Finite-strain viscoelasticity with temperature coupling *

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Calculus of Variations and Applications

A conference to celebrate

Gianni Dal Maso's 65th birthday

January 27 – February 1, 2020

* joint work with Thomás Roubíček (WIAS #2584, March 2019)

Overview

1. **Prologue**

2. Finite-strain elasiticity and temperature

3. Three tools

4. The existence result

5. Sketch of proof

1. Prologue

Happy birthday to Gianni . . .

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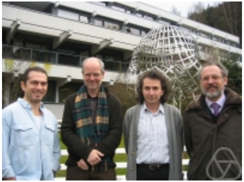
First meeting: MFO July 7–13, 1996 CoV (Ambrosio, Hélein, Müller)

True encounter: Submission January 12, 2004 to ARMA (by Gilles)
Dal Maso, Francfort, Toader: Quasistatic crack growth in finite elasticity

Oberwolfach meeting (March 2007)
Analysis and Numerics of Rate-Independent Processes



Interaction with Gianni and his school produced cross-fertilization

Gianni		my research
BV and CoV		parabolic systems
crack evolutions	1996	nonlinear elasticity
quasistatic evolution	$\approx$	rate-independent processes
vanishing-viscosity approach	$\approx$	Balanced-Viscosity solutions

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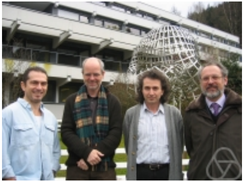

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Fundamental contributions to **finite-strain elasticity**:

Dal Maso, Negri, Percivale: Linearized elasticity as Γ-limit of finite elasticity 2002.
–"–, Francfort, Toader: Quasistatic crack growth in nonlinear elasticity, 2005.
–"–, Lazzaroni: Quasistatic crack growth in finite elasticity with non-interpenetration, 2010

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2. **Finite-strain elasiticity and temperature**

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Describe the interaction between

- viscoelastic deformation $y(t,\cdot) : \Omega \to \mathbb{R}^d$ and
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  - consider relatively slow processes $\Longrightarrow$ ignore inertial terms (quasistatic)
  - fully nonlinear obeying frame indifference (static and dynamic)
  - avoid non-selfinterpenetration (only locally via $\det\nabla y(t,x)>0$)
  - use a second grade material involving $\int_\Omega \mathcal{H}(\nabla^2 y)\,\mathrm{d}x$
  - coupling of temperature and deformation via
    - latent heat and
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Free energy functional $\mathcal{F}(y,\theta)=\int_\Omega \big\{\psi(\nabla y,\theta)+\mathcal{H}(\nabla^2 y)\big\}\,\mathrm{d}x$

Viscous dissipation potential $\mathcal{R}(y,\theta,\dot y)=\int_\Omega \zeta(\nabla y,\theta,\nabla\dot y)\,\mathrm{d}x$

Balance of linear momentum $0=\mathrm{D}_{\dot y}\mathcal{R}(y,\theta,\dot y)+\mathrm{D}_y\mathcal{F}(y,\theta)$

2. Finite-strain elasiticity and temperature

Balance of linear momentum $0 = \mathrm{D}_{\dot{y}}\mathcal{R}(y,\theta,\dot{y}) + \mathrm{D}_y\mathcal{F}(y,\theta)$

Heat equation with entropy $s(t,x) = -\partial_\theta\psi(\nabla y(t,x),\theta(t,x))$

$\theta\dot{s} + \operatorname{div}\boldsymbol{q} = \xi$ with heat flux $\boldsymbol{q} = -\mathbb{K}(\nabla y,\theta)\nabla\theta$

and viscous heating $\xi = \partial_{\nabla\dot{y}}\zeta(\nabla y,\theta,\nabla\dot{y}){:}\nabla\dot{y} \geq 0$

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Today we simplify notation by assuming

- no external forces or heat sources
- simple boundary conditions $y|_{\Gamma_{\mathrm{Dir}}} = y_{\mathrm{Dir}}$ and $\boldsymbol{q}\cdot\nu = 0$ (otherwise natural ones)

Total energy conservation holds with $e(F,\theta) = \psi(F,\theta) - \theta\partial_\theta\psi(F,\theta)$

$\mathcal{E}(y,\theta) = \int_\Omega \big\{ e(\nabla y,\theta) + \mathcal{H}(\nabla^2 y)\big\}\,\mathrm{d}x$

For smooth solutions we have $\mathcal{E}(y(t),\theta(t)) = \mathcal{E}(y(0),\theta(0))$ (energy conserv.).

