Atomic decompositions and two stars theorems for non reflexive Banach function spaces

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joint work with
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Ann. Inst. H. Poincaré An. Non Lin

- Bourgain-Brezis-Mironescu, JEMS, 2015
- Ambrosio-Bourgain-Brezis-Figalli, Comm.P. A.M., 2016
- Ambrosio– Comi, Nonlinear Analysis, 2016
- Fusco–Moscariello– Sbordone, J. Funct. Anal, 2016
- Milman, Ann. Acad. Sci. Fen., 2016
- Ponce– Spector, Nonlinear Analysis, 2017
- De Philippis– Fusco–Pratelli, Rend. Lincei, 2018
- Dafni–Hytönen-Korte–Yue, J. Funct. Anal., 2018
- Fusco–Moscariello– Sbordone, ESAIM, 2018
- Ambrosio-Puglisi Reine Angew. Math. 2019
- Farroni–Fusco–Guarino Lo Bianco– Schiattarella, J. Funct. Anal. 2020
- Angrisani– Ascione–D'Onofrio– Manzo, 2020

E non reflexive Banach space normed by

$$||u||_E = \sup_{L \in \mathcal{L}} ||Lu||_Y$$

where $\mathcal{L}$ is a collection of linear operators

$$L \in \mathcal{L}(X, Y)$$

and X and Y Banach spaces (X reflexive)

Example

BMO, $C^{0,\alpha}$, BV, $L^{q,\infty}$ (Marcinkiewicz), EXP_α, Lip

$$B = \text{space of Bourgain, Brezis and Mironescu}$$

We prove that E is a dual space, showing that the elements of the predual $E_\star$ have atomic decomposition

Bourgain–Brezis-Mironescu (JEMS, 2015)

B = <u>*new*</u> BMO–type space on $Q_0 =]0,1[^n$

$$u \in L^1(Q_0) \qquad \|u\|_B = \sup_{0<\varepsilon<1} [u]_\varepsilon < \infty \qquad \text{(big-O condition)}$$

where $[u]_\varepsilon$ is defined with a suitable <u>maximization procedure</u>.

$B_0 =$ its VMO–type space.

$$[u] = \limsup_{\varepsilon \to 0} [u]_\varepsilon = 0 \qquad \text{(little-o condition)}$$

$$BMO \cup BV \subset B \subset L^{\frac{n}{n-1},\infty}$$

$$VMO \cup W^{1,1} \subset B_0$$

For $n = 1 \qquad B = BMO \qquad B_0 = VMO.$

$0 < \varepsilon < 1$, $u \in L^1(Q_0)$

$$[u]_\varepsilon = \sup_{\mathcal{F}_\varepsilon} \left(\varepsilon^{n-1} \sum_{Q_\varepsilon \in \mathcal{F}_\varepsilon} \fint_{Q_\varepsilon} |u - u_{Q_\varepsilon}| \right)$$

(where $u_Q = \fint_Q u$). The supremum runs among all families $\mathcal{F}_\varepsilon$ of disjoint ε-cubes $Q_\varepsilon \subset Q_0$ with sides parallel to the axes such that $\#\mathcal{F}_\varepsilon \leq \frac{1}{\varepsilon^{n-1}}$

$$u \in B \Leftrightarrow ||u||_B = \sup_{0<\varepsilon<1} [u]_\varepsilon < \infty$$

$$u \in B_0 \Leftrightarrow \limsup_{\varepsilon \to 0} [u]_\varepsilon = 0$$

Separable vanishing subspace

BMO– seminorm of $u : Q_0 \to \mathbb{R}$

John–Nirenberg (1961) $u \in L^1(Q_0)$

$$\|u\|_{BMO} = \sup_{0<\epsilon<1} \sup_{\ell(Q)=\epsilon} \fint_Q |u - u_Q| < \infty$$

Sarason (1975) $u \in BMO$

$$u \in VMO \iff \limsup_{\epsilon \to 0} \sup_{\ell(Q)=\epsilon} \fint_Q |u - u_Q| = 0$$

$VMO(Q_0) =$ closure of $C^\infty(\bar{Q}_0)$ in BMO.

