Singular perturbations of gradient flows and rate-independent evolution

Giuseppe Savaré
http://www-dimat.unipv.it/∼savare

Dipartimento di Matematica “F. Casorati”, Università di Pavia

CALCULUS OF VARIATIONS AND APPLICATIONS
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In collaboration with Virginia Agostiniani, Riccarda Rossi

Outline

1. Rate-independent evolution and singular perturbation of gradient flows

2. Transversality conditions for the critical set

3. Compactness and variational characterization of the limit evolution

4. A useful tool: graph convergence

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1 Rate-independent evolution and singular perturbation of gradient flows

2 Transversality conditions for the critical set

3 Compactness and variational characterization of the limit evolution

4 A useful tool: graph convergence

Evolution by critical/stable points

- $\mathbb{H} := \mathbb{R}^d$ ($\rightsquigarrow$ Hilbert space),
- $\mathcal{E} : [0, T] \times \mathbb{H} \to \mathbb{R}$ is a C^2 time dependent energy with $\mathbb{H}$-differential $D\mathcal{E} : [0, T] \times \mathbb{H} \to \mathbb{H}$.
  Typical example: time dependent linear perturbation
  $$\mathcal{E}(t, x) := E(x) - \langle f(t), x \rangle, \quad D\mathcal{E}(t, x) = DE(x) - f(t).$$
- $u_0 \in \mathbb{H}$.

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- **Critical points**:

$$\mathbf{C} := \Big\{(t, x) : D\mathcal{E}(t, x) = 0\Big\},$$

$$\mathbf{C}(t) := \Big\{x : D\mathcal{E}(t, x) = 0\Big\} = \text{“section of } \mathbf{C} \text{ at time } t\text{”}.$$

- **ρ-critical points**: fix some $\rho > 0$ $\quad \|D\mathcal{E}(t, x)\| \leqslant \rho$
- **Globally ρ-stable points**:

$$\mathcal{E}(t, x) \leqslant \mathcal{E}(t, y) + \rho\|y - x\| \quad \forall y \in \mathbb{H}.$$

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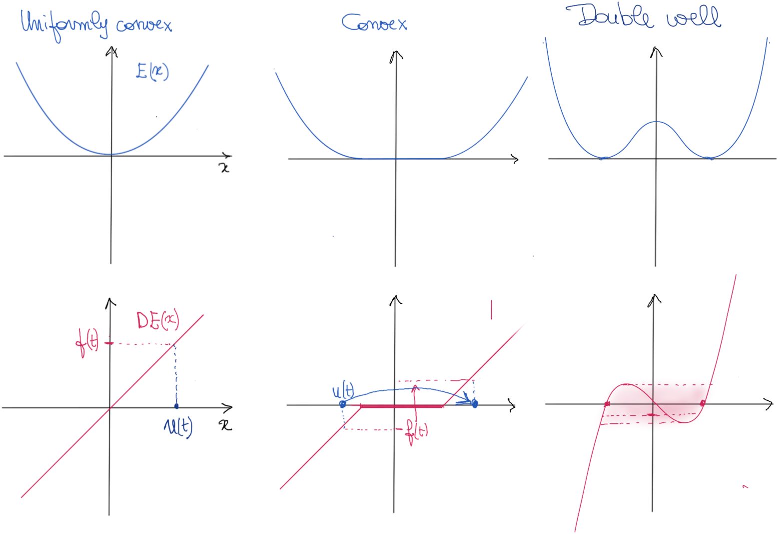
Aim:

Select “reasonable” evolution curves $t \mapsto u(t)$ starting from u_0 such that $u(t)$ is critical/stable for every time $t \in [0, T]$.

Simple examples in 1D



Time Incremental Minimization Scheme

In the case of global ρ-stable evolutions, the main tool to provide existence and to approximate solutions is

The time Incremental Minimization scheme

Fix $\tau := T/N$ (for simplicity), $t_\tau^n := n\tau$, $U_\tau^0 = \mathbf{u}(0)$.

Recursively choose U_τ^n among the minimizers of

$$\boxed{U \mapsto \mathcal{E}(t_\tau^n, U) + \rho\|U - U_\tau^{n-1}\|}$$

$\overline{U}_\tau :=$ the piecewise constant interpolant of the values U_τ^n.

