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Blowup Equations for Topological Strings and Supersymmetric Gauge Theories

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“As you set out for Ithaka
hope the voyage is a long one,
full of adventure, full of discovery.”

Konstantinos Kavafis

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Abstract

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by Kaiwen SUN

Blowup equations and their K-theoretic version were proposed by Nakajima-Yoshioka and Göttsche as functional equations for Nekrasov partition functions of supersymmetric gauge theories in 4d and 5d. We generalize the blowup equations in two directions: one is for the refined topological string theory on arbitrary local Calabi-Yau threefolds, the other is an elliptic version for arbitrary 6d $(1,0)$ superconformal field theories (SCFTs) in the "atomic classification" of Heckman-Morrison-Rudelius-Vafa. In general, blowup equations fall into two types: the unity and the vanishing. We find the unity part of generalized blowup equations can be used to efficiently solve all refined BPS invariants of local Calabi-Yau geometries and the elliptic genera of 6d $(1,0)$ SCFTs, while the vanishing part can derive the compatibility formulas between two quantization schemes of algebraic curves, which are the exact Nekrasov-Shatashivili quantization conditions and the Grassi-Hatsuda-Mariño conjecture. Blowup equations also give many interesting identities among modular forms and Jacobi forms. Furthermore, we study the relation between the elliptic genera of pure gauge 6d $(1,0)$ SCFTs and the superconformal indices of certain 4d $\mathcal{N}=2$ SCFTs. At last, we study the K-theoretic blowup equations on $\mathbb{Z}_2$ orbifold space and their connection with the bilinear relations of q-deformed periodic Toda systems.

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Contents

Chapter 1

Introduction

Blowup formulas originated from the attempt to understand the relation between the Donaldson invariants of a four-manifold X and those of its one-point blowup $\widehat{X}$. Based on the pioneering works (Kronheimer and Mrowka, 1994; Taubes, 1994; Bryan, 1997; Ozsváth et al., 1994), Fintushel and Stern proposed a concise form of the blowup formulas for the $SU(2)$ and $SO(3)$ Donaldson invariants in (Fintushel and Stern, 1996). In Donaldson-Witten theory, the Donaldson invariants are realized as the correlation functions of certain observables in the topologically twisted 4D $\mathcal{N}=2$ supersymmetric Yang-Mills theory (Witten, 1988). After the breakthrough of Seiberg-Witten theory (Seiberg and Witten, 1994a; Seiberg and Witten, 1994b), the generating function for these correlators can be computed by using the low-energy exact solutions (Witten, 1994). Therefore, the blowup formulas can be regarded as certain universal property of 4d $\mathcal{N}=2$ theories. This was extensively studied using the technique of u-plane integral (Moore and Witten, 1997) and soon was generalized to $SU(N)$ cases (Marino and Moore, 1998; Edelstein, Gomez-Reino, and Marino, 2000). In fact, the relation can already be seen in (Fintushel and Stern, 1996) that the Seiberg-Witten curve naturally appears in the setting of blowup formulas. Besides, the blowup fromulas are also closely related to the wall-crossing of Donaldson invariants (Göttsche, 1996; Göttsche and Zagier, 1996), integrable (Whitham) hierarchies (Takasaki, 2000; Takasaki, 1999; Marino, 1999) and contact term equations (Losev, Nekrasov, and Shatashvili, 1998; Lossev, Nekrasov, and Shatashvili, 1999).

In (Nekrasov, 2003), the 4d $\mathcal{N}=2$ gauge theories were formulated on the so called Omega background $\Omega_{\epsilon_1,\epsilon_2}$, which is a two-parameter deformation of $\mathbb{C}^2$. In physics, this means to turn on the graviphoton background field and $\epsilon_{R/L}=\frac{1}{2}(\epsilon_1\pm\epsilon_2)$ denote the self-dual and anti-self-dual parts of the graviphoton field strength respectively. Such background breaks the Poincare symmetry but maximally preserves the supersymmetry. The partition function computable from the localization on instanton moduli space can reproduce the Seiberg-Witten prepotential in the limit $\epsilon_1,\epsilon_2\to 0$, which was conjectured in (Nekrasov, 2003) and independently proved by (Nakajima and Yoshioka, 2005a), (Nekrasov and Okounkov, 2006) and (Braverman and Etingof, 2004) from different viewpoints. In the first approach, a generalization of the blowup formulas containing the two deformation parameters was proposed and proved, which played a crucial role to confirm Nekrasov's conjecture, see also (Nakajima and Yoshioka, 2003). Mathematically, the Nekrasov instanton partition function for gauge group $U(N)$ is defined as the generating function of the integral of the equivariant cohomology class 1 of the moduli space $M(N,n)$ of framed torsion free sheaves E of $\mathbb{P}^2$ with rank N, $c_2 = n$:

$$Z^{\text{inst}}_{\text{Nek}}(\epsilon_1, \epsilon_2, \vec{a}; \mathfrak{q}) = \sum_{n=0}^{\infty} \mathfrak{q}^n \int_{M(N,n)} 1, \tag{1.0.1}$$

where the framing is a trivialization of the restriction of E at the line at infinity ℓ_∞. On the blowup $\widehat{\mathbb{P}}^2$ with exceptional divisor C, one can define similar partition function via the moduli space $\widehat{M}(N,k,n)$, where $\langle c_1(E), [C]\rangle = -k$ and $\langle c_2(E) - \frac{N-1}{2N} c_1(E)^2, [\widehat{\mathbb{P}}^2]\rangle = n$. Based on the localization computation on the fixed point set of $\mathbb{C}^* \times \mathbb{C}^*$ in $\widehat{\mathbb{C}}^2 = \widehat{\mathbb{P}}^2 \backslash \ell_\infty$, such partition function can be represented in terms of the original Nekrasov partition function. The blowup formulas connect the Nekrasov partition functions on $\mathbb{C}^2$ and $\widehat{\mathbb{C}^2}$, which result in a system of functional equations for the original Nekrasov partition function.

Lifted by a circle, we move to the $\mathcal{N} = 1$ supersymmetric gauge theories on the 5d Omega background. The partition function here becomes K-theoretic and relates to the equivariant Donaldson invariants. The K-theoretic Nekrasov partition function is defined mathematically by replacing the integration in the equivariant cohomology by one in equivariant K-theory:

$$Z^{\text{inst}}_{\text{Nek}}(\epsilon_1, \epsilon_2, \vec{a}; \mathfrak{q}, \boldsymbol{\beta}) = \sum_n \left(\mathfrak{q} \boldsymbol{\beta}^{2N} e^{-N\boldsymbol{\beta}(\epsilon_1+\epsilon_2)/2} \right)^n \sum_i (-1)^i \mathrm{ch} H^i \left(M(N,n), \mathcal{O} \right), \tag{1.0.2}$$

where $\boldsymbol{\beta}$ is the radius of the circle. When $\boldsymbol{\beta} \to 0$, the K-theoretic partition function becomes the homological one. It was proved in (Nakajima and Yoshioka, 2005b) that such partition function also satisfies certain blowup formulas. Besides, one can also consider the situation with 5d Chern-Simons term of which the coefficient $m = 0, 1, \ldots, N$ (Intriligator, Morrison, and Seiberg, 1997; Tachikawa, 2004). The corresponding blowup formulas were conjectured in (Gottsche, Nakajima, and Yoshioka, 2009a) and proved in (Nakajima and Yoshioka, 2011), which we call the *Göttsche-Nakajima-Yoshioka K-theoretic blowup equations*. Such equations are one of our starting points of this thesis.

Geometric engineering connects the supersymmetry gauge theory with the topological string theory on certain local Calabi-Yau manifolds (Katz, Klemm, and Vafa, 1997). Such correspondence can be established on classical level $\epsilon_1, \epsilon_2 \to 0$, self-dual level $\epsilon_1 + \epsilon_2 \to 0$, $\epsilon_1 = g_s$, quantum level $\epsilon_1 \to 0$, $\epsilon_2 = \hbar$ and refined level generic ϵ_1, ϵ_2 (Iqbal, Kozcaz, and Vafa, 2009). Each level contains rich structures in mathematical physics. The typical example on refined level is the correspondence between the 5d $\mathcal{N} = 1$ $SU(N)$ gauge theory with Chern-Simons coefficient m on Omega background and the refined topological string theory on local toric Calabi-Yau threefold $X_{N,m}$, which is the resolution of the cone over the $Y^{N,m}$ singularity. The description of such geometries can be found in (Brini and Tanzini, 2009). Physically, one can consider M-theory compactified on local Calabi-Yau threefold X with Kähler moduli t, then the BPS particles in the 5d supersymmetric gauge theory arising from M2-branes wrapping the holomorphic curves in X. Besides the homology class $\beta \in H_2(X, \mathbb{Z})$ which can be represented by a degree vector d, these particles are in addition classified by their spins (j_L, j_R) under the 5d little group $SU(2)_L \times SU(2)_R$. The multiplicities $N^d_{j_L, j_R}$ of the BPS particles are called the *refined BPS invariants*. The instanton partition function of refined topological string can be obtained from the refined Schwinger-loop calculation (Iqbal, Kozcaz, and Vafa, 2009)

$$Z_{\mathrm{ref}}^{\mathrm{inst}}(\epsilon_1,\epsilon_2,t)=\prod_{j_L,j_R,d}\prod_{m_L=-j_L}^{j_L}\prod_{m_R=-j_R}^{j_R}\prod_{m_1,m_2=1}^{\infty}\left(1-q_L^{m_L}q_R^{m_R}q_1^{m_1-\frac{1}{2}}q_2^{m_2-\frac{1}{2}}e^{-d\cdot t}\right)^{(-1)^{2(j_L+j_R)}N^d_{j_L,j_R}},\tag{1.0.3}$$

where $q_{1,2}=e^{\epsilon_{1,2}}$ and $q_{R/L}=e^{\epsilon_{R/L}}$. With appropriate identification of parameters, this is equivalent to the refined Pandharipande-Thomas partition function, which is rigorously defined in mathematics as the generating function of the counting of refined stable pairs on X (Choi, Katz, and Klemm, 2014). See other definitions of the refined invariants in (Nekrasov and Okounkov, 2014) and (Maulik and Toda, 2016). The consequence of geometric engineering is the equivalence between the K-theoretic Nekrasov partition function and the partition function of refined topological string, with appropriate identification among the Coulomb parameters $\vec{a}$ and the Kähler moduli t. Therefore, the blowup formulas satisfied by the K-theoretic Nekrasov partition function can also be regarded as the functional equations of the partition function of refined topological string, at least for those local Calabi-Yau which can engineer supersymmetric gauge theories. One main purpose of the thesis is to generalize such functional equations to arbitrary local Calabi-Yau threefolds.

The other clue of blowup formulas for general local Calabi-Yau came from the recent study on the exact quantization of mirror curves, which is within the framework of B model of topological strings. It is well known that the mirror of a local Calabi-Yau threefold is effectively a Riemann surface, called *mirror curve* (Chiang et al., 1999). On the classical level, the B-model topological string is governed by the special geometry on the mirror curve. All physical quantities in the geometric engineered gauge theory such as Seiberg-Witten differential, prepotential, periods and dual periods have direct correspondences in the special geometry. On the quantum level, the high genus free energy of topological string can be computed by the holomorphic anomaly equations (Bershadsky et al., 1994). For compact Calabi-Yau threefolds, the holomorphic anomaly equations are normally not sufficient to determine the full partition function due to the holomorphic ambiguities, while for local Calabi-Yau, new symmetry emerges whose Ward identities are sufficient to completely determine the partition function at all genera. This is based on the observation on the relations among quantum mirror curves, topological strings and integrable hierarchies (Aganagic et al., 2006). The appearance of integrable hierarchies here is not surprising since the correspondence between the 4d $\mathcal{N}=2$ gauge theories and integrable systems have been proposed in (Gorsky et al., 1995; Martinec and Warner, 1996) and well studied in 1990s, see for example (D'Hoker and Phong, 1999). One can regard the relation web in the context of local Calabi-Yau as certain generalization. In mathematics, the using of mirror curve to construct the B-model partition function on local Calabi-Yau is usually called topological recursion (Eynard and Orantin, 2007) or BKMP remodeling conjecture (Bouchard et al., 2009), which was rigorously proved in (Eynard and Orantin, 2015).

In (Nekrasov and Shatashvili, 2009a), the chiral limit ($\epsilon_1\to 0,\ \epsilon_2=\hbar$) was studied and it was found that the quantization of the underlying integrable systems is governed by the supersymmetric gauge theories in such limit. The Nekrasov-Shatashvili (NS) free energy (effective twisted superpotential) which is the chiral limit of Nekrasov partition function serves as the Yang-Yang function of the quantum integrable systems while the supersymmetric vacua become the eigenstates and the supersymmetric vacua equations become the thermodynamic Bethe ansatz. Mathematically, this equates quantum K-theory of a Nakajima quiver variety with Bethe equations for a certain quantum affine Lie algebra. Via geometric engineering, such correspondence can be rephrased as a direct relation between the quantum phase volumes of the mirror curve of a local Calabi-Yau and the Nekrasov-Shatashvili free energy of topological string. Now the Bethe ansatz is just the traditional Bohr-Sommerfeld or Einstein-Brillouin-Keller quantization conditions for the mirror curves (Aganagic et al., 2012). For certain local toric Calabi-Yau, topological string theory is directly related to 5d gauge theory. Thus certain non-perturbative contributions are expected to appear. The exact quantization conditions were proposed in (Wang, Zhang, and Huang, 2015) for toric Calabi-Yau with genus-one mirror curve, and soon were generalized to arbitrary toric cases in (Franco, Hatsuda, and Mariño, 2016). Such exact quantization conditions were later derived in (Sun, Wang, and Huang, 2017) by replacing the original partition function to the Lockhart-Vafa partition function of non-perturbative topological string (Lockhart and Vafa, 2018). On the other hand, Grassi-Hatsuda-Mariño proposed an entirely different approach to quantize the mirror curve (Grassi, Hatsuda, and Marino, 2016). This approach takes root in the study on the non-perturbative effects in ABJM theories on three sphere, which is dual to topological string on local $\mathbb{P}^1 \times \mathbb{P}^1$ (Marino and Putrov, 2010). The equivalence between the two quantization approaches was established in (Sun, Wang, and Huang, 2017) by introducing the so called r fields and certain compatibility formulas which are constraint equations for the refined free energy of topological string theory. It was later realized in (Grassi and Gu, 2016) that for $SU(N)$ geometries $X_{N,m}$ such compatibility formulas were exactly the Nekrasov-Shatashvili limit of the vanishing part of Göttsche-Nakajima-Yoshioka K-theoretic blowup equations. This inspired that the constraint equations in (Sun, Wang, and Huang, 2017) should be able to generalize to refined level, as was suggested in (Gu et al., 2017) and called generalized blowup equations. It was also shown in (Gu et al., 2017) that the partition function of E-string theory which is equivalent to the refined topological string on local half K3 satisfies the generalized blowup equations. This suggests blowup formulas should exist for non-toric Calabi-Yau as well.

The Göttsche-Nakajima-Yoshioka K-theoretic blowup equations can be divided to two sets of equations. Roughly speaking, the equations in one set indicate that certain infinite bilinear summations of Nekrasov partition function vanish, while those in the other set indicate that certain other infinite bilinear summations give the Nekrasov partition function itself. The former set of equations was generalized to the refined topological string on generic local Calabi-Yau in (Gu et al., 2017), which we call the *vanishing blowup equations* in this thesis. The latter set of equations will be generalized in this thesis, which we call the *unity blowup equations*.

The full partition function of refined topological string $Z_{\text{ref}}(\epsilon_1, \epsilon_2; t)$ is the product of the instanton partition function (1.0.3) and the perturbative contributions which will be given in (3.1.24). To make contact with the quantization of mirror curve, we also need to make a twist to the original partition function, denoted as $\widehat{Z}_{\text{ref}}(\epsilon_1, \epsilon_2; t)$. Such twist which will be defined in (3.1.22) does not lose any information of the partition function, in particular the refined BPS invariants. Then blowup equations are the functional equations of the twisted partition function of refined topological string.

One main result of the thesis is as follows: *For an arbitrary local Calabi-Yau threefold*

X with mirror curve of genus g, suppose there are $b = \dim H_2(X,\mathbb{Z})$ irreducible curve classes corresponding to Kähler moduli t, and denote C as the intersection matrix between the b curve classes and the g irreducible compact divisor classes, then there exist infinite constant integral vectors $r \in \mathbb{Z}^b$ such that the following functional equations for the twisted partition function of refined topological string on X hold:

$$\begin{aligned}&\sum_{n\in\mathbb{Z}^g}(-)^{|n|}\,\widehat{Z}_{\text{ref}}\left(\epsilon_1,\epsilon_2-\epsilon_1;t+\epsilon_1 R\right)\cdot\widehat{Z}_{\text{ref}}\left(\epsilon_1-\epsilon_2,\epsilon_2;t+\epsilon_2 R\right)\\&\qquad=\begin{cases}0, & \text{for } r\in\mathcal{S}_{\text{vanish}},\\ \Lambda(\epsilon_1,\epsilon_2;m,r)\widehat{Z}_{\text{ref}}\left(\epsilon_1,\epsilon_2;t\right), & \text{for } r\in\mathcal{S}_{\text{unity}},\end{cases}\end{aligned}\tag{1.0.4}$$

where $|n| = \sum_{i=1}^{g} n_i$, $R = C\cdot n + r/2$ and Λ only depends on ϵ_1,ϵ_2,r and "mass parameters" m which are part of the Kähler moduli associated to the $b-g$ curve classes that have zero interesection with all g divisor classes. In addition, all the vector r are the representatives of the B field of X, which means for all triples of degree d, spin j_L and j_R such that the refined BPS invariants $N^d_{j_L,j_R}(X)$ is non-vanishing, they must satisfy

$$(-1)^{2j_L+2j_R-1} = (-1)^{r\cdot d}.\tag{1.0.5}$$

Besides, both sets $\mathcal{S}_{\text{vanish}}$ and $\mathcal{S}_{\text{unity}}$ are finite under the quotient of shift $2C\cdot n$ symmetry.

Let us make a few remarks here. For local toric Calabi-Yau, the matrix C is just part of the charge matrix of the toric action. The factor $\Lambda(\epsilon_1,\epsilon_2;m,r)$ in the unity blowup equations normally has very simple expression and can be easily determined from the polynomial part of topological strings. It is important that factor Λ only depends on the mass parameters, but not on true moduli. In addition, the blowup equations (1.0.4) is invariant under the shift $t \to t + 2C\cdot n$, thus we only need to consider the equivalent classes of the r fields. Let us denote the corresponding symmetry group as Γ_C for later use. The condition (1.0.5) actually can be derived from the blowup equations, as well be shown Chapter 4.2.2. Such condition was known as the B field condition which was established in (Hatsuda et al., 2014) for local del Pezzoes and in (Sun, Wang, and Huang, 2017) for arbitrary local toric Calabi-Yau.

Previously, the partition function of refined topological strings on local Calabi-Yau can be computed using the refined topological vertex in the A-model side (Iqbal, Kozcaz, and Vafa, 2009; Taki, 2008; Iqbal and Kozcaz, 2017), or refined holomorphic anomaly equations in the B-model side (Huang and Klemm, 2012; Huang, Klemm, and Poretschkin, 2013; Klemm et al., 2015). We use those results to check the validity of blowup equations. Reversely, we can assume the correctness of blowup equations and use them to determine the refined partition function. We find that blowup equations combined together are sufficient to determine the full partition function of refined topological string on a large class of local Calabi-Yau threefolds. While the holomorphic anomaly equations are directly related to the worldsheet physics and Gromov-Witten formulation, the blowup equations on the other hand are directly related to the target physics and Gopakumar-Vafa (BPS) formulation. Therefore, when refined BPS invariants are the main concern, the blowup equations usually are a more effective technique.

A particular interesting class of local Calabi-Yau threefolds comes from the study on *6d* $(1,0)$ *superconformal field theories* (SCFTs) in the recent decade (Morrison and

Taylor, 2012; Heckman, Morrison, and Vafa, 2014; Heckman et al., 2015). They are certain elliptic fibration over some non-compact base surface S in which all curve classes can be simultaneously shrinkable to zero volume. See an excellent review (Heckman and Rudelius, 2019). The geometry of the base S directly translates into the tensor branch of the 6d SCFTs where the number of tensor multiplets – also called the *rank* of a 6d SCFT – is given by the dimension of $H^{1,1}(S,\mathbb{Z})$ and the intersection form on S gives the couplings of the tensor multiplets to each other. The gauge groups G of a 6d SCFT are given by Kodaira singularity type of the elliptic fibration on each curve in S. One can also add matter – hypermultiplets in representation $\mathfrak{R}$ – at the intersection point between two curves in S. The Calabi-Yau condition puts strong restrictions on the possible intersection form, gauge symmetry and matter representations. The full classification was achieved in (Heckman et al., 2015), called "atomic classification", where some low rank theories called "non-Higgsable clusters" serving as "atoms" are linked together by certain generalized quiver structures, which we will review in Chapter 5.1. Besides, the hypermultiplets can enjoy certain global symmetry – the flavor group F, see (Bertolini, Merkx, and Morrison, 2016; Del Zotto and Lockhart, 2018). For example, when specializing to rank one, the base surface S can only be $\mathcal{O}_{\mathbb{P}^1}(-n)$, with $n = 1,2,3,4,5,6,7,8,12$. Some simple rank one theories called "minimal" 6d $(1,0)$ SCFTs are shown in Table 1.1. The full list of rank one 6d $(1,0)$ SCFTs with their $(n,G,F,\mathfrak{R})$ quadruples (Del Zotto and Lockhart, 2018) will be shown in Table 5.2, 5.3 and 5.4.

$\mathfrak{n}$	1	2	3	4	5	6	7	8	12
gauge G	–	–	$\mathfrak{su}(3)$	$\mathfrak{so}(8)$	F_4	E_6	E_7	E_7	E_8
flavor F	E_8	$\mathfrak{su}(2)$	–	–	–	–	–	–	–
hypers in $\mathfrak{R}$	–	–	–	–	–	–	$\frac{1}{2}\mathbf{56}$	–	–

Table 1.1: List of minimal 6d SCFTs. The $n = 1$ case is also called E-string theory. The $n = 2$ case is also called M-string theory which actually has $(2,0)$ supersymmetry.

Compactifying F theory on the above *ellitpic non-compact Calabi-Yau threefolds*, one obtain tons of 6d $(1,0)$ gauge theories which restore conformal invariance when all the curves in S shrink to zero size. We are interesting in the partition function of these 6d gauge theories – 6d $(1,0)$ SCFTs on tensor branch – on 6d Omega background $\mathbb{C}^2_{\epsilon_1,\epsilon_2} \times T^2_\tau$. The full partition function Z contains three parts: classical, one-loop and the elliptic genera of some BPS strings:

$$Z(\phi,\tau,m_{G,F},\epsilon_{1,2}) = \\ Z^{\text{cls}}(\phi,\tau,m_{G,F},\epsilon_{1,2})Z^{\text{1-loop}}(\tau,m_{G,F},\epsilon_{1,2})\Big(1+\sum_d e^{i2\pi\phi\cdot d}\mathbb{E}_d(\tau,m_{G,F},\epsilon_{1,2})\Big). \tag{1.0.6}$$

Here $e^{i2\pi\phi}$ is the counting parameter for the number of BPS strings, playing a role like the instanton count parameter $\mathfrak{q}$ in 4d and 5d gauge theories. The classical and one-loop part can be easily determined from the gauge theory data $(n,G,F,\mathfrak{R})$, while the elliptic genera – as the natural elliptic lift of K-theoretic Nekrasov partition function – are in general very hard to compute. A lot of methods have been devoleped in the recent decade to compute these elliptic genera of 6d $(1,0)$ SCFTs, including 2d quiver gauge theories (Haghighat et al., 2014; Haghighat et al., 2015a; Kim et al., 2014; Kim, Kim, and Park, 2016; Kim, Kim, and Lee, 2015; Yun, 2016; Kim et al., 2018; Haghighat et al., 2015b; Del Zotto and Lockhart, 2018), modular bootstrap (Del Zotto and Lockhart, 2018; Kim, Lee, and Park, 2018; Del Zotto et al., 2018; Gu et al., 2017; Duan, Gu, and Kashani-Poor, 2018), refined topological vertex (Kim, Taki, and Yagi, 2015; Hayashi et al., 2019a; Hayashi et al., 2019b; Kim, Kim, and Lee, 2019; Hayashi et al., 2015; Hayashi and Ohmori, 2017), domail walls (Haghighat, Lockhart, and Vafa, 2014; Cai, Huang, and Sun, 2015) and twisted H_G theories (Putrov, Song, and Yan, 2016; Agarwal, Maruyoshi, and Song, 2018). There are also other checks for elliptic genera can be made from the topological string computation in B model (Huang, Klemm, and Poretschkin, 2013; Haghighat et al., 2015b) and 5d Nekrasov partition functions in the $q_\tau = e^{i2\pi\tau} \to 0$ limit. Each of the above methods usually works for some special theories or some specially limit. We will give a full review on the current status of each method in Chapter 5.1.3.

We now want to develop a universal method to solve the elliptic genera. Based on the geometric enegineering, the full partition function $Z(\phi, \tau, m_{G,F}, \epsilon_{1,2})$ of 6d $(1,0)$ SCFTs is also the partition function of refined topological string theory on the elliptic non-compact Calabi-Yau. Therefore, we can use blowup equations (1.0.4) and translate into some functional equations of ellitpic genera. Indeed, we found an elegant form of such functional equation for the elliptic genera of arbitrary 6d $(1,0)$ SCFTs in the atomic classification, which we call *elliptic blowup equations*. In such equations, all components are some Jacobi forms and the whole equations enjoy nice modularity under $SL(2,\mathbb{Z})$ transformation of the torus. They are also the natural elliptic lift of the K-theoretic blowup equations in (Nakajima and Yoshioka, 2005b; Gottsche, Nakajima, and Yoshioka, 2009a; Nakajima and Yoshioka, 2011; Keller and Song, 2012; Kim et al., 2019).

Consider a rank one 6d SCFT with tensor branch coefficient $\mathfrak{n}$, gauge symmetry G, flavor symmetry F, and half-hypermultiplets transforming in the representations $\mathfrak{R} = (R_G, R_F)$. The flavor symmetry induces a current algebra of level k_F on the worldsheet of BPS strings. Then the elliptic genera $\mathbb{E}_d(\tau, m_{G,F}, \epsilon_{1,2})$ satisfy the following elliptic blowup equations

$$\begin{aligned}
&\sum_{\lambda_G \in \phi_{\lambda_0}(Q^\vee(G))}^{\frac{1}{2}||\lambda_G||^2+d'+d''=d+\delta} (-1)^{|\phi_{\lambda_0}^{-1}(\lambda_G)|} \\
&\times \theta_i^{[a]}(\mathfrak{n}\tau, -\mathfrak{n}\lambda_G\cdot m_G + k_F\lambda_F\cdot m_F + (y - \frac{n}{2}||\lambda_G||^2)(\epsilon_1+\epsilon_2) - \mathfrak{n}d'\epsilon_1 - \mathfrak{n}d''\epsilon_2) \\
&\times A_V(\tau, m_G, \lambda_G) A_H^{\mathfrak{R}}(\tau, m_G, m_F, \lambda_G, \lambda_F) \\
&\times \mathbb{E}_{d'}(\tau, m_G+\epsilon_1\lambda_G, m_F+\epsilon_1\lambda_F, \epsilon_1, \epsilon_2-\epsilon_1)\mathbb{E}_{d''}(\tau, m_G+\epsilon_2\lambda_G, m_F+\epsilon_2\lambda_F, \epsilon_1-\epsilon_2, \epsilon_2) \\
&= \begin{cases} 0, & \delta > 0, \\ \theta_i^{[a]}(\mathfrak{n}\tau, k_F\lambda_F\cdot m_F + \mathfrak{n}\,y(\epsilon_1+\epsilon_2))\mathbb{E}_d(\tau, m_G, m_F, \epsilon_1, \epsilon_2), & \delta = 0. \end{cases}
\end{aligned} \tag{1.0.7}$$

Here the subscript i of Jacobi theta function is 3 if $\mathfrak{n}$ is even and 4 if $\mathfrak{n}$ is odd, and the characteristic a can be $a = 1/2 - k/\mathfrak{n}$ with $k = 0, 1, \ldots, \mathfrak{n}-1$. The parameter $y = (\mathfrak{n} - 2 + h_{\mathfrak{g}}^\vee)/4 + k_F(\lambda_F\cdot\lambda_F)/2$. The map ϕ_{λ_0} induces a shift of coroot lattice in the coweight lattice. For example when there is no shift at all, $\lambda_G \in Q^\vee(G)$, then $\frac{1}{2}||\lambda_G||^2 \in \mathbb{Z}$ and $\delta = 0$, one obtains unity blowup equations. For the precise definitions of ϕ_{λ_0}, $|\phi_{\lambda_0}^{-1}(\lambda_G)|$ and functions A_V, $A_H^{\mathfrak{R}}$ which are contributions from vector and hypermultiplets in the one-loop part of 6d gauge theories, we refer to

Chapter 5.2. We develop several techniques to solve the elliptic genera. In particular, from the unity part of the above elliptic blowup equations, we are able to solve the elliptic genera for *almost all* rank one 6d SCFTs, except 12 theories with unpaired half-hypermulitplets. We further generalize equation (1.0.7) to arbitrary rank 6d SCFTs in Chapter 6.

Using the elliptic genera we solved from blowup equations, we are able to do lots of checks and study new phenomena. First of all, we checked our results on elliptic genera could recover all previous partial results from each methods in 6d and 5d. We also find some universal behaviors of the elliptic genera which will be given in Chapter 5.4. Besides, we study an interesting conjecture proposed by (Del Zotto and Lockhart, 2017) on the relation between the elliptic genera of pure gauge 6d $(1,0)$ SCFTs and the superconformal indices – to be precise, Hall-Littlewood indices and Schur indices – of some 4d $\mathcal{N}=2$ SCFTs called H_G theories. Benefited the two and three string elliptic genera we obtain, we are able to generalize the conjecture in (Del Zotto and Lockhart, 2017) from rank one and higher rank, which will be given in Chapter 7.

One more interesting direction of generalization is to consider the blowup equations on the asymptotically locally Euclidean spaces (ALE spaces), which are defined as $\mathbb{C}^2/\Gamma$ with Γ a finite subgroup of $SU(2)$. The instanton parition function can be defined on such spaces by a modification of ADHM construction (Kronheimer and Nakajima, 1990; Nakajima, 2018). The Nekrasov partition function on the resolved ALE space can be computed by fixed point theorem just like the $\widehat{\mathbb{C}^2}$ case, see for instance (Bonelli, Maruyoshi, and Tanzini, 2011; Bonelli, Maruyoshi, and Tanzini, 2012a). While the Nekrasov partition function on the orbifold ALE were explicitly computed for instance in (Fucito, Morales, and Poghossian, 2004). Just like the blowup equations on $\mathbb{C}^2$, there exist blowup equations on ALE spaces which connect the resolved and orbifold partition function. For example, lots of 4d blowup equations on A type ALE spaces with gauge group $U(N)$ have been found in (Bonelli, Maruyoshi, and Tanzini, 2011; Bonelli, Maruyoshi, and Tanzini, 2012a; Belavin et al., 2013; Ito, Maruyoshi, and Okuda, 2013; Bruzzo et al., 2016; Bruzzo, Sala, and Szabo, 2015). For a special case of gauge $SU(2)$ theory with $N_f=4$, the conjectural blowup equations in (Ito, Maruyoshi, and Okuda, 2013) was rigoriously proved in (Ohkawa, 2018).

One particular interesting case is the blowup equations on A_1-ALE space, that is $\mathbb{C}^2/\mathbb{Z}_2$ or equivalently $\mathcal{O}_{\mathbb{P}^1}(-2)$. It was first found in (Bershtein and Shchechkin, 2015) that such Z_2 type blowup equations for pure $SU(2)$ gauge theory can be related to the bilinear relations of the Tau functions of Painlevé III_3 system. The relation between the Tau functions of Painlevé systems and Nekrasov parition function for 4d $\mathcal{N}=2$ $SU(2)$ gauge theories dates back to the seminar works (Gamayun, Iorgov, and Lisovyy, 2012; Gamayun, Iorgov, and Lisovyy, 2013). Simply speaking, the Tau functions of Painlevé systems can be expressed by an exact formula from certain conformal blocks, which via AGT correspondence are related to the Nekrasov partition function. This is called *Painlevé/gauge theory correspondence*, or more generally *isomonodromic/CFT correspondence*. See an excellent review in (Bonelli et al., 2016). With this correspondence, the bilinear relations of Tau functions are translated into some interesting functional equations of Nekrasov partition functions. Note this is different from the recent proposal in (Nekrasov, 2020; Jeong and Nekrasov, 2020) where the relation between Tau function and Nekrasov partition function is directly derived from the blowup equations with surface defects. We comment on these blowup equations with defect in the outlook Chapter 9.

The relation found in (Bershtein and Shchechkin, 2015) was later extended to the q-deformed Painlevé III$_3$ system in (Bershtein and Shchechkin, 2017; Shchechkin, 2020), which was also connected to the topological string on local $\mathbb{P}^1 \times \mathbb{P}^1$ in (Bonelli, Grassi, and Tanzini, 2019). It is natural to expect the Tau functions of all q-Painlevé systems are related to the K-theoretic Nekrasov partition functions or more generally topological string partition functions (Bonelli, Grassi, and Tanzini, 2019; Bonelli, Del Monte, and Tanzini, 2020). Besides, the study on the bilinear relations of Tau functions of q-deformed periodic Toda systems also results in some interesting functional equation for $SU(N)$ K-theoretic Nekrasov partition functions (Bershtein, Gavrylenko, and Marshakov, 2019). In this thesis, we propose some new Z_2 type K-theoretic blowup equations for gauge group $SU(N)$ and use them to derive some conjectural relations among K-theoretic Nekrasov partition functions in (Bershtein and Shchechkin, 2017; Bershtein, Gavrylenko, and Marshakov, 2018; Bershtein, Gavrylenko, and Marshakov, 2019).

The thesis is organized as follows. In Chapter 2, we review the Nekrasov partition function and blowup equations in 4d and 5d, which are the starting point of generalized blowup equations we obtained in this thesis. A new result is the full list of K-theoretic blowup equations for pure gauge theories with all simple Lie gauge groups in Chapter 2.3. In Chapter 3, we review the other starting point of generalized blowup equations, which is the compatibility formulas (Chapter 3.5) between two kinds of quantization of mirror curves – the Nekrasov-Shatashivili quantization in Chapter 3.3 and Grassi-Hatsuda-Mariño conjecture in Chapter 3.4. Before this, we review some basics on refined topological string theory, local Calabi-Yau and local mirror symmetry in Chapter 3.1 and 3.2. In Chapter 4, we study in detail the blowup equations for refined topological string on general local Calabi-Yau threefolds. In particular, we study the ϵ_1, ϵ_2 expansion of blowup equations called *component equations* in Chapter 4.1, using which we derive the modularity of blowup equations and the consistency with refined holomorphic/modular anomaly equations in Chapter 4.4. We propose the non-holomorphic version of blowup equations in Chapter 4.4.5 and give a physical picture in M-theory about why blowup equations could exist for general local Calabi-Yau in Chapter 4.5. We further show how to solve refined free energy and refined BPS invariants in Chapter 4.3 and eleborate our theory with some examples in Chapter 4.6.

Chapter 5 and 6 are devoted to elliptic blowup equations for rank one 6d $(1,0)$ SCFTs and arbitrary rank cases respectively. We seperate them because the elliptic genera for almost all rank one theories can be solved from blowup equations while for most higher rank theories can not. Given that existence of every rank one 6d $(1,0)$ SCFT is quite remarkable, we extensively study the elliptic blowup equations for each of them. The full list of both unity and vanishing blowup equations for all rank one theories are given in Table 5.5, 5.6, 5.7, 5.8 and 5.9. We also prove the modularity of our elliptic blowup equations in Chapter 5.2.3 and propose two efficient method to solve the elliptic genera – recursion formula and Weyl orbit expansion in Chapter 5.3. We explicitly present the elliptic blowup equations and our computation results on the elliptic genera of lots of rank one 6d SCFTs in Chapter 5.5 and Appendix D. For higher rank theories, not only we give an *gluing rule* to write down the elliptic blowup equations for arbitrary 6d $(1,0)$ SCFTs in the atomic classification in Chapter 6.1, we also explicitly present all the elliptic blowup equations for a lot of examples including the E–, M–string chain, three higher rank non-Higgsable clusters, ADE chain of -2 curves with gauge symmetry, all conformal matter theories and the blownups of some $-n$ curves in particular $-9, -10, -11$ curves in Chapter 6.2, 6.3, 6.4, 6.5 and 6.6.

In Chapter 7, we take a small detour from blowup equations and use the elliptic genera we solved in Chapter 5 to study an interesting conjecture proposed in (Del Zotto and Lockhart, 2017) on the relation between elliptic genera and the superconformal indices of 4d $\mathcal{N} = 2$ H_G SCFTs. In particular, we compute the Hall-Littlewood indices and Schur indices for lots of rank two and three H_G theories. In Chapter 8, we come back to blowup equations and focus on the K-theoretic blowup equations with $SU(N)$ gauge group on $\mathbb{C}^2/\mathbb{Z}_2$ and their application to bilinear relations of Tau functions of some q-deformed isomonodromic systems.

The relation between this thesis and author's publications is as follows. Chapter 2 is mainly a review on literature except in Chapter 2.3 the full list of K-theoretic blowup equations for all simple Lie gauge groups is based on the Section 3.5 of (Gu et al., 2020b). The contents of Chapter 3 and 4 are largely based on (Huang, Sun, and Wang, 2018), except that Section 4.4 is based on author's unpublished results. Part of the results in (Huang, Sun, and Wang, 2018) were also stated in the doctor thesis of Wang in Universitiy of Science of Technology of China in 2018. The contents of Chapter 5 and 6 are based on a series of publications (Gu et al., 2019a; Gu et al., 2019b; Gu et al., 2020a; Gu et al., 2020b). The contents of Chapter 7 are based on the Section 5 of (Gu et al., 2019b). The contents of Chapter 8 are based on some unpublished results collaborating with Giulio Bonelli and Alessandro Tanzini.

Chapter 2

Nekrasov Partition Function and Blowup Equations

2.1 Nekrasov partition functions and the K-theoretic version

In this section, we give a brief review for the (K-theoretic) Nekrasov partition function. Our conventions are the same with (Nakajima and Yoshioka, 2005a; Nakajima and Yoshioka, 2005b) which is slightly different the physical ones. We begin with the pure $U(N)$ gauge theory. As mentioned in the introduction, the instanton Nekrasov partition function on $\mathbb{P}^2$ is defined as

$$\begin{aligned} Z^{\text{inst}}(\epsilon_1,\epsilon_2,\vec{a};\mathfrak{q}) &= \sum_{n=0}^{\infty} \mathfrak{q}^n Z_n(\epsilon_1,\epsilon_2,\vec{a}), \\ Z_n(\epsilon_1,\epsilon_2,\vec{a}) &= \int_{M(N,n)} 1. \end{aligned} \tag{2.1.1}$$

Here $\mathfrak{q}$ is the instanton counting parameter, $M(N,n)$ is the moduli space of framed torsion free sheaves E of $\mathbb{P}^2$ with rank N, $c_2 = n$. The Cartan generators $\vec{a}$ of $U(N)$ gauge group are often called Coulomb parameters in physics literature.

In the seminar work (Nekrasov, 2003), the instanton partition function was exactly computed using localization formulas. We state the result following the notations in (Nakajima and Yoshioka, 2005a). Let $Y = (\lambda_1 \geq \lambda_2 \geq \cdots)$ be a Young diagram, where λ_i is the length of the ith column. Let $Y' = (\lambda'_1 \geq \lambda'_2 \geq \ldots)$ be the transpose of Y. Thus λ'_j is the length of the jth row of Y. Let $l(Y)$ denote the number of columns of Y, i.e., $l(Y) = \lambda'_1$. Let

$$a_Y(i,j) = \lambda_i - j, \qquad l_Y(i,j) = \lambda'_j - i. \tag{2.1.2}$$

Here we set $\lambda_i = 0$ when $i > l(Y)$. Similarly $\lambda'_j = 0$ when $j > l(Y')$. Then the instanton partition function of 4d pure $U(N)$ gauge theory is given as:

$$Z^{\text{inst}}(\epsilon_1,\epsilon_2,\vec{a};\mathfrak{q}) = \sum_{\vec{Y}} \frac{\mathfrak{q}^{|\vec{Y}|}}{\prod\limits_{\alpha,\beta} n^{\vec{Y}}_{\alpha,\beta}(\epsilon_1,\epsilon_2,\vec{a})}, \tag{2.1.3}$$

where $\vec{Y} = \{Y_1, Y_2, \ldots, Y_N\}$ and $\alpha, \beta = 1, 2, \ldots, N$,

$$n^{\vec{Y}}_{\alpha,\beta}(\epsilon_1,\epsilon_2,\vec{a}) = \prod_{s\in Y_\alpha} (-l_{Y_\beta}(s)\epsilon_1 + (a_{Y_\alpha}(s)+1)\epsilon_2 + a_\beta - a_\alpha)$$

$$\times \prod_{t\in Y_\beta} ((l_{Y_\alpha}(t)+1)\epsilon_1 - a_{Y_\beta}(t)\epsilon_2 + a_\beta - a_\alpha).$$

In the case of gauge group $SU(N)$, one simply requires $\sum_{\alpha=1}^N a_\alpha = 0$.

It was conjectured in (Nekrasov, 2003) that the prepotential of 4d $\mathcal{N}=2$ pure $SU(N)$ Seiberg-Witten theory can be obtained from the above Nekrasov partition function combined with the perturbative and one-loop part as

$$\mathcal{F}_{SW}(\vec{a},\mathfrak{q}) = \lim_{\epsilon_1,\epsilon_2\to 0} \epsilon_1\epsilon_2 \log Z(\epsilon_1,\epsilon_2,\vec{a};\mathfrak{q}). \tag{2.1.4}$$

This was independently proved by (Nakajima and Yoshioka, 2005a), (Nekrasov and Okounkov, 2006) and (Braverman and Etingof, 2004) using rather different methods. Nekrasov partition function can also be computed by localization for BCD type gauge algebras, although there is no explicit combinatorial formula like the $SU(N)$ case (Nekrasov, 2003; Nekrasov and Shadchin, 2004). The Nekrasov partition function is also related to W-algebras as the norm of Gaiotto-Whittaker states or Whittaker vectors (Gaiotto, 2013; Bonelli and Tanzini, 2010; Taki, 2011; Keller et al., 2012), which has been rigorously proved in (Braverman, Finkelberg, and Nakajima, 2014).

Lifted to 5d, the instanton part of K-theoretic $U(N)$ Nekrasov partition function on $\mathbb{P}^2$ is defined as

$$\begin{aligned} Z^{\text{inst}}(\epsilon_1,\epsilon_2,\vec{a};\mathfrak{q},\boldsymbol{\beta}) &= \sum_{n=0}^{\infty} \left(\mathfrak{q}\boldsymbol{\beta}^{2N} e^{-N\boldsymbol{\beta}(\epsilon_1+\epsilon_2)/2}\right)^n Z_n(\epsilon_1,\epsilon_2,\vec{a};\boldsymbol{\beta}), \\ Z_n(\epsilon_1,\epsilon_2,\vec{a};\boldsymbol{\beta}) &= \sum_i (-1)^i \text{ch} H^i(M(N,n),\mathcal{O}). \end{aligned} \tag{2.1.5}$$

Here $\boldsymbol{\beta}$ is the radius of the fifth dimension S^1. In the physics literature, the factor $e^{-N\boldsymbol{\beta}(\epsilon_1+\epsilon_2)/2}$ is usually absorbed into the definition of $\mathfrak{q}$. Similarly, the instanton partition function of 5d pure $U(N)$ gauge theory can be computed from localization as:

$$Z^{\text{inst}}(\epsilon_1,\epsilon_2,\vec{a};\mathfrak{q},\boldsymbol{\beta}) = \sum_{\vec{Y}} \frac{(\mathfrak{q}\boldsymbol{\beta}^{2N}e^{-N\boldsymbol{\beta}(\epsilon_1+\epsilon_2)/2})^{|\vec{Y}|}}{\prod_{\alpha,\beta} n^{\vec{Y}}_{\alpha,\beta}(\epsilon_1,\epsilon_2,\vec{a};\boldsymbol{\beta})}, \tag{2.1.6}$$

where

$$\begin{aligned} n^{\vec{Y}}_{\alpha,\beta}(\epsilon_1,\epsilon_2,\vec{a};\boldsymbol{\beta}) = \prod_{s\in Y_\alpha} &\left(1-e^{-\boldsymbol{\beta}(-l_{Y_\beta}(s)\epsilon_1+(a_{Y_\alpha}(s)+1)\epsilon_2+a_\beta-a_\alpha)}\right) \\ &\times \prod_{t\in Y_\beta}\left(1-e^{-\boldsymbol{\beta}((l_{Y_\alpha}(t)+1)\epsilon_1-a_{Y_\beta}(t)\epsilon_2+a_\beta-a_\alpha)}\right). \end{aligned}$$

It is easy to see by taking the limit $\boldsymbol{\beta}\to 0$, one gets back the 4d Nekrasov partition function.

2.2 K-theoretic blowup equations for $SU(N)$

The blowup equations were firstly proposed for 4d Nekrasov partition function (Nakajima and Yoshioka, 2005a). However, as the 5d i.e K-theoretic blowup equations are the starting point of our generalization in this thesis, here we just review the 5d blowup equations. In fact, the 5d blowup equations (Nakajima and Yoshioka, 2005b) looks more elegant than the 4d ones (Nakajima and Yoshioka, 2005a).[1]

On the blowup $\widehat{\mathbb{P}}^2$ with exceptional divisor C, one can define similar instanton partition function depending on a pair of integers (k,d) by

$$Z^{\text{inst}}_{k,d}(\epsilon_1,\epsilon_2,\vec{a};\mathfrak{q},\beta)=\sum_{n=0}^{\infty}\left(\mathfrak{q}\beta^{2N}e^{-N\beta(\epsilon_1+\epsilon_2)/2}\right)^n(\iota_{0*})^{-1}\left(\widehat{\pi}_*(\mathcal{O}(d\mu(C)))\right), \tag{2.2.1}$$

where $k=0,1,2,\dots,N-1$ is an integer characterizing the blowup with $\langle c_1(E),[C]\rangle=-k$, and d is the degree of certain determinant line bundle μ associated to the exceptional divisor C. We refer the precise definition to (Nakajima and Yoshioka, 2005b). Using Atiyah-Bott-Lefschetz fixed points formula, one can compute it as

$$\begin{aligned}Z^{\text{inst}}_{k,d}(\epsilon_1,\epsilon_2,\vec{a};\mathfrak{q},\beta)=&\sum_{\{\vec{k}\}=-k/N}\frac{(e^{\beta(\epsilon_1+\epsilon_2)(d-N/2)}\mathfrak{q}\beta^{2N})^{(\vec{k},\vec{k})/2}e^{\beta(\vec{k},\vec{a})d}}{\prod_{\vec{\alpha}\in\Delta}l^{\vec{k}}_{\vec{\alpha}}(\epsilon_1,\epsilon_2,\vec{a},\beta)}\times\\ &Z^{\text{inst}}(\epsilon_1,\epsilon_2-\epsilon_1,\vec{a}+\epsilon_1\vec{k};e^{\beta\epsilon_1(d-N/2)}\mathfrak{q},\beta)Z^{\text{inst}}(\epsilon_1-\epsilon_2,\epsilon_2,\vec{a}+\epsilon_2\vec{k};e^{\beta\epsilon_2(d-N/2)}\mathfrak{q},\beta).\end{aligned} \tag{2.2.2}$$

Here Δ is the roots of $\mathfrak{su}(N)$. The vector $\vec{k}$ runs over the coweight lattice

$$\vec{k}\in\{(k_\alpha)=(k_1,k_2,\dots,k_N)\in\mathbb{Q}^N\Big|\sum_\alpha k_\alpha=0,\forall\alpha\,k_\alpha\equiv-k/N\mod\mathbb{Z}\}, \tag{2.2.3}$$

and

$$l^{\vec{k}}_{\vec{\alpha}}(\epsilon_1,\epsilon_2,\vec{a},\beta)=\begin{cases}\prod\limits_{\substack{i,j\geq0\\ i+j\leq-\langle\vec{k},\vec{\alpha}\rangle-1}}\left(1-e^{\beta(i\epsilon_1+j\epsilon_2-\langle\vec{a},\vec{\alpha}\rangle)}\right) & \text{if }\langle\vec{k},\vec{\alpha}\rangle<0,\\ \prod\limits_{\substack{i,j\geq0\\ i+j\leq\langle\vec{k},\vec{\alpha}\rangle-2}}\left(1-e^{\beta(-(i+1)\epsilon_1-(j+1)\epsilon_2-\langle\vec{a},\vec{\alpha}\rangle)}\right) & \text{if }\langle\vec{k},\vec{\alpha}\rangle>1,\\ 1 & \text{otherwise.}\end{cases} \tag{2.2.4}$$

From the geometric argument, the following theorem was established in (Nakajima and Yoshioka, 2005b)

Theorem 1 (Nakajima-Yoshioka). (1)(*$d=0$ case*)

$$Z^{\text{inst}}_{k,0}(\epsilon_1,\epsilon_2,\vec{a};\mathfrak{q},\beta)=(\mathfrak{q}\beta^{2N}e^{-N\beta(\epsilon_1+\epsilon_2)/2})^{\frac{k(N-k)}{2N}}Z^{\text{inst}}(\epsilon_1,\epsilon_2,\vec{a};\mathfrak{q},\beta). \tag{2.2.5}$$

(2)(*$0<d<N$ case*)

$$Z^{\text{inst}}_{k,d}(\epsilon_1,\epsilon_2,\vec{a};\mathfrak{q},\beta)=\begin{cases}Z^{\text{inst}}(\epsilon_1,\epsilon_2,\vec{a};\mathfrak{q},\beta) & \textit{for } k=0,\\ 0 & \textit{for } 0<k<N.\end{cases} \tag{2.2.6}$$

[1]This is not a surprise from the viewpoint of blowup equations for refined topological strings which will be studied in Chapter 4. It is the 5d blowup equations here directly correspond to the blowup equations for topological strings on local toric Calabi-Yau threefolds, while in the 4d, one need take limit both in the Kähler parameters of Calabi-Yau and the string coupling.

(3)($d = N$ *case*)

$$Z^{\text{inst}}_{k,N}(\epsilon_1,\epsilon_2,\vec{a};\mathfrak{q},\beta) = (-1)^{k(N-k)}(t_1t_2)^{k(N-k)/2}(\mathfrak{q}\beta^{2r}e^{-N\beta(\epsilon_1+\epsilon_2)/2})^{\frac{k(N-k)}{2N}}Z^{\text{inst}}(\epsilon_1,\epsilon_2,\vec{a};\mathfrak{q},\beta). \tag{2.2.7}$$

Here $t_1 = e^{\beta\epsilon_1}$, $t_2 = e^{\beta\epsilon_2}$. Basically, the above theorem shows that for $0 < k < N$, $0 < d < N$, $Z^{\text{inst}}_{k,d}$ vanishes. We call this part of the theorem as *vanishing blowup equations*. While for $d = 0$, $0 < k < N$ and $k = 0$, $0 < d < N$ and $d = N$, $0 < k < N$, $Z^{\text{inst}}_{k,d}$ is proportional to Z^{inst}, with the proportionality factor independent from $\vec{a}$! We call this part of theorem as *unity blowup equations*. In fact, if properly defined, $Z^{\text{inst}}_{k,d}$ for $k = N$, $0 < d < N$ is also proportional to Z^{inst} in this sense! In the context of refined topological strings, there is no difficulty to deal with such cases at all. Therefore, we can observe that the (k,d) pair for vanishing cases are within a square and those for unity cases are exactly surrounding the square.

One can also combine the perturbative part and instanton part together to obtain the full 5D Nekrasov partition function. To defined the perturbative part, one need to introduce the following function

$$\begin{aligned}
\gamma_{\epsilon_1,\epsilon_2}(x|\beta;\Lambda) &= \frac{1}{2\epsilon_1\epsilon_2}\left(-\frac{\beta}{6}\left(x+\frac{1}{2}(\epsilon_1+\epsilon_2)\right)^3 + x^2\log(\beta\Lambda)\right) \\
&\quad + \sum_{n\geq 1}\frac{1}{n}\frac{e^{-\beta n x}}{(e^{\beta n\epsilon_1}-1)(e^{\beta n\epsilon_2}-1)}, \\
\widetilde{\gamma}_{\epsilon_1,\epsilon_2}(x|\beta;\Lambda) &= \gamma_{\epsilon_1,\epsilon_2}(x|\beta;\Lambda) + \frac{1}{\epsilon_1\epsilon_2}\left(\frac{\pi^2 x}{6\beta} - \frac{\zeta(3)}{\beta^2}\right) \\
&\quad + \frac{\epsilon_1+\epsilon_2}{2\epsilon_1\epsilon_2}\left(x\log(\beta\Lambda) + \frac{\pi^2}{6\beta}\right) + \frac{\epsilon_1^2+\epsilon_2^2+3\epsilon_1\epsilon_2}{12\epsilon_1\epsilon_2}\log(\beta\Lambda).
\end{aligned} \tag{2.2.8}$$

Here $\Lambda = \mathfrak{q}^{1/2N}$. Then the full Nekrasov partition function is defined by

$$\begin{aligned}
Z(\epsilon_1,\epsilon_2,\vec{a};\mathfrak{q},\beta) &= \exp(-\sum_{\vec{\alpha}\in\Delta}\widetilde{\gamma}_{\epsilon_1,\epsilon_2}(\langle\vec{a},\vec{\alpha}\rangle|\beta;\Lambda))Z^{\text{inst}}(\epsilon_1,\epsilon_2,\vec{a};\mathfrak{q},\beta)\,, \\
\widehat{Z}_{k,d}(\epsilon_1,\epsilon_2,\vec{a};\mathfrak{q},\beta) &= \exp(-\sum_{\vec{\alpha}\in\Delta}\widetilde{\gamma}_{\epsilon_1,\epsilon_2}(\langle\vec{a},\vec{\alpha}\rangle|\beta;\Lambda))\widehat{Z}^{\text{inst}}_{k,d}(\epsilon_1,\epsilon_2,\vec{a};\mathfrak{q},\beta)\,.
\end{aligned} \tag{2.2.9}$$

Using formula (2.2.2), one can obtain the blowup formula for the full partition function:

$$\begin{aligned}
\widehat{Z}_{k,d}(\epsilon_1,\epsilon_2,\vec{a};\mathfrak{q},\beta) &= \exp\left[-\frac{(4d-N)(N-1)}{48}\beta(\epsilon_1+\epsilon_2)\right] \\
&\quad\times\sum_{\{\vec{k}\}=-k/N} Z\left(\epsilon_1,\epsilon_2-\epsilon_1,\vec{a}+\epsilon_1\vec{k};\exp\left(\epsilon_1(d-\tfrac{N}{2})\right)\mathfrak{q},\beta\right) \\
&\qquad\times Z\left(\epsilon_1-\epsilon_2,\epsilon_2,\vec{a}+\epsilon_2\vec{k};\exp\left(\epsilon_2(d-\tfrac{N}{2})\right)\mathfrak{q},\beta\right)\,.
\end{aligned} \tag{2.2.10}$$

Similar as the instanton partition function, the full partition function $\widehat{Z}_{k,d}$ also has same relations with Z as in Theorem 1. For $0 < k < N$, $0 < d < N$, $\widehat{Z}_{k,d}$ just vanishes. For the unity (k,d) pair in the instanton case, $\widehat{Z}_{k,d}$ is proportional to Z, with the proportionality factor independent from $\vec{a}$. These blowup equations for the full 5D Nekrasov partition function are in fact the special cases of the blowup equations for refined topological string theory, where the (k,d) pair is generalized to non-equivalent r fields.

Now we turn to the 5D $\mathcal{N}=1$ $SU(N)$ gauge theories theory with Chern-Simons term. Mathematically, one need to consider the line bundle $\mathcal{L}:=\lambda_{\mathcal{E}}(\mathcal{O}_{\mathbb{P}^2}(-\ell_\infty))^{-1}$ where $\mathcal{E}$ be the universal sheaf on $\mathbb{P}^2\times M(N,n)$. The instanton part of K-theoretic Nekrasov partition functions on $\mathbb{P}^2$ with generic Chern-Simons level m is defined by (Gottsche, Nakajima, and Yoshioka, 2009a)

$$Z_m^{\text{inst}}(\epsilon_1,\epsilon_2,\vec{a};\Lambda,\beta)=\sum_{n=0}^{\infty}((\beta\Lambda)^{2N}e^{-\beta(N+m)(\epsilon_1+\epsilon_2)/2})^n\sum_i(-1)^i\mathrm{ch}H^i(M(N,n),\mathcal{L}^{\otimes m}). \tag{2.2.11}$$

This instanton partition function again can be computed using localization as

$$\begin{aligned}&Z_m^{\text{inst}}(\epsilon_1,\epsilon_2,\vec{a};\Lambda,\beta)\\ =&\sum_{\vec{Y}}\frac{((\beta\Lambda)^{2N}e^{-\beta(N+m)(\epsilon_1+\epsilon_2)/2})^{|\vec{Y}|}}{\prod_{\alpha,\beta}n^{\vec{Y}}_{\alpha,\beta}(\epsilon_1,\epsilon_2,\vec{a};\beta)}\exp\Big(m\beta\sum_\alpha\sum_{s\in Y_\alpha}(a_\alpha-l'(s)\epsilon_1-a'(s)\epsilon_2)\Big),\end{aligned} \tag{2.2.12}$$

On blowup $\widehat{\mathbb{P}}^2$ with exceptional divisor C, just like in the pure gauge cases, the blowup instanton partition function with Chern-Simons terms $Z_{m,k,d}^{\text{inst}}(\epsilon_1,\epsilon_2,\vec{a};\Lambda,\beta)$ can be similarly defined. Indeed, using again Atiyah-Bott-Lefschetz fixed points formula, it was obtained in (Gottsche, Nakajima, and Yoshioka, 2009a) that

$$\begin{aligned}Z_{m,k,d}^{\text{inst}}(\epsilon_1,\epsilon_2,\vec{a};\Lambda,\beta)=&\exp(\frac{k^3m\beta}{6N^2}(\epsilon_1+\epsilon_2))\\ &\times\sum_{\{\vec{l}\}=-k/N}\frac{(\exp\left[\beta(\epsilon_1+\epsilon_2)(d+m\left(-\frac12+\frac{k}{N}\right)-\frac{N}{2})\right](\beta\Lambda)^{2N})^{(\vec{l},\vec{l})/2}}{\prod_{\vec{\alpha}\in\Delta}l^{\vec{l}}_{\vec{\alpha}}(\epsilon_1,\epsilon_2,\vec{a},\beta)}\\ &\times\exp\left[\beta(\vec{l},\vec{a})(d+m(-\frac12+\frac{k}{N}))\right]\\ &\times\exp\left[m\beta\left(\frac16(\epsilon_1+\epsilon_2)\sum_\alpha l_\alpha^3+\frac12\sum_\alpha l_\alpha^2a_\alpha\right)\right]\\ &\times Z_m^{\text{inst}}(\epsilon_1,\epsilon_2-\epsilon_1,\vec{a}+\epsilon_1\vec{l};\exp\left[\frac{\beta\epsilon_1}{2N}\left\{d+m\left(-\frac12+\frac{k}{N}\right)-\frac{N}{2}\right\}\right]\Lambda,\beta)\\ &\times Z_m^{\text{inst}}(\epsilon_1-\epsilon_2,\epsilon_2,\vec{a}+\epsilon_2\vec{l};\exp\left[\frac{\beta\epsilon_2}{2N}\left\{d+m\left(-\frac12+\frac{k}{N}\right)-\frac{N}{2}\right\}\right]\Lambda,\beta).\end{aligned} \tag{2.2.13}$$

Some conjectural blowup equations were proposed in (Gottsche, Nakajima, and Yoshioka, 2009a) and later proved in (Nakajima and Yoshioka, 2011), which are

$$Z_{m,0,d}^{\text{inst}}(\epsilon_1,\epsilon_2,\vec{a};\Lambda,\beta)=Z_m^{\text{inst}}(\epsilon_1,\epsilon_2,\vec{a};\Lambda,\beta)\qquad\text{for } 0\le d\le N,\ |m|\le N. \tag{2.2.14}$$

In fact, these equations are just some special cases of the unity blowup equations. By numerical computation, we find the following full set of blowup equations

$$Z^{\text{inst}}_{m,k,d}(\epsilon_1,\epsilon_2,\vec{a};\Lambda,\boldsymbol{\beta}) = \begin{cases} 0 & \text{for } 0<k<N, 0<d<N, \\ f(m,k,d,N,\epsilon_1,\epsilon_2,\Lambda,\boldsymbol{\beta}) Z^{\text{inst}}_{m}(\epsilon_1,\epsilon_2,\vec{a};\Lambda,\boldsymbol{\beta}) & \text{for } (k,d)\in S_{\text{unity}}, \end{cases} \tag{2.2.15}$$

where $S_{\text{unity}} = \{(k,d)|d=0, 0\le k<N \text{ or } 0<d<N, k=0 \text{ or } d=N, 0\le k<N\}$.[2] It is important that $f(m,k,d,\epsilon_1,\epsilon_2,\Lambda,\boldsymbol{\beta})$ does not depend on $\vec{a}$. In the context of refined topological string theory, this means that this the proportional factor does not depend on the true moduli of the Calabi-Yau. The vanishing cases were previously found in (Sun, Wang, and Huang, 2017; Grassi and Gu, 2019).

2.3 K-theoretic blowup equations for all simple Lie gauge group

The existence of blowup equations for all simple Lie group as gauge group G was conjectured in (Nakajima and Yoshioka, 2005a) and explicitly checked in (Keller and Song, 2012). However the K-theoretic blowup equations given in (Keller and Song, 2012) are not complete, which only contains part of the unity blowup equations and no vanishing blowup equation. Inspired from the elliptic blowup equations in Chapter 5, we find for all simple Lie groups G, there exist $h^\vee_G+1+2(r_c-1)$ non-equivalent unity blowup equations and $(h^\vee_G-1)(r_c-1)$ non-equivalent vanishing blowup equations, where r_c is the rank of the center of G with $r_c=|P^\vee/Q^\vee|$, and $P^\vee$ and $Q^\vee$ are the coweight lattice and coroot lattice of G.

Similar to the $SU(N)$ case, K-theoretic Nekrasov partition function for gauge group G can be formally defined as

$$\begin{aligned} Z^{\text{inst}}(\epsilon_1,\epsilon_2,\vec{a};\mathfrak{q},\boldsymbol{\beta}) &= \sum_{n=0}^{\infty}\left(\mathfrak{q}\boldsymbol{\beta}^{2h^\vee_G}e^{-h^\vee_G\boldsymbol{\beta}(\epsilon_1+\epsilon_2)/2}\right)^n Z_n(\epsilon_1,\epsilon_2,\vec{a};\boldsymbol{\beta}), \\ Z_n(\epsilon_1,\epsilon_2,\vec{a};\boldsymbol{\beta}) &= \sum_i (-1)^i \mathrm{ch} H^i(M(G,n),\mathcal{O}). \end{aligned} \tag{2.3.1}$$

Here $M(G,n)$ is the moduli space of n G-instantons. For more rigorous treatment, we refer to the section 9 of (Nakajima and Yoshioka, 2005a). The n instanton partition function Z_n is also called the *Hilbert series of the moduli spaces of n G-instantons*. For general G, the Nekrasov partition function can be computed by serveral different methods, for example the recursion formula in (Keller and Song, 2012), monopole formulas (Benvenuti, Hanany, and Mekareeya, 2010; Hanany, Mekareeya, and Razamat, 2013; Cremonesi, Hanany, and Zaffaroni, 2014), Hall-Littlewood indices (Gadde et al., 2013; Gaiotto and Razamat, 2012) and so on.

[2]In fact, in the Theorem 2.6 of (Nakajima and Yoshioka, 2011), blowup equations for $Z^{\text{inst}}_{m,k,d}$ with more general (k,d) are established. There an infinite-many parameter extention of Nekrasov partition function was introduced, which is beyond the scope of current thesis.

Similar to the $SU(N)$ case, the K-theoretic Nekrasov partition function on $\widehat{\mathbb{C}^2}$ for pure gauge theory with gauge group G can be computed as

$$Z^{\text{inst}}_{w,d}(\epsilon_1,\epsilon_2,\vec{a};q,\boldsymbol{\beta}) = \sum_{\vec{k}\in Q^\vee_w} \frac{\left(e^{\boldsymbol{\beta}(\epsilon_1+\epsilon_2)(d-h^\vee/2)} q\boldsymbol{\beta}^{2h^\vee}\right)^{(\vec{k},\vec{k})/2} e^{\boldsymbol{\beta}(\vec{k},\vec{a})d}}{\prod_{\vec{\alpha}\in\Delta} l^{\vec{k}}_{\vec{\alpha}}(\epsilon_1,\epsilon_2,\vec{a},\boldsymbol{\beta})}\times$$
$$Z^{\text{inst}}(\epsilon_1,\epsilon_2-\epsilon_1,\vec{a}+\epsilon_1\vec{k};e^{\boldsymbol{\beta}\epsilon_1(d-h^\vee/2)}q,\boldsymbol{\beta})Z^{\text{inst}}(\epsilon_1-\epsilon_2,\epsilon_2,\vec{a}+\epsilon_2\vec{k};e^{\boldsymbol{\beta}\epsilon_2(d-h^\vee/2)}q,\boldsymbol{\beta}).$$

Here $l^{\vec{k}}_{\vec{\alpha}}(\epsilon_1,\epsilon_2,\vec{a},\boldsymbol{\beta})$ is still the one defined in (2.2.4). Besides, w is a vector in the coweight space $P^\vee_G$, and $Q^\vee_w$ is the coroot lattice with a shift w in the coweight lattice. Clearly, there are only $r_c = |(P^\vee/Q^\vee)_G|$ non-equivalent choices of w, and we choice the minimal vector to represent each equivalent class of the quotient group $(P^\vee/Q^\vee)_G$. For example, for $G = SU(3)$, w can be 0, or the two fundamental coweights w_1, w_2. We can also use their associated representations to denote them, for example for $G = SU(3)$, the three possible w can be denoted as $\mathbf{1},\mathbf{3},\bar{\mathbf{3}}$, corresponding to $k = 0,1,2$ in the notation of (Nakajima and Yoshioka, 2005b). We summarize all possible w and their associated representations in Table 2.1.

G	$h^\vee_G$	r_c	w
A_{N-1}	N	N	$\mathbf{1},\mathbf{N},\mathbf{\Lambda^2},\mathbf{\Lambda^3},\ldots,\overline{\mathbf{N}}$
$B_{N\geq3}$	$2N-1$	2	$\mathbf{1},\mathbf{2N+1}$
$C_{N\geq2}$	$N+1$	2	$\mathbf{1},\mathbf{2N}$
$D_{N\geq4}$	$2N-2$	4	$\mathbf{1},\mathbf{2N},S,C$
G_2	4	1	$\mathbf{1}$
F_4	9	1	$\mathbf{1}$
E_6	12	3	$\mathbf{1},\mathbf{27},\overline{\mathbf{27}}$
E_7	18	2	$\mathbf{1},\mathbf{56}$
E_8	30	1	$\mathbf{1}$

Table 2.1: Some useful data for all simple Lie algebras. For A_{N-1}, i.e. $SU(N)$, $\mathbf{N}$ is the fundamental representation, $\mathbf{\Lambda^2}$ is the antisymmetric representation, $\mathbf{\Lambda}^n$ is the n-antisymmetric representation. For B_N, i.e. $SO(2N+1)$, $\mathbf{2N+1}$ is the vector representation. For C_N, i.e. $Sp(2N)$, $\mathbf{2N}$ is the fundamental representation. For D_N, i.e. $SO(2N)$, $\mathbf{2N}, S, C$ is the vector representation, spinor representation and conjugate spinor representation.

By explicit checks for various gauge groups of BCDEFG type, we find the following K-theoretic blowup equations:

Conjecture 1. (1)($d=0$ case)

$$Z^{\text{inst}}_{w,0}(\epsilon_1,\epsilon_2,\vec{a};\mathfrak{q},\boldsymbol{\beta}) = (\mathfrak{q}\boldsymbol{\beta}^{2h^\vee_G}e^{-h^\vee_G\boldsymbol{\beta}(\epsilon_1+\epsilon_2)/2})^{\frac{w\cdot w}{2}} Z^{\text{inst}}(\epsilon_1,\epsilon_2,\vec{a};\mathfrak{q},\boldsymbol{\beta}). \tag{2.3.2}$$

(2)($0<d<h^\vee_G$ case)

$$Z^{\text{inst}}_{w,d}(\epsilon_1,\epsilon_2,\vec{a};\mathfrak{q},\boldsymbol{\beta}) = \begin{cases} Z^{\text{inst}}(\epsilon_1,\epsilon_2,\vec{a};\mathfrak{q},\boldsymbol{\beta}) & \text{for } w\in[Q^\vee], \\ 0 & \text{for } w\in[P^\vee\backslash Q^\vee]. \end{cases} \tag{2.3.3}$$

(3)($d = h_G^\vee$ case)

$$Z^{\text{inst}}_{w,h_G^\vee}(\epsilon_1,\epsilon_2,\vec{a};\mathfrak{q},\boldsymbol{\beta}) = (-1)^{h_G^\vee w\cdot w}(t_1t_2)^{h_G^\vee\frac{w\cdot w}{2}}\left(\mathfrak{q}\boldsymbol{\beta}^{2h_G^\vee}e^{-h_G^\vee\boldsymbol{\beta}(\epsilon_1+\epsilon_2)/2}\right)^{\frac{w\cdot w}{2}}Z^{\text{inst}}(\epsilon_1,\epsilon_2,\vec{a};\mathfrak{q},\boldsymbol{\beta}). \tag{2.3.4}$$

Recall $t_1 = e^{\boldsymbol{\beta}\epsilon_1}$, $t_2 = e^{\boldsymbol{\beta}\epsilon_2}$. It is easy to see for $G = SU(N)$, these blowup equations go back to the K-theoretic blowup equations in Theorem 1. Besides, in the case of w as $\mathbf{1}$ that is $\vec{k}$ take all the vectors in the coroot lattice $Q_G^\vee$, by expanding order by order in the instanton number, one gets the following relation:

$$\begin{aligned} Z_n(\epsilon_1,\epsilon_2,\vec{a};\boldsymbol{\beta}) = &\sum_{\frac{1}{2}(\vec{k},\vec{k})+l+m=n} \exp\left[\boldsymbol{\beta}d\left(l\epsilon_1+m\epsilon_2+(\vec{k},\vec{a})+\frac{(\vec{k},\vec{k})}{2}(\epsilon_1+\epsilon_2)\right)\right] \\ &\times\frac{1}{\prod_{\alpha\in\Delta} l_\alpha^{\vec{k}}(\epsilon_1,\epsilon_2,\vec{a},\boldsymbol{\beta})}Z_l(\epsilon_1,\epsilon_2-\epsilon_1,\vec{a}+\epsilon_1\vec{k};\boldsymbol{\beta})Z_m(\epsilon_1-\epsilon_2,\epsilon_2,\vec{a}+\epsilon_2\vec{k};\boldsymbol{\beta}), \end{aligned}$$

where $d = 0,1,2,\ldots,h_G^\vee$. This results in the recursion formula for $Z_n(\epsilon_1,\epsilon_2,\vec{a};\boldsymbol{\beta})$ in (Keller and Song, 2012), as in the same spirit of Corollary 2.8 of (Nakajima and Yoshioka, 2005b) and the recursion formula for elliptic genera which will be shown in Chapter 5.3.1.

Chapter 3

Quantum Mirror Curves and Refined Topological Strings

3.1 Refined topological strings

Here we first briefly review some well-known definitions in (refined) topological string theory. We follow the notion in (Codesido, Grassi, and Marino, 2017). The Gromov-Witten invariants of a Calabi-Yau X are encoded in the partition function $Z(t)$ of topological string on X. It has a genus expansion $Z(t) = \exp[\sum_{g=0}^{\infty} g_s^{2g-2} F_g(t)]$ in terms of genus g free energies $F_g(t)$. At genus zero,

$$F_0(t) = \frac{1}{6} a_{ijk} t_i t_j t_k + \sum_d N_0^d \mathrm{e}^{-d \cdot t}, \tag{3.1.1}$$

where a_{ijk} denotes the classical triple intersection numbers, and t_i are the Kähler moduli with $i = 1, 2, \ldots, s$ and $s = \dim H_2(X, \mathbb{Z})$.[1] At genus one, one has

$$F_1(t) = b_i t_i + \sum_d N_1^d \mathrm{e}^{-d \cdot t}, \tag{3.1.2}$$

At higher genus, one has

$$F_g(t) = C_g + \sum_d N_g^d \mathrm{e}^{-d \cdot t}, \qquad g \geq 2, \tag{3.1.3}$$

where C_g is the constant map contribution to the free energy. The total free energy of the topological string is formally defined as the sum,[2]

$$F^{\mathrm{WS}}(t, g_s) = \sum_{g \geq 0} g_s^{2g-2} F_g(t) = F_{\mathrm{pert}}(t, g_s) + \sum_{g \geq 0} \sum_d N_g^d \mathrm{e}^{-d \cdot t} g_s^{2g-2}, \tag{3.1.4}$$

where

$$F_{\mathrm{pert}}(t, g_s) = \frac{1}{6 g_s^2} a_{ijk} t_i t_j t_k + b_i t_i + \sum_{g \geq 2} C_g g_s^{2g-2}. \tag{3.1.5}$$

The BPS part of partition function (3.1.4) can be resumed with a new set of enumerative invariants, called Gopakumar-Vafa (GV) invariants n_g^d (Gopakumar and Vafa,

[1] To avoid any possible confusion on the linear coefficients b_i in the genus one free energy, we use s in this chapter rather than the b in the introduction to denote $\dim H_2(X, \mathbb{Z})$.

[2] Here superscript "WS" is to stress the worldsheet nature of topological string theory.

1998) as

$$F^{\mathrm{GV}}(t,g_s)=\sum_{g\geq 0}\sum_{d}\sum_{w=1}^{\infty}\frac{1}{w}n_g^d\left(2\sin\frac{wg_s}{2}\right)^{2g-2}\mathrm{e}^{-wd\cdot t}. \tag{3.1.6}$$

Then,

$$F^{\mathrm{WS}}(t,g_s)=F^{(\mathrm{p})}(t,g_s)+F^{\mathrm{GV}}(t,g_s). \tag{3.1.7}$$

For local Calabi-Yau threefold, topological string have a refinement correspond to the supersymmetric gauge theory in the Omega background. In refined topological string, the GV invariants can be generalized to the refined BPS invariants $N^d_{j_L,j_R}$ which depend on the degree vector d and spins, j_L, j_R (Iqbal, Kozcaz, and Vafa, 2009; Choi, Katz, and Klemm, 2014; Nekrasov and Okounkov, 2014). Refined BPS invariants are positive integers and are closely related with the Gopakumar-Vafa invariants,

$$\sum_{j_L,j_R}\chi_{j_L}(q)(2j_R+1)N^d_{j_L,j_R}=\sum_{g\geq 0}n_g^d\left(q^{1/2}-q^{-1/2}\right)^{2g}, \tag{3.1.8}$$

where q is a formal variable and

$$\chi_j(q)=\frac{q^{2j+1}-q^{-2j-1}}{q-q^{-1}} \tag{3.1.9}$$

is the $SU(2)$ character for the spin j. Using these refined BPS invariants, one can define the NS free energy as

$$F^{\mathrm{NS}}(t,\hbar)=\frac{1}{6\hbar}a_{ijk}t_it_jt_k+b_i^{\mathrm{NS}}t_i\hbar+\sum_{j_L,j_R}\sum_{w,d}N^d_{j_L,j_R}\frac{\sin\frac{\hbar w}{2}(2j_L+1)\sin\frac{\hbar w}{2}(2j_R+1)}{2w^2\sin^3\frac{\hbar w}{2}}\mathrm{e}^{-wd\cdot t}. \tag{3.1.10}$$

in which b_i^{NS} can be obtained by using mirror symmetry as in (Huang and Klemm, 2012). By expanding (3.1.10) in powers of $\hbar$, we find the NS free energies at order n,

$$F^{\mathrm{NS}}(t,\hbar)=\sum_{n=0}^{\infty}F_n^{\mathrm{NS}}(t)\hbar^{2n-1}. \tag{3.1.11}$$

The BPS part of free energy of refined topological string is defined by refined BPS invariants as

$$F^{\mathrm{BPS}}_{\mathrm{ref}}(t,\epsilon_1,\epsilon_2)=\sum_{j_L,j_R}\sum_{w,d_j\geq 1}\frac{1}{w}N^d_{j_L,j_R}\frac{\chi_{j_L}(q_L^w)\chi_{j_R}(q_R^w)}{(q_1^{w/2}-q_1^{-w/2})(q_2^{w/2}-q_2^{-w/2})}\mathrm{e}^{-wd\cdot t}, \tag{3.1.12}$$

where

$$\epsilon_j=2\pi\mathrm{i}\tau_j,\quad q_j=\mathrm{e}^{2\pi\mathrm{i}\tau_j},\quad (j=1,2),\qquad q_L=\mathrm{e}^{\pi\mathrm{i}(\tau_1-\tau_2)},\qquad q_R=\mathrm{e}^{\pi\mathrm{i}(\tau_1+\tau_2)}. \tag{3.1.13}$$

The refined topological string free energy can also be defined by refined Gopakumar-Vafa invariants as

$$F^{\mathrm{BPS}}_{\mathrm{ref}}(t,\epsilon_1,\epsilon_2)=\sum_{g_L,g_R\geq 0}\sum_{w\geq 1}\sum_{d}\frac{1}{w}n^d_{g_L,g_R}\frac{\left(q_L^{w/2}-q_L^{-w/2}\right)^{2g_L}}{q^{w/2}-q^{-w/2}}\frac{\left(q_R^{w/2}-q_R^{-w/2}\right)^{2g_R}}{t^{w/2}-t^{-w/2}}e^{-d\cdot t}, \tag{3.1.14}$$

where

$$q = \mathrm{e}^{\epsilon_1}, \qquad t = \mathrm{e}^{-\epsilon_2}. \tag{3.1.15}$$

The refined Gopakumar-Vafa invariants are related with refined BPS invariants,

$$\sum_{j_L, j_R \geq 0} N^d_{j_L, j_R} \chi_{j_L}(q_L) \chi_{j_R}(q_R) = \sum_{g_L, g_R \geq 0} n^d_{g_L, g_R} \left(q_L^{1/2} - q_L^{-1/2}\right)^{2g_L} \left(q_R^{1/2} - q_R^{-1/2}\right)^{2g_R}. \tag{3.1.16}$$

The refined topological string free energy can be expand as

$$F(t, \epsilon_1, \epsilon_2) = \sum_{n,g=0}^{\infty} (\epsilon_1 + \epsilon_2)^{2n} (\epsilon_1 \epsilon_2)^{g-1} F_{(n,g)}(t) \tag{3.1.17}$$

where $F_{(n,g)}(t)$ can be determined recursively using the refined holomorphic anomaly equations.

With the refined free energy, the traditional topological string free energy can be obtained by taking the unrefined limit,

$$\epsilon_1 = -\epsilon_2 = g_s. \tag{3.1.18}$$

Therefore,

$$F_{\mathrm{GV}}(t, g_s) = F(t, g_s, -g_s). \tag{3.1.19}$$

The NS free energy can be obtained by taking the NS limit in refined topological string,

$$F^{\mathrm{NS}}(t, \hbar) = \lim_{\epsilon_1 \to 0} \epsilon_1 F(t, \epsilon_1, \hbar). \tag{3.1.20}$$

We will also need to specify an s dimensional integral vector B such that non-vanishing BPS invariants $N^d_{j_L, j_R}$ occur only at

$$2j_L + 2j_R + 1 \equiv B \cdot d \qquad \mathrm{mod}\ 2\,. \tag{3.1.21}$$

This condition specifies B only mod 2. The existence of such a vector B is guaranteed by the fact that the non-vanishing BPS invariants follow a so-called checkerboard pattern, as first observed in (Choi, Katz, and Klemm, 2014), and is also important in the pole cancellation in the non-perturbative completion (Hatsuda et al., 2014) .

We define the *twisted refined free energy* $\widehat{F}_{\mathrm{ref}}(t, \epsilon_1, \epsilon_2)$ as

$$\widehat{F}_{\mathrm{ref}}(t; \epsilon_1, \epsilon_2) = F^{\mathrm{pert}}_{\mathrm{ref}}(t; \epsilon_1, \epsilon_2) + F^{\mathrm{inst}}_{\mathrm{ref}}(t + \pi \mathrm{i} B; \epsilon_1, \epsilon_2)\,, \tag{3.1.22}$$

and *twisted partition function*

$$\widehat{Z}_{\mathrm{ref}}(t; \epsilon_1, \epsilon_2) = \exp\left(\widehat{F}_{\mathrm{ref}}(t; \epsilon_1, \epsilon_2)\right). \tag{3.1.23}$$

Here, the perturbative contributions are given by

$$F^{\mathrm{pert}}_{\mathrm{ref}}(t; \epsilon_1, \epsilon_2) = \frac{1}{\epsilon_1 \epsilon_2} \left(\frac{1}{6} \sum_{i,j,k=1}^{s} a_{ijk} t_i t_j t_k + 4\pi^2 \sum_{i=1}^{s} b_i^{\mathrm{NS}} t_i \right) + \sum_{i=1}^{s} b_i t_i - \frac{(\epsilon_1 + \epsilon_2)^2}{\epsilon_1 \epsilon_2} \sum_{i=1}^{s} b_i^{\mathrm{NS}} t_i\,, \tag{3.1.24}$$

where a_{ijk} and b_i are related to the topological intersection numbers in X, and b_i^{NS} can be obtained from the refined genus one holomorphic anomaly equation. The instanton contributions are given by the refined Gopakumar-Vafa formula,

$$F^{\text{inst}}_{\text{ref}}(t,\epsilon_1,\epsilon_2) = \sum_{j_L,j_R\geq 0}\sum_{d}\sum_{w=1}^{\infty}(-1)^{2j_L+2j_R}N^d_{j_L,j_R}\frac{\chi_{j_L}(q_L^w)\chi_{j_R}(q_R^w)}{w(q_1^{w/2}-q_1^{-w/2})(q_2^{w/2}-q_2^{-w/2})}e^{-wd\cdot t}\,, \tag{3.1.25}$$

where

$$q_{1,2} = e^{\epsilon_{1,2}}\,,\quad q_{L,R} = e^{(\epsilon_1\mp\epsilon_2)/2}\,, \tag{3.1.26}$$

and

$$\chi_j(q) = \frac{q^{2j+1}-q^{-2j-1}}{q-q^{-1}}\,. \tag{3.1.27}$$

Apparently, both $F^{\text{pert}}_{\text{ref}}(t;\epsilon_1,\epsilon_2)$ and $F^{\text{inst}}_{\text{ref}}(t;\epsilon_1,\epsilon_2)$ are invariant under the $\epsilon_{1,2}\to -\epsilon_{1,2}$, thus

$$\widehat{F}_{\text{ref}}(t;\epsilon_1,\epsilon_2) = \widehat{F}_{\text{ref}}(t;-\epsilon_1,-\epsilon_2). \tag{3.1.28}$$

3.2 Local Calabi-Yau and local mirror symmetry

A toric Calabi-Yau threefold is a toric variety given by the quotient,

$$M = (\mathbb{C}^{k+3}\backslash\mathcal{SR})/G, \tag{3.2.1}$$

where $G = (\mathbb{C}^*)^k$ is the toric action and $\mathcal{SR}$ is the Stanley-Reisner ideal of G. The quotient is specified by a matrix of toric charges Q_i^α, $i = 0,\cdots,k+2$, $\alpha = 1,\cdots,k$. The group G acts on the homogeneous coordinates x_i as

$$x_i \to \lambda_\alpha^{Q_i^\alpha}x_i,\qquad i = 0,\cdots,k+2, \tag{3.2.2}$$

where $\alpha = 1,\ldots,k$, $\lambda_\alpha\in\mathbb{C}^*$ and $Q_i^\alpha\in\mathbb{Z}$. To make the total space with vanishing first Chern class, one requires the Calabi-Yau condition

$$\sum_{i=1}^{k+3}Q_i^\alpha = 0,\quad \alpha = 1,\ldots,k. \tag{3.2.3}$$

The mirrors to toric Calabi-Yau threefolds were constructed in (Chiang et al., 1999). We define the Batyrev coordinates

$$z_\alpha = \prod_{i=1}^{k+3}x_i^{Q_i^\alpha},\quad \alpha = 1,\ldots,k, \tag{3.2.4}$$

and

$$H = \sum_{i=1}^{k+3}x_i. \tag{3.2.5}$$

The homogeneity allows us to set one of x_i to be one. Eliminate all the x_i in (3.2.5) by using (3.2.4), and choose other two as e^x and e^p, then the mirror geometry is described by

$$uv = H(e^x,e^p;z_\alpha),\quad \alpha = 1,\ldots,k, \tag{3.2.6}$$

where $x, p, u, v \in \mathbb{C}$. Clearly all information of mirror geometry is encoded in function H. The equation

$$H(e^x, e^p; z_\alpha) = 0 \tag{3.2.7}$$

defines a Riemann surface Σ, which is called the *mirror curve* to a toric Calabi-Yau. We denote g_Σ as the genus of the mirror curve.

