Variational fracture with Neumann boundary conditions

Chris Larsen

WPI

January 31, 2020

Problem:

Variational fracture: minimize elastic plus surface energy

$$E_F(u) := \frac{1}{2}\int_\Omega |\nabla u|^2 + \mathcal{H}^{N-1}(\Gamma \cup S_u),$$

Neumann boundary condition, with elastic energy, minimize

$$E_N(u) := \frac{1}{2}\int_\Omega |\nabla u|^2 - \int_{\partial_N \Omega} fu.$$

Naturally, it would seem, for variational fracture with Neumann condition, minimize

$$E_{FN}(u) := \frac{1}{2}\int_\Omega |\nabla u|^2 + \mathcal{H}^{N-1}(\Gamma \cup S_u) - \int_{\partial_N \Omega} fu.$$

Not possible (?). Issue is global minimization?

On the other hand:

Suppose we have a minimizer u (static or quasi-static) of

$$E_F(v) = \frac{1}{2}\int_\Omega |\nabla v|^2 + \mathcal{H}^{N-1}(\Gamma \cup S_v),$$

over $v \in SBV(\Omega)$ satisfying a certain Dirichlet condition

On the other hand:

Suppose we have a minimizer u (static or quasi-static) of

$$E_F(v) = \frac{1}{2}\int_\Omega |\nabla v|^2 + \mathcal{H}^{N-1}(\Gamma \cup S_v),$$

over $v \in SBV(\Omega)$ satisfying a certain Dirichlet condition, and u happens to satisfy

$$\partial_\nu u = f \text{ on } \partial_N \Omega.$$

On the other hand:

Suppose we have a minimizer u (static or quasi-static) of

$$E_F(v) = \frac{1}{2}\int_\Omega |\nabla v|^2 + \mathcal{H}^{N-1}(\Gamma \cup S_v),$$

over $v \in SBV(\Omega)$ satisfying a certain Dirichlet condition, and u happens to satisfy

$$\partial_\nu u = f \text{ on } \partial_N \Omega.$$

Is this not a solution of the Neumann problem?

On the other hand:

Suppose we have a minimizer u (static or quasi-static) of

$$E_F(v) = \frac{1}{2}\int_\Omega |\nabla v|^2 + \mathcal{H}^{N-1}(\Gamma \cup S_v),$$

over $v \in SBV(\Omega)$ satisfying a certain Dirichlet condition,
and u happens to satisfy

$$\partial_\nu u = f \text{ on } \partial_N\Omega.$$

Is this not a solution of the Neumann problem?

Seems to be – main question: how can we find solutions, given f?
Specifically, if we look for u satisfying

$$u = 0 \text{ on } \partial_D\Omega, \;\; \partial_\nu u = f \text{ on } \partial_N\Omega,$$

where $\partial\Omega = \partial_D\Omega \cup \partial_N\Omega$.

1-Dimension

$\Omega = (0,1)$. Solve $u'' = 0$ with $u(0) = 0$, $\partial_\nu u = f$ on $\partial_N \Omega = \{1\}$, get $u(1)$, then minimize

$$\frac{1}{2}\int_0^1 (v')^2 + \mathcal{H}^0(S_v),$$

subject to $v(0) = 0$, $v(1) = u(1)$.

1-Dimension

$\Omega = (0, 1)$. Solve $u'' = 0$ with $u(0) = 0$, $\partial_\nu u = f$ on $\partial_N \Omega = \{1\}$, get $u(1)$, then minimize

$$\frac{1}{2}\int_0^1 (v')^2 + \mathcal{H}^0(S_v),$$

subject to $v(0) = 0$, $v(1) = u(1)$.

If $\mathcal{H}^0(S_v) = 0$, then $v = u$, satisfies minimality and Neumann condition.

1-Dimension

$\Omega = (0,1)$. Solve $u'' = 0$ with $u(0) = 0$, $\partial_\nu u = f$ on $\partial_N\Omega = \{1\}$, get $u(1)$, then minimize

$$\frac{1}{2}\int_0^1 (v')^2 + \mathcal{H}^0(S_v),$$

subject to $v(0) = 0$, $v(1) = u(1)$.

If $\mathcal{H}^0(S_v) = 0$, then $v = u$, satisfies minimality and Neumann condition.

If $\mathcal{H}^0(S_v) \geq 1$, cannot solve Neumann problem, material has failed.

1-Dimension

$\Omega = (0,1)$. Solve $u'' = 0$ with $u(0) = 0$, $\partial_\nu u = f$ on $\partial_N \Omega = \{1\}$, get $u(1)$, then minimize

$$\frac{1}{2}\int_0^1 (v')^2 + \mathcal{H}^0(S_v),$$

subject to $v(0) = 0$, $v(1) = u(1)$.