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Theorem [Mainik-Mielke '05]:
There exists a sequence $k \mapsto \tau(k) \downarrow 0$ and $\mathbf{u} : [0, T] \to \mathbb{H}$ such that and

$$\boxed{\overline{U}_{\tau(k)}(t) \to \mathbf{u}(t), \quad \mathcal{E}(t, \overline{U}_{\tau(k)}(t) \to \mathcal{E}(t, \mathbf{u}(t))} \qquad \text{for every } t \in [0, T]$$

and $\mathbf{u}$ is called an **Energetic solution to the Rate Independent Ssystem (R.I.S.)** $(\mathbb{H}, \mathcal{E}, \rho)$**.**

Energetic solutions

Energetic solution: a curve $\mathbf{u} : [0, T] \to \mathbb{H}$ satisfying for every $t \in [0, T]$ the **ρ-stability condition**

$$\mathcal{E}(t, \mathbf{u}(t)) \leqslant \mathcal{E}(t, v) + \rho \|\mathbf{u}(t) - v\| \quad \text{for every } v \in \mathbb{H}, \tag{S}$$

and the **energy balance**

$$\mathcal{E}(t, \mathbf{u}(t)) + \rho \operatorname{Var}(\mathbf{u}, [0, t]) = \mathcal{E}(0, \mathbf{u}(0)) + \int_0^t \mathcal{P}(r, \mathbf{u}(r))\, dr \tag{E}$$

where

$$\mathcal{P}(t, x) = \frac{\partial}{\partial t} \mathcal{E}(t, x).$$

[Mielke-Theil-Levitas '02, Mielke-Theil '04
Francfort-Marigo '93/'98, DalMaso-Toader '02, Francfort-Larsen '05,
DalMaso-Francfort-Toader '05
Mainik-Mielke '05, Francfort-Mielke '06
. . .
Mielke-Roubicek '15]

The “smooth” finite dimensional case

Energetic solutions provides a variational selection among trajectories satisfying

$$\boxed{\rho\,\mathrm{sign}(\dot{\mathbf{u}}(t)) + D\mathcal{E}(t,\mathbf{u}(t)) \ni 0} \qquad \text{in particular } \|D\mathcal{E}(t,\mathbf{u}(t))\| \leqslant \rho,$$

and at every jump point $t \in J(\mathbf{u})$ the **energetic jump conditions**

$$\mathcal{E}(t,(\mathbf{u}(t-)) - \mathcal{E}(t,\mathbf{u}(t+)) = \rho\|\mathbf{u}(t+) - \mathbf{u}(t+)\|$$

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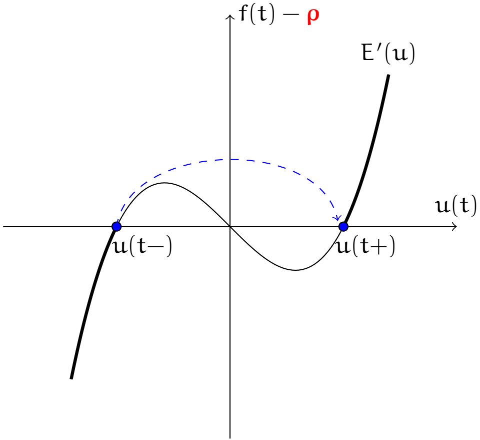

Energetic solution in the 1-dimensional case for a strictly increasing f.

A few technical points

- **Compactness w.r.t. space:** it follows by the compactness of the sublevels of E and the estimate

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- **Compactness w.r.t. time:** it follows by the uniform BV estimate (Helly's Theorem)
$$\rho \, \mathrm{Var}(\overline{U}_\tau, [0, T]) \leqslant C$$
- **Stability:** it follows by the stability of each minimizer and from the

closure of the ρ-stable set $\left\{(t, x) : E(t, x) \leqslant E(t, y) + \rho \|y - x\|\right\}$.