The form of mirror curve can be written down specifically with the vectors in the toric diagram. Given the matrix of charges Q_i^α, we introduce the vectors,

$$\nu^{(i)} = \left(1, \nu_1^{(i)}, \nu_2^{(i)}\right), \qquad i = 0, \cdots, k+2, \tag{3.2.8}$$

satisfying the relations

$$\sum_{i=0}^{k+2} Q_i^\alpha \nu^{(i)} = 0. \tag{3.2.9}$$

In terms of these vectors, the mirror curve can be written as

$$H(\mathrm{e}^x, \mathrm{e}^p) = \sum_{i=0}^{k+2} x_i \exp\left(\nu_1^{(i)} x + \nu_2^{(i)} p\right). \tag{3.2.10}$$

It is easy to construct the holomorphic 3-form for mirror Calabi-Yau as

$$\Omega = \frac{du}{u} \wedge dx \wedge dp \tag{3.2.11}$$

If we integrate out the non-compact directions, the holomorphic 3-forms become meromorphic 1-form on the mirror curve (Katz, Klemm, and Vafa, 1997; Chiang et al., 1999):

$$\lambda = p\, \mathrm{d}x. \tag{3.2.12}$$

The mirror maps and the genus zero free energy $F_0(t)$ are determined by making an appropriate choice of cycles on the curve, α_i, β_i, $i = 1, \cdots, s$, then we have

$$t_i = \oint_{\alpha_i} \lambda, \qquad \frac{\partial F_0}{\partial t_i} = \oint_{\beta_i} \lambda, \qquad i = 1, \cdots, s. \tag{3.2.13}$$

In general, $s \geq g_\Sigma$, where g_Σ is the genus of the mirror curve. The s complex moduli can be divided into two classes, which are g_Σ true moduli, κ_i, $i = 1, \cdots, g_\Sigma$, and r_Σ mass parameters, ξ_j, $j = 1, \cdots, r_\Sigma$, where $r_\Sigma = s - g_\Sigma$. The true moduli can be also be expressed with the chemical potentials μ_i,

$$\kappa_i = \mathrm{e}^{\mu_i}, \qquad i = 1, \cdots, g_\Sigma. \tag{3.2.14}$$

Among the Kähler parameters, there are g_Σ of them which correspond to the true moduli, and their mirror map at large μ_i is of the form

$$t_i \approx \sum_{j=1}^{g_\Sigma} C_{ji} \mu_j + \sum_{j=1}^{r_\Sigma} \alpha_{ij} t_{\xi_j}, \qquad i = 1, \cdots, g_\Sigma, \tag{3.2.15}$$

where t_{ξ_j} is the flat coordinate associated to the mass parameter ξ_j by an algebraic mirror map. For toric cases, the $g_\Sigma \times s$ matrix C_{ij} can be read off from the toric data of X directly. For generic cases (Gu et al., 2017), C_{ij} should be understood as the intersection number of Kähler class C_i to the g_X irreducible compact divisor classes D_j in the geometry,

$$C_{ij} = D_i \cdot C_j \tag{3.2.16}$$

3.3 Nekrasov-Shatashvili quantization

In this and following sections, we review two quantization conditions of mirror curves, and their equivalence condition, which promote the study of blowup equation for refined topological string.

In (Nekrasov and Shatashvili, 2009a), the correspondence between 4D $N = 2$ gauge theories and integrable systems was promoted to the quantum level, see also (Nekrasov and Shatashvili, 2009c; Nekrasov and Shatashvili, 2009b). The correspondence is usually called Bethe/Gauge correspondence. The $SU(2)$ and $SU(N)$ cases were soon checked for the first few orders in (Mironov and Morozov, 2010a; Mironov and Morozov, 2010b). For a proof for the $SU(N)$ cases, see (Kozlowski and Teschner, 2010; Meneghelli and Yang, 2014). This 4D/1D correspondence is also closely related to the AGT conjecture (Alday, Gaiotto, and Tachikawa, 2010), which is a 4D/2D correspondence. In fact, the duality web among $N = 2$ gauge theories, matrix model, topological string and integrable systems (CFT) can be formulated in generic Nekrasov deformation (Dijkgraaf and Vafa, 2009), not just in NS limit.

The observation made in (Nekrasov and Shatashvili, 2009a) is that in the limit $\epsilon_1 \to 0$, $\epsilon_2 = \hbar$, the partition function is closely related to certain quantum integrable systems. The limit is usually called *Nekrasov-Shatashvili (NS) limit* in the context of refined topological string, or classical limit in the context of AGT correspondence. To be precise, the correspondence says that the supersymmetric vacua equation

$$\exp\left(\partial_{a_I} \mathcal{W}(\vec{a};\hbar)\right) = 1\,, \tag{3.3.1}$$

of the Nekrasov-Shatashvili free energy

$$\mathcal{W}(\vec{a};\hbar) = \lim_{\epsilon_1 \to 0} \epsilon_1 \log Z_{\text{Nek}}(\vec{a};\epsilon_1,\epsilon_2 = \hbar) \tag{3.3.2}$$

gives the Bethe ansatz equations for the corresponding integrable system. The NS free energy (3.3.2) also called effective twisted superpotential in the context of super gauge theory serves as the Yang-Yang function of the integrable system. The quantized/deformed Seiberg-Witten curve becomes the quantized spectral curve and the twisted chiral operators become the quantum Hamiltonians. Since it is usually difficult to written down the Bethe ansatz for general integrable systems, this observation provide a brand new perspective to study quantum integrable systems. In particular, the NS limit is studied in (Bonelli, Maruyoshi, and Tanzini, 2018) in the context of quantum Hitchin systems and β-ensemble matrix models related to conformal blocks of Liouville theory on punctured Riemann surfaces.

The physical explanations of Bethe/Gauge correspondence were given in (Nekrasov and Witten, 2010; Aganagic et al., 2012), see also the discussion for β-ensemble in (Bonelli, Maruyoshi, and Tanzini, 2018). Let us briefly review the approach in (Aganagic et al., 2012), which is closely related to refined topological string. In the context of geometric engineering, the NS free energy of supersymmetric gauge theory is just the NS limit of the partition function of topological string (Katz, Klemm, and Vafa, 1997),

$$\mathcal{W}(\vec{a};\hbar) = F^{\rm NS}(t,\hbar). \tag{3.3.3}$$

Consider the branes in unrefined topological string theory, it is well known (Aganagic et al., 2006) that for B-model on a local Calabi-Yau given by

$$uv + H(x,p) = 0 \tag{3.3.4}$$

the wave-function $\Psi(x)$ of a brane whose position is labeled by a point x on the Riemann surface Σ that is $H(x,p) = 0$ classically, satisfies an operator equation

$$H(\mathsf{x},\mathsf{p})\Psi = 0, \tag{3.3.5}$$

with the Heisenberg relation[3]

$$[\mathsf{x},\mathsf{p}] = \mathrm{i}g_s. \tag{3.3.6}$$

In the refined topological string theory, the brane wave equation is generalized to a multi-time dependent Schrödinger equation,[4]

$$H(x,p)\Psi = \epsilon_1\epsilon_2 \sum f_i(t)\frac{\partial\Psi}{\partial t_i}, \tag{3.3.7}$$

where $f_i(t)$ are some functions of the Kähler moduli t_i and the momentum operator is given by either $p = i\epsilon_1\partial_x$ or $p = i\epsilon_2\partial_x$, depending on the type of brane under consideration.

In the NS limit $\epsilon_1 \to 0$, $\epsilon_2 = \hbar$, the time dependence vanishes, and we simply obtain the time-independent Schrödinger equation

$$H(\mathsf{x},\mathsf{p})\Psi = 0, \tag{3.3.8}$$

with

$$[\mathsf{x},\mathsf{p}] = \mathrm{i}\hbar. \tag{3.3.9}$$

To have a well-defined wave function we need the wave function to be single-valued under monodromy. In unrefined topological string, the monodromy is characterized by taking branes around the cycles of a Calabi-Yau shifts the dual periods in units of g_s. While in the NS limit, the shifts becomes derivatives. Therefore, the single-valued conditions now are just the supersymmetric vacua equation (3.3.1). In the context of topological string, we expect

$$C_{ij}\frac{\partial F^{\rm NS}(t,\hbar)}{\partial t_j} = 2\pi\left(n_i + \frac{1}{2}\right),\ i = 1,\cdots,g_\Sigma. \tag{3.3.10}$$

In fact these conditions are just the Einstein-Brillouin-Keller (EBK) quantization, which is a generalization of Bohr-Sommerfeld quantization for high-dimensional integrable systems. Thus we can regard the left side of (3.3.10) as phase volumes corresponding to each periods of the mirror curve,

$$\mathrm{Vol}_i(t,\hbar) = \hbar C_{ij}\frac{F^{\rm NS}(t,\hbar)}{\partial t_j},\ i = 1,\cdots,g_\Sigma. \tag{3.3.11}$$

[3] In general, this relation only holds up to order g_s correction.

[4] This is the case for the mirror curve of genus one, for higher genus there will be many Hamiltonians, see (Bonelli, Maruyoshi, and Tanzini, 2018).

Now the NS quantization conditions for the mirror curve are just the EBK quantization conditions,

$$\mathrm{Vol}_i(t,\hbar) = 2\pi\hbar\left(n_i + \frac{1}{2}\right),\ i = 1,\cdots,g_\Sigma. \tag{3.3.12}$$

As we mentioned, these NS quantization conditions need non-perturbative completions (Kallen and Marino, 2016; Wang, Zhang, and Huang, 2015; Hatsuda and Marino, 2016; Franco, Hatsuda, and Mariño, 2016) and such completion can be obtained by simply substituting Lockhart-Vafa free energy $F_{\rm LV}$ into (3.3.11). Based on the localization calculation on the partition function of superconformal theories on squashed S^5, a non-perturbative definition of refined topological string was proposed by Lockhart and Vafa in (Lockhart and Vafa, 2018) as

$$Z_{\rm LV}(t,\tau_1,\tau_2) = \frac{Z_{\rm ref}(t,\tau_1+1,\tau_2)}{Z_{\rm ref}(t/\tau_1,-1/\tau_1,\tau_2/\tau_1+1)\cdot Z_{\rm ref}(t/\tau_2.\tau_1/\tau_2+1,-1/\tau_2)} \tag{3.3.13}$$

Here $\epsilon_{1,2} = 2\pi\tau_{1,2}$. Then the non-perturbative free energy of refined topological string is given by

$$F_{\rm LV}(t,\tau_1,\tau_2) = F_{\rm ref}(t,\tau_1+1,\tau_2) - F_{\rm ref}\left(\frac{t}{\tau_1},-\frac{1}{\tau_1},\frac{\tau_2}{\tau_1}+1\right) - F_{\rm ref}\left(\frac{t}{\tau_2},\frac{\tau_1}{\tau_2}+1,-\frac{1}{\tau_2}\right). \tag{3.3.14}$$

3.4 Grassi-Hatsuda-Mariño conjecture

The Grassi-Hatsuda-Mariño conjecture, also known as topological string/spectral theory correspondence or TS/ST correspondence reveals a surprising non-perturbative relationship between topological strings on toric Calabi-Yau threefolds and the spectral theory of operators associated to the quantized mirror curve (Grassi, Hatsuda, and Marino, 2016), see also (Codesido, Grassi, and Marino, 2017) for the cases of mirror curve of higher genus and a good review in (Marino, 2018). The GHM conjecture has passed highly nontrivial checks for lots of local toric Calabi-Yau threefolds, but till now there is still no proof even for local $\mathbb{P}^2$ or $\mathbb{P}^1\times\mathbb{P}^1$. The 4d limit of the conjecture was derived in some cases in (Bonelli, Grassi, and Tanzini, 2017; Bonelli, Grassi, and Tanzini, 2018). In the following, we follow (Codesido, Grassi, and Marino, 2017) to give a brief introduction to the GHM conjecture (Grassi, Hatsuda, and Marino, 2016) and the generalized GHM conjecture in which the concept of r fields is introduced (Sun, Wang, and Huang, 2017).

For a mirror curve Σ with genus g_Σ, there are g_Σ different canonical forms for the curve,

$$\mathcal{O}_i(x,y) + \kappa_i = 0, \qquad i = 1,\cdots,g_\Sigma. \tag{3.4.1}$$

Here, κ_i is normally a true modulus x_i of Σ. The different canonical forms of the curves are related by reparameterizations and overall factors,

$$\mathcal{O}_i + \kappa_i = \mathcal{P}_{ij}\left(\mathcal{O}_j + \kappa_j\right), \qquad i,j = 1,\cdots,g_\Sigma, \tag{3.4.2}$$

where $\mathcal{P}_{ij}$ is of form $\mathrm{e}^{\lambda x+\mu y}$. Equivalently, we can write

$$\mathcal{O}_i = \mathcal{O}_i^{(0)} + \sum_{j\neq i}\kappa_j\mathcal{P}_{ij}. \tag{3.4.3}$$

Perform the Weyl quantization of the operators $\mathcal{O}_i(x,y)$, we obtain g_Σ different Hermitian operators O_i, $i=1,\cdots,g_\Sigma$,

$$\mathsf{O}_i = \mathsf{O}_i^{(0)} + \sum_{j\neq i} \kappa_j \mathsf{P}_{ij}. \tag{3.4.4}$$

The operator $\mathsf{O}_i^{(0)}$ is the unperturbed operator, while the moduli κ_j encode different perturbations. It turns out that the most interesting operator was not O, but its inverse ρ. This is because ρ is expected to be of trace class and positive-definite, therefore it has a discrete, positive spectrum, and its Fredholm (or spectral) determinant is well-defined. We have

$$\rho_i = \mathsf{O}_i^{-1}, \qquad i=1,\cdots,g_\Sigma, \tag{3.4.5}$$

and

$$\rho_i^{(0)} = \left(\mathsf{O}_i^{(0)}\right)^{-1}, \qquad i=1,\cdots,g_\Sigma. \tag{3.4.6}$$

For the discussion on the eigenfunctions of ρ, see (Marino and Zakany, 2017). In order to construct the generalized spectral determinant, we need to introduce the following operators,

$$\mathsf{A}_{jl} = \rho_j^{(0)} \mathsf{P}_{jl}, \quad j,l=1,\cdots,g_\Sigma. \tag{3.4.7}$$

Now the generalized spectral determinant is defined as

$$\Xi_X(\kappa;\hbar) = \det\left(1+\kappa_1 \mathsf{A}_{j1} + \cdots + \kappa_{g_\Sigma}\mathsf{A}_{jg_\Sigma}\right). \tag{3.4.8}$$

It is easy to prove this definition does not depend on the index j.

This completes the definitions on quantum mirror curve from the quantum-mechanics side. Let us now turn to the topological string side. The total modified grand potential for CY with arbitrary-genus mirror curve is defined as

$$\mathsf{J}_X(\mu,\xi,\hbar) = \mathsf{J}_X^{\mathrm{WKB}}(\mu,\xi,\hbar) + \mathsf{J}_X^{\mathrm{WS}}(\mu,\xi,\hbar), \tag{3.4.9}$$

where

$$\mathsf{J}_X^{\mathrm{WKB}}(\mu,\xi,\hbar) = \frac{t_i(\hbar)}{2\pi}\frac{\partial F^{\mathrm{NS}}(t(\hbar),\hbar)}{\partial t_i} + \frac{\hbar^2}{2\pi}\frac{\partial}{\partial \hbar}\left(\frac{F^{\mathrm{NS}}(t(\hbar),\hbar)}{\hbar}\right) + \frac{2\pi}{\hbar} b_i t_i(\hbar) + A(\xi,\hbar). \tag{3.4.10}$$

and

$$\mathsf{J}_X^{\mathrm{WS}}(\mu,\xi,\hbar) = F^{\mathrm{GV}}\left(\frac{2\pi}{\hbar}t(\hbar) + \pi \mathrm{i} B, \frac{4\pi^2}{\hbar}\right). \tag{3.4.11}$$

The modified grand potential has the following structure,

$$\mathsf{J}_X(\mu,\xi,\hbar) = \frac{1}{12\pi\hbar} a_{ijk} t_i(\hbar) t_j(\hbar) t_k(\hbar) + \left(\frac{2\pi b_i}{\hbar} + \frac{\hbar b_i^{\mathrm{NS}}}{2\pi}\right) t_i(\hbar) + \mathcal{O}\left(\mathrm{e}^{-t_i(\hbar)}, \mathrm{e}^{-2\pi t_i(\hbar)/\hbar}\right). \tag{3.4.12}$$

$A(\xi,\hbar)$ is some unknown function, which is relevant to the spectral determinant but does not affect the quantum Riemann theta function, therefore does not appear in the quantization conditions.

GHM conjecture (Grassi, Hatsuda, and Marino, 2016; Codesido, Grassi, and Marino, 2017) says that the generalized spectral determinant (3.4.8) is exactly given by

$$\Xi_X(\kappa;\hbar) = \sum_{\mathbf{n}\in\mathbb{Z}^{g_\Sigma}} \exp\left(\mathrm{J}_X(\mu + 2\pi\mathrm{i}n, \xi, \hbar)\right). \tag{3.4.13}$$

As a corollary, the quantization condition for the mirror curve is given by

$$\Xi_X(\kappa;\hbar) = 0. \tag{3.4.14}$$

In (Sun, Wang, and Huang, 2017), the concept of r fields was introduced. The r field characterizes the phase-changing of complex moduli in the way that when one makes a transformation for the Batyrev coordinates

$$(z_1, \ldots, z_n) \to (z_1 e^{r_1\pi\mathrm{i}}, \ldots, z_n e^{r_n\pi\mathrm{i}}), \tag{3.4.15}$$

equivalently, we have the following translation on the Kähler parameters:

$$t \to t + \pi\mathrm{i}r. \tag{3.4.16}$$

This makes the effect of r field just like B field. For some specific choices of r fields, we have the *generalized GHM conjecture* on the quantization of mirror curves as

$$\Xi(t + \pi\mathrm{i}r, \hbar) = 0. \tag{3.4.17}$$

3.5 Compatibility formulas

We can see now the main difference between quantization conditions is that NS quantization condition quantize g_Σ particles of a integrable systems (Franco, Hatsuda, and Mariño, 2016), there are g_Σ constraint equations. But GHM quantization quantize the operators, and the number of operators is usually greater than g_Σ. The r fields stand for the phase of complex parameters z_i, and different r fields quantize the operators in different phase. Because the definition of the generalized spectral determinant involves infinite sum, it is easy to see that different choices of r fields may result in the same functions. We define non-equivalent r fields as those which produce non-equivalent generalized spectral determinant. We denote the number of non-equivalent r fields as w_Σ. This lead to w_Σ different quantization conditions. It is conjectured in (Sun, Wang, and Huang, 2017):

The spectra of quantum mirror curve are solved by the simultaneous equations:

$$\left\{\Theta(t + \mathrm{i}\pi r^a, \hbar) = 0,\ a = 1, \cdots, w_\Sigma.\right\} \tag{3.5.1}$$

This spectra is the same as the spectra of NS quantization conditions:

$$\left\{\Theta(t + \mathrm{i}\pi r^a, \hbar) = 0,\ a = 1, \cdots, w_\Sigma.\right\} \Leftrightarrow \left\{\mathrm{Vol}_i(t, \hbar) = 2\pi\hbar\left(n_i + \frac{1}{2}\right),\ i = 1, \cdots, g_\Sigma.\right\} \tag{3.5.2}$$

In addition, all the vector r^a are the representatives of the B field of X, which means for all triples of degree d, spin j_L and j_R such that the refined BPS invariants $N^d_{j_L,j_R}(X)$ is non-vanishing, they must satisfy

$$(-1)^{2j_L+2j_R-1} = (-1)^{r^a \cdot d}, \quad a = 1, \cdots, w_\Sigma. \tag{3.5.3}$$

Besides, in (Sun, Wang, and Huang, 2017), some novel identities called *compatibility formulas* were found which guarantee the above equivalence:

$$\sum_{n\in\mathbb{Z}^g} \exp\left(i\sum_{i=1}^{g} n_i\pi + F_{\text{unref}}\left(t + \mathrm{i}\hbar n\cdot C + \frac{1}{2}\mathrm{i}\hbar r, \hbar\right) - \mathrm{i}n_j C_{ji}\frac{\partial}{\partial t_i} F_{\text{NS}}\left(t,\hbar\right)\right) = 0. \tag{3.5.4}$$

in which F_{unref} is the traditional topological string partition function, F_{NS} is the Nekrasov-Shatashvili free energy, C is the charge matrix of toric Calabi-Yau and $a = 1, \cdots, w_\Sigma$. This identities is now known as NS limit of vanishing blowup equation (Grassi and Gu, 2016), which we will show in Chapter 4.1.2.

Chapter 4

Blowup Equations for Refined Topological Strings

The compatibility formulas (3.5.4) between the exact Nekrasov-Shatashivili quantization conditions in (Wang, Zhang, and Huang, 2015; Hatsuda and Marino, 2016) and the Grassi-Hatsuda-Mariño quantization condition in (Grassi, Hatsuda, and Marino, 2016; Codesido, Grassi, and Marino, 2017) give one inspiration for the new structures of refined topological string theory. In (Grassi and Gu, 2016), the compatibility formulas are recognized as the NS limit of vanishing blowup equations. The other inspiration comes from the K-theoretic blowup equations in supersymmetric gauge theories we reviewed in Chapter 2. In the spirit of geometric engineering (Katz, Klemm, and Vafa, 1997), the Nekrasov partition function of 5d $\mathcal{N}=1$ gauge theories obtained from compactifying M-theory on local Calabi-Yau X should be equal to the partition function of refined topological string on X, or more precisely the twisted partition function we introduced in (3.1.22), (3.1.23):

$$Z_{\mathrm{Nek}}(\epsilon_1,\epsilon_2,\vec{a},m_f,\mathfrak{q}) = \widehat{Z}_{\mathrm{ref}}(\epsilon_1,\epsilon_2,t). \tag{4.0.1}$$

Here m_f are some masses of possible matter content in the gauge theory. Since blowup equations are known to exist for so many 5d gauge theories with all kinds of gauge group, Chern-Simons level and various matters (Nakajima and Yoshioka, 2005b; Gottsche, Nakajima, and Yoshioka, 2009b; Keller and Song, 2012; Kim et al., 2019), it is natural to expect the structure of blowup equations may exist more generally for refined topological strings on arbitrary local Calabi-Yau threefolds. Note an local Calabi-Yau is not necessarily engineers a 5d gauge theories, or even a gauge theory, for instance local $\mathbb{P}^2$. The key feature of the K-theoretic blowup equations for 5d gauge theories in the above literature is that for unity case, the blowup partition function $Z_{k,d}$ is equal to the original partition function Z up to a factor independent from the Coulomb parameters $\vec{a}$. These Coulomb parameters in geometric engineering correspond to the "true" Kähler moduli of local Calabi-Yau, which have non-zero intersections with the divisor classes. This inspires us to propose the generalized blowup equations for refined topological strings (1.0.4).

We rephrase our generalized blowup equations here. For an arbitrary local Calabi-Yau threefold X with mirror curve of genus g, suppose there are $b=\dim H_2(X,\mathbb{Z})$ irreducible curve classes corresponding to Kähler moduli t in which $b-g$ classes correspond to mass parameters m, and denote C as the intersection matrix between the b curve classes and the g irreducible compact divisor classes, then there exist infinite constant integral vectors $r\in\mathbb{Z}^b$ such that the following functional equations for the twisted partition function of refined topological string on X hold:

$$\begin{aligned}\sum_{n\in\mathbb{Z}^g}(-)^{|n|}\,&\frac{\widehat{Z}_{\text{ref}}\left(\epsilon_1,\epsilon_2-\epsilon_1;t+\epsilon_1R\right)\cdot\widehat{Z}_{\text{ref}}\left(\epsilon_1-\epsilon_2,\epsilon_2;t+\epsilon_2R\right)}{\widehat{Z}_{\text{ref}}\left(\epsilon_1,\epsilon_2;t\right)}\\&=\begin{cases}0, & \text{for } r\in\mathcal{S}_{\text{vanish}},\\ \Lambda(\epsilon_1,\epsilon_2;m,r), & \text{for } r\in\mathcal{S}_{\text{unity}},\end{cases}\end{aligned}\tag{4.0.2}$$

where $|n|=\sum_{i=1}^{g}n_i$, $R=C\cdot n+r/2$ and Λ is a simple factor that is independent from the true moduli and purely determined by the polynomial part of the refined free energy. In addition, all the vector r are the representatives of the B field of X, which means for all triples of degree d, spin j_L and j_R such that the refined BPS invariants $N^d_{j_L,j_R}(X)$ is non-vanishing, they must satisfy

$$(-1)^{2j_L+2j_R-1}=(-1)^{r\cdot d}.\tag{4.0.3}$$

Besides, both sets $\mathcal{S}_{\text{vanish}}$ and $\mathcal{S}_{\text{unity}}$ are finite under the quotient of shift $2C\cdot n$ symmetry.

Note for a specific local Calabi-Yau, unity and vanishing blowup equations do not always both exist. For example, resolved conifold has only unity blowup equations but no vanishing, as will be shown in section 4.6.1. This also happens for some elliptic non-compact Calabi-Yau associated with M-string theory, see Chapter 5.5.2, and some special 6d $(1,0)$ SCFTs for example those with gauge group G_2,F_4,E_8, see Chapter 5.5. There are also some examples for which there only exist vanishing blowup equations but no unity one, for example massless half-K3 Calabi-Yau in Chapter 5.5.1, and all 6d $(1,0)$ SCFTs with unpaired half hypermultiplets. See more in Chapter 5.2. Nevertheless, for all local Calabi-Yau threefolds we have studied, there always exist at least one blowup equation!

As mentioned before, the blowup equations (4.0.2) specializing to toric $X_{N,m}$ Calabi-Yau are equivalent to the K-theoretic blowup equations for $SU(N)$ gauge group and Chern-Simons level m in (Gottsche, Nakajima, and Yoshioka, 2009b). We refer the precise derivation to section 3 of (Grassi and Gu, 2016) and section 3.8 of (Huang, Sun, and Wang, 2018).

4.1 Generalized blowup equations and component equations

In this section, we study blowup equations (4.0.2) in two expansions: the refined BPS expansion which is useful to solve the refined BPS invariants of local Calabi-Yau, and ϵ_1,ϵ_2 expansion which gives many interesting functional equations for free energy $F_{(n,g)}$ which we call *component equations*.

4.1.1 Unity blowup equations

Let us first study the structure of unity blowup equations. For an arbitrary vector $r\in\mathbb{C}^b$, we can define a function Λ by

$$\Lambda(t,\epsilon_1,\epsilon_2,r):=\sum_{n\in\mathbb{Z}^g}(-1)^{|n|}\exp\Big(G(t,R,\epsilon_1,\epsilon_2)\Big),\tag{4.1.1}$$

where $R = C \cdot n + r/2$ and

$$G(t,R,\epsilon_1,\epsilon_2) := \widehat{F}_{\text{ref}}\left(t+\epsilon_1 R,\epsilon_1,\epsilon_2-\epsilon_1\right)+\widehat{F}_{\text{ref}}\left(t+\epsilon_2 R,\epsilon_1-\epsilon_2,\epsilon_2\right)-\widehat{F}_{\text{ref}}\left(t,\epsilon_1,\epsilon_2\right). \tag{4.1.2}$$

For generic vector r, the function $\Lambda(t,\epsilon_1,\epsilon_2,r)$ is complicated and depends on all Kähler moduli t. However, when $r \in \mathcal{S}_{\text{unity}}$, the function $\Lambda(t,\epsilon_1,\epsilon_2,r)$ will be significantly simplified, in particular, independent from the true moduli of local Calabi-Yau. In such situation, we write it as $\Lambda(m,\epsilon_1,\epsilon_2,r)$ where m denote the mass parameters and call the vector r as *unity r fields*. The prerequisite such fields should satisfy is the B field condition

$$r \equiv B \ (\mathrm{mod}\ (2\mathbb{Z})^b). \tag{4.1.3}$$

It is obvious that two different vectors r, r' are equivalent for the vanishing blowup equation if

$$r' = r + 2C\cdot n, \qquad n \in \mathbb{Z}^g. \tag{4.1.4}$$

We denote the number of non-equivalent unity r fields as w_{u} and the set of non-equivalent unity r fields as $\mathcal{S}_{\text{unity}}$.

Separating the perturbative and instanton part of the refined partition function, it is easy to obtain $G = G_{\text{pert}} + G_{\text{inst}}$ with

$$G_{\text{pert}} = \left(b_i + b_i^{\text{NS}}\right)t_i - \frac{1}{2}a_{ijk}t_iR_jR_k + (\epsilon_1+\epsilon_2)\left(-\frac{1}{6}a_{ijk}R_iR_jR_k + b_iR_i - b_i^{\text{NS}}R_i\right), \tag{4.1.5}$$

and

$$\begin{aligned}G_{\text{inst}} =& F_{\text{ref}}^{\text{inst}}\left(t+\mathrm{i}\pi B+\epsilon_1 R,\epsilon_1,\epsilon_2-\epsilon_1\right)+F_{\text{ref}}^{\text{inst}}\left(t+\mathrm{i}\pi B+\epsilon_2 R,\epsilon_1-\epsilon_2,\epsilon_2\right)\\ &-F_{\text{ref}}^{\text{inst}}\left(t+\mathrm{i}\pi B,\epsilon_1,\epsilon_2\right)\\ =&\sum_{g,n,d}(-1)^{d\cdot B}n_{g,n}^d\Big((\epsilon_1(\epsilon_2-\epsilon_1))^{g-1}\epsilon_2^{2n}\mathrm{e}^{-\epsilon_1 d\cdot R}+(\epsilon_2(\epsilon_1-\epsilon_2))^{g-1}\epsilon_1^{2n}\mathrm{e}^{-\epsilon_2 d\cdot R}\\ &-(\epsilon_1\epsilon_2)^{g-1}(\epsilon_1+\epsilon_2)^{2n}\Big)\mathrm{e}^{-d\cdot t}.\end{aligned} \tag{4.1.6}$$

Here $n_{g,n}^d$ is the refined Gromov-Witten invariants defined by

$$F_{\text{ref}}^{\text{inst}}(t,\epsilon_1,\epsilon_2) = \sum_{g,n=0}^{\infty}\sum_d(\epsilon_1+\epsilon_2)^{2n}(\epsilon_1\epsilon_2)^{g-1}n_{g,n}^d\mathrm{e}^{-d\cdot t}. \tag{4.1.7}$$

It is important that G_{pert} is a linear function of t and the coefficients of t_i are quadratic for R, as we will see later. Apparently, the unity blowup equations can be expanded with respect to $Q = e^{-t}$ and the equality of the coefficients at each degree on the both side of the equation gives many constraints among the refined BPS invariants.

The unity blowup equations can also be expanded with respect to ϵ_1 and ϵ_2. On the left hand side of (4.1.1), we have

$$\Lambda(t,\epsilon_1,\epsilon_2,r) = \sum_{g,n=0}^{\infty}\Lambda_{(n,g)}(t,r)(\epsilon_1+\epsilon_2)^{2n}(\epsilon_1\epsilon_2)^{g-1}, \quad 2n\in\mathbb{Z}_{\geq 0}, g\in\mathbb{Z}_{\geq 0}. \tag{4.1.8}$$

While on the right hand side of (4.1.1), one can use the refined free energy expansion

$$F(t,\epsilon_1,\epsilon_2)=\sum_{g,n=0}^{\infty}F_{(n,g)}(t)(\epsilon_1+\epsilon_2)^{2n}(\epsilon_1\epsilon_2)^{g-1},n,g\geq 0. \tag{4.1.9}$$

Here we omit the "widehat" on F to lighten the notations. Then it is easy to find the leading order of the unity blowup equations is

$$\Lambda_{(0,0)}=\sum_{n\in\mathbb{Z}^g}(-1)^{|n|}\exp\left(-\frac{1}{2}R^2F''_{(0,0)}+F_{(0,1)}-F_{(1,0)}\right), \tag{4.1.10}$$

where

$$R^2F''_{(0,0)}=\sum_{i,j=1}^{b}R_iR_j\frac{\partial^2F_{(0,0)}}{\partial t_i\partial t_j}. \tag{4.1.11}$$

In the following, we also use the abbreviation

$$R^mF^{(m)}_{(n,g)}=\sum_{i_1,\dots,i_m}R_{i_1}R_{i_2}\cdots R_{i_m}\frac{\partial}{\partial t_{i_1}}\frac{\partial}{\partial t_{i_2}}\cdots\frac{\partial}{\partial t_{i_m}}F_{(n,g)}. \tag{4.1.12}$$

We call equation (4.1.10) as *generalized contact term equations*, as the analogy of the contact term equations in Seiberg-Witten theory (Marino, 1999). We also denote the summand in the above expression as $\Theta(R)$. Then the leading equations of the unity blowup equations can be simply written as

$$\Lambda_{(0,0)}=\sum\Theta(R). \tag{4.1.13}$$

The subleading equations of (4.1.1) can also be easily derived as

$$\Lambda_{(1,0)}=\sum\Theta(R)\left(-\frac{1}{6}R^3F^{(3)}_{(0,0)}+R\left(F'_{(0,1)}+F'_{(1,0)}\right)\right). \tag{4.1.14}$$

For the subsubleading order, i.e. $n+g=1$, we have

$$\begin{aligned}\Lambda_{(1,0)}=&\sum\Theta(R)\Big(-F_{(0,2)}-3F_{(2,0)}+\frac{1}{2}R^2\left(\left(F'_{(0,1)}+F'_{(1,0)}\right)^2+F''_{(0,1)}\right)\\&+\frac{1}{24}R^4\left(-4F^{(3)}_{(0,0)}F'_{(0,1)}-4F^{(3)}_{(0,0)}F'_{(1,0)}-F^{(4)}_{(0,0)}+\frac{1}{72}R^6\left(F^{(3)}_{(0,0)}\right)^2\right)\Big),\end{aligned} \tag{4.1.15}$$

$$\Lambda_{(0,1)}=\sum\Theta(R)\left(3F_{(0,2)}-2F_{(1,1)}+F_{(2,0)}-R^2F''_{(0,1)}+\frac{1}{2}R^2F''_{(1,0)}+\frac{1}{24}R^4F_0^{(4)}\right). \tag{4.1.16}$$

For $n+g=3/2$, we have

$$\begin{aligned}\Lambda_{(3/2,0)}=&\sum\Theta(R)\Big(R\left(-F'_{(0,2)}+F'_{(2,0)}-(F_{(0,2)}+3F_{(2,0)})\left(F'_{(0,1)}+F'_{(1,0)}\right)\right)\\&+\frac{1}{6}R^3\left(3\left(F'_{(0,1)}+F'_{(1,0)}\right)F''_{(0,1)}+\left(F'_{(0,1)}+F'_{(1,0)}\right)^3+(F_{(0,2)}+3F_{(2,0)})F^{(3)}_{(0,0)}+F^{(3)}_{(0,1)}\right)\\&+\frac{1}{120}R^5\left(-10F^{(3)}_{(0,0)}\left(\left(F'_{(0,1)}+F'_{(1,0)}\right)^2+F''_{(0,1)}\right)-5F^{(4)}_{(0,0)}\left(F'_{(0,1)}+F'_{(1,0)}\right)-F^{(5)}_{(0,0)}\right)\end{aligned}$$

$$+\frac{1}{144}R^7F^{(3)}_{(0,0)}\left(2F^{(3)}_{(0,0)}\left(F'_{(0,1)}+F'_{(1,0)}\right)+F^{(4)}_{(0,0)}\right)-\frac{1}{1296}R^9\left(F^{(3)}_{(0,0)}\right)^3\Bigg),\tag{4.1.17}$$

$$\Lambda_{(1/2,1)}=\sum\Theta(R)\Bigg(R\Big(4F'_{(0,2)}+F'_{(1,1)}-2F'_{(2,0)}+(3F_{(0,2)}-2F_{(1,1)}+F_{(2,0)})(F'_{(0,1)}+F'_{(1,0)})\Big)$$
$$+\frac{1}{6}R^3\left(-3(F'_{(0,1)}+F'_{(1,0)})(2F''_{(0,1)}-F''_{(1,0)})-(3F_{(0,2)}-2F_{(1,1)}+F_{(2,0)})F^{(3)}_{(0,0)}-3F^{(3)}_{(0,1)}\right)$$
$$+\frac{1}{120}R^5\Big(10(2F''_{(0,1)}-F''_{(1,0)})F^{(3)}_{(0,0)}+5(F'_{(0,1)}+F'_{(1,0)})F^{(4)}_{(0,0)}+2F^{(5)}_{(0,0)}\Big)-\frac{1}{144}R^7F^{(3)}_{(0,0)}F^{(4)}_{(0,0)}\Bigg).\tag{4.1.18}$$

We call these equations as the *component equations of blowup equations*.

In general, for the integral order i.e. $n+g=g_{\rm t}$, $n\in\mathbb{Z}$, the component equations only involves $F_{(n_i,g_i)}$ with $n_i+g_i\le g_{\rm t}+1$ and have the following form

$$\Lambda_{(n,g)}=\sum\Theta(R)\left(\sum_{l=0}^{3g_{\rm t}}R^{6g_{\rm t}-2l}\sum_{6g_{\rm t}-2l=\sum_i h_i}c_\star\prod_{n_i+g_i\le g_{\rm t}+1}F^{(h_i)}_{n_i,g_i}\right).\tag{4.1.19}$$

Here $c_\star$ are some rational constants depending on n,g and the set of $\{n_i,g_i,h_i\}$. For the half-integral order i.e. $n+g=g_{\rm t}$, $n\in\mathbb{Z}+1/2$, the component equations only involves $F_{(n_i,g_i)}$ with $n_i+g_i\le g_{\rm t}+1/2$ and have the following form

$$\Lambda_{(n,g)}=\sum\Theta(R)\left(\sum_{l=0}^{3g_{\rm t}-1/2}R^{6g_{\rm t}-2l}\sum_{6g_{\rm t}-2l=\sum_i h_i}c_\star\prod_{n_i+g_i\le g_{\rm t}+1/2}F^{(h_i)}_{n_i,g_i}\right).\tag{4.1.20}$$

Still $c_\star$ are some rational constants depending on n,g and the set of $\{n_i,g_i,h_i\}$. These structures are important for the modularity of blowup equations, as will be shown in Chapter 4.4.2.

4.1.2 Vanishing blowup equations

The vanishing blowup equations for general local Calabi-Yau were already written down in (Gu et al., 2017), which is

$$\sum_{n\in\mathbb{Z}^g}(-1)^{|n|}\exp\left(\widehat{F}_{\rm ref}\left(t+\epsilon_1R,\epsilon_1,\epsilon_2-\epsilon_1\right)+\widehat{F}_{\rm ref}\left(t+\epsilon_2R,\epsilon_1-\epsilon_2,\epsilon_2\right)\right)=0,\tag{4.1.21}$$

where $R=C\cdot n+r/2$. We call the vectors r making the above equation hold as the *vanishing r fields*. The vanishing r fields should also satisfy the B field condition and exhibit obvious shift $2C\cdot n$ symmetry. We denote the number of non-equivalent vanishing r fields as $w_{\rm v}$ and the set of non-equivalent vanishing r fields as $\mathcal{S}_{\rm vanish}$.

It is easy to see the NS limit of the vanishing blowup equations give exactly the compatibility formulas (3.5.4), as mentioned before. Indeed,

$$\lim_{\epsilon_1=\mathrm{i}\hbar,\ \epsilon_2\to0}\widehat{F}_{\rm ref}\left(t+\epsilon_1\left(C\cdot n+r/2\right),\epsilon_1,\epsilon_2-\epsilon_1\right)=F_{\rm unref}\left(t+\mathrm{i}\hbar n\cdot C+\frac{1}{2}\mathrm{i}\hbar r,\hbar\right),\tag{4.1.22}$$

and

$$\lim_{\epsilon_1=\mathrm{i}\hbar,\ \epsilon_2\to 0}\widehat{F}_{\text{ref}}\left(t+\epsilon_2\left(C\cdot n+r/2\right),\epsilon_1-\epsilon_2,\epsilon_2\right)=-\mathrm{i}n_jC_{ji}\frac{\partial}{\partial t_i}F_{\text{NS}}\left(t,\hbar\right). \tag{4.1.23}$$

Now let us have a look at thecomponent equations of the vanishing blowup equations. Since we can divide each side of equation (4.1.21) with $\exp(\widehat{F}_{\text{ref}}(t,\epsilon_1,\epsilon_2))$, obviously the component equations of vanishing blowup equations should have the same expression as the unity ones. The only difference lie as the r fields. Thus we have the leading order component equation as

$$\Lambda_{(0,0)}(t,r)=\sum_{n\in\mathbb{Z}^g}(-1)^{|n|}\exp\left(-\frac{1}{2}R^2F''_{(0,0)}+F_{(0,1)}-F_{(1,0)}\right)=0, \tag{4.1.24}$$

or equivalently

$$\sum_{n\in\mathbb{Z}^g}(-1)^{|n|}\exp\left(-\frac{1}{2}R^2F''_{(0,0)}\right):=\sum\Theta(R)=0. \tag{4.1.25}$$

The higher order component equations look just like those of unity ones presented in the last section. However, there is one new phenomenon, that is for order $n+g=g_{\mathrm{t}}\in\mathbb{Z}$ component equations of vanishing blowup equations, the terms with free energy $F_{(n_i,g_i)}$ with $n_i+g_i=g_{\mathrm{t}}+1$ will dispear due to the leading order vanishing component equation. For example, let us look at $\Lambda_{(0,1)}$ for a vanishing r. As the unity cases, direct expansion with respect to ϵ_1,ϵ_2 gives

$$\Lambda_{(0,1)}=\sum\Theta(R)\left(3F_{(0,2)}-2F_{(1,1)}+F_{(2,0)}-R^2F''_{(0,1)}+\frac{1}{2}R^2F''_{(1,0)}+\frac{1}{24}R^4F_0^{(4)}\right)=0. \tag{4.1.26}$$

However, due to leading order equation (4.1.25), obviously $F_{(0,2)},F_{(1,1)},F_{(2,0)}$ in the above equation can be dropped out. Thus unlike the unity case, here this equation does not give constraints on $F_{(0,2)},F_{(1,1)},F_{(2,0)}$. This fact will be used in section 5.3 for the counting of independent component equations.

Now we consider how the blowup equations and the r fields behave under the reduction of local Calabi-Yau. Here the reduction means to set some of the mass parameters to zero while the genus of mirror curve does not change. Such procedure is quite common. For example, local $\mathbb{P}^2$ can be reduced from local $\mathbb{F}_1$ and resolved $\mathbb{C}^3/\mathbb{Z}_5$ orbifold can be reduced from $SU(3)$ geometry $X_{3,2}$. Since blowup equations can be expanded with respect to $Q=\mathrm{e}^{-t}$, thus under the reduction, one can simply set some $Q_m=e^{-t_m}$ to be zero in the blowup equations. Obviously, all the original vanishing r fields will result in the vanishing r fields of the reduced local Calabi-Yau. Note this is *not* true for unity r fields, where some of the original unity r fields could turn to vanishing r fields after the reduction.

4.2 Properties of the r fields

In this section, we prove an important property of the r fields called the *reflective property*. We also show that the B field condition of refined topological string can actually be derived from blowup equations.

4.2.1 Reflective property

Here we prove the reflective property of r fields, that is if one r makes the vanishing (unity) blowup equation hold, then $-r$ makes the vanishing (unity) blowup equation hold as well.

Let us consider the vanishing and unity blowup equations together:

$$\Lambda(t,\epsilon_1,\epsilon_2,r) = \sum_{n\in\mathbb{Z}^g}(-1)^{|n|}\frac{\widehat{Z}_{\mathrm{ref}}(t+\epsilon_1 R,\epsilon_1,\epsilon_2-\epsilon_1)\,\widehat{Z}_{\mathrm{ref}}(t+\epsilon_2 R,\epsilon_1-\epsilon_2,\epsilon_2)}{\widehat{Z}_{\mathrm{ref}}(t,\epsilon_1,\epsilon_2)}$$
$$=\sum_{n\in\mathbb{Z}^g}(-1)^{|n|}\exp\Big(\widehat{F}_{\mathrm{ref}}(t+\epsilon_1 R,\epsilon_1,\epsilon_2-\epsilon_1)+\widehat{F}_{\mathrm{ref}}(t+\epsilon_2 R,\epsilon_1-\epsilon_2,\epsilon_2)-\widehat{F}_{\mathrm{ref}}(t,\epsilon_1,\epsilon_2)\Big). \tag{4.2.1}$$

Here for $r=r_{\mathrm{v}}$, $\Lambda=0$.

Let us take a substitution $\epsilon_{1,2}\to-\epsilon_{1,2}$ in equation (4.2.1) and use the property of refined free energy (3.1.28). We have

$$\begin{aligned}&\Lambda(t,-\epsilon_1,-\epsilon_2,r)\\ &=\sum_{n\in\mathbb{Z}^g}(-1)^{|n|}\frac{\widehat{Z}_{\mathrm{ref}}(t-\epsilon_1 R,-\epsilon_1,-\epsilon_2+\epsilon_1)\,\widehat{Z}_{\mathrm{ref}}(t-\epsilon_2 R,-\epsilon_1+\epsilon_2,-\epsilon_2)}{\widehat{Z}_{\mathrm{ref}}(t,-\epsilon_1,-\epsilon_2)}\\ &=\sum_{n\in\mathbb{Z}^g}(-1)^{|n|}\frac{\widehat{Z}_{\mathrm{ref}}(t-\epsilon_1 R,\epsilon_1,\epsilon_2-\epsilon_1)\,\widehat{Z}_{\mathrm{ref}}(t-\epsilon_2 R,\epsilon_1-\epsilon_2,\epsilon_2)}{\widehat{Z}_{\mathrm{ref}}(t,\epsilon_1,\epsilon_2)}.\end{aligned} \tag{4.2.2}$$

Consider the blowup equations for $-r$ field and use the invariance of the summation under reflection $n\to-n$,

$$\Lambda(t,\epsilon_1,\epsilon_2,-r)=\sum_{n\in\mathbb{Z}^g}(-1)^{|n|}\frac{\widehat{Z}_{\mathrm{ref}}(t-\epsilon_1 R,\epsilon_1,\epsilon_2-\epsilon_1)\,\widehat{Z}_{\mathrm{ref}}(t-\epsilon_2 R,\epsilon_1-\epsilon_2,\epsilon_2)}{\widehat{Z}_{\mathrm{ref}}(t,\epsilon_1,\epsilon_2)}. \tag{4.2.3}$$

Clearly, if one r field makes the unity (vanishing) blowup equation hold, once we require

$$\Lambda(t,\epsilon_1,\epsilon_2,-r)=\Lambda(t,-\epsilon_1,-\epsilon_2,r), \tag{4.2.4}$$

then $-r$ field makes the unity (vanishing) blowup equation hold as well.

4.2.2 Relation with the B field condition

In this section, we show the B field condition of refined BPS invariants $N^d_{j_L,j_R}$ can actually be derived from the blowup equations. The B field condition is the key of the pole cancellation in both exact NS quantization conditions (Wang, Zhang, and Huang, 2015) and the HMO mechanism (Hatsuda, Moriyama, and Okuyama, 2013b; Hatsuda, Moriyama, and Okuyama, 2013a). This condition was first found in (Hatsuda et al., 2014) for local del Pezzo CY threefolds. A physical explanation was given for the existence of B field and an effective method was proposed to calculate B field for arbitrary toric Calabi-Yau threefold in (Sun, Wang, and Huang, 2017). Here, we show the condition (4.0.3) of the r fields is the result of blowup equations. This implies the existence of the B field and all r fields are the representatives of the B field.

It was shown in (Sun, Wang, and Huang, 2017) that to keep the form of quantum mirror curve unchanged when the Planck constant $\hbar$ is shifted to $\hbar + 2\pi\mathrm{i}$, the complex moduli must have transformation $z_i \to (-1)^{B_i} z_i$, $i = 1, 2, \dots, b$. Accordingly the Kähler moduli are shifted as $t \to t + \pi\mathrm{i}B$. In the refinement of mirror curves or Seiberg-Witten geometries, it is $\epsilon_1 + \epsilon_2$ that plays the role of quantum parameter (Kimura, 2018; Kimura, Mori, and Sugimoto, 2018; Bourgine et al., 2017). We find that when one shifts the deformation parameters in the following way:

$$\epsilon_1 \to \epsilon_1 + 2\pi\mathrm{i}m, \quad \epsilon_2 \to \epsilon_2 + 2\pi\mathrm{i}n, \quad m, n \in \mathbb{Z}, \tag{4.2.5}$$

to keep the refined mirror curve unchanged, the complex moduli must transform as:

$$z_i \to (-1)^{B_i(m+n)} z_i, \qquad i = 1, 2, \dots, b. \tag{4.2.6}$$

Accordingly the Kähler moduli are shifted like:

$$t \to t + \pi\mathrm{i}(m+n)B. \tag{4.2.7}$$

Now let us study how blowup equations (4.2.1) change under the simultaneous transformation (4.2.5) and (4.2.7). We are not interested in the polynomial part here, because the polynomial contribution (4.1.5) from the three refined free energies is linear with respect to t and $\epsilon_1 + \epsilon_2$ and its shift should be absorbed into the phase change of the factor $\Lambda(t, \epsilon_1, \epsilon_2, r)$. We focus on the instanton part of the refined free energy. Using the fact that C is an integral matrix, we deduce under the simultaneous transformation (4.2.5) and (4.2.7), every summand in refined BPS formulation of $\widehat{F}_{\mathrm{ref}}\,(t + \epsilon_1 R, \epsilon_1, \epsilon_2 - \epsilon_1)$ obtains a phase change:

$$(-1)^{nw(2j_L+2j_R-1-B\cdot d)+mw(B+r)\cdot d}. \tag{4.2.8}$$

Every summand in $\widehat{F}_{\mathrm{ref}}\,(t + \epsilon_2 R, \epsilon_1 - \epsilon_2, \epsilon_2)$ obtains a phase change:

$$(-1)^{mw(2j_L+2j_R-1-B\cdot d)+nw(B+r)\cdot d}. \tag{4.2.9}$$

Every summand in $\widehat{F}_{\mathrm{ref}}\,(t, \epsilon_1, \epsilon_2)$ obtains a phase change:

$$(-1)^{(m+n)w(2j_L+2j_R-1-B\cdot d)}. \tag{4.2.10}$$

We know the B model topological string theory is determined by the mirror curve and the refined free energy is determined by the refined mirror curve. Since the refined mirror curve remains the same under the simultaneous transformation (4.2.5) and (4.2.6), the blowup equations must still hold under the simultaneous transformation (4.2.5) and (4.2.7). Clearly the only way to achieve this is to require all three factors in (4.2.8, 4.2.9, 4.2.10) to be identical to one. Requiring (4.2.10) to be one for arbitrary m, n, w means for all non-vanishing refined BPS invariants $N^{d}_{j_L, j_R}$, there are constraints:

$$2j_L + 2j_R - 1 - B \cdot d \equiv 0 \ (\mathrm{Mod}\ 2). \tag{4.2.11}$$

This is exactly the B field condition we introduced in the first place! Substitute this definition into the (4.2.8) and (4.2.9), it is easy to see that to require them to be one for arbitrary m, n, w, there must be constraints:

$$r \equiv B \ (\mathrm{Mod}\ (2\mathbb{Z})^b). \tag{4.2.12}$$

This means all r fields are the representatives of the B field, which is the prerequisite of r fields we introduced in the first place.

4.3 Solving blowup equations

We propose two general methods to solve blowup equations, corresponding to the two universal expansions of refined free energy.

4.3.1 ϵ_1, ϵ_2 expansion

The ϵ_1, ϵ_2 expansion of blowup equations has been studied in the previous section 4.1, where the expansion coefficients give the component equations which are constraint equations for $F_{n,g}$. To solve refined free energy $F_{n,g}$, we need to count the number of unknown functions and the number of constraint equations. Besides, we assume the genus zero free energy $F_{(0,0)}$ is known as the initial input.

Let us first look at a simple situation for $F_{(0,1)}$ and $F_{(1,0)}$. Recall $R = C \cdot n + r/2$, the leading and subleading order component equations of unity blowup equations are:

$$\sum_{n\in\mathbb{Z}^g} \Theta(n,r) := \sum_{n\in\mathbb{Z}^g} (-)^{|n|} \exp\left(-\frac{1}{2}R^2 F''_{(0,0)} + F_{(0,1)} - F_{(1,0)}\right) = \Lambda_{(0,0)}(r), \tag{4.3.1}$$

and

$$\sum_{n\in\mathbb{Z}^g} \left(-\frac{1}{6}R^3 F^{(3)}_{(0,0)} + R\left(F'_{(0,1)} + F'_{(1,0)}\right)\right)\Theta(n,r) = \Lambda_{(1/2,0)}(r). \tag{4.3.2}$$

Consider a local Calabi-Yau with one Kähler parameter and mirror curve of genus one, it is easy to see that if there exists one unity blowup equation, the component equation (4.3.1) can be solved for $F_{(0,1)} - F_{(1,0)}$, while component equation (4.3.2) can be solved for $F_{(0,1)} + F_{(1,0)}$, thus we obtain $F_{(0,1)}$ and $F_{(1,0)}$ at the same time[1]. This simple example shows the spirit of ϵ_1, ϵ_2 expansion method that is to use component equations for $\Lambda_{(n,g)}$ at total order $n + g = g_{\mathrm{t}}$ and $g_{\mathrm{t}} + 1/2$ to solve the refined free energy $F_{(n_i,g_i)}$ with $n_i + g_i = g_{\mathrm{t}} + 1$, $i = 0, 1, 2, \ldots, g_{\mathrm{t}} + 1$.

In general, it is easy to find one unity component equation of $\Lambda_{(n,g)}$ (4.1.19) at order $n + g = g_{\mathrm{t}} \in \mathbb{Z}$ gives $g_{\mathrm{t}} + 1$ algebraic equations for the $g_{\mathrm{t}} + 2$ unknown functions $F_{(n_i,g_i)}$ with $n_i + g_i = g_{\mathrm{t}} + 1$, $i = 0, 1, 2, \ldots, g_{\mathrm{t}} + 1$. For the above g_{t}, one unity component equation of $\Lambda_{(n,g)}$ (4.1.20) at order $n + g = g_{\mathrm{t}} + 1/2$ gives $g_{\mathrm{t}} + 1$ first-order differential equations of those $F_{(n_i,g_i)}$. On other hand, difference for the vanishing cases lies in that the integral g_{t} order component equations do not give constraints on the above $F_{(n_i,g_i)}$, as demonstrated in section 4.1.2.

In summary, by counting the number of independent equations for $F_{(n,g)}$, we can find the following conclusions. Let w_u and w_v be the number of unity and vanishing blowup equations respectively.

[1]We assume the linear coefficients b_{GV} and b_{NS} are already known, which fix the integration constants here.

- For a generic local Calabi-Yau with b Kähler parameters, if $w_{\mathrm{u}} \geq 1$ and $w_{\mathrm{u}} + w_{\mathrm{v}} \geq b$, then given $F_{(0,0)}$, one can solve all $F_{(n,g)}$ with $n, g \geq 0$ from blowup equations.

- For a generic local Calabi-Yau with b Kähler parameters, if $w_{\mathrm{v}} \geq b$, then given NS free energy or the self-dual free energy, i.e. all $F_{(n,0)}$ or all $F_{(0,g)}$, one can solve all $F_{(n,g)}$ with $n, g \geq 0$ from the blowup equations.

4.3.2 Refined BPS expansion

The refined BPS expansion gives a completely different type of expansion and make blowup equations very efficient to solve the refined BPS invariants of local Calabi-Yau threefolds. The instanton partition function Z^{inst} can be expressed in terms of refined BPS invariants (Iqbal, Kozcaz, and Vafa, 2009) as

$$Z^{\text{inst}} = \exp\left[\sum_{j_L,j_R=0}^{\infty}\sum_{\beta}\sum_{w=1}^{\infty}\frac{N^{\beta}_{j_L,j_R}}{w} f_{(j_L,j_R)}(q_1^w,q_2^w)Q^{w\beta}\right]. \tag{4.3.3}$$

Here we define

$$f_{(j_L,j_R)}(q_1,q_2) = \frac{\chi_{j_L}(q_l)\chi_{j_R}(q_r)}{(q_1^{1/2}-q_1^{-1/2})(q_2^{1/2}-q_2^{-1/2})}, \tag{4.3.4}$$

where $\chi_j(q)$ is the $\mathfrak{su}(2)$ character given by

$$\chi_j(q) = \frac{q^{2j+1}-q^{-2j-1}}{q-q^{-1}}. \tag{4.3.5}$$

Let us also define

$$Bl_{(j_L,j_R,R)}(q_1,q_2) = f_{(j_L,j_R)}(q_1,q_2/q_1)q_1^R + f_{(j_L,j_R)}(q_1/q_2,q_2)q_2^R - f_{(j_L,j_R)}(q_1,q_2), \tag{4.3.6}$$

where R is the shift of the Kähler parameter t in the blowup equation. Recall (4.1.5) and (4.1.6), the blowup equations for general topological string can be reformulated as

$$\Lambda(\epsilon_1,\epsilon_2,m) = \sum_{n\in\mathbb{Z}^g}(-1)^{|n|}e^{f_0(n)(\epsilon_1+\epsilon_2)+\sum_{i=1}^b f_i(n)t_i}\exp\left[-\sum_{j_L,j_R,\beta}\sum_{w=1}^{\infty}N^{\beta}_{j_L,j_R}\frac{Q^{w\beta}}{w}Bl_{(j_L,j_R,R)}(q_1^w,q_2^w)\right], \tag{4.3.7}$$

where $f_0(n), f_i(n)$ are respectively some cubic and quadratic polynomials. For any fixed curve degree β, comparison of coefficients on both sides of the equation gives

$$\sum_{j_L,j_R}N^{\beta}_{j_L,j_R}Bl_{(j_L,j_R,R(\beta,n_0))}(q_1,q_2) = I^{\beta}(q_1,q_2), \tag{4.3.8}$$

where $I^{\beta}(q_1,q_2)$ consists only of invariants of lower curve degrees. The BPS invariants $N^{\beta}_{j_L,j_R}$ can be regarded as the coefficients of $Bl_{(j_L,j_R,R(\beta,n_0))}(q_1,q_2)$, and they can be fixed if the decomposition of $I^{\beta}(q_1,q_2)$ in terms of $Bl_{(j_L,j_R,R(\beta,n_0))}(q_1,q_2)$ is unique. In fact, we have the following lemma.

Lemma 2. *$\forall R \in \mathbb{Z}/2$, $Bl_{(j_L,j_R,R)}(q_1,q_2)$ are linearly independent with only exceptions at $Bl_{(0,0,1/2)}(q_1,q_2) = Bl_{(0,0,-1/2)}(q_1,q_2) = Bl_{(0,1/2,0)}(q_1,q_2) = 0$.*

Proof: For a generic fixed R and a finite set J of spin (j_L, j_R) which satisfy $2j_L + 2j_R + 1 \equiv 2R \mod 2$, we need to prove if

$$\sum_{(j_L,j_R)\in J} x_{(j_L,j_R)} Bl_{(j_L,j_R,R)}(q_1,q_2) = 0, \tag{4.3.9}$$

then all coefficients $x_{(j_L,j_R)}$ must vanish. Since J is finite, there exist maximum for j_L and j_R, denoted as $j_L^{\max}$ and $j_R^{\max}$. We expand J to the set of all spins on the rectangle from $(0,0)$ to $(j_L^{\max}, j_R^{\max})$. On such spin rectangle, we can define a strict total order of (j_L, j_R). Then one can use descending method to prove the coefficients $x_{(j_L,j_R)}$ vanish one by one. Such procedure was actually already given in section 6.2 in (Huang, Sun, and Wang, 2018). For $R = 1/2$ or $-1/2$, the lowest spin in such order is $(0,0)$ and the value of Bl function is 0, and for $R = 0$, the lowest spin in such order is $(0,1/2)$ and the value of Bl function is 0 too. These are the only exceptions for linear independence.

Since the refined BPS invariants have B field condition, in each fixed degree d, only one of the two degrees $(0,0)$ and $(0,1/2)$ can appears. Therefore, in each fixed degree, equation (4.3.8) can fix the refined BPS invariants for all spins except for only one spin $(0,0)$ or $(0,1/2)$. If we provide initial data $F_{(0,0)}$, obviously in such degree, the only unknown invariants with spin $(0,0)$ or $(0,1/2)$ can then be fixed. We conclude that if genus zero BPS invariants are provided, we can always use one unity blowup equation to solve all the refined BPS invariants order by order. In practice, if we have a toric construction of the Calabi-Yau geometry, we can use mirror symmetry techniques (Hosono et al., 1995) to compute the genus zero invariants to furnish the necessary input data.

4.4 Blowup equations and holomorphic anomaly equations

In the section, we study the modularity property of blowup equations and elaborate how the two universal approaches to refined topological strings – blowup equations and refined holomorphic/modular anomaly equations – are consistent.

4.4.1 Modular property of refined free energy

Topological string theory is closely related to modular forms and Jacobi forms. Such relation was first systematically studied in (Aganagic, Bouchard, and Klemm, 2008) and soon was used to solve the (refined) holomorphic anomaly equations for local Calabi-Yau in (Huang and Klemm, 2007; Grimm et al., 2007; Haghighat, Klemm, and Rauch, 2008; Huang and Klemm, 2012; Huang, Klemm, and Poretschkin, 2013; Huang et al., 2015). All Calabi-Yau geometries studied in these paper have mirror curve of genus one and there topological string free energies can be expressed by Eisenstein series, Dedekind eta function and Jacobi theta functions. For the geometries with mirror curve of higher genus, the free energies are related to Siegel modular forms (Klemm et al., 2015).

For the B model on a local Calabi-Yau manifold X, there exists a discrete symmetry group Γ, which is generated by the monodromies of the periods. For example, for local $\mathbb{P}^2$, the symmetry group is Γ_3, which is a subgroup of classical modular group $\Gamma_0 = SL(2,\mathbb{Z})$. The main statement of (Aganagic, Bouchard, and Klemm, 2008) is that the genus g topological string free energy, depending on the polarization, is either a holomorphic quasi-modular form or an almost holomorphic modular form of *weight zero* under Γ. This fact can be directly generalized to refined topological string, which means every refined free energy $F_{(n,g)}$ is certain modular form of *weight zero* under certain discrete group Γ.[2]

The idea that free energy is a (quasi-)modular form comes from the observation in (Witten, 1993) that the partition function $Z(t,\bar{t}) = \exp(\sum_{g=0}^{\infty} g_s^{2g-2}\mathcal{F}_g(t,\bar{t}))$ is a wave function in a Hilbert space obtained by quantizing $H_3(X)$. The quantized Hilbert space is parameterized by $x^I = t_i$, $p_I = \frac{\partial F^{(0,0)}}{\partial t_i}$, and the wave function should be invariant under $Sp(2n,\mathbb{Z})$ transformation M

$$\begin{aligned}\tilde{p}_I &= A_I{}^J p_J + B_{IJ}x^J,\\ \tilde{x}^I &= C^{IJ}p_J + D^I{}_J x^J,\end{aligned} \tag{4.4.1}$$

where

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma \subset Sp(2n,\mathbb{Z}). \tag{4.4.2}$$

More precisely, the invariance means a state $|Z\rangle$ is invariant under $Sp(2n,\mathbb{Z})$, but after choosing polarization x, the wave function $\langle x|Z\rangle$ indeed have changed.

In (Aganagic, Bouchard, and Klemm, 2008), two kinds of polarizations are introduced called *holomorphic polarization* and *real polarization*, which can be easily generalized to refined topological strings. In the holomorphic polarization, the refined free energy $\mathcal{F}_{(n,g)}(t,\bar{t})$ is invariant under Γ, which means they are modular forms of Γ of weight zero. Besides, they are almost holomorphic, which means their antiholomorphic dependence can be summarized in a finite power series in $(\tau-\bar{\tau})^{-1}$. While in the real polarization, $F_{(n,g)}(t)$ is holomorphic but quasi-modular which means they are the constant part of the series expansion of $\mathcal{F}_{(n,g)}(t,\bar{t})$ in $(\tau-\bar{\tau})^{-1}$. It is convenient to introduce a holomorphic quasi-modular form of $E_{IJ}(\tau)$ of Γ transform as (Aganagic, Bouchard, and Klemm, 2008)

$$E^{IJ}(\tau) \to (C\tau+D)^I{}_K\,(C\tau+D)^J{}_L\,E^{KL}(\tau) + C^{IL}(C\tau+D)^J{}_L, \tag{4.4.3}$$

such that

$$\hat{E}^{IJ}(\tau,\bar{\tau}) = E^{IJ}(\tau) + \left((\tau-\bar{\tau})^{-1}\right)^{IJ} \tag{4.4.4}$$

is a modular form and transforms as

$$\hat{E}^{IJ}(\tau,\bar{\tau}) \to (C\tau+D)^I{}_K\,(C\tau+D)^J{}_L\,\hat{E}^{KL}(\tau,\bar{\tau}), \tag{4.4.5}$$

under the modular transformation

$$\tau \to (A\tau+B)(C\tau+D)^{-1} \tag{4.4.6}$$

[2]Here the modular parameters come from the period matrix τ_{ij} which is connected to the prepotential F_0. There is fundamental difference between these general cases and the local Calabi-Yau with elliptic fibration, where the refined free energy $F_{(n,g)}$ can be written as the modular form of elliptic fiber moduli τ and the weights are typically non-zero and related to n and g.

where

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma \subset Sp(2n, \mathbb{Z}). \tag{4.4.7}$$

Here E^{IJ} and $\hat{E}^{IJ}$ are just Γ analogues of the second Eisenstein series $E_2(\tau)$ of $SL(2,\mathbb{Z})$, and its modular but non-holomorphic counterpart $E_2(\tau,\bar{\tau})$. In general, E_{IJ} should be obtained from

$$E_{IJ}(\tau) = \frac{\partial}{\partial \tau_{IJ}} \log \phi(\tau) \tag{4.4.8}$$

with $\phi(\tau)$ as certain scalar Siegel cusp form of degree g. For example, for local Calabi-Yau with mirror curve of $g = 1$, the cusp form is just the well-known Ramanujan modular form Δ of weight 12. While for $g = 2$, ϕ is the Igusa cusp form of weight 10. For $g = 3$, such cusp form is of weight 18 and defined in (Tsuyumine, 1986), of which the use in topological string is still not known. For the explicit construction for E^{IJ} at genus two, see (Klemm et al., 2015).

In holomorphic polarization refined free energy $\mathcal{F}_{(n,g)}(t,\bar{t})$ can be written as

$$\begin{aligned} \mathcal{F}_{(n,g)}(t,\bar{t}) =& h^{(0)}_{(n,g)}(\tau) + (h^{(1)}_{(n,g)})_{IJ}\, \hat{E}^{IJ}(\tau,\bar{\tau}) + \dots \\ &+ (h^{(3(n+g)-3)}_{(n,g)})_{I_1 \dots I_{6(n+g)-6}}\, \hat{E}^{I_1 I_2}(\tau,\bar{\tau}) \dots \hat{E}^{I_{6(n+g)-7} I_{6(n+g)-6}}(\tau,\bar{\tau}), \end{aligned} \tag{4.4.9}$$

where $h^{(k)}_{(n,g)}(\tau)$ are holomorphic modular forms of Γ. This property is actually a direct consequence of the refined holomorphic anomaly equations. Sending $\bar{\tau}$ to infinity,

$$F_{(n,g)}(\tau) = \lim_{\bar{\tau}\to\infty} \mathcal{F}_{(n,g)}(\tau,\bar{\tau}), \tag{4.4.10}$$

one obtains the refined free energy in real polarization:

$$\begin{aligned} F_{(n,g)}(t) =& h^{(0)}_{(n,g)}(\tau) + (h^{(1)}_{(n,g)})_{IJ}\, E^{IJ}(\tau) + \dots \\ &+ (h^{(3(n+g)-3)}_{(n,g)})_{I_1 \dots I_{6(n+g)-6}}\, E^{I_1 I_2}(\tau) \dots E^{I_{6(n+g)-7} I_{6(n+g)-6}}(\tau). \end{aligned} \tag{4.4.11}$$

These formulas also show that certain combinations of $F_{(n,g)}$ and their derivatives can be both modular and holomorphic, as is indeed achieved by the component equations of blowup equations!

Note that in our convention, there is B field adding on Kähler parameter, while it doesn't appear in the original paper (Aganagic, Bouchard, and Klemm, 2008). We explain how these two results match. As pointed out in (Sun, Wang, and Huang, 2017), r fields can be obtained by shifting the complex parameter z_i with a phase $(-1)^{r_i}$. This is actually only a change of variable, thus the free energies $F^{(n,g)}(\tau)$ do not change even though the period matrix $\tau(z)$ may be different. The only thing we need to care about is that the genus 0,1 parts indeed have changed. For genus 1 part, only a constant phase emerges. For genus 0 part, we found the genus 0 free energy of local $\mathbb{P}^2$ becomes the same as our computation. We assume this happens quite general and will not mention the difference.

4.4.2 Modular property of blowup equations

In section 4.1, we write down the blowup equations in the real polarization $F_{(n,g)}(t)$. Our main assertion here is that *the unity blowup equations (4.2.1) are modular forms of* Γ *of weight* 0.[3] This claim contains two steps: recall $R = C \cdot n + r/2$,

1, the summation $\Lambda(t,r) =$

$$\sum_{n\in\mathbb{Z}^g}(-1)^{|n|}\exp\left(F\left(t+\epsilon_1 R,\epsilon_1,\epsilon_2-\epsilon_1\right)+F\left(t+\epsilon_2 R,\epsilon_1-\epsilon_2,\epsilon_2\right)-F\left(t,\epsilon_1,\epsilon_2\right)\right) \tag{4.4.12}$$

is a *quasi-modular form of weight 0* for arbitrary r vectors.

2, the above summation becomes *holomorphic modular form of weight 0*, i.e. modular invariant functions for the correct r fields.

This section is devoted to the first step, and the second step is related to refined modular anomaly equations which be elaborated in section 4.4.4. We first argue the infinite summation

$$\sum_R(-1)^{|n|}\exp\left(-\frac{1}{2}R^2F''_{(0,0)}\right) \tag{4.4.13}$$

is a modular form of weight $1/2$. Since the period matrix

$$\tau=\tau_{ij}=-C_{ik}C_{jl}\frac{\partial^2F_{0,0}}{\partial t_k\partial t_l},\qquad i,j=0,1,\ldots,g,\quad k,l=0,1,\ldots,b, \tag{4.4.14}$$

and $R_k = C_{ik}n_i + r_k/2$, we have

$$\sum_R(-1)^{|n|}\exp\left(-\frac{1}{2}R^2F''_{(0,0)}\right)=\vartheta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(\tau_{ij},0)\,, \tag{4.4.15}$$

where ϑ is the Riemann theta function with rational characteristic α and β. Very much similar to the cases of Jacobi theta function, such theta functions at special value should be Siegel modular form of certain modular group Γ of weight $1/2$.

On the other hand, it was found in (Klemm et al., 2015) that $\exp(F_{(0,1)}(t))$ is a quasi-modular form of weight $-1/2$, and $F_{(1,0)}(t)$ is modular invariant. Therefore, we observe that the weight of

$$\sum_R(-1)^{|n|}\exp\left(-\frac{1}{2}R^2F''_{(0,0)}+F_{(0,1)}-F_{(1,0)}\right):=\sum_R\Theta(R) \tag{4.4.16}$$

is zero. In fact, this expression is both holomorphic and modular invariant.

The higher order $\Lambda_{(n,g)}$ consist of many terms with form

$$\sum_R\left(R^m\prod_{m=\sum_i h_i}F^{(h_i)}_{n_i,g_i}\right)\Theta(R). \tag{4.4.17}$$

By taking τ_{ij} derivatives of

$$\sum_R(-1)^{|n|}\exp\left(-\frac{1}{2}R^2F''_{(0,0)}\right), \tag{4.4.18}$$

[3] The vanishing blowup equations can be regarded as the special cases of unity blowup equations.

or

$$\sum_R (-1)^{|n|} R \exp\left(-\frac{1}{2}R^2 F''_{(0,0)}\right), \tag{4.4.19}$$

it is not hard to find

$$\sum_R (-1)^{|n|} R^m \exp\left(-\frac{1}{2}R^2 F''_{(0,0)}\right) \tag{4.4.20}$$

is a (quasi-)modular form of weight $1/2+m$. On the other hand, since the τ_{IJ} derivative of a (quasi-)modular form is a quasi-modular form, $\frac{\partial}{\partial t^I}F_{(n,g)} = C_{IJK}\frac{\partial}{\partial \tau_{JK}}F_{(n,g)}$ is a quasi modular form of weight -1. Recall $\frac{\partial}{\partial \tau_{JK}}$ makes the weight adding two, while C_{IJK} is a modular form of weight -3. Since $R_i\partial_{t_i}$ always appear together, or more precisely since $m = \sum_i h_i$ in (4.4.17), we conclude $\Lambda_{(n,g)}(r)$ is a quasi modular form of weight zero no matter what r vector takes.

4.4.3 Refined holomorphic/modular anomaly equations

Based on a worldsheet interpretation on the refined free energy and inspiration from gauge theories (Walcher, 2009), the *refined holomorphic anomaly equations* were proposed in (Huang, Kashani-Poor, and Klemm, 2013) as

$$\begin{aligned}\frac{\partial}{\partial \bar{t}_{\bar{\imath}}}F^{(n,g)}_{i_1,\ldots,i_m} &= \frac{1}{2}\bar{C}^{jk}_{\bar{\imath}}\left(F^{(n,g-1)}_{i,j,i_1,\ldots,i_m} + \sum_{\substack{g'+g''=g \\ n'+n''=n,\ m'+m''=m}} \frac{1}{m'!m''!}F^{(n',g')}_{i,i_{\sigma(1)},\ldots,i_{\sigma(m')}}F^{(n'',g'')}_{j,i_{\sigma(1)},\ldots,i_{\sigma(m'')}}\right) \\ &\quad -(2g-2+m-1)\sum_{r=1}^{m} G_{\bar{\imath}i_r}F^{(n,g)}_{i_1,\ldots,i_{r-1}i_{r+1}\ldots i_m}. \end{aligned} \tag{4.4.21}$$

Note that in this sum (m',g') run from $(0,0)$ to (m,g) and for $g'=0$ or $g'=g$ either $n>0$ or $m>3$. Here $F^{(n,g)}_{i_1,\ldots,i_m}$ is the covariant differentiation of non-holomorphic refined free energy $D_{i_1}\ldots D_{i_m}F^{(n,g)}(t_i,\bar{t}_{\bar{\imath}})$. Besides, $\bar{C}^{ij}_{\bar{\imath}} = e^{2\mathcal{K}}G^{j\bar{\jmath}}G^{k\bar{k}}C_{\bar{\imath}\bar{\jmath}\bar{k}}$ contains the Kähler potential $\mathcal{K}$, the metric $G^{j\bar{\jmath}}$, and the complex conjugate of three point function C_{ijk}. This is the natural refined generalization of BCOV holomorphic anomaly equations (Bershadsky et al., 1994).

The anti-holomorphic derivatives in above equation (4.4.21) can also be translated to the derivatives with respect to $\hat{E}^{IJ}(\tau,\bar{\tau})$. Then equivalently we can also have the refined modular anomaly equations for $F_{(n,g)}(t)$ in the real polarization. Let us use δ_{IJ} to denote the derivative with respect to modular anomaly generators $E_{IJ}(\tau)$.

To simplify the discussion, let us focus on the case with only one Kähler parameter and mirror curve genus one. Then $\delta = \frac{\partial}{\partial E_2}$, and *refined modular anomaly equation* can be universally written as (Huang, Kashani-Poor, and Klemm, 2013)

$$\delta F^{(m)}_{(n,g)} = \frac{1}{2}\left(F^{(m+2)}_{(n,g-1)} + \sum_{g'+g''=g,n'+n''+n}^{m'+m''=m}\frac{m!}{m'!m''!}F^{(m'+1)}_{(n',g')}F^{(m''+1)}_{(n'',g'')}\right). \tag{4.4.22}$$

The summation (m',g') runs from $(0,0)$ to (m,g) and for $g'=0$ or $g'=g$ either $n>0$ or $m>3$. Besides,

$$\delta F^{(m)}_{(n,g)}(t) = 0, \qquad \text{for} \quad 3(n+g-1)+m \le 0. \tag{4.4.23}$$

In next section, we show how this equation is consistent with blowup equations.

4.4.4 The consistency

As was demonstrated in previous section 4.4.2, $\Lambda_{(n,g)}(t,r)$ are in general quasi-modular functions of monodromy group Γ. The non-triviality of blowup equations lies in that for specially chosen r — the r fields, they are not just quasi-modular but in fact *modular*! In physics language, they contain no modular anomaly: $\delta_{IJ}\Lambda_{(n,g)} = 0$. In the one side of blowup equations, since Λ only depends on mass parameters which remains the same in modular transformation, thus it obviously contain no modular anomaly. On the other side of blowup equations, things become less clear and we get component equations which are complicated combinations of $F^{(m)}_{(n,g)}$ and many theta functions and their derivatives. We want to use refined modular anomaly equation to explicitly compute the modular anomaly of component equations. After highly complicated computations by computer, we find an elegant recursive structure emerges. Our main result is the following recursion equation

$$\delta_{IJ}\Lambda_{(n,g)} = \frac{1}{2}\partial_I\partial_J\Lambda_{(n,g-1)} + \frac{1}{2}\sum\left(\partial_I F_{(n_1,g_1)}\partial_J\Lambda_{(n_2,g_2)} + \partial_J F_{(n_1,g_1)}\partial_I\Lambda_{(n_2,g_2)}\right). \quad (4.4.24)$$

Here the summation is over all non-negative (n_1,g_1,n_2,g_2) such that $n_1+n_2=n$ and $g_1+g_2=g$ excluding $(n_1,g_1)=(0,0)$ or $(n_2,g_2)=(0,0)$. Besides, I,J correspond to the true Kähler moduli, and for simplicity we assume proper orthogonalization is made.[4] Obviously, this shows $\delta_{IJ}\Lambda_{(n,g)}$ is indeed zero if the lower $\Lambda_{(n,g-1)}$ and $\Lambda_{(n_2,g_2)}$ do not depend on the ture moduli! This equation governs the consistency between blowup equations and refined modular anomaly equations.

It is also convenient to write the above equations together as

$$\delta_{IJ}\Lambda = \frac{\epsilon_1\epsilon_2}{2}\Big(\partial_I\partial_J\Lambda + \partial_I\bar{F}\partial_J\Lambda + \partial_J\bar{F}\partial_I\Lambda\Big). \quad (4.4.25)$$

Here we define $\bar{F} = F - F_{(0,0)}/(\epsilon_1\epsilon_2)$. More simply we can write as

$$\delta\Lambda = \frac{\epsilon_1\epsilon_2}{2}\Big(\partial^2\Lambda + \{\partial\bar{F},\partial\Lambda\}\Big). \quad (4.4.26)$$

In the following, we briefly show how we compute the modular anomaly of component equations. To simply the discussion, we focus on the case with one one Kähler parameter. Let us denote

$$f(n) = \sum_R \Theta(t,R)\Big(-\frac{1}{2}R^2\Big)^n, \quad (4.4.27)$$

where

$$\Theta(t,R) = (-)^{|N|}\exp\left(-\frac{1}{2}R^2F''_{(0,0)} + F_{(0,1)} - F_{(1,0)}\right). \quad (4.4.28)$$

It is easy to obtain the following recursion relation

$$F'''_{(0,0)}f(n+1) + (F'_{(0,1)} - F'_{(1,0)})f(n) = f'(n). \quad (4.4.29)$$

[4] In general, on the right hand side of (4.4.24), $\partial_I\partial_J$ should be replaced as $C^i_I C^j_J\partial_i\partial_j$, with i,j takes over all Kähler parameters.

With the initial condition $f(0) = \Lambda_0$, the recursion relation gives all $f(n)$. Remember we assume the generalized contact term equation holds: i.e. Λ_0 is independent from the true moduli t.

Consider for example $\Lambda_{(0,1)}$. Recall in (4.1.16), we have

$$\Lambda_{(0,1)} = \sum \Theta(R)\Big(3F_{(0,2)} - 2F_{(1,1)} + F_{(2,0)} - R^2 F''_{(0,1)} + \frac{1}{2}R^2 F''_{(1,0)} + \frac{1}{24}R^4 F_0^{(4)}\Big). \tag{4.4.30}$$

Using (4.4.29), we obtain

$$\begin{aligned}\Lambda_{(0,1)} = \Lambda_{(0,0)}\Big(&3F_{(0,2)} - 2F_{(1,1)} + F_{(2,0)} + (2F''_{(0,1)} - F''_{(1,0)})\frac{F'_{(1,0)} - F'_{(0,1)}}{F_0'''}\\ &+ \frac{F_0^{(4)}}{6(F_0''')^2}\Big(-\frac{F_0^{(4)}}{F_0'''}(F'_{(1,0)} - F'_{(0,1)}) + (F''_{(1,0)} - F''_{(0,1)}) + (F'_{(1,0)} - F'_{(0,1)})^2\Big)\Big).\end{aligned} \tag{4.4.31}$$

By direct variation using refined modular anomaly equations (4.4.22) and $\delta\Lambda_{(0,0)} = 0$, we have $\delta\Lambda_{(0,1)} = 0$. Similar computation gives $\delta\Lambda_{(1,0)} = 0$.