BMO

- is important because its norm is self improving.
- is dual of the separable Banach space $\mathcal{H}^1$, defined in *Stein-Weiss, Acta 1960*

$$(\mathcal{H}^1)^\star = BMO\ (Fefferman, 1971\ Fefferman - Stein\ 1972)$$

$$\mathcal{H}^1 = \text{ Hardy space } = \left\{f \in L^1 : R_j f \in L^1\right\}$$

Duality properties: BMO, $\mathcal{H}^1$, VMO

1) (Coifman-Weiss) $VMO^\star = \mathcal{H}^1$
2) $VMO^{\star\star}$ isometrically isomorphic with BMO
3) (Sarason, 1975) distance formula

$$\limsup_{\varepsilon\to 0} \sup_{\ell(Q)=\varepsilon} ⨍_Q |u - u_Q| \simeq \operatorname{dist}_{BMO}(u, VMO)$$

Abstract setting

Let $X,\ Y \in L^1(Q_0)$ Banach space (X reflexive) and

$$L_j \in \mathcal{L}(X, Y) \qquad j = 1, \ldots$$

Define

$$E = \{u \in X : \sup_j ||L_j u||_y < \infty\}$$

Suppose E a Banach space , $E \subset X$ continuously, E dense in X and

$$||u||_E = \sup_j ||L_J u||_Y$$

We prove that E is a dual space and we characterize its predual E_* and its dual E^*.

Example ($E = BMO(Q_0)$)

$X = L^p(Q_0)$, $1 < p < \frac{n}{n-1}$, $Y = L^1(Q_0)$, Q_j a sequence of (well chosen) cubes contained in Q_0 we define

$$L_j \in \mathcal{L}(L^p, L^1)$$

$$L_j u(x) = \frac{\chi_{Q_j}}{|Q_j|}(u(x) - u_{Q_j})$$

that implies

$$||L_j u||_{L^1} = \fint_{Q_j} |u - u_{Q_j}|$$

Example ($E = BV(Q_0)$)

$X = L^p(Q_0)$, $1 < p < \frac{n}{n-1}$, $Y = L^1(Q_0)$,
$0 < \epsilon < 1$ $\mathcal{F}_\epsilon = \{Q_\epsilon\}$ finite disjoint collection of ϵ- cubes. define

$$L_{\mathcal{F}_\epsilon} u = \epsilon^{n-1} \sum_{Q_\epsilon \in \mathcal{F}_\varepsilon} \frac{\chi_{Q_\epsilon}}{|Q_\epsilon|}(u - u_{Q_\epsilon})$$

choose $L_j = L_{\mathcal{F}^j_{\epsilon_j}} \in \mathcal{L}(L^p, L^1)$ that implies

$$||L_j u||_{L^1} = \varepsilon_j^{n-1} \sum_j \fint_{Q_{\epsilon_j}} |u - u_{Q_{\epsilon_j}}|$$

Example ($E = B$)

For the space of Bourgain-Brezis-Mironescu we choose only collections $\mathcal{F}_\varepsilon$ with

$$\#\mathcal{F}_\varepsilon \leq \frac{1}{\varepsilon^{n-1}}$$

$$L_{\mathcal{F}_\epsilon} u = \epsilon^{n-1} \sum_{Q_\epsilon \in \mathcal{F}_\varepsilon} \frac{\chi_{Q_\epsilon}}{|Q_\epsilon|}(u - u_{Q_\epsilon})$$

Choosing a suitable $L_j = L_{\mathcal{F}_{\varepsilon_j}} \in (L^p, L^1)$ that implies

$$||L_j u||_{L^1} = \varepsilon_j^{n-1} \sum_j \fint_{Q_{\epsilon_j}} |u - u_{Q_{\epsilon_j}}|$$

With this notations it is obvious that

$$||u||_B \leq ||u||_{BV}$$

Example ($E = C^{0,\alpha}(Q_0)$, $0 < \alpha < 1$)

$X = W^{\ell,p} \setminus A$

Besov spaces where $0 < \ell < \alpha$, $p\ell > n$ where $u \in L^p(\mathbb{R}^n)$