Main assumption: splitting of free-energy density

$$\psi(F,\theta) = \varphi_{\rm el}(F) + \phi_{\rm cpl}(F,\theta)$$

- $\varphi_{\rm el}$ contains main mech. behavior $\varphi_{\rm el}(F) \geq c/(\det F)^\delta + c|F|^p - C$
- $\phi_{\rm cpl}$ is "relatively nice" with respect to F

Mechanical energy $\mathcal{M}(y) = \int_\Omega \big\{\varphi_{\rm el}(\nabla y) + \mathcal{H}(\nabla^2 y)\big\}\,{\rm d}x$

For smooth solutions one obtains the mechanical energy-dissipation balance

$$\mathcal{M}(y(t)) + \int_0^t \Big({\rm D}_{\dot y}\mathcal{R}(y,\theta,\dot y)[\dot y] + \int_\Omega \partial_F \phi_{\rm cpl}(\nabla y,\theta){:}\nabla \dot y\,{\rm d}x \Big){\rm d}s = \mathcal{M}(y(0))$$

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Strategy: Gain good control on y without using any properties of θ:

$$\mathcal{M}(y(t)) + \int_0^t \Big(c_{\rm Korn}\|\dot y\|^2_{\mathrm{H}^1} - \|\partial_F\phi_{\rm cpl}\|_{\mathrm{L}^2}\|\nabla \dot y\|_{\mathrm{L}^2}\Big)\,\mathrm{d}s \leq \mathcal{M}(y(0))$$

Using $|\partial_F\phi_{\rm cpl}(F,\theta)|^2 \leq K\varphi_{\rm el}(F)$ gives

$$\mathcal{M}(y(t)) \leq \mathcal{M}(y(0)) + \int_0^t \frac{K}{4c_{\rm Korn}}\mathcal{M}(y(s))\,\mathrm{d}s.$$

Message: We need some fundamental tools

- We need a generalized Korn inequality (Neff 2002, Pompe 2003)

$$\mathrm{D}_{\dot y}\mathcal{R}(y,\theta,\dot y)[\dot y] = \int_\Omega \partial_{\dot{\nabla} y}\zeta(\nabla y,\theta,\nabla\dot y){:}\nabla\dot y\,\mathrm{d}y \geq c_{\mathrm{Korn}}\int_\Omega |\nabla \dot y|^2\,\mathrm{d}x$$

for all relevant (y,θ) and $\dot y \in \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)$,
where "relevant" means $\mathcal{M}(y) \leq C_M$ and θ arbitrary.

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- From $\mathcal{M}(y) \leq C_M$ we derive invertibility (using Healey-Krömer 2009)

$$\|\nabla y\|_{\mathrm{C}^\alpha} \leq C \quad \text{and} \quad \det \nabla y(x) \geq c_{\mathrm{HeKr}} > 0.$$

2. Finite-strain elasiticity and temperature

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- To control the viscous heating we need to turn weak into strong convergence. This will be done by a chain-rule argument via Λ-convexity of $\mathcal{M}$ on sublevels: If $\mathcal{M}(y),\mathcal{M}(\widehat y) \leq C_M$ and $\|\nabla\widehat y-\nabla y\|_{\mathrm{L}^\infty} \leq \delta(C_M)$, then

$$\mathcal{M}(\widehat y) \geq \mathcal{M}(y) + \mathrm{D}\mathcal{M}(y)[\widehat y-y] - \Lambda(C_M)\|\nabla\widehat y-\nabla y\|^2_{\mathrm{L}^2}.$$