Viscous corrections of the Incremental Minimization Scheme

By introducing a small parameter $\varepsilon = \varepsilon(\tau) > 0$, one may consider the following **modified incremental minimization scheme**

$$\text{minimize} \qquad \boxed{u \mapsto \mathcal{E}(t_\tau^n, u) + \rho\|u - u_\tau^{n-1}\| + \frac{\varepsilon}{2\tau}\|u - u^{n-1}\|^2}$$

and its limit behaviour in three cases:

- **Visco Energetic solutions:** [Minotti-S.]

$$\varepsilon/\tau = \mu > 0 \text{ and } \rho > 0 \text{ are fixed.} \qquad \text{(VE)}$$

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The last two methods correspond to the limit behaviour of

$$\boxed{\rho\,\mathrm{sign}(\dot{u}(t)) + \varepsilon\dot{u}(t) + D\mathcal{E}(t, u(t)) \ni 0}$$

Singular limit of gradient flows ($\rho = 0$)

Main problem

Study the asymptotic behaviour as $\varepsilon \downarrow 0$ of the solution $\mathbf{u}_\varepsilon : [0, T] \to \mathbb{H}$ of the gradient flow

$$\begin{cases} \varepsilon\, \mathbf{u}_\varepsilon'(t) = -D\mathcal{E}(t, \mathbf{u}_\varepsilon(t)) \\ \mathbf{u}_\varepsilon(0) = \mathbf{u}_0. \end{cases}$$

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In order to avoid a transition layer at $t = 0$ we will assume $D\mathcal{E}(0, \mathbf{u}_0) = 0$.

Basic energy estimate

Set $\mathcal{P}(t,x) = \partial_t \mathcal{E}(t,x)$. Chain rule: $\dfrac{d}{dt}\mathcal{E}(t,x(t)) = \langle D\mathcal{E}(t,x), x'(t)\rangle + \mathcal{P}(t,x(t))$.

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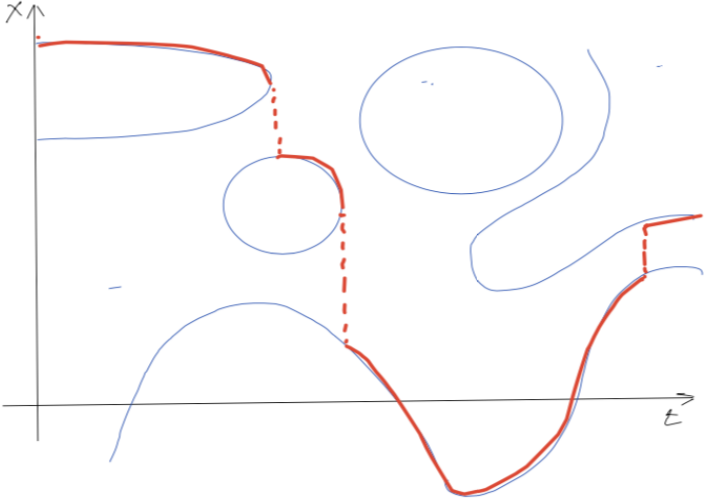
When $\mathcal{E}$ is not convex, one may expect **jumps** and more complex bifurcation behaviour when $\mathbf{u}(t)$ hits a **degenerate critical point** of

$$\mathbf{C}_d := \Big\{(t, x) \in \mathbf{C}(t) : \mathrm{D}^2_{xx}\mathcal{E}(t, x) \quad \text{is not invertible}\Big\}.$$

The critical set (1D)

Jumps between curves



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In $1D$ the above conditions mean that $G(t,x) := \partial_x \mathcal{E}(t,x)$ satisfies