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^p}{|x - y|^{p\ell+n}} \mathrm{d}x \mathrm{d}y < \infty$$

$$A = A_{Q_0} = \{u \in W^{\ell,p} : \exists x_0 \in Q_0 : u(x_0) = 0\}$$

$Y = \mathbb{R}$. $\forall x, y \in Q_0$, $x \neq y$

$$L_{x,y} u = \frac{|u(x) - u(y)|}{|x - y|^\alpha}$$

and a suitable sequence $L_j = L_{x_j,y_j} \in \mathcal{L}(X, \mathbb{R}) = X^*$

$$|L_j u| = \sup_{x_j \neq y_j} \frac{|u(x) - u(y)|}{|x - y|^\alpha}$$

Predual of $E = C^{0,\alpha}$ is $\mathcal{M}(Q_0)$ equipped with KR norm.

A compactly supported function $a = a(x)$ is an L^q-atom with defining cube Q, if $supp\, a \subseteq Q$, $\int_Q adx = 0$ and

$$\left(\fint_Q |a(z)|^q \right)^{\frac{1}{q}} \leq \frac{1}{|Q|}$$

Hardy space $\mathcal{H}^1(\mathbb{R}^n)$ can be characterized in terms of decomposition involving L^q-atoms for each fixed $q > 1$.

Atomic decomposition of $L^1(\mathbb{R}^n)$

$$f = \sum_j \lambda_j a_j(x) + \lambda \chi_{Q_0}(x)$$

$$\sum_j |\lambda_j| < \infty \qquad a_j(x) \text{ is } L^{q_j} - \textit{atom}$$

$$\|f\|_{L^1} \simeq |\lambda| + \inf \sum_j |\lambda_j|$$

$$\lambda = \int_{\mathbb{R}^n} f \, dx$$

2007, Proc. AMS, Torchinski

Atomic decomposition is effective tool to prove boundness of operators such as: the Hardy-Littlewood maximal operator, Hilbert transform, the composition operators acting on these spaces: the boundedness of these operators is reduced to the boundedness on characteristic functions.

Theorem 1 ((DGPSS) Atomic decomposition for B_*)

B has an (isometric) predual B_*. Every $\varphi \in B_*$ is of the form

$$\varphi = \sum_{j=1}^{\infty} \lambda_j g_j, \tag{1}$$

where $(\lambda_j) \in \ell^1(\mathbb{N})$ and each atom *g_j is associated with an $\varepsilon = \varepsilon_j$ and $|\mathcal{F}_\varepsilon| \leq \varepsilon^{1-n}$ and*

- *$\operatorname{supp} g_j \subset \cup \mathcal{F}_\varepsilon$; $\int_{Q_\varepsilon} g_j \, dx = 0$ for every $Q_\varepsilon \in \mathcal{F}_\varepsilon$,*
- *$|g_j| \chi_{Q_\varepsilon} \leq \varepsilon^{n-1} \frac{1}{|Q_\varepsilon|}$ for every $Q_\varepsilon \in \mathcal{F}_\varepsilon$.*

$$f(\varphi) = \sum_{j=1}^{\infty} \lambda_j \int_{(0,1)^n} f g_j \, dx,$$

$$\|\varphi\|_{B_*} \sim \inf \sum_{j=1}^{\infty} |\lambda_j|,$$

and the infimum is taken over all representations of φ.

Hardy's space $\mathcal{H}^1$ (Coifman, 1974)

Theorem

Any $\varphi \in \mathcal{H}^1$ *can be written as*

$$\varphi(x) = \sum_{j=1}^{\infty} \lambda_j a_j(x)$$

where a_j *are q-atoms* $1 < q \leq 2$ *and*

$$\sum_{j=1}^{\infty} |\lambda_j| < \infty$$

furthermore:

$$||\varphi||_{\mathcal{H}^1} \sim \inf \sum_{j=1} |\lambda_j|$$

$BV(Q_0)$ functions of bounded variation, i.e. $u \in L^1(Q_0)$ and

$$|Du|(Q_0) = \sup_{\|\Phi\|_{C^0(Q_0,\mathbb{R}^n)} \leq 1} \int_{Q_0} u \,\mathrm{div}\Phi < \infty$$