Using refined modular anomaly equations (4.4.22), and by computer programs, we find the modular anomaly of component equations $\Lambda_{(n,g)}$ have the following recursion relation:

$$\delta\Lambda_{(n,g)} = \frac{1}{2}\Lambda''_{(n,g-1)} + \sum_{n_1,g_1} F'_{(n_1,g_1)}\Lambda'_{(n-n_1,g-g_1)} \tag{4.4.32}$$

where the summation is n_1 from 0 to n, g_1 from 0 to g, but $(n_1, g_1) \neq (0,0)$ or (n,g). We have checked this equation up to all $\Lambda_{(n,g)}$ with $n + g \leq 3$. For odd orders, $n \in \mathbb{Z} + 1/2$, still we find

$$\delta\Lambda_{(n,g)} = \frac{1}{2}\Lambda''_{(n,g-1)} + \sum_{n_1,g_1} F'_{(n_1,g_1)}\Lambda'_{(n-n_1,g-g_1)} \tag{4.4.33}$$

where the summation is n_1 from 0 to $n - 1/2$, g_1 from 0 to g, but $(n_1, g_1) \neq (0,0)$ or $(n - 1/2, g)$. We have checked this equation up to all $\Lambda_{(n,g)}$ with $n + g \leq 7/2$. Recursion relation (4.4.32) and (4.4.33) are the one Kähler modulus case of equation (4.4.24).

4.4.5 A non-holomorphic version of blowup equations

Inspired from the above discussions on the consistency between blowup equations and refined modular anomaly equations, it is almost obvious that blowup equations should also work for the refined free energy in the holomorphic polarization. Actually, it is more convenient to discuss the non-holomorphic version of blowup equations in terms of component equations. For example, with $\Theta(R)$ still the same as in (4.4.28), we have the non-holomorphic component equation at $\Lambda_{(0,1)}$:

$$\begin{aligned}\Lambda_{(0,1)}(m,r) = \sum \Theta(R)\Big(&3F_{(0,2)}(t,\bar{t}) - 2F_{(1,1)}(t,\bar{t}) + F_{(2,0)}(t,\bar{t}) - R^2 F''_{(0,1)}(t,\bar{t})\\ &+ \frac{1}{2}R^2 F''_{(1,0)}(t,\bar{t}) + \frac{1}{24}R^4 F_0^{(4)}(t,\bar{t})\Big).\end{aligned} \tag{4.4.34}$$

Note all non-holomorphic part in the right hand side will cancel, so that finally $\Lambda_{(0,1)}(m,r)$ is still a holomorphic modular form of weight zero, i.e. only depends on mass parameters m.

We can also make the formula compact if we define the non-holomorphic refined free energy by

$$
\begin{aligned}
F(t+\epsilon,\bar{t}) :=& \frac{1}{\epsilon_1\epsilon_2}\left(\sum_{m=0}^{3}\frac{\epsilon^m}{m!}F^{(m)}_{(0,0)}(t)+\sum_{m=4}^{\infty}\frac{\epsilon^m}{m!}F^{(m)}_{(0,0)}(t,\bar{t})\right)\\
&+F_{(0,1)}(t)+\sum_{m=1}^{\infty}\frac{\epsilon^m}{m!}F^{(m)}_{(0,1)}(t,\bar{t})+\frac{(\epsilon_1+\epsilon_2)^2}{\epsilon_1\epsilon_2}\left(F_{(1,0)}(t)+\sum_{m=1}^{\infty}\frac{\epsilon^m}{m!}F^{(m)}_{(1,0)}(t,\bar{t})\right)\\
&+\sum_{n+g\geq 2}^{\infty}(\epsilon_1+\epsilon_2)^{2n}(\epsilon_1\epsilon_2)^{g-1}\sum_{m=0}^{\infty}\frac{\epsilon^m}{m!}F^{(m)}_{(n,g)}(t,\bar{t}). \qquad (4.4.35)
\end{aligned}
$$

Then the non-holomorphic version of blowup equations can be written as

$$
\Lambda(t,\epsilon_1,\epsilon_2,r)=\sum_{N\in\mathbb{Z}^g}(-)^{|N|}\frac{Z(t+\epsilon_1R,\bar{t},\epsilon_1,\epsilon_2-\epsilon_1)\,Z(t+\epsilon_2R,\bar{t},\epsilon_1-\epsilon_2,\epsilon_2)}{Z(t,\bar{t},\epsilon_1,\epsilon_2)}, \qquad (4.4.36)
$$

For generic r vectors, Λ is almost-holomorphic, while for the correct r fields, Λ is holomorphic!

4.5 Interpretation from M-theory

In this section, we would like to give a speculative interpretation on the generalized blowup equations from M-theory. Before going into M-theory, let us first go back to Nekrasov's master formula for the partition function of $\mathcal{N}=2$ gauge theories on general toric four dimensional manifolds (Nekrasov, 2006). It was shown that any toric four-manifold M admits a natural ϵ_1,ϵ_2 deformation and $\mathcal{N}=2$ gauge theories can be well-defined on them. Using the equivariant version of Atiyah-Singer index theorem, the partition function of $U(N)$ $\mathcal{N}=2$ gauge theories on M can be expressed via the original Nekrasov partition function on $\mathbb{C}^2$:

$$
Z_{\vec{w}}(\vec{a},\vec{\epsilon})=\sum_{\vec{k}_a\in\mathbb{Z}^N,\{\vec{k}_a\}=w_a}\prod_{v}Z(\vec{a}+\sum_a k_a\phi_a^{(v)}(\epsilon),w_{v_1},w_{v_2}), \qquad (4.5.1)
$$

in which $H^2(M)=\mathbb{Z}^d$. In the presence of Higgs vev, the $U(N)$ gauge bundle is reduced to $U(1)^N$ bundle and the d vectors $\vec{k}_a=(k_{a,l})$, $a=1,\dots,d$ classify all equivalence classes of $U(1)^N$ bundle. One need to sum over all equivalence classes and fix the traces : $\{\vec{k}_a\}=w_a=\sum_{l=1}^{N}k_{a,l}$. The simplest example beyond $\mathbb{C}^2$ is just its one-point blowup $\widehat{\mathbb{C}^2}$. For this toric complex surface, the master formula reduces to:

$$
Z_{\widehat{\mathbb{C}^2}}(\vec{a},\epsilon_1,\epsilon_2,\epsilon_3)=\sum_{\vec{k}\in\mathbb{Z}^N,\{\vec{k}\}=w}Z(\vec{a}+\vec{k}\epsilon_1,\epsilon_1+\epsilon_3,\epsilon_2-\epsilon_1)Z(\vec{a}+\vec{k}\epsilon_2,\epsilon_1-\epsilon_2,\epsilon_2+\epsilon_3), \qquad (4.5.2)
$$

where ϵ_3 controls the size of the exceptional divisor $\mathbb{P}^1$. For $\epsilon_3=0$, the above formula goes to Nakajima-Yoshioka's blowup equations.

To understand the blowup equations for general local Calabi-Yau takes two steps. First we want to argue that the situations for $\widehat{\mathbb{C}^2}$ with the exceptional divisor $\mathbb{P}^1$ of vanishing size is very similar to those for $\mathbb{C}^2$. Of course $\widehat{\mathbb{C}^2}$ with the exceptional divisor $\mathbb{P}^1$ of vanishing size itself is almost the same as $\mathbb{C}^2$ except for the singular origin. It is well-known that partition function of M-theory compactified on local Calabi-Yau X and five dimensional Omega background is equivalent to the partition function of refined topological string on X (Dijkgraaf, Vafa, and Verlinde, 2006):

$$Z_{\mathrm{M-theory}}(X \times S^1 \times \mathbb{C}^2_{\epsilon_1,\epsilon_2}) = Z_{\mathrm{ref}}(X, \epsilon_1, \epsilon_2) \tag{4.5.3}$$

In physics, the refined BPS invariants encoded in refined topological string partition function count the refined BPS states on 5D Omega background which comes from M2-branes wrapping the 11th dimensional circle S^1 and the holomorphic curves in X. Let us further consider M-theory compactified on local Calabi-Yau X and Omega deformed $\widehat{\mathbb{C}^2}$, see Figure 4.1. In this case, the M2-branes can either warp S^1 and the holomorphic curves in X or the exceptional divisor $\mathbb{P}^1$ or the both. In the first circumstance, the refined BPS counting should be exactly the same with $\mathbb{C}^2$ case. While in the second and third circumstance, it will contribute to the M-theory partition function with terms relevant to the size of the divisor $\mathbb{P}^1$. However, when we shrink the size of blowup divisor to be zero, it can be expected that the second and third circumstances only contribute to the M-theory partition function an overall factor:

$$Z_{\mathrm{M-theory}}(X \times S^1 \times \widehat{\mathbb{C}^2}_{\epsilon_1,\epsilon_2}) \sim \Lambda(X, \epsilon_1, \epsilon_2) Z_{\mathrm{ref}}(X, \epsilon_1, \epsilon_2). \tag{4.5.4}$$

We expect such factor explains the existence of the Λ factor in the blowup equations (4.2.1).

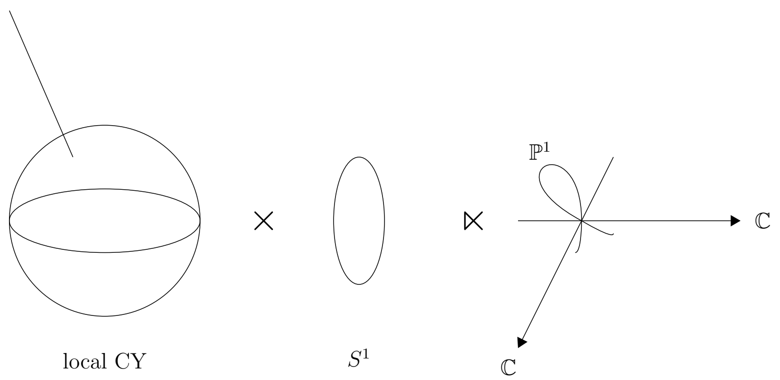


FIGURE 4.1: The background of M-theory.

Now it leaves the question how to actually compute the $Z_{\mathrm{M-theory}}(X \times S^1 \ltimes \widehat{\mathbb{C}^2}_{\epsilon_1,\epsilon_2})$. This relies much on the inspiration from supersymmetric gauge theories. Let us have a close look at $\widehat{\mathbb{C}^2}$, see the toric diagram of $\widehat{\mathbb{C}^2}$ in Figure 4.2. The length of the slash controls the size of the blowup divisor $\mathbb{P}^1$. There are two fixed points of torus $\mathbb{T}^2_{\epsilon_1,\epsilon_2}$ action. Remembering the $\mathbb{T}^2_{\epsilon_1,\epsilon_2}$ acts on $\mathbb{C}^2$ as

$$(z_1, z_2) \;\sim\; (z_1 e^{\beta\epsilon_1}, z_2 e^{\beta\epsilon_2}) \tag{4.5.5}$$

Since the homogeneous coordinates near the fixed points are respectively $(z_1, z_2/z_1)$ and $(z_1/z_2, z_2)$, thus the $\mathbb{T}^2$ weight are $(\epsilon_1, \epsilon_2 - \epsilon_1)$ and $(\epsilon_1 - \epsilon_2, \epsilon_2)$ respectively on the two patches. This actually explains the behavior of ϵ in the blowup equations.

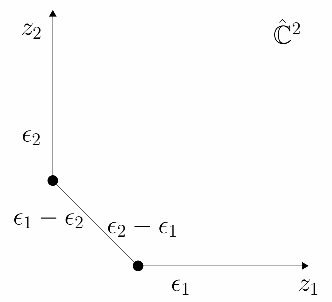


FIGURE 4.2: The toric diagram of $\widehat{\mathbb{C}^2}$.

To express the partition function on $\widehat{\mathbb{C}^2}$ via the partition function on $\mathbb{C}^2$, we need to calculate the partition function on the two patches near the two fixed points. In the blowup circumstance, certain background field emerges and has nontrivial flux through the exceptional divisor $\mathbb{P}^1$. The Kähler moduli in the partition function must receive certain shifts proportional to the flux. This is very much like the complexified Kähler parameters $r^\alpha = \int C^\alpha J + \mathrm{i}B$ where J denotes the Kähler class and B is the Kalb-Ramond field. The flux is quantized, independent of the size of divisor $\mathbb{P}^1$ and can only take some special values. The quantization is reflected in the summation over n in $R = C \cdot n + r/2$ and the r fields characterize the zero-point energy. Although not in refined topological string, similar structure already appeared in the context of traditional topological string theory when I-branes or NS 5-branes are in presence, see (Dijkgraaf, Verlinde, and Vonk, 2002) and the chapter four of (Hollands, 2009). It should be stressed that even when the size of divisor $\mathbb{P}^1$ is shrinked to zero, the flux and the two fixed points still exist. In summary, the partition function on $\widehat{\mathbb{C}^2}$ with vanishing size of exceptional divisor should be the product of the partition function on two patches with the Kähler moduli shifted by the background field flux and summed over all possible flux:

$$\begin{aligned} Z_{\mathrm{M-theory}}(X \times S^1 \times \widehat{\mathbb{C}^2}_{\epsilon_1,\epsilon_2}) \sim \sum_R Z_{\mathrm{M-theory}}(X_{t+\epsilon_1 R} \times S^1 \times \mathbb{C}^2_{\epsilon_1,\epsilon_2-\epsilon_1}) \\ \cdot Z_{\mathrm{M-theory}}(X_{t+\epsilon_2 R} \times S^1 \times \mathbb{C}^2_{\epsilon_1-\epsilon_2,\epsilon_2}). \end{aligned} \tag{4.5.6}$$

Together with (4.5.3) and (4.5.4), we can see why the blowup equations for Z_{ref} exist. This is of course a very rough picture. Nevertheless, we can already see why the structure of blowup equations could exist for general local Calabi-Yau threefolds. In the case of elliptic non-compact Calabi-Yau threefolds that engineer 6d $(1,0)$ SCFTs, a more detailed physical explanation for the elliptic blowup equations can be found in (Gu et al., 2020a; Gu et al., 2020b).

It is worthwhile to point out blowup equations can also work non-perturbatively. It is obvious that if ϵ_1 or ϵ_2 equal to $2\pi\mathrm{i}p/q$, the refined free energy (3.1.12) is the divergent. This means one needs to add non-perturbative contributions in the correspondence to the quantization of mirror curve. This idea was first proposed from the study of ABJM theory in (Hatsuda, Moriyama, and Okuyama, 2013b). We can define the non-perturbative completion for the refined partition function as

$$Z_{\text{ref}}^{(\text{np})}(t,\tau_1,\tau_2) = \frac{Z_{\text{ref}}(t,\tau_1+1,\tau_2)Z_{\text{ref}}(\frac{t}{\tau_1},\frac{1}{\tau_1},\frac{\tau_1}{\tau_2}+1)}{Z_{\text{ref}}(-\frac{t}{\tau_2},-\frac{1}{\tau_2},-\frac{\tau_1}{\tau_2}-1)}. \tag{4.5.7}$$

Here $2\pi\mathrm{i}\tau_{1,2} = \epsilon_{1,2}$, and the polynomial part is not included. Then we can write the non-perturbative blowup equations as

$$\Lambda(t,\tau_1,\tau_2,r) = \sum_{n\in\mathbb{Z}^g}(-1)^{|n|}\frac{Z_{\text{ref}}^{(\text{np})}(t+2\pi\mathrm{i}\tau_1 R,\tau_1,\tau_2-\tau_1)Z_{\text{ref}}^{(\text{np})}(t+2\pi\mathrm{i}\tau_2 R,\tau_1-\tau_2,\tau_2)}{Z_{\text{ref}}^{(\text{np})}(t,\tau_1,\tau_2)}, \tag{4.5.8}$$

where Λ is the same with the one in the original perturbative blowup equations (4.2.1). It is easy to prove once the perturbative blowup equations are satisfied, the non-perturbative ones will be satisfied simultaneously. See the derivation in section 3.10 of (Huang, Sun, and Wang, 2018).

4.6 Examples

In this section, we demonstrate blowup equations with various local toric geometries. Many simple examples like local $\mathbb{P}^2$, Hirzebruch surfaces $\mathbb{F}_n$ and resolved $\mathbb{C}^3/\mathbb{Z}_5$ orbifold can be realized as $X_{N,m}$ geometries or their reduction. For such cases, the blowup equations can be derived from the Göttsche-Nakajima-Yoshioka K-theoretic blowup equations (Gottsche, Nakajima, and Yoshioka, 2009a).

4.6.1 Resolved conifold

The resolved conifold $\mathcal{O}(-1)\oplus\mathcal{O}(-1)\mapsto\mathbb{P}^1$ is a non-compact Calabi-Yau threefold described by the constraint equation $xy - zw = 0$, where the singularity is resolved by a two-sphere $x = \rho z$, $w = \rho y$. There is a single Kähler parameter T measuring the size of base $\mathbb{P}^1$. It is well known the only non-vanishing Gopakumar-Vafa invariant of the resolved conifold is $n_0^1 = 1$, and the only non-vanishing refined BPS invariant is $n_{0,0}^1 = 1$. Clearly the B field is 1.

The resolved conifold involves a lot of interesting physics. For example, the large-N duality, or later known as the open/closed duality originated from the observation that the closed topological string theory on the resolved conifold is exactly dual to the $U(N)$ Chern-Simons theory on S^3 (Gopakumar and Vafa, 1999). In geometric engineering, the compactification of M-theory on resolved conifold gives rise to $U(1)$ supersymmetric gauge theory (Katz, Klemm, and Vafa, 1997). The resolved conifold has the simplest toric diagram, and its refined partition function was computed with the refined topological vertex in (Iqbal, Kozcaz, and Vafa, 2009) as

$$Z(q,t,Q) = \exp\left(-\sum_{n=1}^{\infty}\frac{Q^n}{n(q^{\frac{n}{2}}-q^{-\frac{n}{2}})(t^{\frac{n}{2}}-t^{-\frac{n}{2}})}\right), \tag{4.6.1}$$

where $q = e^{\epsilon_1}$, $t = e^{-\epsilon_2}$ and $Q = e^{-T}$.

It is easy to check that

$$Z(q,qt,\frac{1}{\sqrt{q}}Q)Z(qt,t,\sqrt{t}Q) = Z(q,t,Q), \tag{4.6.2}$$

which means the unity blowup equation holds for $r=1$. This equation is the result of nothing but a simple identity:

$$\frac{\frac{1}{x}}{(x-\frac{1}{x})(\frac{y}{x}-\frac{x}{y})}+\frac{\frac{1}{y}}{(\frac{x}{y}-\frac{y}{x})(y-\frac{1}{y})}=\frac{1}{(x-\frac{1}{x})(y-\frac{1}{y})}. \tag{4.6.3}$$

Here we do not have to make the twist of $t+\mathrm{i}\pi$ in partition function since there is no perturbative part as comparison. Similarly, we can check $r=-1$ is also an unity r field:

$$Z(q,qt,\sqrt{q}Q)Z(qt,t,\frac{1}{\sqrt{t}}Q)=Z(q,t,Q), \tag{4.6.4}$$

which is the result of identity

$$\frac{x}{(x-\frac{1}{x})(\frac{y}{x}-\frac{x}{y})}+\frac{y}{(\frac{x}{y}-\frac{y}{x})(y-\frac{1}{y})}=\frac{1}{(x-\frac{1}{x})(y-\frac{1}{y})}. \tag{4.6.5}$$

It is easy to prove that there is no other unity r fields. The two r fields ±1 are non-equivalent, since there is no Γ_C symmetry for genus zero geometries, but they are related by the reflective property. In fact, it can be further proved that a local Calabi-Yau satisfying blowup equation (4.6.2) can only be the resolved conifold, see the section 6.2 of (Huang, Sun, and Wang, 2018). It is also worthwhile to point out that geometries with genus-zero mirror curve do not have vanishing blowup equations. This is not very surprising since there is no traditional quantization condition for genus-zero curves.

4.6.2 Local $\mathbb{P}^2$

This geometry is the simplest local toric Calabi-Yau with genus-one mirror curve and compact four-cycle. We not only check its blowup equations to high degree of Q from the refined BPS expansion, but also give a rigorous proof for the first two component equations in the ϵ expansion of both vanishing and unity equations. Interestingly, as we will see later, the leading order of the unity blowup equation of local $\mathbb{P}^2$ just gives the pentagonal number theorem, originally due to Euler.

Local $\mathbb{P}^2$ is a geometry of line bundle $\mathcal{O}(-3)\to\mathbb{P}^2$. The toric data are

$$\begin{array}{c|ccc|c} & & v_i & & Q \\ D_u & 1 & 0 & 0 & -3 \\ D_1 & 1 & 1 & 0 & 1 \\ D_2 & 1 & 0 & 1 & 1 \\ D_3 & 1 & -1 & -1 & 1 \end{array} \tag{4.6.6}$$

The moduli space of local $\mathbb{P}^2$ in B-model is described by complex structure parameter z. The moduli space contains three singular points: large radius point, conifold point and orbifold point with $z\sim0$, $z\sim1/27$, $z\sim\infty$ respectively. We can write down the mirror curve as

$$1+x+y+\frac{z}{xy}=0, \tag{4.6.7}$$

which is an elliptic curve, with meromorphic 1-form $\lambda = \log y \frac{dx}{x}$. The periods are defined as

$$t = \int_\alpha \lambda, \quad \frac{\partial F_0}{\partial t} = \int_\beta \lambda, \tag{4.6.8}$$

where F_0 is the genus 0 free energy of topological string. In A-model, the moduli space is described by Kähler parameter t. We can compute mirror map $t(z)$ via Picard-Fuchs equation

$$(\theta^3 + 3z(3\theta - 2)(3\theta - 1)\theta)\Pi = 0, \tag{4.6.9}$$

where $\theta = t\frac{\partial}{\partial t}$. There are three solutions $\Pi_0 = 1, \Pi_1 = t, \Pi_2 = \frac{\partial F_0}{\partial t}$ to this equation. At large radius point, solving from Picard-Fuchs equation, also as is computed in (Aganagic, Bouchard, and Klemm, 2008), we have

$$F_0 = -\frac{1}{18}t^3 + \frac{1}{12}t^2 + \frac{1}{12}t + 3Q - \frac{45}{4}Q^2 + \cdots, \tag{4.6.10}$$

where $Q = e^t$. Define modular parameter $2\pi i\tau = 3\frac{\partial^2}{\partial t^2}F_0$ of elliptic curve (4.6.7), the modular group of local $\mathbb{P}^2$ is $\Gamma(3) \in SL(2,\mathbb{Z})$. It has generators

$$a := \theta^3 \begin{bmatrix} \frac{1}{6} \\ \frac{1}{6} \end{bmatrix}, \quad b := \theta^3 \begin{bmatrix} \frac{1}{6} \\ \frac{1}{2} \end{bmatrix}, \quad c := \theta^3 \begin{bmatrix} \frac{1}{6} \\ \frac{5}{6} \end{bmatrix}, \quad d := \theta^3 \begin{bmatrix} \frac{1}{2} \\ \frac{1}{6} \end{bmatrix}, \tag{4.6.11}$$

all have weight 3/2. The Dedekind η function satisfies the identity $\eta^{12} = \frac{i}{3^{3/2}}abcd$. As in (Aganagic, Bouchard, and Klemm, 2008), the genus one free energy can be compute from holomorphic anomaly equation:

$$F^{(0,1)} = -\frac{1}{6}\log(d\eta^3), \quad F^{(1,0)} = \frac{1}{6}\log(\eta^3/d). \tag{4.6.12}$$

Besides, it is obvious from the toric data that for local $\mathbb{P}^2$, $C = 3$, and from the curve that $B = 1$.

Let us first consider the unity blowup equations. It is easy to find the non-equivalent unity r field for local $\mathbb{P}^2$ are $r = \pm 1$. They are reflexive, so we can merely look at $r = 1$. In this case, $R = 3n + 1/2$ and the leading order of unity blowup equation (4.1.13) gives

$$\sum_{n=-\infty}^{\infty} (-1)^n e^{\frac{1}{2}(n+1/6)^2 3\cdot 2\pi i\tau} = \eta(\tau), \tag{4.6.13}$$

where the right side comes from

$$F^{(0,1)} - F^{(1,0)} = \log(\eta(\tau)). \tag{4.6.14}$$

This is exactly the Euler identity, or the Pentagonal number theorem! We can see both sides of the equation are weight 1/2 modular forms of $\Gamma(3)$. For higher order of the blowup equation, we obtain more such identities. For example, the subleading order requires the following identity:

$$\sum_{n=-\infty}^{\infty} (-1)^n \frac{3n+1/2}{2} e^{\frac{1}{2}(n+1/6)^2 3\cdot 2\pi i\tau} = \frac{b}{2i} + \frac{d}{2\sqrt{3}}. \tag{4.6.15}$$

One can in principle prove the component equations order by order.

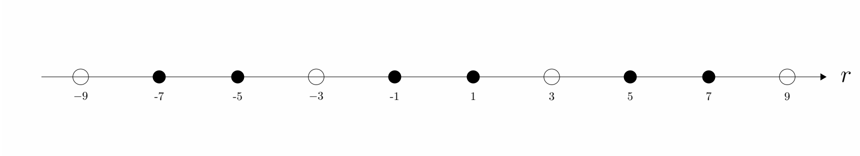


FIGURE 4.3: The r lattice of local $\mathbb{P}^2$.

Now we turn to the vanishing blowup equation. The sole vanishing r field of local $\mathbb{P}^2$ is $r = 3$. Then we have $R = 3(n + \frac{1}{2})$. Because of the symmetry under $n \to -n$, it is easy to see that half of the component equations including the leading order of vanishing blowup equation vanish trivially. The first nontrivial identity is the subleading order equation:

$$\sum_{n=-\infty}^{\infty} (-1)^n \left((n+1/2)^3 F'''_{(0,0)} + 6(n+1/2)\left(F'_{(0,1)} + F'_{(1,0)}\right)\right) e^{\frac{1}{2}(n+1/2)^2 3\cdot 2\pi i\tau} = 0. \tag{4.6.16}$$

Integrate the above equation and fix the integration constant, we obtain the following identity of level three modular forms:

$$\sum_{n=-\infty}^{\infty} (-1)^n (n+1/2) e^{\frac{1}{2}(n+1/2)^2\cdot 6\pi i\tau} = \frac{d}{3\sqrt{3}}. \tag{4.6.17}$$

All the unity and vanishing r fields can be gathered into a lattice, which we call the r *lattice*, as is shown in Figure 4.3. The white dots represent the vanishing r fields, while the black dots represent the unity r fields.

4.6.3 Local $\mathbb{P}^1 \times \mathbb{P}^1$

Local $\mathbb{P}^1 \times \mathbb{P}^1$ is a typical local toric Calabi-Yau threefold with genus-one mirror curve. There are two complex structure parameters z_1, z_2 controling the size of each $\mathbb{P}^1$. The mirror curve $\Sigma(z_1, z_2)$ can be writen as (Huang, Klemm, and Poretschkin, 2013)

$$H(x,y) = y^2 - x^3 - (1-4z_1-4z_2)x^2 - 16z_1z_2x = 0. \tag{4.6.18}$$

We can integrate the mirror curve, which is an elliptic curve, and obtain the genus zero free energy F_0. The relation between complex moduli τ of mirror curve and z_1, z_2 is given by

$$j(\tau) = \frac{\left(16z_1^2 - 8\left(2z_2+1\right)z_1 + \left(1-4z_2\right)^2\right)^3}{1728z_1^2z_2^2\left(16z_1^2 - 8\left(4z_2+1\right)z_1 + \left(1-4z_2\right)^2\right)}, \tag{4.6.19}$$

It is convinient to introduce the following convention:

$$z_1 = z,\ z_2 = zm. \tag{4.6.20}$$

where we separate the true modulus z and mass parameter m. Then the mirror map related to mass parameter is just $t_m = \log m$, which is invariant under modular transformation. While the true Kähler modulus t is related to the mirror map of z.

Unity r fields	Λ
$(0,0)$	1
$(0,2)$	1
$(0,-2)$	1
$(0,4)$	$1-e^{\epsilon_1+\epsilon_2+t_m}$
$(0,-4)$	$1-e^{-\epsilon_1-\epsilon_2+t_m}$
$(2,-2)$	$e^{-(-t_m+\epsilon_1+\epsilon_2)/4}$
$(2,2)$	$-e^{(t_m+\epsilon_1+\epsilon_2)/4}$

Table 4.1: The non-equivalent r fields of (t, t_m) and Λ factor of local $\mathbb{P}^1 \times \mathbb{P}^1$.

To simplify the discussion, we just consider the massless case $m = 1$ in the following. The modular property of this geometry is very well studied, see for example (Aganagic, Bouchard, and Klemm, 2008; Haghighat, Klemm, and Rauch, 2008). The complex moduli τ is related to genus zero free energy as

$$2\pi i\tau = 2\frac{\partial^2 F_0}{\partial t^2}. \tag{4.6.21}$$

From holomorphic anomaly equation, one can fix the genus zero free energy as

$$F^{(1,0)} = -\frac{1}{6}\log(\frac{\theta_2^2}{\theta_3\theta_4}), \tag{4.6.22}$$

and

$$F^{(0,1)} = -\log(\eta(\tau)). \tag{4.6.23}$$

We find for local $\mathbb{P}^1 \times \mathbb{P}^1$ parameterized by (t, t_m), the unique vanishing r field is $(2,0)$. Besides, all non-equivalent unity r fields and the corresponding Λ factors are listed in table 4.1. The r lattice is shown in Figure 4.4.

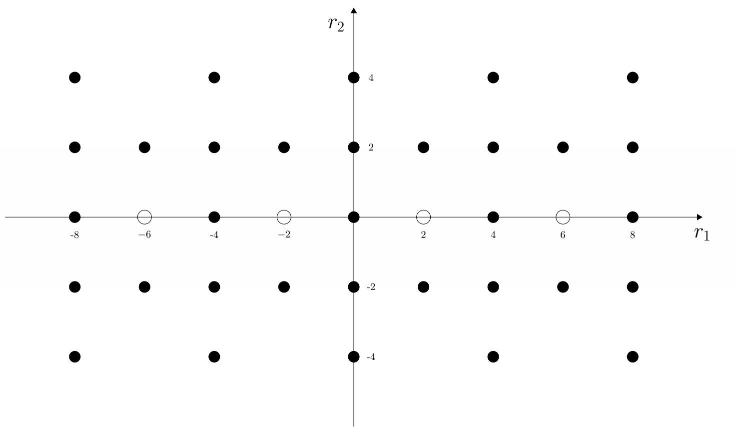


FIGURE 4.4: The r lattice of local $\mathbb{P}^1 \times \mathbb{P}^1$.

For the unity $r = (0,0)$, the leading order of unity blowup equation, i.e. the contact term equation gives the following identity

$$\theta_4(2\tau) = 2^{1/3}\Big(\frac{\theta_3\theta_4}{\theta_2^2}\Big)^{1/6}\eta. \tag{4.6.24}$$

This can be easily proved by the well-known identities $2\eta^3 = \theta_2\theta_3\theta_4$ and $\theta_4(2\tau)^2 = \theta_3\theta_4$. For the vanishing case, the contact term equation $\theta_1(2\tau) = 0$ is trivial. The integral of subleading component equation gives

$$\frac{1}{i}\theta_1'(2\tau) = \frac{1}{2}\left(\frac{\theta_3\theta_4}{\theta_2^2}\right)^{-1/2}\eta^3. \tag{4.6.25}$$

Here the left side is just $\sum_{n=-\infty}^{\infty}(-1)^n(n+1/2)q^{(n+1/2)^2}$. It is easy to prove the above identity. The higher order component equations give more identities among modular forms of Γ_2.

4.6.4 Resolved $\mathbb{C}^3/\mathbb{Z}_5$ orbifold

Resolved $\mathbb{C}^3/\mathbb{Z}_5$ orbifold is the simplest local toric Calabi-Yau with genus-two mirror curve. It has two true complex moduli and no mass parameter. This model has been extensively studied in (Klemm et al., 2015; Codesido, Grassi, and Marino, 2017; Franco, Hatsuda, and Mariño, 2016). In this section, we amplify our theory with this example, determine all r fields and check the blowup equations to high degrees of refined BPS invariants. We find for resolved $\mathbb{C}^3/\mathbb{Z}_5$, there exist three vanishing r fields and two unity r fields.

$$\begin{array}{c|ccc|cc} & & v_i & & Q_1 & Q_2 \\ x_0 & 0 & 0 & 1 & -3 & 1 \\ x_1 & 1 & 0 & 1 & 1 & -2 \\ x_2 & 2 & 0 & 1 & 0 & 1 \\ x_3 & 0 & 1 & 1 & 1 & 0 \\ x_4 & -1 & -1 & 1 & 1 & 0 \end{array} \tag{4.6.26}$$

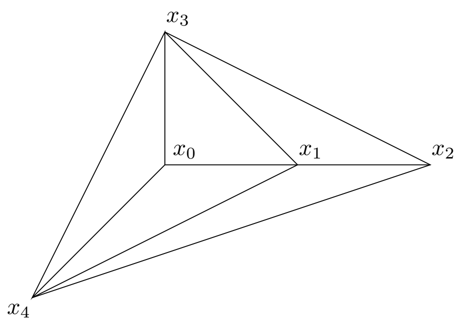


FIGURE 4.5: Fan diagram of resolved $\mathbb{C}^3/\mathbb{Z}_5$ orbifold.

Resolved $\mathbb{C}^3/\mathbb{Z}_5$ can be obtained by taking a limit in the $SU(3)$ geometry with $m = 2$. The toric data of this model is listed in (4.6.26)[5]. The fan diagram is illustrated in Figure 4.5. From the toric data, we can see there are two Batyrev coordinates,

$$z_1 = \frac{x_1x_3x_4}{x_0^3}, \quad z_2 = \frac{x_0x_2}{x_1^2}. \tag{4.6.27}$$

[5]One should not mix the toric charges Q_i here and exponential of Kähler parameter $Q_i = e^{-t_i}$

The true moduli of this model are x_0, x_1 and the C matrix is

$$C = \begin{pmatrix} -3 & 1 \\ 1 & -2 \end{pmatrix}. \tag{4.6.28}$$

The genus zero free energy of this geometry is given in (Klemm et al., 2015) as

$$F_0 = \frac{1}{15}t_1^3 + \frac{1}{10}t_1^2 t_2 + \frac{3}{10}t_1 t_2^2 + \frac{3}{10}t_2^3 + 3Q_1 - 2Q_2 - \frac{45}{8}Q_1^2 + 4Q_1Q_2 - \frac{Q_2^2}{4} + \mathcal{O}(Q_i^3). \tag{4.6.29}$$

The genus one free energy in NS limit is

$$F_1^{\rm NS} = -\frac{1}{12}t_1 - \frac{1}{8}t_2 - \frac{7Q_1}{8} + \frac{Q_2}{6} + \frac{129Q_1^2}{16} - \frac{5Q_1Q_2}{6} + \frac{Q_2^2}{12} + \mathcal{O}(Q_i^3), \tag{4.6.30}$$

and the genus one self-dual free energy is

$$F_1^{\rm GV} = \frac{2}{15}t_1 + \frac{3}{20}t_2 + \frac{Q_1}{4} - \frac{Q_2}{6} - \frac{3Q_1^2}{8} + \frac{Q_1Q_2}{3} - \frac{Q_2^2}{12} + \mathcal{O}(Q_i^3). \tag{4.6.31}$$

We find three vanishing r fields: $r = (-3,2), (-3,0), (-1,2)$ and two unity r fields, whose coresponding Λ factor are listed in table 4.6. The r lattice is shown in Figure 4.6, where black dots represent the unity r fields and colored dots represent the vanishing r fields.

r fields	Λ
$(1,0)$	$\exp\left(-\frac{1}{10}(\epsilon_1+\epsilon_2)\right)$
$(-1,0)$	$\exp\left(\frac{1}{10}(\epsilon_1+\epsilon_2)\right)$

Table 4.2: The non-equivalent unity r fields and Λ factor of resolved $\mathbb{C}^3/\mathbb{Z}_5$ orbifold.

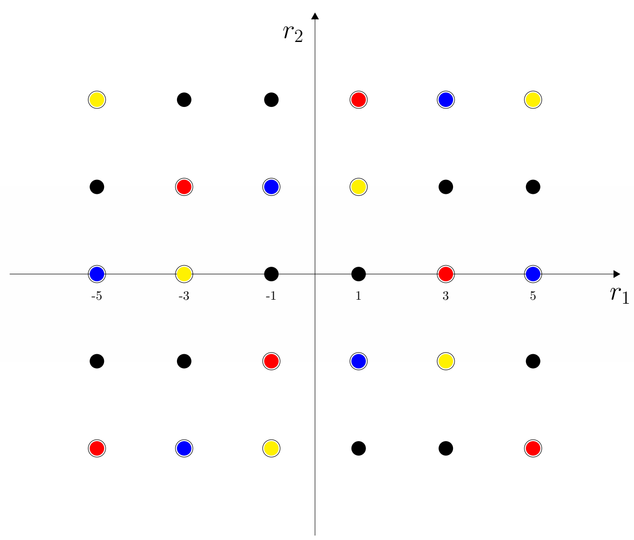


FIGURE 4.6: The r lattice of resolved $\mathbb{C}^3/\mathbb{Z}_5$ orbifold.

Chapter 5

Elliptic Blowup Equations for Rank One 6d $(1,0)$ SCFTs

Since in last chapter the blowup equations are established for general local Calabi-Yau threefolds, we can use them to study a particular subclass called *elliptic non-compact Calabi-Yau threefolds*, which are defined by elliptic fibration over some non-compact base surface S in which all curve classes can be simultaneously shrinkable to zero volume. This type of Calabi-Yau threefolds have been intensively studied in the recent decade. They are of great interest because by compactifying F-theory on such geometries, one can obtain many nontrivial interacting 6d supersymmetric gauge theories. By shrinking all curves in inside base surface to zero volume, one reaches the superconformal point, and the 6d gauge theories become 6d SCFTs.

Six is the highest dimension that superconformal algebra can exist (Nahm, 1978). 6d SCFTs contain two types of supersymmetry – $(2,0)$ and $(1,0)$ (Witten, 1995; Seiberg and Witten, 1996). The 6d $(2,0)$ SCFTs are well studied and have an ADE classification. For example, A_{N-1} $(2,0)$ SCFTs are the worldvolume theory of N M5-branes. The 6d $(1,0)$ SCFTs however are much harder to study. There are tons of them, and the classification was only achieved a few years ago by classifying the elliptic non-compact Calabi-Yau threefolds (Morrison and Taylor, 2012; Heckman, Morrison, and Vafa, 2014; Heckman et al., 2015). See an excellent review (Heckman and Rudelius, 2019).

6d $(1,0)$ SCFTs are the natural elliptic lift of 5d $\mathcal{N}=1$ and 4d $\mathcal{N}=2$ gauge theories. They all have 8 supercharges. The natural elliptic generalization of 4d/5d instanton Nekrasov partition function is called the *elliptic genera* of BPS strings in 6d $(1,0)$ SCFTs. As in the 4d and 5d cases, one main goal to study 6d $(1,0)$ SCFTs is to compute their elliptic genera. In general, this is very hard and lots of methods have been developed to determine the elliptic genera of some theories, which we will review in Chapter 5.1.3. Now we would like to establish the blowup equations for the elliptic genera and use such equations to solve the elliptic genera. Concerning both efficiency and universality, this is perhaps the best method by far.

This chapter is devoted to the blowup equations for *rank one* 6d $(1,0)$ SCFTs, although our review on 6d SCFTs in section 5.1 is for arbitrary rank. We give the full list of blowup equations – both unity and vanishing – for *all* rank one 6d $(1,0)$ SCFTs, and use them to solve the elliptic genera. We find the blowup equations can solve the elliptic genera for *almost all* rank one 6d $(1,0)$ SCFTs except twelve theories involving unpaired half-hypermultiplets.

5.1 Review of 6d $(1,0)$ SCFTs

In this section, we first follow (Heckman and Rudelius, 2019) to give a basic review of 6d $(1,0)$ SCFTs, including in particular a discussion of anomaly cancellation, which is related to the modular index of elliptic genera and will be useful for the formulation of elliptic blowup equations, and the atomic classification. We then discuss the main objects of interest, the elliptic genera of 6d SCFT, and summarize the current status of the computational results. Then we describe the semi-classical part and the one-loop part of the free energy, which are the initial data for elliptic blowup equations.

A (1,0) 6d SCFT has superconformal algebra $\mathfrak{osp}(6,2|1)$, which has a bosonic sub-algebra

$$\mathfrak{osp}(6,2|1) \supset \mathfrak{so}(6,2) \times \mathfrak{sp}(1), \tag{5.1.1}$$

where $\mathfrak{so}(6,2)$ is the conformal algebra in 6d, and $\mathfrak{sp}(1) \cong \mathfrak{su}(2)$ the R-symmetry. Massless states are labeled by representations of the sub-algebra $\mathfrak{so}(4) \cong \mathfrak{su}(2) \times \mathfrak{su}(2) \subset \mathfrak{so}(6,2)$. They can be grouped into the following three types of 6d (1,0) supermultiplets:

- Tensor multiplets: Each has an anti-self-dual tensor field H_i of spin $(1,0)$, a scalar field ϕ_i of spin $(0,0)$, and two fermions of spin $(\frac{1}{2},0)$.
- Vector multiplets: Each has a vector field of spin $(\frac{1}{2},\frac{1}{2})$, and two fermions of spin $(0,\frac{1}{2})$.
- Hypermultiplets: Each has four scalars of spin $(0,0)$, and two fermions of spin $(\frac{1}{2},0)$.

6d $(1,0)$ SCFTs in general do not have a Lorentz covariant formulation i.e. Lagrangian, due to the presence of anti-self dual 3-form field strength H associated to the BPS strings. This resembles the 4d electromagnetism where the 2-form field strength F is associated to particles. The charges of the BPS strings $n = (n_i)$, $i = 1,2,\dots,b$ can be computed by integrating the flux of tensor fields

$$n_i = \int_{M_4^\perp} \mathrm{d}H_i \in \mathbb{Z}_{\geq 0} \tag{5.1.2}$$

over the four dimensional hypersurface $M_4^\perp$ transverse to the worldvolume of the string. The number b of tensor multiplets – the string charges is also called the *rank* of a 6d SCFT The lattice of string charges Λ is equipped with a symmetric pairing

$$\langle n, n' \rangle = \sum_{i,j} A_{ij} n_i n_j' \tag{5.1.3}$$

analogous to the Dirac pairing in 4d electromagnetism. Dirac quantisation condition requires that A_{ij} is integral.

Since only tensor and hyper $(1,0)$ multiplet contain scalar fields, the moduli space of 6d SCFTs splits into two branches: tensor branch and Higgs branch. All interacting 6d $(1,0)$ SCFTs have tensor branch and many also have Higgs branch. In this thesis, we focus on the tensor branch, where the tensor branch moduli play a role as the counting number of BPS strings, corresponding to the instanton counting parameter in 4d/5d gauge theories.

6d SCFT can be geometrically engineered by F-theory compactification on an elliptic non-compact Calabi-Yau threefold. This is called the *top down* approach. The base surface of such Calabi-Yau threefold is an orbifold singularity of the type $\mathcal{B}_{\text{sing}} = \mathbb{C}^2/\Gamma_{\mathfrak{u}(2)}$, where $\Gamma_{\mathfrak{u}(2)}$ is some discrete subgroup of $\mathfrak{u}(2)$ (Heckman, Morrison, and Vafa, 2014). When specializing to $\Gamma_{\mathfrak{su}(2)}$, the supersymmetry gets enhanced from 6d $(1,0)$ to $(2,0)$, which just gives the ADE classification of 6d $(2,0)$ SCFTs.

Moving to the tensor branch of 6d SCFTs means in geometry to resolving the base singularity by successive blow-ups. In a generic point of the tensor branch, the base surface is smooth and the compact curves inside are rational curves which intersect with each other in such a way that the intersection matrix

$$A_{ij} = A_{ji} = \Sigma_i \cap \Sigma_j \tag{5.1.4}$$

is negative definite. In addition, the elliptic fibration over any base curve should be of Kodaira-Tate type. All these conditions allow a geometric classification of 6d SCFTs (Heckman et al., 2015; Heckman, Morrison, and Vafa, 2014) as we will later review in Section 5.1.2[1].

After resolving the base singularity, massless fields and BPS strings have clear geometric origin. Tensor multiplets come from dimensional reduction of type IIB fields on compact base curves. The number of these base curves gives the number of tensor multiplets, i.e. the *rank* of a 6d SCFT, while the volumes of these curves are identified with the tensor moduli. Vector multiples come from string modes on 7-branes wrapping the discriminant loci Δ of elliptic fibration. The irreducible components in Δ can be both compact or non-compact. Correspondingly the associated vector fields are either dynamic or fixed as background fields, and they induce non-Abelian gauge and flavor symmetries respectively. We split $\Delta = \Delta_c \cup \Delta_n$, where $\Delta_{c,n}$ are the unions of compact and non-compact components respectively. There could also be Abelian flavor symmetries, which are not localized but are rather associated to additional sections of the elliptic fibration (Lee, Regalado, and Weigand, 2018; Apruzzi et al., 2020). Furthermore, charged hypermultiplets are localized at intersection loci of two base curves, at least one of which is a compact curve in Δ. They come from the zero modes of strings stretched between the seven branes wrapping the two base curves. Finally, D3 branes of type IIB can wrap compact base curves and give rise to BPS strings. It is clear that the pairing of strings should be identified with intersection matrix of compact base curves (5.1.4). Furthermore the tension of strings is proportional to the volumes of base curves, i.e. the tensor moduli. The BPS strings thus become tensionless precisely at the origin of the tensor branch where all compact base curves shrink to zero volume.

5.1.1 Anomalies

In the last section, we briefly discussed the top down approach to 6d $(1,0)$ SCFTs. There is another purely gauge theory viewpoint called *bottom up* approach. From the bottom up approach, a 6d $(1,0)$ SCFT in tensor branch is simply a weakly coupled 6d gauge $(1,0)$ theory, which must be anomaly free. As is well-known, chiral anomalies exist in any even dimension. The anomalies of a 6d field theory are encoded in a

[1] A handful of 6d SCFTs with the so-called "frozen singularities" do not have valid geometric construction (Witten, 1998; Tachikawa, 2016; Bhardwaj et al., 2016; Apruzzi, Heckman, and Rudelius, 2018; Bhardwaj et al., 2018; Bhardwaj, 2020).

closed and gauge invariant 8-form I_{tot}. There are two types of contributions

$$I_{\text{tot}} = I_{\text{1-loop}} + I_{\text{GS}} \tag{5.1.5}$$

where $I_{\text{1-loop}}$ are 1-loop contributions from massless fields, while I_{GS} are contributions from Green-Schwarz counter-terms (Sadov, 1996; Green, Schwarz, and West, 1985; Sagnotti, 1992). The latter can be written as

$$I_{\text{GS}} = \frac{1}{2}\sum_{i,j} A^{ij} X_i X_j. \tag{5.1.6}$$

where A^{ij} is the inverse of the Dirac pairing A_{ij}. The 4-forms X_i read (Sadov, 1996; Green, Schwarz, and West, 1985; Sagnotti, 1992)

$$X_i = \frac{1}{4}a_i p_1(M_6) + b_i c_2(I) + \sum_{k'} b_{i,k'} c_2(\mathfrak{g}_{k'}) + \frac{1}{2}\sum_{m,n} b_{i,mn} c_1(\mathfrak{u}(1)_m) c_1(\mathfrak{u}(1)_n). \tag{5.1.7}$$

Here $p_1(M_6)$ is the first Pontryagin class of the tangent bundle of the six dimensional spacetime, $c_2(I)$ and $c_2(\mathfrak{g}_{k'})$ are the second Chern classes of the bundle of the $\mathfrak{su}(2)$ R-symmetry, and the bundles of non-Abelian gauge or flavor symmetries respectively. We also include in the last term the contributions from the first Chern classes of the flavor $\mathfrak{u}(1)$ bundles. The anomaly coefficients actually have a beautiful geometric meanings (Grassi and Morrison, 2012; Grassi and Morrison, 2000; Sadov, 1996) and determine the modular index of the elliptic genus.

Every term concerning gauge symmetry in I_{tot} must be canceled. This includes not only pure gauge anomaly, but also mixed gauge-flavor and mixed gauge-gravity anomalies in order to preserve superconformal invariance (Córdova, Dumitrescu, and Intriligator, 2019b; Córdova, Dumitrescu, and Intriligator, 2019a). Let us define the fiducial trace $\text{Tr} = \frac{1}{2\,\text{ind}_\square}\text{Tr}_\square$, where $\square$ is the defining representation, so that

$$\text{Tr}F^2 = \frac{1}{2h^\vee_{\mathfrak{g}}}\text{Tr}_{\mathfrak{adj}}F^2, \tag{5.1.8}$$

and the following Lie algebraic constants

$$\text{Tr}_R F^2 = 2\,\text{ind}_R\,\text{Tr}F^2, \quad \text{Tr}_R F^4 = x_R \text{Tr}F^4 + y_R(\text{Tr}F^2)^2, \quad \text{Tr}_R F^3 = z_R \text{Tr}F^3. \tag{5.1.9}$$

These constants for common representations of simple Lie algebras in our convention can be found in (Grassi and Morrison, 2012; Grassi and Morrison, 2000; Del Zotto and Lockhart, 2018). We then have the following anomaly cancellation conditions (Sadov, 1996; Green, Schwarz, and West, 1985; Sagnotti, 1992):

- Mixed gauge-gravity anomaly cancellation:

$$\text{ind}_{\mathfrak{adj}_i} - \sum_{R_i} n_{R_i}\,\text{ind}_{R_i} = -3(A_{ii}+2). \tag{5.1.10}$$

- Pure gauge anomaly cancellation:

$$x_{\mathfrak{adj}_i} - \sum_{R_i} n_{R_i} x_{R_i} = 0, \tag{5.1.11}$$

$$y_{\mathfrak{adj}_i} - \sum_{R_i} n_{R_i} y_{R_i} = -3A_{ii}, \tag{5.1.12}$$

- Mixed gauge-gauge anomaly cancellation:

$$\sum_{R_i,R_j} n_{R_i,R_j}\,\mathrm{ind}_{R_i}\,\mathrm{ind}_{R_j} = \frac{1}{4}. \tag{5.1.13}$$

- Mixed gauge-flavor anomaly cancellation:

$$\sum_{R_i,R_{\ell'}} n_{R_i,R_{\ell'}}\,\mathrm{ind}_{R_i}\,\mathrm{ind}_{R_{\ell'}} = \frac{1}{4} b_{i,\ell'}\,, \tag{5.1.14}$$

$$\sum_{R_i,q_m,q_n} n_{R_i,q_m,q_n} q_m q_n\,\mathrm{ind}_{R_i} = \frac{1}{2} b_{i,mn}\,, \tag{5.1.15}$$

$$\sum_{R_i,q_m} n_{R_i,q_m} q_m z_{R_i} = 0. \tag{5.1.16}$$

Here i, j label gauge symmetries, and ℓ' a non-Abelian flavor symmetry. n_{R_i}, n_{R_i,R_j}, $n_{R_i,R_{\ell'}}$, n_{R_i,q_m}, n_{R_i,q_m,q_n} are the numbers of charged hypermultiplets respectively transforming in symmetry representations with $\mathfrak{u}(1)$ charges q_m, q_n.

On the other hand, the BPS strings induces additional contribution to I_{tot}, which must be canceled by the anomaly on the world-sheet theory of BPS strings through the anomaly inflow mechanism (Shimizu and Tachikawa, 2016; Kim, Kim, and Park, 2016), see (Del Zotto and Lockhart, 2018) for a good summary. This determines the 't Hooft anomaly four-form I_4 on the worldsheet theory wrapping the base curve $S = \sum_i d_i \Sigma_i$ as

$$\begin{aligned} I_4 = &-\frac{1}{2}\sum_{i,j} A_{ij} d_i d_j (c_2(L) - c_2(R)) + \sum_i d_i \Big(h^\vee_{\mathfrak{g}_i} c_2(I) \\ &- \frac{2 + A_{ii}}{4}(p_1(T_2) - 2c_2(L) - 2c_2(R)) - \frac{1}{4} b_{i,k'} \mathrm{Tr} F_{k'}^2 - \frac{1}{2} b_{i,mn} \mathrm{Tr} F_{\mathfrak{u}(1)_m} F_{\mathfrak{u}(1)_n} \Big). \end{aligned} \tag{5.1.17}$$

Here $c_2(L), c_2(R)$ refer to the second Chern classes of the bundles associated to the global $\mathfrak{su}(2)_L, \mathfrak{su}(2)_R$ symmetry of $\mathbb{R}^4$ perpendicular to the string worldsheet M_2 in 6d. $F_{k'}$ are the field strength of non-Abelian symmetries and we sum over both gauge and flavor symmetries, while $F_{\mathfrak{u}(1)}$ are the field strength of Abelian flavor symmetries. In the case of flavor symmetries, the coefficients $b_{i,k'}$ and $b_{i,mn}$ are interpreted as the levels of the associated current algebras (Del Zotto and Lockhart, 2018), and are sometimes denoted as k_F.

5.1.2 Classification

The constraints on the elliptic non-compact Calabi-Yau threefold associated to 6d SCFTs, as well as the anomaly cancellation conditions discussed previously, allow for a geometric classification of 6d SCFTs, dubbed as "atoms classification" (Heckman et al., 2015; Heckman, Morrison, and Vafa, 2014), see also (Bhardwaj, 2015; Bhardwaj, 2020). We give a brief review here following (Heckman and Rudelius, 2019).

$\mathfrak{n}$	3	4	5	6	7	8	12	3,2	2,3,2	3,2,2
G	$\mathfrak{su}(3)$	$\mathfrak{so}(8)$	F_4	E_6	E_7	E_7	E_8	$G_2 \oplus \mathfrak{su}(2)$	$\mathfrak{su}(2) \oplus \mathfrak{so}(7) \oplus \mathfrak{su}(2)$	$G_2 \oplus \mathfrak{su}(2) \oplus \varnothing$
$\mathfrak{R}$	–	–	–	–	$\frac{1}{2}\mathbf{56}$	–	–	$\frac{1}{2}(\mathbf{7}+\mathbf{1},\mathbf{2})$	$\frac{1}{2}(\mathbf{2},\mathbf{8},\mathbf{1}) \oplus \frac{1}{2}(\mathbf{1},\mathbf{8},\mathbf{2})$	$\frac{1}{2}(\mathbf{7}+\mathbf{1},\mathbf{2},\mathbf{1})$

Table 5.1: All possible non-Higgsible clusters with minus the self-intersection numbers $\mathfrak{n}$ of curves, the gauge algebras G of the minimal singularities of elliptic fibers, and possible charged hypers in representation $\mathfrak{R}$.

The classification is divided into two steps. In the first step, classify all possible bases. There are three types of basic configurations called "atoms"

- A single -1 curve, i.e. a singe rational curve $\mathbb{P}^1$ with self-intersection -1.
- Configuration of -2 curves intersecting according to ADE Dynkin diagrams.
- Non-Higgsible clusters (Morrison and Taylor, 2012), which include: a single $-\mathfrak{n}$ curve, i.e. a rational curve with self-intersection $-\mathfrak{n}$ with $\mathfrak{n} = 3, \ldots, 8, 12$ [2], and three higher rank cases.

The -1 curve in the first category is equipped with an E_8 flavor symmetry and usually called E-string theory. The chain of -2 curves of type A in the second category always has an overall $\mathfrak{u}(1)$ flavor symmetry. The non-Higgsible clusters in the last category distinguish themselves in that elliptic fibers over them have minimal non-trivial singularity (hence the name non-Higgsible), and they are tabulated in Table 5.1. A larger configuration of base is then built by gluing the last two categories of "atomic" configurations using -1 curves subject to certain constraints. The most important of which is the gluing gluing condition that the minimal algebras $\mathfrak{g}_L, \mathfrak{g}_R$ carried by two curves glued by a -1 curve must satisfy $\mathfrak{g}_L \times \mathfrak{g}_R \subset E_8$. The other constraints including i) three curves cannot intersect in a point, ii) two curves cannot intersect tangentially, iii) intersection graphs contain no loops, iv) -1 curves can intersect at most two other curves, v) two -1 curves Σ, Σ' have $\Sigma \cdot \Sigma' = 0$.

All such configurations are classified and in general fit into a generalized quiver structure (Heckman et al., 2015). A "node" in such a quiver is a $-\mathfrak{n}$ curve with $\mathfrak{n} = 4, 6, 7, 8, 9$ which supports a minimal symmetry algebra of D- or E-type. A "link" is an appropriate configuration of curves which do not involve any nodes. All possible links are listed in (Heckman et al., 2015). The simplest links are called the minimal conformal matters, for examples:

$$[\mathfrak{so}(8)]\ 1\ [\mathfrak{so}(8)] \tag{5.1.18}$$

$$[E_6]\ 1,3,1\ [E_6] \tag{5.1.19}$$

$$[E_7]\ 1,2,3,2,1\ [E_7] \tag{5.1.20}$$

$$[E_8]\ 1,2,2,3,1,5,1,3,2,2,1\ [E_8] \tag{5.1.21}$$

Here the symmetry algebras wrapped in square brackets are flavor symmetries, and they are also the symmetry algebras carried by the nodes that can be connected to the links, while the chains of integers $\mathfrak{n}$ in the middle represent intersecting $(-n)$-curves. These configurations are so named because they come from resolving the

[2] For a single $\mathbb{P}^1$ with self-intersection $-9, -10, -11$, the elliptic fiber is not of Kodaira-Tate type, and additional blow-ups are required. See more in Chapter 6.6.

singularity at the intersection of two seven branes. They can also be realized in M-theory as a M5-brane probing D- or E-type singularity $\mathbb{C}^2/\Gamma_{DE}$. A complete list of minimal conformal matters can be found in (Del Zotto et al., 2015). Some of the more complicated link configurations can be obtained by joining two minimal conformal matters and gauging the common flavor symmetry, or by performing Higgs branch RG flow (Del Zotto et al., 2015; Heckman, Rudelius, and Tomasiello, 2016).

The second step of classification is to assign suitable singular fibers so that the total space of fibration is a Calabi-Yau threefold. In particular, one has to make sure that every elliptic fiber is of the Kodaira-Tate type, which is in general equivalent to the condition of gauge anomaly cancellation discussed in Section 5.1.1. This step can also be done in two parts. The first part involves the classification of singular fibers over a single base curve or equivalently the associated symmetry algebra, i.e. the classification of rank one 6d SCFTs. The minimal symmetry algebras have been given in Table 5.1, and they can be enhanced by making worse the singularity of elliptic fibers. At the same time the numbers of charged hypermultiplets increases. Their numbers as well as the representations of symmetry algebras under which they transform are completely determined by the anomaly cancellation conditions (5.1.10), (5.1.11), (5.1.12).

If there are multiple hypermultiplets in the same gauge representation R, they enjoy a non-trivial flavor symmetry F. The type of the flavor symmetry is determined by the number of hypermultiplets and the nature of R. The flavor symmetries including possibly a discrete part can be determined by analyzing the current algebra on the worldsheet of BPS string (Del Zotto and Lockhart, 2018).

A rank one 6d SCFT may also have Abelian flavor symmetry (Heckman, Rudelius, and Tomasiello, 2016), which can be uncovered by either subjecting the candidate Abelian symmetry that accompanies complex representations to the test of the anomaly cancellation condition (5.1.16) (Apruzzi et al., 2020), or by studying the current algebras on the worldsheet theory of BPS strings (Del Zotto and Lockhart, 2018). Once the flavor symmetry is known, the associated anomaly coefficients $b_{i,k'}, b_{i,mn}$ can be computed by (5.1.14), (5.1.15). With all these taken into account, the gauge symmetries and flavor symmetries of all rank one 6d SCFTs are given in (Heckman et al., 2015; Del Zotto and Lockhart, 2018), and we reproduce it in Tables 5.2, 5.3, 5.4.

The second part of fiber classification is to consider bi-representation of two gauge algebras, which are further constrained by the anomaly cancellation condition (5.1.13). There are only five possibilities (Heckman and Rudelius, 2019)

- $\mathfrak{g}_a = \mathfrak{su}(n_a), \mathfrak{g}_b = \mathfrak{su}(n_b), R = (\mathbf{n}_a, \mathbf{n}_b)$
- $\mathfrak{g}_a = \mathfrak{su}(n_a), \mathfrak{g}_b = \mathfrak{sp}(n_b), R = (\mathbf{n}_a, \mathbf{2n}_b)$
- $\mathfrak{g}_a = \mathfrak{sp}(n_a), \mathfrak{g}_b = \mathfrak{so}(n_b), R = \frac{1}{2}(\mathbf{2n}_a, \mathbf{n}_b)$
- $\mathfrak{g}_a = \mathfrak{sp}(n_a), \mathfrak{g}_b = \mathfrak{so}(n_b), n_b = 7, 8, R = \frac{1}{2}(\mathbf{2n}_a, \mathbf{8}_{s,c})$
- $\mathfrak{g}_a = \mathfrak{sp}(n_a), \mathfrak{g}_b = G_2, R = \frac{1}{2}(\mathbf{2n}_a, \mathbf{7})$

5.1.3 Elliptic genera

We are interested in the partition function of 6d SCFT on the tensor branch on the 6d Ω background. Such background is a curved spacetime background, which

$\mathfrak{n}$	G	F	(R_G, R_F)
12	E_8	$-$	$-$
8	E_7	$-$	$-$
7	E_7	$-$	$(\mathbf{56},\mathbf{1})$
6	E_6	$-$	$-$
6	E_7	$\mathfrak{so}(2)_{12}$	$(\mathbf{56},\mathbf{2})$
5	F_4	$-$	$-$
5	E_6	$\mathfrak{u}(1)_6$	$\mathbf{27}_{-1}\oplus c.c.$
5	E_7	$\mathfrak{so}(3)_{12}$	$(\mathbf{56},\mathbf{3})$
4	$\mathfrak{so}(8)$	$-$	$-$
4	$\mathfrak{so}(N\geq 9)$	$\mathfrak{sp}(N-8)_1$	$(\mathbf{N},\mathbf{2(N-8)})$
4	F_4	$\mathfrak{sp}(1)_3$	$(\mathbf{26},\mathbf{2})$
4	E_6	$\mathfrak{su}(2)_6\times\mathfrak{u}(1)_{12}$	$(\mathbf{27},\overline{\mathbf{2}})_{-1}\oplus c.c.$
4	E_7	$\mathfrak{so}(4)_{12}$	$(\mathbf{56},\mathbf{2}\oplus\mathbf{2})$
3	$\mathfrak{su}(3)$	$-$	$-$
3	$\mathfrak{so}(7)$	$\mathfrak{sp}(2)_1$	$(\mathbf{8},\mathbf{4})$
3	$\mathfrak{so}(8)$	$\mathfrak{sp}(1)^a_1\times\mathfrak{sp}(1)^b_1\times\mathfrak{sp}(1)^c_1$	$(\mathbf{8}_v\oplus\mathbf{8}_c\oplus\mathbf{8}_s,\mathbf{2})$
3	$\mathfrak{so}(9)$	$\mathfrak{sp}(2)^a_1\times\mathfrak{sp}(1)^b_2$	$(\mathbf{9},\mathbf{4}^a)\oplus(\mathbf{16},\mathbf{2}^b)$
3	$\mathfrak{so}(10)$	$\mathfrak{sp}(3)^a_1\times(\mathfrak{su}(1)_4\times\mathfrak{u}(1)_4)^b$	$(\mathbf{10},\mathbf{6}^a)\oplus[(\mathbf{16}_s)^b_1\oplus c.c.]$
3	$\mathfrak{so}(11)$	$\mathfrak{sp}(4)^a_1\times\text{Ising}^b$	$(\mathbf{11},\mathbf{8}^a)\oplus(\mathbf{32},\mathbf{1}^b_s)$
3	$\mathfrak{so}(12)$	$\mathfrak{sp}(5)_1$	$(\mathbf{12},\mathbf{10})\oplus(\mathbf{32}_s,\mathbf{1})$
3	G_2	$\mathfrak{sp}(1)_1$	$(\mathbf{7},\mathbf{2})$
3	F_4	$\mathfrak{sp}(2)_3$	$(\mathbf{26},\mathbf{4})$
3	E_6	$\mathfrak{su}(3)_6\times\mathfrak{u}(1)_{18}$	$(\mathbf{27},\overline{\mathbf{3}})_{-1}\oplus c.c.$
3	E_7	$\mathfrak{so}(5)_{12}$	$(\mathbf{56},\mathbf{5})$

Table 5.2: Gauge, flavor symmetries and charged matter contents of rank one 6d SCFTs with $\mathfrak{n}\geq 3$ (Del Zotto and Lockhart, 2018). The subscript in a flavor symmetry algebra indicates the level of the associated current algebra. When a flavor symmetry has multiple simple components, superscripts are used to distinguish them and their representations. Matters are presented as the gauge and flavor representations by which the half-hypermultiplets transform. If there is an Abelian flavor symmetry, the Abelian charge is given as subscript.

is topologically $T^2\times\mathbb{R}^4$ with the metric (Losev, Marshakov, and Nekrasov, 2003)

$$\mathrm{d}s^2=\mathrm{d}z\mathrm{d}\bar{z}+(\mathrm{d}x^\mu+\Omega^\mu\mathrm{d}z+\bar{\Omega}^\mu\mathrm{d}\bar{z})^2,\quad \mu=1,2,3,4 \tag{5.1.22}$$

where $z,\bar{z}$ are coordinates on T^2 and x^μ coordinates on $\mathbb{R}^4$. The Ω^μ satisfy

$$\mathrm{d}\Omega=\epsilon_1\,\mathrm{d}x^1\wedge\mathrm{d}x^2-\epsilon_2\,\mathrm{d}x^3\wedge\mathrm{d}x^4, \tag{5.1.23}$$

and $\epsilon_{L,R}=(\epsilon_1\mp\epsilon_2)/2$ are the background field strengths for the spacetime symmetry $\mathfrak{su}(2)_L\times\mathfrak{su}(2)_R$ acting on $\mathbb{R}^4$. The compactification on T^2 allows access to the BPS states on BPS strings, encoded in the Ramond-Ramond elliptic genera, which are the generalized Witten index on the worldsheet 2d $(0,4)$ theory of BPS strings. For more details about these 2d $(0,4)$ theories and the 2d definitions on RR elliptic genera, we refer to (Del Zotto and Lockhart, 2017). The BPS strings wrapped on T^2 appear as instantons on $\mathbb{R}^4$. The partition function of 6d SCFT can be written as

$\mathfrak{n}$	G	F	(R_G, R_F)
2	$\mathfrak{su}(1)$	$\mathfrak{su}(2)_1$	$-$
2	$\mathfrak{su}(2)$	$\mathfrak{so}(7)_1 \times \text{Ising}$	$(\mathbf{2}, \mathbf{8}_s \times \mathbf{1}_s)$
2	$\mathfrak{su}(N \geq 3)$	$\mathfrak{su}(2N)_1$	$(\mathbf{N}, \overline{\mathbf{2N}}) \oplus c.c.$
2	$\mathfrak{so}(7)$	$\mathfrak{sp}(1)_1^a \times \mathfrak{sp}(4)_1^b$	$(\mathbf{7}, \mathbf{2}^a) \oplus (\mathbf{8}, \mathbf{8}^b)$
2	$\mathfrak{so}(8)$	$\mathfrak{sp}(2)_1^a \times \mathfrak{sp}(2)_1^b \times \mathfrak{sp}(2)_1^c$	$(\mathbf{8}_v, \mathbf{4}^a) \oplus (\mathbf{8}_s, \mathbf{4}^b) \oplus (\mathbf{8}_c, \mathbf{4}^c)$
2	$\mathfrak{so}(9)$	$\mathfrak{sp}(3)_1^a \times \mathfrak{sp}(2)_2^b$	$(\mathbf{9}, \mathbf{6}^a) \oplus (\mathbf{16}, \mathbf{4}^b)$
2	$\mathfrak{so}(10)$	$\mathfrak{sp}(4)_1^a \times (\mathfrak{su}(2)_4 \times \mathfrak{u}(1)_8)^b$	$(\mathbf{10}, \mathbf{8}^a) \oplus [(\mathbf{16}_s, \mathbf{2}^b)_1 \oplus c.c.]$
2	$\mathfrak{so}(11)$	$\mathfrak{sp}(5)_1^a \times ?^b$	$(\mathbf{11}, \mathbf{10}^a) \oplus (\mathbf{32}, \mathbf{2}^b)$
2	$\mathfrak{so}(12)_a$	$\mathfrak{sp}(6)_1^a \times \mathfrak{so}(2)_8$	$(\mathbf{12}, \mathbf{12}^a) \oplus (\mathbf{32}_s, \mathbf{2}^b)$
2	$\mathfrak{so}(12)_b$	$\mathfrak{sp}(6)_1^a \times \text{Ising}^b \times \text{Ising}^c$	$(\mathbf{12}, \mathbf{12}^a) \oplus (\mathbf{32}_s, \mathbf{1}_s^b) \oplus (\mathbf{32}_c, \mathbf{1}_s^c)$
2	$\mathfrak{so}(13)$	$\mathfrak{sp}(7)_1$	$(\mathbf{13}, \mathbf{14}) \oplus (\mathbf{64}, \mathbf{1})$
2	G_2	$\mathfrak{sp}(4)_1$	$(\mathbf{7}, \mathbf{8})$
2	F_4	$\mathfrak{sp}(3)_3$	$(\mathbf{26}, \mathbf{6})$
2	E_6	$\mathfrak{su}(4)_6 \times \mathfrak{u}(1)_{24}$	$(\mathbf{27}, \overline{\mathbf{4}})_{-1} \oplus c.c.$
2	E_7	$\mathfrak{so}(6)_{12}$	$(\mathbf{56}, \mathbf{6})$

Table 5.3: Gauge, flavor symmetries and charged matter contents of rank one 6d SCFTs with $\mathfrak{n} = 2$ (Del Zotto and Lockhart, 2018). ? means the flavor symmetry predicted by field theoretic considerations cannot be realized consistently on the worldsheet of BPS strings.

follows:

$$Z(\phi, \tau, m_{G,F}, \epsilon_{1,2}) = \\ Z^{\text{cls}}(\phi, \tau, m_{G,F}, \epsilon_{1,2}) Z^{\text{1-loop}}(\tau, m_{G,F}, \epsilon_{1,2}) \Big(1 + \sum_d e^{i2\pi\phi\cdot d} \mathbb{E}_d(\tau, m_{G,F}, \epsilon_{1,2})\Big). \tag{5.1.24}$$

Here $Z^{\text{cls}}, Z^{\text{1-loop}}$ are semi-classical contributions, and one-loop contributions from tensor, vector and hypermultiplets respectively. $\mathbb{E}_d$ is the RR elliptic genus of the BPS strings with string charge $d = (d_i) \in \Lambda$ associated to the base curve $S = \sum_i d_i \Sigma_i$. $\phi = (\phi_i), \tau$ are respectively the tensor moduli and geometrically the complex structure of T^2. $\mathbb{E}_d$ is also called d-string elliptic genus. We have turned on the vevs $m_{G,F}$ of Wilson loops of gauge and flavor vector fields along 1-cycles in T^2, also called the gauge and flavor fugacities. They take value in the complexified Cartan subalgebra of the corresponding symmetry algebra, where a Weyl invariant bilinear form $(\bullet, \bullet)$ is defined. See Appendix A for our Lie algebraic convention. We will also use the notation of the *reduced* d-string elliptic genus:

$$\mathbb{E}_d^{\text{red}}(\tau, m_{G,F}, \epsilon_1, \epsilon_2) = \mathbb{E}_d(\tau, m_{G,F}, \epsilon_1, \epsilon_2) / \mathbb{E}_{c.m.}(\tau, \epsilon_1, \epsilon_2), \tag{5.1.25}$$

where the the contribution from the center of mass free hypermultiplet

$$\mathbb{E}_{c.m.}(\tau, \epsilon_1, \epsilon_2) = \frac{\eta(\tau)^2}{\theta_1(\tau, \epsilon_1)\theta_1(\tau, \epsilon_2)} \tag{5.1.26}$$

is factored out (Del Zotto and Lockhart, 2017). This brings certain simplification for elliptic genera especially for the one-string case.

Gauge theories on 6d Ω background have connection with refined topological

$\mathfrak{n}$	G	F	(R_G, R_F)
1	$\mathfrak{sp}(0)$	$(E_8)_1$	$-$
1	$\mathfrak{sp}(N\geq 1)$	$\mathfrak{so}(4N+16)_1$	$(\mathbf{2N},\mathbf{4N+16})$
1	$\mathfrak{su}(3)$	$\mathfrak{su}(12)_1$	$(\mathbf{3},\overline{\mathbf{12}})_1\oplus c.c.$
1	$\mathfrak{su}(4)$	$\mathfrak{su}(12)_1^a\times\mathfrak{su}(2)_1^b$	$[(\mathbf{4},\overline{\mathbf{12}}_1^a)\oplus c.c.]\oplus(\mathbf{6},\mathbf{2}^b)$
1	$\mathfrak{su}(N\geq 5)$	$\mathfrak{su}(N+8)_1\times\mathfrak{u}(1)_{2N(N-1)(N+8)}$	$[(\mathbf{N},\overline{\mathbf{N+8}})_{-N+4}\oplus(\mathbf{\Lambda^2},\mathbf{1})_{N+8}]\oplus c.c.$
1	$\mathfrak{su}(6)_*$	$\mathfrak{su}(15)_1$	$[(\mathbf{6},\overline{\mathbf{15}})\oplus c.c.]\oplus(\mathbf{20},\mathbf{1})$
1	$\mathfrak{so}(7)$	$\mathfrak{sp}(2)_1^a\times\mathfrak{sp}(6)_1^b$	$(\mathbf{7},\mathbf{4}^a)\oplus(\mathbf{8},\mathbf{12}^b)$
1	$\mathfrak{so}(8)$	$\mathfrak{sp}(3)_1^a\times\mathfrak{sp}(3)_1^b\times\mathfrak{sp}(3)_1^c$	$(\mathbf{8}_v,\mathbf{6}^a)\oplus(\mathbf{8}_s,\mathbf{6}^b)\oplus(\mathbf{8}_c,\mathbf{6}^c)$
1	$\mathfrak{so}(9)$	$\mathfrak{sp}(4)_1^a\times\mathfrak{sp}(3)_2^b$	$(\mathbf{9},\mathbf{8}^a)\oplus(\mathbf{16},\mathbf{6}^b)$
1	$\mathfrak{so}(10)$	$\mathfrak{sp}(5)_1^a\times(\mathfrak{su}(3)_4\times\mathfrak{u}(1)_{12})^b$	$(\mathbf{10},\mathbf{10}^a)\oplus[(\mathbf{16}_s,\mathbf{3}^b)_1\oplus c.c.]$
1	$\mathfrak{so}(11)$	$\mathfrak{sp}(6)_1^a\times ?^b$	$(\mathbf{11},\mathbf{12}^a)\oplus(\mathbf{32},\mathbf{3}^b)$
1	$\mathfrak{so}(12)_a$	$\mathfrak{sp}(7)_1^a\times\mathfrak{so}(3)_8^b$	$(\mathbf{12},\mathbf{14}^a)\oplus(\mathbf{32}_s,\mathbf{3}^b)$
1	$\mathfrak{so}(12)_b$	$\mathfrak{sp}(7)_1^a\times ?^b\times ?^c$	$(\mathbf{12},\mathbf{14}^a)\oplus(\mathbf{32}_s,\mathbf{2}^b)\oplus(\mathbf{32}_c,\mathbf{1}^c)$
1	G_2	$\mathfrak{sp}(7)_1$	$(\mathbf{7},\mathbf{14})$
1	F_4	$\mathfrak{sp}(4)_3$	$(\mathbf{26},\mathbf{8})$
1	E_6	$\mathfrak{su}(5)_6\times\mathfrak{u}(1)_{30}$	$(\mathbf{27},\overline{\mathbf{5}})_{-1}\oplus c.c.$
1	E_7	$\mathfrak{so}(7)_{12}$	$(\mathbf{56},\mathbf{7})$

Table 5.4: Gauge, flavor symmetries and charged matter contents of rank one 6d SCFTs with $\mathfrak{n}=1$ (Del Zotto and Lockhart, 2018). Λ^2 is the rank-two anti-symmetric representation. ? means the flavor symmetry predicted by field theoretic considerations cannot be realized consistently on the worldsheet of BPS strings.

string theory. F-theory compactified on an elliptic Calabi-Yau threefold X and T^2 is dual to M-theory compactified on the same threefold X and the M-theory circle S^1, where the volume of elliptic fiber in X is inversely proportional to the volume of T^2. Turning on Wilson loops of gauge and flavor vector fields amounts to resolving singular elliptic fibers so that the threefold X is smooth. M-theory BPS states are computed in this setup by topological string theory which encodes in particular the numbers of BPS states of M2-branes wrapping 2-cycles in X. One can therefore use topological string theory techniques to get information about the $\mathbb{E}_d$ and in particular initial data for the blow up equations.

The modular property of the elliptic genera measures how they transform under the modular group $SL(2,\mathbb{Z})$ of the torus T^2. In fact, the elliptic genera are not invariant, but transform as meromorphic Jacobi forms of weight zero with non-trivial index, where both the gauge/flavor fugacities and the parameters of the Ω background transform as elliptic parameters.

$$\mathbb{E}_d\left(\frac{a\tau+b}{c\tau+d},\frac{m_{G,F}}{c\tau+d},\frac{\epsilon_{1,2}}{c\tau+d}\right)=\mathrm{e}^{\frac{c}{c\tau+d}\,\mathrm{Ind}\,\mathbb{E}_d(m_{G,F},\epsilon_{1,2})}\mathbb{E}_d(\tau,m_{G,F},\epsilon_{1,2}). \tag{5.1.27}$$

Here $\mathrm{Ind}\,\mathbb{E}_d(m_{G,F},\epsilon_{1,2})$, called the modular *index polynomial*, is a quadratic polynomial. The index polynomial can be computed from the integral of 't Hooft anomaly four-form (Bobev, Bullimore, and Kim, 2015), or equivalently by the following simple replacements rules (Del Zotto et al., 2018; Del Zotto and Lockhart, 2017)

$$\begin{aligned} p_1(M_2)\to 0,\quad c_2(L)\to -\epsilon_L^2,\quad c_2(R),c_2(I)\to -\epsilon_R^2,\\ \mathrm{Tr}F_{k'}^2\to -2(m_{k'},m_{k'}),\quad \mathrm{Tr}F_{\mathfrak{u}(1)}\to \mathrm{i}\,m_{\mathfrak{u}(1)}. \end{aligned} \tag{5.1.28}$$

Applying these rules on (5.1.17) yields the following useful formula for the index polynomial

$$\begin{aligned}\mathrm{Ind}\,\mathbb{E}_d = & -\frac{(\epsilon_1+\epsilon_2)^2}{4}\sum_i (2+A_{ii}+h^\vee_{\mathfrak{g}_i})d_i + \frac{\epsilon_1\epsilon_2}{2}\Big(\sum_i (2+A_{ii})d_i - \sum_{i,j} A_{ij}d_id_j\Big) \\ & + \frac{1}{2}\sum_{i,k'} b_{i,k'}d_i(m_{k'},m_{k'}) + \frac{1}{2}\sum_{i,\ell,n} b_{i,\ell n}m_\ell m_n. \end{aligned} \tag{5.1.29}$$

Known computational methods

In this section, we summarize all known results on the elliptic genera of 6d $(1,0)$ SCFTs, especially all rank one theories. Three methods with relatively wide range of application are 2d quiver gauge theories, modular ansatz and refined topological vertex. In the following, we briefly introduce each method, list the theories it can solve and comment on its advantages and disadvantages.

In the spirit of the ADHM construction for 4d/5d instantons, certain 6d $(1,0)$ SCFTs are known to correspond to 2d quiver gauge theories. Once the 2d quiver construction is found, one can use localization – Jeffrey-Kirwan residue (Jeffrey and Kirwan, 1995) to exactly compute the elliptic genera to arbitrary number of strings (Benini et al., 2014; Benini et al., 2015). However, like in the ADHM construction, such correspondence normally just exists for classical gauge groups with simple matter contents, but difficult to generalize to exceptional gauge groups. In particular, all rank one $(1,0)$ theories with known 2d quiver construction are listed below:

- $n=1$, $G=\mathfrak{sp}(N)$ (Kim, Kim, and Lee, 2015; Yun, 2016)
- $n=1$, $G=\mathfrak{su}(N)$ (Kim, Kim, and Lee, 2015)
- $n=2$, $G=\mathfrak{su}(N)$ (Haghighat et al., 2014)
- $n=3$, $G=\mathfrak{su}(3), G_2$ and $\mathfrak{so}(7)$ (Kim et al., 2018)
- $n=4$, $G=\mathfrak{so}(8+N)$ (Haghighat et al., 2015b; Del Zotto and Lockhart, 2018)

For all these theories, we use the known elliptic genera from quiver formulas to check against our elliptic blowup equations and find perfect agreement.

The modular ansatz method exploits the Jacobi-form transformations of the elliptic genera as well as their pole structures and can be very constraining sometimes. For the reduced one string elliptic genus with all gauge and flavor fugacities turned off, the modular ansatz has a particularly simple form and was extensively studied in (Del Zotto and Lockhart, 2018). For example, using the constraints from the spectral flow relation between RR and NSR elliptic genus, such ansatz were fixed in (Del Zotto and Lockhart, 2018) for all rank-one theories except for

- $n=1,2,3,4$, $G=E_7$
- $n=1,2$, $G=E_6$, $\mathfrak{so}(11)$ and $\mathfrak{so}(12)_b$

These results provide an excellent testing ground for our blowup equations. Indeed, for all the theories we have studied where the modular ansatz is fixed in (Del Zotto and Lockhart, 2018), we find agreement for the one-string elliptic genera. Besides, we are able to use blowup equations to further determine the modular ansatz for

$n=2,4$ E_7 theories, $n=1,2$ E_6 theories and $n=2$ $\mathfrak{so}(11)$ theory and make cross checks. The modular ansatz method also extends to the situation with gauge and flavor fugacities turned on, where Weyl-invariant Jacobi forms are involved and the computation becomes much more complicated. Still, the ansatz for the one-string elliptic genus with gauge fugacities turned on for $n=3$ $\mathfrak{su}(3)$ and $n=4$ $\mathfrak{so}(8)$ theories was determined in (Del Zotto et al., 2018), and for $n=1$ $\mathfrak{sp}(1)$, $n=2$ $\mathfrak{su}(2)$, $n=3$ $\mathfrak{su}(3)$ and $n=3$ G_2 theories was determined in (Kim, Lee, and Park, 2018). Besides, the modular ansatz for E-string and M-string elliptic genera has been studied in (Gu et al., 2017; Duan, Gu, and Kashani-Poor, 2018).

The refined topological vertex and the brane-webs can also compute the elliptic genera of some 6d theories with matters. For example, the brane web construction was known for E-string theory (Kim, Taki, and Yagi, 2015), M-string theory (Haghighat et al., 2015a), minimal pure gauge theories (Hayashi and Ohmori, 2017), $n=1, G=\mathfrak{sp}(N)$ theories (Hayashi et al., 2019a), $n=1, G=\mathfrak{su}(N)$ theories (Hayashi et al., 2019a), $n=1, G=\mathfrak{su}(6)_*$ theory (Hayashi et al., 2019b), a family of $n=2,3$ $\mathfrak{so}(N)$ theories (Kim, Kim, and Lee, 2019), the D-type conformal matter theories (Hayashi et al., 2015). See also (Hayashi et al., 2016; Hayashi et al., 2017). The brane construction for theories with non $\mathfrak{su}$ type gauge symmetry or complicated matter representations typically involves orientifold 7-plane and O5-planes.

The domain wall method uses the Hořava-Witten picture of $E_8\times E_8$ heterotic string theory, which is M-theory on $S^1/\mathbb{Z}_2$ orbifold (Horava and Witten, 1996). There are two M9-branes called M9 domain walls in the fixed points of $\mathbb{Z}_2$ orbifold action. There are also dynamical M2-branes and M5-branes as usual. In the limit that the distance of two M9 domain walls goes to zero, the E-strings can be realized by M2-branes stretched between a M5-brane and a M9-brane and M-strings by M2-branes stretched between two M5-branes. Using the picture, (Haghighat, Lockhart, and Vafa, 2014) computed the elliptic genera of arbitrary number of M-strings, and of one and two E-strings, (Cai, Huang, and Sun, 2015) computed the elliptic genus of three E-strings.