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3. Three tools: I. Invertibility via second gradient

$\Omega \subset \mathbb{R}^d$ bounded, Lipschitz domain and $\Gamma_{\mathrm{Dir}} \subset \partial\Omega$ with $\mathscr{H}^{d-1}(\Gamma_{\mathrm{Dir}}) > 0$

$\boldsymbol{Y} := \left\{ y \in \mathrm{H}^1(\Omega;\mathbb{R}^d) \,\middle|\, y|_{\Gamma_{\mathrm{Dir}}} = y_{\mathrm{Dir}} \right\}$ set of admissible deformations

Theorem (Healey-Krömer 2009)* Assume $\varphi_{\mathrm{el}}(F) = \infty$ for $\det F \leq 0$,

$\varphi_{\mathrm{el}}(F) \geq c/(\det F)^\delta - C$, $\quad \mathcal{H}(A) \geq c|A|^r - C$, $\quad$ and $\quad \dfrac{1}{r} + \dfrac{1}{\delta} < \dfrac{1}{d}$.

Then, for all $C_M > 0$ there exists $C^*, c_{\mathrm{HeKr}} > 0$ such that for all $y \in \boldsymbol{Y}$ with $\mathcal{M}(y) \leq C_M$ we have

$$\|y\|_{\mathrm{W}^{2,r}(\Omega)} \leq C^* \text{ and } \det \nabla y(x) \geq c_{\mathrm{HeKr}} \text{ on } \Omega.$$

This gives **uniform invertibility** on sublevels, in particular

$$\|\nabla y\|_{\mathrm{C}^\alpha} + \|(\nabla y)^{-1}\|_{\mathrm{C}^\alpha} \leq K \quad \text{ with } \alpha = 1 - r/d \in \,]0,1[.$$

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* Healey, Krömer: *Injective weak solutions in second-gradient nonlinear elasticity.* ESAIM COCV 15, 863–871, 2009

Time-dependent frame-indifference asks for $\zeta(F,\theta,\dot F) = \widehat{\zeta}(C,\theta,\dot C)$ with $C = F^\top F$ and $\dot C = F^\top \dot F + \dot F^\top F$ (Antman 1998)

We assume **linear** viscoelasticity, i.e. $\widehat{\zeta}(C,\theta,\dot C) = \frac{1}{2}\dot C{:}\mathbb{D}(C,\theta)\dot C$ and assume upper and lower bounds $\frac{1}{K}|\dot C|^2 \le \dot C{:}\mathbb{D}(C,\theta)\dot C \le K|\dot C|^2$ for all $C,\theta,\dot C$

Time-dependent frame-indifference asks for $\zeta(F,\theta,\dot F)=\widehat\zeta(C,\theta,\dot C)$ with $C=F^\top F$ and $\dot C=F^\top\dot F+\dot F^\top F$ (Antman 1998)

We assume **linear** viscoelasticity, i.e. $\widehat\zeta(C,\theta,\dot C)=\frac12\dot C{:}\mathbb{D}(C,\theta)\dot C$ and assume upper and lower bounds $\frac1K|\dot C|^2\le\dot C{:}\mathbb{D}(C,\theta)\dot C\le K|\dot C|^2$ for all $C,\theta,\dot C$

Thus, viscoelastic dissipation only controls $\dot C=F^\top\dot F+\dot F^\top F=\nabla y^\top\nabla\dot y+\nabla\dot y^\top\nabla y$

Theorem (Neff 2002, Pompe 2003*) Let $\Omega\subset\mathbb{R}^d$ be bdd, Lipschitz, $\mathcal{H}^{d-1}(\Gamma_{\mathrm{Dir}})>0$, and $F\in\mathrm{C}^0(\overline\Omega;\mathbb{R}^{d\times d})$ with $\min\{\det F(x)\,|\,x\in\overline\Omega\}\gneqq 0$.
Then, there exists $c_{\mathrm{Korn}}(F)>0$ such that

$$\forall\,V\in\mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega;\mathbb{R}^d):\ \int_\Omega\big|F^\top\nabla V+\nabla V^\top F\big|^2\,\mathrm{d}x\ge c_{\mathrm{Korn}}(F)\|V\|^2_{\mathrm{H}^1}.$$

Neff: *On Korn's first inequality with non-constant coefficients*, Proc. Roy. Soc. Edinburgh Sect. A 132, 221–243, 2002.