with the norm

$$\|u\|_{BV} = \|u\|_{L^1} + |Du|(Q_0)$$

$BV(Q_0)$

- non separable
- smooth compactly supported functions fail to be norm-dense
- is dual of a separable Banach space

$$BV_\star = \left\{ T \in \mathcal{D}' : T = \varphi_0 + \sum_{j=1}^{n} \frac{\partial \varphi_j}{\partial x_j},\ \varphi_i \in \mathcal{C}_0(Q_0) \right\}$$

with integral representation of duality pair.
Fusco–Spector (2018 JMAA) integral characterization of the dual $BV^\star$.
We find an atomic decomposition for $BV_\star$ like $B_\star$ but without the limitation

$$|\mathcal{F}_\varepsilon| \leq \varepsilon^{1-n}$$

$$\varphi = \sum_j \lambda_j g_j$$

Kantorovich (1942)
Let (K, ρ) be a compact metric space

$$Lip(K)$$

- is dual of the space normed space $\mathcal{M} = \mathcal{M}(K, \mathcal{B})$ of Borel measures μ on K with finite total variation , $\mathcal{B} =$ Borel σ-algebra of (K, ρ), equipped with the Kantorovich-Rubinstein norm $|| \; ||_{KR}$ important in optimal transport (notice that $\mathcal{M}$ equipped with KR norm is not a Banach space):

$$\mathcal{M}^* \backsimeq Lip$$

Angrisani-Ascione-D'Onofrio-Manzo: *atomic decomposition* of predual of Lip_α (2020).

Relations between $Lip(K,\rho)$ defined by $u \in Lip$ iff (Big-O condition)

$$\sup_{x \neq y} \frac{|u(x) - u(y)|}{\rho(x,y)} < \infty$$

and $lip(K,\rho)$ defined by $v \in lip$ iff (little-o condition)

$$\limsup_{\rho(x,y) \to 0} \frac{|v(x) - v(y)|}{\rho(x,y)} = 0$$

similar to sequence spaces ℓ^∞ and c^0.

Problem

The question is to know when the second dual to the ‘small’Lipschitz space is isometrically isomorphic to the ‘big ’Lipschitz space i.e.

$$lip^{\star\star} \cong Lip \tag{2}$$

Equivalently, when the completion of $\mathcal{M}$ is isomorphically isomorphic to $lip^{\star}$

$$\mathcal{M}^c \cong lip^{\star} \tag{3}$$

Best results in case

$$C^{0,\alpha}(K,d) \simeq Lip(K,\rho^\alpha) \qquad 0 < \alpha < 1.$$

Here $lip(K,\rho^\alpha) = c^{0,\alpha}$ is a "rich "subspace of $C^{0,\alpha}$ While $lip[0,1] = \{\text{constants}\}$ trivial

$$Lip \subset c^{0,\alpha}$$

$c^{0,\alpha}$ "rich "

The idea is to identify $lip([0,1], d^{\alpha})$ as a subspace of

$$C_0(W) \qquad W \subset \mathbb{R}^2$$

and then use the Representation Theorem of Riesz

$$u \in lip_{\alpha} \to \tilde{u} \in C_0(W)$$

$$\|u\|_{lip_{\alpha}} \simeq \|\tilde{u}\|_{C_0}$$

Let (K, ρ) be a compact metric space, the space $\mathcal{M}(K)$ of Borel measures $\mu : \mathcal{B}(K) \to \mathbb{R}$ endowed with *the total variation norm*

$$||\mu||_{TV} = |\mu|(K) \tag{4}$$

is a Banach space isometric to the dual space of $C_0(K)$. The role of *weak-star* convergence in $\mathcal{M}(K)$ as dual of $C_0(K)$ is much more relevant then the role of strong convergence. Given a sequence $(\mu_j) \subset \mathcal{M}(K)$, recall that

$$\mu_j \overset{*}{\rightharpoonup} \mu \tag{5}$$

if and only if

$$\int_K u d\mu_j \to \int_K u d\mu \qquad \forall u \in C_0(K) \tag{6}$$

We will see that on the subset

$$\mathcal{M}_0(K) = \{\mu \in \mathcal{M}(K) : \mu(K) = 0\}$$

besides *strong convergence* of μ_j to μ

$$||\mu_j - \mu||_{TV} \to 0 \tag{7}$$

under the norm (4) and the *weak star* convergence (5), (6) one can consider the KR-norm convergence