If $\mathcal{H}^0(S_v) = 0$, then $v = u$, satisfies minimality and Neumann condition.

If $\mathcal{H}^0(S_v) \geq 1$, cannot solve Neumann problem, material has failed.

In higher dimensions, it is more interesting.

Higher dimensions:

Variationally, looking for u that solves *two* problems:

i) Minimize

$$E_F(v) := \frac{1}{2}\int_\Omega |\nabla v|^2 + \mathcal{H}^{N-1}(S_u \cup S_v),$$

over $v \in SBV(\Omega)$ with $v = 0$ on $\partial_D\Omega$ and $v = u$ on $\partial_N\Omega$, and

ii) Minimize

$$E_N(v) := \frac{1}{2}\int_\Omega |\nabla v|^2 - \int_{\partial_N\Omega} fv$$

over $v \in SBV(\Omega)$ with $S_v \subset S_u$, $v = 0$ on $\partial_D\Omega$.

Quick note on global minimization

What is wrong with global minimization?

Static:

Quasi-static:

Quick note on global minimization

What is wrong with global minimization?

Static:

- nothing
- study of certain stable states
- no implication that other states are not stable

Quasi-static:

Quick note on global minimization

What is wrong with global minimization?

Static:

- nothing
- study of certain stable states
- no implication that other states are not stable

Quasi-static:

- can imply no other states are stable
- can still lead to progress, e.g., new methods
- can correspond to other stable states

Existence?

We can start similarly to 1-D:

Preexisting Γ_0; Minimize

$$E_N(u) := \frac{1}{2}\int_\Omega |\nabla u|^2 - \int_{\partial_N \Omega} fu,$$

$u \in H^1_{0(D)}(\Omega \setminus \Gamma_0)$. Get u_1. Then

minimize

$$E_F(u) := \frac{1}{2}\int_\Omega |\nabla u|^2 + \mathcal{H}^{N-1}(\Gamma_0 \cup S_u),$$

$u \in SBV_{0(D)}(\Omega)$, $u = u_1$ on $\partial_N \Omega$, get v_1. $\Gamma_1 := \Gamma_0 \cup S_{v_1}$. Repeat:

u_n minimizes E_N over $u \in H^1_{0(D)}(\Omega \setminus \Gamma_{n-1})$,

v_n minimizes $E_{F(\Gamma_{n-1})}$ with second term $\mathcal{H}^{N-1}(\Gamma_{n-1} \cup S_u)$, over $u \in SBV_{0(D)}(\Omega)$, $u = u_n$ on $\partial_N \Omega$. $\Gamma_n := \Gamma_{n-1} \cup S_{v_n}$.

Idea:

The material fails, or:

$$u_n \rightharpoonup u_\infty, \quad v_n \rightharpoonup v_\infty, \quad \Gamma_\infty := \cup \Gamma_n,$$

Idea:

The material fails, or:

$$u_n \rightharpoonup u_\infty, \quad v_n \rightharpoonup v_\infty, \quad \Gamma_\infty := \cup \Gamma_n,$$

u_∞ minimizes E_N over $u \in H^1_{0(D)}(\Omega \setminus \Gamma_\infty)$

v_∞ minimizes $E_{F(\Gamma_\infty)}$ over $u \in SBV_{0(D)}(\Omega)$ with $u = u_\infty$ on $\partial_N \Omega$

$u_\infty = v_\infty$.

Idea:

The material fails, or:

$$u_n \rightharpoonup u_\infty, \quad v_n \rightharpoonup v_\infty, \quad \Gamma_\infty := \cup \Gamma_n,$$

u_∞ minimizes E_N over $u \in H^1_{0(D)}(\Omega \setminus \Gamma_\infty)$

v_∞ minimizes $E_{F(\Gamma_\infty)}$ over $u \in SBV_{0(D)}(\Omega)$ with $u = u_\infty$ on $\partial_N \Omega$

$u_\infty = v_\infty$.

In fact, enough to just work with u_∞, show it satisfies both minimality conditions.

First step: for n large, u_n almost has the minimality of v_n.

Before that, what is failure?

Failure

If a piece of the Neumann boundary breaks off at some stage n or in the limit, material has failed, solutions blow up, Neumann problem has no solution.