Pompe: *Korn's first inequality with variable coefficients and its generalization*, Comment. Math. Univ. Carolinae 44(1) 57-70, 2003.

3. Three tools: II. Generalized Korn inequality

- The Neff-Pompe result is wrong for general $F \in \mathrm{L}^\infty(\Omega)$, even for $F = \nabla y$ with $y \in \mathrm{W}^{1,\infty}(\Omega)$
- The mapping $F \mapsto c_{\mathrm{Korn}}(F)$ is norm continuous on $\mathrm{C}^0(\overline{\Omega}; \mathbb{R}^{d\times d})$.

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Combining this with the invertibility provides a uniform generalized Korn inequality on sublevels of $\mathcal{M}$

Proposition (Uniform generalized Korn inequality on sublevesl of $\mathcal{M}$)
For each $C_M > 0$ there exists $\widetilde{c}_{\mathrm{Korn}}(C_M) > 0$ such that

$$\forall\, y \in \boldsymbol{Y} \text{ with } \mathcal{M}(y) \leq C_M \ \forall\, \theta \in \mathrm{L}^1(\Omega)\ \forall\, V \in \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega):$$
$$K\,\mathrm{D}_v\mathcal{R}(y,\theta,V)[V] \geq \|\nabla y^\top \nabla V + \nabla V^\top \nabla y\|^2_{\mathrm{L}^2} \geq \widetilde{c}_{\mathrm{Korn}}(C_M)\|V\|^2_{\mathrm{H}^1}.$$

Proof: Combine Neff-Pompe with compact embedding
$\mathrm{W}^{2,r}(\Omega) \subset \mathrm{C}^{1,\alpha}(\overline{\Omega}) \Subset \mathrm{C}^1(\overline{\Omega})$,
the uniform Healey-Krömer invertibility, and Weierstraß' extremum principle.

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$$K\, \mathrm{D}_v \mathcal{R}(y,\theta,V)[V] \ge \|\nabla y^\top \nabla V + \nabla V^\top \nabla y\|^2_{\mathrm{L}^2} \ge \widetilde{c}_{\mathrm{Korn}}(C_M)\|V\|^2_{\mathrm{H}^1}.$$

Can this result be derived from **rigidity estimates**
as a kind of **“infinitesimal rigidity” ?**

$\boldsymbol{X}$ reflexive Banach space

$\mathcal{M} : \boldsymbol{X} \to \mathbb{R}\cup\{+\infty\}$ is weakly lower semicontinuous and Λ-convex for some $\Lambda \in \mathbb{R}$, i.e. for all $y_0, y_1 \in \boldsymbol{X}$ and all $\theta \in \,]0,1[$ we have

$$\mathcal{M}((1-\theta)y_0 + \theta y_1) \leq (1-\theta)\mathcal{M}(y_0) + \theta\mathcal{M}(y_1) - \frac{\Lambda}{2}(1-\theta)\theta\|y_1-y_0\|^2_{\boldsymbol{X}}.$$

Theorem (RS'06). Assume $u \in \mathrm{W}^{1,p}([0,T];X)$ and $\eta \in \mathrm{L}^{p^*}([0,T];\boldsymbol{X}^*)$ such that $\sup_{[0,T]}\mathcal{M}(u(t)) < \infty$ and $\eta(t) \in \partial\mathcal{M}(u(t))$ a.e. in $[0,T]$, then

$$\mathcal{M}(u(t)) = \mathcal{M}(u(0)) + \int_0^t \langle \eta(s), \dot{u}(s)\rangle \, \mathrm{d}s.$$