$$||\mu_j - \mu||_{KR} \to 0 \tag{8}$$

and the *weak-convergence*

$$\int_K u d\mu_j \to \int_K u d\mu \qquad \forall u \in Lip(K) \tag{9}$$

This last weak convergence is of course weaker then the weak-star convergence (6) but it is possible to prove the surprising equivalence with the KR-norm convergence.
On subsets of $\mathcal{M}_0(K)$ which are uniformly bounded in total variation the *KR*-norm convergence induced by equivalent norm

$$\|\mu\|_{KR} = \sup_{\varphi \in Lip_1} \int_K \varphi \, d\mu$$

is equivalent to weak-star (6).

The classical KR-norm $\| \; \|_{KR}$

$$\mathcal{M}_0(K) = \{\mu \in \mathcal{M}(K) : \mu(K) = 0\}$$

$$\mu \to \Psi_\mu \subset \mathcal{M}_+(K \times K)$$

$$\Psi_\mu = \{\psi \in \mathcal{M}_+(K \times K) : \psi(K, A) - \psi(A, K) = \mu(A)\}$$

$A \in \mathcal{B}(K)$, Borel set
$\psi(A_1, A_2)$ represents transport with given mass μ_- and required mass μ_+

The classical KR norm of $\mu \in \mathcal{M}_0(K)$

$$||\mu||_{KR} = \inf_{\psi \in \Psi_\nu} \int \int_{K \times K} \rho(x,y) d\psi(x,y)$$

The extended KR norm of $\nu \in \mathcal{M}(K)$

$$||\nu||_{KR} = \inf_{\mu \in \mathcal{M}_0(K)} \{||\mu||_\rho + Var(\nu - \mu)\}$$

The normed space $(\mathcal{M}, ||\ ||_\tau)$ in general is not complete

$$||\nu||_{KR} \leq c Var(\nu)$$

Theorem (KR)

$$\mathcal{M}_0(K)^\star \backsimeq Lip(K)/\mathbb{R}$$

$$L \in \mathcal{M}_0(K)^\star \to \varphi(x) = \langle L, \delta_x - \delta_a \rangle \qquad a \in K$$

$$\mathcal{M}_0(K)^c \backsimeq \overline{\mathcal{M}_0(K)}^{Lip_0^\star}$$

$$Lip_0(K) = Lip(K)\big/\mathbb{R}$$

Theorem

$$f \in \mathcal{M}_0(K)^c \Longleftrightarrow \exists \text{ max in dual Kantorovich problem}$$

$$\sup\{\langle f,\varphi\rangle : \varphi \in Lip_1(K)\} = \max\{\langle f,\varphi\rangle : \varphi \in Lip_1(K)\} = \|f\|_{KR} = \langle f,\varphi_f\rangle$$

The case $K = \bar{\Omega}$ has been studied by G.Bouchitté- T.Champion- C. Jimenez "*Completion of the space of measures in the Kantorovich norm.*"(see also Bouchitte'-Buttazzo- De Pascale 2003).
This is a case were double star theorem does not hold, since the vanishing space is trivial.
Theorem
Let $\mu \in \mathcal{M}_0(K)$, then the problem

$$\begin{aligned} &\inf\left\{\int_K |\underline{\lambda}| : \underline{\lambda} \in \mathcal{M}(K, \mathbb{R}^n) \ \text{ div } \underline{\lambda} = \mu\right\} \\ =&\min\left\{\int_K |\underline{\lambda}| : \underline{\lambda} \in \mathcal{M}(K, \mathbb{R}^n) \ \text{ div } \underline{\lambda} = \mu\right\} \\ =&\int_K |\underline{\lambda}_\mu| = \|\mu_+ - \mu_-\|_{KR} \end{aligned}$$