Lack of failure:

- u_n and v_n bounded ($\Rightarrow \mathcal{H}^{N-1}(\Gamma_\infty) < \infty$)
- $\Gamma_\infty \cap \partial_N\Omega = \emptyset$ (strengthened to $\mathrm{dist}(\Gamma_\infty, \partial_N\Omega) > 0$)

So, failure or $u_n \rightharpoonup u_\infty$. Show u_∞ has desired properties.

Theorem

If the material does not fail under the boundary load f*, then there exists* $u_\infty \in SBV(\Omega)$ *such that*

$$u_n \rightharpoonup u_\infty \text{ (up to a subsequence)}$$

$$\Gamma_\infty := \cup \Gamma_n,$$

u_∞ *minimizes* E_N *over*

$$\{u \in SBV(\Omega) : S_u \subset \Gamma_\infty, u = 0 \text{ on } \partial_D \Omega\},$$

and it minimizes $E_{F(\Gamma_\infty)}$ *over*

$$\{u \in SBV(\Omega) : u = u_\infty \text{ on } \partial\Omega\}.$$

Easy: v_∞ would minimize E_F – variations for v_∞ are variations for v_n
Pretty easy: u_∞ minimizes E_F
Less easy: u_∞ minimizes E_N – variations for u_∞ are not variations for u_n.

Lemma

u_∞ minimizes

$$E_F(u) := \frac{1}{2}\int_\Omega |\nabla u|^2 + \mathcal{H}^{N-1}(\Gamma_\infty \cup S_u)$$

over $u \in SBV(\Omega)$, $u = 0$ on $\partial_D\Omega$, $u = u_\infty$ on $\partial_N\Omega$.

Furthermore, for all $\psi \in SBV(\Omega)$ satisfying $\psi = 0$ on $\partial\Omega$ with $S_\psi \subset \Gamma_\infty$, we have

$$\int_\Omega \nabla u_n \cdot \nabla\psi \to 0.$$

Why not obvious: u_∞, not v_∞? Plus, Neumann sieve-type problems ($\mathrm{Cap}(\Gamma_\infty \setminus \Gamma_n) \not\to 0$). But u_n almost has the minimality of v_n.

Proof:

From the minimality of u_n and v_{n-1},

$$\frac{1}{2}\int_\Omega |\nabla u_n|^2 - \int_{\partial_N\Omega} fu_n \leq \frac{1}{2}\int_\Omega |\nabla v_{n-1}|^2 - \int_{\partial_N\Omega} fv_{n-1},$$

$$\frac{1}{2}\int_\Omega |\nabla v_{n-1}|^2 - \int_{\partial_N\Omega} fv_{n-1} \leq$$

$$\frac{1}{2}\int_\Omega |\nabla u_{n-1}|^2 - \int_{\partial_N\Omega} fu_{n-1} - \mathcal{H}^{N-1}(\Gamma_{n-1} \setminus \Gamma_{n-2}).$$

Furthermore, since u_n is an admissible variation for its minimality, we get

$$\int_\Omega \nabla u_n \cdot \nabla u_n = \int_{\partial_N\Omega} fu_n,$$

and so

$$\frac{1}{2}\int_\Omega |\nabla u_n|^2 - \int_{\partial_N\Omega} fu_n = -\frac{1}{2}\int_\Omega |\nabla u_n|^2 = -\frac{1}{2}\int_{\partial_N\Omega} fu_n. \qquad (1)$$

By monotonicity and boundedness, all energies of u_n converge. Then the elastic energies of v_n also converge, to the same limit:

$$\frac{1}{2}\int_\Omega |\nabla u_n|^2 - \int_{\partial_N\Omega} f u_n$$

$$\leq \frac{1}{2}\int_\Omega |\nabla v_{n-1}|^2 - \int_{\partial_N\Omega} f v_{n-1}$$

$$\leq \frac{1}{2}\int_\Omega |\nabla u_{n-1}|^2 - \int_{\partial_N\Omega} f u_{n-1} - \mathcal{H}^{N-1}(\Gamma_{n-1}\setminus\Gamma_{n-2}).$$

So

$$\mathcal{H}^{N-1}(\Gamma_\infty\setminus\Gamma_n)\to 0$$

and

$$\frac{1}{2}\int_\Omega |\nabla u_n|^2 - \frac{1}{2}\int_\Omega |\nabla v_n|^2 \to 0.$$

Suppose u_∞ does not minimize E_F, i.e., $\exists\psi = 0$ on $\partial\Omega$ s.t.

$$\int_\Omega \nabla u_\infty \cdot \nabla\psi + \frac{1}{2}\int_\Omega |\nabla\psi|^2 + \mathcal{H}^{N-1}(S_\psi \setminus \Gamma_\infty) =: \eta < 0.$$

