t) + Q_\varepsilon(t)\,, \tag{3.1}$$

where

$$D_\varepsilon(t) := \frac{\|v''_\varepsilon(t)\|^2}{2\varepsilon^2} \quad \text{and} \quad Q_\varepsilon(t) := \frac{Q(v_\varepsilon(t))}{2}\,. \tag{3.2}$$

Moreover, we define the kinetic energy function K_ε as

$$K_\varepsilon(t) := \frac{\|v'_\varepsilon(t)\|^2}{2\varepsilon^2}\,. \tag{3.3}$$

We shall use the following result, which can be proven as in [9, Lemma 3.4].

Lemma 3.1. *There exists a constant $C > 0$ (depending only on $\|u^0\|_0$, $\|u^1\|$, and C_1 in (1.9)) such that for every $\varepsilon \in (0,1)$ the minimizer v_ε of $\mathcal{G}_\varepsilon$ in $\mathcal{V}^{0,2}_\varepsilon(u^0, \varepsilon u^1_\varepsilon)$ satisfies*

$$\int_0^\infty e^{-t} D_\varepsilon(t)\,\mathrm{d}t = \int_0^\infty e^{-t}\frac{\|v''_\varepsilon(t)\|^2}{2\varepsilon^2}\,\mathrm{d}t \le C\,, \tag{3.4}$$

$$\int_0^\infty e^{-t} K_\varepsilon(t)\,\mathrm{d}t = \int_0^\infty e^{-t}\frac{\|v'_\varepsilon(t)\|^2}{2\varepsilon^2}\,\mathrm{d}t \le C\,. \tag{3.5}$$

In particular, in view of Lemma 3.1, we have $K_\varepsilon \in W^{1,1}(0,T)$ for all $T > 0$ and

$$K'_\varepsilon(t) = \frac{1}{\varepsilon^2}(v'_\varepsilon(t), v''_\varepsilon(t)) \qquad \text{for a.e. } t > 0\,. \tag{3.6}$$

Following the approach in [9], we introduce the *average operator* $\mathcal{A}$, defined by

$$(\mathcal{A}f)(s) := \int_s^\infty e^{-(t-s)} f(t)\,\mathrm{d}t\,, \qquad s \ge 0\,.$$

for every measurable function $f\colon [0,\infty) \to [0,\infty]$.

We note that $\mathcal{A}f$ is well defined (possibly ∞) since $f \ge 0$. Moreover, the equaity

$$\mathcal{A}f(0) = \int_0^\infty e^{-t} f(t)\,\mathrm{d}t\,, \tag{3.7}$$

implies that, if $\mathcal{A}f(0) < \infty$, then $\mathcal{A}f$ is absolutely continuous on all intervals $[0,T]$ and

$$(\mathcal{A}f)' = \mathcal{A}f - f \quad \text{a.e. in } [0,\infty)\,. \tag{3.8}$$

In any case, since $\mathcal{A}f \ge 0$, starting from $f \ge 0$ one can iterate $\mathcal{A}$, and a simple computation gives

$$(\mathcal{A}^2 f)(s) = \int_s^\infty e^{-(t-s)}(t-s) f(t)\,\mathrm{d}t\,, \tag{3.9}$$

thus in particular

$$(\mathcal{A}^2 f)(0) = \int_0^\infty e^{-t} t f(t)\,\mathrm{d}t\,. \tag{3.10}$$

Finally, we define the approximate energy

$$E_\varepsilon(t) := K_\varepsilon(t) + (\mathcal{A}^2 Q_\varepsilon)(t)\,. \tag{3.11}$$

The key ingredient in order to prove Theorem 1.6 is given by the following proposition.

Proposition 3.2. *The function E_ε is uniformly bounded and monotonically nonincreasing. More precisely, there exists $C_1' > 0$, depending only on $\|u^0\|_0$, $\|u^1\|$, and C_1 in* (1.9)*, such that*

$$E_\varepsilon(t) \le C_1' \quad \text{for every } t \ge 0. \tag{3.12}$$

Moreover, if $\varepsilon \|u^1_\varepsilon\|_0 \to 0$ as $\varepsilon \to 0+$, then

$$E_\varepsilon(t) \le \frac{1}{2}\|u^1_\varepsilon\|^2 + \frac{1}{2}Q(u^0) + \widetilde{r}_\varepsilon\,, \tag{3.13}$$

where $\widetilde{r}_\varepsilon \to 0$ as $\varepsilon \to 0+$.