One more interesting methods is to use the duality to the so called twisted H_G theories, see more descriptions in Chapter 7.1. This method is of limited use, currently it only has been used to compute the one-string elliptic genus of pure gauge D_4, E_6 theory (Putrov, Song, and Yan, 2016) and the one-string elliptic genus of pure gauge E_7 theory (Agarwal, Maruyoshi, and Song, 2018).

Computation in topological string theory such as the genus zero free energy or even higher genus free energy from holomorphic anomaly equations also provide useful information on the elliptic genera. This requires one first construct the elliptic non-compact Calabi-Yau threefold associated to a 6d $(1,0)$ SCFTs. For example, the topological string free energy has been computed to certain genus and order in (Huang, Klemm, and Poretschkin, 2013) for E-strings and in (Haghighat et al., 2015b) for the minimal 6d $(1,0)$ SCFTs. The Calabi-Yau geometries associated to lots of interesting 6d SCFTs with matters have been constructed in (Kashani-Poor, 2019; Gu et al., 2020b).

It is also worthwhile to point out some relevant 5d results. For example, many methods to compute 5d Nekrasov paritition function for pure gauge theories have been pointed out in Chapter 2.3. The 5d Nekrasov partition functions of $n=2, G=\mathfrak{su}(N)$ theories were well known long time ago, see (Nekrasov and Shadchin, 2004;

Shadchin, 2004; Benvenuti, Hanany, and Mekareeya, 2010). The 5d blowup equations with matters were initially studied in (Nakajima and Yoshioka, 2011). Recently, the 5d unity blowup equations for all possible gauge and matter content were studied in (Kim et al., 2019). For a lot of 5d theories, their Nekrasov partition functions can be solved from these blowup equations recursively with respect to the instanton numbers. Such blowup equations can be regarded as the 5d limit of our elliptic blowup equations. Besides, the brane web construction for 5d G_2 theories with a fundamental matter was also obtained recently in (Hayashi et al., 2018). These results provide consistency checks for the elliptic genera we solved from elliptic blowup equations when taking $q \to 0$ limit.

For higher rank 6d SCFTs, the known results on elliptic genera are only for some special theories. For example, the 2d quiver constructions are known for the three higher-rank non-Higgsable clusters (Kim et al., 2018), ADE chain of (-2) curves with gauge symmetry (Gadde et al., 2018) and (E_6, E_6) conformal matter theory (Kim, Kim, and Park, 2016). The modular ansatz has been studied for higher rank E-string and M-string theories in (Gu et al., 2017). Beside, the elliptic genera of of A-type chain of (-2) curves can be computed by refined topological vertex (Haghighat et al., 2014). The recently proposed elliptic topological vertex can also compute the partition function of these theories (Foda and Zhu, 2018; Kimura and Zhu, 2019).

5.1.4 Semiclassical and one-loop free energy

The semiclassical prepotential for a rank one theory with $-n$ base curve and gauge symmetry G as well as flavor symmetry $F = \otimes_j F_j$ is easy to compute following the 5d results in (Intriligator, Morrison, and Seiberg, 1997). In (Gu et al., 2020b), we obtain

$$\begin{aligned} F^{\text{cls}}_{(0,0)} =& -\frac{1}{6}\sum_{\alpha\in\Delta^+}(\alpha\cdot m_G)^3 + \frac{1}{12}\sum_{\omega_{G,F}\in R^{m+}_{G,F}}(\omega_G\cdot m_G+\omega_F\cdot m_F)^3 \\ &+ \frac{t_{\text{ell}}-(\mathfrak{n}-2)\frac{\tau}{2}}{2\mathfrak{n}}(-\mathfrak{n}m_G\cdot m_G + k_F m_F\cdot m_F) - \frac{1}{2\mathfrak{n}}t_{\text{ell}}^2\tau + \mathcal{O}(\tau^3). \end{aligned} \tag{5.1.30}$$

where

$$R^{m+}_{G,F} = \{\omega_G\in R_G, \omega_F\in R_F\,;\, \omega_G\cdot m_G+\omega_F\cdot m_F\geq 0\}. \tag{5.1.31}$$

We ignore all terms only in m_G, m_F, τ, as they depend on the embedding of the associated Calabi-Yau in a compact geometry and thus are not inherent properties of the 6d SCFT. In addition, we can also fix the semiclassical pieces of genus one free energies from 5d results (Nekrasov, 2003; Shadchin, 2005; Kim et al., 2019), as well as the modularity of elliptic blowup equations in Section 5.2.3. We find

$$F^{\text{cls}}_{(0,1)} = -\frac{1}{12}\sum_{\alpha\in\Delta^+}\alpha\cdot m_G + \frac{1}{24}\sum_{\omega_{G,F}\in R^{m+}_{G,F}}(\omega_G\cdot m_G+\omega_F\cdot m_F) + \frac{\mathfrak{n}-2}{2\mathfrak{n}}t_{\text{ell}}, \tag{5.1.32}$$

$$F^{\text{cls}}_{(1,0)} = \frac{1}{12}\sum_{\alpha\in\Delta^+}\alpha\cdot m_G + \frac{1}{48}\sum_{\omega_{G,F}\in R^{m+}_{G,F}}(\omega_G\cdot m_G+\omega_F\cdot m_F) + \frac{\mathfrak{n}-2-h^\vee_G}{4\mathfrak{n}}t_{\text{ell}}. \tag{5.1.33}$$

Note here all the summations over roots and weights only sum over half sets of them, and the one loop contributions of BPS particles have to have the same half sets of them. The choices of the half weights do not have effects on our final result, since they are the same under analytic continuation.

The one-loop part of partition function $Z^{\text{1-loop}}$ contains the contribution of the Kaluza-Klein modes on the 6d S^1 of the 6d particle multiplets. In general, there are three types of contributions: tensor multiplet, vector multiplet in gauge group G and hypermultiplets in representation R. The corresponding contribution can be written down like the one-loop part of 5d gauge theories as (Hayashi et al., 2017)

$$\begin{aligned}
Z_{\text{tensor}} &= \mathrm{PE}\left[-\frac{q_L^{-1/2}+q_L^{1/2}}{(q_1^{1/2}-q_1^{-1/2})(q_2^{1/2}-q_2^{-1/2})}\left(\frac{q_\tau}{1-q_\tau}\right)\right],\\
Z_{\text{vector}}^{G} &= \mathrm{PE}\left[-\frac{q_R^{-1/2}+q_R^{1/2}}{(q_1^{1/2}-q_1^{-1/2})(q_2^{1/2}-q_2^{-1/2})}\sum_{\alpha\in\Delta_+(G)}\left(Q_\alpha+\frac{q_\tau}{Q_\alpha}\right)\left(\frac{1}{1-q_\tau}\right)\right],\\
Z_{\text{hyper}}^{R} &= \mathrm{PE}\left[+\frac{1}{(q_1^{1/2}-q_1^{-1/2})(q_2^{1/2}-q_2^{-1/2})}\sum_{\omega\in R_+}\left(Q_\omega+\frac{q_\tau}{Q_\omega}\right)\left(\frac{1}{1-q_\tau}\right)\right].
\end{aligned}\tag{5.1.34}$$

Here the plethystic exponential is defined as

$$\mathrm{PE}\left[f(x)\right]=\exp\left[\sum_{i=1}^{\infty}\frac{1}{n}f(x^n)\right]. \tag{5.1.35}$$

5.2 Elliptic blowup equations

We present the elliptic blowup equations for all rank one 6d SCFTs on the 6d Omega background and discuss various properties of these equations in this section. The additional information required in this process includes the semiclassical free energy and one-loop partition function, which we have discussed in Section 5.1.4.

Consider a rank one 6d SCFT with tensor branch coefficient $\mathfrak{n}$, gauge symmetry G, flavor symmetry F, and half-hypermultiplets transforming in the representations (R_G, R_F). The flavor symmetry induces a current algebra of level k_F on the worldsheet of BPS strings. Then the elliptic genera $\mathbb{E}_d(\tau, m_{G,F}, \epsilon_{1,2})$ satisfy the following elliptic blowup equations[3]

$$\begin{aligned}
&\sum_{\lambda_G\in\phi_{\lambda_0}(Q^\vee(G))}^{\frac{1}{2}||\lambda_G||^2+d'+d''=d+\delta}(-1)^{|\phi_{\lambda_0}^{-1}(\lambda_G)|}\\
&\times\theta_i^{[a]}(\mathfrak{n}\tau,-\mathfrak{n}\lambda_G\cdot m_G+k_F\lambda_F\cdot m_F+(y-\frac{n}{2}||\lambda_G||^2)(\epsilon_1+\epsilon_2)-\mathfrak{n}d'\epsilon_1-\mathfrak{n}d''\epsilon_2)\\
&\times A_V(\tau,m_G,\lambda_G)A_H(\tau,m_G,m_F,\lambda_G,\lambda_F)\\
&\times\mathbb{E}_{d'}(\tau,m_G+\epsilon_1\lambda_G,m_F+\epsilon_1\lambda_F,\epsilon_1,\epsilon_2-\epsilon_1)\mathbb{E}_{d''}(\tau,m_G+\epsilon_2\lambda_G,m_F+\epsilon_2\lambda_F,\epsilon_1-\epsilon_2,\epsilon_2)
\end{aligned}$$

[3] Roughly speaking, this elliptic blowup equation can be derived by putting the full partition function (5.1.24) with the known classical and one-loop partition function in section 5.1.4 into the generalization blowup equations (4.0.2). The contribution from tensor multiplet in the one-loop part can always be factored out because τ modulus never gets shifts due to modularity. The contributions A_V and A_H from vector and hypermultiplet in the one-loop part can be derived from the (B.0.16) and (B.0.20). The $\theta_i^{[a]}$ comes from the semiclassical free energy (5.1.30), (5.1.32) and (5.1.33). In practice, we actually conjectured this universal form after studying many examples. We refer to (Gu et al., 2019a; Gu et al., 2019b) for detailed derivations from blowup equations of local Calabi-Yau to elliptic blowup equations of pure gauge 6d $(1,0)$ SCFTs.

$$= \Lambda(\delta)\, \theta_i^{[a]}(\mathfrak{n}\tau, k_F\lambda_F \cdot m_F + ny(\epsilon_1+\epsilon_2))\mathbb{E}_d(\tau, m_G, m_F, \epsilon_1, \epsilon_2), \qquad d = 0,1,2,\ldots \tag{5.2.1}$$

where[4]

$$y = \frac{\mathfrak{n} - 2 + h^\vee_{\mathfrak{g}}}{4} + \frac{k_F}{2}(\lambda_F \cdot \lambda_F), \tag{5.2.2}$$

and

$$\Lambda(\delta) = \begin{cases} 1, & \delta = 0, \\ 0, & \delta > 0. \end{cases} \tag{5.2.3}$$

In the Jacobi theta function $\theta_i^{[a]}$,[5] the subscript i is 3 if $\mathfrak{n}$ is even and 4 if $\mathfrak{n}$ is odd, and the characteristic a of the theta function can be one of the following $\mathfrak{n}$ numbers

$$a = \frac{1}{2} - \frac{k}{\mathfrak{n}}, \quad k = 0,1,\ldots,\mathfrak{n}-1. \tag{5.2.4}$$

Besides, if there is an Abelian factor in the flavor symmetry, the argument $k_F\lambda_F \cdot m_F$ should be extended to

$$k_F\lambda_F \cdot m_F \to k_F\lambda_F \cdot m_F + k_{\mathfrak{u}(1)}\lambda_{\mathfrak{u}(1)}m_{\mathfrak{u}(1)}. \tag{5.2.5}$$

The summation index λ_G is a coweight vector of G; to be more precise, it takes value in the shifted coroot lattice defined by the embedding through a coweight vector λ_0

$$\begin{aligned} \phi_{\lambda_0} :& Q^\vee \hookrightarrow P^\vee \\ & \alpha^\vee \to \alpha^\vee + \lambda_0, \quad \lambda_0 \in P^\vee. \end{aligned} \tag{5.2.6}$$

The index λ_G in fact consists of components of the so-called r-field[6] in the blowup equations of topological string, and different λ_G correspond to r-fields which are equivalent to each other. On the other hand, there can be different embeddings. The number of different embeddings is the index of $Q^\vee$ as an Abelian subgroup of $P^\vee$, which is also the determinant of the Cartan matrix of G. There is a special embedding where the shift λ_0 is a coroot vector. δ is the smallest norm in the shifted coroot lattice; it is zero in the special embedding and positive otherwise. The inverse $\phi^{-1}_{\lambda_0}$ pulls back the coweight λ_G to the coroot lattice, and $|\bullet|$ in the sign factor sums up the coefficients in its decomposition in terms of simple coroots. We say the blowup equation is of the *unity* type if the embedding is the special embedding so that Λ is unity. Otherwise, the r.h.s. of the blowup equation vanishes identically and we say the blowup equations are of the *vanishing* type. Clearly if $P^\vee \cong Q^\vee$, which happens for G_2, F_4 and E_8, there can be no vanishing blowup equation. See Appendix A for our Lie algebraic convention.

The components A_V and A_H are contributions from vector and hypermultiplets respectively. They have the form

$$A_V(\tau, m_G, \lambda_G) = \prod_{\beta\in\Delta_+} \breve{\theta}_V(\beta \cdot m_G, \beta \cdot \lambda_G)\,, \tag{5.2.7}$$

[4]We set $h^\vee_{\mathfrak{g}} = 1$ if gauge symmetry is trivial.
[5]See definitions in Appendix D.
[6]Up to a factor of $1/2$.

$$A_H(\tau, m_{G,F}, \lambda_{G,F}) = \prod_{\omega_{G,F}\in R^+_{G,F}} \breve{\theta}_H(\omega_G\cdot m_G + \omega_F\cdot m_F, \omega_G\cdot\lambda_G + \omega_F\cdot\lambda_F)\,. \tag{5.2.8}$$

Here $R^+_{G,F}$ is half of the total weight space. For unity blowup equations

$$R^+_{G,F} = \{\omega_G\in R_G, \omega_F\in R_F\,|\,\omega_F\cdot\lambda_F = +1/2\}, \tag{5.2.9}$$

and for vanishing blowup equations

$$R^+_{G,F} = \{\omega_G\in R_G, \omega_F\in R_F\,|\,\omega_G\cdot\lambda_G + \omega_F\cdot\lambda_F > 0\}. \tag{5.2.10}$$

Furthermore, the $\breve{\theta}$ functions are defined as

$$\breve{\theta}_V(z,R) := \prod_{\substack{m,n\geq 0\\ m+n\leq |R|-1}} \frac{\eta}{\theta_1(z+sm\epsilon_1+sn\epsilon_2)} \prod_{\substack{m,n\geq 0\\ m+n\leq |R|-2}} \frac{\eta}{\theta_1(z+s(m+1)\epsilon_1+s(n+1)\epsilon_2)}, \tag{5.2.11}$$

with $R\in\mathbb{Z}$, and s the sign of R,

$$\breve{\theta}_H(z,R) := \prod_{\substack{m,n\geq 0\\ m+n\leq |R|-3/2}} \frac{\theta_1(z+s(m+1/2)\epsilon_1+s(n+1/2)\epsilon_2)}{\eta}\,, \quad R\in\frac{1}{2}+\mathbb{Z}, \tag{5.2.12}$$

There is still one free parameter λ_F, which is a coweight vector of the flavor symmetry. The value of λ_F can be determined by the following constraints:

- Checker board pattern: The second arguments on the r.h.s. of (5.2.7), (5.2.8) are one half of the r-field component associated to the refined BPS states of vector and hypermultipets, and thus they must satisfy the conditions

$$\beta\cdot\lambda_G\in\mathbb{Z}, \quad \beta\in\Delta, \tag{5.2.13}$$

$$\omega_G\cdot\lambda_G+\omega_F\cdot\lambda_F\in\frac{1}{2}+\mathbb{Z}, \quad \omega_G\in R_G, \omega_F\in R_F. \tag{5.2.14}$$

  The first condition only confirms that λ_G is a coweight vector of G. The second condition constrains that λ_F takes value in a subset of the coweight lattice of F depending on the domain of ϕ_{λ_0}. In the case of unity equations, λ_G is a coroot vector of G and $\omega_G\cdot\lambda_G$ is an integer, (5.2.14) reduces to

$$\omega_F\cdot\lambda_F\in\frac{1}{2}+\mathbb{Z}, \quad \omega_F\in R_F. \tag{5.2.15}$$

  The above conditions are also called the *B field condition*.

- Modularity: We observe that the elliptic blowup equations (5.2.1) are identities of Jacobi forms. An important consistency condition for (5.2.1) is that every term on the l.h.s. should have the same modular weight and modular index, and when the r.h.s. does not vanish, they coincide with the modular weight and modular index of the r.h.s. as well. The condition on modular weight is trivially satisfied as every term has weight one half. The condition on modular index, on the other hand, is highly nontrivial and very constraining. As we will see in Section 5.2.3, this condition puts strong constraints on λ_F, especially in the case of unity blowup equations.

- *Leading degree identities*: The degree d is the degree of the shifted base curve $t_{\rm ell}$. In the leading degree with $d = d' = d'' = 0$, the elliptic genera do not contribute and the blowup equations become identities of Jacobi theta functions. For unity blowup equations the leading degree identities are trivial, while for vanishing blowup equations the leading degree identities are very non-trivial and they can be used to constrain the parameter λ_F.

We list below the values of the parameter λ_F satisfying all the four constraints for each rank one model and the corresponding y parameter. The coweight vectors λ_F are presented by their Dynkin labels. Note that such a coweight vector can be mapped to a weight vector by the isomorphism φ defined in (A.0.11). We are sometimes sloppy in the main text and refer to λ_F as weights, by which we actually mean the images of φ. Besides in the main text we often directly write the factor $(-1)^{|\varphi^{-1}_{\lambda_0}(\lambda_G)|}$ in (5.2.1) as $(-1)^{|\lambda_G|}$. We will later test the corresponding elliptic blowup equations in Sections 5.3 and 5.5 by checking them explicitly with known results of elliptic genera, and by solving unknown elliptic genera as well as refined BPS invariants from them.

There is another convenient form of elliptic blowup equations, in which we replace $(y - \frac{n}{2}||\lambda_G||^2)$ by $(\bar{y} - nd)$ in (5.2.1). The advantage is that d is always integer for both unity and vanishing cases, and $\bar{y}$ are typically simpler numbers than y. On the other hand, the merit of the current form (5.2.1) is that the modularity proof of both unity and vanishing blowup equations can be combined together. We will also use the notion of d and $\bar{y}$ in the example sections, where $\bar{y}$ and y are related by $y = \bar{y} + n\delta$ in the vanishing cases and naturally $y = \bar{y}$ in the unity cases.

5.2.1 Unity blowup equations

We tabulate in Tables 5.5, 5.6 the coweight vectors λ_F and the associated parameter y for unity blowup equations which satisfy the four constraints discussed above. We note that if a coweight vector λ_F is valid, all the vectors in the same Weyl orbit should be valid as well, and we only list in Tables 5.5, 5.6 the dominant coweight vectors. We commment that for ease of computation, we have used the isomorphism of algebras

$$\mathfrak{so}(2) \cong \mathfrak{sp}(1), \quad \mathfrak{so}(4) \cong \mathfrak{sp}(1) \times \mathfrak{sp}(1). \tag{5.2.16}$$

Whenever possible, we prefer the notation $\mathfrak{sp}(1)$ instead of $\mathfrak{su}(2)$ as it is more similar to other C-algebras instead of A-algebras.

Note that the following theories have unpaired half-hypers and they do not have unity blowup equations. Technically this is because their flavor weight spaces have zero weight, with which the checker board pattern constraint (5.2.15) cannot be satisfied.

- $\mathfrak{n} = 1$: $G = \mathfrak{su}(6)_*, \mathfrak{so}(11), \mathfrak{so}(12)_{a,b}, E_7$;
- $\mathfrak{n} = 2$: $G = \mathfrak{so}(12)_b, \mathfrak{so}(13)$;
- $\mathfrak{n} = 3$: $G = \mathfrak{so}(11), \mathfrak{so}(12), E_7$;
- $\mathfrak{n} = 5, 7$: $G = E_7$.

$\mathfrak{n}$	G	F	#	y	λ_F
12	E_8	$-$	1	10	$\varnothing$
8	E_7	$-$	1	6	$\varnothing$
7	E_7	$-$	0	$-$	$-$
6	E_6	$-$	1	4	$\varnothing$
6	E_7	$\mathfrak{so}(2)_{12} = \mathfrak{sp}(1)_6$	2	7	(1)
5	F_4	$-$	1	3	$\varnothing$
5	E_6	$\mathfrak{u}(1)_6$	2	$9/2$	$\pm 1/2$
5	E_7	$\mathfrak{so}(3)_{12}$	0	$-$	$-$
4	$\mathfrak{so}(8)$	$-$	1	2	$\varnothing$
4	$\mathfrak{so}(N \geq 9)$	$\mathfrak{sp}(N-8)_1$	2^{N-8}	$(N-4)/2$	$(0\ldots01)$
4	F_4	$\mathfrak{sp}(1)_3$	2	$7/2$	(1)
4	E_6	$\mathfrak{su}(2)_6 \times \mathfrak{u}(1)_{12}$	4	5	$(1)_0$ or $(0)_{\pm\frac{1}{2}}$
4	E_7	$\mathfrak{so}(4)_{12} = \mathfrak{sp}(1)_6 \times \mathfrak{sp}(1)_6$	4	8	(1),(1)
3	$\mathfrak{su}(3)$	$-$	1	1	$\varnothing$
3	$\mathfrak{so}(7)$	$\mathfrak{sp}(2)_1$	4	2	(01)
3	$\mathfrak{so}(8)$	$\mathfrak{sp}(1)_1^a \times \mathfrak{sp}(1)_1^b \times \mathfrak{sp}(1)_1^c$	8	$5/2$	(1),(1),(1)
3	$\mathfrak{so}(9)$	$\mathfrak{sp}(2)_1^a \times \mathfrak{sp}(1)_2^b$	8	3	(01),(1)
3	$\mathfrak{so}(10)$	$\mathfrak{sp}(3)_1^a \times (\mathfrak{su}(1)_4 \times \mathfrak{u}(1)_4)^b$	16	$7/2$	(001),$\pm 1/2$
3	$\mathfrak{so}(11)$	$\mathfrak{sp}(4)_1^a \times \text{Ising}^b$	0	$-$	$-$
3	$\mathfrak{so}(12)$	$\mathfrak{sp}(5)_1$	0	$-$	$-$
3	G_2	$\mathfrak{sp}(1)_1$	2	$3/2$	(1)
3	F_4	$\mathfrak{sp}(2)_3$	4	4	(01)
3	E_6	$\mathfrak{su}(3)_6 \times \mathfrak{u}(1)_{18}$	8	$11/2$	$\pm(01)_{\frac{1}{6}}$ or $(00)_{\pm\frac{1}{2}}$
3	E_7	$\mathfrak{so}(5)_{12}$	0	$-$	$-$

Table 5.5: The parameters y, λ_F of unity blowup equations for rank one models with $\mathfrak{n} \geq 3$. # is the number of unity equations with fixed characteristic a.

5.2.2 Vanishing blowup equations

We tabulate in Tables 5.7, 5.8, 5.9 the values of λ_F and the associated parameter y for vanishing equations that satisfy the constraints discussed in the beginning of this section. In particular, we have tested the leading degree identities for all the vanishing blowup equations up to order 20 in $q_\tau = \exp(2\pi i\tau)$. We find unlike the unity λ_F fields which form Weyl orbits, the admissible vanishing λ_F fields typically form representations rather than just Weyl orbits. To be precise, the admissible vanishing λ_F fields are all coweight vectors inside the representation whose highest coweight is given in Dynkin label in Tables 5.7, 5.8, 5.9. Note one representation in general contains many Weyl orbits. Besides, different Weyl orbits inside one representation in general have different associated y which are easily computable with equation (5.2.2). Thus for the situation where several values of y are involved, we leave $\cdots$ in Tables 5.7, 5.8, 5.9.

In the following we discuss some special cases in Tables 5.7, 5.8, 5.9 in more detail.

- $G = \mathfrak{so}(7)$: For $\mathfrak{n} = 3$, the representation [10] of $\lambda_{\mathfrak{sp}(2)}$ has two Weyl orbits generated by coweights (00) and (10), whose associated y are $3/2$ and $5/2$ respectively. For $\mathfrak{n} = 2$, the representation [1000] of $\lambda_{\mathfrak{sp}(4)}$ has two Weyl orbits generated by coweights (0000) and (1000), whose associated y are $3/2$ and $5/2$ respectively. For $\mathfrak{n} = 1$, the representation [100000] of $\lambda_{\mathfrak{sp}(6)}$ has two Weyl

$\mathfrak{n}$	G	F	#	y	λ_F
2	$\mathfrak{su}(1)$	$\mathfrak{su}(2)_1$	2	1/2	(1)
2	$\mathfrak{su}(2)$	$\mathfrak{so}(8)_1 \to \mathfrak{so}(7)_1 \times$ Ising	6	1	(100)
2	$\mathfrak{su}(N \geq 3)$	$\mathfrak{su}(2N)_1$	$\binom{2N}{N}$	N/2	$(0\ldots010\ldots0)$
2	$\mathfrak{so}(7)$	$\mathfrak{sp}(1)_1^a \times \mathfrak{sp}(4)_1^b$	32	5/2	(1),(0001)
2	$\mathfrak{so}(8)$	$\mathfrak{sp}(2)_1^a \times \mathfrak{sp}(2)_1^b \times \mathfrak{sp}(2)_1^c$	64	3	(01),(01),(01)
2	$\mathfrak{so}(9)$	$\mathfrak{sp}(3)_1^a \times \mathfrak{sp}(2)_2^b$	32	7/2	(001),(01)
2	$\mathfrak{so}(10)$	$\mathfrak{sp}(4)_1^a \times (\mathfrak{su}(2)_4 \times \mathfrak{u}(1)_8)^b$	64	4	(0001) and $(1)_0$ or $(0)_{\pm\frac{1}{2}}$
2	$\mathfrak{so}(11)$	$\mathfrak{sp}(5)_1^a \times (? \to \mathfrak{so}(2)_8)^b$	64	9/2	(00001),(1)
2	$\mathfrak{so}(12)_a$	$\mathfrak{sp}(6)_1^a \times \mathfrak{so}(2)_8^b$	128	5	(000001),(1)
2	$\mathfrak{so}(12)_b$	$\mathfrak{sp}(6)_1^a \times \mathrm{Ising}^b \times \mathrm{Ising}^c$	0	–	–
2	$\mathfrak{so}(13)$	$\mathfrak{sp}(7)_1$	0	–	–
2	G_2	$\mathfrak{sp}(4)_1$	16	2	(0001)
2	F_4	$\mathfrak{sp}(3)_3$	8	9/2	(001)
2	E_6	$\mathfrak{su}(4)_6 \times \mathfrak{u}(1)_{24}$	16	6	$(010)_0$ or $\pm(001)_{\frac{1}{4}}$ or $(000)_{\pm\frac{1}{2}}$
2	E_7	$\mathfrak{so}(6)_{12}$	8	9	(001) or (010)
1	$\mathfrak{sp}(0)$	$(E_8)_1$	240	1	$(10\ldots0)$
1	$\mathfrak{sp}(N \geq 1)$	$\mathfrak{so}(4N+16)_1$	2^{2N+7}	$(N+2)/2$	$(0\ldots01)$
1	$\mathfrak{su}(3)$	$\mathfrak{su}(12)_1$	924	2	$(0\ldots010\ldots0)$
1	$\mathfrak{su}(4)$	$\mathfrak{su}(12)_1^a \times \mathfrak{su}(2)_1^b$	1848	5/2	$(0\ldots010\ldots0)$, (1)
1	$\mathfrak{su}(N \geq 5)$	$\mathfrak{su}(N+8)_1 \times \mathfrak{u}(1)_{2N(N-1)(N+8)}$	$2\binom{N+8}{6}$	$(N+1)/2$	$(0000010\ldots)_{-\frac{1}{2(N+8)}}$ or minus
1	$\mathfrak{su}(6)_*$	$\mathfrak{su}(15)_1$	0	–	–
1	$\mathfrak{so}(7)$	$\mathfrak{sp}(2)_1^a \times \mathfrak{sp}(6)_1^b$	256	3	(01),(000001)
1	$\mathfrak{so}(8)$	$\mathfrak{sp}(3)_1^a \times \mathfrak{sp}(3)_1^b \times \mathfrak{sp}(3)_1^c$	512	7/2	(001),(001),(001)
1	$\mathfrak{so}(9)$	$\mathfrak{sp}(4)_1^a \times \mathfrak{sp}(3)_2^b$	128	4	(0001),(001)
1	$\mathfrak{so}(10)$	$\mathfrak{sp}(5)_1^a \times (\mathfrak{su}(3)_4 \times \mathfrak{u}(1)_{12})^b$	256	9/2	(00001), and $\pm(01)_{\frac{1}{6}}$ or $(00)_{\pm\frac{1}{2}}$
1	$\mathfrak{so}(11)$	$\mathfrak{sp}(6)_1^a \times ?^b$	0	–	–
1	$\mathfrak{so}(12)_a$	$\mathfrak{sp}(7)_1^a \times \mathfrak{so}(3)_8^b$	0	–	–
1	$\mathfrak{so}(12)_b$	$\mathfrak{sp}(7)_1^a \times ?^b \times ?^c$	0	–	–
1	G_2	$\mathfrak{sp}(7)_1$	128	5/2	$(0\ldots01)$
1	F_4	$\mathfrak{sp}(4)_3$	16	5	$(0\ldots01)$
1	E_6	$\mathfrak{su}(5)_6 \times \mathfrak{u}(1)_{30}$	32	13/2	$\pm(0001)_{\frac{3}{10}}$ or $\pm(0010)_{\frac{1}{10}}$ or $(0000)_{\pm\frac{1}{2}}$
1	E_7	$\mathfrak{so}(7)_{12}$	0	–	–

Table 5.6: The parameters y, λ_F of unity blowup equations for rank one models with $\mathfrak{n} = 1, 2$. # is the number of unity equations with fixed characteristic a.

orbits generated by coweights (000000) and (100000), whose associated y are 3/2 and 5/2 respectively.

- $G = \mathfrak{so}(8)$: For $\mathfrak{n} = 3$, the representation [2] of $\lambda_{\mathfrak{sp}(1)}$ has two Weyl orbits (0) and (2), whose associated y are 2 and 3 respectively. For $\mathfrak{n} = 2$, the representation [10] of $\lambda_{\mathfrak{sp}(2)}$ has two Weyl orbits generated by coweights (00) and (10), whose associated y are 2 and 3 respectively. For $\mathfrak{n} = 1$, the representation [100] of $\lambda_{\mathfrak{sp}(3)}$ has two Weyl orbits generated by coweights (000) and (100), whose associated y are 2 and 3 respectively.

- $G = \mathfrak{so}(10)$: For $\mathfrak{n} = 3$, the possibilities are

$$\begin{aligned} &\ell = 0, \quad j = 2, \\ &\ell = -1, \quad j = 0, \\ &\ell = 1, \quad j = 1. \end{aligned} \tag{5.2.17}$$

n	G	F	λ_0	y	λ_F
12	E_8	$-$	$-$	$-$	$-$
8	E_7	$-$	(0000010)	6	$\emptyset$
7	E_7	$-$	(0000010)	23/4	$\emptyset$
6	E_6	$-$	(100000)	4	$\emptyset$
			(000010)	4	$\emptyset$
6	E_7	$\mathfrak{so}(2)_{12} = \mathfrak{sp}(1)_6$	(0000010)	11/2	(0)
5	F_4	$-$	$-$	$-$	$-$
5	E_6	$\mathfrak{u}(1)_6$	(100000)	23/6 or 35/6	$-1/6$ or $5/6$
			(000010)	23/6 or 35/6	$1/6$ or $-5/6$
5	E_7	$\mathfrak{so}(3)_{12}$	(0000010)	21/4	(0)
4	$\mathfrak{so}(8)$	$-$	all three	2	$\emptyset$
4	$\mathfrak{so}(2N), N \geq 5$	$\mathfrak{sp}(2N-8)_1$	$(10\ldots0)$	$N-2$	$(0\ldots01)$
			$(\ldots010)$	$\cdots$	$[N-2, 0\ldots00]$
			$(\ldots001)$	$\cdots$	$[N-2, 0\ldots00]$
4	$\mathfrak{so}(2N-1), N \geq 5$	$\mathfrak{sp}(2N-9)_1$	$(10\ldots0)$	$(2N-5)/2$	$(0\ldots01)$
4	F_4	$\mathfrak{sp}(1)_3$	$-$	$-$	$-$
4	E_6	$\mathfrak{su}(2)_6 \times \mathfrak{u}(1)_{12}$	(100000)	11/3	$(0), -1/6$
			(000010)	11/3	$(0), 1/6$
4	E_7	$\mathfrak{so}(4)_{12} = \mathfrak{sp}(1)_6 \times \mathfrak{sp}(1)_6$	(0000010)	5	(0),(0)
3	$\mathfrak{su}(3)$	$-$	(10) or (01)	1	$\emptyset$
3	$\mathfrak{so}(7)$	$\mathfrak{sp}(2)_1$	(100)	$\cdots$	[10]
3	$\mathfrak{so}(8)$	$\mathfrak{sp}(1)_1^a \times \mathfrak{sp}(1)_1^b \times \mathfrak{sp}(1)_1^c$	(1000)	$\cdots$	(1),(0),[2] or (1),[2],(0)
			(0010)	$\cdots$	[2],(1),(0) or (0),(1),[2]
			(0001)	$\cdots$	(0),[2],(1) or [2],(0),(1)
3	$\mathfrak{so}(9)$	$\mathfrak{sp}(2)_1^a \times \mathfrak{sp}(1)_2^b$	(1000)	5/2	(01),(0)
3	$\mathfrak{so}(10)$	$\mathfrak{sp}(3)_1^a \times (\mathfrak{su}(1)_4 \times \mathfrak{u}(1)_4)^b$	(10000)	3	(001),(0),0
			(00010)	$\cdots$	$[j00], (0), -1/4+\ell$: see text
			(00001)	$\cdots$	$[j00], (0), 1/4-\ell$: see text
3	$\mathfrak{so}(11)$	$\mathfrak{sp}(4)_1^a \times \text{Ising}^b$	(10000)	7/2	(0001)
3	$\mathfrak{so}(12)$	$\mathfrak{sp}(5)_1$	(100000)	4	(00001)
			(000001)	$\cdots$	[30000]
			(000010)	$-$	$-$
3	G_2	$\mathfrak{sp}(1)_1$	$-$	$-$	$-$
3	F_4	$\mathfrak{sp}(2)_3$	$-$	$-$	$-$
3	E_6	$\mathfrak{su}(3)_6 \times \mathfrak{u}(1)_{18}$	(100000)	7/2	$(00), -1/6$
			(000010)	7/2	$(00), 1/6$
3	E_7	$\mathfrak{so}(5)_{12}$	(0000010)	19/4	(00)

Table 5.7: The parameters y, λ_F of vanishing blowup equations for rank one models with $\mathfrak{n} \geq 3$. In the column of λ_F, the representations are labeled by their highest coweights. When a representation is composed by many Weyl orbits, we use $[*]$ instead of $(*)$ to stress the difference.

For $\mathfrak{n} = 2$, the possibilities are

$$\ell = 0, \quad j = 2,$$
$$\ell = 1, \quad j = 0. \tag{5.2.18}$$

For $\mathfrak{n} = 1$, the possibilities are

$$\ell = 0, \quad j = 2. \tag{5.2.19}$$

- $\mathfrak{n} = 2, G = \mathfrak{su}(N), N \geq 3$: when $\lambda_0 = \omega_j^{\vee \mathfrak{su}(N)}, j = 1, 2, \ldots, N-1$,

$$\lambda_F^{\mathfrak{su}(2N)} \in [j-1, 0, \ldots, 0, (N-1-j)]. \tag{5.2.20}$$

For fixed N and j, this is a very large representation which contains many Weyl orbits, each of which has its own associated y. We do not list all of them since

$\mathfrak{n}$	G	F	λ_0	y	λ_F
2	$\mathfrak{su}(1)$	$\mathfrak{su}(2)_1$	–	–	–
2	$\mathfrak{su}(2)$	$\mathfrak{so}(8)_1 \to \mathfrak{so}(7)_1 \times$ Ising	(1)	1/2	(000)
2	$\mathfrak{su}(N \geq 3)$	$\mathfrak{su}(2N)_1$	see text		
2	$\mathfrak{so}(7)$	$\mathfrak{sp}(1)^a_1 \times \mathfrak{sp}(4)^b_1$	(100)	$\cdots$	(1),[1000]
2	$\mathfrak{so}(8)$	$\mathfrak{sp}(2)^a_1 \times \mathfrak{sp}(2)^b_1 \times \mathfrak{sp}(2)^c_1$	(1000)	$\cdots$	(01),(00),[10] or (01),[10],(00)
			(0010)	$\cdots$	[10],(01),(00) or (00),(01),[10]
			(0001)	$\cdots$	(00),[10],(01) or [10],(00),(01)
2	$\mathfrak{so}(9)$	$\mathfrak{sp}(3)^a_1 \times \mathfrak{sp}(2)^b_2$	(1000)	5/2	(001),(00)
2	$\mathfrak{so}(10)$	$\mathfrak{sp}(4)^a_1 \times (\mathfrak{su}(2)_4 \times \mathfrak{u}(1)_8)^b$	(10000)	3	(0001),(0),0
			(00010)	$\cdots$	$[j000], (0), -1/4+\ell$: see text
			(00001)	$\cdots$	$[j000], (0), 1/4-\ell$: see text
2	$\mathfrak{so}(11)$	$\mathfrak{sp}(5)^a_1 \times (? \to \mathfrak{so}(2)_8)^b$	(10000)	7/2	(00001),(0)
2	$\mathfrak{so}(12)_a$	$\mathfrak{sp}(6)^a_1 \times \mathfrak{so}(2)^b_8$	(100000)	4	(000001),(0)
			(000001)	$\cdots$	$[300000], (0)$
			(000010)	$\cdots$	$[200000], (1)$
2	$\mathfrak{so}(12)_b$	$\mathfrak{sp}(6)^a_1 \times \text{Ising}^b \times \text{Ising}^c$	(100000)	4	(000001)
			(000001)	–	–
			(000010)	–	–
2	$\mathfrak{so}(13)$	$\mathfrak{sp}(7)_1$	(100000)	9/2	(0000001)
2	G_2	$\mathfrak{sp}(4)_1$	–	–	–
2	F_4	$\mathfrak{sp}(3)_3$	–	–	–
2	E_6	$\mathfrak{su}(4)_6 \times \mathfrak{u}(1)_{24}$	(100000)	10/3	$(000), -1/6$
			(000010)	10/3	$(000), 1/6$
2	E_7	$\mathfrak{so}(6)_{12}$	(0000010)	9/2	(000)

Table 5.8: The parameters y, λ_F of vanishing blowup equations for rank one models with $\mathfrak{n} = 2$. In the column of λ_F, the representations are labeled by their highest coweights. When a representation is composed by many Weyl orbits, we use $[*]$ instead of $(*)$ to stress the difference. See the main text for more discussion.

they are easily computable from equation (5.2.2). Instead, we just point out one particular simple Weyl orbit in inside such representation. For example, if N is odd,

$$\lambda_0 = \omega_j^{\vee \mathfrak{su}(N)}, \quad \lambda_F \in \mathcal{O}_{N+2j}{}^{\mathfrak{su}(2N)}, \quad y = \frac{N^2 - 2j^2}{2N}, \quad j = 1, \ldots, (N-1)/2, \tag{5.2.21}$$

and

$$\lambda_0 = \omega_{N-j}^{\vee}{}^{\mathfrak{su}(N)}, \quad \lambda_F \in \mathcal{O}_{N-2j}{}^{\mathfrak{su}(2N)}, \quad y = \frac{N^2 - 2j^2}{2N}, \quad j = 1, \ldots, (N-1)/2, \tag{5.2.22}$$

where $\mathcal{O}_i$ is the Weyl orbit generated by the i-th fundamental coweight. If N is even, j runs from 1 up to $N/2$ in the equations (5.2.21) and (5.2.22). We will explicitly show the leading degree vanishing identities for these Weyl orbits in Section (5.5.5).

- $\mathfrak{n} = 1, G = \mathfrak{su}(N), N \geq 5$: For $k \leq \lfloor N/2 \rfloor$, we have

$$\lambda_0 = \omega_k^{\vee \mathfrak{su}(N)}, \quad \lambda_F^{U(1)} = \frac{4k - N}{2N(N+8)}, \quad \lambda_F^{\mathfrak{su}(N+8)} \in [k-1, 0, \ldots, 0, N+1-2k], \tag{5.2.23}$$

n	G	F	λ_0	y	λ_F
1	$\mathfrak{sp}(0)$	$(E_8)_1$	$\varnothing$	0	$(0\dots0)$
1	$\mathfrak{sp}(N\geq1)$	$\mathfrak{so}(4N+16)_1$	$(0\dots01)$	$\cdots$	$[N,0\dots0]$, $[N-3,0\dots0]$
1	$\mathfrak{su}(3)$	$\mathfrak{su}(12)_1$	(10) or (01)	$\cdots$	$[\dots02]$ or $[20\dots]$
1	$\mathfrak{su}(4)$	$\mathfrak{su}(12)^a_1\times\mathfrak{su}(2)^b_1$	(100) or (001)	$\cdots$	$[\dots03]$ or $[30\dots0]$,(0)
			(010)	$\cdots$	$[10\dots01]$,(1)
1	$\mathfrak{su}(N\geq5)$	$\mathfrak{su}(N{+}8)_1\times\mathfrak{u}(1)_{2N(N-1)(N+8)}$	see text		
1	$\mathfrak{su}(6)_*$	$\mathfrak{su}(15)_1$	(10000) or (00001)	$\cdots$	$[\dots05]$ or $[50\dots]$
			(00100)	$\cdots$	$[20\dots02]$
1	$\mathfrak{so}(7)$	$\mathfrak{sp}(2)^a_1\times\mathfrak{sp}(6)^b_1$	(100)	$\cdots$	(01),[100000]
1	$\mathfrak{so}(8)$	$\mathfrak{sp}(3)^a_1\times\mathfrak{sp}(3)^b_1\times\mathfrak{sp}(3)^c_1$	(1000)	$\cdots$	(001),(000),[100] or (001),[100],(000)
			(0010)	$\cdots$	[100],(001),(000) or (000),(001),[100]
			(0001)	$\cdots$	(000),[100],(001) or [100],(000),(001)
1	$\mathfrak{so}(9)$	$\mathfrak{sp}(4)^a_1\times\mathfrak{sp}(3)^b_2$	(1000)	$5/2$	(0001),(000)
1	$\mathfrak{so}(10)$	$\mathfrak{sp}(5)^a_1\times(\mathfrak{su}(3)_4\times\mathfrak{u}(1)_{12})^b$	(10000)	3	(00001),(0),0
			(00010)	$\cdots$	$[20000],(0),-1/4$
			(00001)	$\cdots$	$[20000],(0),1/4$
1	$\mathfrak{so}(11)$	$\mathfrak{sp}(6)^a_1\times(?\to\mathfrak{sp}(1)_6)^b$	(10000)	$7/2$	(000001),(0)
1	$\mathfrak{so}(12)_a$	$\mathfrak{sp}(7)^a_1\times\mathfrak{so}(3)^b_8$	(100000)	4	(0000001),(0)
			(000001)	$\cdots$	$[3000000],(0)$
			(000010)	–	–
1	$\mathfrak{so}(12)_b$	$\mathfrak{sp}(7)^a_1\times(?\to\mathfrak{sp}(1)_4)^b\times?^c$	(100000)	4	(0000001),(0)
			(000001)	–	–
			(000010)	$\cdots$	$[2000000],(1)$
1	G_2	$\mathfrak{sp}(7)_1$	–	–	–
1	F_4	$\mathfrak{sp}(4)_3$	–	–	–
1	E_6	$\mathfrak{su}(5)_6\times\mathfrak{u}(1)_{30}$	(100000)	$19/6$	$(0000),-1/6$
			(000010)	$19/6$	$(0000),1/6$
1	E_7	$\mathfrak{so}(7)_{12}$	(0000010)	$17/4$	(000)

Table 5.9: The parameters y,λ_F of vanishing blowup equations for rank one models with $\mathfrak{n}=1$. In the column of λ_F, the representations are labeled by their highest coweights. When a representation is composed by many Weyl orbits, we use $[*]$ instead of $(*)$ to stress the difference. We make assumption for flavor symmetries of the $G=\mathfrak{so}(11)$ and $\mathfrak{so}(12)_b$ models which allow for vanishing blowup equations; see the main text for more discussion.

The associated parameters y are computed by (5.2.2) as,

$$y=\frac{N-1}{4}+\frac{1}{2}(\lambda_F^{\mathfrak{su}(N+8)},\lambda_F^{\mathfrak{su}(N+8)})+N(N-1)(N+8)(\lambda_F^{U(1)})^2. \tag{5.2.24}$$

The cases of $k>\lfloor N/2\rfloor$ can be obtained by complex conjugation.

- $\mathfrak{n}=1, G=\mathfrak{so}(11),\mathfrak{so}(12)_b$: the flavor symmetries consistent at the level of worldsheet theory are not known for these two models (Del Zotto and Lockhart, 2018), especially the component governing the three half-hypers in spinor representation of $\mathfrak{so}(11)$ in the first model, and the component governing the two half-hypers in spinor and and one half-hyper in conjugate spinor representations of $\mathfrak{so}(12)$ in the second model. We find that if we assume the three half-hypers in the first model transform as **3** of flavor symmetry $\mathfrak{sp}(1)$, and the two half-hypers in the second model transform as **2** of flavor symmetry $\mathfrak{sp}(1)$, we can find λ_F of vanishing equations which satisfy all the constraints discussed in the beginning of this section. In particular, we have checked the leading base degree identities also up to order 20 in q.

Let us give a simple example of the leading base degree identities. Consider $n=1, G=\mathfrak{su}(3), F=\mathfrak{su}(12)$ theory with matter representation $(\mathbf{3},\overline{\mathbf{12}})$. Let us look at the situation with $\lambda_G=(10)_{\mathfrak{su}(3)}=\mathbf{3}$. The admissible λ_F form representation

$[\ldots 002]_{\mathfrak{su}(12)}$ which has two Weyl orbits $(\ldots 010)$ and $(\ldots 002)$. The first Weyl orbit itself is a representation $\overline{\mathbf{66}}$. In this case, the leading base degree of the vanishing blowup equations gives the following identity: $\forall \lambda \in \overline{\mathbf{66}}$,

$$\sum_{w\in\mathbf{3}}(-1)^{|w|}\theta_1(-m_w+m_\lambda+2\epsilon_+)\prod_{\beta\in\Delta(\mathfrak{su}(3))}^{w\cdot\beta=1}\frac{1}{\theta_1(m_\beta)}\prod_{\mu\in\overline{\mathbf{12}}}^{\mu\cdot\lambda=5/6}\theta_1(m_w+m_\mu+\epsilon_+)=0. \tag{5.2.25}$$

We have checked this identity to $\mathcal{O}(q^{20})$. To write it more explicitly, we have

$$\begin{aligned}&\frac{\theta_1(-a_1+x_i+x_j)\theta_1(a_1+x_i)\theta_1(a_1+x_j)}{\theta_1(a_1-a_2)\theta_1(a_1-a_3)}+\frac{\theta_1(-a_2+x_i+x_j)\theta_1(a_2+x_i)\theta_1(a_2+x_j)}{\theta_1(a_2-a_1)\theta_1(a_2-a_3)}\\&+\frac{\theta_1(-a_3+x_i+x_j)\theta_1(a_3+x_i)\theta_1(a_3+x_j)}{\theta_1(a_3-a_1)\theta_1(a_3-a_2)}=0,\qquad \text{for } a_1+a_2+a_3=0.\end{aligned} \tag{5.2.26}$$

Here $a_k, k=1,2,3$ are the $\mathfrak{su}(3)$ fugacities and $x_i=m_i+\epsilon_+, i=1,2,\ldots,12$ where m_i are the symmetric $\mathfrak{su}(12)$ fugacities. The modularity here means that each among the three terms in the above equation has the same index $-(a_1^2+a_2^2+a_3^2)/2$. The leading base degree identities from the other set of vanishing blowup equations are just similar.

Some more simple examples come from pure gauge theories with $\mathfrak{n}=3,4,6,8$ and $G=\mathfrak{su}(3),\mathfrak{so}(8),E_6,E_7$ repsectively. Denote the fundamental representation of G as $\square_G$, then the leading base degree identities can be universally written as

$$\sum_{\omega\in\square_G}(-1)^{|\omega|}\,\theta_i^{[a]}(\mathfrak{n}\tau,\mathfrak{n}m_\omega)\prod_{\beta\in\Delta(G)}^{\omega\cdot\beta=1}\frac{1}{\theta_1(m_\beta)}=0\,. \tag{5.2.27}$$

We have tons of vanishing theta identities like these involving the root and weight lattices of Lie algebras from the leading base degree of vanishing blowup equations. We present some of them in Chapter 5.5 and make a summary in Appendix D.

5.2.3 Modularity

In this section we discuss the modularity constraint. To make sure the elliptic blowup equation transform as a whole Jacobi form, every term on the l.h.s. of (5.2.1) should have the same modular index independent from λ_G, d', d'' themselves but only depending on the combination $\frac{1}{2}||\lambda_G||^2+d'+d''$. The modular index polynomial for the generalized theta function $\theta_i^{[a]}(\mathfrak{n}\tau,z)$ is

$$\mathrm{Ind}\,\theta_i(n\tau,z)=\frac{1}{2\mathfrak{n}}z^2. \tag{5.2.28}$$

The modular index polynomial for d-string elliptic genus for a rank one model can be deduced from (5.1.29) (Del Zotto and Lockhart, 2017; Del Zotto et al., 2018)

$$\begin{aligned}\mathrm{Ind}\,\mathbb{E}_d(\epsilon_1,\epsilon_2,m_G,m_F)=&-\left(\frac{\epsilon_1+\epsilon_2}{2}\right)^2(2-\mathfrak{n}+h^\vee_{\mathfrak{g}})d+\frac{\epsilon_1\epsilon_2}{2}(\mathfrak{n}d^2+(2-\mathfrak{n})d)\\&+\frac{d}{2}(-\mathfrak{n}\,m_G\cdot m_G+k_F\,m_F\cdot m_F)\,.\end{aligned}$$

Here if the flavor symmetry has a $\mathfrak{u}(1)$ factor, we should replace

$$k_F m_F \cdot m_F \to k_F m_F \cdot m_F + k_{\mathfrak{u}(1)} m_{\mathfrak{u}(1)}^2. \tag{5.2.29}$$

Finally the index polynomials of A_V, A_H can be calculated from their definitions (5.2.7), (5.2.8), (5.2.11), (5.2.12); in particular, the following results are useful

$$\begin{aligned}\operatorname{Ind} \check{\theta}_V(z,R) =& -\frac{R^2z^2}{2} - \frac{(R-1)R(R+1)}{3} z(\epsilon_1+\epsilon_2) & (5.2.30)\\ &- \frac{(R-1)R^2(R+1)}{12}(\epsilon_1^2+\epsilon_1\epsilon_2+\epsilon_2^2), & (5.2.31)\\ \operatorname{Ind} \check{\theta}_H(z,R) =& \frac{(R+1/2)(R-1/2)}{4}z^2 + \frac{R(R-1/2)(R+1/2)}{6}z(\epsilon_1+\epsilon_2)\\ &+ \frac{(R-1/2)(R+1/2)(R^2-3/4)}{24}(\epsilon_1^2+\epsilon_2^2)\\ &+ \frac{(R-1/2)(R+1/2)(R^2+3/4)}{24}\epsilon_1\epsilon_2. & (5.2.32)\end{aligned}$$

If we compute the modular index polynomial of an arbitrary term on the l.h.s. subtracted by that of the r.h.s., we find that the dependence on λ_G, d', d'' cancel completely thanks to the choice of y (5.2.2) and the constraints on the number of hypermultiplets imposed by the anomaly cancellation conditions (5.1.10), (5.1.11), (5.1.12), (5.1.14), (5.1.16), (5.1.15). What remains is a quadratic polynomial of the following form

$$\operatorname{Ind}(\delta,d,m_G,\epsilon_{1,2},\lambda_F) = \delta^2 P_2(d,m_G,\epsilon_{1,2},\lambda_F) + \delta P_1(d,m_G,\epsilon_{1,2},\lambda_F) + P_0(d,m_G,\epsilon_{1,2},\lambda_F), \tag{5.2.33}$$

where

$$\begin{aligned}P_0(d,m_G,\epsilon_{1,2},\lambda_F) =& \frac{1}{8} n_{R_G,R_F}(2\operatorname{ind}_{R_G}(m_G\cdot m_G)\,\mathrm{id}_1 + \dim R_G\,\mathrm{id}_2)\\ &+ \frac{1}{12} n_{R_G,R_F} \dim R_G\,\mathrm{id}_3(\epsilon_1+\epsilon_2)\\ &+ \frac{1}{48} n_{R_G,R_F} \dim R_G(\mathrm{id}_4-\mathrm{id}_1)(\epsilon_1+\epsilon_2)^2\\ &- \Big(\frac{1}{2} n_{R_G,R_F}\operatorname{ind}_{R_G} d\,\mathrm{id}_1 + \frac{1}{96} n_{R_G,R_F}\dim R_G(2\mathrm{id}_4-5\mathrm{id}_1)\Big)\epsilon_1\epsilon_2\end{aligned} \tag{5.2.34}$$

and

$$\mathrm{id}_1 = \sum_{\omega\in R_F}\Big((\omega\cdot\lambda_F)^2 - \frac{1}{4}\Big), \tag{5.2.35}$$

$$\mathrm{id}_2 = \sum_{\omega\in R_F}\Big((\omega\cdot\lambda_F)^2(\omega\cdot m_F)^2 - \frac{1}{4}(\omega\cdot m_F)^2\Big), \tag{5.2.36}$$

$$\mathrm{id}_3 = \sum_{\omega\in R_F}\Big((\omega\cdot\lambda_F)^3(\omega\cdot m_F) - \frac{1}{4}(\omega\cdot\lambda_F)(\omega\cdot m_F)\Big), \tag{5.2.37}$$

$$\mathrm{id}_4 = \sum_{\omega\in R_F}\Big((\omega\cdot\lambda_F)^4 - \frac{1}{4}(\omega\cdot\lambda_F)^2\Big). \tag{5.2.38}$$

In the case of unity blowup equations where $\delta = 0$, the index $\mathrm{Ind}(0, d, m_G, \epsilon_{1,2}, \lambda_F)$ should vanish identically for arbitrary values of $d, \epsilon_{1,2}, m_G$. This implies the additional conditions

$$\mathrm{id}_1 = \mathrm{id}_2 = \mathrm{id}_3 = \mathrm{id}_4 = 0. \tag{5.2.39}$$

The checker board pattern condition (5.2.15) together with the first condition (5.2.35) above lead to the identity

$$\omega \cdot \lambda_F = \pm\frac{1}{2}, \quad \omega \in R_F, \tag{5.2.40}$$

with which the other three conditions above (5.2.36),(5.2.37),(5.2.38) are automatically satisfied. The identity (5.2.40) fixes the norm of λ_F

$$\lambda_F \cdot \lambda_F = \frac{1}{2\,\mathrm{ind}_{R_F}} \sum_{\omega \in R_F} (\omega \cdot \lambda_F)^2 = \frac{h^\vee_{\mathfrak{g}} - 3\mathfrak{n} + 6}{2k_F}, \tag{5.2.41}$$

where we have used (5.1.10),(5.1.14). The expression (5.2.2) can then be simplified to

$$y = \frac{h^\vee_{\mathfrak{g}} - \mathfrak{n} + 2}{2}. \tag{5.2.42}$$

The identity (5.2.40) turns out to completely fix the coweight vectors λ_F for unity blowup equations. We have listed them in Table 5.5 and 5.6.

5.2.4 K-theoretic limit

When taking the K-theoretic limit $q_\tau \to 0$, the elliptic genera of a 6d theory in general reduce to the K-theoretic instanton partition function of the 5d theory with the same gauge, flavor group and the same matter contents on a circle of radius one,[7] and the elliptic blowup equations in general reduce to the K-theoretic blowup equations. For example, it is easy to find that the unity elliptic blowup equations (5.2.1) with $n \geq 3$ in the $q_\tau \to 0$ limit naturally reduce to the 5d blowup equations with matters proposed in (Kim et al., 2019). However, there are several subtle points.

- The elliptic blowup equation with characteristic $a = -1/2$ could split to *two* K-theoretic blowup equations in the $q_\tau \to 0$ limit. For other characteristics a, each elliptic blowup equation will reduce to one K-theoretic blowup equation.
- For $n = 2$ theories, the elliptic genera in the $q_\tau \to 0$ limit give the 5d Nekrasov partition function with an extra term which are neutral with respect to G. To obtain the precise 5d blowup equations from 6d, one needs to factor out the extra term which possibly contributes to the Λ factor.
- All $n = 1$ theories in the $q_\tau \to 0$ limit just reduce to the theory of a free hypermultiplet, whose associated Calabi-Yau space is simply the resolved conifold (Del Zotto and Lockhart, 2018). The reduced one-string elliptic genera all have leading q_τ order as just q^{-1}, and the 5d gauge theory information are encoded in the q^0 coefficient. It is easy to see this works along well with elliptic blowup equations. In fact, the unity elliptic blowup equations for all $n = 1$ theories at

[7] The radius can be easily recovered to arbitrary β by dimensional analysis.

the q_τ leading order just give the blowup equation of the resolved conifold. After factoring out the q^{-1} term and a gauge natural term similar with the $n=2$ situation, one can obtain the 5d blowup equations from the order q^0 of 6d ones.

- More importantly, we find in general, *not all* K-theoretic blowup equations are reduced from elliptic blowup equations. In particular, the admissible range of the shifts for the 5d instanton counting parameter $\mathfrak{q}$ can be larger than the admissible range of the shifts for the 6d string number counting parameter Q_{ell}. This makes some $n=2$ theories such as $G=\mathfrak{su}(N), F=\mathfrak{su}(2N)$ theory not recursively solvable in 6d, but recursively solvable in 5d.

Let us discuss the pure gauge minimal 6d $(1,0)$ SCFTs as example. In the $q_\tau \to 0$ limit, the elliptic genera directly reduce to the 5d Nekrasov-partition functions. We find all possible 5d blowup equations for the pure gauge theory with $G = A_2, D_4, F_4, E_{6,7,8}$. The 5d r fields and Λ factors and their 6d origins are listed in Table 5.10. Note the first three rows were given by the K-theoretic blowup equations in (Keller and Song, 2012).

	5d r	5d Λ	from 6d r	6d Λ
$j=1,2,\ldots,n-1$	$(0,-n+2j)$	1	$(0,0,-n+2j)$	$\Lambda^{[1/2-j/n]}$
	$(0,\pm n)$	1	$(0,0,n)$	$\Lambda^{[-1/2]}$
$j=1,2,\ldots,n-3$	$(0,\pm(n+2j))$	1		
	$(0,\pm(3n-4))$	$1-(-1)^n e^{\pm\frac{3(n-2)}{2}(\epsilon_1+\epsilon_2)}\mathfrak{q}$		
$j=1,2,\ldots,n-1$	$(2w,-n+2j)$	0	$(0,2w,-n+2j)$	0
	$(2w,\pm n)$	0	$(0,2w,n)$	0
$j=1,2,\ldots,n-4$	$(2w,\pm(n+2j))$	0		
	$(2w,\pm(3n-6))$	$e^{\pm\frac{3(n-2)^2}{2n}(\epsilon_1+\epsilon_2)}\mathfrak{q}^{\frac{n-2}{n}}$		

Table 5.10: The 5d and 6d blowup equations for pure gauge theories with $G=A_2,D_4,F_4,E_{6,7,8}$ and corresponding $n=3,4,5,6,8,12$. The 5d r fields are denoted as $(r_{m_G}, r_{\log\mathfrak{q}})$, with weight $w\in(P^\vee\backslash Q^\vee)_G$ and 6d r fields are denoted as $(r_\tau, r_{m_G}, r_{\log Q_{\text{ell}}})$.

5.3 Solving elliptic blowup equations

In this section, we discuss how to solve the blowup equations. By solving we mean extracting refined BPS invariants of the local Calabi-Yau threefold, or equivalently computing elliptic genera in the case of 6d theories and instanton partition functions in the case of 5d theories. In Chapter 4.3, we already discussed two methods – ϵ_1,ϵ_2 expansion and refined BPS expansion, which of course also apply to the current elliptic non-compact Calabi-Yau. Here we will propose two more methods – recursion formula and Weyl orbit expansion which are designed to compute elliptic genera.

To discuss solving the blowup equations for 6d $(1,0)$ SCFTs, it is convenient to divide all these theories into three classes according to the difficulty of solving their associated blowup equations:

A These theories have unity blowup equations and possibly vanishing blowup equations as well; there are enough unity equations so that recursion formulas in the same spirit as in (Nakajima and Yoshioka, 2005b; Keller and Song, 2012) can be written down and the blowup equations can be solved immediately.

- These are the rank one theories with $\mathfrak{n} \geq 3$ and without unpaired half-hypermultiplets[8]. There are infinitely many theories in this class.

B These theories have unity blowup equations and possibly also vanishing blowup equations; the number of unity blowup equations is not sufficient to allow for recursion formulas. Nevertheless in practice it is still possible to solve blowup equations order by order using other methods.

- In the case of rank one 6d SCFTs, these are the theories with $\mathfrak{n} = 1, 2$ and without unpaired half-hypermultiplets. There are also infinitely many theories in this class.

C These theories have only vanishing blowup equations but no unity blowup equations. There is currently no algorithm to solve these equations completely.

- In the case of rank one theories, these are the theories with unpaired half-hypermultiplets. There are in total 12 theories in this class which are $n = 1, 3, 5, 7$ $G = E_7$ theories, $n = 1, 3$ $G = \mathfrak{so}(11)$ theories, $n = 3$ $G = \mathfrak{so}(12)$ theory, $n = 2$ $G = \mathfrak{so}(12)_b, \mathfrak{so}(13)$ and $n = 1$ $G = \mathfrak{su}(6)_\star, \mathfrak{so}(12)_{a,b}$ theories.

In this section and the later section of examples, we will focus on rank one theories. In the next Chapter, we will see that all *higher-rank* theories belong to classes **B** or **C**.

We discuss four methods to solve blowup equations, summarized in Table 5.11. The first two methods – the recursion formulas and the Weyl orbit expansion – are based on elliptic blowup equations, they require implicitly the semiclassical and the one-loop partition function as input data. The recursion formulas has the least scope of applicability among the first two methods; but when it applies, it is the most powerful, as it calculates explicitly elliptic genera to arbitrary numbers of strings. The Weyl orbit expansion has a wider range of applicability. The last two methods, the refined BPS expansion and the ϵ_1, ϵ_2 expansion, are designed to compute refined BPS invariants or refined free energies. We comment that although theories in class **C** cannot be solved completely, there are some examples, for instance the $\mathfrak{n} = 7, G = E_7$ model, where one can use the BPS expansion method to solve the majority of refined BPS invariants below any degree bound, see (Gu et al., 2020b). We also need to point out that although the method of ϵ_1, ϵ_2 expansion seems to apply to all three classes, the necessary initial data are sometimes rather difficult to come by. Here it is only used to discuss the solvability of the blowup equations associated to different classes of theories. For example, According to the counting of component equations in , all rank one theories in class **B** can be solved if free energy $F_{(0,0)}$ is provided, and all rank one theories in class **C** except for the four $G = E_7$ theories can be solved if all $F_{(n,0)}$ or all $F_{(0,g)}$ are provided. Since the full NS free energy or the full self-dual free energy are themselves difficult to compute, and besides for 6d SCFTs, we are more interested in the elliptic genera and BPS invariants rather than $F_{n,g}$ themselves, we only use this method to discuss the solvability of 6d theories in different classes, but do not further explore this method.

[8] The theory of $\mathfrak{n} = 3, G = \mathfrak{su}(3)$ is a bit special. The recursion formulas do not work for the *one*-string elliptic genus, as the latter enjoys an enhanced symmetry so that the number of independent unity equations is reduced; the recursion formulas, nevertheless, still work for elliptic genera of more than one string (Gu et al., 2019b).

methods	solvable classes	input data	output results
recursion formulas	**A**	semiclassical, one-loop	elliptic genera
Weyl orbit expansion	**A**, **B**	semiclassical, one-loop	elliptic genera
refined BPS expansion	**A**, **B**, partially **C**	semiclassical, prepotential	BPS invariants
ϵ_1, ϵ_2 expansion	**A**, **B**, **C**	depends, see Section 5.3	free energies

Table 5.11: Summary of methods to solve blowup equations.

We explain the first two methods in the following subsections. We give an inventory of all our results on elliptic genera in Appendix D, and present some of these results explicitly in Section 5.5 and Appendix D, more results can be found on the website. The refined BPS invariants of the elliptic non-compact Calabi-Yau threefolds associated to some 6d SCFTs can be found in (Gu et al., 2020b) and the website.

5.3.1 Recursion formula

From (5.2.1), the unity blowup equation can be written as

$$\begin{aligned}
&\theta_i^{[a]}(\mathfrak{n}\tau, k_F\lambda_F\cdot m_F+\mathfrak{n}y(\epsilon_1+\epsilon_2))\mathbb{E}_d(m_G,m_F,\epsilon_1,\epsilon_2)\\
&-\theta_i^{[a]}(\mathfrak{n}\tau, k_F\lambda_F\cdot m_F+\mathfrak{n}(y(\epsilon_1+\epsilon_2)-d\epsilon_1))\mathbb{E}_d(m_G,m_F+\epsilon_1\lambda_F,\epsilon_1,\epsilon_2-\epsilon_1)\\
&-\theta_i^{[a]}(\mathfrak{n}\tau, k_F\lambda_F\cdot m_F+\mathfrak{n}(y(\epsilon_1+\epsilon_2)-d\epsilon_2))\mathbb{E}_d(m_G,m_F+\epsilon_2\lambda_F,\epsilon_1-\epsilon_2,\epsilon_2)\\
=&\sum_{\lambda_G,d_1,d_2}{}'(-1)^{|\alpha^\vee|}\theta_i^{[a]}(\mathfrak{n}\tau,-\mathfrak{n}\,\lambda_G\cdot m_G+k_F\,\lambda_F\cdot m_F+\mathfrak{n}((y-d_0)(\epsilon_1+\epsilon_2)-d_1\epsilon_1-d_2\epsilon_2))\\
&\qquad\times A_V(\tau,m_G,\lambda_G)A_H(\tau,m_{G,F},\lambda_{G,F})\\
&\times\mathbb{E}_{d_1}(\tau,m_{G,F}+\epsilon_1\lambda_{G,F},\epsilon_1,\epsilon_2-\epsilon_1)\mathbb{E}_{d_2}(\tau,m_{G,F}+\epsilon_2\lambda_{G,F},\epsilon_1-\epsilon_2,\epsilon_2),
\end{aligned}\tag{5.3.1}$$

where $\sum'_{\lambda_G,d_1,d_2}$ means the summation over all $\lambda_G\in Q^\vee, d_0=\frac{1}{2}||\lambda_G||_G^2$ and $0\le d_{1,2}<d$ with $d_0+d_1+d_2=d$. All the instances of d-string elliptic genus are collected on the l.h.s., and there are only less than d-string elliptic genera on the r.h.s.. With the characteristic a taking value as in (5.2.4), the number of such equations with fixed d and λ_F is $\mathfrak{n}$. For models with $\mathfrak{n}\ge 3$, we can choose three arbitrary characteristics a_1,a_2,a_3 and solve the d-string elliptic genus from (5.3.1) as

$$\begin{aligned}
&\mathbb{E}_d(\tau,m_G,m_F,\epsilon_1,\epsilon_2)=\\
&\sum_{\lambda_G,d_1,d_2}{}'(-1)^{|\lambda_G|}[D/D]_{\mathfrak{n},(a_1,a_2,a_3)}(m_{G,F},\epsilon_{1,2},\lambda_{G,F})A_V(\tau,m_G,\lambda_G)A_H(\tau,m_{G,F},\lambda_{G,F})\\
&\qquad\times\mathbb{E}_{d_1}(\tau,m_{G,F}+\epsilon_1\lambda_{G,F},\epsilon_1,\epsilon_2-\epsilon_1)\mathbb{E}_{d_2}(\tau,m_{G,F}+\epsilon_2\lambda_{G,F},\epsilon_1-\epsilon_2,\epsilon_2)
\end{aligned}$$

where we define

$$\begin{aligned}
&[D/D]_{\mathfrak{n},(a_1,a_2,a_3)}(m_{G,F},\epsilon_{1,2},\lambda_{G,F})=\\
&\frac{D_{\mathfrak{n},(a_1,a_2,a_3)}(\lambda_G\cdot m_G-d_0(\epsilon_1+\epsilon_2)-d_1\epsilon_1-d_2\epsilon_2,-d\epsilon_1,-d\epsilon_2;k_F\lambda_F\cdot m_F+\mathfrak{n}y(\epsilon_1+\epsilon_2))}{D_{\mathfrak{n},(a_1,a_2,a_3)}(0,-d\epsilon_1,-d\epsilon_2;k_F\lambda_F\cdot m_F+\mathfrak{n}y(\epsilon_1+\epsilon_2))}
\end{aligned}\tag{5.3.2}$$

with determinant

$$D_{\mathfrak{n},(a_1,a_2,a_3)}(z_1,z_2,z_3;z_\mathfrak{n})=\det\left(\theta_i^{[a_j]}(\mathfrak{n}\tau,z_\mathfrak{n}+\mathfrak{n}z_k)_{j,k=1,2,3}\right).\tag{5.3.3}$$

In particular, the one-string elliptic genus can be simply obtained as

$$\mathbb{E}_1(\tau, m_{G,F}, \epsilon_{1,2}) = \sum_{||\lambda_G||^2=2} (-1)^{|\lambda_G|} A_V(\tau, m_G, \lambda_G) A_H(\tau, m_{G,F}, \lambda_{G,F})$$
$$\times \frac{D_{\mathfrak{n},(a_1,a_2,a_3)}(\lambda_G \cdot m_G - (\epsilon_1+\epsilon_2), -\epsilon_1, -\epsilon_2; k_F\lambda_F \cdot m_F + \mathfrak{n}y(\epsilon_1+\epsilon_2))}{D_{\mathfrak{n},(a_1,a_2,a_3)}(0, -\epsilon_1, -\epsilon_2; k_F\lambda_F \cdot m_F + \mathfrak{n}y(\epsilon_1+\epsilon_2))}. \tag{5.3.4}$$

Note the final result of $\mathbb{E}_d$ does not depend on the choice of $a_{1,2,3}$.

5.3.2 Weyl orbit expansion

For class **B** theories, without sufficient number of unity blowup equations with different characteristics, one can not have explicit recursion formulas for $\mathbb{E}_d$. Here we develop new method based on Weyl orbit expansion of elliptic genera for all theories in class **A** and **B**. Although we do not have a proof, explicit computation for many examples shows that the unity blowup equations are sufficient to uniquely determine the elliptic genera.[9] This method is particularly efficient for *one*-string elliptic genus and *small* flavor group such as $\mathfrak{su}(2)$ or $\mathfrak{u}(1)$. The reason is that the reduced one-string elliptic genus only depends on $v = \mathrm{e}^{(\epsilon_1+\epsilon_2)/2}$ but not on $x = \mathrm{e}^{(\epsilon_1-\epsilon_2)/2}$, while reduced higher string elliptic genera do depend on x thus the flavor group is effectively $F \times \mathfrak{su}(2)_x$.

Let us focus on the reduced one-string elliptic genus $\mathbb{E}_1^{\mathrm{red}}(v, q, m_G, m_F)$. We can always write it in the following Weyl orbit expansion of F, or v expansion in other words:[10]

$$\mathbb{E}_1^{\mathrm{red}}(v, q, m_G, m_F) = \sum c_{n,m,p,k}(m_G) q^n v^m \mathcal{O}^F_{p,k}. \tag{5.3.5}$$

Here $c_{n,m,p,k}(m_G)$ are some G Weyl-invariant rational functions of $\exp(m_G)$. Note the order n of q has a known *lower* bound, and for each n, the order m of v also has a *lower* bound, while for each (n, m) pair, the lengths of Weyl orbit elements of flavor group p have an *upper* bound, i.e. there are only finitely many different flavor Weyl orbits. One can further decompose $c_{n,m,p,k}(m_G)$ into the Weyl orbit expansion w.r.t. the gauge group. In this case, for fixed (n, m) and flavor Weyl orbit $\mathcal{O}^F_{p,k}$, there could be in general infinitely many gauge Weyl orbits.[11] Our strategy here is to use the unity blowup equations to solve the coefficient functions $c_{n,m,p,k}$.[12] Remember that the full one-string elliptic genus is

$$\mathbb{E}_1(q, m_G, m_F, \epsilon_1, \epsilon_2) = \frac{\eta^2}{\theta_1(\epsilon_1)\theta_1(\epsilon_2)} \mathbb{E}_1^{\mathrm{red}}(\mathrm{e}^{(\epsilon_1+\epsilon_2)/2}, q, m_G, m_F). \tag{5.3.6}$$

[9] We exclude E-string theory in the discussion in this subsection. As there is no gauge symmetry, the unity blowup equations can only determine the elliptic genus up to a free function of q_τ.

[10] For a Weyl orbit $\mathcal{O}_{p,k}$, we adopt the common notation that p is the length of its elements and k is the number of its elements.

[11] These properties can also be easily seen from the unity elliptic blowup equations. For example, the flavor parameters only appear in the nominators of blowup equations, which determine that there exist only finitely many different flavor Weyl orbits for elliptic genus at each fixed order of q and v.

[12] It is proposed in (Del Zotto and Lockhart, 2018) that one should be able to $\mathbb{E}_1^{\mathrm{red}}(v, q, m_G, m_F)$ in terms of representations of G and F rather than just Weyl orbits. Nevertheless, from the viewpoint of solving blowup equations, the most natural setting is Weyl orbit expansion.

For class **A** and **B** theories, the unity blowup equation for one-string elliptic genus reads

$$\begin{aligned}
&\theta_i^{[a]}(\mathfrak{n}\tau, k_F\lambda_F\cdot m_F+\mathfrak{n}y(\epsilon_1+\epsilon_2))\mathbb{E}_1(m_G,m_F,\epsilon_1,\epsilon_2)\\
&-\theta_i^{[a]}(\mathfrak{n}\tau, k_F\lambda_F\cdot m_F+\mathfrak{n}(y(\epsilon_1+\epsilon_2)-\epsilon_1))\mathbb{E}_1(m_G,m_F+\epsilon_1\lambda_F,\epsilon_1,\epsilon_2-\epsilon_1)\\
&-\theta_i^{[a]}(\mathfrak{n}\tau, k_F\lambda_F\cdot m_F+\mathfrak{n}(y(\epsilon_1+\epsilon_2)-\epsilon_2))\mathbb{E}_1(m_G,m_F+\epsilon_2\lambda_F,\epsilon_1-\epsilon_2,\epsilon_2)\\
=&\sum_{\lambda_G\in Q_G^\vee}(-1)^{|\lambda_G|}\theta_i^{[a]}(\mathfrak{n}\tau,-\mathfrak{n}\lambda_G\cdot m_G+k_F\lambda_F\cdot m_F+\mathfrak{n}(y-1)(\epsilon_1+\epsilon_2))\\
&\qquad\times A_V(\tau,m_G,\lambda_G)A_H(\tau,m_{G,F},\lambda_{G,F}).
\end{aligned}\tag{5.3.7}$$

By substituting the v expansion ansatz (5.3.5) into the above equation, it is expected that all the coefficient functions $c_{n,m,p,k}(m_G)$ can be determined. Note the shift such as $m_F+\epsilon_1\lambda_F$ in the elliptic genus will break the flavor Weyl orbits into pieces, and the blowup equations work in a miraculous way such that all such pieces in each term of the l.h.s. are reorganized again into a Weyl invariant on the r.h.s..

The complexity of solving Weyl orbit expansion in blowup equations increases with the complexity of the Weyl orbits of the flavor group F. Therefore, the solution process can be complicated (but still feasible) when the flavor group is large. Practically, one can normally just turn on a subgroup $\mathfrak{su}(2)$ or $\mathfrak{u}(1)$ of the flavor group to make use of the blowup equations and still obtain the elliptic genera with lots of useful information. In particular, if one does not need the gauge fugacities, the computation can be even easier where one can firstly turn off the gauge fugacities in unity blowup equations (5.3.7) and then solve $c_{*,*,*,*}$ as numbers. In such a situation, we can even withdraw the v expansion and arrive in the following useful ansatz for the reduced one-string elliptic genus of a 6d SCFT from a $-n$ curve:

$$\mathbb{E}_1^{\text{red}}(v,q,m_G=0,m_F)=\delta_{n,1}\,q^{-\frac{1}{3}}+q^{\frac{1}{6}-\frac{n-2}{2}}\sum_{m,p,k}\frac{P_{m,p,k}(v^2)}{(1-v^2)^{2h_G^\vee-2}}q^m\mathcal{O}^F_{p,k}.\tag{5.3.8}$$

Here the $P_{m,p,k}(v^2)$ are, up to an overall factor like v^N, $N\in\mathbb{Z}$, *palindromic polynomial* functions of v^2 with integral coefficients. We use this ansatz to solve many class **B** theories.

The leading q order of $\mathbb{E}_1^{\text{red}}(v,q,m_G,m_F)$,[13] i.e. the 5d reduced one-instanton partition function is expected to yield *exact* formulas as v expansions. For example, it is well-known that the reduced one G-instanton Nekrasov partition function in 5d, i.e. the Hilbert series of the reduced moduli space of one G-instanton has the following exact formula (Benvenuti, Hanany, and Mekareeya, 2010; Keller et al., 2012)

$$\sum_{n=0}^{\infty}v^{h_G^\vee-1+2n}\chi^G_{n\theta},\qquad\text{where }\theta\text{ is the Dynkin label for }\mathfrak{adj}_G.\tag{5.3.9}$$

For theories with matter and flavor group F, the one-instanton Hilbert series should still have an exact but more complicated v expansion formula as follows

$$\sum_{i\in I}\pm v^{k_i}\chi^G_{a_i}\chi^F_{b_i}+\sum_{j\in J}\Big(\pm\sum_{n=0}^{\infty}v^{h_j+2n}\chi^G_{c_j+n\theta}\chi^F_{d_j}\Big).\tag{5.3.10}$$

[13] The subleading order for $n=1$ theories.

Here both I and J are finite sets, k_i and h_j are integers, a_i and c_j are Dynkin labels of G, while b_i and d_j are Dynkin labels of F. Besides, $\pm$ means the coefficient can only be either $+1$ or -1. For lots of 6d $(1,0)$ SCFTs, this type of exact formulas were conjectured or found in (Del Zotto and Lockhart, 2018) and (Kim et al., 2019). Benefiting from the results of one-string elliptic genera solved from the blowup equations, we are able to confirm them and obtain such exact formulas for more theories, see Section 5.5 and Appendix D. We hope these exact v expansion formulas could have an interpretation from 3d monopole formulas (Benvenuti, Hanany, and Mekareeya, 2010) in the future.

5.4 Universal behaviors of elliptic genera

In this section, we focus on pure gauge minimal 6d $(1,0)$ SCFTs, i.e. $G = \mathfrak{su}(3)$, $\mathfrak{so}(8)$, F_4, $E_{6,7,8}$ and study the behaviors of their elliptic genera in q_τ expansion. Benefited from the higher-string elliptic genera we solved from recursion formula, we find some very interesting universal features. We propose the universal expansion of reduced one-, two- and three-string elliptic genera. These formulas will also be useful later in Chapter 7.

5.4.1 Universal expansion

For all possible gauge group G, recall $v \equiv \exp(\pi\mathrm{i}(\epsilon_1+\epsilon_2))$ and $x \equiv \exp(\pi\mathrm{i}(\epsilon_1-\epsilon_2))$, we propose the following general ansatz for the reduced k-string elliptic genera

$$\mathbb{E}_{h_G^{(k)}}(v,x,q_\tau,Q_{m_i}) = v^{kh_G^\vee-1}q_\tau^{-(kh_G^\vee-1)/6}\sum_{n=0}^{\infty} q_\tau^n\, g_{k,G}^{(n)}(v,x,Q_{m_i}). \tag{5.4.1}$$

Here all $g_{k,G}^{(n)}(v,x,Q_{m_i})$ are rational functions. In particular, $g_{1,G}^{(n)}$ is independent from x. One obvious symmetry for all $g_{k,G}^{(n)}$ is

$$g_{k,G}^{(n)}(v,x,Q_{m_i}) = g_{k,G}^{(n)}(v,x^{-1},Q_{m_i}), \tag{5.4.2}$$

which comes from the symmetry between ϵ_1 and ϵ_2 in the Omega background, and can be understood as the Weyl symmetry of $\mathfrak{su}(2)_x$. From on on we use $\mathfrak{su}(2)_x$ to denote $\mathfrak{su}(2)_l$ symmetry to stress the associated fugacity is x. We can further compute the v-expansion of each $g_{k,G}^{(n)}$ function where the coefficients are finite sum of products between the characters of $\mathfrak{su}(2)_x$ and characters of G which respect Weyl symmetries of both groups. For example, $g_{k,G}^{(0)} = 1 + \dots$ gives the Hilbert series of the reduced k G-instanton moduli space. In fact, we find plenty of universal coefficients for the first a few order v-expansion of $g_{k,G}^{(n)}$.

It is known that the Hilbert series of the reduced one-instanton moduli space for any simple gauge group G has the expansion (Benvenuti, Hanany, and Mekareeya, 2010)

$$g_{1,G}^{(0)}(v,Q_{m_i}) = \sum_{k=0}^{\infty} \chi_{n\theta}^G v^{2n}\,, \tag{5.4.3}$$

where $\chi_{k\theta}$ is the character of the representation whose highest weight is k-multiple of the longest root θ; in particular χ_θ is the character of the adjoint representation of G. In particular this is true for $G=\mathfrak{su}(3),\mathfrak{so}(8),F_4,E_{6,7,8}$ when $g^{(0)}_{1,G}$ serves as the leading contribution to one-string elliptic genus. As for sub- and subsub-leading contributions, we find that except for $G=SU(3)$[14]

$$\begin{aligned} g^{(1)}_{1,G}(v,Q_{m_i}) =&1+\chi_\theta+\Big(1+\chi_\theta+\chi_{2\theta}+\chi_{\mathrm{Alt}^2\theta}\Big)v^2\\ &+\Big(2\chi_{2\theta}+\chi_{\mathrm{Alt}^2\theta}+\chi_{3\theta}+B_2(G)\Big)v^4+\mathcal{O}(v^6), \end{aligned} \tag{5.4.4}$$

while except for $G=\mathfrak{su}(3),\mathfrak{so}(8)$,

$$g^{(2)}_{1,G}(v,Q_{m_i})=2+2\chi_\theta+\chi_{\mathrm{Sym}^2\theta}+\mathcal{O}(v^2)\,. \tag{5.4.5}$$

Here $B_2(G)$ are characters of some representations for which we do not find any universal expressions, and we list them in Table 5.12.[15] The exceptions of $\mathfrak{su}(3)$ and $\mathfrak{so}(8)$ can be explained by the higher structures of $\mathbb{E}_1$ revealed by its intriguing relation with the Schur indices of certain rank one 4d SCFTs discovered in (Del Zotto and Lockhart, 2017), which we will review and extend in section 7.