Brézis: *Opérateurs maximaux monotones et semi-groupes dans espaces Hilbert*. 1973 (convex!)
Rossi, Savaré: *Gradient flows of non convex functionals in Hilbert spaces and applications*, ESAIM COCV 12, 564–614, 2006. (Λ-convex)
M., Rossi, Savaré: *Nonsmooth analysis of doubly nonlinear evolution equations*, Calc. Var. PDE 46, 253-310, 2013. (even more general)

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We first collect the assumptions for

free-energy density $\psi(F,\theta)=\varphi_{\rm el}(F)+\phi_{\rm cpl}(F,\theta)$ $F=\nabla y$

Hyperstress density $\mathcal{H}(A)=\mathcal{H}(\nabla^2 y)$

dissipation potential $\zeta(F,\theta,\dot F)=\widehat\zeta(C,\theta,\dot C)$

- $\widehat\zeta$ quadratic in $\dot C$ and bounded above on with $\mathbb{D}$ continuous
- $\mathcal{H}$ convex, C^1, and $\frac{1}{K}|A|^r-K\leq\mathcal{H}(A)\leq K(1+|A|)^r$
- $(F,\theta)\mapsto\mathbb{K}(F,\theta)$ continuous, bounded and uniformly positive definite
- initial conditions $y^0\in\boldsymbol{Y}$ and $\theta^0\in\mathrm{L}^1_{\geq0}(\Omega)$ with $\mathcal{E}(y^0,\theta^0)<\infty$
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- $\psi(F,\theta) = \varphi_{\mathrm{el}}(F) + \phi_{\mathrm{cpl}}(F,\theta)$ with $\phi_{\mathrm{cpl}}(F,0) = 0$ and
  - $\varphi_{\mathrm{el}} \in \mathrm{C}^2(\mathrm{GL}^+(\mathbb{R}^d))$ and $\varphi_{\mathrm{el}}(F) = \infty$ for $\det F \leq 0$
  - $\varphi_{\mathrm{el}}(F) \geq \frac{1}{K}(\det F)^{-\delta}$ with $1/\delta + 1/r < 1/d$
  - $|\partial_F \phi_{\mathrm{cpl}}(F,\theta)|^2 \leq K\varphi_{\mathrm{el}}(F)$
  - $\frac{1}{K} \leq -\theta\partial_\theta^2\phi_{\mathrm{cpl}}(F,\theta) \leq K$ and $\partial_F^2\phi_{\mathrm{cpl}}(F,\theta) \leq K$

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Decomposition of free energy $\psi(F,\theta) = \varphi_{\rm el}(F) + \phi_{\rm cpl}(F,\theta)$

leads to a corresponding decomposition of internal energy

$e(F,\theta) = \psi(F,\theta) - \theta\partial_\theta\psi(F,\theta)$

$e(F,\theta) = \varphi_{\rm el}(F) + \mathfrak{w}(F,\theta)$ with $\mathfrak{w}(F,\theta) = \phi_{\rm cpl}(F,\theta) - \theta\partial_\theta\phi_{\rm cpl}(F,\theta)$

Assumptions give $0 = \mathfrak{w}(F,0) \le \mathfrak{w}(F,\theta)$ and $\partial_\theta\mathfrak{w}(F,\theta) \in \big[\frac{1}{K}, K\big]$

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Three possible formulations of the "heat equation"

(equivalent for smooth solutions)

$\theta\dot{s} + \operatorname{div}\boldsymbol{q} = \xi = 2\zeta$ $\qquad e = \psi + \theta s$

$\dot{e} + \operatorname{div}\boldsymbol{q} = \xi + \underbrace{\partial_F\psi(\nabla y,\theta)}_{\text{sing. at } \det\nabla y = 0} : \nabla\dot{y}$ $\qquad$ full mechanical power

reduced heat equation for the "thermal energy" $\mathfrak{w} = e - \varphi_{\text{el}}$ only