Proof. The proof of Proposition 3.2 closely follows the strategy adopted in [9] to prove [9, Theorem 4.8]. We briefly sketch the main steps, underlining the main differences with respect to the case treated in [9]. The proof is divided into four steps.

Step 1. For every $g \in C^{1,1}(\mathbb{R};[0,\infty))$, with $g(0) = 0$ and $g(t)$ affine for t sufficiently large, there exists a constant $C_1(g) > 0$, depending on g, $\|u^0\|_0$, and C_1 in (1.9)*, such that*

$$\int_0^\infty e^{-s}(g'(s) - g(s))L_\varepsilon(s)\,\mathrm{d}s - \int_0^\infty e^{-s}(4D_\varepsilon(s)g'(s) + K_\varepsilon'(s)g''(s))\,\mathrm{d}s + R_\varepsilon \ge 0\,, \tag{3.14}$$

where

$$R_\varepsilon := \varepsilon g'(0)\int_0^\infty e^{-s}s\,a(v_\varepsilon(s), u^1_\varepsilon)\,\mathrm{d}s$$

satisfies

$$|R_\varepsilon| < C_1(g)\,. \tag{3.15}$$

In particular, if $\varepsilon\|u^1_\varepsilon\|_0 \to 0$ as $\varepsilon \to 0+$, then

$$|R_\varepsilon| \to 0 \qquad \text{as } \varepsilon \to 0+\,. \tag{3.16}$$

Using the approximation argument in [9, Corollary 4.5], it is enough to prove (3.14) for $g \in C^2(\mathbb{R};[0,\infty))$ with $g(0) = 0$ and $g(t)$ constant for t large enough.

For $\delta \ge 0$ small enough, the function $\varphi_\delta(t) := t - \delta g(t)$ is a C^2-diffeomorphism of $[0,\infty)$ into itself. We consider the function $v_{\varepsilon,\delta}(t) := v_\varepsilon(\varphi_\delta(t)) + t\delta\varepsilon g'(0)u^1_\varepsilon$. By construction $\varphi_\delta(t) \le t$ so that, in view of (H2), $v_{\varepsilon,\delta} \in \mathcal{V}^{0,2}_\varepsilon$. Note that in the proof of this property the condition $\delta \ge 0$ is crucial. Moreover, $v_{\varepsilon,\delta}(0) = v_\varepsilon(0) = u^0$ and

$$v'_{\varepsilon,\delta}(t)_{|t=0} = v'_\varepsilon(0)(1 - \delta g'(0)) + \delta\varepsilon g'(0)u^1_\varepsilon = \varepsilon u^1_\varepsilon\,,$$

whence $v_{\varepsilon,\delta} \in \mathcal{V}^{0,2}_\varepsilon(u^0, \varepsilon u^1_\varepsilon)$.

Set $\psi_\delta(s) := \varphi_\delta^{-1}(s)$ for every $s \ge 0$. By the change of variables $t = \psi_\delta(s)$, it is straightforward to check that

$$\begin{aligned}\mathcal{G}_\varepsilon(v_{\varepsilon,\delta}) =& \frac{1}{2\varepsilon^2}\int_0^\infty \psi_\delta'(s)e^{-\psi_\delta(s)}\|v''_\varepsilon(s)|\varphi'_\delta(\psi_\delta(s))|^2 + v'_\varepsilon(s)\varphi''_\delta(\psi_\delta(s))\|^2\,\mathrm{d}s\\ &+ \frac{1}{2}\int_0^\infty \psi_\delta'(s)e^{-\psi_\delta(s)}Q(v_\varepsilon(s) + \delta\varepsilon g'(0)\psi_\delta(s)u^1_\varepsilon)\,\mathrm{d}s\,.\end{aligned} \tag{3.17}$$