G	χ_θ	$\chi_{2\theta}$	$\chi_{3\theta}$	B_2	C_6	C_7	C_8
A_2	**8**	**27**	**64**	$2\cdot$**35**	**1**	**27**	**8**
D_4	**28**	**300**	**1925**	**4096**	$2\cdot$**28**	$3\cdot$**567**	$2\cdot($**300**$+$**350**$+$**1**$)+3\cdot$**35**
F_4	**52**	**1053**	**12376**	**29172**	**273**	**10829**	**8424**$+$**4096**$+$**324**$+$**26**
E_6	**78**	**2430**	**43758**	**105600**	**650**	**34749**	**34749**$+2\cdot$**5824**$+$**650**$+$**78**
E_7	**133**	**7371**	**238602**	**573440**	**1463**	**152152**	**150822**$+$**40755**$+$**1539**
E_8	**248**	**27000**	**1763125**	**4096000**	0	**779247**	**147250**

Table 5.12: Certain representations appearing in the expansion of $g^{(n)}_{k,G}$ functions.

Furthermore, we find the Hilbert series of reduced two-instanton moduli space for any simple gauge group G has the expansion

$$\begin{aligned} g^{(2)}_{0,G}(v,x,Q_{m_i}) =&\, 1+(\chi_\theta+\chi_3)v^2+\chi_\theta\chi_2v^3+(\chi_5+\chi_\theta\chi_3+\chi_{\mathrm{Sym}^2\theta})v^4+(\chi_\theta\chi_4\\ &+(\chi_{2\theta}+\chi_{\mathrm{Alt}^2\theta})\chi_2)v^5+\Big(\chi_7+\chi_5\chi_\theta+\chi_3(\chi_{\mathrm{Sym}^2\theta}+\chi_{2\theta})+\chi_{\mathrm{Sym}^3\theta}-C_6(G)\Big)\,v^6\\ &+\Big(\chi_\theta\chi_6+(\chi_{2\theta}+\chi_{\mathrm{Alt}^2\theta})\chi_4+(\chi_{2\theta}+\chi_{3\theta}+\chi_{\mathrm{Alt}^2\theta}+B_2(G)+C_7(G))\chi_2\Big)v^7\\ &+\Big(\chi_9+\chi_7\chi_\theta+\chi_5(\chi_{\mathrm{Sym}^2\theta}+\chi_{2\theta})+\chi_3(\chi_{3\theta}+\chi_{2\theta}+B_2(G)+\chi_{\mathrm{Sym}^3\theta}-C_6(G))\\ &\quad+\chi_{\mathrm{Sym}^4\theta}-C_8(G)\Big)v^8+\Big(\chi_\theta\chi_8+(\chi_{2\theta}+\chi_{\mathrm{Alt}^2\theta})\chi_6+(\chi_{2\theta}+2\chi_{3\theta}+\chi_{\mathrm{Alt}^2\theta}\\ &\quad+B_2(G)+C_7(G))\chi_4+\dots\Big)v^9+\dots \end{aligned} \tag{5.4.6}$$

[14]From now on, to shorten formulas, we do not explicitly write G in each character.

[15]The bold numbers mean the character of representations with dimension of such number. Note different representations can have the same dimension sometimes, for instance, the representations $\mathbf{35}_v$, $\mathbf{35}_s$ and $\mathbf{35}_c$ of $\mathfrak{so}(8)$. To lighten the notation, we do not distinguish them in the table. Nevertheless, they can be recovered by taking into account the symmetry of Dynkin diagrams.

Here χ_n is the character of n-dimensional representation of $\mathfrak{su}(2)_x$. The expansion coefficients up to v^6 were already observed in (Keller and Song, 2012), and we further push the observation up to v^8. We have checked this expression to be consistent with all the results on Hilbert series of reduced two G instanton moduli space in (Hanany, Mekareeya, and Razamat, 2013). In particular it is true for $G = \mathfrak{su}(3), \mathfrak{so}(8), F_4, E_{6,7,8}$ when $g^{(0)}_{2,G}$ is the leading contribution to the two-string elliptic genera. Note that in this expression, $C_6(G), C_7(G)$ are characters of certain representations of G for which universal expressions are not found. They are collected for individual G in Table 5.12. As for the subleading and subsubleading contribution to the two-string elliptic genera, we find there exists the following universal v-expansion: except for $G = \mathfrak{su}(3)$,

$$\begin{aligned} g^{(2)}_{1,G}(v,x,Q_{m_i}) = &1 + \chi_\theta + \chi_3 + (\chi_\theta+1)\chi_2 v + (\chi_5 + (2\chi_\theta+3)\chi_3 + (\chi_\theta+1)^2)v^2 \\ &+ \Big((2\chi_\theta+1)\chi_4 + (\chi_{2\theta} + (\chi_\theta+1)^2 + \chi_{\mathrm{Sym}^2\theta})\chi_2\Big)\, v^3 \\ &+ (\chi_7 + (2\chi_\theta+3)\chi_5 + \ldots)v^4 + \mathcal{O}(v^5)\,, \end{aligned} \tag{5.4.7}$$

while except for $G = SU(3)$ and $SO(8)$,

$$\begin{aligned} g^{(2)}_{2,G}(v,x,Q_{m_i}) = &(\chi_5 + (\chi_\theta+2)\chi_3 + \chi_{\mathrm{Sym}^2\theta} + 2\chi_\theta + 4) \\ &+ \Big((\chi_\theta+1)\chi_4 + ((\chi_\theta+1)^2 + 2(\chi_\theta+1))\chi_2\Big)v \\ &+\Big(\chi_7 + (2\chi_\theta+4)\chi_5 + ((\chi_\theta+1)^2 + 2\chi_{\mathrm{Sym}^2\theta} + 6\chi_\theta + 9)\chi_3 + \ldots\Big)v^2 + \mathcal{O}(v^3)\,. \end{aligned} \tag{5.4.8}$$

For the reduced three string elliptic genus $\mathbb{E}_{h^{(3)}_G}$, although we have not checked for all six G due to the complexity of computation, still we propose the following universal expansion:

$$\begin{aligned} g^{(3)}_{0,G}(v,x,Q_{m_i}) = &1 + (\chi_3+\chi_\theta)v^2 + (\chi_4 + \chi_\theta\chi_2)v^3 + \Big(\chi_5 + \chi_\theta\chi_3 + \chi_{\mathrm{Sym}^2\theta} + 1\Big)\, v^4 \\ &+ \Big(\chi_6 + (2\chi_\theta+1)\chi_4 + 2\chi_{\mathrm{Sym}^2\theta}\Big)v^5 \\ &+\Big(2\chi_7 + 3\chi_\theta\chi_5 + (\chi_{2\theta} + 3\chi_{\mathrm{Sym}^2\theta} + 1)\chi_3 + \chi_{\mathrm{Sym}^3\theta} + \chi_{\mathrm{Sym}^2\theta}\Big)v^6 + \mathcal{O}(v^7)\,. \end{aligned} \tag{5.4.9}$$

We have checked this against the three-instanton Hilbert series for $\mathfrak{su}(2)$, G_2, $\mathfrak{so}(7)$, $\mathfrak{sp}(2)$, $\mathfrak{sp}(3)$ in (Cremonesi et al., 2014; Hanany and Kalveks, 2014), and against the three-string elliptic genus for $\mathfrak{su}(3)$ (Kim, Kim, and Park, 2016). Note the first two terms also agree with the rank three E_6 Hall-Littlewood index ((A.14) in (Gaiotto and Razamat, 2012)). For the subleading q_τ order, again except $SU(3)$, we propose

$$\begin{aligned} g^{(3)}_{1,G}(v,x,Q_{m_i}) = &(\chi_3 + \chi_\theta + 1) + (\chi_4 + (\chi_\theta+1)\chi_2)v \\ &+(\chi_5 + (3\chi_\theta+4)\chi_3 + 2\chi_{\mathrm{Sym}^2\theta} + \chi_\theta + 2)v^2 + \mathcal{O}(v^3)\,. \end{aligned} \tag{5.4.10}$$

As in rank one and two cases, for $\mathfrak{su}(3)$, the higher contributions begin to merge in at q_τ subleading order.

All above v-expansion coefficients can be easily obtained by setting $Q_m = 1$ in $g^{(k)}_{n,G}(v,x,Q_m)$. Thus the rational functions $g^{(k)}_{n,G}(v,x,1)$ are very useful as they encode most information. For large k or n, such rational functions with generic x turn out to be too length. One can take the unrefined limit $x=1$ in $g^{(k)}_{n,G}(v,x,1)$ to still store meaning information on arbitrary order coefficients of v-expansion. Indeed, when the fugacities of flavor as well as $SU(2)_x$ are turned off, we find

$$g^{(n)}_{1,G}(v)=\frac{1}{(1-v^2)^{2(h^\vee_G-1)}}\times P^{(n)}_{1,G}(v)\,, \tag{5.4.11}$$

$$g^{(n)}_{2,G}(v)=\frac{1}{(1-v^2)^{2(h^\vee_G-1)}(1+v)^{2b_G}(1+v+v^2)^{2h^\vee_G-1}}\times P^{(n)}_{2,G}(v)\,. \tag{5.4.12}$$

The exponents b_G are given by

G	$SU(3)$	$SO(8)$	F_4	E_6	E_7	E_8
b	3	6	11	16	26	46

We notice that $b=5h^\vee_G/3-4$ except for $\mathfrak{su}(3)$. The numerators $P^{(n)}_{1,G}(v)$ and $P^{(n)}_{2,G}(v)$ are palindromic Laurent polynomials. They have negative powers of v when n is large. Nevertheless $P^{(0)}_{1,G}(v),P^{(0)}_{2,G}(v)$ are both polynomials and their maximum degrees are $h^\vee_G-1$ and $2(2h^\vee_G-1)+2b_G$ respectively. The explicit expressions of $P^{(n)}_{k,G}(v)$ for the minimal SCFTs with $G=\mathfrak{su}(3),\mathfrak{so}(8),F_4,E_6,E_7,E_8$ are presented in Section 5.5 and also Appendix D.

5.4.2 Symmetric product approximation

It was noticed both in (Hanany, Mekareeya, and Razamat, 2013) and (Gaiotto and Razamat, 2012) that the reduced two G-instanton Hilbert series can be realized as certain symmetric product of two one G-instantons as approximation:

$$\frac{1}{1-vx^{\pm1}}g^{(2)}_{0,G}(v,x,a)=\frac{1}{2}\left(\left(g^{(1)}_{0,G}(v,a)\frac{1}{1-vx^{\pm1}}\right)^2+g^{(1)}_{0,G}(v^2,a^2)\frac{1}{1-v^2x^{\pm2}}\right)+\mathcal{O}(v^4)\,. \tag{5.4.13}$$

Here we adopt their notation $a=Q_{m_G}$ to lighten the notation. It also was noticed in (Cremonesi et al., 2014) that the reduced three G-instanton Hilbert series can be realized as certain symmetric product of three one G-instantons as approximation:

$$\begin{aligned}\frac{1}{1-vx^{\pm1}}g^{(3)}_{0,G}(v,x,a)=\frac{1}{6}\Bigg(&\left(\frac{1}{1-vx^{\pm1}}g^{(1)}_{0,G}(v,a)\right)^3\\ &+\frac{3}{(1-vx^{\pm1})(1-v^2x^{\pm2})}g^{(1)}_{0,G}(v,a)g^{(1)}_{0,G}(v^2,a^2)+\frac{2}{1-v^3x^{\pm3}}g^{(1)}_{0,G}(v^3,a^3)\Bigg)+\mathcal{O}(v^4)\,.\end{aligned} \tag{5.4.14}$$

The above formulas have clear physical meaning. For example in (5.4.14), the first term represents the configuration that three instantons are far from each other, the second term represents the configuration that two instantons sit on the same site and the third one are far from them, while the third term represents the configuration that all three instantons sit on the same site. Note the triple symmetric product would give the coefficient of v^4 of $g^{(3)}_{0,G}$ as $\chi_5+\chi_\theta\chi_3+\chi_{2\theta}+\chi_{\mathrm{Sym}^2\theta}+1$, one can see the difference with (5.4.9) begins to appear.

In fact, it is reasonable that arbitrary k G-instanton Hilbert series can be realized as symmetric product of k G-instantons as approximation:

$$\frac{1}{1-vx^{\pm1}} g_{0,G}^{(k)}(v,x,a) = \mathrm{Sym}^k_{\mathcal{M}_{G,1}}(v,x,a) + \mathcal{O}(v^4)\,, \tag{5.4.15}$$

where $\mathrm{Sym}^k_{G,1}(v,x,a)$ can be obtain from generating function

$$\sum_{k=1}^{\infty} \mathrm{Sym}^k_{\mathcal{M}_{G,1}}(v,x,a) Q^k = \mathrm{PE}\left[\frac{Q}{1-vx^{\pm1}} g_{0,G}^{(1)}(v,a)\right] \equiv \mathrm{PE}\left[\widetilde{g}_{0,G}^{(1)}(v,x,a)Q\right]. \tag{5.4.16}$$

For example,

$$\begin{aligned}\mathrm{Sym}^4_{\mathcal{M}_{G,1}}(v,x,a) =& \frac{1}{24}\bigg(\left(\widetilde{g}_{0,G}^{(1)}(v,x,a)\right)^4 + 6\left(\widetilde{g}_{0,G}^{(1)}(v,x,a)\right)^2 \widetilde{g}_{0,G}^{(1)}(v^2,x^2,a^2) \\ &+3\left(\widetilde{g}_{0,G}^{(1)}(v^2,x^2,a^2)\right)^2 + 8\,\widetilde{g}_{0,G}^{(1)}(v,x,a)\widetilde{g}_{0,G}^{(1)}(v^3,x^3,a^3) + 6\,\widetilde{g}_{0,G}^{(1)}(v^4,x^4,a^4)\bigg) \\ &= 1 + (\chi_3+\chi_\theta)v^2 + (\chi_4+\chi_\theta\chi_2)v^3 + \mathcal{O}(v^4)\,.\end{aligned} \tag{5.4.17}$$

It is not hard to find that for all $k \geq 3$, the leading coefficients in v expansion of symmetric product are the same:

$$\begin{aligned}\mathrm{Sym}^k_{\mathcal{M}_{G,1}}(v,x,a) =& \frac{1}{k!}\bigg(\left(\widetilde{g}_{0,G}^{(1)}(v,x,a)\right)^k + C_k^2\left(\widetilde{g}_{0,G}^{(1)}(v,x,a)\right)^{k-2} \widetilde{g}_{0,G}^{(1)}(a^2,v^2,x^2) + \dots\bigg) \\ =& 1 + (\chi_3+\chi_\theta)v^2 + (\chi_4+\chi_\theta\chi_2)v^3 + \mathcal{O}(v^4)\,.\end{aligned} \tag{5.4.18}$$

Here the first term represents all k instantons are far from each other, while the second term represents two instantons sit at the same site and the rest $k-2$ instanton are far from them and each other... From v^4, the interaction among instantons will contribute in.

We can also include $g_{0,G}^{(k)}$ into the elliptic genus to write down the above symmetric product approximation. For example in the reduced three-string elliptic genus, since

$$\mathbb{E}_{\widetilde{h}_G^{(3)}}(v,x,a,q_\tau) = \frac{v^{3h_G^\vee}}{1-vx^{\pm1}} g_{0,G}^{(3)}(v,x,a) q_\tau^{-h_G^\vee/2} + \mathcal{O}(q_\tau^{-h_G^\vee/2+1}), \tag{5.4.19}$$

combining (5.4.14), we obtain

$$\begin{aligned}\mathbb{E}_{\widetilde{h}_G^{(3)}}(v,x,a,q_\tau) = \frac{1}{6}\Big(& \mathbb{E}_{\widetilde{h}_G^{(1)}}(v,x,a,q_\tau)^3 + 3\mathbb{E}_{\widetilde{h}_G^{(1)}}(v,x,a,q_\tau)\mathbb{E}_{\widetilde{h}_G^{(1)}}(v^2,x^2,a^2,q_\tau^2) \\ &+ 2\mathbb{E}_{\widetilde{h}_G^{(1)}}(v^3,x^3,a^3,q_\tau^3)\Big) + \dots\,,\end{aligned} \tag{5.4.20}$$

which holds for the leading q_τ order and the first four v-expansion coefficients. For arbitrary k-strings elliptic genus, it is better to use Hecke transformation. Neglecting the interaction among strings, the resulting k-strings elliptic genus $\mathbb{E}_{\mathrm{Sym}_G^{(k)}}(v,x,a,q_\tau)$

can be generated from

$$\sum_{k=0} \mathbb{E}_{\mathrm{Sym}^{(k)}_G}(v,x,a,q_\tau)Q^k = \exp\left[\sum_{n\geq 0} Q^n \frac{1}{n} \sum_{\substack{cd=n\\ c,d>0}} \sum_{b(\mathrm{mod}\ d)} \mathbb{E}_{\widetilde{h}^{(1)}_G}\left(\frac{c\tau+b}{d}, c\epsilon_i, c\, m_G\right)\right]. \tag{5.4.21}$$

Note this relies on the Jacobi form nature of $\mathbb{E}_{\widetilde{h}^{(1)}_G}(\tau,\epsilon_i,m_G)$. Also take $d = 1$ in (5.4.21), one will go back to instanton formula (5.4.16) where this is no modularity. Finally, we obtain

$$\mathbb{E}_{\widetilde{h}^{(k)}_G}(v,x,a,q_\tau) = \mathbb{E}_{\mathrm{Sym}^{(k)}_G}(v,x,a,q_\tau) + \mathcal{O}(q_\tau^{-kh^\vee_G/6+1}) + \mathcal{O}(v^{kh^\vee_G+4}). \tag{5.4.22}$$

As we have checked this symmetric product approximation does not give exact subleading q_τ orders $g^{(k)}_{1,G}$ even for its leading v-expansion coefficient. This means all subleading q_τ orders involves interaction among strings.

5.4.3 Symmetries

Besides the obvious symmetry

$$\mathbb{E}_{h^{(k)}_G}(v,x,q_\tau,Q_m) = \mathbb{E}_{h^{(k)}_G}(v,x^{-1},q_\tau,Q_m), \tag{5.4.23}$$

which comes from the symmetry between ϵ_1 and ϵ_2 in Omega background, it was found in (Del Zotto and Lockhart, 2017) that the reduced one-string elliptic genus $\mathbb{E}_{h^{(1)}_G}(v,q_\tau)$ satisfy an additional symmetry

$$\mathbb{E}_{h^{(1)}_G}(q_\tau^{1/2}/v,q_\tau) = (-1)^{\mathfrak{n}+1} v^{2(1-h^\vee_G/3)} Q_\tau^{(h^\vee_G/3-1)/2} \mathbb{E}_{h^{(1)}_G}(v,q_\tau). \tag{5.4.24}$$

Here the dependence on m_G is implicit. This symmetry was later interpreted as a spectral flow symmetry in (Del Zotto and Lockhart, 2018). The left hand side of (5.4.24) actually computes the NS-R elliptic genus, which should be equal to the R-R elliptic genus on the right hand side due to the lack of chiral fermions in the minimal SCFT in consideration. See section 6.4 of (Del Zotto and Lockhart, 2017) for a detailed discussion.

We extend the symmetry (5.4.24) to arbitrary k-string elliptic genus $\mathbb{E}_{h^{(k)}_G}(v,x,q_\tau)$:

$$\mathbb{E}_{h^{(k)}_G}\left(\frac{q_\tau^{1/2}}{v},\frac{q_\tau^{1/2}}{x},q_\tau\right) = (-1)^{\mathfrak{n}k+1} v^{\frac{k(k-5)}{6}h^\vee_G+k^2+1} x^{-\frac{k(k-1)}{6}h^\vee_G-k^2+1} Q_\tau^{(kh^\vee_G-3)/6} \mathbb{E}_{h^{(k)}_G}(v,x,q_\tau), \tag{5.4.25}$$

which can be derived by the modular indices of elliptic genera. For the situation where 2d quiver description is known, i.e. $G = \mathfrak{su}(3)$ and $\mathfrak{so}(8)$, the above symmetry can also be obtained by looking into the transformation of integrand of localization with the quasi-periodicity of Jacobi theta function. Note symmetry (5.4.25) is a non-perturbative symmetry, which can not be seen from the q_τ expansion of the elliptic genus, except for the one-string case that is (5.4.24).[16] This means (5.4.25) should

[16]Practically, we find that for the two-string elliptic genus, when q_τ order is enough high, for one order of q_τ goes up, the leading v order goes down for 3. Thus, if one naively does the transformation for the left hand side of (5.4.25) in q_τ expansion, one would get infinite negative order of q_τ. Similar situation also happens for three-string elliptic genus. But for one-string elliptic genus, luckily for one

be seen as the symmetry of the chiral algebra associated to the underlying $(0,4)$ 2d CFT, as suggested in (Del Zotto and Lockhart, 2017).

5.5 Examples

In this section, we choose some of the most interesting rank one theories to explicitly show the λ_F parameter and the elliptic blowup equations. The chosen theories with gauge symmetry of classical type all have known 2d quiver theory correspondence, therefore the elliptic genera are exactly computable via Jeffrey-Kirwan residue of localization. We checked against them our results from blowup equations and found perfect agreement, mostly for one-string elliptic genera and some up to two-string. For theories with exceptional gauge symmetries, we explicitly show our computational results on the elliptic genera for most of them.[17] Sometimes to specify a theory with base curve $-n$ and gauge group G, we denote the reduced k-string elliptic genus as

$$\mathbb{E}_{h_{n,G}^{(k)}}(q,v,x,m_G,m_F) = \frac{\theta_1(\tau,\epsilon_1)\theta_1(\tau,\epsilon_2)}{\eta(\tau)^2}\mathbb{E}_k(\tau,m_G,m_F,\epsilon_1,\epsilon_2). \tag{5.5.1}$$

Recall $v = \mathrm{e}^{(\epsilon_1+\epsilon_2)/2} = \mathrm{e}^{\epsilon_+}$, $x = \mathrm{e}^{(\epsilon_1-\epsilon_2)/2}$ and reduced one-string elliptic genus does not depend on x.

We also show some interesting theta identities coming from the leading degree of vanishing blowup equations. Although we have checked the leading degree identities for all the vanishing blowup equations in Tables 5.7, 5.8, 5.9, here we only explicitly written down a small part of them, in particular with λ_F in small representations.

5.5.1 E-string theory

A typical example of non-toric local Calabi-Yau threefolds is the local half K3 where the half K3 surface can be realized as nine-point blowup of $\mathbb{P}^2$. It is well-known that the blowup surfaces $\mathfrak{B}_i(\mathbb{P}^2)$ are non-toric for $i > 3$ and not even del Pezzo for $i > 8$. Therefore it will be a strong support for the universality of the blowup equations if they can apply to local half K3.

Local half K3 Calabi-Yau can also be identified as the elliptic fibration over the total space of the bundle $\mathcal{O}(-1) \to \mathbb{P}^1$. Such geometry is described by ten parameters, in which t_b controls the size of base $\mathbb{P}^1$ and τ controls the elliptic fiber and there are eight mass parameters m_i, $i = 1,2,\ldots,8$ which give a global E_8 symmetry. For details on the geometry of local half K3, see for example (Gu et al., 2017). In physics, the topological string theory on local half K3 corresponds to the E-string theory which is the simplest 6d $(1,0)$ SCFT (Witten, 1996; Ganor and Hanany, 1996; Seiberg and Witten, 1996). In the Hořava-Witten picture of $E_8 \times E_8$ heterotic string theory, E-strings can be realized by M2-branes stretched between a M5-brane and a M9-brane. The E-string elliptic genera $\mathbb{E}_k(\tau, m_{E_8}, \epsilon_1, \epsilon_2)$ have been computed by many methods including 2d quiver gauge theories (Kim et al., 2014; Kim, Kim, and

order of q_τ goes up, the leading v order goes down for 2, which only result in finite negative order of q_τ.

[17] We usually only show the elliptic genera with gauge and flavor fugacities turned off or partially turned on. More detailed results for lots of theories can be found on the website.

Lee, 2015), refined topological vertex (Kim, Taki, and Yagi, 2015), modular ansatz (Gu et al., 2017; Duan, Gu, and Kashani-Poor, 2018), refined holomorphic anomaly equations (Huang, Klemm, and Poretschkin, 2013) and domain walls (Haghighat, Lockhart, and Vafa, 2014; Cai, Huang, and Sun, 2015). The last three approaches deeply rely on the structure of E_8 Weyl invariant Jacobi forms (Sakai, 2017; Wang, 2018; Wang, 2020).

We find there exist 240 unity blowup equations for E–string theory, corresponding to the 240 roots of flavor E_8. To be precise, for any $\alpha \in \Delta(E_8)$, we have

$$\begin{aligned}\sum_{k_1+k_2=k} &\theta_1(\tau, m_\alpha+\epsilon_1+\epsilon_2-k_1\epsilon_1-k_2\epsilon_2)\mathbb{E}_{k_1}(\tau, m+\epsilon_1\alpha,\epsilon_1,\epsilon_2-\epsilon_1)\\ &\times \mathbb{E}_{k_2}(\tau, m+\epsilon_2\alpha,\epsilon_1-\epsilon_2,\epsilon_2)\\ &=\theta_1(\tau, m_\alpha+\epsilon_1+\epsilon_2)\mathbb{E}_k(\tau,m,\epsilon_1,\epsilon_2).\end{aligned} \tag{5.5.2}$$

These equations can be checked explicitly. Using the expression of the one-string elliptic genus

$$\mathbb{E}_1(\tau,m,\epsilon_1,\epsilon_2)=\left(\frac{\Theta_{E_8}(\tau,m)}{\eta^8}\right)\frac{\eta^2}{\theta_1(\epsilon_1)\theta_1(\epsilon_2)}, \tag{5.5.3}$$

where $\Theta_{E_8}(\tau,m)$ is the theta function defined on E_8 lattice. The unity blowup equation at base degree one reads

$$\frac{\theta_1(m_\alpha+\epsilon_2)\Theta_{E_8}(m+\epsilon_1\alpha)}{\theta_1(\epsilon_1)\theta_1(\epsilon_2-\epsilon_1)}+\frac{\theta_1(m_\alpha+\epsilon_1)\Theta_{E_8}(m+\epsilon_2\alpha)}{\theta_1(\epsilon_1-\epsilon_2)\theta_1(\epsilon_2)}=\frac{\theta_1(m_\alpha+\epsilon_1+\epsilon_2)\Theta_{E_8}(m)}{\theta_1(\epsilon_1)\theta_1(\epsilon_2)}, \tag{5.5.4}$$

which we have verified to very high orders of q_τ. Besides, we also verified the unity blowup equation at base degree $k=2$.

On the other hand, there is one unique vanishing blowup equation

$$\sum_{k_1+k_2=k}\theta_1(\tau,k_1\epsilon_1+k_2\epsilon_2)\mathbb{E}_{k_1}(\tau,m,\epsilon_1,\epsilon_2-\epsilon_1)\mathbb{E}_{k_2}(\tau,m,\epsilon_1-\epsilon_2,\epsilon_2)=0. \tag{5.5.5}$$

We have verified this equation up to base degree $k=3$ for high orders of q_τ. Since there is no shift for the E_8 parameters, it is easy to see the above equation is also the vanishing blowup equations for massless E–string theory, or in other word massless half-K3 Calabi-Yau. In fact, it is the unique blowup equation for massless E–string. Another form of vanishing blowup equation was also obtained in (Gu et al., 2017).

Weyl orbit expansion

In this section, we show how to solve the one E-string elliptic genus from blowup equations by Weyl orbit expansion. The reduced one E–string elliptic genus is well-known to be

$$\begin{aligned}\mathbb{E}_1^{\mathrm{red}}&=\frac{\eta^2}{\theta_1(\epsilon_1)\theta_1(\epsilon_2)}\mathbb{E}_1=\eta^{-8}\Theta_{E_8}(m)=\eta^{-8}\sum_{\mathcal{O}_{p,k}}q^{p/2}\cdot\mathcal{O}_{p,k}\\ &=q^{-\frac{1}{3}}(1+\mathbf{248}q+(\mathbf{3875}+\mathbf{248}+1)q^2+(\mathbf{30380}+\mathbf{3875}+2\times\mathbf{248}+1)q^3\\ &\quad+(\mathbf{147250}+2\times\mathbf{30380}+\mathbf{3875}+5\times\mathbf{248}+1)q^4+\dots)\end{aligned} \tag{5.5.6}$$

where

$$\Theta_{E_8}(\tau, m) = \sum_{k \in \Gamma_{E_8}} \exp(\pi \mathrm{i} \tau k \cdot k + 2\pi \mathrm{i} m \cdot k) = \frac{1}{2} \sum_{k=1}^{4} \prod_{\ell=1}^{8} \theta_k(\tau, m_\ell). \tag{5.5.7}$$

The first few E_8 Weyl orbits are as follows:

$$\begin{aligned}
&\mathcal{O}_{0,1}, \quad \mathcal{O}_{2,240}, \quad \mathcal{O}_{4,2160}, \quad \mathcal{O}_{6,6720}, \quad \mathcal{O}_{8,240}, \mathcal{O}_{8,17280}, \quad \mathcal{O}_{10,30240} \\
&\mathcal{O}_{12,60480}, \quad \mathcal{O}_{14,13440}, \mathcal{O}_{14,69120}, \quad \mathcal{O}_{16,2160}, \mathcal{O}_{16,138240}, \quad \mathcal{O}_{18,240}, \mathcal{O}_{18,181440}, \\
&\mathcal{O}_{20,30240}, \mathcal{O}_{20,241920}, \quad \mathcal{O}_{22,138240}, \mathcal{O}_{22,181440}, \quad \mathcal{O}_{24,6720}, \mathcal{O}_{24,483840}, \\
&\mathcal{O}_{26,13440}, \mathcal{O}_{26,30240}, \mathcal{O}_{26,483840}, \quad \cdots .
\end{aligned} \tag{5.5.8}$$

In unity blowup equations, each Weyl orbit breaks down due to the shifts proportional to a root. For example, for $\mathcal{O}_{2,240}$,

$$\sum_{w \in \mathcal{O}_{2,240}} \mathrm{e}^{w \cdot (m + \epsilon_1 \alpha)} = q_1^{-2} \mathrm{e}^{-\alpha \cdot m} + \sum_{\alpha \cdot w = -1} q_1^{-1} \mathrm{e}^{w \cdot m} + \sum_{\alpha \cdot w = 0} \mathrm{e}^{w \cdot m} + \sum_{\alpha \cdot w = 1} q_1 \mathrm{e}^{w \cdot m} + q_1^2 \mathrm{e}^{\alpha \cdot m}. \tag{5.5.9}$$

This forces us to look into how every Weyl orbit splits under the shift of a root. Due to the Weyl symmetry, all the elements in one Weyl orbit intersect with any of the roots in the same way, i.e. for any root the distribution of intersection numbers $R = \alpha \cdot w$ between the root and all Weyl orbit elements is the same. For example, for any positive root α, we list the distribution for some Weyl orbits in Table 5.13. Note the elements are all Weyl orbits of E_7. Knowing (5.5.6), it is easy to check the unity blowup equations (5.5.2) are correct.

$R = \alpha \cdot w$	-5	-4	-3	-2	-1	0	1	2	3	4	5
$\mathcal{O}_{0,1}$						1					
$\mathcal{O}_{2,240}$				1	56	126	56	1			
$\mathcal{O}_{4,2160}$				126	576	756	576	126			
$\mathcal{O}_{6,6720}$			56	756	1512	2072	1512	756	56		
$\mathcal{O}_{8,240}$		1		56		126		56		1	
$\mathcal{O}_{8,17280}$			576	2016	4032	4032	4032	2016	576		
$\mathcal{O}_{10,30240}$		126	1512	4158	5544	7560	5544	4158	1512	126	
$\mathcal{O}_{12,60480}$		756	4032	7560	12096	11592	12096	7560	4032	756	
$\mathcal{O}_{14,13440}$	56	56	1512	1512	1568	4032	1568	1512	1512	56	56
$\mathcal{O}_{14,69120}$		2016	4032	10080	12096	12672	12096	10080	4032	2016	
$\mathcal{O}_{16,2160}$		126		576		756		576		126	
$\mathcal{O}_{16,138240}$	576	4032	12096	16128	24192	24192	24192	16128	12096	4032	576

Table 5.13: Intersection numbers between roots and elements of E_8 Weyl orbits.

Conversely it is possible to solve (5.5.6) from the blowup equations. Let us first write $\mathbb{E}_1^{\mathrm{red}} = f(q, v, m)/\eta^8$.[18] The vanishing blowup equation (5.5.5) gives

$$f(q, \frac{\epsilon_1}{2}, m) = f(q, \frac{\epsilon_2}{2}, m). \tag{5.5.10}$$

[18] The denominator η^8 can be later determined by requiring that $\mathbb{E}_1^{\mathrm{red}}$ can be decomposed as representations of E_8, rather than just Weyl orbits. Besides, there is an overall constant in front of the whole elliptic genus $\mathbb{E}_1$ that can not determined by blowup equations due to the lack of gauge symmetry. This is of course not surprising. Here we assume the overall constant is 1.

Thus $f(q,v,m)$ is independent from v. We can simply write it as

$$f(q,m)=\sum_{n=0}^{\infty}q^n\sum_{\mathcal{O}_{p,k}}x_{n,p,k}\mathcal{O}_{p,k}. \tag{5.5.11}$$

The task is to determine all $x_{n,p,k}$. It is convenient to write the unity blowup equations as

$$\begin{aligned}&\theta_1(\epsilon_2)\theta_1(\alpha\cdot m+\epsilon_2)f(q,m+\epsilon_1\alpha)-\theta_1(\epsilon_1)\theta_1(\alpha\cdot m+\epsilon_1)f(q,m+\epsilon_2\alpha)\\&\quad=\theta_1(\epsilon_2-\epsilon_1)\theta_1(\alpha\cdot m+\epsilon_1+\epsilon_2)f(q,m),\end{aligned} \tag{5.5.12}$$

where $\alpha\in\Delta(E_8)$. We conjecture the solution is uniquely $f(\tau,m)=\eta^{-8}\Theta_{E_8}(\tau,m)$ under the conditions:

- The q expansion coefficients of $\mathbb{E}_1$ can be decomposed as sums of irreducible representations of E_8;
- The leading q expansion coefficient is 1, i.e. the trivial E_8 orbit $\mathcal{O}_{0,1}$.

Note the blowup equations themselves only determine $f(q,m)$ up to a free function of τ. The two assumptions assure the prefactor is η^{-8}.[19] In fact, it is proved by Don Zagier that (5.5.12) has a unique solution which is the E_8 theta function up to a free function of τ, and similar statements can be made for arbitrary positive definite even unimodular lattices generated by roots, such as the $E_8\times E_8$ lattice and the Barnes-Wall lattice Λ_{16} in dimension 16 and the 23 Niemeier lattices in dimension 24. We give the proof in Appendix C.

Now we briefly show how the Weyl orbit recursion works. Given that we have assumed the leading q order of $f(q,m)$ is the trivial orbit $\mathcal{O}_{0,1}$, we find that in order for the subleading order of (5.5.12) to be satisfied, the subleading order of $f(q,m)$ should have two $\mathcal{O}_{0,1}$ with $R=\pm2$. Looking up in Table 5.13, one finds that in order to store the E_8 symmetry, one has to add two E_7 orbits of length 56 at $R=\pm1$ and one E_7 orbit of length 126 at $R=0$. Thus in the subleading order, $f(q,m)$ has the E_8 orbit $\mathcal{O}_{2,240}$. Next, for the subsubleading order of (5.5.12) to be satisfied, one needs to add two E_7 orbits of length 126 at $R=\pm2$ in the subsubleading order of $f(q,m)$ to cancel the effect of the previous E_7 orbit of length 56. Then to restore the E_8 symmetry, one needs to add two E_7 orbits of length 576 at $R=\pm1$ and one E_7 orbit of length 756 at $R=0$. Repeating this process, we find each sub E_7 Weyl orbit in Table 5.13 is in an infinite series of the ones in the larger Weyl orbits. Moreover, the contributions from each infinite series can be organized into one of the following two identities:

$$\begin{aligned}&\theta_1(\epsilon_2)\theta_1(\lambda+\epsilon_2)\theta_j(2\tau,\lambda+2\epsilon_1)-\theta_1(\epsilon_1)\theta_1(\lambda+\epsilon_1)\theta_j(2\tau,\lambda+2\epsilon_2)\\&\quad=\theta_1(\epsilon_2-\epsilon_1)\theta_1(\lambda+\epsilon_1+\epsilon_2)\theta_j(2\tau,\lambda),\qquad j=2,3.\end{aligned} \tag{5.5.13}$$

With this in mind, one can directly write down the Weyl orbit expansion satisfying the unity blowup equations by the following rule: Each sub E_7 orbit in an E_8 Weyl orbit $\mathcal{O}_{p,k}$ with intersection number R generates an infinite series of sub E_7 orbits in E_8 Weyl orbits $\mathcal{O}_{p',k'}$ with intersection numbers R' where p' grows quadratically and $|R'|$ grows linearly. To be more explicit,

[19] This agrees with the refined BPS invariants of local half K3 Calabi-Yau.

- If R is even, the growth is based on $\theta_3(2\tau,2z)$, i.e. p' increases by $2n^2$ and $|R'|$ increases by $2n$;
- If R is odd, the growth is based on $\theta_2(2\tau,2z)$, i.e. p' increases by $2n(n+1)$ and $|R'|$ increases by $2n$.

We have marked some sub E_7 orbits in the same series with the same color in Table 5.13. It turns out all E_8 Weyl orbits appear and just appear once in $f(q,m)$, which means it is indeed the E_8 theta function.

5.5.2 M-string theory

Following the physical picture of E–strings in the last section, the M-strings can be realized by M2-branes stretched between two M5-branes (Haghighat et al., 2015a). The M-string elliptic genera $\mathbb{E}_k(\tau,m,\epsilon_1,\epsilon_2)$ have been computed by many methods. For example, the following nice formulas were given in (Haghighat et al., 2015a) as

$$\mathbb{E}_k(\tau,m,\epsilon_1,\epsilon_2)=\sum_{|\nu|=k}\prod_{(i,j)\in\nu}\frac{\theta_1(z_{ij})\theta_1(v_{ij})}{\theta_1(w_{ij})\theta_1(u_{ij})},\tag{5.5.14}$$

where

$$\begin{aligned}
z_{ij}&=-m+(\nu_i-j+1/2)\epsilon_1+(i-1/2)\epsilon_2,\\
v_{ij}&=-m-(\nu_i-j+1/2)\epsilon_1-(i-1/2)\epsilon_2,\\
w_{ij}&=(\nu_i-j+1)\epsilon_1-(\nu_j^t-i)\epsilon_2,\\
u_{ij}&=(\nu_i-j)\epsilon_1-(\nu_j^t-i+1)\epsilon_2.
\end{aligned}\tag{5.5.15}$$

In particular, the one M–string elliptic genus is (Haghighat et al., 2015a)

$$\mathbb{E}_1(\tau,m,\epsilon_1,\epsilon_2)=\frac{\theta_1(\frac{1}{2}(\epsilon_1+\epsilon_2)+m)\theta_1(\frac{1}{2}(\epsilon_1+\epsilon_2)-m)}{\theta_1(\epsilon_1)\theta_1(\epsilon_2)}.\tag{5.5.16}$$

We find M–string theory has four unity blowup equations and no vanishing blowup equation. The unity blowup equations can be written as

$$\sum_{k_1+k_2=k}\theta_3^{[a]}(2\tau,2(\pm\frac{m}{2}+\frac{\epsilon_1+\epsilon_2}{4}-k_1\epsilon_1-k_2\epsilon_2))\tag{5.5.17}$$

$$\times\,\mathbb{E}_{k_1}(\tau,m\pm\frac{\epsilon_1}{2},\epsilon_1,\epsilon_2-\epsilon_1)\mathbb{E}_{k_2}(\tau,m\pm\frac{\epsilon_2}{2},\epsilon_1-\epsilon_2,\epsilon_2)$$

$$=\theta_3^{[a]}(2\tau,2(\pm\frac{m}{2}+\frac{\epsilon_1+\epsilon_2}{4}))\mathbb{E}_k(\tau,m,\epsilon_1,\epsilon_2),\qquad a=0,-1/2.\tag{5.5.18}$$

Note $\theta_3^{[-1/2]}$ is just the Jacobi theta function θ_2. These equations can be checked very explicitly. Substituting (5.5.16) into (5.5.18), we find the unity blowup equations at base degree one is equivalent to

$$\frac{\theta_3^{[a]}(2\tau,\mp m+(3\epsilon_1-\epsilon_2)/2)\theta_1(\epsilon_2/2+(m\pm\epsilon_1/2))\theta_1(\epsilon_2/2-(m\pm\epsilon_1/2))}{\theta_1(\epsilon_1)\theta_1(\epsilon_2-\epsilon_1)}$$

$$+\frac{\theta_3^{[a]}(2\tau,\mp m+(-\epsilon_1+3\epsilon_2)/2)\theta_1(\epsilon_1/2+(m\pm\epsilon_2/2))\theta_1(\epsilon_1/2-(m\pm\epsilon_2/2))}{\theta_1(\epsilon_1-\epsilon_2)\theta_1(\epsilon_2)}$$

$$= \frac{\theta_3^{[a]}(2\tau, \mp m - (\epsilon_1+\epsilon_2)/2)\theta_1((\epsilon_1+\epsilon_2)/2+m)\theta_1((\epsilon_1+\epsilon_2)/2-m)}{\theta_1(\epsilon_1)\theta_1(\epsilon_2)}, \tag{5.5.19}$$

which we have checked up to q_τ^{10}. Using (5.5.14), we have also checked the unity blowup equations of base degrees two and three up to q_τ^{10}.

5.5.3 $n = 1$ $\mathfrak{sp}(N)$ theories

The $n = 1$ $G = \mathfrak{sp}(N)$ theories have $8+2N$ fundamental hypermultiplets and flavor symmetry $\mathfrak{so}(16+4N)$. For $N = 0$, it specializes to E-string theory, with flavor symmetry $\mathfrak{so}(16)$ enhanced to E_8. The 2d quiver description for these theories was proposed in (Kim, Kim, and Lee, 2015; Yun, 2016), therefore their elliptic genera can be exactly computed from localization. For example, the reduced one-string elliptic genus can be computed as

$$\mathbb{E}_1^{\mathrm{red}}(v,q,m_G,m_F) = \frac{1}{2}\sum_{j=1,2,3,4}\left(\prod_{i=1}^{8+2N}\frac{\theta_j(m_F^i)}{\eta}\right)\left(\prod_{i=1}^{N}\frac{\eta^2}{\theta_j(\epsilon_+ + m_G^i)\theta_j(\epsilon_+ - m_G^i)}\right). \tag{5.5.20}$$

The index of d-string elliptic genus of $\mathfrak{sp}(N)$ theory is known to be

$$\mathrm{Ind}_{\mathbb{E}_d} = -\frac{N+2}{4}(\epsilon_1+\epsilon_2)^2 d + \epsilon_1\epsilon_2\frac{d^2+d}{2} - \frac{d}{2}(m,m)_{\mathfrak{sp}(N)} + \frac{d}{2}(m,m)_{\mathfrak{so}(16+4N)}. \tag{5.5.21}$$

Let us first discuss the vanishing blowup equations. Since for C type Lie algebra $(P^\vee/Q^\vee)_{C_n} \cong \mathbb{Z}_2$, there could exist one vanishing equation when the paramter λ_F and the characteristic a are fixed with λ_G taking value in $(P^\vee\backslash Q^\vee)_{C_n}$. Denote the smallest Weyl orbit in $(P^\vee\backslash Q^\vee)_{C_n}$ as $\mathcal{O}_{\min}$, which is just $\mathcal{O}^{\mathfrak{sp}(N)}_{[00\cdots01]}$. Note $|\mathcal{O}_{\min}| = 2^N$. Then for $N \geq 2$, the leading base degree of the vanishing blowup equations with $\lambda_F = 0$ can be universally written as

$$\sum_{w\in\mathcal{O}_{\min}}(-1)^{|w|}\theta_1(\tau,m_w)\times\prod_{\beta\in\Delta(C_N)}^{w\cdot\beta=1}\frac{1}{\theta_1(\tau,m_\beta)} = 0, \qquad N\geq 2. \tag{5.5.22}$$

We have checked this identity up to $\mathcal{O}(q^{20})$ for $N = 2,3,4,5$. Note there are $(N+1)(N+2)/2$ Jacobi θ_1 functions in the denominator.

The $G = \mathfrak{sp}(1), F = \mathfrak{so}(20)$ case is peculiar due to the Lie algebra isomorphism $C_1 \cong A_1$. In fact, it is easy to check (5.5.22) *does not* hold for $N = 1$. The correct λ_F in this case belongs to vector representation $\mathbf{20_v}$ of $\mathfrak{so}(20)$. The leading base degree of the vanishing blowup equations turn out to be the following trivial identity

$$\theta_1(m_w + \lambda_F\cdot m_F + \epsilon_+)\theta_1(-m_w + \lambda_F\cdot m_F + \epsilon_+) - (w\to -w) = 0. \tag{5.5.23}$$

Here w is the fundamental weight of $\mathfrak{sp}(1)$, the first θ_1 comes from the contribution of perturbative part, the second θ_1 comes from the contribution of hypermultiplet and we have factored out the contribution from vector multiplet.

Now let us turn to unity blowup equations. All the unity λ_F fields are just the weights of the spinor representation $S = [0,0,\dots,0,1]$ of $\mathfrak{so}(16+4N)$. There are 2^{7+2N} of them. The matters are in representation $([1,0,\dots,0,0],[1,0,\dots,0,0])$ of $\mathfrak{sp}(N)\times\mathfrak{so}(16+4N)$, i.e. $(\mathbf{2N}, \mathbf{16+4N})$. The following fact about the weight system of S is crucial for blowup equations to hold: $\forall w \in [0,0,\dots,0,1]$, there are precisely $8+2N$ weights $w' \in \mathbf{16}+\mathbf{4N}$ such that $w\cdot w' = 1/2$, and the rest $8+2N$ weights $w' \in \mathbf{16}+\mathbf{4N}$ are such that $w\cdot w' = -1/2$. Besides, $w\cdot w = 2+N/2$. Note the conjugate spinor representation $C=[0,0,\dots,1,0]$ of $\mathfrak{so}(16+4N)$ if serving as λ_F is not correct, although they satisfy the modularity of unity blowup equations!

The unity elliptic blowup equations for $G=\mathfrak{sp}(N)$ theory with $\lambda_F=\lambda\in S_{\mathfrak{so}(16+4N)}$ can be written as

$$\begin{aligned}
&\sum_{d_0=\frac{1}{2}||\alpha^\vee||_{\mathfrak{sp}(N)}}^{d_0+d_1+d_2=d}(-1)^{|\alpha^\vee|}\\
&\quad\times\theta_1\Big(\tau,-\alpha^\vee\cdot m_{\mathfrak{sp}(N)}+\lambda\cdot m_{\mathfrak{so}(16+4N)}+\Big(\frac{N+2}{2}-d_0\Big)(\epsilon_1+\epsilon_2)-d_1\epsilon_1-d_2\epsilon_2\Big)\\
&\quad\times A_V^{\mathfrak{sp}(N)}(\alpha^\vee,\tau,m_{\mathfrak{sp}(N)})A_H^{\frac{1}{2}(\mathbf{2N},\mathbf{4N+16})}(\alpha^\vee,\tau,m_{\mathfrak{sp}(N)},m_{\mathfrak{so}(16+4N)},\lambda)\\
&\quad\times\mathbb{E}_{d_1}(\tau,m_{\mathfrak{sp}(N)}+\epsilon_1\alpha^\vee,m_{\mathfrak{so}(16+4N)}+\epsilon_1\lambda,\epsilon_1,\epsilon_2-\epsilon_1)\\
&\quad\times\mathbb{E}_{d_2}(\tau,m_{\mathfrak{sp}(N)}+\epsilon_2\alpha^\vee,m_{\mathfrak{so}(16+4N)}+\epsilon_2\lambda,\epsilon_1-\epsilon_2,\epsilon_2)\\
&=\theta_1\Big(\tau,\lambda\cdot m_{\mathfrak{so}(16+4N)}+\frac{N+2}{2}(\epsilon_1+\epsilon_2)\Big)\,\mathbb{E}_d(\tau,m_{\mathfrak{sp}(N)},m_{\mathfrak{so}(16+4N)},\epsilon_1,\epsilon_2).
\end{aligned}\tag{5.5.24}$$

For $N=0$ case, there is no summation over coroots, and the above equation goes back to the unity blowup equations of E-strings (5.5.2). For $N=1$ case, using the 2d quiver formula for one-string elliptic genus (5.5.20), we have verified the unity blowup equations for all possible λ_F up to $\mathcal{O}(q_\tau^{10})$.

Let us have a closer look at the $N=1$ case. From the 2d quiver formula (5.5.20), it is easy to find the following expansion

$$\begin{aligned}
\mathbb{E}_{h^{(1)}_{1,\mathfrak{sp}(1)}}=&\,q^{-1/3}+\Big(\frac{\chi^{\mathfrak{sp}(1)}_{(2)}}{v^2}-\frac{\chi^{\mathfrak{sp}(1)}_{(1)}\cdot\mathbf{20}_{\rm v}}{v}+\chi^{\mathfrak{sp}(1)}_{(2)}+1+\mathbf{190}-\chi^{\mathfrak{sp}(1)}_{(1)}\cdot\mathbf{20}_{\rm v}v+\chi^{\mathfrak{sp}(1)}_{(2)}v^2\\
&+\sum_{n=0}^{\infty}\Big[\chi^{\mathfrak{sp}(1)}_{(2n)}\cdot\mathbf{512}_{\rm s}\,v^{1+2n}-\chi^{\mathfrak{sp}(1)}_{(1+2n)}\cdot\mathbf{512}_{\rm c}\,v^{2+2n}\Big]\Big)q^{2/3}+\mathcal{O}(q^{5/3}).
\end{aligned}\tag{5.5.25}$$

Here the bold numbers are the representations for flavor symmetry $\mathfrak{so}(20)$ or its character based on the context. We can also check the unity blowup equation using this expansion or use the Weyl orbit expansion method to solve elliptic genus in this form from unity blowup equation (5.5.24). We summarize some useful information on the intersection distribution relevant to Weyl orbit expansion in Table 5.14. Combining equation (5.5.25) and Table 5.14, one can already understand why weights in the spinor representation $\mathbf{512_s}$ can serve as λ_F fields while weights in the conjugate spinor representation $\mathbf{512_c}$ cannot. This is because in (5.5.25) the coefficients of each v^n should have λ_F shifts all even or all odd to preserve the B field condition.

Similarly, from (5.5.20), the reduced one-string elliptic genus of $G=\mathfrak{sp}(2), F=\mathfrak{so}(24)$ theory has the following expansion

$$\mathbb{E}_{h^{(1)}_{1,\mathfrak{sp}(2)}}=q^{-1/3}+\Big(\frac{\chi^{\mathfrak{sp}(2)}_{(20)}}{v^2}-\frac{\chi^{\mathfrak{sp}(2)}_{(10)}\cdot\mathbf{24}_{\rm v}}{v}+\chi^{\mathfrak{sp}(2)}_{(20)}+1+\mathbf{276}-\chi^{\mathfrak{sp}(2)}_{(10)}\cdot\mathbf{24}_{\rm v}v+\chi^{\mathfrak{sp}(2)}_{(20)}v^2$$

λ	w	$-5/2$	-2	$-3/2$	-1	$-1/2$	0	$1/2$	1	$3/2$	2	$5/2$
$\mathbf{512_c}$	$\mathbf{512_s}$		10		120		252		120		10	
$\mathbf{512_c}$	$\mathbf{20_v}$					10		10				
$\mathbf{512_c}$	$\mathbf{512_c}$	1		45		210		210		45		1
$\mathbf{512_s}$	$\mathbf{512_s}$	1		45		210		210		45		1
$\mathbf{512_s}$	$\mathbf{20_v}$					10		10				
$\mathbf{512_s}$	$\mathbf{512_c}$		10		120		252		120		10	

Table 5.14: For any λ in a fixed reprentation, the numbers of weights w in another representation with $\lambda \cdot w$ equal to a given number.

$$+\sum_{n=0}^{\infty}\left[\chi^{\mathfrak{sp}(2)}_{(2n,0)}\cdot \mathbf{2048}_{\mathrm{s}}\, v^{2+2n}-\chi^{\mathfrak{sp}(2)}_{(1+2n,0)}\cdot \mathbf{2048}_{\mathrm{c}}\, v^{3+2n}\right]\Big)q^{2/3}+\mathcal{O}(q^{5/3}), \tag{5.5.26}$$

which we also reconfirm by solving the unity blowup equations in Weyl orbit expansion with the fugacity of one subalgebra $\mathfrak{so}(3)$ of the flavor symmetry turned on.

5.5.4 $n = 1$ $\mathfrak{su}(N)$ theories

All $n = 1$ $\mathfrak{su}(N)$ theories with $N \geq 2$ have known universal 2d quiver gauge constructions in (Kim, Kim, and Lee, 2015), therefore the elliptic genera are exactly computable via Jeffrey-Kirwan residue. For example, the reduced one-string elliptic genus can be universally written as

$$\begin{aligned}&-\sum_{i=1}^{N}\frac{\prod_{j=1}^{N+8}\theta_1(m_i-\epsilon_+-\mu_j)}{\eta^8\theta_1(2m_i-3\epsilon_++\mu_{N+9})}\prod_{1\leq j\leq N}^{j\neq i}\frac{\theta_1(m_i+m_j-\epsilon_++\mu_{N+9})}{\theta_1(m_i-m_j)\theta_1(2\epsilon_+-(m_i-m_j))}\\&-\frac{1}{2\eta^8}\left(\frac{\prod_{j=1}^{N+8}\theta_1(\frac{\epsilon_+-\mu_{N+9}}{2}-\mu_j)}{\prod_{i=1}^{N}\theta_1(\frac{3\epsilon_+-\mu_{N+9}}{2}-m_i)}+(-1)^N\sum_{k=2}^{4}\frac{\prod_{j=1}^{N+8}\theta_k(\frac{\epsilon_+-\mu_{N+9}}{2}-\mu_j)}{\prod_{i=1}^{N}\theta_k(\frac{3\epsilon_+-\mu_{N+9}}{2}-m_i)}\right).\end{aligned} \tag{5.5.27}$$

Here $m_i, i = 1,2,\dots,N$ are the symmetric $\mathfrak{su}(N)$ fugacities and $\mu_j, j = 1,2,\dots,N+9$ are the symmetric $\mathfrak{su}(N+9)$ fugacities. Note this formula is from UV 2d gauge theory, where the IR global symmetry i.e. the true flavor symmetry does not manifest itself. One can convert μ_j into the fugacities of the true flavor symmetry according to the matter representations. Note all these theories are on one single branch of the E-string Higgsing tree, which is also easy to see from the above elliptic genus formula. Besides, for the $N = 2$ case, the flavor symmetry is enhanced to $\mathfrak{so}(20)$ which is just the $n = 1, G = \mathfrak{sp}(1)$ theory we have discussed in the last subsection.

One additional case is the $G = \mathfrak{su}(6)_*, F = \mathfrak{su}(15)$ theory, where there is a half hypermultiplet in the 3-antisymmetric representation $\Lambda^3_{\mathfrak{su}(6)} = \mathbf{15}$. This theory does not have known 2d quiver gauge construction, but has a brane web construction, thus the topological string partition function can be computed by refined topological vertex (Hayashi et al., 2019b). Due to the presence of half hypermultiplet, this theory does not have unity blowup equation. This is the single case with $\mathfrak{su}$ gauge symmetry where we could not solve elliptic genera from blowup equations.

Let us first discuss the vanishing blowup equations. We have shown some leading degree vanishing identities for $G = \mathfrak{su}(3), F = \mathfrak{su}(15)$ theory in (5.2.25) and

(5.2.26) with $\lambda_G = \mathbf{3}$. In fact, the vanishing theta identity (5.2.26) can be generalized to all $N \geq 2$:

$$\sum_{i=1}^{N} \frac{\theta_1(-a_i + \sum_{k=1}^{N-1} x_k) \prod_{k=1}^{N-1} \theta_1(a_i + x_k)}{\prod_{1\leq j\leq N}^{j\neq i} \theta_1(a_i - a_j)} = 0, \qquad \text{for } \sum_{i=1}^{N} a_i = 0. \tag{5.5.28}$$

These identities come from the leading base degree of vanishing blowup equations for $G = \mathfrak{su}(N)$ theories with $\lambda_G = \omega_1 \in \mathbf{N}$, i.e. the first fundamental weight that induces the fundamental representation. Similarly, for $\lambda_G = \omega_2 \in \Lambda^2$ i.e. the second fundamental weight that induces the anti-symmetric representation, we find the leading degree of vanishing blowup equations result in the following identities for arbitrary $N \geq 4$,

$$\sum_{1\leq i<j\leq N} \frac{\theta_1(-a_i - a_j + y + \sum_{k=1}^{N-4} x_k)\theta_1(a_i + a_j + y) \prod_{k=1}^{N-4} \theta_1(a_i + x_k)\theta_1(a_j + x_k)}{\prod_{1\leq k\leq N}^{k\neq i,j} \theta_1(a_i - a_k)\theta_1(a_j - a_k)} = 0. \tag{5.5.29}$$

More generally, for $\lambda_G = \omega_k$, we find the leading degree of vanishing blowup equations result in the following identities for arbitrary $N \geq 3k-2$,

$$\begin{aligned}\sum_{1\leq i_1<i_2<\cdots<i_k\leq N} &\frac{\theta_1(-\sum_{s=1}^{k} a_{i_s} + (k-1)y + \sum_{h=1}^{N-3k+2} x_h) \prod_{1\leq s<s'\leq k} \theta_1(a_{i_s} + a_{i_{s'}} + y)}{\prod_{1\leq l\leq N}^{l\neq i_1,\ldots,i_k} \prod_{s=1}^{k} \theta_1(a_{i_s} - a_l)} \\ &\times \prod_{h=1}^{N-3k+2} \prod_{s=1}^{k} \theta_1(a_{i_s} + x_h) = 0.\end{aligned} \tag{5.5.30}$$

Here still $\sum_{i=1}^{N} a_i = 0$ and y and x_k are arbitrary numbers. We have checked this identity for many different (N,k) up to very high order of q. Note the second line in (5.5.30) comes from the contribution of hypermultiplets in $(\mathbf{N}, \overline{\mathbf{N}+\mathbf{8}})_{-N+4}$, while the product $\prod_{1\leq s<s'\leq k} \theta_1(a_{i_s} + a_{i_{s'}} + y)$ in the first line comes from the contribution of hypermultiplets in $(\mathbf{\Lambda}^2, \mathbf{1})_{N+8}$. Besides, we also find the following identity for arbitrary $N \geq 1$:

$$\sum_{1\leq i_1<i_2<\cdots<i_N\leq 2N} \frac{\theta_1(-\sum_{s=1}^{N} a_{i_s} + \epsilon_+)\theta_1(\sum_{s=1}^{N} a_{i_s} + \epsilon_+)}{\prod_{1\leq l\leq 2N}^{l\neq i_1,\ldots,i_N} \prod_{s=1}^{N} \theta_1(a_{i_s} - a_l)} = 0, \qquad \text{for } \sum_{i=1}^{2N} a_i = 0. \tag{5.5.31}$$

This identity is related to the situation where matters are in the middle representation of gauge group such as the $\mathfrak{su}(6)_*$ theory with 3-antisymmetric representation. For example, taking $N = 2$ it gives the leading base degree of vanishing blowup equation of $n = 1, G = \mathfrak{su}(4), F = \mathfrak{su}(12) \times \mathfrak{su}(2)$ theory with $(\lambda_G, \lambda_F) = (\mathbf{6}, \mathbf{1})$, and taking $N = 3$ gives the one of $n = 1, G = \mathfrak{su}(6)_*, F = \mathfrak{su}(15)$ theory with $(\lambda_G, \lambda_F) = (\mathbf{20}, \mathbf{1})$.

Now we turn to the unity blowup equations for all $\mathfrak{su}(N)$ theories with $N \geq 3$. Since flavor enhancement does not matter here, let us use the symmetric $\mathfrak{su}(N+9)$ fugacities $\mu_j, j = 1, 2, \ldots, N+9$ in (5.5.27) to make the form of blowup equations universal. Consider the flavor decomposition $\mathfrak{su}(N+8) \oplus \mathfrak{u}(1) \subset \mathfrak{su}(N+9)$ according to

$$(\nu_1 + \nu_0, \nu_2 + \nu_0, \ldots, \nu_{N+8} + \nu_0, -(N+8)\nu_0) \tag{5.5.32}$$

such that $\nu_j, j = 1,2,\dots N+8$ are the symmetric $\mathfrak{su}(N+8)$ fugacities and ν_0 is the $\mathfrak{u}(1)$ fugacity. Then the unity r fields have two possibilities

$$\lambda = (\lambda^{\mathfrak{su}(N+8)}, \lambda^{\mathfrak{u}(1)}) = \Big(\omega_6, -\frac{1}{2(N+8)}\Big) \text{ or } \Big(\omega_{N+2}, \frac{1}{2(N+8)}\Big). \tag{5.5.33}$$

The unity blowup equations can be universally written as

$$\begin{aligned}
&\sum_{d_0=\frac{1}{2}||\alpha^\vee||_{\mathfrak{su}(N)}}^{d_0+d_1+d_2=d} (-1)^{|\alpha^\vee|}\\
&\times \theta_1\Big(\tau, -\alpha^\vee\cdot m_{\mathfrak{su}(N)} + \lambda\cdot\mu_{\mathfrak{su}(N+9)} + \Big(\frac{N+1}{2} - d_0\Big)(\epsilon_1+\epsilon_2) - d_1\epsilon_1 - d_2\epsilon_2\Big)\\
&\quad\times A_V^{\mathfrak{su}(N)}(\alpha^\vee,\tau,m_{\mathfrak{su}(N)})A_H^{\mathfrak{R}}(\alpha^\vee,\tau,m_{\mathfrak{su}(N)},\mu_{\mathfrak{su}(N+9)},\lambda)\\
&\quad\times \mathbb{E}_{d_1}(\tau, m_{\mathfrak{su}(N)}+\epsilon_1\alpha^\vee, \mu_{\mathfrak{su}(N+9)}+\epsilon_1\lambda, \epsilon_1, \epsilon_2-\epsilon_1)\\
&\quad\times \mathbb{E}_{d_2}(\tau, m_{\mathfrak{su}(N)}+\epsilon_2\alpha^\vee, \mu_{\mathfrak{su}(N+9)}+\epsilon_2\lambda, \epsilon_1-\epsilon_2, \epsilon_2)\\
=\ &\theta_1\Big(\tau, \lambda\cdot\mu_{\mathfrak{su}(N+9)} + \frac{N+1}{2}(\epsilon_1+\epsilon_2)\Big)\,\mathbb{E}_d(\tau, m_{\mathfrak{su}(N)}, \mu_{\mathfrak{su}(N+9)}, \epsilon_1, \epsilon_2).
\end{aligned} \tag{5.5.34}$$

For $N=4$, it is easy to find the two copies of λ combined together form the middle representation $\chi_{(00000001000000)} = \mathbf{924}$ of flavor $\mathfrak{su}(12)$. Indeed, for arbitrary one of the 924 λ fields, we have used the quiver formula (5.5.27) to check the above unity blowup equations up to $\mathcal{O}(q^{10})$. Conversely, we also used blowup equation (5.5.34) to solve elliptic genus independently. In the following we show two examples. As the quiver formulas are powerful enough for computational purposes in these cases, we only turn on a small subgroup of the flavor to solve blowup equations and only to the subleading q order which contains the information of 5d one-instanton partition functions.

$\mathbf{n=1}$, $\mathbf{G}=\mathfrak{su}(\mathbf{3})$, $\mathbf{F}=\mathfrak{su}(\mathbf{12})$

Using the Weyl orbit expansion, we turn on a subgroup $\mathfrak{su}(2)$ of the flavor group to compute the elliptic genus. We obtain the reduced one-string elliptic genus as

$$\mathbb{E}_{h^{(1)}_{1,\mathfrak{su}(3)}}(q_\tau, v, m_{\mathfrak{su}(3)}=0, m_{\mathfrak{su}(12)}=0) = q_\tau^{-1/3} + q_\tau^{2/3}v^{-2}\sum_{n=0}^{\infty} q_\tau^n \frac{P_n(v)}{(1+v)^4}, \tag{5.5.35}$$

where

$$P_0(v) = 8(1-5v-11v^2+81v^3+364v^4+81v^5-11v^6-5v^7+v^8).$$

This agrees with the modular ansatz in (Del Zotto and Lockhart, 2018). Using the result with flavor fugacities turned on, we obtain the following exact v expansion formula for the subleading q order coefficient, which contains the 5d one-instanton Nekrasov partition function:

$$\begin{aligned}
&\chi^{\mathfrak{su}(3)}_{(1,1)}v^{-2} - (\chi^{\mathfrak{su}(3)}_{(1,0)}\chi^{\mathfrak{su}(12)}_{(00000000001)} + c.c.)v^{-1} + \chi^{\mathfrak{su}(3)}_{(1,1)} + 1 + \chi^{\mathfrak{su}(12)}_{(10000000001)}\\
&+ (\chi^{\mathfrak{su}(12)}_{(00100000000)}v - \chi^{\mathfrak{su}(3)}_{(1,0)}\chi^{\mathfrak{su}(12)}_{(01000000000)}v^2 + \chi^{\mathfrak{su}(3)}_{(2,0)}\chi^{\mathfrak{su}(12)}_{(10000000000)}v^3 - \chi^{\mathfrak{su}(3)}_{(3,0)}v^4 + c.c\,)
\end{aligned}$$

$$+\sum_{n=0}^{\infty}\Big[\chi^{\mathfrak{su}(3)}_{(n,n)}\chi^{\mathfrak{su}(12)}_{(00000100000)}v^{2+2n}+\Big(-\chi^{\mathfrak{su}(3)}_{(n+1,n)}\chi^{\mathfrak{su}(12)}_{(00001000000)}v^{3+2n}$$
$$+\chi^{\mathfrak{su}(3)}_{(n+2,n)}\chi^{\mathfrak{su}(12)}_{(00010000000)}v^{4+2n}-\chi^{\mathfrak{su}(3)}_{(n+3,n)}\chi^{\mathfrak{su}(12)}_{(00100000000)}v^{5+2n}$$
$$+\chi^{\mathfrak{su}(3)}_{(n+4,n)}\chi^{\mathfrak{su}(12)}_{(01000000000)}v^{6+2n}-\chi^{\mathfrak{su}(3)}_{(n+5,n)}\chi^{\mathfrak{su}(12)}_{(10000000000)}v^{7+2n}+\chi^{\mathfrak{su}(3)}_{(n+6,n)}v^{8+2n}+c.c.\Big)\Big]. \tag{5.5.36}$$

Here *c.c.* means complex conjugate. We have checked this agrees with the localization formula (5.5.27) from 2d quiver gauge theory.

5.5.5 $n=2$ $\mathfrak{su}(N)$ theories

The $n=2$ $\mathfrak{su}(N)$ theories with flavor $\mathfrak{su}(2N)$ and matter in bi-representation $(R^G,R^F)=(\mathbf{N},\overline{\mathbf{2N}})$ are the most familiar SCFTs. The theory at $N=2$ is special, as the flavor symmetry is enhanced to $\mathfrak{so}(7)$. Nevertheless, as the flavor enhancement does not affect blowup equations, one can still use $\mathfrak{su}(4)$ effectively. Besides, the $N=1$ case is just the M-string. All these theories are on one single Higgsing tree, and the elliptic genus of $\mathfrak{su}(N)$ theory can be obtained by Higgsing from the elliptic genus of $\mathfrak{su}(N+1)$ theory. The 2d quiver construction is a slight modification of the A_1 string chain with $\mathfrak{su}(N)$ gauge group proposed in (Gadde et al., 2018). By Jeffrey-Kirwan residue, the n-string elliptic genus can be computed as[20]

$$\mathbb{E}^N_n=\sum_{\sum_{\ell=1}^N|Y_\ell|=n}\left(\prod_{\ell,m=1}^{N}\prod_{\substack{(x_1,y_1)\in Y_\ell\\(x_2,y_2)\in Y_m}}\frac{\theta_1(\frac{s_\ell}{s_m}t^{x_1-x_2}d^{y_1-y_2})\theta_1(\frac{s_\ell}{s_m}t^{x_1-x_2+1}d^{y_1-y_2+1})}{\theta_1(\frac{s_\ell}{s_m}t^{x_1-x_2+1}d^{y_1-y_2})\theta_1(\frac{s_\ell}{s_m}t^{x_1-x_2}d^{y_1-y_2+1})}\right)$$
$$\times\left(\prod_{\ell,m=1}^{N}\prod_{(x,y)\in Y_\ell}\theta_1(\frac{s_\ell}{s_m}t^xd^y)\theta_1(\frac{s_\ell}{s_m}t^{x+1}d^{y+1})\right)^{-1}\left(\prod_{\ell=1}^{N}\prod_{m=1}^{2N}\prod_{(x,y)\in Y_\ell}\theta_1(\frac{s_\ell}{f_m}t^{x+\frac12}d^{y+\frac12})\right). \tag{5.5.37}$$

Here $s_\ell,\ell=1,\dots,N$ are gauge parameters for $\mathfrak{su}(N)$, and $f_m,m=1,\dots,2N$ are the flavor parameters for $\mathfrak{su}(2N)$. Note the products include various factors of $\theta_1(1)$, which however completely cancel against each other. The index of d-string elliptic genera of $\mathfrak{su}(N)$ theory is known to be

$$\mathrm{Ind}_{\mathbb{E}_d}=-\frac{Nd}{4}(\epsilon_1+\epsilon_2)^2+d^2\epsilon_1\epsilon_2-d(s,s)_{su(N)}+\frac{d}{2}(f,f)_{su(2N)}. \tag{5.5.38}$$

For A type Lie algebra $(P^\vee/Q^\vee)_{A_n}\cong\mathbb{Z}_{n+1}$. The zero element, i.e. the coroot lattice $Q^\vee$ is labeled by trivial representation and results in unity blowup equations, while the n other elements each labeled by one of the n fundamental weights ω_i, $i=1,2,\dots,n$ result in vanishing blowup equations. The checker board pattern condition of blowup equations is guaranteed by the following Lie algebra facts. For $\mathfrak{su}(N)$ algebra, i.e. A_{N-1}, we denote by $\mathcal{O}_{\omega_i},i=1,2,\dots,N-1$ the Weyl orbit containing the fundamental weight ω_i. Note $|\mathcal{O}_{\omega_i}|=\binom{N}{i}$. Then $\forall w'\in\mathcal{O}_{\omega_i}$, w' intersects with i weights and $(N-i)$ weights of $\mathbf{N}=\mathcal{O}_{\omega_1}$ with intersection numbers

[20] Here we adopt the same notation as in (Gadde et al., 2018) to make the formula simple. The variable of theta functions are multiplicative. Deformation parameters $t,d=\mathrm{e}^{\epsilon_{1,2}}$. The coordinates of the boxes in a Young diagram start from 0 rather than 1.

$(N-i)/N$ and $-i/N$ respectively. Similarly, w' intersects with i weights and $(N-i)$ weights of $\overline{\mathbf{N}} = \mathcal{O}_{\omega_{N-1}}$ with intersection numbers $-(N-i)/N$ and i/N respectively.

Let us first discuss some vanishing blowup equations. For odd N and $i = 1, 2, \dots, (N-1)/2$, the leading base degree of the vanishing blowup equations with λ_F as $\lambda \in \mathcal{O}_{\omega_{N+2i}}(\mathfrak{su}(2N))$ can be universally written as

$$\sum_{w'\in\mathcal{O}_{\omega_i}(\mathfrak{su}(N))}(-1)^{|w'|}\theta_3^{[a]}(2\tau,-2m_{w'}+m_\lambda+(N-2i)\epsilon_+)\times\prod_{\beta\in\Delta(\mathfrak{su}(N))}^{w'\cdot\beta=1}\frac{1}{\theta_1(m_\beta)}$$
$$\times\prod_{\mu\in\mathbf{N}}^{\mu\cdot w'=1-\frac{i}{N}}\ \prod_{\nu\in\overline{\mathbf{2N}}}^{\nu\cdot\lambda=\frac{1}{2}+\frac{i}{N}}\theta_1(m_\mu+m_\nu+\epsilon_+)=0,\qquad a=-1/2,0. \tag{5.5.39}$$

For $i = (N+1)/2, \dots, N-2, N-1$, the leading base degree of the vanishing blowup equations with λ_F as $\lambda \in \mathcal{O}_{\omega_{2N-1-2i}}(\mathfrak{su}(2N))$ can be universally written as

$$\sum_{w'\in\mathcal{O}_{\omega_i}(\mathfrak{su}(N))}(-1)^{|w'|}\theta_3^{[a]}(2\tau,-2m_{w'}+m_\lambda+(2i-N)\epsilon_+)\times\prod_{\beta\in\Delta(\mathfrak{su}(N))}^{w'\cdot\beta=1}\frac{1}{\theta_1(m_\beta)}$$
$$\times\prod_{\mu\in\mathbf{N}}^{\mu\cdot w'=-\frac{i}{N}}\ \prod_{\nu\in\overline{\mathbf{2N}}}^{\nu\cdot\lambda=-\frac{3}{2}+\frac{i}{N}}\theta_1(m_\mu+m_\nu-\epsilon_+)=0,\qquad a=-1/2,0. \tag{5.5.40}$$

Note in the denominator there are $i(N-i)$ Jacobi θ_1, while in nominator there are $i(N-2i)$ Jacobi θ_1 if $i \le N/2$ or $(N-i)(2i-N)$ Jacobi θ_1 if $i \ge N/2$. For even N, the leading base degree of the vanishing blowup equations look almost the same with the above formulas, except the two cases are divided by $i = N/2$. In fact, we find for all integers $N \ge 2$ and $1 \le i \le N/2$, the leading base degree of vanishing blowup equations result in the following mathematical identity:

$$\sum_{\substack{\sigma\subset I_N\\|\sigma|=i}}\frac{\theta_3^{[a]}(2\tau,-2\sum_{j=1}^i m_{\sigma_j}+\sum_{k=1}^{N-2i}y_k)\prod_{j=1}^i\prod_{k=1}^{N-2i}\theta_1(m_{\sigma_j}+y_k)}{\prod_{j=1}^i\prod_{s\in I_N\backslash\sigma}\theta_1(m_{\sigma_j}-m_s)}=0,\qquad\text{for}\ \sum_{i=1}^N m_i=0. \tag{5.5.41}$$

Here $\sigma = (\sigma_1, \dots, \sigma_i)$ runs over all unordered subsets of size i of $I_N = (1, 2, \dots, N)$. Note y_k are arbitrary numbers. We have verified this identity for lots of N and i pair up to $\mathcal{O}(q^{20})$. For example, for $i = 1$, the above identity gives

$$\sum_{i=1}^N\frac{\theta_3^{[a]}(2\tau,-2m_i+\sum_{k=1}^{N-2}y_k)\prod_{k=1}^{N-2}\theta_1(m_i+y_k)}{\prod_{j\neq i}\theta_1(m_i-m_j)}=0,\qquad\text{for}\ \sum_{i=1}^N m_i=0. \tag{5.5.42}$$

All the unity λ_F fields are just the weights of representation $[0,\dots,0,1,0,\dots,0]$ of $\mathfrak{su}(2N)$, which is the largest representation generated by fundamental weights. There are $\binom{2N}{N} = \frac{(2N)!}{N!N!}$ of them, i.e. the sums of arbitrary N fundamental weights among the total $2N$ fundamental weights. Note $\forall w \in \mathbf{2N}$ and $w' \in \chi^{\mathfrak{su}(2N)}_{[0,\dots,0,1,0,\dots,0]}$,

$$w\cdot w'=\begin{cases}1/2 & \text{if } w \text{ is among the } N \text{ weights that sum up to } w',\\ -1/2 & \text{otherwise.}\end{cases} \tag{5.5.43}$$

Besides, for $\mathfrak{su}(N)$, any vector $\alpha^\vee$ in the coroot lattice and any fundamental weight

w, there always is $\alpha^\vee \cdot w \in \mathbb{Z}$. These two properties are necessary for A_H to have correct R shift.

The unity elliptic blowup equations for $G = \mathfrak{su}(N), F = \mathfrak{su}(2N)$ theory with $\lambda_F \in \chi^{\mathfrak{su}(2N)}_{[0,\dots,0,1,0,\dots,0]}$ can be written as

$$\begin{aligned}
&\sum_{d_0=\frac{1}{2}||\alpha^\vee||_{\mathfrak{su}(N)}}^{d_0+d_1+d_2=d} (-1)^{|\alpha^\vee|}\theta_3^{[a]}\Big(2\tau, 2\Big(-m^G_{\alpha^\vee}+m^F_\lambda+(\frac{N}{4}-d_0)\,(\epsilon_1+\epsilon_2)-d_1\epsilon_1-d_2\epsilon_2\Big)\Big)\\
&\qquad\qquad\qquad \times A_V(\alpha^\vee,\tau,m_G)A_H^{\mathfrak{R}}(\alpha^\vee,\tau,m_G,m_F,\lambda_F)\\
&\times\mathbb{E}_{d_1}(\tau,m_G+\epsilon_1\alpha^\vee,m_F+\epsilon_1\lambda_F,\epsilon_1,\epsilon_2-\epsilon_1)\mathbb{E}_{d_2}(\tau,m_G+\epsilon_2\alpha^\vee,m_F+\epsilon_2\lambda_F,\epsilon_1-\epsilon_2,\epsilon_2)\\
&\qquad\qquad = \theta_3^{[a]}\Big(2\tau,2m^F_\lambda+\frac{N}{2}(\epsilon_1+\epsilon_2)\Big)\,\mathbb{E}_d(\tau,m_G,m_F,\epsilon_1,\epsilon_2). \qquad (5.5.44)
\end{aligned}$$

Using the quiver formula (5.5.37) for one-string elliptic genus, we have checked the above unity blowup equations hold for $G = \mathfrak{su}(3)$ theory for all fifteen λ_F and $a = -1/2, 0$ up to $\mathcal{O}(q^{10})$. The $G = \mathfrak{su}(2)$ case is more subtle, we leave the check of blowup equations later. Conversely, we also used the Weyl orbit expansion method to solve one-string elliptic genus from above unity blowup equations at $a = 0$ for $G = \mathfrak{su}(2), \mathfrak{su}(3)$ and obtained consistent results with the quiver formulas.

$\mathbf{n = 2}$, $\mathbf{G = \mathfrak{su}(2)}$, $\mathbf{F = \mathfrak{so}(7)}$

The $G = \mathfrak{su}(2)$ case is special because the flavor symmetry $\mathfrak{su}(4)$ is enhanced to $\mathfrak{so}(7)$. In (Del Zotto and Lockhart, 2018), an inspiring exact formula for the reduced one-string elliptic genus was proposed in which it is found the flavor fugacities are even naturally arranged in $\mathfrak{so}(8)$ characters:

$$\begin{aligned}
\mathbb{E}_{h^{(1)}_{2,\mathfrak{su}(2)}}(q,v,m_{\mathfrak{su}(2)},m_{\mathfrak{so}(8)}) &= \widehat{\chi}^{\mathfrak{so}(8)}_{\mathbf{0}}(m_{\mathfrak{so}(8)},q)\xi^{2,\mathfrak{su}(2)}_{\mathbf{0}}(m_{\mathfrak{su}(2)},v,q)\\
&+\widehat{\chi}^{\mathfrak{so}(8)}_{\mathbf{c}}(m_{\mathfrak{so}(8)},q)\xi^{2,\mathfrak{su}(2)}_{\mathbf{c}}(m_{\mathfrak{su}(2)},v,q)\\
&+\widehat{\chi}^{\mathfrak{so}(8)}_{\mathbf{v}}(m_{\mathfrak{so}(8)},q)\xi^{2,\mathfrak{su}(2)}_{\mathbf{v}}(m_{\mathfrak{su}(2)},v,q), \qquad (5.5.45)
\end{aligned}$$

where the affine characters of $\mathfrak{so}(8)$ representations are defined as

$$\begin{aligned}
&\widehat{\chi}^{\mathfrak{so}(8)}_{\mathbf{1}}(m_{\mathfrak{so}(8)}) = \frac{1}{2}\sum_{j=3}^4\prod_{i=1}^4\frac{\theta_j(m_i)}{\eta}, &&\widehat{\chi}^{\mathfrak{so}(8)}_{\mathbf{v}}(m_{\mathfrak{so}(8)}) = \frac{1}{2}\sum_{j=3}^4(-1)^{j+1}\prod_{i=1}^4\frac{\theta_j(m_i)}{\eta},\\
&\widehat{\chi}^{\mathfrak{so}(8)}_{\mathbf{s}}(m_{\mathfrak{so}(8)}) = \frac{1}{2}\sum_{j=1}^2\prod_{i=1}^4\frac{\theta_j(m_i)}{\eta}, &&\widehat{\chi}^{\mathfrak{so}(8)}_{\mathbf{c}}(m_{\mathfrak{so}(8)}) = \frac{1}{2}\sum_{j=1}^2(-1)^{j}\prod_{i=1}^4\frac{\theta_j(m_i)}{\eta}, \qquad (5.5.46)
\end{aligned}$$

and ξ functions are defined as

$$\begin{aligned}
\xi^{2,\mathfrak{su}(2)}_{\mathbf{0}} &= \frac{1}{q^{1/6}\prod_{j=1}^\infty(1-q^j)\widetilde{\Delta}_{\mathfrak{su}(2)}(m_{\mathfrak{su}(2)},q)}\sum_{k\geq 0}\frac{q^{k+1/2}(v^{2k+1}+v^{-2k-1})}{1-q^{2k+1}}\chi^{\mathfrak{su}(2)}_{(2k)}(m_{\mathfrak{su}(2)}),\\
\xi^{2,\mathfrak{su}(2)}_{\mathbf{c}} &= -\frac{1}{q^{1/6}\prod_{j=1}^\infty(1-q^j)\widetilde{\Delta}_{\mathfrak{su}(2)}(m_{\mathfrak{su}(2)},q)}\sum_{k\geq 0}\frac{v^{2k+1}+q^{2k+1}v^{-2k-1}}{1-q^{2k+1}}\chi^{\mathfrak{su}(2)}_{(2k)}(m_{\mathfrak{su}(2)}),\\
\xi^{2,\mathfrak{su}(2)}_{\mathbf{v}} &= \frac{1}{q^{1/6}\prod_{j=1}^\infty(1-q^j)\widetilde{\Delta}_{\mathfrak{su}(2)}(m_{\mathfrak{su}(2)},q)}\sum_{k\geq 0}\frac{v^{2k+2}-q^{k+1}v^{-2k-2}}{1+q^{k+1}}\chi^{\mathfrak{su}(2)}_{(2k+1)}(m_{\mathfrak{su}(2)}),
\end{aligned}$$

with a modified version of Weyl-Kac determinant

$$\widetilde{\Delta}_G(m_G,q)=\prod_{j=1}^{\infty}(1-q^j)^{\mathrm{rank}(G)}\prod_{\alpha\in\Delta_+^G}(1-q^jQ_{m_\alpha})(1-q^jQ_{m_\alpha}^{-1}). \tag{5.5.47}$$

Using the above formula for one string elliptic genus, we have checked the unity elliptic blowup equations (5.5.44) hold only for $F=\mathfrak{so}(7)$ but not $\mathfrak{so}(8)$. For arbitrary $m_{\mathfrak{so}(7)}$, we checked the 6×2 unity blowup equations up to $\mathcal{O}(q_\tau^{10})$.