$\dot{\mathfrak{w}} + \operatorname{div}\boldsymbol{q} = \xi + \underbrace{\partial_F\phi_{\text{cpl}}(\nabla y,\theta)}_{\text{well-behaved}} : \nabla\dot{y}$ $\qquad$ only power of "coupling energy"

We use the simpler reduced heat equation for the "thermal energy" $\mathfrak{w}$

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only power of "coupling energy"

Together with the mechanical power balance

$$\mathcal{M}(y(t)) + \int_0^t \int_\Omega \Big(\xi(\cdots) + \partial_F \phi_{\text{cpl}}(\nabla y, \theta) : \nabla \dot{y} \Big) \mathrm{d}x \Big) \, \mathrm{d}s = \mathcal{M}(y(0))$$

we obtain the conservation of the total energy

$$\mathcal{E}(y(t), \theta(t)) = \mathcal{M}(y(t)) + \int_\Omega \mathfrak{w}(\nabla y(t), \theta(t)) \, \mathrm{d}x = \mathcal{E}(y^0, \theta^0).$$

4. The existence result

Theorem (Global existence) Under the above assumptions there exists for all $T > 0$ a weak solution $(y, \theta) : [0, T] \to \boldsymbol{Y} \times \mathrm{L}^1(\Omega)$ with $(y(0), \theta(0)) = (y^0, \theta^0)$ and

$$y \in \mathrm{C}^0_\mathrm{w}([0,T]; \mathrm{W}^{2,r}(\Omega)) \cap \mathrm{H}^1([0,T]; \mathrm{H}^1(\Omega)) \quad \text{and}$$
$$\theta \in \mathrm{L}^1([0,T]; \mathrm{W}^{1,1}(\Omega)) \cap \mathrm{L}^q([0,T]; \mathrm{W}^{1,q}(\Omega)) \text{ for all } q \in [1, \tfrac{d+2}{d+1}[$$

Moreover, this solution satisfies energy balance $\mathcal{E}(y(t), \theta(t)) = \mathcal{E}(y^0, \theta^0)$ and the mechanical energy-dissipation balance.

A pair (y, θ) is called weak solution if

$$\int_0^T\!\!\int_\Omega \Big(\big(\partial_{\dot F}\zeta(\nabla y, \theta, \nabla \dot y) + \partial_F \psi(\nabla y, \theta)\big) : \nabla z + \partial_A \mathcal{H}(\nabla^2 y) \vdots \nabla^2 z \Big) \mathrm{d}x \, \mathrm{d}t = 0$$
$$\text{for all } z \in \mathrm{C}^0\big([0,T]; (\mathrm{W}^{2,\infty} \cap \mathrm{H}^1_{\Gamma_\mathrm{Dir}})(\Omega)\big)$$

$$\int_0^T\!\!\int_\Omega \Big(\mathfrak{w}(\nabla y, \theta)\dot v - \nabla v \cdot \mathbb{K}(\nabla y, \theta)\nabla\theta + \big(2\zeta(\cdots) + \partial_F \phi_\mathrm{cpl}(\nabla y, \theta) : \nabla \dot y\big) v \Big) \, \mathrm{d}x \, \mathrm{d}t$$
$$= \int_\Omega \mathfrak{w}(\nabla y^0, \theta^0) \, \mathrm{d}x \quad \text{for all } v \in \mathrm{C}^1([0,T]; \mathrm{W}^{1,\infty}(\Omega)) \text{ with } v(T) = 0.$$

Overview

1. Prologue

2. Finite-strain elasiticity and temperature

3. Three tools

4. The existence result

5. Sketch of proof

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We first solve a regularized problem with $\varepsilon > 0$