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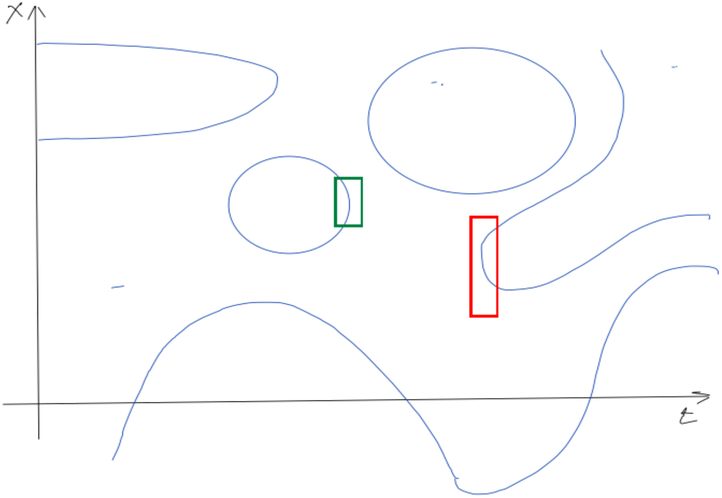
Strong transversality implies that $\mathbf{C}(t)$ **is discrete for every** $t \in [0,T]$**.** [Agostiniani-Rossi, Scilla-Solombrino].
It is a "generic" condition, in the following sense: if $\mathcal{E}$ is C^4, for a G_δ dense set of $g \in \mathbb{H}$ and $Q \in \mathscr{L}(\mathbb{H},\mathbb{H})$ the perturbed energy

$$\mathcal{E}_{g,Q}(t,x) := \mathcal{E}(t,x) - \langle g, x\rangle - \langle Qx, x\rangle$$

satisfies the strong transversality conditions.

Transversality of the critical set



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We consider only linear perturbations

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$$\mathrm{d}D\mathcal{E}_g(t, x) \in \mathscr{L}(\mathbb{R} \times \mathbb{H}; \mathbb{H}) \quad \text{is surjective for every } (t, x) \in \mathbf{C}$$

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- $\mathcal{E}(t,\cdot)$ is constant on every connected component of $\mathbf{C}(t)$.
- There is an at most countable set of times $N \subset [0,T]$ such that $\mathbf{C}(t)$ is not totally disconnected:
  - if $t \in [0,T]\setminus N$ then every connected component of $\mathbf{C}(t)$ is reduced to a point,
  - if $t \in N$ then $\mathbf{C}(t)$ has “larger” connected components.

A simple example in infinite dimension

Let Ω be a bounded connected open set of $\mathbb{R}^3$, $\mathbb{H} := L^2(\Omega)$, $D(E) := H^2(\Omega) \cap H^1_0(\Omega)$. Consider

$$E(u) := \int_\Omega \Big(\frac{1}{2}|\nabla u(x)|^2 + W(u(x))\Big)\,dx, \quad f \in C^2([0,T]; L^2(\Omega)),$$

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One can also consider genericity w.r.t. Ω or w.r.t. the coefficients of the elliptic operator.

Outline

1. Rate-independent evolution and singular perturbation of gradient flows
2. Transversality conditions for the critical set
3. **Compactness and variational characterization of the limit evolution**
4. A useful tool: graph convergence

The main compactness result

Theorem (Agostiniani-Rossi-S.)

Suppose that $\mathbf{C}$ *is countably* $(\mathscr{H}^1, 1)$*-rectifiable (it is sufficient that* $\mathscr{H}^1$ *is* σ*-finite on* $\mathbf{C}$*)*

Then there exists:

- *a subsequence* $n \mapsto \varepsilon(n) \downarrow 0$
- *a limit curve* $\mathbf{u} : [0, T] \to \mathbb{H}$ *such that*

$$\boxed{\lim_{n\to\infty} \mathbf{u}_{\varepsilon(n)}(t) = \mathbf{u}(t) \quad \textit{for every } t \in [0, T].}$$

Properties of limit solutions

Let $u : [0, T] \to \mathbb{H}$ be a limit solution arising from the previous compactness result.

For the sake of simplicity, we suppose that $\mathbf{C}(t)$ is totally disconnected for every $t \in [0, T]$, i.e. $N = \emptyset$.

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- **u is regulated**, i.e. for every $t \in [0, T]$
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- $\mathbf{u}(t) \in \mathbf{C}(t)$ for every $t \in [0, T] \setminus J(\mathbf{u})$; if $t \in J(u)$ then $\mathbf{u}(t-), \mathbf{u}(t+) \in \mathbf{C}(t)$.