5.5.6 $n=3$ $\mathfrak{so}(7)$ and $\mathfrak{su}(3)$ theories

The $n=3$, $G=\mathfrak{so}(7)$ theory has flavor symmetry $F=\mathfrak{sp}(2)$ and matter representation $\mathbf{8}$. This theory is particularly interesting because it has a known 2d quiver description and can be Higgsed to the $n=3, G=G_2$ theory, making which the first exactly computable exceptional 6d SCFT (Kim et al., 2018). The elliptic genera of this theory were computed via Jeffrey-Kirwan residue of localization in (Kim et al., 2018). For example, the reduced one-string elliptic genus can be expressed as

$$\mathbb{E}_{h^{(1)}_{3,\mathfrak{so}(7)}}(\tau,\epsilon_{1,2},m_i,\mu_k)=\sum_{i=1}^{3}\frac{\theta(4\epsilon_+-2m_i)\prod_{k=1}^{2}\theta(\mu_k\pm(m_i-\epsilon_+))}{\prod_{j\neq i}\theta(m_{ij})\theta(2\epsilon_+-m_{ij})\theta(2\epsilon_+-m_i-m_j)}, \tag{5.5.48}$$

where $\theta(z)=\theta_1(\tau,z)/\eta(\tau)$, $m_{ij}\equiv m_i-m_j$, and $m_i, i=1,2,3$ are the $\mathfrak{so}(7)$ fugacities such that $\mathbf{7}_{\mathbf{v}}^{\mathfrak{so}(7)}=1+\sum_{i=1}^{3}(Q_{m_i}+Q_{m_i}^{-1})$ and $\mu_k, k=1,2$ are associated to each $\mathfrak{sp}(1)$ in flavor decomposition $\mathfrak{sp}(2)\to\mathfrak{sp}(1)\times\mathfrak{sp}(1)$.

Let us first discuss the vanishing blowup equations. Since $(P^\vee/Q^\vee)_{\mathfrak{so}(7)}\cong\mathbb{Z}_2$, there should exist vanishing blowup equations with λ_G taking value in $(P^\vee\backslash Q^\vee)_{\mathfrak{so}(7)}$. For flavor fugacities, we find λ_F has five possible values, weights of representation $\mathbf{1}$ or $\mathbf{4}$ of $\mathfrak{sp}(2)$. The checker board pattern condition of A_V is guaranteed by the Lie algebra fact $\forall\alpha\in\Delta(\mathfrak{so}(7)), w\in(P^\vee\backslash Q^\vee)_{\mathfrak{so}(7)}$, the intersection $\alpha\cdot w\in\mathbb{Z}$. On the other hand, the checker board pattern condition of A_H is guaranteed by the Lie algebra fact $\forall\omega'\in\mathbf{8}, w\in(P^\vee\backslash Q^\vee)_{\mathfrak{so}(7)}$, the intersection $\omega'\cdot w\in\mathbb{Z}+1/2$.

Note the smallest Weyl orbit in $(P^\vee\backslash Q^\vee)_{\mathfrak{so}(7)}$ is $\mathcal{O}_{1/2,6}$, which is contained in the weight space of the vector representation $\mathbf{7}_{\mathbf{v}}^{\mathfrak{so}(7)}=1+\mathcal{O}_{1/2,6}$. We find the leading base degree of the vanishing blowup equations with $\lambda_F=0$ can be written as

$$\sum_{w\in\mathcal{O}_{1/2,6}}(-1)^{|w|}\theta_4^{[a]}(3\tau,3m_w)\times\prod_{\beta\in\Delta(\mathfrak{so}(7))}^{w\cdot\beta=1}\frac{1}{\theta_1(\tau,m_\beta)}=0, \tag{5.5.49}$$

where $a=-1/2$ and $\pm1/6$. We have checked this identity up to $\mathcal{O}(q^{20})$. Here the hypermultiplets do not contribute to the leading base degree equation, since $\forall w\in\mathcal{O}_{1/2,6}, w'\in\mathbf{8}$, $w\cdot w'=\pm1/2$. On the other hand, the leading base degree of the vanishing blowup equations with $\lambda_F\in\mathbf{4}$ is

$$\sum_{w\in\mathcal{O}_{1/2,6}}(-1)^{|w|}\theta_4^{[a]}(3\tau,3m_w+2x)\prod_{\beta\in\Delta(\mathfrak{so}(7))}^{w\cdot\beta=1}\frac{1}{\theta_1(m_\beta)}\prod_{\omega'\in\mathbf{8}}^{w\cdot\omega'=-1/2}\theta_1(m_{\omega'}+x)=0, \tag{5.5.50}$$

where $x = \pm m_{\mathfrak{sp}(1)} + \epsilon_+$ is an arbitrary number. We also checked this identity up to $\mathcal{O}(q^{20})$. For higher base degrees, we checked all five vanishing blowup equations from the viewpoint of Calabi-Yau.

For unity blowup equations, λ_F has four choices which are just the four short roots of $\mathfrak{sp}(2)$, or explicitly $(\pm 1, \pm 1)$ if we view the effective flavor group as $\mathfrak{sp}(1)_a \times \mathfrak{sp}(1)_b$. Therefore, all the 12 unity blowup equations with $\lambda_F = \lambda \in \mathcal{O}^{\mathfrak{sp}(2)}_{[01]}$ can be written as

$$\begin{aligned}\sum_{d_0=\frac{1}{2}||\alpha^\vee||_{\mathfrak{so}(7)}}^{d_0+d_1+d_2=d} & (-1)^{|\alpha^\vee|}\theta_4^{[a]}\Big(3\tau, 3\Big(-m_{\alpha^\vee}^{\mathfrak{so}(7)} + m_\lambda^{\mathfrak{sp}(2)} + (\frac{2}{3} - d_0)\,(\epsilon_1+\epsilon_2) - d_1\epsilon_1 - d_2\epsilon_2\Big)\Big) \\ & \times A_V^{\mathfrak{so}(7)}(\alpha^\vee, \tau, m_{\mathfrak{so}(7)})A_H^{(\mathbf{8},\frac{1}{2}\mathbf{4})}(\alpha^\vee, \tau, m_{\mathfrak{so}(7)}, m_{\mathfrak{sp}(2)}, \lambda) \\ & \times \mathbb{E}_{d_1}(\tau, m_{\mathfrak{so}(7)} + \epsilon_1\alpha^\vee, m_{\mathfrak{sp}(2)} + \epsilon_1\lambda, \epsilon_1, \epsilon_2 - \epsilon_1) \\ & \times \mathbb{E}_{d_2}(\tau, m_{\mathfrak{so}(7)} + \epsilon_2\alpha^\vee, m_{\mathfrak{sp}(2)} + \epsilon_2\lambda, \epsilon_1 - \epsilon_2, \epsilon_2) \\ = & \ \theta_4^{[a]}\Big(3\tau, 3m_\lambda^{\mathfrak{sp}(2)} + 2(\epsilon_1+\epsilon_2)\Big)\,\mathbb{E}_d(\tau, m_{\mathfrak{so}(7)}, m_{\mathfrak{sp}(2)}, \epsilon_1, \epsilon_2). \qquad (5.5.51)\end{aligned}$$

Here $a = -1/2, \pm 1/6$. All four possible λ just give $\lambda \cdot m_{\mathfrak{sp}(2)} = \pm m_{\mathfrak{sp}(1)_a} \pm m_{\mathfrak{sp}(1)_b}$. Fix arbitrary one λ, there are three unity blowup equations with different characteristics from which one can solve elliptic genera recursively. For example, using the recursion formula, we computed the one-string elliptic genus to $\mathcal{O}(q_\tau^3)$. Our result agrees precisely with the quiver formula in (Kim et al., 2018) and the modular ansatz in (Del Zotto and Lockhart, 2018), therefore we just present the first few q_τ orders with all gauge and flavor fugacities turned off. For example, denote the reduced one-string elliptic genus as

$$\mathbb{E}_{h^{(1)}_{3,\mathfrak{so}(7)}}(q_\tau, v) = q_\tau^{-1/3} v^4 \sum_{n=0}^{\infty} q_\tau^n \frac{P_n(v)}{(1-v)^4(1+v)^8}. \qquad (5.5.52)$$

We obtain

$$\begin{aligned}P_0(v) &= -(5 - 12v + 22v^2 - 12v^3 + 5v^4), \\ P_1(v) &= v^{-6}(1 + 4v + 2v^2 - 12v^3 - 18v^4 + 4v^5 + 158v^6 - 316v^7 + 418v^8 - \cdots + v^{16}).\end{aligned}$$

We also computed the two-string elliptic genus using the recursion formula and find agreement with the quiver formula in (Kim et al., 2018). For example,

$$\mathbb{E}_{h^{(2)}_{3,\mathfrak{so}(7)}}(q_\tau, v) = q_\tau^{-5/6} v^9 \sum_{n=0}^{\infty} q_\tau^n \frac{P_n^{(2)}(v)}{(1-v)^{10}(1+v)^{10}(1+v+v^2)^9}, \qquad (5.5.53)$$

where

$$\begin{aligned}P_0^{(2)}(v) = & \ 14 + 18v - 3v^2 + 69v^3 + 298v^4 + 295v^5 + 175v^6 + 684v^7 + 1426v^8 \\ & + 1132v^9 + 660v^{10} + \cdots + 14v^{20}, \\ P_1^{(2)}(v) = & \ v^{-6}(5 + 23v + 68v^2 + 135v^3 + 216v^4 + 273v^5 + 649v^6 + 838v^7 - 117v^8 \\ & - 407v^9 + 3496v^{10} + 6341v^{11} + 6252v^{12} + 12839v^{13} + 24595v^{14} \\ & + 23918v^{15} + 19272v^{16} + \cdots + 5v^{32}).\end{aligned}$$

$\mathbf{G} = \mathfrak{su}(\mathbf{3})$

The elliptic genus of pure $\mathfrak{su}(3)$ theory can be directly obtained by Higgsing from the one of above $\mathfrak{so}(7)$ theory by realizing $m_i, i = 1,2,3$ as the symmetric fugacities of $\mathfrak{su}(3)$ with $m_1 + m_2 + m_3 = 0$ and setting $\mu_k = \epsilon_+, k = 1,2$. For example, denote the reduced one-string elliptic genus with gauge fugacities turned off as

$$\mathbb{E}_{h^{(1)}_{3,\mathfrak{su}(3)}}(q_\tau, v) = q_\tau^{-1/3} v^2 \sum_{n=0}^{\infty} q_\tau^n \frac{P_n(v)}{(1-v^2)^4}, \tag{5.5.54}$$

we obtain

$$\begin{aligned} P_0(v) =& -(1+4v^2+v^4),\\ P_1(v) =& v^{-2}(1-4v^2+15v^4+24v^6+15v^8-4v^{10}+v^{12}). \end{aligned}$$

We also computed the two-string elliptic genus using the recursion formula and find agreement with the quiver formula in (Kim, Kim, and Park, 2016). For example,

$$\mathbb{E}_{h^{(2)}_{3,\mathfrak{su}(3)}}(q_\tau, v) = q_\tau^{-5/6} v^5 \sum_{n=0}^{\infty} q_\tau^n \frac{P_n^{(2)}(v)}{(1-v)^{10}(1+v)^6(1+v+v^2)^5}, \tag{5.5.55}$$

where

$$\begin{aligned} P_0^{(2)}(v) =& 1+v+6v^2+17v^3+31v^4+52v^5+92v^6+110v^7+112v^8+110v^9+\ldots+v^{16},\\ P_1^{(2)}(v) =& v^{-4}(1+3v+8v^2+11v^3+18v^4+13v^5+55v^6+238v^7+601v^8+1121v^9\\ &+1777v^{10}+2262v^{11}+2424v^{12}+2262v^{13}+\cdots+v^{24}). \end{aligned}$$

5.5.7 $n = 4$ $\mathfrak{so}(N+8)$ theories

The $n = 4$, $G = \mathfrak{so}(N+8)$ theories have flavor group $F = \mathfrak{sp}(N)$ and matter representation $(R^G, R^F) = (\mathbf{N+8}, \mathbf{2N})$. For even $N = 2p$, such theories can be realized by type IIB superstring theory with orientfold. The Kodaira elliptic singularity of type $I_p^\star$ here is due to the presence of $4+p$ D7-branes wrapping the base $\mathbb{P}^1$ together with an orientifold 7-plane. This picture results in a quiver gauge theory description which makes the elliptic genera exactly computable via Jeffrey-Kirwan residues (Haghighat et al., 2015b). For example, the reduced one-string elliptic genus can be computed as

$$\mathbb{E}_{h^{(1)}_{4,\mathfrak{so}(8+2p)}} = \frac{1}{2}\sum_{i=1}^{4+p}\left[\frac{\theta(2\epsilon_+ + 2m_i)\theta(4\epsilon_+ + 2m_i)\prod_{j=1}^{2p}\theta(\epsilon_+ + m_i \pm \mu_j)}{\prod_{j\neq i}\theta(m_i \pm m_j)\theta(2\epsilon_+ + m_i \pm m_j)} + (m_i \to -m_i)\right]. \tag{5.5.56}$$

Here $\theta(z) = \theta_1(\tau, z)/\eta(\tau)$, m_i and μ_j are fugacities of gauge $\mathfrak{so}(8+2p)$ and flavor $\mathfrak{sp}(2p)$. For odd N cases, the 2d quiver description also exists similarly and was discussed in Appendix D of (Del Zotto and Lockhart, 2018). For example, the reduced one-string elliptic genus for $G = \mathfrak{so}(9+2p)$, $F = \mathfrak{sp}(1+2p)$ theory is

$$\mathbb{E}_{h^{(1)}_{4,\mathfrak{so}(9+2p)}} = \frac{1}{2}\sum_{i=1}^{4+p}\left[\frac{\theta(2\epsilon_+ + 2m_i)\theta(4\epsilon_+ + 2m_i)\prod_{j=1}^{2p+1}\theta(\epsilon_+ + m_i \pm \mu_j)}{\theta(m_i)\theta(2\epsilon_+ + m_i)\prod_{j\neq i}\theta(m_i \pm m_j)\theta(2\epsilon_+ + m_i \pm m_j)} + (m_i \to -m_i)\right]. \tag{5.5.57}$$

Still m_i and μ_j are gauge and flavor fugacities respectively.

Let us first discuss the vanishing blowup equations. As is well-known in Lie algebra, $(P^\vee/Q^\vee)_{B_n} \cong \mathbb{Z}_2$ and $(P^\vee/Q^\vee)_{D_n} \cong \mathbb{Z}_4$. Consider the vanishing blowup equations with λ_G taking value in $\mathcal{O}^{\mathfrak{so}(8+N)}_{[10\cdots00]}$, i.e. the Weyl orbit associated to the vector representation. For flavor fugacities, we find λ_F can always take value in Weyl orbit $\mathcal{O}^{\mathfrak{sp}(N)}_{[00\cdots01]}$. Let us denote the smallest Weyl orbit in $(P^\vee\backslash Q^\vee)_{\mathfrak{so}(8+N)}$ as $\mathcal{O}_{\text{min}}$. It has relation with the vector representation of $\mathfrak{so}(8+N)$ as

$$(\mathbf{8}+\mathbf{N})_{\mathbf{v}} = \begin{cases} \mathcal{O}_{\text{min}}, & \text{for even } N, \\ \mathbf{1}+\mathcal{O}_{\text{min}}, & \text{for odd } N, \end{cases} \tag{5.5.58}$$

Then the leading base degree of the vanishing blowup equations of $G = \mathfrak{so}(8+N)$ theory with $\lambda_F \in \mathcal{O}^{\mathfrak{sp}(N)}_{[00\cdots01]}$ can be universally written as

$$\sum_{w\in\mathcal{O}_{\text{min}}}(-1)^{|w|}\theta_3^{[a]}(4\tau,4m_w+Nx)\theta_1(-m_w+x)^N \times \prod_{\beta\in\Delta(\mathfrak{so}(8+N))}^{w\cdot\beta=1}\frac{1}{\theta_1(\tau,m_\beta)} = 0. \tag{5.5.59}$$

Here $a=-1/2,-1/4,0,1/4$ and $x=\lambda_F\cdot m_F+\epsilon_+$. We have checked this identity up to $\mathcal{O}(q^{20})$ for several N. Note there are $N+6$ Jacobi θ_1 functions in the denominator.

For even N cases, there exist more vanishing blowup equations with λ_G taking value in $\mathcal{O}^{\mathfrak{so}(8+N)}_{[00\cdots01]}$ and $\mathcal{O}^{\mathfrak{so}(8+N)}_{[00\cdots10]}$, which coincide with the spinor and conjugate spinor representations. For example, the leading base degree of the vanishing blowup equations with $\lambda_F=0$ can be universally written as

$$\sum_{w\in S}(-1)^{|w|}\theta_3^{[a]}(4\tau,4m_w) \times \prod_{\beta\in\Delta(\mathfrak{so}(8+N))}^{w\cdot\beta=1}\frac{1}{\theta_1(\tau,m_\beta)} = 0, \qquad N\geq 0,\, N\equiv 0\,(\mathrm{mod}\,2). \tag{5.5.60}$$

Here S is the spinor representation of $\mathfrak{so}(8+N)$ which can also be replaced by its conjugate representation. We have checked this identity up to $\mathcal{O}(q^{20})$ for several even N. Note there are $(N+6)(N+8)/8$ Jacobi θ_1 functions in the denominator.

The unity λ_F fields of $\mathfrak{so}(N+8)$ theories all take value in the Weyl orbit $\mathcal{O}^{\mathfrak{sp}(N)}_{[00\cdots01]}$. There are 2^N of them. The unity elliptic blowup equations for $G=\mathfrak{su}(8+N), F=\mathfrak{sp}(N)$ theory with λ short for λ_F can be written as

$$\begin{aligned}
&\sum_{d_0=\frac{1}{2}||\alpha^\vee||_{\mathfrak{so}(8+N)}}^{d_0+d_1+d_2=d}(-1)^{|\alpha^\vee|}\\
&\times\theta_3^{[a]}\Big(4\tau,4(-\alpha^\vee\cdot m_{\mathfrak{so}(8+N)}+\lambda\cdot m_{\mathfrak{sp}(N)}+(\frac{N+4}{8}-d_0)(\epsilon_1+\epsilon_2)-d_1\epsilon_1-d_2\epsilon_2)\Big)\\
&\qquad\times A_V^{\mathfrak{so}(8+N)}(\alpha^\vee,\tau,m_{\mathfrak{so}(8+N)})A_H^{\frac{1}{2}(\mathbf{8+N},\mathbf{2N})}(\alpha^\vee,\tau,m_{\mathfrak{so}(8+N)},m_{\mathfrak{sp}(N)},\lambda)\\
&\qquad\times\mathbb{E}_{d_1}\big(\tau,m_{\mathfrak{so}(8+N)}+\epsilon_1\alpha^\vee,m_{\mathfrak{sp}(N)}+\epsilon_1\lambda,\epsilon_1,\epsilon_2-\epsilon_1\big)\\
&\qquad\times\mathbb{E}_{d_2}\big(\tau,m_{\mathfrak{so}(8+N)}+\epsilon_2\alpha^\vee,m_{\mathfrak{sp}(N)}+\epsilon_2\lambda,\epsilon_1-\epsilon_2,\epsilon_2\big)\\
=\;&\theta_3^{[a]}\Big(4\tau,4\lambda\cdot m_{\mathfrak{sp}(N)}+\frac{N+4}{2}(\epsilon_1+\epsilon_2)\Big)\,\mathbb{E}_d\big(\tau,m_{\mathfrak{so}(8+N)},m_{\mathfrak{sp}(N)},\epsilon_1,\epsilon_2\big).
\end{aligned} \tag{5.5.61}$$

Here $a = -1/2, -1/4, 0, 1/4$. Fix arbitrary one λ and choose arbitrary three characteristics a, one can use the three unity blowup equations to solve elliptic genera recursively.

In the following, we present some of our computational results on one-string and two-string elliptic genera from recursion formula. To save space, we turn off both gauge and flavor fugacities.

$\mathbf{G} = \mathfrak{so}(\mathbf{8})$

Let us denote the reduced one-string elliptic genus with gauge fugacities turned off as

$$\mathbb{E}_{h^{(1)}_{4,\mathfrak{so}(8)}}(q_\tau, v) = q_\tau^{-5/6} v^5 \sum_{n=0}^{\infty} q_\tau^n \frac{P_n(v)}{(1-v^2)^{10}}. \tag{5.5.62}$$

We obtain

$$\begin{aligned} P_0(v) &= 1 + 18v^2 + 65v^4 + 65v^6 + 18v^8 + v^{10}, \\ P_1(v) &= 29 + 417v^2 + 1234v^4 + 1234v^6 + 417v^8 + 29v^{10}. \end{aligned}$$

This agrees precisely with the quiver formula (5.5.56) and the modular ansatz result in (Del Zotto and Lockhart, 2018). Using recursion formula, we also computed the reduced two-string elliptic genus with all gauge and flavor fugacities turned off. Denote

$$\mathbb{E}_{h^{(2)}_{4,\mathfrak{so}(8)}}(q_\tau, v) = q_\tau^{-11/6} v^{11} \sum_{n=0}^{\infty} q_\tau^n \frac{P_n^{(2)}(v)}{(1-v)^{22}(1+v)^{12}(1+v+v^2)^{11}},$$

we obtain

$$\begin{aligned} P_0^{(2)}(v) \ &= 1 + v + 20v^2 + 65v^3 + 254v^4 + 841v^5 + 2435v^6 + 6116v^7 + 14290v^8 \\ &+ 29700v^9 + 55947v^{10} + 96519v^{11} + 152749v^{12} + 220408v^{13} + 293226v^{14} \\ &+ 359742v^{15} + 406014v^{16} + 421960v^{17} + 406014v^{18} + \cdots + v^{34}, \\ P_1^{(2)}(v) \ &= (1+v^2)(32 + 90v + 697v^2 + 2913v^3 + 10582v^4 + 34415v^5 + 97961v^6 \\ &+ 242492v^7 + 540749v^8 + 1085137v^9 + 1958185v^{10} + 3205774v^{11} + 4789888v^{12} \\ &+ 6522178v^{13} + 8110633v^{14} + 9248825v^{15} + 9668450v^{16} + 9248825v^{17} + \cdots + v^{32}). \end{aligned} \tag{5.5.63}$$

$\mathbf{G} = \mathfrak{so}(\mathbf{9})$

For $G = \mathfrak{so}(9)$, $F = \mathfrak{sp}(1)$ theory, let us denote the reduced one-string elliptic genus as

$$\mathbb{E}_{h^{(1)}_{4,\mathfrak{so}(9)}}(q_\tau, v) = q_\tau^{-5/6} v^6 \sum_{n=0}^{\infty} q_\tau^n \frac{P_n(v)}{(1-v)^{10}(1+v)^{12}}. \tag{5.5.64}$$

We obtain

$$\begin{aligned} P_0(v) &= 2 - 5v + 36v^2 - 46v^3 + 130v^4 - 90v^5 + 130v^6 - 46v^7 + 36v^8 - 5v^9 + 2v^{10}, \\ P_1(v) &= 4(19 - 52v + 270v^2 - 368v^3 + 815v^4 - 648v^5 + \cdots + 19v^{10}). \end{aligned}$$

This agrees precisely with the quiver formula (5.5.57) and the modular ansatz result in (Del Zotto and Lockhart, 2018). Using recursion formula, we also computed the reduced two-string elliptic genus with all gauge and flavor fugacities turned off. Denote

$$\mathbb{E}_{h^{(2)}_{4,\mathfrak{so}(9)}}(q_\tau, v) = q_\tau^{-11/6} v^{13} \sum_{n=0}^{\infty} q_\tau^n \frac{P_n^{(2)}(v)}{(1-v)^{22}(1+v)^{16}(1+v+v^2)^{13}},$$

we obtain

$$\begin{aligned}
P_0^{(2)}(v) &= 3 + 5v + 41v^2 + 184v^3 + 623v^4 + 1987v^5 + 6119v^6 + 16024v^7 + 38003v^8 \\
&\quad + 84127v^9 + 170974v^{10} + 315783v^{11} + 541464v^{12} + 864989v^{13} + 1277738v^{14} \\
&\quad + 1747831v^{15} + 2235019v^{16} + 2666784v^{17} + 2956416v^{18} + 3054876v^{19} \\
&\quad + \cdots + 3v^{38}, \\
P_1^{(2)}(v) &= 2(62 + 193v + 1031v^2 + 4553v^3 + 16024v^4 + 49985v^5 + 146893v^6 + 383794v^7 \\
&\quad + 904569v^8 + 1962488v^9 + 3926557v^{10} + 7208099v^{11} + 12237790v^{12} \\
&\quad + 19308839v^{13} + 28304443v^{14} + 38563232v^{15} + 49018799v^{16} + 58173759v^{17} \\
&\quad + 64417144v^{18} + 66611780v^{19} + \cdots + 62v^{38}).
\end{aligned} \tag{5.5.65}$$

$\mathbf{G} = \mathfrak{so}(\mathbf{10})$

For $G = \mathfrak{so}(10)$, $F = \mathfrak{sp}(2)$ theory, let us denote the reduced one-string elliptic genus as

$$\mathbb{E}_{h^{(1)}_{4,\mathfrak{so}(10)}}(q_\tau, v) = q_\tau^{-5/6} v^7 \sum_{n=0}^{\infty} q_\tau^n \frac{P_n(v)}{(1-v)^{10}(1+v)^{14}}. \tag{5.5.66}$$

We obtain

$$\begin{aligned}
P_0(v) &= -(5 - 20v + 99v^2 - 184v^3 + 370v^4 - 360v^5 + \cdots + 5v^{10}), \\
P_1(v) &= v^{-2}(1 + 4v - 249v^2 + 1024v^3 - 3873v^4 + 7172v^5 - 12223v^6 \\
&\quad + 12688v^7 - \cdots + v^{14}).
\end{aligned} \tag{5.5.67}$$

This agrees precisely with the quiver formula in (5.5.56) and the modular ansatz in (Del Zotto and Lockhart, 2018). Using recursion formula, we also computed the reduced two-string elliptic genus. Denote

$$\mathbb{E}_{h^{(2)}_{4,\mathfrak{so}(10)}}(q_\tau, v) = q_\tau^{-11/6} v^{15} \sum_{n=0}^{\infty} q_\tau^n \frac{P_n^{(2)}(v)}{(1-v)^{22}(1+v)^{20}(1+v+v^2)^{15}},$$

we obtain

$$\begin{aligned}
P_0^{(2)}(v) &= 14 + 42v + 174v^2 + 840v^3 + 3180v^4 + 9606v^5 + 28723v^6 + 80545v^7 \\
&\quad + 200547v^8 + 453260v^9 + 967049v^{10} + 1923811v^{11} + 3524339v^{12} + 6005020v^{13} \\
&\quad + 9637502v^{14} + 14497632v^{15} + 20342110v^{16} + 26767114v^{17} + 33232318v^{18} \\
&\quad + 38795360v^{19} + 42443836v^{20} + 43677620v^{21} + \cdots + 14v^{42},
\end{aligned}$$

$$
\begin{aligned}
P_1^{(2)}(v) = & -v^{-2}(5+35v-566v^2-2413v^3-9796v^4-43257v^5-166563v^6-516948v^7\\
&-1493092v^8-4045182v^9-9976992v^{10}-22346950v^{11}-46615056v^{12}\\
&-90796062v^{13}-164272366v^{14}-276641406v^{15}-437103585v^{16}-648567657v^{17}\\
&-902450252v^{18}-1179498629v^{19}-1452843842v^{20}-1686000677v^{21}\\
&-1841747735v^{22}-1895883244v^{23}+\cdots+5v^{46}).
\end{aligned}
\tag{5.5.68}
$$

5.5.8 G_2 theories

$G=G_2$ theories on base curve $(-n)$, $n=1,2,3$ have flavor group $F=\mathfrak{sp}(10-3n)$ and $n_f=(10-3n)$ hypermultiplets in fundamental representation $\mathbf{7}$ of gauge symmetry. There only exist unity blowup equations but no vanishing due to the Lie algebra fact $Q^\vee \cong P^\vee$ for G_2. The unity λ_F fields are just all the elements of the Weyl orbit $[0,0,\ldots,0,1]$ of $\mathfrak{sp}(10-3n)$ or in other word take value ±1 for each $\mathfrak{sp}(1)$ with decomposition $\mathfrak{sp}(10-3n)\to\mathfrak{sp}(1)^{10-3n}$. There are in total $n\times 2^{10-3n}$ unity blowup equations when different choices of the characteristic are also taken into account.

$\mathbf{n=3,\ G=G_2,\ F=\mathfrak{sp}(1)}$

This theory can be Higgsed from the $n=3$, $G=\mathfrak{so}(7)$, $F=\mathfrak{sp}(2)$ theory and to the $n=3$, $G=\mathfrak{su}(3)$ minimal SCFT. The 2d quiver description was found in (Kim et al., 2018), therefore the elliptic genus can be computed exactly via localization. For example, the reduced one-string elliptic genus of such theory is given in (Kim et al., 2018) as

$$
\mathbb{E}_{h^{(1)}_{3,G_2}} = \sum_{i=1}^{3} \frac{\theta(2m_i-4\epsilon_+)\theta(m_{\mathfrak{sp}(1)}\pm(m_i-\epsilon_+))}{\theta(m_i-2\epsilon_+)\prod_{j\neq i}\theta(m_{ij})\theta(2\epsilon_+-m_{ij})\theta(2\epsilon_++m_j)},
\tag{5.5.69}
$$

where $\theta(z)=\theta_1(\tau,z)/\eta(\tau)$ and $m_{1,2,3}$ are the embedding of G_2 into $\mathfrak{su}(3)$ with $m_1+m_2+m_3=0$ and $m_{ij}=m_i-m_j$.

Using the recursion formula from blowup equations, we computed the one-string elliptic genus to $\mathcal{O}(q_\tau^3)$. Our result agrees precisely with the quiver formula in (Kim et al., 2018) and the modular ansatz in (Kim, Lee, and Park, 2018) and (Del Zotto and Lockhart, 2018), therefore we just present the first few q_τ orders with all gauge and flavor fugacities turned off. For example, denote the reduced one-string elliptic genus as

$$
\mathbb{E}_{h^{(1)}_{3,G_2}}(q_\tau,v) = q_\tau^{-1/3}v^3\sum_{n=0}^{\infty} q_\tau^n \frac{P_n(v)}{(1-v)^4(1+v)^6}.
\tag{5.5.70}
$$

We obtain

$$
P_0(v) = 2-3v+8v^2-3v^3+2v^4,
\tag{5.5.71}
$$

$$
P_1(v) = v^{-5}(1+2v-3v^2-8v^3+2v^4+44v^5-60v^6+92v^7+\cdots+v^{14}).
\tag{5.5.72}
$$

We also compute the two-string elliptic genus using the recursion formula and find perfect agreement with the quiver formula in (Kim et al., 2018). For example,

$$\mathbb{E}_{h^{(2)}_{3,G_2}}(q_\tau, v) = q_\tau^{-5/6} \sum_{n=0}^{\infty} q_\tau^n \frac{P_n^{(2)}(v)}{(1-v)^{10}(1+v)^6(1+v+v^2)^7}, \tag{5.5.73}$$

where

$$\begin{aligned} P_0^{(2)}(v) &= v^7(3-3v+8v^2+21v^3+17v^4+16v^5+89v^6+71v^7+42v^8+\cdots+3v^{16}), \\ P_1^{(2)}(v) &= v^2(2+3v+11v^2+9v^3+20v^4+46v^5-24v^6+19v^7+313v^8+442v^9 \\ &\quad +569v^{10}+1364v^{11}+1473v^{12}+1226v^{13}+\cdots+2v^{26}). \end{aligned} \tag{5.5.74}$$

$\mathbf{n=2}$, $\mathbf{G=G_2}$, $\mathbf{F=\mathfrak{sp}(4)}$

Using the Weyl orbit expansion method elaborated in section 5.3.2 and the unity blowup equation with characteristic $a=0$, we solved the reduced one-string elliptic genus as

$$\mathbb{E}_{h^{(1)}_{2,G_2}}(q_\tau, v) = q_\tau^{1/6} v^{-1} \sum_{n=0}^{\infty} q_\tau^n \frac{(1-v)^2 P_n(v)}{(1+v)^6}, \tag{5.5.75}$$

where

$$P_0(v) = 1+8v+30v^2+64v^3+30v^4+8v^5+v^6, \tag{5.5.76}$$

$$P_1(v) = v^{-2}(14+56v+23v^2-216v^3-305v^4+288v^5+\cdots+14v^{10}). \tag{5.5.77}$$

The above result agrees with the modular ansatz in (Del Zotto and Lockhart, 2018). With flavor fugacities turned on, we found the leading q_τ order agrees with the exact expression for reduced 5d one-instanton partition function proposed in (Del Zotto and Lockhart, 2018)

$$\begin{aligned} & v^{-1} - \chi^{\mathfrak{sp}(4)}_{(1000)} v^2 + \chi^{G_2}_{(10)} v^3 + \sum_{n=0}^{\infty} \Big[-\chi^{G_2}_{(0n)} \chi^{\mathfrak{sp}(4)}_{(0001)} v^{3+2n} + \chi^{G_2}_{(1n)} \chi^{\mathfrak{sp}(4)}_{(0010)} v^{4+2n} \qquad (5.5.78) \\ & \quad - \chi^{G_2}_{2n} \chi^{\mathfrak{sp}(4)}_{(0100)} v^{5+2n} + \chi^{G_2}_{(3n)} \chi^{\mathfrak{sp}(4)}_{(1000)} v^{6+2n} - \chi^{G_2}_{(4n)} v^{7+2n} \Big] \\ & = v^{-1} - \mathbf{8}^{\mathfrak{sp}(4)} v^2 + (\mathbf{7}^{G_2} - \mathbf{42}^{\mathfrak{sp}(4)}) v^3 + \mathbf{7}^{G_2} \cdot \mathbf{48}^{\mathfrak{sp}(4)} v^4 + \mathcal{O}(v^5), \qquad (5.5.79) \end{aligned}$$

We also obtained the subleading q order of the reduced one-string elliptic genus as

$$\mathbf{14} v^{-3} - \mathbf{7} \cdot \chi^{\mathfrak{sp}(4)}_{(1000)} v^{-2} + (\mathbf{14} + 1 + \chi^{\mathfrak{sp}(4)}_{(2000)}) v^{-1} + \chi^{\mathfrak{sp}(4)}_{(0100)} + \mathbf{7} v + \mathcal{O}(v^2). \tag{5.5.80}$$

Here and below bold letters in the v expansion represent characters of representations of gauge symmetry.

$\mathbf{n=1}$, $\mathbf{G=G_2}$, $\mathbf{F=\mathfrak{sp}(7)}$

We study this theory from the viewpoint of Weyl orbit expansion. Let us just turn on a subgroup $\mathfrak{sp}(1)$ of the flavor $\mathfrak{sp}(7)$. Using the Weyl orbit expansion method and the unity blowup equation with characteristic $a = 1/2$, we solved the reduced one-string elliptic genus with flavor subgroup $\mathfrak{sp}(1)$ at leading q order as

$$\mathbb{E}_{h^{(1)}_{1,G_2}}(q_\tau, v) = q_\tau^{-1/3} + q_\tau^{2/3} v^{-2} \sum_{n=0}^{\infty} q_\tau^n \frac{P_n(v)}{(1+v)^6}, \tag{5.5.81}$$

where

$$P_0(v) = 2(7 - 7v - 129v^2 - 60v^3 + 1530v^4 + 5254v^5 + \cdots + 7v^{10}). \tag{5.5.82}$$

The above result agrees with the modular ansatz in (Del Zotto and Lockhart, 2018). Besides, turning on both gauge and flavor fugacities, we find the following v expansion for the subleading q_τ order of reduced one-string elliptic genus:

$$\begin{aligned} &\mathbf{14}v^{-2} - \mathbf{7} \cdot \chi^{\mathfrak{sp}(7)}_{(1000000)} v^{-1} + \mathbf{14} + \chi^{\mathfrak{sp}(7)}_{(2000000)} + 1 + \chi^{\mathfrak{sp}(7)}_{(0010000)} v + \chi^{\mathfrak{sp}(7)}_{(0001000)} v^2 \\ &+ (\chi^{\mathfrak{sp}(7)}_{(0000001)} - \mathbf{7} \cdot \chi^{\mathfrak{sp}(7)}_{(0010000)} - \mathbf{14} \cdot \chi^{\mathfrak{sp}(7)}_{(1000000)}) v^3 + \mathcal{O}(v^4). \end{aligned} \tag{5.5.83}$$

In fact, we find the following exact formula of the v expansion:

$$\begin{aligned} &\chi^{\mathfrak{sp}(7)}_{(0001000)} v^2 + \chi^{\mathfrak{sp}(7)}_{(0010000)} (v - \chi^{G_2}_{(10)} v^3) + \chi^{G_2}_{(20)} \chi^{\mathfrak{sp}(7)}_{(0100000)} v^4 - \chi^{\mathfrak{sp}(7)}_{(1000000)} (\chi^{G_2}_{(10)} v^{-1} \\ &+ \chi^{G_2}_{(01)} v^3 + \chi^{G_2}_{(30)} v^5) + \chi^{G_2}_{(01)} v^{-2} + \chi^{G_2}_{(01)} + \chi^{\mathfrak{sp}(7)}_{(2000000)} + 1 + \chi^{G_2}_{(11)} v^4 + \chi^{G_2}_{(40)} v^6 \\ &+ \sum_{n=0}^{\infty} \Big[\chi^{G_2}_{(0n)} \chi^{\mathfrak{sp}(7)}_{(0000001)} v^{3+2n} - \chi^{G_2}_{(1n)} \chi^{\mathfrak{sp}(7)}_{(0000010)} v^{4+2n} + \chi^{G_2}_{(2n)} \chi^{\mathfrak{sp}(7)}_{(0000100)} v^{5+2n} \\ &\quad - \chi^{G_2}_{(3n)} \chi^{\mathfrak{sp}(7)}_{(0001000)} v^{4+2n} + \chi^{G_2}_{(4n)} \chi^{\mathfrak{sp}(7)}_{(0010000)} v^{7+2n} - \chi^{G_2}_{(5n)} \chi^{\mathfrak{sp}(7)}_{(0100000)} v^{8+2n} \\ &\quad + \chi^{G_2}_{(6n)} \chi^{\mathfrak{sp}(7)}_{(1000000)} v^{9+2n} - \chi^{G_2}_{(7n)} v^{10+2n} \Big]. \end{aligned} \tag{5.5.84}$$

5.5.9 F_4 theories

$G = F_4$ theories on base curve $(-n)$, $n = 1, 2, 3, 4, 5$ have flavor group $F = \mathfrak{sp}(5-n)$ and $n_f = (5-n)$ hypermultiplets in the fundamental representation **26** of gauge symmetry. There only exist unity blowup equations but no vanishing equations due to the Lie algebra fact $Q^\vee \cong P^\vee$ for F_4. The corresponding Calabi-Yau geometries with flavor fugacities turned off were constructed in (Haghighat et al., 2015b; Kashani-Poor, 2019). The unity λ_F fields of these theories are just all the elements of the Weyl orbit $[0, 0, \ldots, 0, 1]$ of $\mathfrak{sp}(5-n)$. For $n = 3, 4, 5$ cases, we can use the recursion formula to exactly compute the elliptic genera to arbitrary numbers of strings. For $n = 1, 2$ cases, we used the Weyl orbit expansion to compute them. In the following, we discuss the $n = 1, 2, 3, 4, 5$ cases individually.

$\mathbf{n = 5, G = F_4}$

There exist 5 unity blowup equations in total. Using the recursion formula, we computed the one-string elliptic genus to $\mathcal{O}(q_\tau^3)$. Our result when turning off all gauge fugacities agrees precisely with the modular ansatz in (Del Zotto and Lockhart, 2018), therefore we just present the first few q_τ orders. Denote the reduced one-string elliptic genus as

$$\mathbb{E}_{h^{(1)}_{5,F_4}}(q_\tau,v)=q_\tau^{-4/3}v^8\sum_{n=0}^{\infty}q_\tau^n\frac{P_n(v)}{(1-v^2)^{16}}. \tag{5.5.85}$$

We obtain

$$\begin{aligned}P_0(v)&=1+36v^2+341v^4+1208v^6+1820v^8+1208v^{10}+341v^{12}+36v^{14}+v^{16},\\P_1(v)&=(1+v^2)^2(53+1478v^2+9419v^4+18036v^6+9419v^8+1478v^{10}+53v^{12}).\end{aligned} \tag{5.5.86}$$

One can also keep all flavor and gauge fugacities in the recursion formula to compute the full elliptic genus.

Using the recursion formula, we also computed the two-string elliptic genus to the subleading order of q_τ. For example, denote the reduced two-string elliptic genus as

$$\mathbb{E}_{h^{(2)}_{5,F_4}}(q_\tau,v)=q_\tau^{-17/6}v^{17}\sum_{n=0}^{\infty}q_\tau^n\frac{P^{(2)}_n(v)}{(1-v)^{34}(1+v)^{22}(1+v+v^2)^{17}},$$

we obtain

$$\begin{aligned}P_0(v)\ =\ &1+5v+48v^2+287v^3+1560v^4+7503v^5+32316v^6+125355v^7+444325v^8\\&+1443572v^9+4322993v^{10}+11989241v^{11}+30913094v^{12}+74321701v^{13}\\&+167106519v^{14}+352245510v^{15}+697557618v^{16}+1300152932v^{17}+2284606168v^{18}\\&+3790004228v^{19}+5943020899v^{20}+8818128233v^{21}+12392104012v^{22}\\&+16505926853v^{23}+20851379873v^{24}+24994963144v^{25}+28442119825v^{26}\\&+30731161887v^{27}+31533797982v^{28}+30731161887v^{29}+\cdots+v^{56},\\P_1(v)\ =\ &(1+v^2)(56+386v+3217v^2+20295v^3+110327v^4+529286v^5+2266151v^6\\&+8718327v^7+30479449v^8+97433532v^9+286304088v^{10}+777049966v^{11}\\&+1956035588v^{12}+4581942186v^{13}+10017235514v^{14}+20492637094v^{15}\\&+39315499928v^{16}+70871529676v^{17}+120240591034v^{18}+192278945658v^{19}\\&+290168035137v^{20}+413676858801v^{21}+557641624668v^{22}\\&+711294838217v^{23}+859008747683v^{24}+982638991174v^{25}+1065069893896v^{26}\\&+1094033908456v^{27}+1065069893896v^{28}+\cdots+v^{56}).\end{aligned} \tag{5.5.87}$$

$\mathbf{n=4}$, $\mathbf{G=F_4}$, $\mathbf{F}=\mathfrak{sp}(\mathbf{1})$

There exist 8 unity blowup equations in total with $\lambda_F=\pm1$. Using the recursion formula, we computed the one-string elliptic genus to $\mathcal{O}(q_\tau^3)$. Our result agrees precisely with the modular ansatz in (Del Zotto and Lockhart, 2018), therefore we just present the first few q_τ orders. Denote the reduced one-string elliptic genus as

$$\mathbb{E}_{h^{(1)}_{4,F_4}}(q_\tau,v) = q_\tau^{-5/6}v^7\sum_{n=0}^{\infty} q_\tau^n \frac{P_n(v)}{(1-v)^{10}(1+v)^{16}}. \tag{5.5.88}$$

We obtain

$$\begin{aligned} P_0(v) &= 1+10v-49v^2+266v^3-549v^4+1068v^5-1110v^6+\cdots+v^{12},\\ P_1(v) &= 2(28+277v-1552v^2+6305v^3-13020v^4+21834v^5-23904v^6+\cdots+28v^{12}). \end{aligned} \tag{5.5.89}$$

One can also keep all flavor and gauge fugacities in the recursion formula to compute the full elliptic genus. Indeed, as the leading q order of elliptic genus, we confirm the conjectural formula of the reduced 5d one-instanton partition function in (H.36) of (Del Zotto and Lockhart, 2018):

$$v^7+\sum_{n=0}^{\infty}\Big[-\chi^{F_4}_{(n000)}\chi^{\mathfrak{sp}(1)}_{(3)}v^{8+2n}+\chi^{F_4}_{(n001)}\chi^{\mathfrak{sp}(1)}_{(2)}v^{9+2n}-\chi^{F_4}_{(n010)}\chi^{\mathfrak{sp}(1)}_{(1)}v^{10+2n}+\chi^{F_4}_{(n100)}v^{11+2n}\Big].$$

For the subleading q order of the reduced one-string elliptic genus, we obtain the following v expansion

$$\begin{aligned} &(\mathbf{52}+1+\chi^{\mathfrak{sp}(1)}_{(2)})v^7+((\mathbf{52}+2)\chi^{\mathfrak{sp}(1)}_{(3)}+\chi^{\mathfrak{sp}(1)}_{(1)})v^8\\ -&(\mathbf{26}\cdot\chi^{\mathfrak{sp}(1)}_{(4)}+(\chi^{F_4}_{(1001)}+\mathbf{273}+3\cdot\mathbf{26})\chi^{\mathfrak{sp}(1)}_{(2)}+\mathbf{324}+\mathbf{26})v^9+\mathcal{O}(v^{10}) \end{aligned}$$

Using the recursion formula, we also computed the two-string elliptic genus to the subleading order of q_τ. For example, denote the reduced two-string elliptic genus as

$$\mathbb{E}_{h^{(2)}_{4,F_4}}(q_\tau,v) = q_\tau^{-11/6}v^{15}\sum_{n=0}^{\infty} q_\tau^n \frac{P^{(2)}_n(v)}{(1-v)^{22}(1+v)^{16}(1+v+v^2)^{17}},$$

we obtain

$$\begin{aligned} P_0(v) = &\,1+15v+34v^2+97v^3+715v^4+2022v^5+4997v^6+15039v^7+41395v^8\\ &+87572v^9+180994v^{10}+376306v^{11}+700157v^{12}+1152469v^{13}+1848360v^{14}\\ &+2846743v^{15}+3983439v^{16}+5139498v^{17}+6428973v^{18}+7611291v^{19}\\ &+8253543v^{20}+8388168v^{21}+\cdots+v^{42},\\ P_1(v) = &\,2(30+480v+1478v^2+4015v^3+20963v^4+63895v^5+157718v^6+414969v^7\\ &+1079969v^8+2315076v^9+4619079v^{10}+9059109v^{11}+16530696v^{12}\\ &+27157331v^{13}+42451387v^{14}+63499177v^{15}+88251928v^{16}+113833998v^{17}\\ &+140332628v^{18}+163891834v^{19}+178266540v^{20}+182276136v^{21}+\cdots+v^{42}). \end{aligned} \tag{5.5.90}$$

$\mathbf{n = 3}$, $\mathbf{G = F_4}$, $\mathbf{F = \mathfrak{sp}(2)}$

Using the recursion formula, we computed the one-string elliptic genus to $\mathcal{O}(q_\tau^3)$. Our result agrees precisely with the modular ansatz in (Del Zotto and Lockhart, 2018), therefore we just present the first few q_τ orders with all gauge and flavor fugacities turned off. Denote the reduced one-string elliptic genus as

$$\mathbb{E}_{h^{(1)}_{3,F_4}}(q_\tau, v) = q_\tau^{-1/3} v^6 \sum_{n=0}^{\infty} q_\tau^n \frac{P_n(v)}{(1-v)^4(1+v)^{16}}, \tag{5.5.91}$$

we obtain

$$\begin{aligned} P_0(v) &= 5 + 80v + 268v^2 - 1232v^3 + 2142v^4 - 1232v^5 + 268v^6 + 80v^7 + 5v^8, \\ P_1(v) &= v^{-8}(1 + 12v + 62v^2 + 172v^3 + 237v^4 - 20v^5 - 722v^6 - 1472v^7 - 1357v^8 \\ &\quad + 4812v^9 + 21908v^{10} - 72624v^{11} + 101054v^{12} + \cdots + v^{24}). \end{aligned} \tag{5.5.92}$$

Keeping all flavor and gauge fugacities in the recursion formula to compute the full elliptic genus. Indeed, as the leading q order of elliptic genus, we confirm the conjectural formula of reduced 5d one-instanton partition function in (H.36) of (Del Zotto and Lockhart, 2018). For example, the first few terms are

$$\begin{aligned} &\chi^{\mathfrak{sp}(2)}_{(01)} v^6 + \chi^{\mathfrak{sp}(2)}_{(30)} v^7 + (\chi^{\mathfrak{sp}(2)}_{(03)} - \mathbf{52} - \mathbf{26} \cdot \chi^{\mathfrak{sp}(2)}_{(20)}) v^8 + (\mathbf{273} \cdot \chi^{\mathfrak{sp}(2)}_{(10)} - \mathbf{26} \cdot \chi^{\mathfrak{sp}(2)}_{(12)}) v^9 \\ &+ (\mathbf{52} \cdot \chi^{\mathfrak{sp}(2)}_{(03)} + \mathbf{273} \cdot \chi^{\mathfrak{sp}(2)}_{(21)} + \mathbf{324} \cdot \chi^{\mathfrak{sp}(2)}_{(02)} - \mathbf{1274}) v^{10} + \mathcal{O}(v^{11}). \end{aligned}$$

For the subleading q order the reduced one-string elliptic genus, we obtain the following v expansion

$$\begin{aligned} &v^{-2} - \chi^{\mathfrak{sp}(2)}_{(10)} v^3 - \chi^{\mathfrak{sp}(2)}_{(20)} v^4 + \left(\chi^{\mathfrak{sp}(2)}_{(21)} + \chi^{\mathfrak{sp}(2)}_{(20)} + (\mathbf{52} + \mathbf{26} + 2)\chi^{\mathfrak{sp}(2)}_{(01)}\right) v^6 \\ &- (\chi^{\mathfrak{sp}(2)}_{(31)} + \chi^{\mathfrak{sp}(2)}_{(12)} + \chi^{\mathfrak{sp}(2)}_{(11)} + \chi^{\mathfrak{sp}(2)}_{(10)} + (\mathbf{52} + 2)\chi^{\mathfrak{sp}(2)}_{(30)}) v^7 + \mathcal{O}(v^8) \end{aligned}$$

We also computed the two-string elliptic genus to the subleading order of q_τ. For example, denote the reduced two-string elliptic genus as

$$\mathbb{E}_{h^{(2)}_{3,F_4}}(q_\tau, v) = q_\tau^{-5/6} v^{13} \sum_{n=0}^{\infty} q_\tau^n \frac{P^{(2)}_n(v)}{(1-v)^{10}(1+v)^{16}(1+v+v^2)^{17}},$$

we have

$$\begin{aligned} P^{(2)}_0(v) &= 15 + 449v + 5327v^2 + 30906v^3 + 101183v^4 + 187889v^5 + 183238v^6 \\ &+ 180121v^7 + 820970v^8 + 2527029v^9 + 3954101v^{10} + 3268018v^{11} + 2502062v^{12} \\ &+ 6631296v^{13} + 14672455v^{14} + 17834663v^{15} + 12802905v^{16} + 8758778v^{17} \\ &+ \cdots + 15v^{34}, \\ P^{(2)}_1(v) &= v^{-8}(5 + 145v + 1763v^2 + 11722v^3 + 53549v^4 + 182991v^5 + 493575v^6 \\ &+ 1078556v^7 + 1935972v^8 + 2865208v^9 + 3665294v^{10} + 5010010v^{11} + 8956794v^{12} \\ &+ 15093412v^{13} + 14295923v^{14} - 2110395v^{15} - 13976451v^{16} + 18409580v^{17} \end{aligned}$$

$$+78794748v^{18}+85716318v^{19}+44817687v^{20}+102304199v^{21}+290636920v^{22}$$
$$+388309453v^{23}+271239229v^{24}+167708226v^{25}+\cdots+5v^{50}). \qquad (5.5.93)$$

$\mathbf{n=2}$, $\mathbf{G=F_4}$, $\mathbf{F}=\mathfrak{sp}(\mathbf{3})$

Using the unity blowup equation with characteristic $a=0$ in Weyl orbit expansion, we solved the reduced one-string elliptic genus as

$$\mathbb{E}_{h^{(1)}_{2,F_4}}(q_\tau,v)=q_\tau^{1/6}v^{-1}\sum_{n=0}^{\infty}q_\tau^n\frac{(1-v)^2P_n(v)}{(1+v)^{16}}, \qquad (5.5.94)$$

where

$$P_0(v)=1+18v+155v^2+852v^3+3369v^4+10240v^5+24825v^6$$
$$+47834v^7+66180v^8+\cdots+v^{16}). \qquad (5.5.95)$$

The above result agrees with the modular ansatz in (Del Zotto and Lockhart, 2018). When turning on all fugacities, we find the leading q_τ order coefficient agrees with the exact formula of reduced 5d one-instanton partition function conjectured in (H.26) of (Del Zotto and Lockhart, 2018). For example, the first few terms are

$$v^{-1}-v^4\chi^{\mathfrak{sp}(3)}_{(001)}-v^5\chi^{\mathfrak{sp}(3)}_{(101)}-v^6\chi^{\mathfrak{sp}(3)}_{(201)}+v^7(\mathbf{52}\cdot\chi^{\mathfrak{sp}(3)}_{(010)}+\mathbf{26}\cdot\chi^{\mathfrak{sp}(3)}_{(101)}-\chi^{\mathfrak{sp}(3)}_{(030)})$$
$$+v^8(\mathbf{52}\cdot\chi^{\mathfrak{sp}(3)}_{(300)}-\mathbf{273}\cdot\chi^{\mathfrak{sp}(3)}_{(001)}+\mathbf{26}\cdot\chi^{\mathfrak{sp}(3)}_{(120)})+\mathcal{O}(v^9). \qquad (5.5.96)$$

One can also turn on full flavor fugacity and gauge fugacity and push the computation to higher q_τ orders and higher number of strings. For example, for the subleading q order of reduced one-string elliptic genus, we obtain

$$\begin{aligned}&\mathbf{52}v^{-3}-\mathbf{26}\cdot\chi^{\mathfrak{sp}(3)}_{(100)}v^{-2}+(\mathbf{52}+\chi^{\mathfrak{sp}(3)}_{(200)}+1)v^{-1}+\chi^{\mathfrak{sp}(3)}_{(300)}\\&+\chi^{\mathfrak{sp}(3)}_{(020)}v+\chi^{\mathfrak{sp}(3)}_{(011)}v^2+(\chi^{\mathfrak{sp}(3)}_{(002)}+\mathbf{26}\cdot\chi^{\mathfrak{sp}(3)}_{(010)})v^3+\mathcal{O}(v^4)\end{aligned} \qquad (5.5.97)$$

$\mathbf{n=1}$, $\mathbf{G=F_4}$, $\mathbf{F}=\mathfrak{sp}(\mathbf{4})$

Using the Weyl orbit expansion method and the unity blowup equation with characteristic $a=1/2$, we solved the reduced one-string elliptic genus as

$$\mathbb{E}_{h^{(1)}_{1,F_4}}(q_\tau,v)=q_\tau^{-1/3}+q_\tau^{2/3}v^{-2}\sum_{n=0}^{\infty}q_\tau^n\frac{P_n(v)}{(1+v)^{16}}, \qquad (5.5.98)$$

where

$$P_0(v)=52+624v+3001v^2+5704v^3-8932v^4-81464v^5-210244v^6-145256v^7$$
$$+896624v^8+3964136v^9+7404438v^{10}+\cdots+52v^{20}. \qquad (5.5.99)$$

We checked this agrees with the modular ansatz in (Del Zotto and Lockhart, 2018). Turning on all $\mathfrak{sp}(4)$ flavor fugacities, we obtain the first few terms are

$$\begin{aligned} &\mathbf{52}\, v^{-2} - \mathbf{26}\cdot\chi^{\mathfrak{sp}(4)}_{(1000)} v^{-1} + \mathbf{52} + \chi^{\mathfrak{sp}(4)}_{(2000)} + 1 + \chi^{\mathfrak{sp}(4)}_{(3000)} v + \chi^{\mathfrak{sp}(4)}_{(0200)} v^2 \\ &+ \chi^{\mathfrak{sp}(4)}_{(0110)} v^3 + (\chi^{\mathfrak{sp}(4)}_{(0020)} + \chi^{\mathfrak{sp}(4)}_{(2001)} - \mathbf{26}\cdot\chi^{\mathfrak{sp}(4)}_{(0001)}) v^4 + \mathcal{O}(v^5). \end{aligned} \tag{5.5.100}$$

This contains the information of the 5d Nekrasov partition function of the $G = F_4, F = \mathfrak{sp}(4)$ theory.

5.5.10 E_6 theories

$G = E_6$ theories on base curve $(-n)$ have flavor symmetry $F = \mathfrak{su}(6-n)_6 \times \mathfrak{u}(1)_{6(6-n)}$ and $n_f = (6-n)$ hypermultiplets in the bi-representation $(\mathbf{27}, (\mathbf{6}-\mathbf{n})_1)$. Note $\mathbf{6}-\mathbf{n}$ is the fundamental representation of flavor symmetry $\mathfrak{su}(6-n)$, and $n = 1,2,3,4,5,6$. There are $2n$ vanishing blowup equations with $\lambda_{\mathfrak{su}(6-n)} = 0$ and $\lambda_{\mathfrak{u}(1)} = \pm 1/6$.

The reason there are two copies of vanishing equations is that the Dynkin diagram of E_6 is axisymmetric, in particular there exist two fundamental representations of E_6: $\mathbf{27}$ and $\overline{\mathbf{27}}$. For any two weights $w_1, w_2 \in \mathbf{27}$, $w_1 \cdot w_2 = 4/3, 1/3, -2/3$. The same for $\overline{\mathbf{27}}$. While for $w_1 \in \mathbf{27}$ and $w_2 \in \overline{\mathbf{27}}$, one has $w_1 \cdot w_2 = -4/3, -1/3, 2/3$. Since $(P^\vee/Q^\vee)_{E_6} = \mathbb{Z}_3$, accordingly let us denote $P^\vee = Q^\vee \oplus \Lambda \oplus \overline{\Lambda}$, such that $\mathbf{27} \subset \Lambda$ and $\overline{\mathbf{27}} \subset \overline{\Lambda}$. For any $w_1 \in \mathbf{27}$, $w_2 \in \overline{\mathbf{27}}$, $\lambda_1 \in \Lambda$ and $\lambda_2 \in \overline{\Lambda}$, always

$$\begin{aligned} &w_1 \cdot \lambda_1 \in \mathbb{Z} + 1/3, \qquad w_1 \cdot \lambda_2 \in \mathbb{Z} - 1/3, \\ &w_2 \cdot \lambda_1 \in \mathbb{Z} - 1/3, \qquad w_2 \cdot \lambda_2 \in \mathbb{Z} + 1/3. \end{aligned} \tag{5.5.101}$$

It is easy to find the leading base degree of one copy of the vanishing blowup equations

$$\begin{aligned} &\sum_{w \in \mathbf{27}} (-1)^{|w|} \theta_i^{[a]}(n\tau, n m_w^{E_6} + (6-n)\epsilon'_+) \prod_{w' \in \mathbf{6}-\mathbf{n}} \theta_1(m_w^{E_6} + m_{w'}^{su(6-n)} - \epsilon'_+) \prod_{\alpha \in \Delta(E_6)}^{w\cdot\alpha=1} \frac{1}{\theta_1(m_\alpha^{E_6})} \\ &= 0, \end{aligned} \tag{5.5.102}$$

where we denote $\epsilon'_+ = m_{u(1)} + \epsilon_+$. We have verified this identity up to q_τ^{10} for all $n = 1,2,\ldots,6$. Note this identity contains m_{E_6}, $m_{su(6-n)}$, $m_{u(1)}$ and ϵ_+ as free parameters, thus are highly nontrivial. By setting m_F as zero, one obtains an easier identity:

$$\sum_{w \in \mathbf{27}} (-1)^{|w|} \theta_i^{[a]}(n\tau, n m_w + (6-n)\epsilon_+) \big(\theta_1(m_w - \epsilon_+)\big)^{6-n} \prod_{\alpha \in \Delta(E_6)}^{w\cdot\alpha=1} \frac{1}{\theta_1(m_\alpha)} = 0. \tag{5.5.103}$$

For unity blowup equations, there are 2^{6-n} choices for λ_F fields. In fact, they form the Weyl orbit $\mathcal{O}^{\mathfrak{sp}(6-n)}_{[00\ldots01]}$ if we embed $\mathfrak{su}(6-n) \times \mathfrak{u}(1)$ into $\mathfrak{sp}(6-n)$. Note there always exist λ_F fields $(\lambda_{su(6-n)}, \lambda_{u(1)}) = (0, \pm 1/2)$. For $n = 3,4,5,6$, one can choose arbitrary one λ_F and three unity blowup equations with different characteristics a to solve elliptic genera recursively.

$\mathbf{n=6,\ G=E_6}$

There are 6 unity blowup equations. Using the recursion formula, we computed the one-string elliptic genus with all fugacities turned off to $\mathcal{O}(q_\tau^1)$. Our result agrees precisely with the modular ansatz in (Del Zotto and Lockhart, 2018), therefore we just present the first few q_τ orders. Denote the reduced one-string elliptic genus as

$$\mathbb{E}_{h^{(1)}_{6,E_6}}(q_\tau,v)=q_\tau^{-11/6}v^{11}\sum_{n=0}^{\infty}q_\tau^n\frac{P_n(v)}{(1-v^2)^{22}},\tag{5.5.104}$$

We obtain

$$P_0(v)=1+56v^2+945v^4+6776v^6+23815v^8+43989v^{10}+\cdots+v^{22},$$
$$P_1(v)=79+3774v^2+54206v^4+337457v^6+1067286v^8+1862806v^{10}+\cdots+79v^{22}.$$

Using the recursion formula, we also computed the two-string elliptic genus to the subleading order of q_τ which will be given in Appendix D.

$\mathbf{n=5,\ G=E_6,\ F=\mathfrak{u}(1)}$

There exist 5 unity blowup equations with $r_F=0$. Using the recursion formula, we computed the one-string elliptic genus to $\mathcal{O}(q_\tau)$. Our result agrees precisely with the modular ansatz in (Del Zotto and Lockhart, 2018), therefore we just present the first few q_τ orders. For example, denote the reduced one-string elliptic genus with all gauge and flavor fugacities turned off as

$$\mathbb{E}_{h^{(1)}_{5,E_6}}(q_\tau,v)=q_\tau^{-4/3}v^{10}\sum_{n=0}^{\infty}q_\tau^n\frac{P_n(v)}{(1-v)^{16}(1+v)^{22}},\tag{5.5.105}$$

we obtain

$$\begin{aligned}P_0(v)=&\,1+8v-43v^2+456v^3-1436v^4+5116v^5-9848v^6+19504v^7-24164v^8\\&+30016v^9+\cdots+v^{18},\\P_1(v)=&\,2(40+320v-2072v^2+16128v^3-51094v^4+155036v^5-297317v^6\\&+530598v^7-670889v^8+785764v^9-\cdots+40v^{18}).\end{aligned}\tag{5.5.106}$$

By keeping the gauge and flavor fugacities in the recursion formula and taking the leading q_τ order, we confirm the conjectural formula of reduced 5d one-instanton partition function in (H.38) of (Del Zotto and Lockhart, 2018):

$$\begin{aligned}v^{10}+\sum_{n=0}^{\infty}\Big[&\chi^{E_6}_{(00000n)}\chi^{u(1)}_{(3)\oplus(-3)}v^{11+2n}-(\chi^{E_6}_{(00001n)}\chi^{u(1)}_{(-2)}+\chi^{E_6}_{(10000n)}\chi^{u(1)}_{(2)})v^{12+2n}\\&+(\chi^{E_6}_{(00010n)}\chi^{u(1)}_{(-1)}+\chi^{E_6}_{(01000n)}\chi^{u(1)}_{(1)})v^{13+2n}-\chi^{E_6}_{(00100n)}v^{14+2n}\Big].\end{aligned}\tag{5.5.107}$$

For the subleading q_τ order, we obtain

$$\begin{aligned}&(\mathbf{78}+2)v^{10}+(\mathbf{78}+2)\chi^{u(1)}_{(3)\oplus(-3)}v^{11}-\Big((\chi^{E_6}_{(100000)}\chi^{u(1)}_{(-4)}\\&+\chi^{E_6}_{(000011)\oplus(010000)\oplus 3(000010)}\chi^{u(1)}_{(-2)}+c.c.)+\chi^{E_6}_{(100010)}\Big)v^{12}+\mathcal{O}(v^{13}).\end{aligned}$$

Using recursion formula, we also computed the two-string elliptic genus to the subleading order of q_τ which will be given in Appendix D.

$\mathbf{n=4,\ G=E_6,\ F=\mathfrak{su}(2)\times\mathfrak{u}(1)}$

Using the recursion formula, we computed the one-string elliptic genus to $\mathcal{O}(q_\tau^2)$. Our result agrees precisely with the modular ansatz in (Del Zotto and Lockhart, 2018), therefore we just present the first few q_τ orders. Denote the reduced one-string elliptic genus as

$$\mathbb{E}_{h^{(1)}_{4,E_6}}(q_\tau,v)=q_\tau^{-5/6}v^9\sum_{n=0}^{\infty}q_\tau^n\frac{P_n(v)}{(1-v)^{10}(1+v)^{22}},\tag{5.5.108}$$

We obtain

$$\begin{aligned}P_0(v)=&-(3+44v+33v^2-1052v^3+6513v^4-17404v^5+31905v^6\\&-37432v^7+\cdots+3v^{14}),\\P_1(v)=&\,v^{-2}(3+36v-135v^2-4000v^3-3894v^4+106168v^5-500700v^6\\&+1239080v^7-2078322v^8+2430488v^9-\cdots+3v^{18}).\end{aligned}\tag{5.5.109}$$

One can also keep all flavor and gauge fugacities in blowup equations to compute the full elliptic genus. In (Del Zotto and Lockhart, 2018), the Weyl orbit expansion of reduced 5d one-instanton partition function was conjectured up to v^{11}. Using the recursion formula from blowup equations, we find the following exact formula where $F=\mathfrak{su}(2)_a\times\mathfrak{u}(1)_b$:

$$\begin{aligned}&-v^9\chi^F_{(2)_a}-v^{10}\chi^F_{(3)_a\otimes((3)_b\oplus(-3)_b)}+v^{11}(\chi^{E_6}_{(100000)}\chi^F_{(2)_a\otimes(2)_b}+c.c.)+v^{11}\chi^{E_6}_{(000001)}\\&-v^{12}(\chi^{E_6}_{(010000)}\chi^F_{(1)_a\otimes(1)_b}+c.c)+v^{13}\chi^{E_6}_{(001000)}+\sum_{n=0}^{\infty}\Big[-v^{11+2n}\chi^{E_6}_{(00000n)}(\chi^F_{(6)_b\oplus(-6)_b}+\chi^F_{(6)_a})\\&+v^{12+2n}(\chi^{E_6}_{(10000n)}\chi^F_{(1)_a\otimes(5)_b}+\chi^{E_6}_{(00001n)}\chi^F_{(5)_a\otimes(1)_b}+c.c.)\\&-v^{13+2n}\Big((\chi^{E_6}_{(01000n)}\chi^F_{(2)_a\otimes(4)_b}+\chi^{E_6}_{(00010n)}\chi^F_{(4)_a\otimes(2)_b}+\chi^{E_6}_{(20000n)}\chi^F_{(4)_b}+c.c.)+\chi^G_{(10001n)}\chi^F_{(4)_a}\Big)\\&+v^{14+2n}\Big(\chi^{E_6}_{(00100n)}\chi^F_{(3)_a\otimes((3)_b\oplus(-3)_b)}+(\chi^{E_6}_{(11000n)}\chi^F_{(1)_a\otimes(3)_b}+\chi^{E_6}_{(10010n)}\chi^F_{(3)_a\otimes(1)_b}+c.c.)\Big)\\&-v^{15+2n}\Big((\chi^{E_6}_{(10100n)}\chi^F_{(2)_a\otimes(2)_b}+\chi^{E_6}_{(02000n)}\chi^F_{(-2)_b}+c.c.)+\chi^{E_6}_{(01010n)}\chi^F_{(2)_a}\Big)\\&+v^{16+2n}(\chi^{E_6}_{(01100n)}\chi^F_{(1)_a\otimes(1)_b}+c.c.)-v^{17+2n}\chi^{E_6}_{(00200n)}\Big].\end{aligned}\tag{5.5.110}$$

This formula can be reconfirmed by the Weyl dimension formula of representation of E_6 and $\mathfrak{su}(2)$, where one can obtain the rational function of v as in (5.5.108). For the subleading q_τ order of reduced one-string elliptic genus, we obtain

$$\begin{aligned}&\chi^F_{(2)_a}v^7-(\chi^F_{(4)_a}+(\mathbf{78}+3)\chi^F_{(2)_a}+\chi^{E_6}_{(000010)}\chi^F_{(-2)_b}+\chi^{E_6}_{(100000)}\chi^F_{(2)_b}+1)v^9\\&-(\chi^F_{(5)_a}+(\mathbf{78}+3)\chi^F_{(3)_a}+\chi^F_{(1)_a})\chi^F_{(-3)_b\oplus(3)_b}v^{10}+\mathcal{O}(v^{11}).\end{aligned}$$

Using the recursion formula, we also computed the two-string elliptic genus to the subleading order of q_τ which will be given in Appendix D.

$\mathbf{n=3}$, $\mathbf{G=E_6}$, $\mathbf{F=\mathfrak{su}(3)\times\mathfrak{u}(1)}$

Using the recursion formula, we computed the one-string elliptic genus to $\mathcal{O}(q_\tau^3)$. Our result agrees precisely with the modular ansatz in (Del Zotto and Lockhart, 2018), therefore we just present the first few q_τ orders. Denote the reduced one-string elliptic genus as

$$\mathbb{E}_{h^{(1)}_{3,E_6}}(q_\tau,v)=q_\tau^{-1/3}v^7\sum_{n=0}^{\infty}q_\tau^n\frac{P_n(v)}{(1-v)^4(1+v)^{12}},\tag{5.5.111}$$

We obtain

$$\begin{aligned}P_0(v)&=2(1+28v+356v^2+2045v^3+1583v^4-19638v^5+36572v^6-\cdots+v^{12}),\\P_1(v)&=v^{-9}(1+18v+149v^2+744v^3+2454v^4+5412v^5+7230v^6+2216v^7-14256v^8\\&\quad-39160v^9-61154v^{10}-18988v^{11}+372829v^{12}+642294v^{13}-3309245v^{14}\\&\quad+4904064v^{15}+\cdots+v^{30}).\end{aligned}\tag{5.5.112}$$

We can also turn on all gauge and flavor fugacities. Using recursion formula from blowup equations, we find the exact formula for the leading q_τ order of reduced one-string elliptic genus with $F=\mathfrak{su}(3)_a\times\mathfrak{u}(1)_b$, which will be presented in Appendix (D.0.34). The first few terms are

$$\begin{aligned}&v^7\chi^F_{(3)_b\oplus(-3)_b}+v^8(\chi^F_{(30)_a}+\chi^F_{(03)_a})+v^9\chi^F_{(22)_a\otimes((3)_b\oplus(-3)_b)}\\&-v^{10}(\chi^G_{(100000)}\chi^F_{(12)_a\otimes(2)_b}+\chi^G_{(000010)}\chi^F_{(21)_a\otimes(-2)_b}+\chi^G_{(000001)}\chi^F_{(11)_a}\\&-\chi^F_{(30)_a\otimes(-6)_b}-\chi^F_{(03)_a\otimes(6)_b}-\chi^F_{(33)_a})+\mathcal{O}(v^{12}),\end{aligned}\tag{5.5.113}$$

which were already conjectured in (Del Zotto and Lockhart, 2018). For the subleading q_τ order of reduced one-string elliptic genus, we obtain

$$v^{-2}-\chi^F_{(11)_a}v^4-\chi^F_{(11)_a\otimes((3)_b\oplus(-3)_b)}v^5-\chi^F_{(22)_a}v^6+(\mathbf{78}+\chi^F_{(11)_a}+2)\chi^F_{(3)_b\oplus(-3)_b}v^7+\mathcal{O}(v^8).$$

Using recursion formula, we also computed the two-string elliptic genus to the leading order of q_τ which will be given in Appendix D.

$\mathbf{n=2}$, $\mathbf{G=E_6}$, $\mathbf{F=\mathfrak{su}(4)\times\mathfrak{u}(1)}$

We use Weyl orbit expansion to solve elliptic genus for this theory. Let us first turn off the $\mathfrak{su}(4)$ fugacities and only keep $\mathfrak{u}(1)$ and make use of the unity blowup equations with nonzero λ_F only on $\mathfrak{u}(1)$. Then the reduced one-string elliptic genus can be computed as

$$\mathbb{E}_{h^{(1)}_{2,E_6}}(q_\tau,v)=q_\tau^{1/6}v^{-1}\sum_{n=0}^{\infty}q_\tau^n\frac{P_n(v)}{(1+v)^{22}},\tag{5.5.114}$$

where

$$
\begin{aligned}
P_0(v) = &(1-v)^2(1+24v+278v^2+2072v^3+11181v^4+46624v^5+156660v^6\\
&+436728v^7+1030043v^8+2066568v^9+3435967v^{10}+4315392v^{11}\\
&+3435967v^{12}+\cdots+v^{22}),\\
P_1(v) = &v^{-2}(78+1500v+13361v^2+72354v^3+260839v^4+631520v^5+910434v^6\\
&+142972v^7-2884243v^8-7465814v^9-7830327v^{10}+5820340v^{11}\\
&+30116822v^{12}+14704216v^{13}-68988104v^{14}+14704216v^{15}+\cdots+78v^{28}).
\end{aligned}
\tag{5.5.115}
$$

We have cross-checked our result against the modular ansatz in (Del Zotto and Lockhart, 2018).[21] Let us denote

$$\mathbb{E}_{h^{(1)}_{2,E_6}}(q_\tau, v) = q_\tau^{1/6} v^{-1} \sum_{i,j} c_{i,j} v^j (q_\tau/v^2)^i. \tag{5.5.116}$$

Then we have the following table 5.15 for the coefficients c_{ij}. Note the red numbers in the first column are just the dimensions of representations $k\theta$ of E_6 where θ is the adjoint representation. The blue numbers in the second column are eight times of the dimensions of representations $\square + k\theta$ of E_6, where the eight is the double of the dimension of matter representation **4** of flavor $\mathfrak{su}(4)$. The orange number 95 in the third column is given by $\dim(E_6)+\dim(\mathfrak{su}(4)\times\mathfrak{u}(1))+1 = 78+16+1 = 95$. These are the constraints predicted in (Del Zotto and Lockhart, 2018) by analyzing the spectral flow to Neveu-Schwarz-Ramond elliptic genus.

i,j	0	1	2	3	4	5	6	7	8
0	1	0	0	0	0	0	-20	-72	-319
1	78	-216	95	40	84	120	195	1248	-2155
2	2430	-13824	28392	-20520	-1555	-3760	3102	12264	17277
3	43758	-370656	1334745	-2526856	2380950	-587824	-213080	-601120	-339398

Table 5.15: Series coefficients $c_{i,j}$ for the one-string elliptic genus of $\mathfrak{n} = 2$ E_6 model.

We also computed the elliptic genus with all flavor $\mathfrak{su}(4)_a \times \mathfrak{u}(1)_b$ fugacities turned on and gauge fugacities turned off. For example, the q_τ leading order of

[21] In (Del Zotto and Lockhart, 2018), the modular ansatz for this theory is determined up to three unfixed parameters. Using our result from blowup equations, we are able to determine their three unfixed parameters as

$$a_1 = \frac{6581939}{638959998741245853696}, a_2 = -\frac{12286901}{5111679989929966829568}, a_3 = \frac{16984805}{5750639988671212683264}.$$

reduced one-string elliptic genus has v expansion as

$$
\begin{aligned}
&v^{-1} - \chi^F_{(020)_a} v^5 - (\chi^F_{(102)_a\otimes(3)_b} + c.c.)v^6 - \Big((\chi^F_{(200)_a\oplus(6)_b} + 27\chi^F_{(4)_b} + c.c.) \\
&+ \chi^F_{(400)_a\oplus(004)_a\oplus(121)_a} \Big) v^7 + (27\chi^F_{(100)_a\otimes(5)_b} + 78\chi^F_{(001)_a\otimes(3)_b} - \chi^F_{((130)_a\oplus(203)_a)\otimes(3)_b} + c.c.)v^8 \\
&- \Big((\chi^F_{(022)_a\otimes(6)_b} + 351\chi_{(4)_b} - 27\chi^F_{((030)_a\oplus(103)_a)\otimes(2)_b} + c.c) + \chi^F_{(222)_a} - 78\chi^F_{(210)_a\oplus(012)_a} \Big) v^9 \\
&\quad + \mathcal{O}(v^{10}),
\end{aligned}
$$

or in the descending order of the absolute value of $\mathfrak{u}(1)$ charge as

$$
\sum_{n=0}^{\infty} \Big[-\chi^F_{(-12)_b\oplus(12)_b} \chi^{E_6}_{(00000n)} v^{11+2n} + (\chi^F_{(001)a\otimes(11)_b}\chi^{E_6}_{(10000n)} + c.c.)v^{12+2n} + \dots \Big]. \tag{5.5.117}
$$

$\mathbf{n=1}$, $\mathbf{G=E_6}$, $\mathbf{F}=\mathfrak{su}(\mathbf{5})\times\mathfrak{u}(\mathbf{1})$

We use Weyl orbit expansion to solve elliptic genus for this theory. Let us first turn off the $\mathfrak{su}(5)$ fugacities and only keep $\mathfrak{u}(1)$ and make use of the unity blowup equations with nonzero λ_F only on $\mathfrak{u}(1)$. Then the reduced one-string elliptic genus with all gauge and flavor fugacities turned off can be computed as

$$
\mathbb{E}_{h^{(1)}_{1,E_6}}(q_\tau, v) = q_\tau^{-1/3} + q_\tau^{2/3} v^{-2} \sum_{n=0}^{\infty} q_\tau^n \frac{P_n(v)}{(1+v)^{22}}, \tag{5.5.118}
$$

where

$$
\begin{aligned}
P_0(v) =& 78 + 1446v + 12182v^2 + 60108v^3 + 180534v^4 + 260152v^5 - 365242v^6 \\
&- 3157324v^7 - 9013936v^8 - 13246110v^9 + 3729696v^{10} + 83186464v^{11} \\
&+ 255829040v^{12} + 405233216v^{13} + \cdots + 78v^{26}, \\
P_1(v) =& v^{-2}(2430 + 36180v + 222432v^2 + 630204v^3 + 69266v^4 - 5565632v^5 \\
&- 17594496v^6 - 11700192v^7 + 74362142v^8 + 245593684v^9 + 202313896v^{10} \\
&- 730064340v^{11} - 2618359266v^{12} - 2448587624v^{13} + 5677163436v^{14} \\
&+ 16560265456v^{15} + \cdots + 2430v^{30}).
\end{aligned} \tag{5.5.119}
$$

We have cross-checked our result against the modular ansatz in (Del Zotto and Lockhart, 2018).[22] Let us further denote

$$
\mathbb{E}_{h^{(1)}_{1,E_6}}(q_\tau, v) = q_\tau^{-1/3} \sum_{i,j} c_{i,j} v^j (q_\tau/v^2)^i. \tag{5.5.120}
$$

[22] In (Del Zotto and Lockhart, 2018), the modular ansatz for this theory is determined up to three unfixed parameters. Using our result from blowup equations, we are able to determine their three unfixed parameters as

$$
a_1 = -\frac{14389465}{35941499929195079270 4}, a_2 = -\frac{227027173}{115012799773424253665 28}, a_3 = \frac{146734631}{345038399320272760995 84}.
$$

Then we have the following table 5.16 for the coefficients c_{ij}. Note the red numbers in the first column are just the dimensions of representations $k\theta$ of E_6 where θ is the adjoint representation. The blue numbers in the second column are 10 times of the dimensions of representations $\square + k\theta$ of E_6, where the 10 is the double of the dimension of matter representation $\mathbf{5}$ of flavor $\mathfrak{su}(5)$. The orange number 104 in the third column is given by $\dim(E_6) + \dim(\mathfrak{su}(5) \times \mathfrak{u}(1)) + 1 = 78 + 25 + 1 = 104$. These are the constraints predicted in (Del Zotto and Lockhart, 2018) by analyzing the spectral flow to Neveu-Schwarz-Ramond elliptic genus.

i,j	0	1	2	3	4	5	6	7	8
0	1	0	0	0	0	0	0	0	0
1	78	-270	104	70	200	420	1124	5220	3468
2	2430	-17280	41262	-28080	-8746	-18640	-10490	7680	35296
3	43758	-463320	1999296	-4254770	3930732	-200322	-14660	-1987042	-3198410

Table 5.16: Series coefficients $c_{i,j}$ for the one-string elliptic genus of $\mathfrak{n} = 1$ E_6 model.

Let us also show some results with all flavor $\mathfrak{su}(5)_a \times \mathfrak{u}(1)_b$ fugacities turned on. For example, the q_τ subleading order of reduced one-string elliptic genus with $m_{E_6} = 0$ is

$$78\,v^{-2} - (27\chi^F_{(1000)_a\oplus(1)_b} + c.c.)v^{-1} + \chi^F_{(1001)_a} + 80 + (\chi^F_{(3000)_a\oplus(3)_b} + c.c.)v \\ + \chi^F_{(2002)_a} v^2 + (\chi^F_{(0201)_a\oplus(3)_b} + c.c.)v^3 + \mathcal{O}(v^4),$$

or in the descending order of the absolute value of $\mathfrak{u}(1)$ charge as

$$\sum_{n=0}^{\infty} \Big[\chi^F_{(-15)_b\oplus(15)_b}\chi^{E_6}_{(00000n)} v^{11+2n} - (\chi^F_{(0001)a\otimes(14)_b}\chi^{E_6}_{(10000n)} + c.c.)v^{12+2n} + \dots\Big]. \tag{5.5.121}$$

5.5.11 E_7 theories

$G = E_7$ theories on base curve $(-n)$ have flavor symmetry $F = \mathfrak{so}(8-n)$ and $n_f = (8-n)/2$ hypermultiplets in bi-representation $\frac{1}{2}(\mathbf{56}, \mathbf{8-n})$. Note $\mathbf{8-n}$ is the fundamental representation of flavor group and $n = 1,2,3,\dots,7,8$. There are n vanishing blowup equations with $\lambda_F = 0$. Using the fact that the minimal Weyl orbit of $(P^\vee \backslash Q^\vee)_{E_7}$ consists just of weights of $\mathbf{56}$, it is easy to find the leading base degree of the vanishing blowup equations can be written as

$$\sum_{w\in\mathbf{56}} (-1)^{|w|}\theta_i^{[a]}(n\tau, nm_w^{E_7} + (8-n)\epsilon_+) \prod_{w'\in\mathbf{8-n}} \theta_1(m_w^{E_7} + m_{w'}^{\mathfrak{so}(8-n)} - \epsilon_+) \prod_{\alpha\in\Delta(E_7)}^{w\cdot\alpha=1} \frac{1}{\theta_1(m_\alpha^{E_7})} \\ = 0, \tag{5.5.122}$$

which we have checked to be correct up to q_τ^{20} for all n. Note these identities contain m_{E_7}, $m_{\mathfrak{so}(8-n)}$ and ϵ_+ as free parameters, thus are highly nontrivial. By setting

$m_{\mathfrak{so}(8-n)}$ as zero, one obtains an easier identity:

$$\sum_{w\in\mathbf{56}}(-1)^{|w|}\theta_i^{[a]}(n\tau,nm_w+(8-n)\epsilon_+)\big(\theta_1(m_w-\epsilon_+)\big)^{8-n}\prod_{\alpha\in\Delta(E_7)}^{w\cdot\alpha=1}\frac{1}{\theta_1(m_\alpha)}=0. \tag{5.5.123}$$

The unity blowup equations for $G=E_7$ theories only exist for even n, because for odd n the theory involves half-hyper. In the following, we discuss the cases $n=8,6,4,2$ individually.