Notice that

$$s = \varphi_\delta(\psi_\delta(s)) = \psi_\delta(s) - \delta g(\psi_\delta(s)) \tag{3.18}$$

so that, in view of the assumptions on g, we have $e^{-\psi_\delta(s)} \leq e^{\delta\|g\|_{L^\infty}} e^{-s}$. Moreover, since

$$\psi_\delta'(s) = 1 + \delta g'(\psi_\delta(s))\psi_\delta'(s) \quad \text{and} \quad \psi_\delta''(s) = \delta(g''(\psi_\delta(s))(\psi_\delta'(s))^2 + g'(\psi_\delta(s))\psi_\delta''(s)),$$

for δ sufficiently small both $\psi_\delta'(s)$ and $\psi_\delta''(s)$ are bounded uniformly with respect to s. This fact, together with Lemma 3.1, implies that the first integral in (3.17) is finite. As for the second integral we have

$$\frac{1}{2}\int_0^\infty \psi_\delta'(s)e^{-\psi_\delta(s)}Q(v_\varepsilon(s) + \delta\varepsilon g'(0)\psi_\delta(s)u_\varepsilon^1)\,\mathrm{d}s \leq \frac{1}{2}\|\psi_\delta'\|_{L^\infty} e^{\delta\|g\|_{L^\infty}}(A_1 + A_2 + A_3)\,, \tag{3.19}$$

where

$$\begin{aligned} A_1 &:= \int_0^\infty e^{-s}Q(v_\varepsilon(s))\,\mathrm{d}s \\ A_2 &:= \delta^2(g'(0))^2\varepsilon^2 Q(u_\varepsilon^1)\int_0^\infty e^{-s}(\psi_\delta(s))^2\,\mathrm{d}s \\ A_3 &:= 2\delta\varepsilon g'(0)\int_0^\infty e^{-s}\psi_\delta(s)a(v_\varepsilon(s), u_\varepsilon^1)\,\mathrm{d}s\,. \end{aligned}$$

Now, $A_1 < \infty$ by (2.3) and $A_2 < +\infty$ in view of (3.18). Finally, by (H5) and the Cauchy inequality, we have $A_3 \leq A_1 + A_2 < \infty$. It follows $\mathcal{G}_\varepsilon(v_{\varepsilon,\delta}) < \infty$ for δ sufficiently small. Analogously, one can show that differentiation under the integral sign in (3.17) is possible.

Since $v_{\varepsilon,0} = v_\varepsilon$ and $v_{\varepsilon,\delta} \in \mathcal{V}_\varepsilon^{0,2}(u^0, \varepsilon u_\varepsilon^1)$ only for $\delta \geq 0$, the minimality of v_ε implies

$$\frac{\mathrm{d}}{\mathrm{d}\delta}\mathcal{G}_\varepsilon(v_{\varepsilon,\delta})\Big|_{\delta=0} \geq 0\,,$$

while in [9] the equality holds. One can compute this derivative as in [9, pages 2031-2032] and one can check that it coincides with the left-hand side of (3.14).

As for R_ε, by assumptions (H3) and (H5) and by (1.9) and (2.2), we have

$$\begin{aligned} |R_\varepsilon| &= \varepsilon|g'(0)|\int_0^\infty e^{-s}s\,|a(v_\varepsilon(s), u_\varepsilon^1)|\,\mathrm{d}s \\ &\leq \varepsilon|g'(0)|\Big(\int_0^\infty e^{-s}Q(v_\varepsilon(s))\,\mathrm{d}s + M_0\|u_\varepsilon^1\|_0\int_0^\infty e^{-s}s^2\,\mathrm{d}s\Big) \\ &\leq |g'(0)|(2\varepsilon\mathcal{G}_\varepsilon(v_\varepsilon) + 2M_0\varepsilon\|u_\varepsilon^1\|_0) \leq 2g'(0)(\varepsilon\bar{C} + C_1) =: C_1(g)\,, \end{aligned} \tag{3.20}$$

thus proving (3.15). By the last but one inequality in (3.20) and by (2.2), it follows that, if $\varepsilon\|u_\varepsilon^1\|_0 \to 0$ as $\varepsilon \to 0+$, then $R_\varepsilon \to 0$ as $\varepsilon \to 0+$.

Step 2. $(\mathcal{A}^2L_\varepsilon)(0) \leq (\mathcal{A}L_\varepsilon)(0) - 4(\mathcal{A}D_\varepsilon)(0) + R_\varepsilon$.