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- The map $t \mapsto \mathcal{E}(t, \mathbf{u}(t))$ has **bounded variation**,
  its jump set coincides with the jump set of $\mathbf{u}$
  for every $0 \leqslant s < t \leqslant T$ the **energy balance** holds:

$$\mathcal{E}(t, \mathbf{u}(t-)) + \boxed{\sum_{r \in J(u) \cap (s,t)} c(r; \mathbf{u}(r-), \mathbf{u}(r+))} = \mathcal{E}(s, \mathbf{u}(s+)) + \int_s^t \mathcal{P}(r, \mathbf{u}(r))\, dr$$

The transition cost c

At every jump $t \in J(\mathbf{u})$, the energy dissipation corresponds to the **optimal transition cost:**

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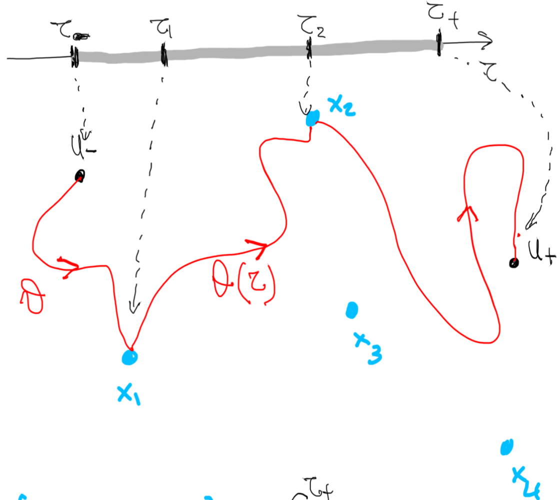
We always have

$$\boxed{\mathcal{E}(t, u_-) - \mathcal{E}(t, u_+) \leqslant \mathsf{c}(t; u_-, u_+)}$$

Transitions



$$C(t)=\{x_1,x_2,x_3,x_4\}$$

$$\int_{\tau_-}^{\tau_+} |D\mathcal{E}(t,\theta(\tau))|\cdot|\theta'(\tau)|\,d\tau \to \min$$

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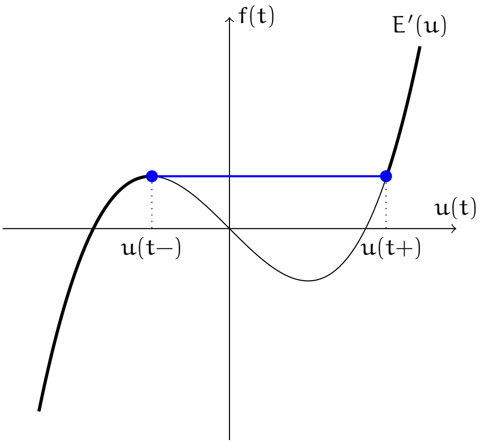
- in each connected component of $\Omega(\vartheta) \subset [0,1]$ where $\vartheta \notin \mathbf{C}(t)$ there is an increasing change of variable $\tau = \tau(r)$, $r \in (a,b)$, such that the reparametrized transition $\theta(r) := \vartheta(\tau(r))$ satisfies **the gradient flow equation**

$$\boxed{\frac{d}{dr}\theta(r) = -D\mathcal{E}(t, \theta(r))}$$

  at the "frozen time" t.

Jumps in the smooth 1-dimensional case: double-well potential



Limit solution in the 1-dimensional case for a strictly increasing f. The blue line represents the graph of the jump transition ϑ.

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Main idea

Instead of studying the convergence of u_ε we consider the **limit of their graphs:**

$$\boxed{\mathbf{G}_\varepsilon := \Big\{(t, u_\varepsilon(t)) : t \in [0, T]\Big\} \subset [0, T] \times \mathbb{H}},$$

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Equivalently,

$$\lim_{n\to\infty} D_H(\mathsf{K}_n, \mathsf{K}) = 0,$$

D_H **is the Hausdorff distance.**

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$$\mathrm{Ls}_{n\to\infty} \mathbf{K}_n := \Big\{ y : \exists\, y_{n(k)} \in \mathbf{K}_{n(k)},\ y_{n(k)} \to y \quad \text{as } k \uparrow \infty \Big\}$$

$$\mathrm{Li}_{n\to\infty} \mathbf{K}_n := \Big\{ y : \exists\, y_n \in \mathbf{K}_n : y_n \to y \quad \text{as } n \uparrow \infty \Big\}$$

$$\mathbf{K}_n \stackrel{K}{\to} \mathbf{K} \quad \Leftrightarrow \quad \mathbf{K} = \mathrm{Ls}_{n\to\infty} \mathbf{K}_n = \mathrm{Li}_{n\to\infty} \mathbf{K}_n$$