$\mathbf{n=8,\ G=E_7}$

There are 8 unity blowup equations. Using the recursion formula, we computed the one-string elliptic genus with all fugacities turned off to $\mathcal{O}(q_\tau^1)$. Our result agrees precisely with the modular ansatz in (Del Zotto and Lockhart, 2018), therefore we just present the first few q_τ orders. Denote the reduced one-string elliptic genus as

$$\mathbb{E}_{h^{(1)}_{8,E_7}}(q_\tau,v)=q_\tau^{-17/6}v^{17}\sum_{n=0}^{\infty}q_\tau^n\frac{P_n(v)}{(1-v^2)^{34}}, \tag{5.5.124}$$

We obtain

$$\begin{aligned}P_0(v)=&(1+v^2)(1+98v^2+3312v^4+53305v^6+468612v^8+2421286v^{10}+7664780v^{12}\\&+15203076v^{14}+19086400v^{16}+15203076v^{18}+\cdots+v^{32}),\\P_1(v)=&(1+v^2)(134+11593v^2+345521v^4+4931707v^6+38850151v^8+182614170v^{10}\\&+536726278v^{12}+1014596958v^{14}+1252490096v^{16}+1014596958v^{18}+\cdots+v^{32}).\end{aligned}$$

Using the recursion formula, we also computed the two-string elliptic genus to the subleading order of q_τ which will be given in Appendix D.

$\mathbf{n=6,\ G=E_7,\ F=\mathfrak{so}(2)}$

There are 12 unity blowup equations with $\lambda_F=(\pm1)$. Using the recursion formula, we computed the one-string elliptic genus with all fugacities turned off to $\mathcal{O}(q_\tau^1)$. Our result agrees precisely with the modular ansatz in (Del Zotto and Lockhart, 2018), therefore we just present the first few q_τ orders. Denote the reduced one-string elliptic genus as

$$\mathbb{E}_{h^{(1)}_{6,E_7}}(q_\tau,v)=q_\tau^{-11/6}v^{15}\sum_{n=0}^{\infty}q_\tau^n\frac{P_n(v)}{(1-v)^{22}(1+v)^{34}}, \tag{5.5.125}$$

We obtain

$$
\begin{aligned}
P_0(v) = &-(2+24v-43v^2+52v^3+8027v^4-53360v^5+279039v^6-950972v^7\\
&+2698740v^8-5898532v^9+10988680v^{10}-16600348v^{11}+21616127v^{12}\\
&-23243264v^{13}+\cdots+v^{26}),\\
P_1(v) = &\, v^{-2}(1+12v-226v^2-3284v^3+8157v^4+28752v^5-1098207v^6+6964508v^7\\
&-32103023v^8+103825488v^9-273840598v^{10}+575865704v^{11}-1024745731v^{12}\\
&+1517074676v^{13}-1931373701v^{14}+2077804192v^{15}+\cdots+v^{30}).
\end{aligned}
$$

With gauge and flavor fugacities turned on, we confirm the conjectural exact formula for the reduced 5d one-instanton partition function in (H.40) of (Del Zotto and Lockhart, 2018). For example, the leading q order of (5.5.125) is

$$
\begin{aligned}
&-\chi^F_{(2)\oplus(-2)}v^{15}-(\chi^F_{(6)\oplus(-6)}-\mathbf{133})v^{17}-(\mathbf{912}\cdot\chi^F_{(1)\oplus(-1)}-\mathbf{56}\cdot\chi^F_{(5)\oplus(-5)})v^{18}\\
&\quad+(\mathbf{8645}-\mathbf{133}\cdot\chi^F_{(6)\oplus(-6)}-\mathbf{1539}\cdot\chi^F_{(4)\oplus(-4)})v^{19}+\mathcal{O}(v^{20}),
\end{aligned}
\tag{5.5.126}
$$

and the subleading q order is

$$
\begin{aligned}
&v^{13}-(\mathbf{133}+2)\chi^F_{(4)\oplus(-4)}v^{15}+(-(\mathbf{133}+2)\chi^F_{(6)\oplus(-6)}+\mathbf{1539}\cdot\chi^F_{(2)\oplus(-2)}\\
&+\mathbf{8645}+\mathbf{7371}+\mathbf{1539}+3\cdot\mathbf{133}+1)v^{17}+\mathcal{O}(v^{18}).
\end{aligned}
\tag{5.5.127}
$$

$\mathbf{n=4,\ G=E_7,\ F=\mathfrak{so}(4)}$

There are 16 unity blowup equations with $\lambda_F=(\pm1,\pm1)$ if we regard $\mathfrak{so}(4)\cong\mathfrak{su}(2)\times\mathfrak{su}(2)$. Using the recursion formula, we computed the one-string elliptic genus with flavor fugacities turned off to $\mathcal{O}(q_\tau^4)$. Denote the reduced one-string elliptic genus as

$$
\mathbb{E}_{h^{(1)}_{4,E_7}}(q_\tau,v)=q_\tau^{-5/6}v^{11}\sum_{n=0}^{\infty}q_\tau^n\frac{P_n(v)}{(1-v)^{10}(1+v)^{34}},
\tag{5.5.128}
$$

We obtain

$$
\begin{aligned}
P_0(v) = &-(1+24v+305v^2+2720v^3+14385v^4+10328v^5-213107v^6+227936v^7\\
&+3681535v^8-15349240v^9+32121373v^{10}-40005232v^{11}+32121373v^{12}+\cdots+v^{22}).\\
P_1(v) = &\, v^{-2}(9+216v+2296v^2+13704v^3+35681v^4-191536v^5-2195202v^6\\
&-3469024v^7+34360924v^8+12656096v^9-543596903v^{10}+1892316824v^{11}\\
&-3595032965v^{12}+4390454000v^{13}+\cdots+9v^{26}).
\end{aligned}
$$

The leading q order exactly agrees with the reduced 5d one-instanton partition function in (A.20) of (Kim et al., 2019). Let us denote

$$
\mathbb{E}_{h^{(1)}_{4,E_7}}(q_\tau,v,m_{E_7}=0,m_{\mathfrak{so}(4)}=0)=q_\tau^{-5/6}v^{11}\sum_{i,j}c_{i,j}v^j(q_\tau/v^2)^i.
\tag{5.5.129}
$$

Then we have the following Table 5.17 for the coefficients c_{ij}. Note the red numbers in the first column are just the dimensions of representations $k\theta$ of E_7 where θ is the adjoint representation. The blue numbers in the second column are four times the dimensions of representations $\square + k\theta$ of E_7, where the four is the dimension of matter representation **4** of flavor $\mathfrak{so}(4)$. The orange number 140 in the third column is given by $\dim(E_7)+\dim(\mathfrak{so}(4))+1 = 133+6+1 = 140$. These are the constraints given in (Del Zotto and Lockhart, 2018) by analyzing the spectral flow to Neveu-Schwarz-Ramond elliptic genus which our result satisfies perfectly. By combining our result and the constraints from NSR elliptic genus at even higher q order, we are able to determine the modular ansatz of $\mathbb{E}_{h^{(1)}_{4,E_7}}(q_\tau, v)$, which will be given in the `Mathematica` file on the website.

i,j	−10	−9	−8	−7	−6	−5	−4	−3	−2	−1	0	1
0	0	0	0	0	0	0	0	0	0	0	−1	0
1	0	0	0	0	0	0	0	0	0	0	9	0
2	1	0	0	0	0	0	0	0	−9	0	0	0
3	133	−224	140	0	25	0	14	0	−42	224	−1463	0
4	7371	−25920	41249	−31360	10010	−2688	3500	0	2050	2688	−7419	31360

Table 5.17: Series coefficients $c_{i,j}$ for the one-string elliptic genus of the $\mathfrak{n} = 4$ E_7 model.

If turning on all gauge E_7 and flavor $\mathfrak{so}(4) \cong \mathfrak{su}(2) \times \mathfrak{su}(2)$ fugacities, we find the leading q_τ order of reduced one-string elliptic genus, i.e. the reduced 5d Nekrasov partition function has the following expansion

$$-v^{11} - \chi^F_{(60)\oplus(06)\oplus(44)} v^{13} + (\mathbf{133}\cdot\chi^F_{(42)\oplus(24)} - \chi^F_{(48)\oplus(84)})v^{15} \\ - (\mathbf{912}\cdot\chi^F_{(33)} - \mathbf{56}\cdot\chi^F_{(73)\oplus(37)})v^{16} + \mathcal{O}(v^{17}),$$

which agrees with the (A.20) of (Kim et al., 2019). In fact, we find an exact formula for the reduced 5d Nekrasov partition function which will be given in Appendix (D). For the subleading q_τ order we obtain the following expansion

$$\chi^F_{(22)}v^9 + (\chi^F_{(26)\oplus(62)} - \chi^F_{(02)\oplus(20)} - 1 - \mathbf{133})v^{11} - (\chi^F_{(64)\oplus(46)\oplus(80)\oplus(08)\oplus(62)\oplus(26)\oplus(40)\oplus(04)} \\ + (\mathbf{133}+3)\chi^F_{(44)} + (\mathbf{133}+2)\chi^F_{(60)\oplus(06)} + (\mathbf{133}+1)\chi^F_{(42)\oplus(24)})v^{13} + \mathcal{O}(v^{15}).$$

$\mathbf{n = 2,\ G = E_7,\ F = \mathfrak{so}(6)}$

There are 16 unity blowup equations with $\lambda_F \in \mathbf{4}$ or $\bar{\mathbf{4}}$. Noticing the flavor symmetry $\mathfrak{so}(6) \cong \mathfrak{su}(4)$, we can turn on the fugacity of a sub-algebra $\mathfrak{su}(2)$ to perform the computation on elliptic genus easily. Using the Weyl orbit expansion method, we computed the one-string elliptic genus with $\mathfrak{su}(2)$ flavor fugacities to $\mathcal{O}(q_\tau^2)$. For example, denote the reduced one-string elliptic genus as

$$\mathbb{E}_{h^{(1)}_{2,E_7}}(q_\tau, v) = q_\tau^{1/6} v^{-1} \sum_{n=0}^{\infty} q_\tau^n \frac{(1-v)^2 P_n(v)}{(1+v)^{34}}, \qquad (5.5.130)$$

we obtain

$$P_0(v) = (1 + 36v + 632v^2 + 7212v^3 + 60168v^4 + 391380v^5 + 2067496v^6 + 9123228v^7$$

$$+343335094v^8+111995836v^9+320744719v^{10}+815144896v^{11}+1854166712v^{12}$$
$$+3796415104v^{13}+6997399845v^{14}+11475775012v^{15}+16204920073v^{16}$$
$$+18551114752v^{17}+\cdots+v^{34}),$$
$$P_1(v)=v^{-2}(133+4452v+72109v^2+752208v^3+5673385v^4+32915460v^5$$
$$+152504980v^6+577794348v^7+1815737068v^8+4761819476v^9$$
$$+10385374307v^{10}+18472471608v^{11}+25278998607v^{12}+21455489108v^{13}$$
$$-5924034231v^{14}-61899269488v^{15}-122152636908v^{16}-122341883440v^{17}$$
$$-16307972890v^{18}+84187540856v^{19}+\cdots+133v^{38}). \tag{5.5.131}$$

If we turn on all gauge E_7 and flavor $\mathfrak{su}(4)$ fugacities, we find the leading q_τ order of reduced one-string elliptic genus has the following expansion

$$v^{-1}-(\chi^{\mathfrak{su}(4)}_{(400)}+\chi^{\mathfrak{su}(4)}_{(004)})v^7-\chi^{\mathfrak{su}(4)}_{(222)}v^9-\mathbf{56}\cdot\chi^{\mathfrak{su}(4)}_{(030)}v^{10}$$
$$-(\chi^{\mathfrak{su}(4)}_{(602)}+\chi^{\mathfrak{su}(4)}_{(206)}+\chi^{\mathfrak{su}(4)}_{(323)}+\chi^{\mathfrak{su}(4)}_{(060)}-\mathbf{133}\cdot\chi^{\mathfrak{su}(4)}_{(121)}+\mathbf{1463})v^{11}+\mathbf{6480}\cdot\chi^{\mathfrak{su}(4)}_{(010)}v^{12}+\mathcal{O}(v^{13}).$$

The subleading q_τ order has expansion as

$$\mathbf{133}\,v^{-3}-\mathbf{56}\cdot\chi^{\mathfrak{su}(4)}_{(010)}v^{-2}+(\mathbf{133}+\chi^{\mathfrak{su}(4)}_{(101)}+1)v^{-1}+\chi^{\mathfrak{su}(4)}_{(040)}v$$
$$+\chi^{\mathfrak{su}(4)}_{(303)}v^3+(\chi^{\mathfrak{su}(4)}_{(420)}+\chi^{\mathfrak{su}(4)}_{(024)})v^5+\mathcal{O}(v^6).$$

Let us further denote

$$\mathbb{E}_{h^{(1)}_{2,E_7}}(q_\tau,v)=q_\tau^{1/6}\sum_{i,j}c_{i,j}v^j(q_\tau/v^2)^i. \tag{5.5.132}$$

Then we have the following Table 5.18 for the coefficients c_{ij}. Note the red numbers in the first column are just the dimensions of representations $k\theta$ of E_7 where θ is the adjoint representation. The blue numbers in the second column are six times the dimensions of representations $\square+k\theta$ of E_7, where the six is the dimension of the matter representation $\mathbf{6}$ of flavor symmetry $\mathfrak{so}(6)$. The orange number 149 in the third column is given by $\dim(E_7)+\dim(\mathfrak{so}(6))+1=133+15+1=149$. These are the constraints given in (Del Zotto and Lockhart, 2018) by analyzing the spectral flow to NSR elliptic genus, which our result satisfies perfectly. By combining our result and the constraints from NSR elliptic genus at even higher q order, we are able to determine the modular ansatz of $\mathbb{E}_{h^{(1)}_{2,E_7}}(q_\tau,v)$, which will be given in the `Mathematica` file on the website.

i,j	-1	0	1	2	3	4	5	6	7	8	9
0	1	0	0	0	0	0	0	0	-70	0	-729
1	133	-336	149	0	105	0	300	0	720	7840	-20777
2	7371	-38880	72542	-50064	11324	-21504	15645	-30240	47340	-146106	1938800

Table 5.18: Series coefficients $c_{i,j}$ for the one-string elliptic genus of the $\mathfrak{n}=2$ E_7 model.

5.5.12 E_8 theory

The $n = 12, G = E_8$ pure gauge theory has 12 unity blowup equations and no vanishing blowup equation. We use the recursion formula to compute the one-string and two-string elliptic genera and convert them to reduced versions. The one-string reduced elliptic genus in q_τ expansion with all gauge fugacities turned off reads

$$\mathbb{E}_{h^{(1)}_{12,E_8}}(q_\tau, v) = q_\tau^{-29/6} v^{29} \sum_{n=0}^{\infty} q_\tau^n \frac{P_n(v)}{(1-v^2)^{58}}, \tag{5.5.133}$$

where the leading orders are

$$\begin{aligned} P_0(v) = {} & (1+v^2)(1+189v^2+14080v^4+562133v^6+13722599v^8+220731150v^{10} \\ & +2454952400v^{12}+19517762786v^{14}+113608689871v^{16}+492718282457v^{18} \\ & +1612836871168v^{20}+4022154098447v^{22}+7692605013883v^{24} \\ & +11332578013712v^{26}+12891341012848v^{28}+11332578013712v^{30}+\cdots+v^{56}), \\ P_1(v) = {} & 249+43435v^2+2998484v^4+111587988v^6+2558096217v^8+38985250263v^{10} \\ & +415090167480v^{12}+3197400818096v^{14}+18281159666407v^{16} \\ & +79099752469353v^{18}+262872507223458v^{20}+678620928038790v^{22} \\ & +1372471431431505v^{24}+2187800775100695v^{26}+2759575276449180v^{28} \\ & +2759575276449180v^{30}+\cdots+v^{58}. \end{aligned} \tag{5.5.134}$$

Note the leading q_τ order indeed agrees with the Hilbert series of reduced one E_8-instanton moduli space in (Benvenuti, Hanany, and Mekareeya, 2010). Higher order contributions agree with the modular ansatz in (Del Zotto and Lockhart, 2017).

The two-string reduced elliptic genus in q_τ expansion reads

$$\mathbb{E}_{h^{(2)}_{E_8}}(q_\tau, v) = v^{59} q_\tau^{-59/6} \sum_{n=0}^{\infty} q_\tau^n \frac{1}{(1-v)^{118}(1+v)^{92}(1+v+v^2)^{59}} \times P_n^{(2)}(v)\,. \tag{5.5.135}$$

We have computed $P_0^{(2)}(v)$ which indeed agrees with the Hilbert series of two E_8-instanton reduced moduli space in (Hanany, Mekareeya, and Razamat, 2013).

Chapter 6

Elliptic Blowup Equations for Arbitrary Rank 6d $(1,0)$ SCFTs

6.1 Arbitrary rank

In this section we study the most general 6d $(1,0)$ SCFTs in the atomic classification. We first propose a simple set of rules to glue together the blowup equations of rank one theories to the blowup equations of higher-rank theories. With these gluing rules at hand, we write down the precise form of the elliptic blowup equations for arbitrary 6d $(1,0)$ SCFTs. We then present the admissible blowup equations for a lot of examples including the E–, M–string chain, three higher rank non-Higgsable clusters, ADE chain of -2 curves with gauge symmetry, all conformal matter theories and the blownups of some $-n$ curves in particular $-9,-10,-11$ curves. The prominent feature here is that for higher-rank theories, most of their blowup equations are of vanishing type.

6.1.1 Gluing rules

One of the key steps to write down the blowup equations for a higher rank theory is to fix the parameters λ_G and λ_F in the gauge and the flavor symmetry sectors. They are in fact both components of the r-field in the blowup equations of refined topological string theory. Besides constructing higher rank theories from rank one theories involves gauging the flavor symmetry. Therefore we can view λ_G, λ_F on an equal footing, and here we consider them collectively as the r-field (λ_G, λ_F).

Based on the gluing rules of higher rank 6d $(1,0)$ SCFTs (Heckman, Morrison, and Vafa, 2014; Heckman et al., 2015), we propose the following gluing rules for higher rank elliptic blowup equations, which are simple criteria to determine which r fields of one node can be coupled to which r fields of the adjacent nodes.

- For a node (G,F) with blowup equations labeled by the r-field (λ,ω) and all adjacent nodes (G_i,F_i) with blowup equations labeled by the r-field (λ_i,ω_i), $i=1,2,\ldots,s$, $s\leq 3$ and possibly an adherent free hyper with flavor F_f with r-field λ_f, the admissible coupling for the node (G,F) is such that $\mathcal{O}_{\lambda_1}\times\mathcal{O}_{\lambda_2}\times\ldots\times\mathcal{O}_{\lambda_s}\times\mathcal{O}_{\lambda_f}\subset\mathcal{O}_w$ according to decomposition $\prod_i^s G_i\times F_f\subset F$, where $\mathcal{O}_w$ is the Weyl orbit containing w.
- The admissible blowup equations for a higher-rank theory is such that all its nodes satisfy the above criteria.

A few comments are in order. Note that a node may bear no gauge group such as the E-string theory, in which case $G=\emptyset$ and $\lambda\in\mathbf{1}$. The concept of nodes in the criteria can be generalized to molecules in the atomic classification, which makes it easier to find all admissible blowup equations when lots of molecules are involved. These criteria actually guarantee the consistency with the blowup equations of lower-rank theories when decoupling nodes.

Also note that in this section, we will use the notation $\mathbf{n}_p$ to denote a Weyl orbit consisting of n weights which all have norm square p. Very often we will suppress the subscript p if p is minimal and there is no cause for confusion. We sometimes also use the conjugate bar and subscripts s, c to distinguish orbits of the same lenght just like in the notation of irreducible representations.

Now let us demonstrate the above criteria for NHC $2,3,2$. We recall the r-fields of the individual nodes from Tables 5.5, 5.6, 5.7, 5.8 and 5.9:

$$\mathfrak{n}=2,(G,F)=(\mathfrak{su}(2),\mathfrak{so}(7)): \begin{cases}\text{unity } r\text{-fields} \in (\mathbf{1}_0,\mathbf{6}_1)\\ \text{vanishing } r\text{-fields} \in (\mathbf{2}_{1/2},\mathbf{1}_0)\end{cases} \tag{6.1.1}$$

$$\mathfrak{n}=3,(G,F)=(\mathfrak{so}(7),\mathfrak{sp}(2)): \begin{cases}\text{unity } r\text{-fields} \in (\mathbf{1}_0,\mathbf{4}_1)\\ \text{vanishing } r\text{-fields} \in (\mathbf{6}_1,\mathbf{1}_0),(\mathbf{6}_1,\mathbf{4}_{1/2})\end{cases} \tag{6.1.2}$$

First, to couple the central node $G=\mathfrak{so}(7), F=\mathfrak{sp}(2)$ of the NHC 2,3,2 with the two side nodes $G_{1,2}=\mathfrak{su}(2), F_{1,2}=\mathfrak{so}(7)$, the flavor group F must decompose as $\mathfrak{sp}(2)\to\mathfrak{su}(2)\times\mathfrak{su}(2)$. As we have seen, the unity r fields $(\lambda_{\mathfrak{so}(7)},\omega_{\mathfrak{sp}(2)})$ of the central node $3_{\mathfrak{so}(7)}$ are elements of $(\mathbf{1}_0,\mathbf{4}_1)$. Under the flavor F decomposition, we have $\mathbf{4}_1\to(\mathbf{2}_{1/2},\mathbf{2}_{1/2})$. Since $(\lambda_{\mathfrak{su}(2)},\omega_{\mathfrak{so}(7)})\in(\mathbf{2}_{1/2},\mathbf{1}_0)$ is indeed a correct set of vanishing r fields of the node $2_{\mathfrak{su}(2)}$, we find one set of admissible r fields for the NHC $2,3,2$ with the parameter λ_F of the entire $2,3,2$ chain belonging to $(\mathbf{2}_{1/2},\mathbf{1}_0,\mathbf{2}_{1/2})$, which give rise to vanishing blowup equations. On the other hand, the vanishing r fields of the central node $3_{\mathfrak{so}(7)}$ are elements of $(\mathbf{6}_1,\mathbf{1}_0)$ or $(\mathbf{6}_1,\mathbf{4}_{1/2})$. Under flavor F decomposition, $\mathbf{1}_0\to(\mathbf{1}_0,\mathbf{1}_0)$ and $\mathbf{4}_{1/2}\to(\mathbf{2}_{1/2},\mathbf{1}_0)+(\mathbf{1}_0,\mathbf{2}_{1/2})$. The combination $(\mathbf{2}_{1/2},\mathbf{6}_1)$ does not contain r fields of the node $2_{\mathfrak{su}(2)}$, but the combination $(\mathbf{1}_0,\mathbf{6}_1)$ does contain valid r-fields of the unity type. Clearly, the overall λ_F parameters belonging to $(\mathbf{1}_0,\mathbf{6}_1,\mathbf{1}_0)$ give rise to the other set of vanishing blowup equations for NHC $2,3,2$ and there is no other possible admissible λ_F. Later in Section 6.3.3, we explicitly show these two types of elliptic blowup equations.

6.1.2 Arbitrary rank elliptic blowup equations

Consider F-theory compactifications on an elliptic non-compact Calabi-Yau three-fold, whose non-compact base contains r compact curves with a negative definite intersection matrix $-\Omega_{ij}=A_{ij}$. Recall the symmetry algebras and the massless fields which can arise in this theory. Over the i-th compact curve C_i there could be singular elliptic fibers corresponding to a symmetry algebra G_i. In addition C_i could intersect with a non-compact curve N_i with intersection number k_{F_i}, and the latter supports singular elliptic fibers corresponding to symmetry algebra F_i.

The resulting field theory is a 6d SCFT in its r dimensional tensor branch with total gauge symmetry $\prod_i G_i$ and flavor symmetry $\prod_i F_i$. If we compactify the 6d SCFT on a torus, we can also turn on the gauge and flavor fugacities m_{G_i}, m_{F_i}. There are also charged matter fields localized at intersections of curves. At the intersection locus of two compact base curves C_i, C_j there are hypermultiplets charged under both gauge groups G_i, G_j. We also consider hypermultiplets localized at the intersection locus of compact and non-compact curves. Finally BPS strings arise from D3-branes wrapping compact base curves. The number of times a string wraps each base curve is interpreted as the charge of this string. The string charges form a rank r lattice Λ with the negative definite bilinear form defined by $-\Omega_{ij} = A_{ij}$.

We first introduce a special kind of higher dimension Riemann theta function associated to a $N \times N$ matrix Ω. It turns out that the polynomial part of the higher rank 6d theories contributes to the blowup equation as this type of Riemann theta function. We define

$$\Theta_{\Omega}^{[a]}(\tau, z) = \sum_{k \in \mathbb{Z}^N + a} (-1)^{k \cdot \mathrm{diag}(\Omega)} \exp\left(\frac{1}{2} k \cdot \Omega \cdot k\, \tau + k \cdot \Omega \cdot z\right). \tag{6.1.3}$$

Here the characteristic $a = (a_1, a_2, \ldots, a_N)$ takes the following values

$$a_i = \sum_j (\Omega^{-1})_{ij} \Big(\frac{1}{2}\Omega_{jj} + m_j\Big), \quad m_j \in \mathbb{Z}. \tag{6.1.4}$$

The number of different such characteristics is $\mathrm{Det}(\Omega)$. This kind of Riemann theta function is the proper generalization of $\theta_i^{[a]}(n\tau, nz)$ appearing in rank one elliptic blowup equations. As in the rank one cases, when the characteristic a is trivial, we suppress the superscript $\Theta_\Omega = \Theta_\Omega^{[0]}$.

The elliptic blowup equations for the elliptic genera $\mathbb{E}_{d_i}(\tau, m_{G_i}, m_{F_i}, \epsilon_1, \epsilon_2)$ of arbitrary 6d SCFT can be written as:

$$\begin{aligned}
&\sum_{\alpha_i \in \phi_i(Q^\vee(G_i)), d_i', d_i'' \in \mathbb{N}}^{||\alpha_i||^2/2 + d_i' + d_i'' = d_i + \delta_i/2} (-1)^{\sum_i |\phi_i^{-1}(\alpha_i)|} \\
&\quad \times \Theta_\Omega^{[a_i]}\big(\tau, -\alpha_i \cdot m_{G_i} + \sum_j (\Omega^{-1})_{ij} k_{F_j}(\lambda_j \cdot m_{F_j}) + (y_i - \frac{1}{2}(\alpha_i \cdot \alpha_i))(\epsilon_1 + \epsilon_2) - d_i'\epsilon_1 - d_i''\epsilon_2\big) \\
&\quad \times \prod_i A_{V_i}(\tau, m_{G_i}, \alpha_i) \prod_{ij} A_{H_{ij}}(\tau, m_{G_i}, \mu_j, \alpha_i, \alpha_j) \prod_i A_{H_i}(\tau, m_{G_i}, m_{F_i}, \alpha_i, \lambda_i) \\
&\quad \times \mathbb{E}_{d_i'}(\tau, m_{G_i} + \alpha_i\epsilon_1, m_{F_i} + \lambda_i\epsilon_1, \epsilon_1, \epsilon_2 - \epsilon_1)\mathbb{E}_{d_i''}(\tau, m_{G_i} + \alpha_i\epsilon_2, m_{F_i} + \lambda_i\epsilon_2, \epsilon_1 - \epsilon_2, \epsilon_2) \\
&= \Lambda(\delta_i)\Theta_\Omega^{[a_i]}\big(\tau, \sum_j (\Omega^{-1})_{ij} k_{F_j}(\lambda_j \cdot \nu_j) + y_i(\epsilon_1 + \epsilon_2)\big)\mathbb{E}_{d_i}(\tau, m_{G_i}, m_{F_i}, \epsilon_1, \epsilon_2),
\end{aligned} \tag{6.1.5}$$

with

$$y_i = \sum_j (\Omega^{-1})_{ij} \big(\frac{1}{4}(-2 + \Omega_{jj} + h_{G_j}^\vee) + \frac{1}{2} k_{F_j}(\lambda_j \cdot \lambda_j)\big), \tag{6.1.6}$$

and

$$\Lambda(\delta_i) = \begin{cases} 1, & \forall i,\ \delta_i = 0, \\ 0, & \exists i,\ \delta_i > 0\,. \end{cases} \tag{6.1.7}$$

Here as in the rank one case, ϕ_i is an embedding of the coroot lattice $Q^\vee(G_i)$ in the coweight lattice of G_i by an overall shift of a coweight vector. δ_i is the smallest norm in the image $\phi_i(Q^\vee(G_i))$; δ_i is zero if the embedding is unshifted so that $\phi_i(Q^\vee(G_i)) = Q^\vee(G_i)$ and positive otherwise. $\phi_i^{-1}(\alpha_i)$ gives back a coroot vector, and $|\bullet|$ is the sum of the coefficients in its decomposition in terms of simple coroots. A_{V_i} is the contribution of vector multiplets transforming in the adjoint representation of G_i, and $A_{H_{ij}}, A_{H_i}$ are respectively the contributions of hypermultiplets charged in the mixed representation of two gauge groups, and in the representation of one gauge group. Their expressions have been given in (5.2.7),(5.2.8). Finally the parameters λ_i are the components of r-fields associated to the flavor symmetries. They take value in the coweight lattice and they are determined by the gluing rules discussed in the previous subsection. One important consistency condition is that if we turn off all string charges except for the one indexed by i, i.e. we set $d_j = 0$ for $j \neq i$, and consequently $d'_j = d''_j = 0$, $\alpha_j = 0$ for $j \neq i$ as well, (6.1.5) must reduce to the blowup equations for a rank one 6d SCFT with $\mathfrak{n} = \Omega_{ii}$, gauge group G_i, and the surviving λ_i should be the λ_F parameter worked out in Section 5.2.1 and 5.2.2.

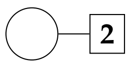
The modularity of higher rank elliptic blowup equations (6.1.5) can be proved in a similar way as in the rank one cases in Chapter 5.2.3. For the general proof, we refer to the section 7.3 of (Gu et al., 2020b). As an example, we explicitly show the modularity proof for NHC 3,2 in Section 6.3.1.

With the gluing rules given in Section 6.1.1, we can efficiently write down all admissible blowup equations for any higher-rank theory once the gauge groups, flavor groups and matter representation are known. We use a simple quiver diagram to denote blowup equations with the following rules:

- We use a circle for a compact base curve and a rectangle for a non-compact one.
- For each base curve with associated gauge/flavor symmetry G, we mark it with a Weyl orbit $\mathbf{n}_p$ (p is often suppressed if it is minimal) of G to denote the r field of G fugacities. If a compact curve has no associated gauge symmetry, we leave the circle blank.

For example, the unity and vanishing blowup equations of E-string theory can be simply denoted as

The unity blowup equations of M-string theory can be denoted as



The unity and two types of vanishing blowup equations of $n = 3, G = \mathfrak{su}(3)$ theory can be denoted respectively as

In the following, we will present and check blowup equations for some most interesting examples of higher-rank theories including E– and M–string chains, three higher rank non-Higgsable clusters which are NHC 3,2, NHC 3,2,2 and NHC 2,3,2, the ADE chains of -2 curves, conformal matter theories and the blowups of $(-n)$-curves in particular $-9, -10, -11$ curves.

6.2 E– and M–string chains

E– and M–string chains are some typical higher rank theories without gauge symmetry (Gadde et al., 2018). In rank r M-string chain theory, also denoted as

M^r, the strings originate from M2 branes that are suspended among $r+1$ parallel M5 branes. In rank r E-string chain theory, also denoted as E–M^{r-1}, the strings originate from M2 branes that are suspended among r parallel M5 branes probing an M9 brane. These theories all have simple 2d quiver discription, and their elliptic genera have been computed in (Gadde et al., 2018).

6.2.1 M-string chain

Consider the rank $r>1$ M–string theory whose matrix Ω is the Cartan matrix of $\mathfrak{su}(r+1)$. Let m be the $\mathfrak{su}(2)$ flavor symmetry, and $\mathbb{E}_k(\tau,m,\epsilon_1,\epsilon_2)$ be the elliptic genus with wrapping numbers $k=(k_1,\ldots,k_r)$ of the base curves. The idea to construct blowup equations of this theory is to "glue'" the blowup equations for each individual (-2) base curves by merging the theta functions $\theta_3^{[a]}$ in those equations into Θ_Ω. The type of the resulting new equations can be determined by the following simple rule. We obtain a unity blowup equation if all the constituent blowup equations are of the unity type, and a vanishing blowup equation if one of the constituent blowup equations is of the vanishing type. Schematically we have

$$\mathrm{U}\star\mathrm{U}=\mathrm{U},\quad \mathrm{U}\star\mathrm{V}=\mathrm{V},\quad \mathrm{V}\star\mathrm{V}=\mathrm{V}. \tag{6.2.1}$$

Since the rank one M–string theory has only unity blowup equations, the higher rank M–string has also only unity blowup equations, which can be written as

$$\begin{aligned}\sum_{k'+k''=k}&\Theta_\Omega^{[a]}(\tau,M_u-k'\epsilon_1-k''\epsilon_2)\mathbb{E}_{k'}(\tau,m+\frac{s}{2}\epsilon_1,\epsilon_1,\epsilon_2-\epsilon_1)\mathbb{E}_{k''}(\tau,m+\frac{s}{2}\epsilon_2,\epsilon_1-\epsilon_2,\epsilon_2)\\ &=\Theta_\Omega^{[a]}(\tau,M_u-k'\epsilon_1-k''\epsilon_2)\mathbb{E}_k(\tau,m,\epsilon_1,\epsilon_2),\end{aligned} \tag{6.2.2}$$

where

$$M_u=\Omega^{-1}\cdot(sm+\frac{\epsilon_1+\epsilon_2}{2},\ldots,sm+\frac{\epsilon_1+\epsilon_2}{2}), \tag{6.2.3}$$

with $s=\pm1$. The characteristic a takes the value in (5.2.4), and their total number is $\det(\Omega)=r+1$.

These blowup equations can be checked in various ways. Using the modular index polynomial of $\mathbb{E}_k$ of the higher rank M–string (Haghighat et al., 2015a; Gu et al., 2017; Haghighat, Yan, and Yau, 2018)

$$\mathrm{ind}_k^{\mathrm{M}^r}=-\frac{(\epsilon_1+\epsilon_2)^2}{4}\sum_{i=1}^r k_i+\frac{\epsilon_1\epsilon_2}{2}k\cdot\Omega\cdot k+m^2\sum_{i=1}^r k_i, \tag{6.2.4}$$

one can find easily that the modularity condition is satisfied. Furthermore, we have verified these equations at $k=(1,1)$ to high degrees of q_τ with the explicit expressions of $\mathbb{E}_k$ in (Haghighat et al., 2015a). Finally, it is possible to demonstrate that these equations reduce properly to the blowup equations of rank one M–string theory. We will use the shorthand notation that for a theory T,

$$\mathrm{V}_T^{[a]}=0,\quad \mathrm{U}_T^{[a]}=0 \tag{6.2.5}$$

denote the vanishing and the unity blowup equations with characteristic a respectively, where in the latter case we have moved the two sides of the equation together. Let us consider the M-M chain and decompactify the (-2) curve on the right. We can choose the non-equivalent characteristics a of the unity blowup equations to be $a = (0,0)$, $(1/3, 2/3)$, $(2/3, 1/3)$, with the corresponding equations denoted by

$$\mathrm{U}_{\mathrm{MM}}^{[0,0]} = 0, \quad \mathrm{U}_{\mathrm{MM}}^{[\frac{1}{3},\frac{2}{3}]} = 0, \quad \mathrm{U}_{\mathrm{MM}}^{[\frac{2}{3},\frac{1}{3}]} = 0. \tag{6.2.6}$$

We can decompactify the (-2) curve on the right by setting k_2, k_2', k_2'' to zero. Then the two dimensions in the summation in $\Theta_\Omega^{[a]}$ decouple. It is easy to deduce that in this limit

$$\begin{aligned} 0 = \mathrm{U}_{\mathrm{MM}}^{[0,0]} &= \theta_3^{[0]}(6\tau, 3z)\mathrm{U}_{\mathrm{M}}^{[0]} + \theta_3^{[-\frac{1}{2}]}(6\tau, 3z)\mathrm{U}_{\mathrm{M}}^{[\frac{1}{2}]}, \\ 0 = \mathrm{U}_{\mathrm{MM}}^{[\frac{1}{3},\frac{2}{3}]} &= \theta_3^{[\frac{1}{3}]}(6\tau, 3z)\mathrm{U}_{\mathrm{M}}^{[0]} + \theta_3^{[-\frac{1}{6}]}(6\tau, 3z)\mathrm{U}_{\mathrm{M}}^{[\frac{1}{2}]}, \\ 0 = \mathrm{U}_{\mathrm{MM}}^{[\frac{2}{3},\frac{1}{3}]} &= \theta_3^{[\frac{2}{3}]}(6\tau, 3z)\mathrm{U}_{\mathrm{M}}^{[0]} + \theta_3^{[\frac{1}{6}]}(6\tau, 3z)\mathrm{U}_{\mathrm{M}}^{[\frac{1}{2}]}, \end{aligned} \tag{6.2.7}$$

where $z = sm + (\epsilon_1 + \epsilon_2)/2$, $s = \pm 1$. Since this is clearly a full-rank system for $\mathrm{U}_{\mathrm{M}}^{[0]}$ and $\mathrm{U}_{\mathrm{M}}^{[1/2]}$, we conclude

$$\mathrm{U}_{\mathrm{M}}^{[0]} = 0, \qquad \mathrm{U}_{\mathrm{M}}^{[\frac{1}{2}]} = 0. \tag{6.2.8}$$

These are exactly the unity blowup equations for M–string, as we already know. Similar situation happens when M^r chain reduces to M^{r-1} chain.

6.2.2 E–M string chain

Let us move onto the rank $r > 1$ E–string theory. The matrix Ω is

$$\Omega = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & \cdots & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & \cdots & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}. \tag{6.2.9}$$

where the lower right $(r-1) \times (r-1)$ submatrix is the Cartan matrix of $\mathfrak{su}(r)$, which will be denoted by $\hat{\Omega}$. Let m and m be the E_8 and $\mathfrak{su}(2)$ flavor masses respectively, and $\mathbb{E}_k(\tau, m, m, \epsilon_1, \epsilon_2)$ with $k = (k_0, k_1, \ldots, k_{r-1})$ be the elliptic genus with wrapping number k_0 on the (-1) base curve and wrapping numbers $\hat{k} = (k_1, \ldots, k_{r-1})$ on the (-2) curves. The blowup equations of this theory is again constructed by merging the theta functions in the constituent blowup equations of rank one E–,M–string theories to Θ_Ω. Following the rule (6.2.1), we expect vanishing blowup equations constructed from vanishing equations of the E–string theory and unity equations of the M–string theory, and unity blowup equations constructed from unity equations of both the E–,M–string theories. The vanishing blowup equations of the rank r E–string theory read

$$\begin{aligned} \sum_{k'+k''=k} &\Theta_\Omega^{[a]}(\tau, M_v - k'\epsilon_1 - k''\epsilon_2)\mathbb{E}_{k'}(\tau, m, m + \frac{s}{2}\epsilon_1, \epsilon_1, \epsilon_2 - \epsilon_1) \\ &\times \mathbb{E}_{k''}(\tau, m, m + \frac{s}{2}\epsilon_2, \epsilon_1 - \epsilon_2, \epsilon_2) = 0 \end{aligned} \tag{6.2.10}$$

where $s = \pm 1$ and

$$M_v = \Omega^{-1} \cdot (0, sm + \frac{\epsilon_1+\epsilon_2}{2}, \dots, sm + \frac{\epsilon_1+\epsilon_2}{2}). \tag{6.2.11}$$

The unity blowup equations read

$$\sum_{k'+k''=k} \Theta_{\Omega}^{[a]}(\tau, M_u - k'\epsilon_1 - k''\epsilon_2)\mathbb{E}_{k'}(\tau, m+\alpha\epsilon_1, m+\frac{s}{2}\epsilon_1, \epsilon_1, \epsilon_2-\epsilon_1) \\ \times \mathbb{E}_{k''}(\tau, m+\alpha\epsilon_2, m+\frac{s}{2}\epsilon_2, \epsilon_1-\epsilon_2, \epsilon_2) = \Theta_{\Omega}^{[a]}(\tau, M_u)\mathbb{E}_{k'}(\tau, m, m\epsilon_1, \epsilon_2) \tag{6.2.12}$$

where $s = \pm 1$ and α is one of the 240 roots of E_8, and

$$M_v = \Omega^{-1} \cdot (\alpha \cdot m + \epsilon_1 + \epsilon_2, sm + \frac{\epsilon_1+\epsilon_2}{2}, \dots, sm + \frac{\epsilon_1+\epsilon_2}{2}). \tag{6.2.13}$$

In both equations, a is unique and it can be written as

$$a = \Omega^{-1} \cdot (\frac{1}{2}, 0, \dots, 0). \tag{6.2.14}$$

We verify these blowup equations in the following ways. First of all, using the modular index polynomial of $\mathbb{E}_k$

$$\text{ind}_k^{\text{EM}^{r-1}} = -\frac{(\epsilon_1+\epsilon_2)^2}{4}(2k_0 + \sum_{i=1}^{r-1} k_i) + \frac{\epsilon_1\epsilon_2}{2}(\hat{k}\cdot\hat{\Omega}\cdot\hat{k} + k_0) + \frac{k_0}{2}(m,m)_{E_8} + m^2\sum_{i=1}^{r-1} k_i, \tag{6.2.15}$$

we find (6.2.10), (6.2.12) satisfy the modularity condition. Furthermore, we verified these equations at $k = (1,1)$ up to high orders of q_τ with the explicit expressions of $\mathbb{E}_k$ given in (Gadde et al., 2018). Finally, we demonstrate that the blowup equations of the rank two E–string, or the E–M chain, can be reduced to the blowup equations of E–, M–string theories by decompactifying base curves. The blowup equations of the E–M chain all have a unique characteristic which we choose to be $a = (0, 1/2)$. Let us first decompactify the (-1) curve by setting $k_0 = 0$. The vanishing blowup equations of the E–M chain become in this limit

$$0 = \mathrm{V}_{\mathrm{EM}}^{[0,\frac{1}{2}]} = \theta_3(2\tau, sm + (\epsilon_1+\epsilon_2)/2)\cdot \mathrm{U}_{\mathrm{M}}^{[-\frac{1}{2}]} - \theta_3(2\tau, sm + (\epsilon_1+\epsilon_2)/2)\cdot \mathrm{U}_{\mathrm{M}}^{[0]}, \tag{6.2.16}$$

while the unity blowup equations become

$$0 = \mathrm{U}_{\mathrm{EM}}^{[0,\frac{1}{2}]} = \theta_2(2\tau, 2m\cdot\alpha + sm + 5(\epsilon_1+\epsilon_2)/2)\cdot \mathrm{U}_{\mathrm{M}}^{[-\frac{1}{2}]} \\ - \theta_3(2\tau, 2m\cdot\alpha + sm + 5(\epsilon_1+\epsilon_2)/2)\cdot \mathrm{U}_{\mathrm{M}}^{[0]}. \tag{6.2.17}$$

Since $s = \pm 1$ and $\alpha \in \Delta(E_8)$, we have a full rank system for $\mathrm{U}_{\mathrm{M}}^{[-\frac{1}{2}]}$ and $\mathrm{U}_{\mathrm{M}}^{[0]}$, and therefore $\mathrm{U}_{\mathrm{M}}^{[-\frac{1}{2}]} = 0$ and $\mathrm{U}_{\mathrm{M}}^{[0]} = 0$, which are the unity blowup equations of the M–string as we know. Next we decompactify the (-2) curve by setting $k_1 = 1$. The vanishing blowup equations of the E–M chain become in this limit

$$0 = \mathrm{V}_{\mathrm{EM}}^{[0,\frac{1}{2}]} = \theta_2(\tau, sm + (\epsilon_1+\epsilon_2)/2)\cdot \mathrm{V}_{\mathrm{E}}^{[-\frac{1}{2}]}. \tag{6.2.18}$$

Thus $\mathrm{V}_{\mathrm{E}}^{[-\frac{1}{2}]} = 0$, which is the vanishing elliptic blowup equation for E–string as we know. The unity blowup equations of E–M chain become

$$0 = \mathrm{U}_{\mathrm{EM}}^{[0,\frac{1}{2}]} = \theta_2(\tau, m\cdot\alpha + sm + 3(\epsilon_1+\epsilon_2)/2)\cdot \mathrm{U}_{\mathrm{E}}^{[-\frac{1}{2}]}. \tag{6.2.19}$$

Thus $\mathrm{U}_{\mathrm{E}}^{[-\frac{1}{2}]} = 0$, which are the 240 unity elliptic blowup equations for E–string.

6.3 Three higher rank non-Higgsable clusters

The three non-Higgsable clusters in Table 6.1 are some simple higher-rank 6d $(1,0)$ SCFTs and building blocks for more complicated higher-rank theories (Morrison and Taylor, 2012). The 2d quiver gauge theories corresponding to these three NHCs have been constructed in (Kim et al., 2018). Using Jeffrey-Kirwan residue, the elliptic genera can be explicitly computed as formulas involving Jacobi theta functions. It is interesting to see how blowup equations work for these higher rank theories. The most prominent feature here is that there only exist *vanishing* blowup equations for these three NHCs. The toric constructions for the elliptic non-compact Calabi-Yau threefolds associated with these three NHCs were given in (Gu et al., 2020b)

base	$3,2$	$3,2,2$	$2,3,2$
gauge symmetry	$G_2\times\mathfrak{su}(2)$	$G_2\times\mathfrak{su}(2)\times\{\,\}$	$\mathfrak{su}(2)\times\mathfrak{so}(7)\times\mathfrak{su}(2)$
matter	$\frac{1}{2}(\mathbf{7}+\mathbf{1},\mathbf{2})$	$\frac{1}{2}(\mathbf{7}+\mathbf{1},\mathbf{2})$	$\frac{1}{2}(\mathbf{2},\mathbf{8},\mathbf{1})+\frac{1}{2}(\mathbf{1},\mathbf{8},\mathbf{2})$

Table 6.1: Three higher-rank NHCs.

6.3.1 NHC $3,2$

Elliptic blowup equations

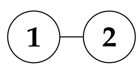
Let us denote the intersection matrix between the two base curves as $-\Omega$, i.e.

$$\Omega = \begin{pmatrix} 3 & -1 \\ -1 & 2 \end{pmatrix}. \tag{6.3.1}$$

Note $\det\Omega = 5$. It turns out there are in total five vanishing λ_F fields and no unity λ fields. To see this, one can simply look at the matter representation $(\mathbf{7}+\mathbf{1},\frac{1}{2}\mathbf{2})$. Note G_2 can only bear unity equations due to the Lie algebra fact $P^\vee\cong Q^\vee$. On the other hand, the unpaired half-hyper on the $\mathfrak{su}(2)$ node indicates only vanishing equations. Thus combining unity and vanishing equations naturally results in vanishing equations. The idea can be roughly expressed as

$$\mathrm{U}\star\mathrm{V} = \mathrm{V}. \tag{6.3.2}$$

Using the quiver diagram introduced in previous section 6.1.2, the five vanishing blowup equations can be denoted as



To be precise, the five vanishing elliptic blowup equations can be written as

$$
\begin{aligned}
&\sum_{\substack{\alpha^\vee\in Q^\vee_{G_2},d_{1,2}\\ d_0=\frac{1}{2}||\alpha^\vee||^2}}^{d_0+d_1+d_2=k_1}\ \sum_{\substack{\lambda\in(P^\vee\backslash Q^\vee)_{\mathfrak{su}(2)},d'_{1,2}\\ d'_0+1/4=\frac{1}{2}||\lambda||^2}}^{d'_0+d'_1+d'_2=k_2}(-1)^{|\alpha^\vee|+|\lambda|}\\
&\Theta^{[a]}_{\Omega}\left(\tau,\begin{pmatrix}\alpha^\vee\cdot m_{G_2}+(\bar{y}_1-d_0)(\epsilon_1+\epsilon_2)-d_1\epsilon_1-d_2\epsilon_2\\ \lambda\cdot m_{\mathfrak{su}(2)}+(\bar{y}_2-d'_0)(\epsilon_1+\epsilon_2)-d'_1\epsilon_1-d'_2\epsilon_2\end{pmatrix}\right)\\
&\times A_V^{G_2}(\alpha^\vee,\tau,m_{G_2})A_V^{\mathfrak{su}(2)}(\lambda,\tau,m_{\mathfrak{su}(2)})A_H^{(\mathbf{7}+\mathbf{1},\frac{1}{2}\mathbf{2})}(\alpha^\vee,\lambda,\tau,m_{G_2},m_{\mathfrak{su}(2)})\\
&\times \mathbb{E}_{d_1,d'_1}(\tau,m_{G_2}-\epsilon_1\alpha^\vee,m_{\mathfrak{su}(2)}-\epsilon_1\lambda,\epsilon_1,\epsilon_2-\epsilon_1)\\
&\times \mathbb{E}_{d_2,d'_2}(\tau,m_{G_2}-\epsilon_2\alpha^\vee,m_{\mathfrak{su}(2)}-\epsilon_2\lambda,\epsilon_1-\epsilon_2,\epsilon_2)\ =\ 0,
\end{aligned}
\tag{6.3.3}
$$

where the summation indices $d_{0,1,2},d'_{0,1,2}\in\mathbb{Z}_{\geq 0}$, and $\bar{y}_1=3/5,\bar{y}_2=3/10$, and

$$
a=\begin{pmatrix}a_k\\ a_l\end{pmatrix}=\begin{pmatrix}2j/5\\ -1/2+j/5\end{pmatrix},\qquad j=-2,-1,0,1,2. \tag{6.3.4}
$$

This is our starting point to prove the modularity. Note $\bar{y}_1,\bar{y}_2$ satisfy the following relation

$$
\Omega\begin{pmatrix}\bar{y}_1\\ \bar{y}_2\end{pmatrix}=\begin{pmatrix}3/2\\ 0\end{pmatrix}=\begin{pmatrix}\bar{y}_{\mathrm{u}}\text{ of rank one }n=3\ G_2\text{ theory}\\ \bar{y}_{\mathrm{v}}\text{ of rank one }n=2\ \mathfrak{su}(2)\text{ theory}\end{pmatrix}. \tag{6.3.5}
$$

It can be shown this is necessary to be consistent with the established elliptic blowup equations for rank one theories when decompactifying one of the base curves.

The leading base degree of the vanishing blowup equations, i.e. $d_0=d_1=d_2=d'_0=d'_1=d'_2=0$ can be simply written as[1]

$$
\Theta^{[a]}_{\Omega}\left(\tau,\begin{pmatrix}6\epsilon_+/5\\ m_{\mathfrak{su}(2)}+3\epsilon_+/5\end{pmatrix}\right)-\Theta^{[a]}_{\Omega}\left(\tau,\begin{pmatrix}6\epsilon_+/5\\ -m_{\mathfrak{su}(2)}+3\epsilon_+/5\end{pmatrix}\right)=0. \tag{6.3.6}
$$

It is easy to check that the above identity is correct. For higher base degrees, the vanishing blowup equations (6.3.3) involve nontrivial elliptic genera. We have checked them from the Calabi-Yau setting to high degrees of Kähler classes. Besides, we find the five vanishing blowup equations are not sufficient to solve all refined BPS invariants. This is not surprising since vanishing blowup equations give less constraints just like in the rank one theories.

Modularity

The index of the elliptic genus $\mathbb{E}_{k_1,k_2}$ is known to be

$$
\begin{aligned}
\mathrm{Ind}_{\mathbb{E}_{k_1,k_2}}=&-\frac{(\epsilon_1+\epsilon_2)^2}{4}(3k_1+2k_2)+\frac{\epsilon_1\epsilon_2}{2}(3k_1^2+2k_2^2-2k_1k_2-k_1)\\
&+(-3k_1+k_2)\frac{(m,m)_{G_2}}{2}+(-2k_2+k_1)\frac{(m,m)_{\mathfrak{su}(2)}}{2}\,.
\end{aligned}
\tag{6.3.7}
$$

[1]The $\mathfrak{su}(2)$ vector multiplet does contribute to the blowup equation here. We omit here because their contribution can be factored out.

Let us use this to prove the modularity of (6.3.3). First, it is easy to derive from the general theory of Riemann theta functions that the index quadratic form of $\Theta_{\Omega}^{[a]}(\tau,z)$ under special modular transformation $\tau\to -1/\tau$ is just

$$\frac{1}{2}z\cdot\Omega\cdot z. \tag{6.3.8}$$

This fact is useful when computing the index of the polynomial contribution. Indeed, the index of polynomial part in (6.3.3) is

$$\begin{aligned}\mathrm{Ind}_{\mathrm{poly}}=&\frac{3}{2}(\alpha^\vee\cdot m_{G_2}+(y_1-d_0)(\epsilon_1+\epsilon_2)-d_1\epsilon_1-d_2\epsilon_2)^2\\&+(\lambda\cdot m_{\mathfrak{su}(2)}+(y_2-d_0')(\epsilon_1+\epsilon_2)-d_1'\epsilon_1-d_2'\epsilon_2)^2\\&-(\alpha^\vee\cdot m_{G_2}+(y_1-d_0)(\epsilon_1+\epsilon_2)-d_1\epsilon_1-d_2\epsilon_2)\\&\times(\lambda\cdot m_{\mathfrak{su}(2)}+(y_2-d_0')(\epsilon_1+\epsilon_2)-d_1'\epsilon_1-d_2'\epsilon_2).\end{aligned}$$

The G_2 vector multiplet contributes to the index as

$$\begin{aligned}\mathrm{Ind}_V^{G_2}=&-\frac{5}{3}\Big((\alpha^\vee\cdot m_{G_2})^2+d_0(m,m)_{G_2}\Big)+\frac{2}{3}(5d_0-2)(\epsilon_1+\epsilon_2)(\alpha^\vee\cdot m_{G_2})\\&-\frac{1}{3}(5d_0^2-2d_0)(\epsilon_1^2+\epsilon_1\epsilon_2+\epsilon_2^2).\end{aligned}$$

and the $\mathfrak{su}(2)$ vector multiplet contributes to the index as

$$\begin{aligned}\mathrm{Ind}_V^{\mathfrak{su}(2)}=&-\frac{4}{3}\Big((\lambda\cdot m_{\mathfrak{su}(2)})^2+(d_0'+\frac{1}{4})(m,m)_{\mathfrak{su}(2)}\Big)+\frac{8}{3}d_0'(\epsilon_1+\epsilon_2)(\lambda\cdot m_{\mathfrak{su}(2)})\\&-\frac{1}{3}(4d_0'^2+d_0')(\epsilon_1^2+\epsilon_1\epsilon_2+\epsilon_2^2).\end{aligned}$$

The hypermultiplet in the representation $\mathfrak{R}=(\mathbf{7}+\mathbf{1},\frac{1}{2}\mathbf{2})$ contributes to the index as

$$\begin{aligned}\mathrm{Ind}_H^{\mathfrak{R}}=&\frac{1}{4}\bigg(\frac{2}{3}\Big((\alpha^\vee\cdot m_{G_2})^2+d_0(m,m)_{G_2}\Big)+2d_0(m,m)_{\mathfrak{su}(2)}+2(d_0'+\frac{1}{4})(m,m)_{G_2}\\&+\frac{4}{3}\Big((\lambda\cdot m_{\mathfrak{su}(2)})^2+(d_0'+\frac{1}{4})(m,m)_{\mathfrak{su}(2)}\Big)+4(\alpha^\vee\cdot m_{G_2})(\lambda\cdot m_{\mathfrak{su}(2)})\bigg)\\&-\frac{1}{8}((m,m)_{G_2}+2(m,m)_{\mathfrak{su}(2)})-\frac{1}{6}\bigg(2d_0\alpha^\vee\cdot m_{G_2}+6(d_0'+\frac{1}{4})\alpha^\vee\cdot m_{G_2}\\&+6d_0\lambda\cdot m_{\mathfrak{su}(2)}+4(d_0'+\frac{1}{4})\lambda\cdot m_{\mathfrak{su}(2)}\bigg)+\frac{1}{12}(\alpha^\vee\cdot m_{G_2}+2\lambda\cdot m_{\mathfrak{su}(2)})+\cdots.\end{aligned}$$

Using (6.3.7), we can also easily compute the index of $\mathbb{E}_{d_1,d_1'}(\tau,m_{G_2}-\epsilon_1\alpha^\vee,m_{\mathfrak{su}(2)}-\epsilon_1\lambda,\epsilon_1,\epsilon_2-\epsilon_1)$ as

$$\begin{aligned}\mathrm{Ind}_{\mathbb{E}_{d_1,d_1'}}=&-\frac{\epsilon_2^2}{4}(3d_1+2d_1')+\frac{\epsilon_1(\epsilon_2-\epsilon_1)}{2}(3d_1^2+2d_1'^2-2d_1d_1'-d_1)\\&+(-3d_1+d_1')\Big(\frac{(m,m)_{G_2}}{2}-\epsilon_1\alpha^\vee\cdot m_{G_2}+d_0\epsilon_1^2\Big)\\&+(d_1-2d_1')\Big(\frac{(m,m)_{A_1}}{2}-\epsilon_1\lambda\cdot m_{\mathfrak{su}(2)}+d_0\epsilon_1^2\Big),\end{aligned}$$

and the index of $\mathbb{E}_{d_2,d_2'}(\tau, m_{G_2} - \epsilon_2\alpha^\vee, m_{\mathfrak{su}(2)} - \epsilon_2\lambda, \epsilon_1 - \epsilon_2, \epsilon_2)$ as

$$\begin{aligned}\mathrm{Ind}_{\mathbb{E}_{d_2,d_2'}} =& -\frac{\epsilon_1^2}{4}(3d_2 + 2d_2') + \frac{(\epsilon_1-\epsilon_2)\epsilon_2}{2}(3d_2^2 + 2{d_2'}^2 - 2d_2d_2' - d_2)\\ &+ (-3d_2 + d_2')\Big(\frac{(m,m)_{G_2}}{2} - \epsilon_2\alpha^\vee \cdot m_{G_2} + d_0\epsilon_2^2\Big)\\ &+ (d_1 - 2d_1')\Big(\frac{(m,m)_{\mathfrak{su}(2)}}{2} - \epsilon_2\lambda \cdot m_{\mathfrak{su}(2)} + d_0\epsilon_2^2\Big) .\end{aligned}$$

Finally, by directly adding all contributions together and using the constraints $d_0 + d_1 + d_2 = k_1$ and $d_0' + d_1' + d_2' = k_2$, we obtain

$$\begin{aligned}&\mathrm{Ind}_{\mathrm{poly}} + \mathrm{Ind}_V^{G_2} + \mathrm{Ind}_V^{\mathfrak{su}(2)} + \mathrm{Ind}_H^{(\mathbf{7}+\mathbf{1},\frac{1}{2}\mathbf{2})} + \mathrm{Ind}_{\mathbb{E}_{d_1,d_1'}} + \mathrm{Ind}_{\mathbb{E}_{d_2,d_2'}}\\ =& -\frac{1}{2}\begin{pmatrix} k_1 & k_2\end{pmatrix}\Omega\begin{pmatrix} m_{G_2}\cdot m_{G_2}\\ m_{\mathfrak{su}(2)}\cdot m_{\mathfrak{su}(2)}\end{pmatrix} - \frac{\epsilon_1^2+\epsilon_2^2}{4}(3k_1 + 2k_2)\\ &+ \frac{\epsilon_1\epsilon_2}{2}(3k_1^2 - 2k_1k_2 + 2k_2^2 - 4k_1) + \frac{9}{5}\epsilon_+^2.\end{aligned} \tag{6.3.9}$$

The final sum is *independent* from $\alpha^\vee, \lambda, d_1, d_1', d_2, d_2'$ themselves, but only depends on their combination (k_1, k_2)! This concludes the modularity of elliptic blowup equations, which serves as the most nontrivial check to arbitrary base degrees.

Limit to rank one theories

By taking the node 2 to zero limit, one obtains the $n = 3$ G_2 theory with $n_{\mathbf{7}} = 1$. The ungauged $\mathfrak{su}(2)$ becomes the $\mathfrak{sp}(1)$ flavor symmetry, thus $t_{\mathfrak{su}(2)}$ becomes the mass m of matter $\mathbf{7}$. As shown in Chapter 5.5.8, there are six unity elliptic blowup equations for the $n = 3$ G_2 theory. In the following, we analyze how they can be obtained from the five vanishing blowup equations of $3,2$ NHC. In fact, it is not hard to find that under the limit $Q_{\mathrm{ell}_2} \to 0$, the vanishing blowup equation (6.3.3) with characteristic (6.3.4) labeled with j reduces to

$$\theta_4^{[\frac{1}{6}+\frac{2j}{5}]}(15\tau, 3\epsilon_+)\mathcal{U}_{G_2}^{[-\frac{1}{6}]} + \theta_4^{[-\frac{1}{6}+\frac{2j}{5}]}(15\tau, 3\epsilon_+)\mathcal{U}_{G_2}^{[\frac{1}{6}]} + \theta_4^{[-\frac{1}{2}+\frac{2j}{5}]}(15\tau, 3\epsilon_+)\mathcal{U}_{G_2}^{[\frac{1}{2}]} = 0. \tag{6.3.10}$$

where we define

$$\mathcal{U}_{G_2}^{[a]} = \mathrm{U}_{G_2}^{[a]}(r_{\mathfrak{su}(2)} = 1) - \mathrm{U}_{G_2}^{[a]}(r_{\mathfrak{su}(2)} = -1), \tag{6.3.11}$$

and $\mathrm{U}_{G_2}^{[a]}$ denotes the l.h.s of unity blowup equations of the $n = 3$ G_2 theory with characteristic a. Since $j = -2, -1, 0, 1, 2$, clearly, one can conclude

$$\mathcal{U}_{G_2}^{[a]} = 0, \quad \text{for} \quad a = -1/6, 1/6, 1/2, \tag{6.3.12}$$

which are

$$\mathrm{U}_{G_2}^{[a]}(r_{\mathfrak{su}(2)} = 1) = \mathrm{U}_{G_2}^{[a]}(r_{\mathfrak{su}(2)} = -1), \quad \text{for} \quad a = -1/6, 1/6, 1/2. \tag{6.3.13}$$

By adding the r.h.s of the unity blowup equations, these give exactly the six unity blowup equations as we already knew.

On the other hand, by taking the node 3 to zero limit, one obtains the $n=2$ $\mathfrak{su}(2)$ theory with 8 half-hypers transforming in $\mathbf{2}$ of $\mathfrak{su}(2)$. There are two vanishing elliptic blowup equations for the $n=2$ $\mathfrak{su}(2)$ theory. In fact, it is not hard to find that under the limit $Q_{\mathrm{ell}_1}\to 0$, the vanishing blowup equation (6.3.3) with characteristic (6.3.4) labeled with j reduces to

$$\theta_3^{[\frac{j}{5}]}(10\tau,6\epsilon_+)\mathrm{V}_{\mathfrak{su}(2)}^{[-\frac{1}{2}]}-\theta_3^{[-\frac{1}{2}+\frac{2j}{5}]}(10\tau,6\epsilon_+)\mathrm{V}_{\mathfrak{su}(2)}^{[0]}=0, \tag{6.3.14}$$

where $\mathrm{V}_{\mathfrak{su}(2)}^{[a]}$ denotes the l.h.s of vanishing blowup equations of the $n=2$ $\mathfrak{su}(2)$ theory. Since $j=-2,-1,0,1,2$, clearly, one can conclude

$$\mathrm{V}_{\mathfrak{su}(2)}^{[-\frac{1}{2}]}=\mathrm{V}_{\mathfrak{su}(2)}^{[0]}=0. \tag{6.3.15}$$

These are just the two vanishing blowup equations of the $n=2$ $\mathfrak{su}(2)$ theory as we already knew.

6.3.2 NHC $3,2,2$

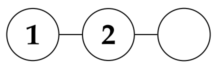
NHC $3,2,2$ can be understood as coupling a M-string node 2 to NHC $3,2$ from the right. The 2d quiver construction was conjectured in (Kim et al., 2018), therefore the elliptic genera are exactly computable.

Elliptic blowup equations

There are in total seven vanishing blowup equations and no unity blowup equations, which is as expected since the M-string only have unity blowup equations, while the NHC $3,2$ has only vanishing equations. The idea can be roughly expressed as

$$\mathrm{V}\star\mathrm{U}=\mathrm{V}. \tag{6.3.16}$$

Using the quiver diagram introduced in previous section 6.1.2, the seven vanishing blowup equations can be denoted as



Let us denote the intersection matrix between the three base curves as $-\Omega$, i.e.

$$\Omega=\begin{pmatrix} 3 & -1 & 0\\ -1 & 2 & -1\\ 0 & -1 & 2\end{pmatrix}. \tag{6.3.17}$$

Note $\det\Omega = 7$ gives the number of non-equivalent vanishing blowup equations. We find the seven vanishing elliptic blowup equations can be written as

$$
\begin{aligned}
0 = &\sum_{\substack{\alpha^\vee\in Q^\vee_{G_2},d_{1,2}\\ d_0=\frac{1}{2}||\alpha^\vee||^2}}^{d_0+d_1+d_2=k_1}\ \sum_{\substack{\lambda\in(P^\vee\backslash Q^\vee)_{\mathfrak{su}(2)},d'_{1,2}\\ d'_0+1/4=\frac{1}{2}||\lambda||^2}}^{d'_0+d'_1+d'_2=k_2}\ \sum_{d''_{1,2}}^{d''_1+d''_2=k_3} (-1)^{|\alpha^\vee|+|\lambda|}\\
&\times\Theta_\Omega^{[a]}\left(\tau,\begin{pmatrix}\alpha^\vee\cdot m_{G_2}+(\bar{y}_1-d_0)(\epsilon_1+\epsilon_2)-d_1\epsilon_1-d_2\epsilon_2\\ \lambda\cdot m_{\mathfrak{su}(2)}+(\bar{y}_2-d'_0)(\epsilon_1+\epsilon_2)-d'_1\epsilon_1-d'_2\epsilon_2\\ \bar{y}_3(\epsilon_1+\epsilon_2)-d'_1\epsilon_1-d'_2\epsilon_2\end{pmatrix}\right)\\
&\times A_V^{G_2}(\alpha^\vee,\tau,m_{G_2})A_V^{\mathfrak{su}(2)}(\lambda,\tau,m_{\mathfrak{su}(2)})A_H^{(\mathbf{7}+\mathbf{1},\frac{1}{2}\mathbf{2},\emptyset)}(\alpha^\vee,\lambda,\tau,m_{G_2},m_{\mathfrak{su}(2)})\\
&\times\mathbb{E}_{d_1,d'_1,d''_1}\big(\tau,m_{G_2}-\epsilon_1\alpha^\vee,m_{\mathfrak{su}(2)}-\epsilon_1\lambda,\epsilon_1,\epsilon_2-\epsilon_1\big)\\
&\times\mathbb{E}_{d_2,d'_2,d''_2}\big(\tau,m_{G_2}-\epsilon_2\alpha^\vee,m_{\mathfrak{su}(2)}-\epsilon_2\lambda,\epsilon_1-\epsilon_2,\epsilon_2\big),
\end{aligned}
\tag{6.3.18}
$$

where the summation indices $d_{0,1,2},d'_{0,1,2},d''_{1,2}\in\mathbb{Z}_{\geq 0}$. The parameters $(\bar{y}_1,\bar{y}_2,\bar{y}_3)=(5/7,9/14,4/7)$, and

$$
a=\begin{pmatrix}a_k\\ a_l\\ a_s\end{pmatrix}=\begin{pmatrix}3j/7\\ -1/2+2j/7\\ j/7\end{pmatrix},\qquad j=-3,-2,-1,0,1,2,3. \tag{6.3.19}
$$

Note $\bar{y}_1,\bar{y}_2,\bar{y}_3$ satisfy the following relation

$$
\Omega\begin{pmatrix}\bar{y}_1\\ \bar{y}_2\\ \bar{y}_3\end{pmatrix}=\begin{pmatrix}3/2\\ 0\\ 1/2\end{pmatrix}=\begin{pmatrix}\bar{y}_{\mathrm{u}}\text{ of rank one }n=3\ G_2\text{ theory}\\ \bar{y}_{\mathrm{v}}\text{ of rank one }n=2\ \mathfrak{su}(2)\text{ theory}\\ \bar{y}_{\mathrm{u}}\text{ of }n=2\text{ M-string theory}\end{pmatrix}. \tag{6.3.20}
$$

This is necessary to be consistent with the rank one elliptic blowup equations when decompactifying one of the base curves.

The leading base degree of the vanishing blowup equations, i.e. $d_0=d_1=d_2=d'_0=d'_1=d'_2=0$ can be simply written as

$$
\Theta_\Omega^{[a]}\left(\tau,\begin{pmatrix}10\epsilon_+/7\\ m_{\mathfrak{su}(2)}+9\epsilon_+/7\\ 8\epsilon_+/7\end{pmatrix}\right)-\Theta_\Omega^{[a]}\left(\tau,\begin{pmatrix}10\epsilon_+/7\\ -m_{\mathfrak{su}(2)}+9\epsilon_+/7\\ 8\epsilon_+/7\end{pmatrix}\right)=0. \tag{6.3.21}
$$

It is easy to check the above identity is correct. For higher base degrees, we have checked the seven vanishing blowup equations from the Calabi-Yau setting to substantial degrees of Kähler classes.

Modularity

The index of the elliptic genus $\mathbb{E}_{k_1,k_2,k_3}$ is known to be

$$
\begin{aligned}
\mathrm{Ind}_{\mathbb{E}_{k_1,k_2,k_3}}=&-\frac{(\epsilon_1+\epsilon_2)^2}{4}(3k_1+2k_2+k_3)+\frac{\epsilon_1\epsilon_2}{2}(3k_1^2+2k_2^2+2k_3^2-2k_1k_2-2k_2k_3-k_1)\\
&+(-3k_1+k_2)\frac{(m,m)_{G_2}}{2}+(-2k_2+k_1+k_3)\frac{(m,m)_{\mathfrak{su}(2)}}{2}.
\end{aligned}
\tag{6.3.22}
$$

To prove modularity, we need to calculate the index of each term in the elliptic blowup equations (6.3.18). After lengthy computations similar with the NHC $3,2$ case, by directly adding all contributions together and using the constraints $d_0+d_1+d_2=k_1$ and $d'_0+d'_1+d'_2=k_2$ and $d''_1+d''_2=k_3$, we obtain

$$\begin{aligned}
&\mathrm{Ind}_{\mathrm{poly}}+\mathrm{Ind}_V^{G_2}+\mathrm{Ind}_V^{\mathfrak{su}(2)}+\mathrm{Ind}_H^{(\mathbf{7}+\mathbf{1},\frac{1}{2}\mathbf{2},\varnothing)}+\mathrm{Ind}_{\mathbb{E}_{d_1,d'_1,d''_1}}+\mathrm{Ind}_{\mathbb{E}_{d_2,d'_2,d''_2}}\\
=&-\frac{1}{2}\begin{pmatrix} k_1 & k_2 & k_3\end{pmatrix}\Omega\begin{pmatrix} m_{G_2}\cdot m_{G_2}\\ m_{\mathfrak{su}(2)}\cdot m_{\mathfrak{su}(2)}\\ 0\end{pmatrix}-\frac{\epsilon_1^2+\epsilon_2^2}{4}(3k_1+2k_2+k_3)\\
&+\frac{\epsilon_1\epsilon_2}{2}(3k_1^2-2k_1k_2+2k_2^2-2k_2k_3+2k_3^2-4k_1)+\frac{19}{28}(\epsilon_1+\epsilon_2)^2.
\end{aligned}\tag{6.3.23}$$

This final sum is *independent* from $\alpha^\vee,\lambda,d_1,d'_1,d_2,d'_2,d''_1,d''_2$ themselves, but only depends on their combination (k_1,k_2,k_3)! This concludes the modularity of elliptic blowup equations, which serves as the most nontrivial check to arbitrary base degrees.

Limits

It is well-known by dropping the last -2 base curve, i.e. taking $k_3=0$, one goes back to the $-3,-2$ NHC. By dropping the left $-3,-2$ base curves, i.e. taking $k_1=k_2=0$, one obtains the M-string theory. By dropping the left -3 base curve, i.e. taking $k_1=0$, one obtains a rank-two Higgsable theory with three vanishing blowup equations. Such theory has

$$\Omega=\begin{pmatrix} 2 & -1\\ -1 & 2\end{pmatrix}.\tag{6.3.24}$$

This theory can be obtained in the following way: one can take the $n=2, G=\mathfrak{su}(2)$ theory, restrict the flavor $\mathfrak{so}(7)\to G_2$, and make the gauge $\mathfrak{su}(2)$ coincide with the flavor $\mathfrak{su}(2)$ of an M-string theory. It is easy to write down the three vanishing blowup equations at degree (k_2,k_3) as

$$\begin{aligned}
\sum_{d'_0+1/4=\frac{1}{2}||\lambda||^2}^{d'_0+d'_1+d'_2=k_2,d''_1+d''_2=k_3}&(-1)^{|\lambda|}\Theta_\Omega^{[a]}\left(\tau,\begin{pmatrix}\lambda\cdot m_{\mathfrak{su}(2)}+(\bar{y}_2-d'_0)(\epsilon_1+\epsilon_2)-d'_1\epsilon_1-d'_2\epsilon_2\\ \bar{y}_3(\epsilon_1+\epsilon_2)-d'_1\epsilon_1-d'_2\epsilon_2\end{pmatrix}\right)\\
&\times A_V^{\mathfrak{su}(2)}(\lambda,\tau,m_{\mathfrak{su}(2)})A_H^{(\mathbf{7}+\mathbf{1},\frac{1}{2}\mathbf{2},\varnothing)}(\lambda,\tau,m_{G_2},m_{\mathfrak{su}(2)})\\
&\times\mathbb{E}_{d'_1,d''_1}(\tau,m_{G_2},m_{\mathfrak{su}(2)}-\epsilon_1\lambda,\epsilon_1,\epsilon_2-\epsilon_1)\\
&\times\mathbb{E}_{d'_2,d''_2}(\tau,m_{G_2},m_{\mathfrak{su}(2)}-\epsilon_2\lambda,\epsilon_1-\epsilon_2,\epsilon_2)=0,
\end{aligned}\tag{6.3.25}$$

where $\bar{y}_2=1/6,\bar{y}_3=1/3$.

6.3.3 NHC $2,3,2$

NHC $2,3,2$ can be understood as coupling two $2_{\mathfrak{su}(2)}$ theories to the rank one theory $3_{\mathfrak{so}(7)}$. The 2d quiver construction of this theory was given in (Kim et al., 2018). Besides, this model has an orbifold construction (Del Zotto, Vafa, and Xie, 2015), where the underlying geometry $T^2\times\mathbb{C}^2/\Gamma$ has discrete action Γ generated by

$(\omega^{-6}, \omega, \omega^5)$, where ω is a root of unity with $\omega^8 = 1$. The S^1 compactification to 5d has been studied with topological vertex in (Hayashi and Ohmori, 2017).

Elliptic blowup equations

Let us denote the intersection matrix between the three base curves as $-\Omega$, i.e.

$$\Omega = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & 2 \end{pmatrix}. \tag{6.3.26}$$

Note $\det \Omega = 8$. It turns out there exist in total 16 vanishing blowup equations and no unity blowup equation. These vanishing equations are divided to two types, each consists of eight equations. One type comes from the configuration

$$\mathrm{V} \star \mathrm{U} \star \mathrm{V} = \mathrm{V}, \tag{6.3.27}$$

which means the unity equations of $3_{\mathfrak{so}(7)}$ theory coupled with the vanishing equations of two $2_{\mathfrak{su}(2)}$ theories. The other comes from the configuration

$$\mathrm{U} \star \mathrm{V} \star \mathrm{U} = \mathrm{V}, \tag{6.3.28}$$

which means the vanishing equations of $3_{\mathfrak{so}(7)}$ theory coupled with the unity equations of two $2_{\mathfrak{su}(2)}$ theories. Using the quiver diagram introduced in previous section 6.1.2, these 16 vanishing blowup equations can be denoted as

$$\textcircled{\textbf{2}}-\textcircled{\textbf{1}}-\textcircled{\textbf{2}} \qquad\qquad \textcircled{\textbf{1}}-\textcircled{\textbf{6}}-\textcircled{\textbf{1}}$$

We find the eight VUV type vanishing elliptic blowup equations can be written as

$$\begin{aligned}
0 = & \sum_{\substack{\lambda\in(P^\vee\backslash Q^\vee)_{\mathfrak{su}(2)},d_{1,2}\\ d_0+1/4=\frac{1}{2}||\lambda||^2}}^{d_0+d_1+d_2=k_1} \ \sum_{\substack{\alpha^\vee\in Q^\vee_{\mathfrak{so}(7)},d'_{1,2}\\ d'_0=\frac{1}{2}||\alpha^\vee||^2}}^{d'_0+d'_1+d'_2=k_2} \ \sum_{\substack{\lambda'\in(P^\vee\backslash Q^\vee)_{\mathfrak{su}(2)},d''_{1,2}\\ d''_0+1/4=\frac{1}{2}||\lambda'||^2}}^{d''_0+d''_1+d''_2=k_3} (-1)^{|\alpha^\vee|+|\lambda|+|\lambda'|} \\
& \times \Theta^{[a]}_{\Omega}\left(\tau, \begin{pmatrix} \lambda\cdot m_{\mathfrak{su}(2)} + (\bar{y}_1 - d_0)(\epsilon_1+\epsilon_2) - d_1\epsilon_1 - d_2\epsilon_2 \\ \alpha^\vee\cdot m_{\mathfrak{so}(7)} + (\bar{y}_2 - d'_0)(\epsilon_1+\epsilon_2) - d'_1\epsilon_1 - d'_2\epsilon_2 \\ \lambda'\cdot m'_{\mathfrak{su}(2)} + (\bar{y}_3 - d''_0)(\epsilon_1+\epsilon_2) - d''_1\epsilon_1 - d''_2\epsilon_2 \end{pmatrix}\right) \\
& \times A_V^{\mathfrak{so}(7)}(\alpha^\vee,\tau,m_{\mathfrak{so}(7)}) A_V^{\mathfrak{su}(2)}(\lambda,\tau,m_{\mathfrak{su}(2)}) A_V^{\mathfrak{su}(2)}(\lambda',\tau,m'_{\mathfrak{su}(2)}) \\
& \times A_H^{\mathfrak{R}}(\lambda,\alpha^\vee,\lambda',\tau,m_{\mathfrak{so}(7)},m_{\mathfrak{su}(2)},m'_{\mathfrak{su}(2)}) \\
& \times \mathbb{E}_{d_1,d'_1,d''_1}(\tau, m_{\mathfrak{su}(2)} - \epsilon_1\lambda, m_{\mathfrak{so}(7)} - \epsilon_1\alpha^\vee, m'_{\mathfrak{su}(2)} - \epsilon_1\lambda', \epsilon_1, \epsilon_2-\epsilon_1) \\
& \times \mathbb{E}_{d_2,d'_2,d''_2}(\tau, m_{\mathfrak{su}(2)} - \epsilon_2\lambda, m_{\mathfrak{so}(7)} - \epsilon_2\alpha^\vee, m'_{\mathfrak{su}(2)} - \epsilon_2\lambda', \epsilon_1-\epsilon_2, \epsilon_2),
\end{aligned} \tag{6.3.29}$$

where the summation indices $d_{0,1,2}, d'_{0,1,2}, d''_{0,1,2} \in \mathbb{Z}_{\geq 0}$. The parameters $\bar{y}_{1,2,3}$ are $\bar{y}_1 = 1/2, \bar{y}_2 = 1, \bar{y}_3 = 1/2$, $\mathfrak{R} = (\mathbf{1},\mathbf{8},\frac{1}{2}\mathbf{2}) + (\frac{1}{2}\mathbf{2},\mathbf{8},\mathbf{1})$, and

$$a = \begin{pmatrix} (2j-1)/8 \\ (2j-1)/4 \\ (2j-1)/8 \end{pmatrix}, \qquad j = -3,-2,-1,0,1,2,3,4. \tag{6.3.30}$$

Note $\bar{y}_1,\bar{y}_2,\bar{y}_3$ satisfy the following relation

$$\Omega\begin{pmatrix}\bar{y}_1\\ \bar{y}_2\\ \bar{y}_3\end{pmatrix}=\begin{pmatrix}0\\2\\0\end{pmatrix}=\begin{pmatrix}\bar{y}_{\mathrm{v}}\text{ of rank one } n=2\ \mathfrak{su}(2)\text{ theory}\\ \bar{y}_{\mathrm{u}}\text{ of rank one } n=3\ \mathfrak{so}(7)\text{ theory}\\ \bar{y}_{\mathrm{v}}\text{ of rank one } n=2\ \mathfrak{su}(2)\text{ theory}\end{pmatrix}. \tag{6.3.31}$$

This is necessary to be consistent with the established elliptic blowup equations for rank one theories when decompactifying some of the base curves.

The leading base degree of the vanishing blowup equations (6.3.29), i.e. $d_0=d_1=d_2=d_0'=d_1'=d_2'=0$ can be simply written as

$$\begin{aligned}&\Theta_\Omega^{[a]}\left(\tau,\begin{pmatrix}m'_{\mathfrak{su}(2)}+\epsilon_+\\ 2\epsilon_+\\ m'_{\mathfrak{su}(2)}+\epsilon_+\end{pmatrix}\right)+\Theta_\Omega^{[a]}\left(\tau,\begin{pmatrix}-m'_{\mathfrak{su}(2)}+\epsilon_+\\ 2\epsilon_+\\ -m'_{\mathfrak{su}(2)}+\epsilon_+\end{pmatrix}\right)\\ -&\Theta_\Omega^{[a]}\left(\tau,\begin{pmatrix}m_{\mathfrak{su}(2)}+\epsilon_+\\ 2\epsilon_+\\ -m'_{\mathfrak{su}(2)}+\epsilon_+\end{pmatrix}\right)-\Theta_\Omega^{[a]}\left(\tau,\begin{pmatrix}-m_{\mathfrak{su}(2)}+\epsilon_+\\ 2\epsilon_+\\ m'_{\mathfrak{su}(2)}+\epsilon_+\end{pmatrix}\right)=0.\end{aligned} \tag{6.3.32}$$

It is easy to check the above identity is correct.