The claim follows by applying (3.14) with $g(t) = t$.

Step 3. $K_\varepsilon'(t) \leq (\mathcal{A}L_\varepsilon)(t) - (\mathcal{A}^2L_\varepsilon)(t) - 4(\mathcal{A}D_\varepsilon)(t)$ *for almost every* $t > 0$.

The proof closely resembles the one of [9, Corollary 4.7]. Fix $t > 0$ and for every $\delta > 0$ let $g_{t,\delta}$ be defined by

$$g_{t,\delta}(s) := \begin{cases} 0 & \text{if } s \leq t \\ \frac{(s-t)^2}{2\delta} & \text{if } s \in [t, t+\delta] \\ s - t - \frac{\delta}{2} & \text{if } s \geq t + \delta\,. \end{cases} \tag{3.21}$$

The claim follows by considering $g = g_{t,\delta}$ in (3.14) and sending $\delta \to 0$.

Step 4. (3.12) *holds true.*

In view of Step 2 and (3.6), $\mathcal{A}^2 Q_\varepsilon$ and K_ε are absolutely continuous on the intervals $[0,T]$ for every $T>0$. Therefore, we can differentiate E_ε and, using Step 3, (3.8), and the very definition of L_ε in (3.1), we get

$$\begin{aligned} E_\varepsilon' &= K_\varepsilon' + (\mathcal{A}^2 Q_\varepsilon)' = K_\varepsilon' + \mathcal{A}^2 Q_\varepsilon - \mathcal{A} Q_\varepsilon \\ &\le \mathcal{A} L_\varepsilon - \mathcal{A}^2 L_\varepsilon - 4\mathcal{A} D_\varepsilon + \mathcal{A}^2 Q_\varepsilon - \mathcal{A} Q_\varepsilon = -\mathcal{A}^2 D_\varepsilon - 3\mathcal{A} D_\varepsilon \le 0\,, \end{aligned}$$

and hence $E_\varepsilon(t) \le E_\varepsilon(0)$ for a.e. $t \ge 0$. Moreover, by the very definition of E_ε and L_ε, together with (2.3), Step 2, and (3.15), it follows that

$$\begin{aligned} E_\varepsilon(0) &= K_\varepsilon(0) + (\mathcal{A}^2 Q_\varepsilon)(0) = \frac{1}{2}\|u_\varepsilon^1\|^2 + (\mathcal{A}^2 Q_\varepsilon)(0) \\ &\le \frac{1}{2}\|u_\varepsilon^1\|^2 + (\mathcal{A}^2 L_\varepsilon)(0) \le \frac{1}{2}\|u_\varepsilon^1\|^2 + (\mathcal{A} L_\varepsilon)(0) + R_\varepsilon \\ &= \frac{1}{2}\|u_\varepsilon^1\|^2 + \mathcal{G}_\varepsilon(v_\varepsilon) + R_\varepsilon < C_1'\,, \end{aligned} \tag{3.22}$$

where C_1' depends on $\|u^0\|_0$, $\|u^1\|$, and C_1 in (1.9). This concludes the proof of (3.12). Finally, by using (3.16) and (2.4) in the last line in (3.22), we obtain that, if $\varepsilon\|u_\varepsilon^1\|_0 \to 0$ as $\varepsilon \to 0+$, then

$$E_\varepsilon(0) \le \frac{1}{2}\|u_\varepsilon^1\|^2 + \frac{1}{2}Q(u^0) + r_\varepsilon + R_\varepsilon \le \frac{1}{2}\|u_\varepsilon^1\|^2 + \frac{1}{2}Q(u^0) + \widetilde{r}_\varepsilon\,,$$

where $\widetilde{r}_\varepsilon \to 0$ as $\varepsilon \to 0+$. Therefore also (3.13) holds true. □