(this destroys frame indifference and energy conservation, but estimates work even better)

$$\mathrm{div}\,\Big(\partial_{\dot F}\zeta(\cdots) + \varepsilon\nabla\dot y_\varepsilon + \partial_F\psi(\nabla y_\varepsilon,\theta_\varepsilon) - \mathrm{div}\,\partial_A\mathcal{H}(\nabla^2 y_\varepsilon)\Big) = 0$$

$$\dot{\mathfrak{w}}_\varepsilon = \mathrm{div}\,\Big(\mathbb{K}(\nabla y_\varepsilon,\theta_\varepsilon)\nabla\theta_\varepsilon\Big) + \frac{2\zeta(\cdots)}{1+2\varepsilon\zeta(\cdots)} + \partial_F\phi_{\mathrm{cpl}}(\nabla y_\varepsilon,\theta_\varepsilon){:}\nabla\dot y_\varepsilon$$

- the additional dissipation provides a simple but ε-dependent a priori bound that avoids Korn's inequality
- the source term in the reduced heat equation lies in L^∞ and is bounded from above by the dissipation

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Existence & a priori estimates by a staggered scheme with time step $\tau > 0$

$$\mathcal{M}(y_{\varepsilon\tau}(t)) + \int_0^t \big(\varepsilon\|\nabla\dot y_{\varepsilon\tau}\|^2_{\mathrm{L}^2} - \|\partial_F\phi_{\mathrm{cpl}}(\cdot)\|_{\mathrm{L}^2}\|\nabla\dot y_{\varepsilon\tau}\|_{\mathrm{L}^2}\big)\,\mathrm{d}t \le \mathcal{M}(y^0)$$

$$\overset{\text{Gronw}}{\Longrightarrow}\ \mathcal{M}(y_{\varepsilon\tau}(t)) \le \mathrm{e}^{Kt/\varepsilon}\mathcal{M}(y^0) \le \mathrm{e}^{KT/\varepsilon}\mathcal{M}(y^0) \quad\Longrightarrow\quad \int_0^T \|\nabla\dot y_{\varepsilon\tau}\|^2_{\mathrm{L}^2}\,\mathrm{d}t \le C_\varepsilon$$

For fixed $\varepsilon > 0$ and $\tau \to 0$ we obtain $(y_{\varepsilon\tau}, \theta_{\varepsilon\tau}) \to (y_\varepsilon, \theta_\varepsilon)$

- a limit pair $(y_\varepsilon, \theta_\varepsilon) : [0, T] \to \boldsymbol{Y} \times \mathrm{L}^1_{\geq 0}(\Omega)$ solving the ε-problem
- the time-continuous mechanical energy-dissipation inequality

$$\mathcal{M}(y_\varepsilon(t)) + \int_0^t \int_\Omega \Big(\frac{1}{2} \xi(\cdots) + \frac{\varepsilon}{2} |\nabla \dot{y}_\varepsilon|^2 + \partial_F \phi_{\mathrm{cpl}}(\nabla y_\varepsilon, \theta_\varepsilon) : \nabla \dot{y}_\varepsilon \Big) \mathrm{d}x \Big) \,\mathrm{d}s \leq \mathcal{M}(y(0))$$

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- the time-continuous energy control $\mathcal{E}(y_\varepsilon(t), \theta_\varepsilon(t)) \leq \mathcal{E}(y^0, \theta^0)$

For the last statement we note that in the time-discrete staggered scheme there is no cancellation of the two different coupling terms

$\partial_F \phi_{\mathrm{cpl}}(\nabla y_k, \theta_{k-1}) : \frac{1}{\tau}\nabla(y_k - y_{k-1})$ and $\partial_F \phi_{\mathrm{cpl}}(\nabla y_k, \theta_k) : \frac{1}{\tau}\nabla(y_k - y_{k-1})$

In the time-continuous setting the cancellation works and gives "$\leq$". This provides new a priori estimates that are independent of ε:

$\mathcal{M}(y_\varepsilon(t)) \leq \mathcal{M}(y_\varepsilon(t)) + \int_\Omega \mathfrak{w}_\varepsilon \,\mathrm{d}x = \mathcal{E}(y_\varepsilon(t), \theta_\varepsilon(t)) \leq \mathcal{E}(y^0, \theta^0)$