Equivalently,

$$\lim_{n\to\infty} D_H(\mathbf{K}_n, \mathbf{K}) = 0, \qquad D_H \textbf{ is the Hausdorff distance.}$$

For every $\varepsilon > 0$ there exists $\bar{n} \in \mathbb{N}$ such that for every $n \geqslant \bar{n}$

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A general compactness property

Blashke compactness theorem

If $\mathbf{G}_\varepsilon$ are contained in a common compact set, there exists a subsequence $n \mapsto \varepsilon(n) \downarrow 0$ and a limit set $\mathbf{G}$ such that $\mathbf{G}_{\varepsilon(n)} \xrightarrow{K} \mathbf{G}$.

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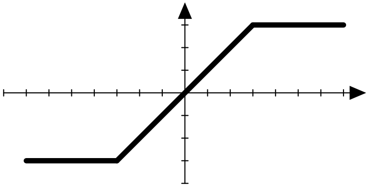
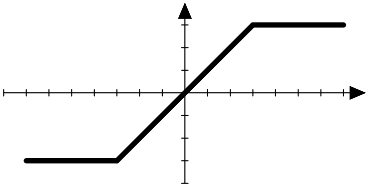
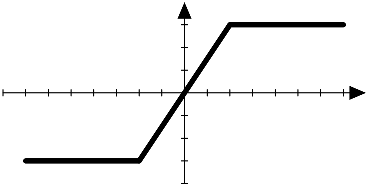
If moreover the sets $\mathbf{G}_\varepsilon$ are connected then also the limit $\mathbf{G}$ is connected.

Examples



Examples





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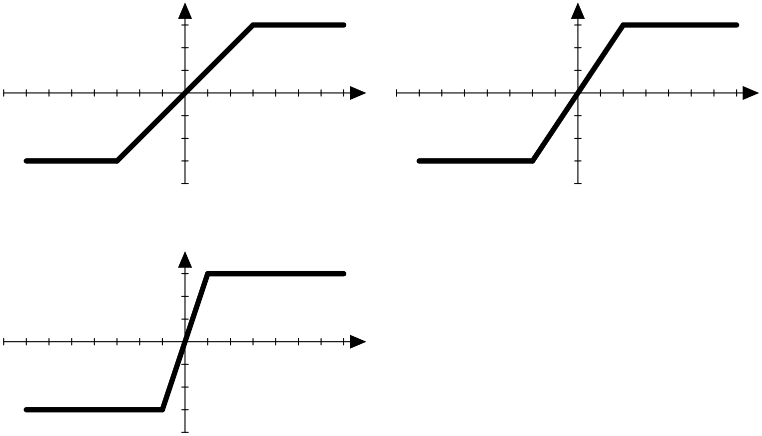

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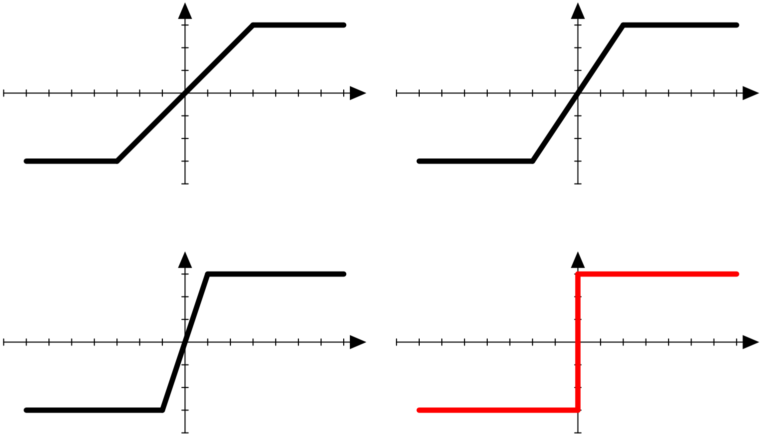