Theorem

$$\{T_f : f \in \mathcal{M}_0^c(K)\} = \left\{ - \operatorname{div} \underline{\sigma} : \underline{\sigma} \in \frac{L^1(K, R^n)}{V_0} \right\}$$

$$V_0 = \{\underline{\sigma} : \operatorname{div} \underline{\sigma} = 0\}$$

Moreover,

$$\|\underline{\sigma}\|_{L^1} = \| \operatorname{div} \underline{\sigma}\|_{KR}$$

Lemma

$$\mu \in \mathcal{M}_0^c(K), \varepsilon > 0 \Longrightarrow \underline{\sigma} \in L^1 : - \operatorname{div} \underline{\sigma} = \mu$$

$$\int_K |\underline{\sigma}| \leq \|\mu\|_{KR} + \varepsilon.$$

Optimal mass transfer always exists $\forall \mu \in \mathcal{M}_0(K)$. A measure $\psi \in \Psi_\mu$ is optimal if and only if there exists $u \in Lip$:

$$\frac{u(x) - u(y)}{\rho(x,y)} = \begin{cases} \leq 1 & \forall (x,y) \in K \\ = 1 & \forall (x,y) \in supp\psi \end{cases}$$

(dual problem).
While the total variation norm $||\nu|| = Var(\nu)$ satisfies for $x, y \in K$

$$||\delta_x - \delta_y||_{TV} = 2$$

where $\delta_x(A) = 1$ if $x \in A$, $\delta_x(A) = 0$ otherwise. For the KR-norm we have

$$||\delta_x - \delta_y||_{KR} = \rho(x,y)$$

it is well related to the existing distance ρ on K.

Theorem (Popoli-Sbordone)

Let (K, ρ) be a compact metric space. Then

$$lip(K, \rho)^{\star\star} \backsimeq Lip(K, \rho)$$

if and only if

the closed unit ball in *lip* is dense in the closed unit ball of *Lip* with respect to the topology of pointwise convergence.

The set of measures μ with finite support is dense in $\mathcal{M}(K)$ with $|| \, ||_{KR}$.
For infinite K $(\mathcal{M}(K), || \, ||_{KR})$ is incomplete.

Theorem (Kantorovich-Rubinstein)

The duality $\mu \in \mathcal{M}(K)$, $u \in Lip(K, \rho)$

$$\langle u, \mu \rangle = \int_K u d\mu$$

defines an isometric isomorphism between $(\mathcal{M}(K))^*$ *and* $Lip(K, \rho)$

Proof.

$u \in Lip$, $\mu \in \mathcal{M}_0$, $\psi \in \Psi_\mu$

$$\begin{aligned} L_u(\mu) &= \int_K u d\mu \\ &= \int_K u(t) d\psi(K,t) - \int_K u(t) d\psi(t,K) \\ &= \int_{K\times K} u(s) d\psi(t,s) - \int_{K\times K} u(t) d\psi(t,s) \\ &= \int_{K\times K} (u(s)-u(t)) d\psi(s,t) \\ &\leq ||u||_{Lip} \int_{K\times K} \rho(t,s) d\psi(t,s) \\ &= ||u||_{Lip} ||\mu||_\tau \end{aligned}$$

□

The *separation property* below, (true for $C^{0,\alpha}, 0 < \alpha < 1$) allows uniform approximation of big *Lip* functions u by little lip functions v_j.

Theorem (Hanin, separation property)

$\mathcal{M}(K)^c \simeq lip(K)^*$
if and only if
$\forall A \subset K$, *A finite* $\forall u : A \to \mathbb{R}, \forall > 1\ \exists g \in lip(K)$:

$$g_{|A} = u \quad ||g||_{K,\tau} \leq C||u||_{A,\rho}$$

if and only if

$$LipK \simeq (lipK)^{**}$$

This implies

$$\forall u \in Lip(K,\rho) \exists v_j \in lip(K,\rho) : v_j \to u(x)$$

$\forall x \in K$ and $\sup ||v_j||_{Lip} \leq ||u||_{Lip}$ (Angrisani, Ascione, D'Onofrio, Manzo)