4. Proof of Theorem 1.7

Before proving Theorem 1.7, we introduce a suitable subset of $\mathcal{V}^{0,2}_{\varepsilon,T}$, which is dense in $\{\eta \in C^2_c((0,T);V_T) : \eta(t) \in V_t \text{ for every } t \in (0,T)\}$. For every $\varepsilon > 0$ and $T > 0$, we define $\mathcal{D}_T$ as the set of all functions $\eta \in C^2_c((0,T);V_T)$ of the form

$$\eta(t) = \sum_{i=2}^{N-2} \sum_{j=0}^{2} \varphi_{i,j}(t) h_{i,j}$$

for some $N \in \mathbb{N}$, $0 = t_0 < t_1 < \ldots < t_N = T$, $\varphi_{i,j} \in C^2(\mathbb{R})$ with $\operatorname{supp} \varphi_{i,j} \subset [t_{i-1}, t_{i+1}]$, and $h_{i,j} \in V_{t_{i-1}}$ for $i = 2, \ldots, N-2$ and $j = 0,1,2$. By (H2) the last two conditions imply that $\eta(t) \in V_t$ for every $t \in [0,T]$. We now prove the density.

Lemma 4.1. *Let $T > 0$. For every $\eta \in C^2_c((0,T);V_T)$. with $\eta(t) \in V_t$ for every $t \in (0,T)$, there exists a sequence $\{\eta_N\} \subset \mathcal{D}_T$ such that*

$$\|\eta - \eta_N\|_{C^2([0,T];V_T)} \to 0 \qquad \text{as } N \to \infty\,. \tag{4.1}$$

Proof. Let $\eta \in C^2_c((0,T);V_T)$, with $\eta(t) \in V_t$ for every $t \in (0,T)$. In order to construct the approximating sequence $\{\eta_N\} \subset \mathcal{D}_T$ we make use of quintic Hermite interpolants, that we construct here through the Bernstein polynomials. Let $N \in \mathbb{N}$ and set $t_i = i\frac{T}{N}$ for $i = 0,1,\ldots,N$. Fix $i = 0,\ldots,N$. For $n \in \mathbb{N}$, we define the Bernstein polynomials in the interval $[t_i, t_{i+1}]$ as

$$B^i_{k,n}(t) := \begin{cases} \binom{n}{k}(t-t_i)^k (t_{i+1}-t)^{n-k} & \text{for } k = 0,\ldots,n\,, \\ 0 & \text{for } k < 0 \text{ or } k > n\,, \end{cases}$$

and we define the polynomials of the spline basis as follows

$$\psi_{i,0,+}(t) := \frac{N^5}{T^5}(B^i_{0,5}(t)+B^i_{1,5}(t)+B^i_{2,5}(t))\,, \qquad \psi_{i,0,-}(t) := \frac{N^5}{T^5}(B^i_{3,5}(t)+B^i_{4,5}(t)+B^i_{5,5}(t))\,,$$
$$\psi_{i,1,+}(t) := \frac{N^4}{5T^4}(B^i_{1,5}(t)+2B^i_{2,5}(t))\,, \qquad \psi_{i,1,-}(t) := -\frac{N^4}{5T^4}(2B^i_{3,5}(t)+B^i_{4,5}(t))\,,$$
$$\psi_{i,2,+}(t) := \frac{N^3}{20T^3}B^i_{2,5}(t)\,, \qquad \psi_{i,2,-}(t) := \frac{N^3}{20T^3}B^i_{3,5}(t)\,.$$

By construction, it is easy to see that

$$\psi_{i,0,+}(t)+\psi_{i,0,-}(t)=1 \qquad \text{for } t\in[t_i,t_{i+1}]\,. \tag{4.2}$$

Moreover, by using that

$$\frac{\mathrm{d}}{\mathrm{d}t}B^i_{k,n}(t) = n(B^i_{k-1,n-1}(t)-B^i_{k,n-1}(t))\,,$$

one can easily show that

$$-\frac{T}{N}\psi'_{i,0,+}(t)+\psi'_{i,1,+}(t)+\psi'_{i,1,-}(t)=1\,, \tag{4.3}$$

$$-\frac{T^2}{2N^2}\psi''_{i,0,+}(t)+\frac{T}{N}\psi''_{i,1,-}(t)+\psi''_{i,2,+}(t)+\psi''_{i,2,-}(t)=1\,. \tag{4.4}$$

For every $i=1,\dots,N-1$ and $j=0,1,2$ we set

$$\varphi_{i,j}(t) := \begin{cases} \psi_{i-1,j,-}(t) & \text{if } t\in[t_{i-1},t_i]\,,\\ \psi_{i,j,+}(t) & \text{if } t\in[t_i,t_{i+1}]\,,\\ 0 & \text{elsewhere}\,. \end{cases}$$