But this is the limit of

$$\int_\Omega \nabla u_{n_k} \cdot \nabla\psi + \frac{1}{2}\int_\Omega |\nabla\psi|^2 + \mathcal{H}^{N-1}(S_\psi \setminus \Gamma_{n_k}),$$

which means that, for k large enough,

$$\frac{1}{2}\int_\Omega |\nabla(u_{n_k} + \psi)|^2 + \mathcal{H}^{N-1}(\Gamma_{n_k} \cup S_\psi) < \frac{1}{2}\int_\Omega |\nabla v_{n_k}|^2 + \mathcal{H}^{N-1}(\Gamma_{n_k}),$$

contradicting the minimality of v_{n_k} since $u_{n_k} + \psi$ is a competitor for v_{n_k}.

Similarly, if

$$\psi = 0 \text{ on } \partial\Omega,$$

$$S_\psi \subset \Gamma_\infty,$$

and

$$\int_\Omega \nabla u_n \cdot \nabla \psi \to \eta \neq 0,$$

then

$$\frac{1}{2}\int_\Omega |\nabla(u_n + \lambda\psi)|^2 + \mathcal{H}^{N-1}(\Gamma_n \cup S_\psi) < \frac{1}{2}\int_\Omega |\nabla v_n|^2 + \mathcal{H}^{N-1}(\Gamma_n),$$

for $|\lambda|$ small enough, and with the correct sign, again contradicting the minimality of v_n for n large enough.

□

Lemma

u_∞ minimizes

$$E_N(u) := \frac{1}{2}\int_\Omega |\nabla u|^2 - \int_{\partial_N \Omega} fu$$

over $u \in SBV(\Omega)$ with $u = 0$ on $\partial_D \Omega$ and $S_u \subset \Gamma_\infty$.

Lemma

u_∞ minimizes

$$E_N(u) := \frac{1}{2}\int_\Omega |\nabla u|^2 - \int_{\partial_N\Omega} fu$$

over $u \in SBV(\Omega)$ with $u = 0$ on $\partial_D\Omega$ and $S_u \subset \Gamma_\infty$.

Issue: $\mathrm{Cap}(\Gamma_\infty \setminus \Gamma_n) \not\to 0$, solution to the Dirichlet problem (min E_F) does not care, but solution to the Neumann problem (min E_N) does care.

Lemma

u_∞ minimizes

$$E_N(u) := \frac{1}{2}\int_\Omega |\nabla u|^2 - \int_{\partial_N \Omega} fu$$

over $u \in SBV(\Omega)$ with $u = 0$ on $\partial_D \Omega$ and $S_u \subset \Gamma_\infty$.

Issue: $\mathrm{Cap}(\Gamma_\infty \setminus \Gamma_n) \not\to 0$, solution to the Dirichlet problem (min E_F) does not care, but solution to the Neumann problem (min E_N) does care.

Intuition: not possible. To prove: convert variations for Neumann problem to Dirichlet problem.

Lemma

u_∞ minimizes

$$E_N(u) := \frac{1}{2}\int_\Omega |\nabla u|^2 - \int_{\partial_N \Omega} fu$$

over $u \in SBV(\Omega)$ with $u = 0$ on $\partial_D\Omega$ and $S_u \subset \Gamma_\infty$.

Issue: $\mathrm{Cap}(\Gamma_\infty \setminus \Gamma_n) \not\to 0$, solution to the Dirichlet problem (min E_F) does not care, but solution to the Neumann problem (min E_N) does care.

Intuition: not possible. To prove: convert variations for Neumann problem to Dirichlet problem.

Proof:
Let $\psi \in SBV(\Omega)$ with $\psi = 0$ on $\partial_D\Omega$ and $S_\psi \subset \Gamma_\infty$. Suppose $E_N(u_\infty + \psi) < E_N(u_\infty)$, so

$$\int_\Omega \nabla u_\infty \cdot \nabla \psi = \gamma + \int_{\partial_N\Omega} f\psi,$$

for some $\gamma < 0$.

Choose $\phi \in H^1(\Omega)$ such that $\phi = 0$ on $\partial_D\Omega$ and $\phi = \psi$ on $\partial_N\Omega$.

From the minimality of u_n, we have

$$\int_\Omega \nabla u_n \cdot \nabla\phi = \int_{\partial_N\Omega} f\phi.$$

Since $\psi - \phi \in SBV(\Omega)$ with $(\psi - \phi) = 0$ on $\partial\Omega$ and $S_{\psi-\phi} \subset \Gamma_\infty$, we have

$$0 \leftarrow \int_\Omega \nabla u_n \cdot \nabla(\psi - \phi) \to \gamma + \int_{\partial_N\Omega} f\psi - \int_{\partial_N\Omega} f\phi = \gamma,$$

a contradiction.