5. Sketch of proof

The a priori bound $\mathcal{M}(y_\varepsilon(t)) \leq \mathcal{E}(y^0, \theta^0)$ allows for the limit passage $\varepsilon \to 0$

- uniform invertibility (Healey-Krömer) and C^α bounds
  $\|\nabla y_\varepsilon(t)\|_{\mathrm{C}^\alpha} \leq C$ and $\det \nabla y_\varepsilon(t,x) \geq 1/C_*$ on $[0,T] \times \overline{\Omega}$
- With this Neff/Pompe provide a uniform Korn's constant $c_{\mathrm{Korn}} > 0$.
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- using the abstract chain rule[†] for the $\Lambda(C_{\mathcal{E}(y^0,\theta^0)})$-convex functional $\mathcal{M}$ we obtain the energy-dissipaton balance

$$\mathcal{M}(y(t)) + \int_0^t\!\!\int_\Omega \Big(2\zeta(\nabla y,\theta,\nabla\dot y) + \partial_F\phi_{\mathrm{cpl}}(\nabla y,\theta){:}\nabla\dot y \,\mathrm{d}x\Big)\mathrm{d}s = \mathcal{M}(y(0))$$

[†] Rossi, Savaré: *Gradient flows of non convex functionals in Hilbert spaces and applications*, ESAIM COCV 12, 564–614, 2006

- We have the mechanical energy-dissipation balance for $\varepsilon > 0$ and for $\varepsilon = 0$.

$$\mathcal{M}(y_\varepsilon(T)) + \int_0^T\!\!\int_\Omega\Big(2\zeta(\cdots_\varepsilon) + \varepsilon|\nabla\dot y_\varepsilon|^2 + \partial_F\phi_{\rm cpl}(\nabla y_\varepsilon,\theta_\varepsilon){:}\nabla\dot y_\varepsilon\,\mathrm{d}x\Big)\mathrm{d}s = \mathcal{M}(y^0)$$

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This implies convergence of the total dissipation

$$\int_0^T\!\!\int_\Omega\Big(2\zeta(\nabla y_\varepsilon,\theta_\varepsilon,\nabla\dot y_\varepsilon) + \varepsilon|\nabla\dot y_\varepsilon|^2\Big)\,\mathrm{d}x\,\mathrm{d}t \ \to \ \int_0^T\!\!\int_\Omega 2\zeta(\nabla y,\theta,\nabla\dot y)\,\mathrm{d}x\,\mathrm{d}t$$

which implies strong convergence $\nabla\dot y_\varepsilon \to \nabla\dot y$ in $\mathrm{L}^2([0,T];\mathrm{L}^2(\Omega))$

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(weak convergence plus convergence of norm $\Longrightarrow$ strong convergence)

- limit the limit passage in the ε-regularized heat equation is possible because $\dfrac{2\zeta(\nabla y_\varepsilon,\theta_\varepsilon,\nabla\dot y_\varepsilon)}{1+2\varepsilon\zeta(\nabla y_\varepsilon,\theta_\varepsilon,\nabla\dot y_\varepsilon)} \to 2\zeta(\nabla y,\theta,\nabla\dot y)$ in $\mathrm{L}^1([0,T];\mathrm{L}^1(\Omega))$

Conclusion

- Second gradient materials allow us to cope with determinant constraints
- Coupling to a heat equation is possible after suitably splitting the free or internal energy
- Generalized Korn inequalities (infinitesimal rigidity) are needed to treat frame-indifferent dissipation
- Chain rules and Λ-convexity allow to establish energy-dissipation balances

Thank you for your attention

and

Happy Birthday to Gianni

A.M., Roubíček: *Thermoviscoelasticity in Kelvin-Voigt rheology at large strains.* WIAS preprint 2584, 2019. Archive Rat. Mech. Analysis, acc. Jan. 26, 2020.

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