Finally, we define the function

$$\eta_N(t) := \sum_{i=2}^{N-2}\big(\varphi_{i,0}(t)\eta(t_{i-1})+\varphi_{i,1}(t)\eta'(t_{i-1})+\varphi_{i,2}(t)\eta''(t_{i-1})\big)\,.$$

By (H2) we have $\eta(t_{i-1})$, $\eta'(t_{i-1})$, $\eta''(t_{i-1})\in V_{t_{i-1}}$, hence $\eta_N\in\mathcal{D}_T$ for every $N\in\mathbb{N}$.

It remains to prove (4.1). Let $t\in\operatorname{supp}\eta$. For $N\in\mathbb{N}$ large enough there exists $i=2,\dots,N-3$ such that $t\in[t_i,t_{i+1})$, so that by (4.2) and by the very definition of η_N, $\psi_{i,1,\pm}$, and $\psi_{i,2,\pm}$, we have

$$\begin{aligned}\|\eta_N(t)-\eta(t)\|_T &\le \|\psi_{i,0,+}(t)\eta(t_{i-1})+\psi_{i,0,-}(t)\eta(t_i)-\eta(t)\|_T+\mathrm{O}(1/N)\\ &\le \|\eta(t_{i-1})-\eta(t)\|_T+\|\eta(t_i)-\eta(t)\|_T+\mathrm{O}(1/N),\end{aligned}$$

and hence η_N converges to η in V_T uniformly in $[0,T]$. Analogously, by (4.3), we obtain

$$\begin{aligned}\|\eta'_N(t)-\eta'(t)\|_T &\le \Big\|\psi'_{i,0,+}(t)\eta(t_{i-1})+\psi'_{i,0,-}(t)\eta(t_i)+\frac{T}{N}\psi'_{i,0,+}(t)\eta'(t)\Big\|_T\\ &\quad+\|\psi'_{i,1,+}\|_{L^\infty}\|\eta'(t_{i-1})-\eta'(t)\|_T+\|\psi'_{i,1,-}\|_{L^\infty}\|\eta'(t_i)-\eta'(t)\|_T+\mathrm{O}(1/N)\,,\end{aligned}$$

which, using that (by (4.2)) the first term on the right-hand side is bounded by

$$\frac{T}{N}\|\psi'_{i,0,+}(t)\|_{L^\infty}\Big\|-\frac{\eta(t_i)-\eta(t_{i-1})}{T/N}+\eta'(t)\Big\|_T\,,$$

implies that η'_N converges to η' in V_T uniformly in $[0,T]$. Analogously, using (4.2), (4.3), and (4.4), one can show that η''_N converges uniformly to η'' in $[0,T]$. □

Lemma 4.2. *Let $\varepsilon > 0$ and $T > 0$. For every $\eta \in C_c^2((0,T);V_T)$, with $\eta(t) \in V_t$ for every $t \in (0,T)$, we have*

$$\int_0^T e^{-s/\varepsilon}\Big(\varepsilon^2\left(u_\varepsilon''(s),\eta''(s)\right) + a(u_\varepsilon(s),\eta(s))\Big)\,\mathrm{d}s = 0\,. \tag{4.5}$$

Proof. In view of Lemma 4.1, it is sufficient to prove (4.5) for $\eta \in \mathcal{D}_T$. The proof is analogous to the one of [9, Lemma 5.1]. Let $\delta \in [-1,1]$ and set $u_{\varepsilon,\delta} := u_\varepsilon + \delta\eta$. By construction, $u_{\varepsilon,\delta} \in \mathcal{V}_T^{0,2}$ and, since η has compact support, also the initial conditions are satisfied. Therefore $u_{\varepsilon,\delta} \in \mathcal{V}^{0,2}(u^0,u_\varepsilon^1)$, and, again by construction, $\mathcal{F}_\varepsilon(u_{\varepsilon,\delta})$ is finite. Then the Euler-Lagrange equation (4.5) easily follows by differentiating $\mathcal{F}_\varepsilon(u_{\varepsilon,\delta})$ with respect to δ at $\delta = 0$. □

We are now in a position to prove Theorem 1.7.