□

This proves the theorem.

We also get, from monotonicity,

$$E_{FN}(u_\infty, \Gamma_\infty) \leq E_{FN}(u_0, \Gamma_0).$$

Also have...

Immediate that we can replace Γ_∞ with S_{u_∞}:

Lemma

u_∞ minimizes

$$E_F(u) := \frac{1}{2}\int_\Omega |\nabla u|^2 + \mathcal{H}^{N-1}(S_u \setminus S_{u_\infty})$$

over $u \in SBV(\Omega)$, $u = 0$ on $\partial_D\Omega$, $u = u_\infty$ on $\partial_N\Omega$.

Lemma

u_∞ minimizes

$$E_N(u) := \frac{1}{2}\int_\Omega |\nabla u|^2 - \int_{\partial_N\Omega} fu$$

over $u \in SBV(\Omega)$ with $u = 0$ on $\partial_D\Omega$ and $S_u \subset S_{u_\infty}$.

Since this has no effect on the energy of u_∞, but increases the energy of competitors or limits competitors.

In the end...

u_∞ does minimize

$$E_{FN}(u) := \frac{1}{2}\int_\Omega |\nabla u|^2 + \mathcal{H}^{N-1}(\Gamma_\infty \cup S_u) - \int_{\partial_N\Omega} fu$$

In the end...

u_∞ does minimize

$$E_{FN}(u) := \frac{1}{2}\int_\Omega |\nabla u|^2 + \mathcal{H}^{N-1}(\Gamma_\infty \cup S_u) - \int_{\partial_N\Omega} fu$$

over

$$\{u \in SBV(\Omega) : u = u_\infty \text{ on } \partial\Omega\}$$

$$\bigcup \{u \in SBV(\Omega) : S_u \subset \Gamma_\infty, u = 0 \text{ on } \partial_D\Omega\}$$

That is, competitors are not allowed to simultaneously vary both their boundary data and the crack. But the Griffith idea is cracks compete with elastic energy, not boundary data...

There is no compelling reason to allow both to vary at the same time.

In the end...

u_∞ does minimize

$$E_{FN}(u) := \frac{1}{2}\int_\Omega |\nabla u|^2 + \mathcal{H}^{N-1}(\Gamma_\infty \cup S_u) - \int_{\partial_N \Omega} fu$$

over

$$\{u \in SBV(\Omega) : u = u_\infty \text{ on } \partial\Omega\}$$

$$\bigcup \{u \in SBV(\Omega) : S_u \subset \Gamma_\infty, u = 0 \text{ on } \partial_D \Omega\}$$

That is, competitors are not allowed to simultaneously vary both their boundary data and the crack. But the Griffith idea is cracks compete with elastic energy, not boundary data...

There is no compelling reason to allow both to vary at the same time.

Natural question: Why E_F minimality with Dirichlet data? Variations are $\phi \in SBV_0(\Omega)$. Is this class too big or too small?

An alternative

Instead, could try to minimize

$$\left\{ \frac{1}{2}\int_\Omega |\nabla u|^2 + \mathcal{H}^{N-1}(S_u) : u \text{ minimizes} \right.$$

$$\left. v \mapsto \frac{1}{2}\int_\Omega |\nabla v|^2 - \int_{\partial_N \Omega} fv, \; v \text{ in } H^1_{0(D)}(\Omega \setminus S_u) \right\}.$$

An alternative

Instead, could try to minimize

$$\left\{ \frac{1}{2}\int_\Omega |\nabla u|^2 + \mathcal{H}^{N-1}(S_u) : u \text{ minimizes} \right.$$

$$\left. v \mapsto \frac{1}{2}\int_\Omega |\nabla v|^2 - \int_{\partial_N \Omega} fv, \;\; v \text{ in } H^1_{0(D)}(\Omega \setminus S_u) \right\}.$$

Or (?)

$$\left\{ \frac{1}{2}\int_\Omega |\nabla u|^2 + \mathcal{H}^{N-1}(S_u) : u \text{ minimizes} \right.$$

$$\left. v \mapsto \frac{1}{2}\int_\Omega |\nabla v|^2 - \int_{\Omega} fv, \;\; v \text{ in } H^1_{0(D)}(\Omega \setminus S_u) \right\}.$$

Thank you

Thank you

and

Happy Birthday Gianni!