To characterize all metric spaces (K, ρ) such that the space

$$E = Lip(K) \text{ big space}$$

equipped with the norm

$$||u||_{K,\rho} = \max\{||u||_\infty, |u|_{K,\rho}\}$$

with

$$|u|_{K,\rho} = \sup_{x \neq y} \frac{|u(x) - u(y)|}{\rho(x, y)} < \infty$$

is isometrically isomorphic to the *second dual* E_0^{**} of

$$E_0 = lip(K) \text{ little space}$$

defined by the *vanishing condition*

$$\lim_{\rho(x,y) \to 0} \frac{|u(x) - u(y)|}{\rho(x, y)} = 0$$

Abstract Theorem for E: dual E^*, predual E_*

Isometric embedding

$$V : E \to \ell^\infty(Y)$$

$$Vu(j) = L_j u$$

Theorem (DGPSS)

$$E^* \simeq \frac{ba(\mathbb{N}, Y^*)}{V(E)^\perp}$$

where $ba(\mathbb{N}, Y^*)$ denotes the space of finitely additive Y^*-valued set functions on $\mathbb{N}$, with bounded variation.

Theorem

E has predual E_*

$$E_* = \frac{\ell^1(Y^*)}{P}$$

where $P = V(E)^\perp \cap \ell^1(Y^*)$.
Moreover E_* admits atomic decomposition

Proof.

Every $u \in E$ corresponds to a linear functional on

$$F = \frac{\ell^1(Y^*)}{P}$$

given by

$$(y_j^*) \in F \to u((y_j^*)) = \sum_{j=1}^{\infty} (y_j^*, L_j u)$$

and conversely. The range of canonical embedding

$$W : u \in E \to (i(L_j u)) \in \ell^\infty(Y^{**})$$

$$i : Y \to Y^{**}$$

is weak-* closed in $\ell^\infty(Y^{**})$ and then E is a dual space. □

Proof of Theorem 1 (DGPSS)

Choose a dense sequence $(\mathcal{F}^j_{\varepsilon_j}) \subset \mathcal{L}$. Then for $\varphi \in B_*$ there is $y^*_j \in \ell^1(L^\infty((0,1)^n))$ of comparable norm such that

$$\varphi = \sum_j L^*_{\mathcal{F}^j_{\varepsilon_j}} y^*_j = \sum_j L_{\mathcal{F}^j_{\varepsilon_j}} y^*_j$$

Then $\lambda_j = 2||y^*_j||_{L^\infty}$ and

$$g_j = \frac{L_{\mathcal{F}^j_{\varepsilon_j}} y^*_j}{2||y^*_j||_{L^\infty}}$$

satisfy

$$\varphi = \sum_j \lambda_j g_j\,; \quad \sum_j |\lambda_j| \le c||\varphi||_{B_*}$$

and conversely.

Example ($E = BMO(Q_0)$)

We know that $\forall u \in E$, $(y_j^*) \in E_*$, $x((y_j^*)) = \sum_j \langle y_j^*, L_j u \rangle$, and so being $Y^* = L^\infty(Q_0)$, $\forall \varphi \in E_*$, $\exists (y_j^*)$:

$$||\varphi||_{E_*} \sim \sum_j ||y_j^*||_{L^\infty(Q_0)}$$

Defining

$$\lambda_j = ||y_j^*||_{L^\infty(Q_0)} \quad a_j = \frac{L_j^* y_j^*}{||y_j^*||_{L^\infty(Q_0)}} = \frac{L_j y_j^*}{\lambda_j}$$

hence

$$\varphi = \sum \lambda_j a_j \quad ||\varphi||_{E_*} \sim \sum_j |\lambda_j|$$

$$supp\, a_j \subset Q_j, \quad |a_j| \le 2\chi_{Q_j} \frac{1}{|Q_j|} \quad \int_{Q_j} a_j = 0$$

$$\fint_{Q_\varepsilon} |u - u_{Q_\varepsilon}| \leq \frac{c}{\varepsilon^{n-1}} |\nabla u|(Q_\varepsilon) \quad \Rightarrow \quad [u]_\varepsilon \leq c|\nabla u|(Q)$$

$$B \subset L^{\frac{n}{n-1},\infty}$$

DGGPS: Atomic decomposition of preduals of

$$BMO(Q_0) \qquad BV(Q_0) \qquad B(Q_0) \qquad L^{\frac{n}{n-1},\infty}(Q_0)$$

as Corollary

General abstract framework

Let X be a reflexive and separable Banach space and Y a Banach space. Given a collection $\mathcal{L}$ of linear operators

$$\mathcal{L} \subset \mathcal{L}(X; Y)$$

equipped with topology τ which is σ-compact, locally compact, Hausdorff, such that

$$L \in (\mathcal{L}, \tau) \to Lu \in (Y, || \cdot ||_Y)$$

is continuous $\forall u \in X$.