Proof of Theorem 1.7. Let us fix a sequence $\{\varepsilon_n\} \subset (0,1)$, with $\varepsilon_n \to 0$ as $n \to \infty$. We divide the proof into five steps.

Step 1: There exist a subsequence, not relabeled, and a function $u \in \mathcal{V}^{0,1}$ such that

$$u_{\varepsilon_n} \rightharpoonup u \quad \textit{in } \mathcal{W}_T^{0,1} \qquad \textit{for every } T > 0\,. \tag{4.6}$$

Moreover, $u' \in L^\infty((0,\infty);H)$ and $u \in L^\infty((0,T);V_T)$ for every $T > 0$.

Let $T > 0$. By (1.15) and (1.16),

$$\sup_{n\in\mathbb{N}} \|u_{\varepsilon_n}\|_{H^1((0,T);H)} < \infty\,.$$

This inequality, together with (H4) and (1.14), implies that there exists $C_T < \infty$ such that

$$\nu_0\|u_{\varepsilon_n}\|^2_{L^2((0,T);V_T)} \le \int_0^T Q(u_{\varepsilon_n}(t))\,\mathrm{d}t + \lambda_0\|u_{\varepsilon_n}\|^2_{L^2((0,T);H)} \le C_T\,.$$

As a result $\{u_{\varepsilon_n}\}$ is equibounded in $\mathcal{W}_T^{0,1}$ and hence there exist a subsequence, not relabeled, and a function $u \in \mathcal{W}_T^{0,1}$ such that $u_{\varepsilon_n} \rightharpoonup u$ weakly in $\mathcal{W}_T^{0,1}$. Moreover, since $\{u_{\varepsilon_n}\} \subset \mathcal{V}_T^{0,2} \subset \mathcal{V}_T^{0,1}$ and $\mathcal{V}_T^{0,1}$ is a closed subspace of $\mathcal{W}_T^{0,1}$, we have that $u \in \mathcal{V}_T^{0,1}$. By the arbitrariness of T, the function u belongs to $\mathcal{V}^{0,1}$ and (4.6) holds true. Furthermore, in view of (4.6), inequality (1.16) implies $u' \in L^\infty((0,\infty);H)$ and (1.15) gives $u \in L^\infty((0,T);V_T)$ for every $T > 0$.

Step 2: Let $T > 0$. For every $\psi \in C_c^\infty((0,T);V_T)$, with $\psi(t) \in V_t$ for every $t \in (0,T)$, we have

$$\int_0^T (u_{\varepsilon_n}'(t), \varepsilon_n^2\psi'''(t) + 2\varepsilon_n\psi''(t) + \psi'(t))\,\mathrm{d}t = \int_0^T a(u_{\varepsilon_n}(t),\psi(t))\,\mathrm{d}t\,. \tag{4.7}$$

The claim follows by considering $\eta(t) = e^{t/\varepsilon_n}\psi(t)$ in (4.5) and integrating by parts.

Step 3: u is a weak solution of (1.7). By [4, Lemma 2.8], it is enough to prove the claim for $\psi \in C_c^\infty((0,T);V_T)$ with $\psi(t) \in V_t$ for every $t \in (0,T)$. In view of (4.6), one can pass to the limit as $n \to \infty$ in (4.7), thus obtaining (1.8).

Step 4: u satisfies (a) and (b). Since $u' \in L^\infty((0,\infty);H)$ and $u \in L^\infty((0,T);V_T)$ for every $T > 0$ by Step 1, property (a) follows from Step 3, thanks to Remark 1.4. Claim (b) is obtained by combining (a), (1.9), and (4.6), together with the fact that $u_{\varepsilon_n} \in \mathcal{V}^{0,1}(u^0,u^1_{\varepsilon_n})$.

Step 5: u satisfies the energy inequality (1.17). By using [9, Lemma 6.1] and (3.13), one can argue as in [9, Section 6] to obtain that the energy inequality (1.17) is satisfied for almost every $t > 0$. Actually, in view of (a), this inequality is satisfied for every $t > 0$. □

Acknowledgments

This material is based on work supported by the Italian Ministry of Education, University, and Research through the Project "Calculus of Variations" (PRIN 2015). The authors are members of the Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM).

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