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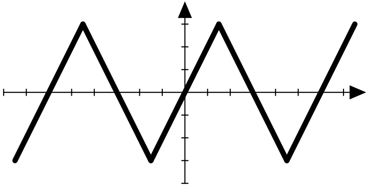

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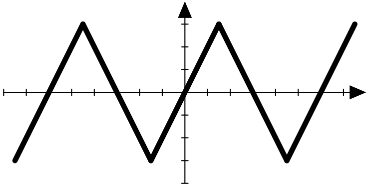

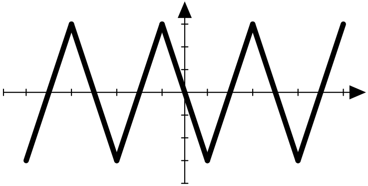

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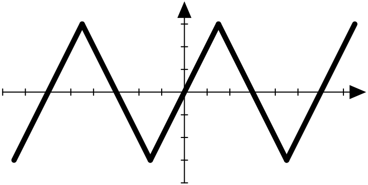

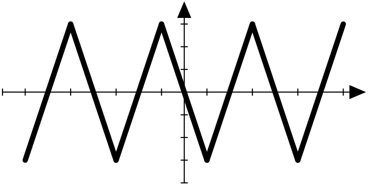

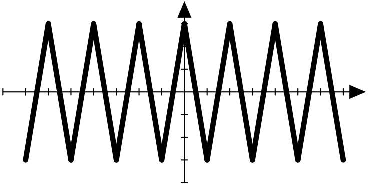

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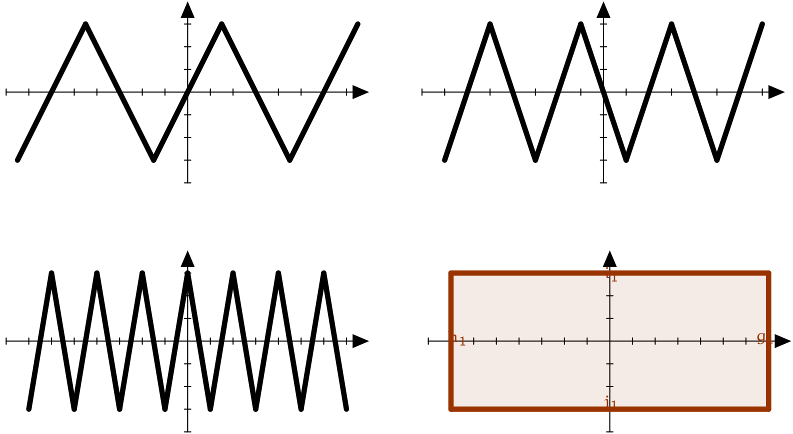

Properties of the limit graph

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There exists a subsequence $n \mapsto \varepsilon(n) \downarrow 0$ and a limit set $\mathbf{G}$ such that $\mathbf{G}_{\varepsilon(n)} = \mathsf{Graph}(\mathbf{u}_{\varepsilon(n)} \overset{K}{\to} \mathbf{G}$.

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- $\mathbf{G}(t) \not\subset \mathbf{C}(t)$: then $t \in J(\mathbf{u})$ and $\mathbf{G}(t)$ contains an optimal transition ϑ connecting $\mathbf{u}(t-)$ with $\mathbf{u}(t+)$ along which

$$\mathcal{E}(t, \mathbf{u}(t-)) - \mathcal{E}(t, \mathbf{u}(t+)) \geqslant c(t; \mathbf{u}(t-), \mathbf{u}(t+))$$

Extensions

- The assumption that $\mathbf{C}(t)$ is totally disconnected for every $t \in [0, T]$ can be removed, by working with possibly discontinuous functions at a countable set of times $N \subset [0, T]$. For each $t \in N$ we can only say that $\mathbf{u}(s)$ approaches a connected component of $\mathbf{C}(t)$ as $s \to t$.

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- A finer description is possible when $\mathcal{E}$ is subanalytic.