we define

$$E = \{u \in X : \sup_{L \in \mathcal{L}} ||Lu||_Y < \infty\}$$

and suppose that, equipped with

$$||u||_E = \sup_{L \in \mathcal{L}} ||Lu||_Y$$

E is a Banach space, continuously contained and dense in X. Define

$$E_0 = \{u \in E : \limsup_{L \in \mathcal{L},\, L \to \infty} ||Lu||_Y = 0\}$$

Here $L \to \infty$ is in the usual sense of escaping all compacts.

(E_0 sufficiently rich vanishing space)
Additionally we assume an Approximation Property

AP) $\forall u \in E$ there is $(v_j) \subset E_0$ such that

$$||v_j||_E \leq ||u||_E$$

and

$$v_j \to u \textit{ in } X$$

Remark

(AP) property can be proved for BMO, B, $L^{q,\infty}$, $C^{0,\alpha} = Lip_\alpha$ $0 < \alpha < 1$, $L^{q)}$. *Not for* L^∞, $Lip(Q_0)$, *BV.*

Theorem

Suppose (AP) holds then isometric

$$(E_0)^* \sim E_*$$

$$(E_*)^* \sim E$$

$$E_0^{**} \sim E$$

$$E^* \sim E_0^* \oplus E_0^\perp$$

$$\forall u \in E \quad \min_{v \in E_0} ||u - v||_E = \limsup_{L \to \infty} ||Lu||_Y$$

Proof

E_0 embeds isometrically into

$C_0(\mathcal{L}; Y) = \text{the space of vanishing continuous Y-valued functions}$

E embeds isometrically into

$C_b(\mathcal{L}; Y) = \text{the space of bounded continuous Y-valued functions}$

equipped with the *sup* norm

$$||T||_{C_b} = \sup_{L \in \mathcal{L}} ||T(L)||_Y$$

Explicitly

$$V : E \to C_b(\mathcal{L}; Y)$$

$$Vu(L) = Lu \qquad u \in E,\ L \in \mathcal{L}$$

$$V : E_0 \to C_0(\mathcal{L}; Y)$$

$ca(\mathcal{L}, Y^*) = \text{the space of countably additive } Y^*\text{-valued Baire measures}$

equipped with norm

$$||\mu||_{ca} = \sup \sum_i ||\mu(\mathcal{E}_i)||_{Y^*}$$

over all pairwise disjoint partitions into sets $\mathcal{E}_i$

Riesz Theorem isometrically isomorphic

$$ca(\mathcal{L}, Y^*) \sim (C_0(\mathcal{L}, Y))^*$$

with pairing

$$\langle T, \mu \rangle = \int_{\mathcal{L}} T(L) d\mu(L)$$

$T \in C_b(\mathcal{L}, Y)$, $\mu \in ca(\mathcal{L}, Y^*)$

Theorem

$\forall T \in C_b(\mathcal{L}, Y) \; \exists k \in ca(\mathcal{L}, Y^*)^*$ *defined by*

$$k(\mu) = \langle T, \mu \rangle$$

$$||k||_{(ca)^*} = ||T||_{C_b}$$

The isometric embedding

$$\begin{array}{lll} C_b(\mathcal{L}, Y) & into & ca(\mathcal{L}, Y^*)^* \\ T & \rightarrow & k \end{array}$$

extends the canonical embedding of $C_0(\mathcal{L}, Y)$ into $(C_0(\mathcal{L}, Y))^{**}$. Use the canonical decomposition

$$(ca(\mathcal{L}, Y^*))^{**} = ca(\mathcal{L}, Y^*) \oplus C_0(\mathcal{L}, Y)$$

that implies

$$E^* \simeq (E_0)^{**} = E_0^* \oplus E_0^{\perp}$$