On the other hand, the eight UVU type vanishing elliptic blowup equations can be written as

$$\begin{aligned}0=&\sum_{d_0=\frac{1}{2}||\alpha||^2}^{d_0+d_1+d_2=k_1}\ \sum_{d_0'+1/2=\frac{1}{2}||\lambda||^2}^{d_0'+d_1'+d_2'=k_2}\ \sum_{d_0''=\frac{1}{2}||\alpha'||^2}^{d_0''+d_1''+d_2''=k_3}(-1)^{|\alpha|+|\lambda|+|\alpha'|}\\ &\times\Theta_\Omega^{[a]}\left(\tau,\begin{pmatrix}\alpha\cdot m_{\mathfrak{su}(2)}+(\bar{y}_1-d_0)(\epsilon_1+\epsilon_2)-d_1\epsilon_1-d_2\epsilon_2\\ \lambda\cdot m_{\mathfrak{so}(7)}+(\bar{y}_2-d_0')(\epsilon_1+\epsilon_2)-d_1'\epsilon_1-d_2'\epsilon_2\\ \alpha'\cdot m'_{\mathfrak{su}(2)}+(\bar{y}_3-d_0'')(\epsilon_1+\epsilon_2)-d_1''\epsilon_1-d_2''\epsilon_2\end{pmatrix}\right)\\ &\times A_V^{\mathfrak{so}(7)}(\lambda,\tau,m_{\mathfrak{so}(7)})A_V^{\mathfrak{su}(2)}(\alpha,\tau,m_{\mathfrak{su}(2)})A_V^{\mathfrak{su}(2)}(\alpha',\tau,m'_{\mathfrak{su}(2)})\\ &\times A_H^{\mathfrak{R}}(\alpha,\lambda,\alpha',\tau,m_{\mathfrak{so}(7)},m_{\mathfrak{su}(2)},m'_{\mathfrak{su}(2)})\\ &\times\mathbb{E}_{d_1,d_1',d_1''}(\tau,m_{\mathfrak{su}(2)}-\epsilon_1\alpha,m_{\mathfrak{so}(7)}-\epsilon_1\lambda,m'_{\mathfrak{su}(2)}-\epsilon_1\alpha',\epsilon_1,\epsilon_2-\epsilon_1)\\ &\times\mathbb{E}_{d_2,d_2',d_2''}(\tau,m_{\mathfrak{su}(2)}-\epsilon_2\alpha,m_{\mathfrak{so}(7)}-\epsilon_2\lambda,m'_{\mathfrak{su}(2)}-\epsilon_2\alpha',\epsilon_1-\epsilon_2,\epsilon_2),\end{aligned} \tag{6.3.33}$$

where $\lambda\in(P^\vee\backslash Q^\vee)_{\mathfrak{so}(7)}$ and $\alpha,\alpha'\in Q^\vee_{\mathfrak{su}(2)}$, and $\bar{y}_1=\bar{y}_3=3/4,\bar{y}_2=1/2$. The characteristics a are still those defined in (6.3.30). Note $\bar{y}_1,\bar{y}_2,\bar{y}_3$ satisfy the following relation

$$\Omega\begin{pmatrix}\bar{y}_1\\ \bar{y}_2\\ \bar{y}_3\end{pmatrix}=\begin{pmatrix}1\\0\\1\end{pmatrix}=\begin{pmatrix}\bar{y}_{\mathrm{u}}\text{ of rank one } n=2\ \mathfrak{su}(2)\text{ theory}\\ \bar{y}_{\mathrm{v}}\text{ of rank one } n=3\ \mathfrak{so}(7)\text{ theory}\\ \bar{y}_{\mathrm{u}}\text{ of rank one } n=2\ \mathfrak{su}(2)\text{ theory}\end{pmatrix}. \tag{6.3.34}$$

Since the smallest Weyl orbit in $(P^\vee \backslash Q^\vee)_{\mathfrak{so}(7)}$ is $\mathcal{O}_{1/2,6}$,[2] the leading base degree of the vanishing blowup equations (6.3.33) can be simply written as

$$\sum_{w\in\mathcal{O}_6}(-1)^{|w|}\Theta_\Omega^{[a]}\left(\tau,\begin{pmatrix}3\epsilon_+/2\\ m_\omega+\epsilon_+\\ 3\epsilon_+/2\end{pmatrix}\right)\times\prod_{\beta\in\Delta(\mathfrak{so}(7))}^{w\cdot\beta=1}\frac{1}{\theta_1(\tau,m_\beta)}=0. \tag{6.3.35}$$

We have checked this identity up to $\mathcal{O}(q^{10})$. For higher base degrees, we have checked all the 16 vanishing blowup equations from the Calabi-Yau setting to substantial degrees of Kähler classes.

Modularity

The index of the elliptic genus $\mathbb{E}_{k_1,k_2,k_3}$ is known to be

$$\begin{aligned}\mathrm{Ind}_{\mathbb{E}_{k_1,k_2,k_3}}=&-\frac{(\epsilon_1+\epsilon_2)^2}{2}(k_1+2k_2+k_3)+\frac{\epsilon_1\epsilon_2}{2}(2k_1^2+3k_2^2+2k_3^2-2k_1k_2-2k_2k_3-k_2)\\&+(-2k_1+k_2)\frac{(m_1,m_1)_{\mathfrak{su}(2)}}{2}+(-3k_2+k_1+k_3)\frac{(m_2,m_2)_{\mathfrak{so}(7)}}{2}\\&+(-2k_3+k_2)\frac{(m_3,m_3)_{\mathfrak{su}(2)}}{2}.\end{aligned} \tag{6.3.36}$$

Let us just show the modularity of the VUV type equations here. We need to calculate the index of each term in the vanishing elliptic blowup equations (6.3.29). After lengthy computations similar with the NHC 3, 2 case, by directly adding all contributions together and using the constraints $d_0+d_1+d_2=k_1$ and $d_0'+d_1'+d_2'=k_2$ and $d_0''+d_1''+d_2''=k_3$, we obtain

$$\begin{aligned}&\mathrm{Ind}_{\mathrm{poly}}+\mathrm{Ind}_V^{\mathfrak{so}(7)}+\mathrm{Ind}_V^{\mathfrak{su}(2)}+\mathrm{Ind}_V^{\mathfrak{su}(2)'}+\mathrm{Ind}_H^{\mathfrak{R}}+\mathrm{Ind}_{\mathbb{E}_{d_1,d_1',d_1''}}+\mathrm{Ind}_{\mathbb{E}_{d_2,d_2',d_2''}}\\=&-\frac{1}{2}\begin{pmatrix}k_1 & k_2 & k_3\end{pmatrix}\Omega\begin{pmatrix}m_{\mathfrak{su}(2)}\cdot m_{\mathfrak{su}(2)}\\ m_{\mathfrak{so}(7)}\cdot m_{\mathfrak{so}(7)}\\ m_{\mathfrak{su}(2)'}\cdot m_{\mathfrak{su}(2)'}\end{pmatrix}-\frac{\epsilon_1^2+\epsilon_2^2}{4}(k_1+2k_2+k_3)\\&+\frac{\epsilon_1\epsilon_2}{2}(2k_1^2+3k_2^2+2k_3^2-2k_1k_2-2k_2k_3-5k_2)+\frac{7}{8}(\epsilon_1+\epsilon_2)^2.\end{aligned} \tag{6.3.37}$$

This final sum is *independent* from $\alpha^\vee,\lambda_1,\lambda_2,d_1,d_1',d_2,d_2',d_1'',d_2''$ themselves, but only depends on their combination (k_1,k_2,k_3)! This concludes the modularity of elliptic blowup equations, which serves as the most nontrivial check to arbitrary base degrees.

6.4 ADE chains of (-2)-curves

The 2d quiver construction and elliptic genera are given in (Gadde et al., 2018), see also another form in (Haghighat, Yan, and Yau, 2018). A crucial property of simply-laced Dynkin diagrams is needed in order to achieve admissible gluing of the blowup equations of individual nodes: the mark of each node has to be the average of the marks of all its adjacent nodes. Besides, when a node is at the end, its

[2]Note the vector representation $\mathbf{7}_{\mathbf{v}}^{\mathfrak{so}(7)}=1+\mathcal{O}_{1/2,6}$.

mark is half of the mark of its adjacent node. The problem of finding all admissible blowup equations then reduces to the decomposition of Weyl orbits of the special unitary algebra to its subalgebras.

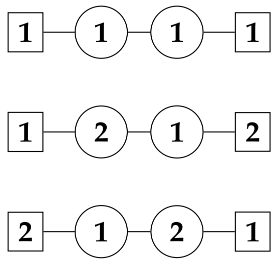
In the following we demonstrate the application of gluing rules for some typical examples including $A_{2,3}$, $D_{4,5}$ and $E_{6,7,8}$ quivers.

- We first demonstrate the gluing for a simple example which is an A type quiver with gauge group $\mathfrak{su}(2)$. Note when two $n = 2$ $\mathfrak{su}(2)$ gauge theories are coupled together, the flavor symmetry $\mathfrak{su}(4)$ (or equivalently $\mathfrak{so}(7)$) breaks down to $\mathfrak{su}(2) \times \mathfrak{su}(2)$. Then one of the flavor symmetry $\mathfrak{su}(2)$ becomes the gauge symmetry $\mathfrak{su}(2)$ for the other theory. For rank one $n = 2$ $\mathfrak{su}(2)$ theory, the unity λ_F is in $\mathbf{1}$, while the vanishing λ_F is in $\mathbf{6}$. Under the flavor group splitting, $\mathbf{6} = 2(\mathbf{1},\mathbf{1}) + (\mathbf{2},\mathbf{2})$. Note also $\mathbf{2} \subset (P^\vee \backslash Q^\vee)_{\mathfrak{su}(2)}$. This means for a unity -2 node, the adjacent two -2 nodes must be both unity or both vanishing. On the other hand, for a vanishing -2 node, the adjacent two -2 nodes can only be both unity.

  For example, for the A_2 quiver, we find the following structure or the blowup equations

$$
\begin{aligned}
&\mathrm{U} \star \mathrm{U} = \mathrm{U}, \\
&\mathrm{V} \star \mathrm{U} = \mathrm{V}, \\
&\mathrm{U} \star \mathrm{V} = \mathrm{V}.
\end{aligned}
\tag{6.4.1}
$$

  Keep in mind there are two $\mathfrak{su}(2)$ fundamental matters at the two ends of the A_2 quiver. Therefore, the above blowup equations can be expressed in quiver diagrams as



  The first quiver diagram represents unity equations, while the other two represent vanishing equations. The number of equations with fixed characteristic represented by each quiver diagram is the product of numbers in square nodes, while the number of characteristics is the determinant of the Cartan matrix C of the quiver diagram. We find $\det(C_{A_2}) = 3$. Thus there are in total $3 \times 1 = 3$ unity equations and $3 \times (2+2) = 12$ vanishing equations.

  For A_3 quiver, there are following blowup equations

$$
\begin{aligned}
&\mathrm{U} \star \mathrm{U} \star \mathrm{U} = \mathrm{U}, \\
&\mathrm{U} \star \mathrm{V} \star \mathrm{U} = \mathrm{V}, \\
&\mathrm{V} \star \mathrm{U} \star \mathrm{V} = \mathrm{V},
\end{aligned}
\tag{6.4.2}
$$

  or in quiver diagrams as

$$\boxed{\mathbf{1}} - \mathbf{1} - \mathbf{1} - \mathbf{1} - \boxed{\mathbf{1}}$$

$$\boxed{\mathbf{2}} - \mathbf{1} - \mathbf{2} - \mathbf{1} - \boxed{\mathbf{2}}$$

$$\boxed{\mathbf{1}} - \mathbf{2} - \mathbf{1} - \mathbf{2} - \boxed{\mathbf{1}}$$

Note $\det(C_{A_3}) = 4$. Thus there are in total 4 unity equations and $4 \times (4+1) = 20$ vanishing equations.

- Consider A type quiver theories with $\mathfrak{su}(3)$ symmetry. When two $n = 2$ $\mathfrak{su}(3)$ gauge theories are coupled together, the flavor symmetry $\mathfrak{su}(6)$ breaks down to $\mathfrak{su}(3) \times \mathfrak{su}(3)$. Then one of the flavor symmetry $\mathfrak{su}(3)$ becomes the gauge symmetry $\mathfrak{su}(3)$ for the other theory. We summarize the r fields behavior of rank one $n = 2$ $\mathfrak{su}(3)$ theory under the flavor group splitting in the following table, where $\mathcal{O}_{\omega_i}$ is the Weyl orbit generated by the i-th fundamental coweight.

	λ_G	λ_F	branching rules of λ_F
	$\mathfrak{su}(3)$	$\mathfrak{su}(6)$	$\mathfrak{su}(3) \times \mathfrak{su}(3)$
unity	$\mathbf{1}$	$\mathcal{O}_{\omega_3} = \mathbf{20}$	$2(\mathbf{1},\mathbf{1}) + (\mathbf{3},\bar{\mathbf{3}}) + (\bar{\mathbf{3}},\mathbf{3})$
vanishing	$\mathbf{3}$	$\mathcal{O}_{\omega_5} = \bar{\mathbf{6}}$	$(\bar{\mathbf{3}},\mathbf{1}) + (\mathbf{1},\bar{\mathbf{3}})$
vanishing	$\bar{\mathbf{3}}$	$\mathcal{O}_{\omega_1} = \mathbf{6}$	$(\mathbf{3},\mathbf{1}) + (\mathbf{1},\mathbf{3})$

For example, for A_2 quiver, read from the table above and gluing rules, we find there are following blowup equations

$$\boxed{\mathbf{1}} - \mathbf{1} - \mathbf{1} - \boxed{\mathbf{1}}$$

$$\boxed{\bar{\mathbf{3}}} - \mathbf{1} - \mathbf{3} - \boxed{\bar{\mathbf{3}}}$$

$$\boxed{\bar{\mathbf{3}}} - \mathbf{3} - \mathbf{1} - \boxed{\bar{\mathbf{3}}}$$

$$\boxed{\mathbf{3}} - \mathbf{1} - \bar{\mathbf{3}} - \boxed{\mathbf{3}}$$

$$\boxed{\mathbf{3}} - \bar{\mathbf{3}} - \mathbf{1} - \boxed{\mathbf{3}}$$

The first quiver represents unity equations, while all the other quivers represent vanishing equations. Note $\det(C_{A_2}) = 3$. Thus there are in total 3 unity equations and 108 vanishing equations.

For A_3 quiver, there are following blowup equations

$$\boxed{\mathbf{1}} - \mathbf{1} - \mathbf{1} - \mathbf{1} - \boxed{\mathbf{1}}$$

$$\boxed{\bar{\mathbf{3}}} - \mathbf{1} - \mathbf{3} - \bar{\mathbf{3}} - \boxed{\mathbf{1}}$$

$$\boxed{\mathbf{1}} - \bar{\mathbf{3}} - \mathbf{3} - \mathbf{1} - \boxed{\bar{\mathbf{3}}}$$

$$\boxed{\mathbf{3}} - \mathbf{1} - \bar{\mathbf{3}} - \mathbf{3} - \boxed{\mathbf{1}}$$

$$\boxed{\mathbf{1}} - \mathbf{3} - \bar{\mathbf{3}} - \mathbf{1} - \boxed{\mathbf{3}}$$

$$\boxed{\bar{\mathbf{3}}} - \mathbf{3} - \mathbf{1} - \bar{\mathbf{3}} - \boxed{\mathbf{3}}$$

$$\boxed{\mathbf{3}} - \bar{\mathbf{3}} - \mathbf{1} - \mathbf{3} - \boxed{\bar{\mathbf{3}}}$$

The first quiver represents unity equations, while the remaining quivers remain vanishing equations. Note $\det(C_{A_3}) = 4$. Thus there are in total 4 unity equations and 120 vanishing equations.

- Consider A type quiver theories with $\mathfrak{su}(4)$ symmetry. When two $n = 2$ $\mathfrak{su}(4)$ gauge theories are coupled together, the flavor symmetry $\mathfrak{su}(8)$ breaks down to $\mathfrak{su}(4) \times \mathfrak{su}(4)$. Then one of the flavor $\mathfrak{su}(4)$ becomes the gauge $\mathfrak{su}(4)$ for the other theory. Note $(P^\vee/Q^\vee)_{A_3} = \mathbb{Z}_4$. We summarize the r fields behavior under the flavor group splitting in the following table, where $\mathcal{O}_{\omega_i}$ is the Weyl orbit generated by the i-th fundamental coweight. Note $\mathbf{6} = \bar{\mathbf{6}}$ so we do not

	λ_G	λ_F	branching rules of λ_F
	$\mathfrak{su}(4)$	$\mathfrak{su}(8)$	$\mathfrak{su}(4) \times \mathfrak{su}(4)$
u	$\mathbf{1}$	$\mathcal{O}_{\omega_4} = \mathbf{70}$	$2(\mathbf{1},\mathbf{1}) + (\mathbf{4},\bar{\mathbf{4}}) + (\bar{\mathbf{4}},\mathbf{4}) + (\mathbf{6},\mathbf{6})$
v	$\mathbf{4}$	$\mathcal{O}_{\omega_6} = \overline{\mathbf{28}}$	$(\mathbf{6},\mathbf{1}) + (\mathbf{1},\mathbf{6}) + (\bar{\mathbf{4}},\bar{\mathbf{4}})$
v	$\mathbf{6}$	$\mathcal{O}_0 = \mathbf{1}$	$(\mathbf{1},\mathbf{1})$
v	$\bar{\mathbf{4}}$	$\mathcal{O}_{\omega_2} = \mathbf{28}$	$(\mathbf{6},\mathbf{1}) + (\mathbf{1},\mathbf{6}) + (\mathbf{4},\mathbf{4})$

write $\bar{\mathbf{6}}$ in the table.

Now based on the general gluing procedure, we can directly write down all admissible blowup equations. For example, for A_2 quiver, there are following blowup equations

$$\boxed{\mathbf{1}} - \mathbf{1} - \mathbf{1} - \boxed{\mathbf{1}}$$

$$\boxed{\bar{\mathbf{4}}} - \mathbf{1} - \mathbf{4} - \boxed{\mathbf{6}}$$

$$\boxed{\mathbf{4}} - \mathbf{1} - \bar{\mathbf{4}} - \boxed{\mathbf{6}}$$

$$\boxed{\mathbf{6}} - \mathbf{4} - \mathbf{1} - \boxed{\bar{\mathbf{4}}}$$

$$\boxed{\mathbf{6}} - \bar{\mathbf{4}} - \mathbf{1} - \boxed{\mathbf{4}}$$

$$\boxed{\mathbf{6}} - \mathbf{1} - \mathbf{6} - \boxed{\mathbf{1}}$$

$$\boxed{\mathbf{1}} - \mathbf{6} - \mathbf{1} - \boxed{\mathbf{6}}$$

The first quiver diagram represents unity equations, while the other quiver diagrams represent vanishing equations. Note $\det(C_{A_2}) = 3$. Thus there are in total 3 unity equations and $3 \times (4 \times 4 \times 6 + 2 \times 6) = 324$ vanishing equations.

For the A_3 quiver, there are the following blowup equations:

$$\boxed{\mathbf{1}} - \mathbf{1} - \mathbf{1} - \mathbf{1} - \boxed{\mathbf{1}}$$

$$\boxed{\mathbf{6}} - \mathbf{4} - \mathbf{1} - \bar{\mathbf{4}} - \boxed{\mathbf{6}}$$

$$\boxed{\mathbf{6}} - \bar{\mathbf{4}} - \mathbf{1} - \mathbf{4} - \boxed{\mathbf{6}}$$

$$\boxed{\mathbf{1}} - \mathbf{6} - \mathbf{1} - \mathbf{6} - \boxed{\mathbf{1}}$$

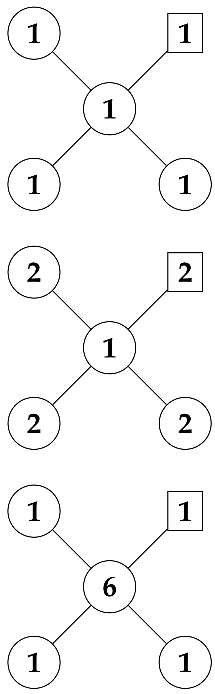
$$\boxed{\mathbf{6}} - \mathbf{1} - \mathbf{6} - \mathbf{1} - \boxed{\mathbf{6}}$$

The first quiver diagram represents unity equations, while the other quiver diagrams represent vanishing equations. Note $\det(C_{A_3}) = 4$. Thus there are in total 4 unity equations and 336 vanishing equations.

- Now consider a D_4 quiver with gauge group $\mathfrak{su}(2d_i)$. We want to couple a $n = 2$ $\mathfrak{su}(4)$ gauge theory with three $n = 2$ $\mathfrak{su}(2)$ gauge theories and a extra $\mathfrak{su}(2)$ fundamental. Note the flavor symmetry $\mathfrak{su}(8)$ of the center node breaks down to $\mathfrak{su}(2)^4$. Note $(P^\vee/Q^\vee)_{A_3} = \mathbb{Z}_4$. We summarize the r fields behavior under the flavor group splitting in the following table, where $\mathcal{O}_{\omega_i}$ is the Weyl orbit of the i-th fundamental coweight.

	λ_G	λ_F	branching rules of λ_F
	$\mathfrak{su}(4)$	$\mathfrak{su}(8)$	$\mathfrak{su}(2)^4$
u	$\mathbf{1}$	$\mathcal{O}_{\omega_4} = \mathbf{70}$	$6(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1}) + 2((\mathbf{2},\mathbf{2},\mathbf{1},\mathbf{1})$ and permutations$) + (\mathbf{2},\mathbf{2},\mathbf{2},\mathbf{2})$
v	$\mathbf{4}$	$\mathcal{O}_{\omega_6} = \overline{\mathbf{28}}$	$4(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1}) + ((\mathbf{2},\mathbf{2},\mathbf{1},\mathbf{1})$ and permutations$)$
v	$\mathbf{6}$	$\mathcal{O}_0 = \mathbf{1}$	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$
v	$\bar{\mathbf{4}}$	$\mathcal{O}_{\omega_2} = \mathbf{28}$	$4(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1}) + ((\mathbf{2},\mathbf{2},\mathbf{1},\mathbf{1})$ and permutations$)$

Now based on the general gluing procedure, we can directly write down all admissible blowup equations as:



The first quiver diagram represents unity equations, while the remaining two diagrams represent vanishing equations. Note $\det(C_{D_4}) = 4$. Thus there are in total 4 unity and $4 \times (2+1) = 12$ vanishing blowup equations. Let us show the leading base degree identities for the two types of vanishing blowup equations. The intersection matrix among base curves $-\Omega$ is just the negative of the Cartan matrix of D_4, i.e.

$$\Omega = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & -1 \\ 0 & -1 & 2 & 0 \\ 0 & -1 & 0 & 2 \end{pmatrix}. \tag{6.4.3}$$

Then we find the first type of vanishing blowup equations has the following leading degree vanishing identities

$$\sum_{\lambda_{a,b,c}=\pm 1/2} (-1)^{\lambda_a+\lambda_b+\lambda_c} \Theta_{\Omega}^{[a]} \left(\tau, \begin{pmatrix} -\lambda_a m_a + 2(\epsilon_1+\epsilon_2) \\ 4(\epsilon_1+\epsilon_2) \\ -\lambda_b m_b + 2(\epsilon_1+\epsilon_2) \\ -\lambda_c m_c + 2(\epsilon_1+\epsilon_2) \end{pmatrix} \right) = 0. \tag{6.4.4}$$

where $m_{a,b,c}$ are the fugacities associated to the three $\mathfrak{su}(2)$ gauge node. Here the contributions from vector and hyper multiplets do not depend on the summation indices $\lambda_{a,b,c}$ and thus we have factored them out. The four possible characteristics a are defined according to (6.1.4). The second type of vanish blowup equation has leading base degree as

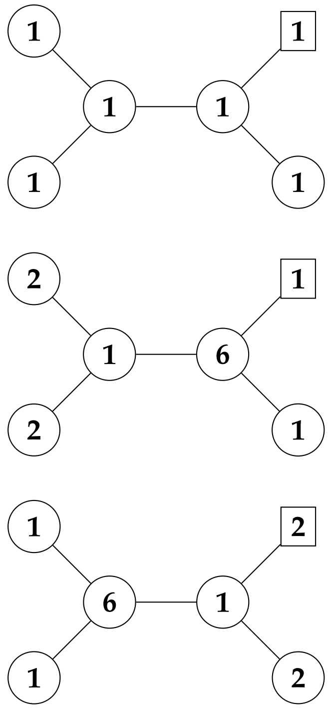
$$\sum_{1\le i<j\le 4} \Theta_{\Omega}^{[a]} \left(\tau, \begin{pmatrix} 2(\epsilon_1+\epsilon_2) \\ -m_i - m_j + 3(\epsilon_1+\epsilon_2) \\ 2(\epsilon_1+\epsilon_2) \\ 2(\epsilon_1+\epsilon_2) \end{pmatrix} \right) \frac{1}{\prod_{k\neq i,j} \theta_1(m_i-m_k)\theta_1(m_j-m_k)} = 0. \tag{6.4.5}$$

Here $m_i, i = 1, 2, 3, 4$ are the $\mathfrak{su}(4)$ fugacities of the central node with $\sum_{i=1}^{4} m_i = 0$. We have checked these identities up to order $\mathcal{O}(q^{10})$.

- Consider a D_5 quiver with gauge group $\mathfrak{su}(2d_i)$. We want to couple two $n = 2$ $\mathfrak{su}(4)$ gauge theories together with three $n = 2$ $\mathfrak{su}(2)$ gauge theories and an extra $\mathfrak{su}(2)$ fundamental. Note the flavor symmetry $\mathfrak{su}(8)$ of the $\mathfrak{su}(4)$ node breaks down to $\mathfrak{su}(4) \times \mathfrak{su}(2)^2$. Note also $(P^\vee/Q^\vee)_{A_3} = \mathbb{Z}_4$. We summarize the r fields behavior under the flavor group splitting in the following table, where $\mathcal{O}_{\omega_i}$ is the Weyl orbit generated by the i-th fundamental coweight.

	λ_G	λ_F	branching rules of λ_F
	$\mathfrak{su}(4)$	$\mathfrak{su}(8)$	$\mathfrak{su}(4) \times \mathfrak{su}(2) \times \mathfrak{su}(2)$
u	$\mathbf{1}$	$\mathcal{O}_{\omega_4} = \mathbf{70}$	$2(\mathbf{1},\mathbf{1},\mathbf{1}) + (\mathbf{6},\mathbf{2},\mathbf{2}) + 2(\mathbf{6},\mathbf{1},\mathbf{1}) + (\mathbf{4},\mathbf{2},\mathbf{1}) + (\mathbf{4},\mathbf{1},\mathbf{2}) + (\bar{\mathbf{4}},\mathbf{2},\mathbf{1}) + (\bar{\mathbf{4}},\mathbf{1},\mathbf{2})$
v	$\mathbf{4}$	$\mathcal{O}_{\omega_6} = \overline{\mathbf{28}}$	$2(\mathbf{1},\mathbf{1},\mathbf{1}) + (\mathbf{6},\mathbf{1},\mathbf{1}) + (\bar{\mathbf{4}},\mathbf{2},\mathbf{1}) + (\bar{\mathbf{4}},\mathbf{1},\mathbf{2}) + (\mathbf{1},\mathbf{2},\mathbf{1}) + (\mathbf{1},\mathbf{1},\mathbf{2})$
v	$\mathbf{6}$	$\mathcal{O}_0 = \mathbf{1}$	$(\mathbf{1},\mathbf{1},\mathbf{1})$
v	$\bar{\mathbf{4}}$	$\mathcal{O}_{\omega_2} = \mathbf{28}$	$2(\mathbf{1},\mathbf{1},\mathbf{1}) + (\mathbf{6},\mathbf{1},\mathbf{1}) + (\mathbf{4},\mathbf{2},\mathbf{1}) + (\mathbf{4},\mathbf{1},\mathbf{2}) + (\mathbf{1},\mathbf{2},\mathbf{1}) + (\mathbf{1},\mathbf{1},\mathbf{2})$

Now based on the general gluing procedure, we can directly write down all admissible blowup equations as:



The first quiver diagram represents unity equations, while the remaining two diagrams represent vanishing equations. Note $\det(C_{D_5}) = 4$. Thus there are in total 4 unity and $4 \times (1 + 2) = 12$ vanishing blowup equations.

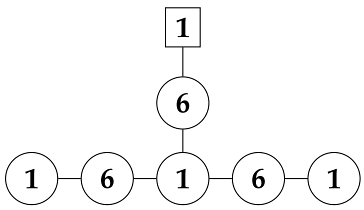
- Consider the E_6 quiver with gauge group $\mathfrak{su}(2d_i)$. We want to couple an $n = 2$ $\mathfrak{su}(6)$ gauge theory with three $n = 2$ $\mathfrak{su}(4)$ gauge theories and two of the $\mathfrak{su}(4)$ theories each with an $\mathfrak{su}(2)$ theory and the other $\mathfrak{su}(4)$ theory to an extra $\mathfrak{su}(2)$ fundamental hypermultiplet. All the nodes together then form the Dynkin diagram of affine E_6. Note the flavor symmetry $\mathfrak{su}(12)$ of the center node breaks down to $\mathfrak{su}(4)^3$, and the flavor symmetry $\mathfrak{su}(8)$ of the $\mathfrak{su}(4)$ node breaks down

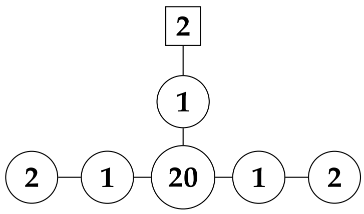
	λ_G	λ_F	branching rules of λ_F
	$\mathfrak{su}(6)$	$\mathfrak{su}(12)$	$\mathfrak{su}(4)^3$
u	$\mathbf{1}$	$\mathcal{O}_{\omega_6} = \mathbf{924}$	$2(\mathbf{6},\mathbf{1},\mathbf{1}) + (\mathbf{4},\mathbf{4},\mathbf{1}) + (\bar{\mathbf{4}},\bar{\mathbf{4}},\mathbf{1}) + (\mathbf{4},\mathbf{6},\bar{\mathbf{4}}) + (\mathbf{6},\mathbf{6},\mathbf{6}) +$ permutations
v	$\mathbf{6}$	$\mathcal{O}_{\omega_8} = \overline{\mathbf{495}}$	$3(\mathbf{1},\mathbf{1},\mathbf{1}) + (\mathbf{4},\bar{\mathbf{4}},\mathbf{1}) + (\mathbf{6},\mathbf{6},\mathbf{1}) + (\bar{\mathbf{4}},\bar{\mathbf{4}},\mathbf{6}) +$ permutations
v	$\mathbf{15}$	$\mathcal{O}_{\omega_{10}} = \overline{\mathbf{66}}$	$(\mathbf{6},\mathbf{1},\mathbf{1}) + (\bar{\mathbf{4}},\bar{\mathbf{4}},\mathbf{1}) +$ permutations
v	$\mathbf{20}$	$\mathcal{O}_0 = \mathbf{1}$	$(\mathbf{1},\mathbf{1},\mathbf{1})$
v	$\overline{\mathbf{15}}$	$\mathcal{O}_{\omega_2} = \mathbf{66}$	$(\mathbf{6},\mathbf{1},\mathbf{1}) + (\mathbf{4},\mathbf{4},\mathbf{1}) +$ permutations
v	$\bar{\mathbf{6}}$	$\mathcal{O}_{\omega_4} = \mathbf{495}$	$3(\mathbf{1},\mathbf{1},\mathbf{1}) + (\mathbf{4},\bar{\mathbf{4}},\mathbf{1}) + (\mathbf{6},\mathbf{6},\mathbf{1}) + (\mathbf{4},\mathbf{4},\mathbf{6}) +$ permutations

	λ_G	λ_F	branching rules of λ_F
	$\mathfrak{su}(4)$	$\mathfrak{su}(8)$	$\mathfrak{su}(2) \times \mathfrak{su}(6)$
u	$\mathbf{1}$	$\mathcal{O}_{\omega_4} = \mathbf{70}$	$(\mathbf{1},\overline{\mathbf{15}}) + (\mathbf{2},\mathbf{20}) + (\mathbf{1},\mathbf{15})$
v	$\mathbf{4}$	$\mathcal{O}_{\omega_6} = \overline{\mathbf{28}}$	$(\mathbf{1},\mathbf{1}) + (\mathbf{2},\mathbf{6}) + (\mathbf{1},\overline{\mathbf{15}})$
v	$\mathbf{6}$	$\mathcal{O}_0 = \mathbf{1}$	$(\mathbf{1},\mathbf{1})$
v	$\bar{\mathbf{4}}$	$\mathcal{O}_{\omega_2} = \mathbf{28}$	$(\mathbf{1},\mathbf{1}) + (\mathbf{2},\mathbf{6}) + (\mathbf{1},\mathbf{15})$

to $\mathfrak{su}(2) \times \mathfrak{su}(6)$. Besides, $(P^\vee/Q^\vee)_{A_5} = \mathbb{Z}_6$. We summarize the r fields behavior under the flavor group splitting in the following tables.

Now based on the general gluing procedure, we can directly write down all admissible blowup equations as:





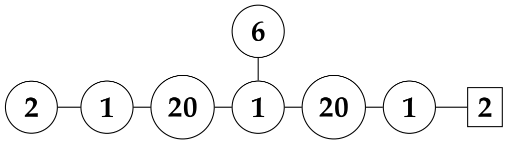
Both quiver diagrams represent vanishing equations. Note $\det(C_{E_6}) = 3$. Thus there are in total $3 \times (1+2) = 9$ vanishing blowup equations.

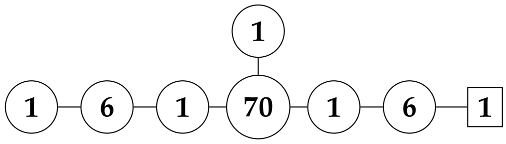
- Consider the E_7 quiver with gauge group $\mathfrak{su}(2d_i)$. In this case, the flavor symmetry $\mathfrak{su}(16)$ of the center node breaks down to $\mathfrak{su}(6)^2 \times \mathfrak{su}(4)$, and the flavor symmetry $\mathfrak{su}(12)$ of the $\mathfrak{su}(6)$ node breaks down to $\mathfrak{su}(8) \times \mathfrak{su}(4)$, and the flavor symmetry $\mathfrak{su}(8)$ of the $\mathfrak{su}(4)$ node breaks down to $\mathfrak{su}(6) \times \mathfrak{su}(2)$. Besides, $(P^\vee/Q^\vee)_{A_7} = \mathbb{Z}_8$. We summarize the r fields behavior under the flavor group splitting in the following tables. Note here the $\ldots$ means conjugate representations and permutations over the first two $\mathfrak{su}(6)$.

  Now based on the general gluing procedure, we can directly write down all admissible blowup equations as:

	λ_G	λ_F	branching rules of λ_F
	$\mathfrak{su}(8)$	$\mathfrak{su}(16)$	$\mathfrak{su}(6)\times\mathfrak{su}(6)\times\mathfrak{su}(4)$
u	$\mathbf{1}$	$\mathcal{O}_{\omega_8}=\mathbf{12870}$	$(\mathbf{1},\mathbf{15},\mathbf{1})+(\bar{\mathbf{6}},\mathbf{20},\mathbf{1})+(\overline{\mathbf{15}},\overline{\mathbf{15}},\mathbf{1})+(\mathbf{1},\mathbf{6},\mathbf{4})+(\bar{\mathbf{6}},\mathbf{15},\mathbf{4})+(\overline{\mathbf{15}},\mathbf{20},\mathbf{4})$ $+(\mathbf{1},\mathbf{1},\mathbf{6})+(\bar{\mathbf{6}},\mathbf{6},\mathbf{6})+(\overline{\mathbf{15}},\mathbf{15},\mathbf{6})+(\mathbf{20},\mathbf{20},\mathbf{6})+\cdots$
v	$\mathbf{8}$	$\mathcal{O}_{\omega_{10}}=\overline{\mathbf{8008}}$	$(\mathbf{1},\mathbf{1},\mathbf{1})+(\bar{\mathbf{6}},\mathbf{6},\mathbf{1})+(\overline{\mathbf{15}},\mathbf{15},\mathbf{1})+(\mathbf{20},\mathbf{20},\mathbf{1})+(\mathbf{6},\mathbf{1},\bar{\mathbf{4}})+(\mathbf{15},\bar{\mathbf{6}},\bar{\mathbf{4}})$ $+(\mathbf{20},\overline{\mathbf{15}},\bar{\mathbf{4}})+(\mathbf{15},\mathbf{1},\mathbf{6})+(\mathbf{20},\bar{\mathbf{6}},\mathbf{6})+(\overline{\mathbf{15}},\overline{\mathbf{15}},\mathbf{6})+\cdots$
v	$\mathbf{28}$	$\mathcal{O}_{\omega_{12}}=\overline{\mathbf{1820}}$	$(\mathbf{15},\mathbf{1},\mathbf{1})+(\mathbf{20},\bar{\mathbf{6}},\mathbf{1})+(\overline{\mathbf{15}},\overline{\mathbf{15}},\mathbf{1})+(\mathbf{20},\mathbf{1},\bar{\mathbf{4}})+(\mathbf{15},\bar{\mathbf{6}},\bar{\mathbf{4}})$ $+(\overline{\mathbf{15}},\mathbf{1},\mathbf{6})+(\bar{\mathbf{6}},\bar{\mathbf{6}},\mathbf{6})+\cdots$
v	$\mathbf{56}$	$\mathcal{O}_{\omega_{14}}=\overline{\mathbf{120}}$	$(\overline{\mathbf{15}},\mathbf{1},\mathbf{1})+(\bar{\mathbf{6}},\bar{\mathbf{6}},\mathbf{1})+(\bar{\mathbf{6}},\mathbf{1},\bar{\mathbf{4}})+(\mathbf{1},\mathbf{1},\mathbf{6})+\cdots$
v	$\mathbf{70}$	$\mathcal{O}_0=\mathbf{1}$	$(\mathbf{1},\mathbf{1},\mathbf{1})$
v	$\overline{\mathbf{56}}$	$\mathcal{O}_{\omega_2}=\mathbf{120}$	conjugate
v	$\overline{\mathbf{28}}$	$\mathcal{O}_{\omega_4}=\mathbf{1820}$	conjugate
v	$\bar{\mathbf{8}}$	$\mathcal{O}_{\omega_6}=\mathbf{8008}$	conjugate

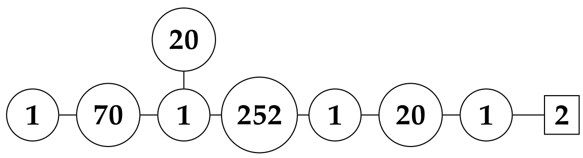
	λ_G	λ_F	branching rules of λ_F
	$\mathfrak{su}(6)$	$\mathfrak{su}(12)$	$\mathfrak{su}(8)\times\mathfrak{su}(4)$
u	$\mathbf{1}$	$\mathcal{O}_{\omega_6}=\mathbf{924}$	$(\overline{\mathbf{28}},\mathbf{1})+(\overline{\mathbf{56}},\mathbf{4})+(\mathbf{70},\mathbf{6})+(\mathbf{56},\bar{\mathbf{4}})+(\mathbf{28},\mathbf{1})$
v	$\mathbf{6}$	$\mathcal{O}_{\omega_8}=\overline{\mathbf{495}}$	$(\mathbf{70},\mathbf{1})+(\overline{\mathbf{56}},\bar{\mathbf{4}})+(\overline{\mathbf{28}},\bar{\mathbf{6}})+(\bar{\mathbf{8}},\mathbf{4})+(\mathbf{1},\mathbf{1})$
v	$\mathbf{15}$	$\mathcal{O}_{\omega_{10}}=\overline{\mathbf{66}}$	$(\overline{\mathbf{28}},\mathbf{1})+(\bar{\mathbf{8}},\bar{\mathbf{4}})+(\mathbf{1},\bar{\mathbf{6}})$
v	$\mathbf{20}$	$\mathcal{O}_0=\mathbf{1}$	$(\mathbf{1},\mathbf{1},\mathbf{1})$
v	$\overline{\mathbf{15}}$	$\mathcal{O}_{\omega_2}=\mathbf{66}$	$(\mathbf{28},\mathbf{1})+(\mathbf{8},\mathbf{4})+(\mathbf{1},\mathbf{6})$
v	$\bar{\mathbf{6}}$	$\mathcal{O}_{\omega_4}=\mathbf{495}$	$(\mathbf{70},\mathbf{1})+(\mathbf{56},\mathbf{4})+(\mathbf{28},\mathbf{6})+(\mathbf{8},\bar{\mathbf{4}})+(\mathbf{1},\mathbf{1})$

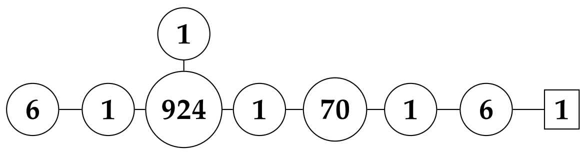




Both quiver diagrams represent vanishing equations. Note $\det(C_{E_7})=2$. Thus there are in total $2\times(2+1)=6$ vanishing blowup equations.

- Consider the E_8 quiver with gauge group $\mathfrak{su}(2d_i)$. After a long but elementary computation on the representation decomposition like in the cases above, and based on the general gluing procedure, we can directly write down all admissible blowup equations as:





	λ_G	λ_F	branching rules of λ_F
	$\mathfrak{su}(4)$	$\mathfrak{su}(8)$	$\mathfrak{su}(2)\times\mathfrak{su}(6)$
u	$\mathbf{1}$	$\mathcal{O}_{\omega_4}=\mathbf{70}$	$(\mathbf{1},\overline{\mathbf{15}})+(\mathbf{2},\mathbf{20})+(\mathbf{1},\mathbf{15})$
v	$\mathbf{4}$	$\mathcal{O}_{\omega_6}=\overline{\mathbf{28}}$	$(\mathbf{1},\mathbf{1})+(\mathbf{2},\mathbf{6})+(\mathbf{1},\overline{\mathbf{15}})$
v	$\mathbf{6}$	$\mathcal{O}_0=\mathbf{1}$	$(\mathbf{1},\mathbf{1})$
v	$\bar{\mathbf{4}}$	$\mathcal{O}_{\omega_2}=\mathbf{28}$	$(\mathbf{1},\mathbf{1})+(\mathbf{2},\mathbf{6})+(\mathbf{1},\mathbf{15})$

Both quiver diagrams represent vanishing equations. Note $\det(C_{E_8})=1$. Thus there are in total $2+1=3$ vanishing blowup equations.

6.5 Conformal matter theories

6d conformal matter theories are interesting SCFTs coming from M5-branes probing an ADE singularity in M-theory or intersecting an ADE singularity with a Horava-Witten M9-wall (Del Zotto et al., 2015). The elliptic genera of these theories are rarely known except for a few cases such as (D_N,D_N) and (E_6,E_6) models. In the following, we present the blowup equations for all notable conformal matter theories. Note for all conformal matter theories except for $(\mathfrak{sp}(n),\mathfrak{sp}(n))$ theory, the determinant of the intersection matrix of base curves is $\det(\Omega)=1$.[3] Therefore the number of non-equivalent blowup equations for each of these theories is just the number of non-equivalent admissible r fields for the nodes.

- (D_4,D_4) conformal matter theory is often denoted as $[D_4],1,[D_4]$. The elliptic genera of this theory can be computed from 2d quiver gauge theory (Hayashi et al., 2019c). This model is actually a special case of the E-string theory. The E_8 flavor group of node 1 splits to $\mathfrak{so}(8)\times\mathfrak{so}(8)$. Since the vanishing r field of E-string theory decomposes as $\mathbf{1}\to(\mathbf{1},\mathbf{1})$, we obtain the following vanishing equation for (D_4,D_4):

$$\boxed{\mathbf{1}}-\bigcirc-\boxed{\mathbf{1}}$$

On the other hand, the unity r fields of E-string theory decompose as

$$\mathbf{240}_2\to(\mathbf{24}_2,\mathbf{1})+(\mathbf{1},\mathbf{24}_2)+(\mathbf{8}_\mathrm{v},\mathbf{8}_\mathrm{v})+(\mathbf{8}_\mathrm{c},\mathbf{8}_\mathrm{s})+(\mathbf{8}_\mathrm{s},\mathbf{8}_\mathrm{c}). \tag{6.5.1}$$

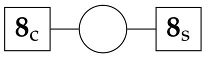
Apply the gluing rules, we find the following five types of unity blowup equations:

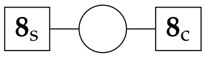
$$\boxed{\mathbf{1}}-\bigcirc-\boxed{\mathbf{24}}$$

$$\boxed{\mathbf{24}}-\bigcirc-\boxed{\mathbf{1}}$$

$$\boxed{\mathbf{8}_\mathrm{v}}-\bigcirc-\boxed{\mathbf{8}_\mathrm{v}}$$

[3] This property can be easily deduced from the fact that all these conformal matter theories can be blown down successively to one single -1 curve.

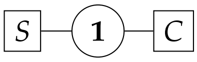


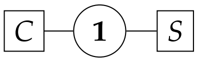


- (D_{N+4}, D_{N+4}) theories are often denoted as $[D_{N+4}], 1_{\mathfrak{sp}(N)}, [D_{N+4}]$. For $N \geq 1$, the $D_{2(N+4)}$ flavor group of the $n = 1$ node splits to $D_{N+4} \times D_{N+4}$. Under splitting $D_{2(N+4)} \to D_{N+4} \times D_{N+4}$,

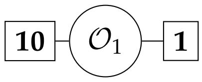
$$S_{D_{2(N+4)}} \to (S_{D_{N+4}}, C_{D_{N+4}}) + (C_{D_{N+4}}, S_{D_{N+4}}). \tag{6.5.2}$$

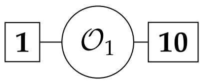
Denote $\mathcal{O}_N$ as the Weyl orbit of $\mathfrak{sp}(N)$ generated by weight $[0, 0, \ldots, 0, 1]$, and V, S, C as the Weyl orbits of $\mathfrak{so}(N+4)$ generated by weights $[1, 0, \ldots, 0, 0]$, $[0, 0, \ldots, 0, 1]$ and $[0, 0, \ldots, 1, 0]$. Apply the gluing rules, we find two types of unity blowup equations



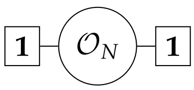


There also exist numerous vanishing blowup equations. For example, for $N = 1$ case, i.e. the $(\mathfrak{so}(10), \mathfrak{so}(10))$ model, the vanishing blowup equations are





For $N \geq 2$, there exist many vanishing blowup equations including



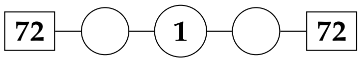
- (E_6, E_6) conformal matter theory is often denoted as $[E_6], 1, 3_{\mathfrak{su}(3)}, 1, [E_6]$. The base curve intersection matrix $-\Omega$ has

$$\Omega = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & 1 \end{pmatrix}. \tag{6.5.3}$$

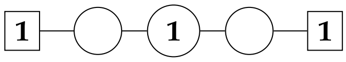
Note $\mathrm{Det}(\Omega) = 1$. The E_8 flavor group of node 1 splits to $E_6 \times \mathfrak{su}(3)$ when coupled with $n = 6$ E_6 gauge theory and $n = 3$ $\mathfrak{su}(3)$ gauge theory. Since $\mathbf{1} \to (\mathbf{1}, \mathbf{1})$ and

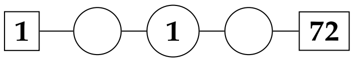
$$\mathbf{240}_2 \to (\mathbf{72}_2, \mathbf{1}) + (\mathbf{27}_{4/3}, \mathbf{3}_{2/3}) + (\overline{\mathbf{27}}_{4/3}, \overline{\mathbf{3}}_{2/3}) + (\mathbf{1}, \mathbf{6}_2), \tag{6.5.4}$$

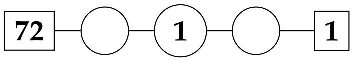
apply the gluing rule, we find one type of unity blowup equations

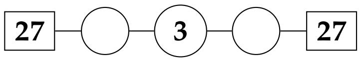


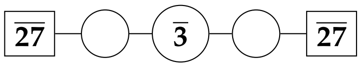
and five types of vanishing blowup equations











One can easily check the leading degree vanishing identities. For example, the first vanish blowup equation has leading base degree as

$$\Theta_{\Omega}^{[a]}\left(\tau,\begin{pmatrix}\epsilon_1+\epsilon_2\\ \epsilon_1+\epsilon_2\\ \epsilon_1+\epsilon_2\end{pmatrix}\right)=0, \tag{6.5.5}$$

while the second vanish blowup equation has leading base degree as

$$\Theta_{\Omega}^{[a]}\left(\tau,\Omega^{-1}\begin{pmatrix}0\\ \epsilon_1+\epsilon_2\\ m_{\alpha}^{E_6}+\epsilon_1+\epsilon_2\end{pmatrix}\right)=0, \tag{6.5.6}$$

and the forth vanish blowup equation has leading base degree as

$$\sum_{i=1,2,3}\Theta_{\Omega}^{[a]}\left(\tau,\begin{pmatrix}0\\ -m_i\\ 0\end{pmatrix}+\Omega^{-1}\begin{pmatrix}m_w^{E_6}+\epsilon_1+\epsilon_2\\ 0\\ m_{w'}^{E_6}+\epsilon_1+\epsilon_2\end{pmatrix}\right)\frac{1}{\prod_{j\neq i}\theta_1(m_i-m_j)}=0. \tag{6.5.7}$$

Here the characteristic $a=(0,1/2,0)$ and α,α' are arbitrary roots of E_6, and w,w' are arbitrary weights of the fundamental representation **27**. Besides, $m_i, i=1,2,3$ are the $\mathfrak{su}(3)$ fugacities satisfying $m_1+m_2+m_3=0$. It is easy to check these identities are correct.

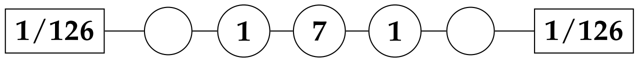
- (E_7,E_7) conformal matter theory is often denoted as $[E_7],1,2_{\mathfrak{su}(2)},3_{\mathfrak{so}(7)},2_{\mathfrak{su}(2)},1,[E_7]$. The base curve intersection matrix $-\Omega$ has

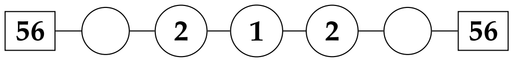
$$\Omega=\begin{pmatrix}1&-1&0&0&0\\ -1&2&-1&0&0\\ 0&-1&3&-1&0\\ 0&0&-1&2&-1\\ 0&0&0&-1&1\end{pmatrix}. \tag{6.5.8}$$

Note $\mathrm{Det}(\Omega) = 1$. The E_8 flavor group of node 1 splits to $E_7 \times \mathfrak{su}(2)$ when coupled with $n = 8$ E_7 gauge theories and $n = 2$ $\mathfrak{su}(2)$ gauge theory. Since

$$\mathbf{240}_2 \to (\mathbf{126}_2, \mathbf{1}) + (\mathbf{56}_{3/2}, \mathbf{2}_{1/2}) + (\mathbf{1}, \mathbf{3}_2), \tag{6.5.9}$$

apply the gluing rules, we find the following possible blowup equations which are all vanishing:



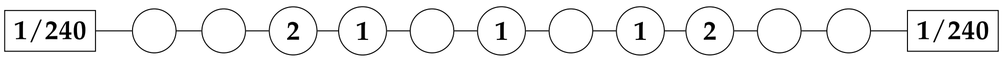


For example, the second type vanish blowup equation has leading base degree as

$$\sum_{\lambda_{a,b}=\pm 1/2} (-1)^{\lambda_a+\lambda_b} \Theta_{\Omega}^{[a]} \left(\tau, \begin{pmatrix} 0 \\ -\lambda_a m_a^{\mathfrak{su}(2)} \\ 0 \\ -\lambda_b m_b^{\mathfrak{su}(2)} \\ 0 \end{pmatrix} + \Omega^{-1} \begin{pmatrix} m_w + \epsilon_1 + \epsilon_2 \\ 0 \\ 2(\epsilon_1 + \epsilon_2) \\ 0 \\ m_{w'} + \epsilon_1 + \epsilon_2 \end{pmatrix} \right) = 0. \tag{6.5.10}$$

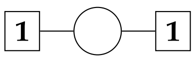
Here the characteristic $a = (1/2, 0, 1/2, 0, 1/2)$, and $w, w' \in \mathbf{56}$ of E_7, and $\mathcal{O}_{1/2,6}$ is the Weyl orbit $\mathcal{O}_{(100)}^{\mathfrak{so}(7)}$. We have checked this identity is correct.

- (E_8, E_8) theory is often denoted as $[E_8], 1, 2, 2_{\mathfrak{su}(2)}, 3_{G_2}, 1, 5_{F_4}, 1, 3_{G_2}, 2_{\mathfrak{su}(2)}, 2, 1, [E_8]$. The base curve intersection matrix $-\Omega$ has $\mathrm{Det}(\Omega) = 1$. Apply the gluing rule, we find the following possible vanishing blowup equations:

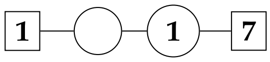


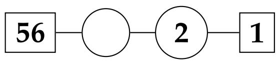
Thus there is no unity and just one type of vanishing blowup equations.

- (G_2, F_4) conformal matter theory is often denoted as $[G_2], 1, [F_4]$. Apply the gluing rule, we find the following possible unity blowup equations:

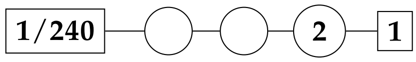


- $(E_7, \mathfrak{so}(7))$ conformal matter theory is often denoted as $[E_7], 1, 2_{\mathfrak{su}(2)}, [\mathfrak{so}(7)]$. Apply the gluing rule, we find the following possible vanishing blowup equations:

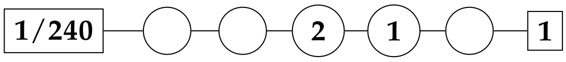




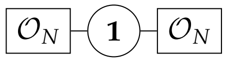
- (E_8, G_2) conformal matter theory is often denoted as $[E_8], 1, 2, 2_{\mathfrak{su}(2)}, [G_2]$. Apply the gluing rule, we find the following possible vanishing blowup equations:



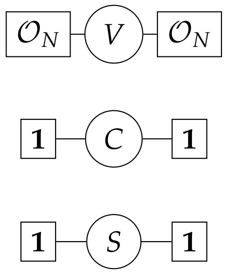
- (E_8, F_4) conformal matter theory is often denoted as $[E_8], 1, 2, 2_{\mathfrak{su}(2)}, 3_{G_2}, 1, [F_4]$. Apply the gluing rule, we find the following possible blowup equations:



- $(\mathfrak{sp}(N), \mathfrak{sp}(N))$ conformal matter is often denoted as $[\mathfrak{sp}(N)], 4_{\mathfrak{so}(2N+8)}, [\mathfrak{sp}(N)]$. The flavor $\mathfrak{sp}(2N)$ of node 4 splits to $\mathfrak{sp}(N) \times \mathfrak{sp}(N)$. Apply the gluing rule, we find the following possible blowup equations: one type of unity equation



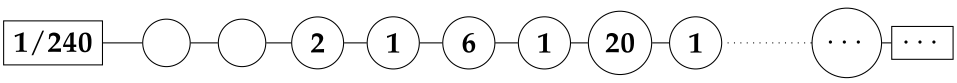
and lots of vanishing ones including



- $(E_8, \mathfrak{su}(N))$ conformal matter theory is often denoted as

$$[E_8]\ 1\ \overset{\mathfrak{su}(1)}{2}\ \overset{\mathfrak{su}(2)}{2}\ \ldots\ \overset{\mathfrak{su}(N-1)}{2}\ [\mathfrak{su}(N)]\,. \tag{6.5.11}$$

Apply the gluing rule, we find the following possible blowup equations:

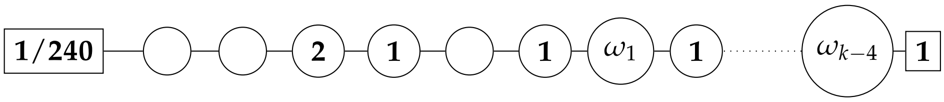


The λ_G/λ_F field associated to the circular/rectangular node carrying gauge/flavor symmetry $\mathfrak{su}(k)$ $(k = 1, \ldots, N)$ is trivial if k is odd and is a non-trivial weight vector belonging to the Weyl orbit $\mathcal{O}_{k/2}$ if k is even.

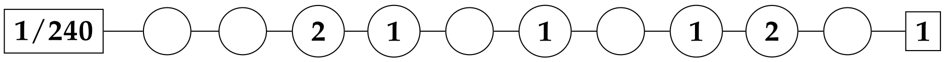
- $(E_8, B_k/D_k)$ conformal matter theory is often denoted as

$$[E_8]\ 1\ 2\ \overset{\mathfrak{su}(2)}{2}\ \overset{\mathfrak{g}_2}{3}\ 1\ \overset{\mathfrak{so}(9)}{4}\ \overset{\mathfrak{sp}(1)}{1}\ \overset{\mathfrak{so}(11)}{4}\ \ldots\ \overset{\mathfrak{sp}(k-4)}{1}\ [\mathfrak{so}(2k)/\mathfrak{so}(2k+1)]\,. \tag{6.5.12}$$

Apply the gluing rule, we find the following possible blowup equations:



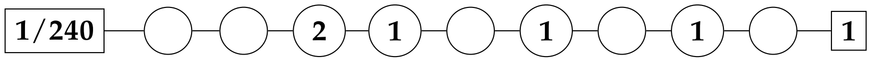
- (E_8, E_7) theory is often denoted as $[E_8], 1, 2, 2_{\mathfrak{su}(2)}, 3_{G_2}, 1, 5_{F_4}, 1, 3_{G_2}, 2_{\mathfrak{su}(2)}, 1, [E_7]$. Apply the gluing rule, we find the following possible blowup equations:



- (E_8, E_6) conformal matter theory is often denoted as

$$[E_8]\ 1\ 2\ \overset{\mathfrak{su}(2)}{2}\ \overset{\mathfrak{g}_2}{3}\ 1\ \overset{\mathfrak{f}_4}{5}\ 1\ \overset{\mathfrak{su}(3)}{3}\ 1\ [E_6]\,. \tag{6.5.13}$$

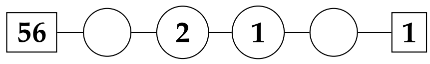
Apply the gluing rule, we find the following possible blowup equations:



- (E_7, D_4) conformal matter theory is often denoted as

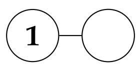
$$[E_7]\ 1\ \overset{\mathfrak{su}_2}{2}\ \overset{\mathfrak{g}_2}{3}\ 1\ [\mathfrak{so}(8)]\,. \tag{6.5.14}$$

Apply the gluing rule, we find the following possible blowup equations:

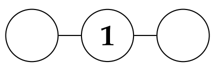


6.6 Blowups of $(-n)$-curves

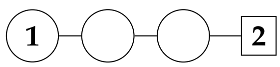
The rank one theories with $n = 9, 10, 11$ do not admit Kodaira-Tate elliptic fibers. One needs to do further blowups which result in higher dimensional tensor branches. There are normally several ways to do this, see for example (Heckman and Rudelius, 2019). The toric construction of some blown-up Calabi-Yau geometries were given in (Haghighat et al., 2015b). For $n = 11$ curve, one blows up once and gets theory $12_{E_8}, 1$. It is easy to find the following vanishing blowup equation for it:



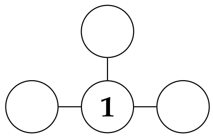
For $n = 10$, one blows up twice and gets $1, 12_{E_8}, 1$ with vanishing blowup equation



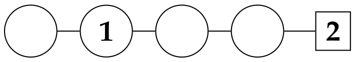
or $12_{E_8}, 1, 2$ with vanishing blowup equations



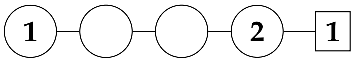
For $n = 9$, one blows up twice and gets $1, \overset{1}{12}_{E_8}, 1$ with vanishing blowup equation



or $1, 12_{E_8}, 1, 2$ with vanishing blowup equations



or $12_{E_8}, 1, 2, 2$ with vanishing blowup equation



Let us now take a closer look at the first example the $12_{E_8}, 1$ theory. The intersection matrix between the two base curves is just

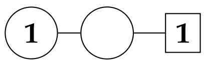
$$\Omega = \begin{pmatrix} 12 & -1 \\ -1 & 1 \end{pmatrix}, \tag{6.6.1}$$

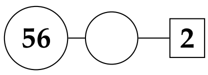
thus we have $\det(\Omega) = 11$ vanishing blowup equations. Since there is only one E_8 vector multiplet and no hypermultiplet, the leading base degree of the vanishing blowup equations can be simply written as

$$\Theta_{\Omega}^{[a]}\left(\tau, \begin{pmatrix} 5/33 \\ 5/33 \end{pmatrix} \epsilon_+\right) = 0. \tag{6.6.2}$$

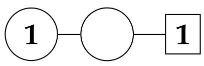
We have checked this identity up to q_τ^{30}. Remember here characteristics a are associated to Ω as defined in (6.1.4).

As a similar example, we consider the blown-up of a -7 curve, which can be represented as $8_{E_7}, 1, [\mathfrak{su}(2)]$. There are two types of vanishing blowup equations:





In fact, for any of $-2, -3, -4, -5, -7, -11$ curves, one can blowup once and obtain a rank two theory which is the coupling between a pure gauge minimal 6d SCFT and the E-string theory. For these rank two theories, there always exists one type of vanishing blowup equations represented as



The leading base degree of the vanishing equations are due to the following identity:

$$\Theta_{\Omega}^{[a]}\left(\tau, \begin{pmatrix} z \\ z \end{pmatrix}\right) = 0, \tag{6.6.3}$$

where

$$\Omega = \begin{pmatrix} n & -1 \\ -1 & 1 \end{pmatrix}. \tag{6.6.4}$$

In fact, this identity holds for arbitrary $n \geq 2$.

6.7 Remarks on solving elliptic genera

For higher rank theories, there in general seems to be no efficient way to solve elliptic genera from elliptic blowup equations. The main reason as mentioned before is that there usually only exist vanishing blowup equations for higher rank theories which do not give enough constraints. Besides, even in the rare cases where exist unity blowup equations, we can hardly make use of the equations to solve elliptic genera.[4] Naively, one may think there could exist some explicit higher dimensional recursion formulas analogous to the rank one cases as long as there exist three or more unity blowup equations. Unfortunately, because any such higher-rank theory involves -2 or -1 curves, the recursion fails when one of these curves is left but all other base curves are decompactified. Therefore, in some sense,all higher-rank theories with unity blowup equations are in class **B** as in Section 5.3, and all those with only vanishing blowup equations are in class **C**. Let us consider a good example, the A_2 chain with gauge symmetry $\mathfrak{su}(N)$ on each node. For arbitrary N, there always exist unity blowup equations:

$$\boxed{\mathbf{1}} - (\mathbf{1}) - (\mathbf{1}) - \boxed{\mathbf{1}}$$

Since the intersection matrix between the base classes is

$$\Omega = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}, \tag{6.7.1}$$

we have in total $\det(\Omega) = 3$ non-equivalent unity blowup equations. To solve elliptic genus say $\mathbb{E}_{2,1}$ by recursion, one need to know $\mathbb{E}_{2,0}, \mathbb{E}_{1,1}, \mathbb{E}_{1,0}, \mathbb{E}_{0,1}$ as initial data. However, all the essentially rank one elliptic genera $\mathbb{E}_{n,0}$ and $\mathbb{E}_{0,n}$ are not possible to solve by recursion as they are in class **B** of rank one theories. In fact, when one decompactifies the right -2 curve, the three unity blowup equations will reduce to just two non-equivalent unity equations of the left -2 curve which are just the two unity equations of the $n = 2$, $G = \mathfrak{su}(N)$ theory. Thus there are not enough unity equations to proceed with the recursion. See the detailed analysis for the degeneration of M-M string chain in Chapter 6.2.1. Nevertheless, from the perspective of the ϵ_1, ϵ_2 expansion, the refined BPS expansion or the Weyl orbit expansion, one can still get some constraints. We do not pursue this direction further since the perfect 2d quiver description were already found for these higher-rank theories.

[4] The higher rank theories with unity blowup equations include for example all A, D type chain of -2 curves with gauge symmetry and (E_6, E_6) conformal matter theory.

Chapter 7

Elliptic Genera and Superconformal Indices

In the chapter, we take a detour from blowup equations and focus on the elliptic genera we solved from recursion formula in Chapter 5. The purpose of this chapter is to connect the k-string elliptic genera $\mathbb{E}_{h_G^{(k)}}$ of the minimal $\mathcal{N} = (1,0)$ 6d SCFTs with $G = A_2, D_4, F_4, E_{6,7,8}$ discussed before to the superconformal indices of the 4d $\mathcal{N} = 2$ SCFTs of rank k denoted by $H_G^{(k)}$. The simplest series of $\mathcal{N} = 2$ SCFTs namely $H_G^{(1)}$ can be obtained by geometric engineering on non-compact del Pezzo geometries and contains the Minahan-Nemeschansky theories. Our main result is an extension of a surprising conjecture proposed in (Del Zotto and Lockhart, 2017) from the rank one cases to the higher rank cases. To be precise, it was found in (Del Zotto and Lockhart, 2017) that the one-string elliptic genus $\mathbb{E}_{h_G^{(1)}}(q_\tau, v)$ can be decomposed in terms of a seemingly more fundamental function $L_G(q_\tau, v)$, which for special choices of q_τ and v specialises to the Hall-Littlewood index or the Schur index of the $H_G^{(1)}$ theories. With the two string elliptic genera computed in Chapter 5, we are able to study this conjectural relation at rank two and in principle at arbitrary rank, and find indeed that similar striking relations exist.

We first review some basic properties of 4d rank k type $H_G^{(k)}$ – and $\widetilde{H}_G^{(k)}$ theories, including their class $\mathcal{S}$ theory construction, and then review the superconformal indices of 4d SCFTs in various physically motivated limits as well as the methods to computed them. Next we state the conjectural relation at rank one from (Del Zotto and Lockhart, 2017), and explain in some detail the new relations at rank two for all G. We also extend the analysis to some rank three cases. For all choices of rank and G we analyzed, the surprising relation between elliptic genera and superconformal indices exists. We define an intermediate function at rank k called $L_G^{(k)}$[1]. This function is on the one hand the ingredient of k-string elliptic genus, on the other hand gives the Hall-Littelwood index and Schur index of $H_G^{(k)}$ theories at special choices of parameters. This general structure allows us to calculate the latter indices efficiently from the $\mathbb{E}_{h_G^{(k)}}$ that are determined from the elliptic blowup equations.

[1]The L_G function in (Del Zotto and Lockhart, 2017) becomes $L_G^{(1)}$ here

7.1 Rank k H_G theories

The 4d $\mathcal{N} = 2$ SCFTs $H_G^{(k)}$ are well known to exist for $G = \emptyset, A_1, A_2, D_4, E_{6,7,8}$ and $k = 1, 2, 3 \ldots$ (Argyres et al., 1996; Banks, Douglas, and Seiberg, 1996; Douglas, Lowe, and Schwarz, 1997; Minahan and Nemeschansky, 1996; Minahan and Nemeschansky, 1997)[2]. In type IIB superstring theory, they are realized as the worldvolume theory for k multiple D3-branes probing a stack of exotic seven-branes. Such seven-branes in F-theory are defined as codimension one singularities with Kodaira type $II, III, IV, I_0^*, IV^*, III^*$, and II^*, which give the gauge symmetries G for the low energy 8d SYM theories. The number k is usually called the rank of H_G theories. For example, the rank one $H_{\emptyset,A_1,A_2}$ theories appear as certain limit of $SU(2)$ gauge theory with $N_f = 1, 2, 3$ respectively (Argyres et al., 1996). The rank one H_{D_4} theory is well known to be the $SU(2)$ gauge theory with $N_f = 4$, while the higher rank cases with $k > 1$ are equivalent to $USp(2k)$ gauge theories with four fundamental hypermultiplets and one antisymmetric hypermultiplet, which are all Lagrangian theories. The rank one $H_{E_{6,7,8}}$ are also known as the Minahan-Nemeschansky theories (Minahan and Nemeschansky, 1996; Minahan and Nemeschansky, 1997), where the simplest example rank one E_6 theory is in S-duality with $SU(3)$, $N_f = 6$ theory (Argyres and Seiberg, 2007).

All $H_G^{(k)}$ theory can be coupled with a free hypermultiplet associated to the center of mass motion of the instantons. We follow (Del Zotto and Lockhart, 2017) and denote these theories as $\widetilde{H}_G^{(k)}$. As was observed in (Gaiotto and Razamat, 2012), for higher rank cases, $\widetilde{H}_G$ are sometimes more natural than H_G theories. One major difference between rank one and higher rank H_G theories is the flavour symmetry. Besides the flavour G given by the strings stretched between D3-branes and exotic seven-brane, for $k > 1$ there is one more $SU(2)$ symmetry coming from the transverse space in the seven-brane. By coupling a free hypermultiplet, all $\widetilde{H}_G^{(k)}$ theories share flavour symmetry $G \times SU(2)$.

The $H_G^{(k)}$ theories of interest here are $G = A_2, D_4, E_{6,7,8}$ as they are directly related to 6d minimal $(1, 0)$ SCFTs with corresponding gauge group G. To be precise, the RR elliptic genus is identified as the β-twisted $T^2 \times S^2$ partition function of the 4d SCFTs:

$$\mathbb{E}_{h_G^{(k)}} = Z_{(T^2 \times S^2)_\beta}(H_G^{(k)}), \tag{7.1.1}$$

Adding the "tildes", one can also obtain the equality with the free hypermultiplet coupled. Here the β-twist was introduced in (Kapustin, 2006) to preserve half of the supersymmetries on the backgrounds such as $T^2 \times S^2$. See a good description of such twist in for example section 3.2 of (Del Zotto and Lockhart, 2017). The identification (7.1.1) makes it sometimes possible to compute the elliptic genus from 4d setting, in which cases the S-duality with a Lagrangian theory is invoked and one can use certain analogy of Spiridonov-Warnaar inverse formula (Spiridonov and Warnaar, 2006) to compute the $T^2 \times S^2$ partition function. This was indeed achieved for one string elliptic genus with $G = D_4, E_{6,7}$ (Putrov, Song, and Yan, 2016; Del Zotto and Lockhart, 2017; Gadde, Razamat, and Willett, 2015; Agarwal, Maruyoshi, and Song, 2018). For example, the elliptic genus of one E_7 instanton string was obtained in

[2]The $G = \emptyset, A_1, A_2$ type theories are also traditionally denoted as $H_{0,1,2}$ theories. Here we follow the notations in (Del Zotto and Lockhart, 2017).

(Agarwal, Maruyoshi, and Song, 2018) via $SU(4)$ gauge theory $N_f = 8$ and appropriate Higgsing as

$$\begin{aligned} Z_{(T^2\times S^2)_\beta}(H^{(1)}_{E_7}) = 1 &+ \chi^{E_7}_{133}v^2 + \chi^{E_7}_{7371}v^4 + \chi^{E_7}_{238602}v^6 + \chi^{E_7}_{5248750}v^8 + \dots \\ &+ q_\tau\Big(1 + \chi^{E_7}_{133} + (1 + 2\chi^{E_7}_{133} + \chi^{E_7}_{7331} + \chi^{E_7}_{8645})v^2 \\ &\quad + (\chi^{E_7}_{133} + 2\chi^{E_7}_{7371} + \chi^{E_7}_{8645} + \chi^{E_7}_{238602} + \chi^{E_7}_{573440})v^4 + \dots\Big) \\ &+ q_\tau^2\Big(3 + 2\chi^{E_7}_{133} + \chi^{E_7}_{1539} + \chi^{E_7}_{7371} + \dots\Big) + \mathcal{O}(q_\tau^3)\,, \end{aligned} \tag{7.1.2}$$

which completely agrees with our universal expansion formula (5.4.3), (5.4.4) and (5.4.5).[3] We also checked for D_4 and E_6, where the agreement holds to all known orders.

Another important feature of $H^{(k)}_G$ theories is that they all admit 6d construction. It is well known all rank k $H_{D_4,E_{6,7,8}}$ theories can be realized by compactifying a 6d A_{N-1} (2,0) SCFT on some punctured sphere with regular singularities (Benini, Benvenuti, and Tachikawa, 2009), i.e. they are class $\mathcal{S}$ theories. The regular singularities are classified by embeddings of $SU(2)$ in $SU(N)$, thus can be denoted as Young diagrams. Such punctures with associated Young diagram represent how the $SU(N)$ decomposes and what is the residual flavour symmetry. For example, the rank one $H_{SO(8)}$ theory is obtained by compactifying 6d A_1 (2,0) SCFT on a sphere with four full punctures $\{1^2\}$, i.e. the residual flavour symmetry is $SU(2)$. Thus the resulting 4d theory has gauge symmetry $SU(2)$ and four fundamentals, as was mentioned already above. We summarize the gauge algebras and punctures for the 6d construction of all $H^{(k)}_G$ theories with $G = D_4, E_{6,7,8}$ in Table 7.1. The 6d construction for

G	6d $(2,0)$ A_{N-1}	punctures Λ_i
D_4	A_{2k-1}	four $\{k^2\}$
E_6	A_{3k-1}	three $\{k^3\}$
E_7	A_{4k-1}	$\{(2k)^2\}$ and two $\{k^4\}$
E_8	A_{6k-1}	$\{(3k)^2\}$,$\{(2k)^3\}$ and $\{k^6\}$

Table 7.1: 6d construction for rank k H_G theory

rank k H_{A_2} theories however involves irregular punctures. For example, they can be realized by compactifying 6d A_{2k-1} theory on a sphere with one regular puncture with Young diagram $\{k^2\}$ and one irregular puncture of form

$$\Phi = \frac{1}{z^3}\mathrm{diag}(1,\dots,1_{k_{\mathrm{th}}},-1,\dots,-1_{k_{\mathrm{th}}}) + \dots \tag{7.1.3}$$

with the coefficient of z^{-2} and z^{-1} have the same type of matrix (Xie, 2013). In particular, the rank one H_{A_2} theory coincides with (A_1, D_4) Argyres-Douglas theory. See also the 6d construction involving irregular punctures in (Bonelli, Maruyoshi, and Tanzini, 2012b).

Class $\mathcal{S}$ 4d SCFTs are also known to be connected to 2d vertex operator algebra, i.e. chiral algebra (Beem et al., 2015b; Beem et al., 2015a). This correspondence relies

[3] In the coefficients of q_τ^2, one also need to use the Joseph relation $\mathrm{Sym}^2\mathbf{133} = 1 + \mathbf{133} + \mathbf{7371}$ to obtain the identification.

directly on the class $\mathcal{S}$ construction and can be understood from certain generalized TQFT structure on the punctured Riemann surface. This relation sometimes gives a new approach to compute the indices of 4d SCFT by realizing them as the vacuum character of associated chiral algebra. For example, the chiral algebras associated to rank one H_{D_4} and H_{E_6} theories are identified as $\mathfrak{so}(8)$ affine Lie algebra at level $k_{2d} = -2$ and E_6 affine Lie algebra at level $k_{2d} = -3$ in (Beem et al., 2015b). See some recent works trying to explain VOA/SCFT correspondence (Pan and Peelaers, 2018; Pan and Peelaers, 2019; Oh and Yagi, 2019; Dedushenko and Fluder, 2019; Jeong, 2019). Besides, the rank one H_{D_4,E_6,E_7} theories are also connected with the curved $\beta\gamma$ systems on cones over the complex Grassmannian $\mathrm{Gr}(2,4)$, the complex orthogonal Grassmannian $\mathrm{OG}^+(5,10)$, and the complex Cayley plane $\mathbb{OP}^2$ respectively in (Eager, Lockhart, and Sharpe, 2019).

7.2 Hall-Littlewood and Schur indices

The superconformal index of 4d $\mathcal{N} = 2$ SCFT is defined as (Kinney et al., 2007; Romelsberger, 2006)

$$\mathcal{I}(p,q,t) = \mathrm{Tr}\,(-1)^F \left(\frac{t}{pq}\right)^r p^{j_{12}}\, q^{j_{34}}\, t^R \prod_i a_i^{f_i}\,, \tag{7.2.1}$$

where $j_{12} = j_2 + j_1$ and $j_{34} = j_2 - j_1$ denote the rotation generators in $\mathbb{C}^2$ with $j_{1,2}$ representing each $SU(2)$ Lorentz symmetry, and r and R denote the $U(1)_r$ and $SU(2)_R$ generators respectively. Besides, a_i are the fugacities for the flavour generators f_i which sometimes are set to be zero for simplicity. For generic 4d SCFT, the full superconformal indices with (p,q,t) are difficult to compute. For example, among all $H_G^{(k)}$ theories, the full superconformal indices to our knowledge are only computable so far for $H_{SO(8)}$ with arbitrary rank owing to their Lagrangian nature and H_{E_6,E_7} for rank one owing to the existence of certain $\mathcal{N} = 1$ Lagrangian flow (Gadde, Razamat, and Willett, 2015; Agarwal, Maruyoshi, and Song, 2018).

Certain limits of superconformal index are particularly interesting due to symmetry enhancement. The name of limit comes from the observation that the resulting indices involve corresponding symmetric polynomial known in mathematics literature. Following (Gadde et al., 2013), we list three of them here:

- (Macdonald) $p \to 0$. Superconformal index when taking the Macdonald limit is computable for all class $\mathcal{S}$ theory with regular punctures. For a genus $\mathfrak{g}$ theory with s punctures compactified from 6d A_{N-1} (2,0) SCFT, the Macdonald index is given in (Gadde et al., 2013) as

$$\begin{aligned}\mathcal{I}^{\mathrm{M}}_{\mathfrak{g},s}(\mathbf{a},q,t) = &\prod_{j=2}^{N}(t^j;q)^{2\mathfrak{g}-2+s}\frac{(t;q)^{(k-1)(1-\mathfrak{g})+s}}{(q;q)^{(k-1)(1-\mathfrak{g})}}\\ &\times\sum_{\lambda}\frac{\prod_{i=1}^{s}\hat{\mathcal{K}}_{\Lambda_i}(\mathbf{a}_i)\;P^{\lambda}(\mathbf{a}_i(\Lambda_i)|q,t)}{\left[P^{\lambda}(t^{\frac{k-1}{2}},t^{\frac{k-3}{2}},\dots,t^{\frac{1-k}{2}}|q,t)\right]^{2\mathfrak{g}-2+s}}\,.\end{aligned} \tag{7.2.2}$$

Here $P^{\lambda}(\mathbf{a}_i(\Lambda_i)|q,t)$ are Macdonald polynomials and the summation is over all possible Young diagrams $\lambda = \{\lambda_1,\lambda_2,\dots,\lambda_{N-1},0\}$. The Pochhammer symbol

$(a;b)$ is defined by

$$(a;b) = \prod_{i=0}^{\infty}(1 - ab^i)\,. \tag{7.2.3}$$

The $\hat{\mathcal{K}}_{\Lambda_i}$ factors are defined by

$$\hat{\mathcal{K}}_{\Lambda}(\mathbf{a}) = \prod_{i=1}^{row(\Lambda)} \prod_{j,k=1}^{l_i} \mathrm{PE}\left[\frac{a^i_j \bar{a}^i_k}{1-q}\right]_{\mathfrak{a}_i,q}\,, \tag{7.2.4}$$

with the coefficients a^i_k associated to the Young diagram as

$$a^i_j = c_j v^{\lambda_j+1-i} \quad \text{and} \quad \bar{a}^i_k = c_k^{-1} v^{\lambda_k+1-i}\,, \tag{7.2.5}$$

with $v^2 = t$. Here these c_j parameterize the residual flavour symmetry and are subject to constrain $\prod_{i=1}^{row(\Lambda)} \prod_j^{l_i} c_j = 1$ to preserve the traceless condition of $SU(N)$. The association of the flavour fugacities for a puncture $a(\Lambda)$ in Macdonald polynomial is defined similarly as $c_j v^{-\lambda_j-1+2i}$. Some good figures to visualize these definitions can be found in (Gadde et al., 2013; Gaiotto and Razamat, 2012).

- (Hall-Littlewood) $p, q \to 0$. By taking limit in (7.2.2), it is easy to obtain the Hall-Littlewood index for all class $\mathcal{S}$ theories. As only genus zero theories are of concern here, we only write down the formulas with $\mathfrak{g} = 0$. For example, the Hall-Littlewood index of 4d SCFT compactified from 6d A_{N-1} theory is

$$\mathcal{I}^{\mathrm{HL}} = \mathcal{N}_{N,s} \sum_{\lambda} \frac{\prod_{i=1}^{s} \hat{\mathcal{K}}_{\Lambda_i}(\mathbf{a}_i)\, \psi^{\lambda}(\mathbf{a}_i(\Lambda_i)|v)}{[\psi^{\lambda}(v^{N-1}, v^{N-3}, \ldots, v^{1-N}|v)]^{s-2}}\,, \tag{7.2.6}$$

where

$$\mathcal{N}_{N,s} = (1-v^2)^{N-1+s} \prod_{j=2}^{N} (1-v^{2j})^{s-2}, \tag{7.2.7}$$

and ψ^{λ} is the Hall-Littlewood polynomials defined as

$$\psi^{\lambda}(x_1, \ldots, x_N|v) = \mathcal{N}_{\lambda}(v) \sum_{\sigma \in S_N} x_{\sigma(1)}^{\lambda_1} \cdots x_{\sigma(N)}^{\lambda_N} \prod_{i<j} \frac{x_{\sigma(i)} - v^2 x_{\sigma(j)}}{x_{\sigma(i)} - x_{\sigma(j)}}\,, \tag{7.2.8}$$

with

$$\mathcal{N}_{\lambda}(v) = \prod_{i=0}^{\infty} \prod_{j=1}^{m(i)} \left(\frac{1-v^{2j}}{1-v^2}\right)^{-1/2}\,, \tag{7.2.9}$$

where $m(i)$ is the number of rows in the Young diagram $\lambda = (\lambda_1, \ldots, \lambda_N)$ of length i. Here we have made the substitution $t = v^2$ for convenience.

It is argued in (Gadde et al., 2013) that for linear quiver theories the HL index is equivalent to the Hilbert series of the Higgs branch. In particular, this is true for all H_G theories. It is well-known the Higgs branch of $H_G^{(k)}$ theories are the reduced moduli space of k G-instantons, which can be understood from the probing picture that the k D3-branes dissolving into the seven-branes resemble k instantons in the transverse space. Thus the HL index of $H_G^{(k)}$ theory are supposed to be equal to the Hilbert series of reduced moduli space of k G-instantons. On the other hand, the Hilbert series can also be obtained from the 5d Nekrasov partition function with pure gauge group G, which are just the 5d limit of elliptic genus of 6d minimal $(1,0)$ SCFT with type G. Therefore, we arrive at the relation:

$$\mathcal{I}^{\mathrm{HL}}_{H_G^{(k)}} = \mathrm{Hilb}^k_G = g^{(0)}_{k,G}, \tag{7.2.10}$$

where $g^{(0)}_{k,G}$ as we defined previously in (5.4.1) is the coefficient of leading q_τ order of k-string elliptic genus $\mathbb{E}_{h_G^{(k)}}$. One can also add "tildes" to get the equality with a free hypermultiplet coupled, in which situation one encounters the full Hilbert series other than the reduced. We have checked relation (7.2.10) for $k = 1,2$ for all possible G and $k = 3$ for $SU(3)$.[4]

- (Schur) $q = t$ with p arbitrary. In fact, it can be shown in such specialization the index is independent of p. Thus, taking $p \to 0$, Schur index is actually a limit of Macdonald index. Using (7.2.2), the Schur index for a class $\mathcal{S}$ theory is given by

$$\mathcal{I}^{\mathrm{Schur}} = \hat{\mathcal{N}}_{N,s} \frac{\prod_{i=1}^s \hat{\mathcal{K}}_{\Lambda_i}(\mathbf{a}_i)\, \chi^\lambda(\mathbf{a}_i(\Lambda_i))}{[\chi^\lambda(v^{N-1}, v^{N-3}, \ldots, v^{1-N})]^{s-2}}, \tag{7.2.11}$$

where[5]

$$\hat{\mathcal{N}}_{N,s} = (v^2; v^2)^s \prod_{j=2}^N (v^{2j}; v^2)^{s-2}, \tag{7.2.12}$$

and χ^λ is the Schur polynomials defined as

$$\chi_\lambda(\mathbf{a}) = \frac{\det(a_i^{\lambda_j+k-j})}{\det(a_i^{k-j})}. \tag{7.2.13}$$

At last, one replaces back $v^2 \to q$.

The Schur indices in some sense are more interesting than the Hall-Littlewood indices. For instance, for class $\mathcal{S}$ theories, Schur indices equal the q-deformed topological 2d Yang-Mills parition function on the punctured Riemann surface (Gadde et al., 2011), and also equal the vacuum character of the associated chiral algebra (Beem et al., 2015b; Beem et al., 2015a). Furthermore, Schur indices can be computed in IR via wall crossing for theories even beyond class $\mathcal{S}$, such as certain Argyres-Douglas theories (Cordova and Shao, 2016) including rank one H_{A_2} theory.

The full superconformal indices of rank one $H_{D_4,E_{6,7}}$ theories have been computed in (Gadde et al., 2010; Putrov, Song, and Yan, 2016; Agarwal, Maruyoshi, and Song, 2018). The Schur index of rank one H_{E_8} was given in (Del Zotto and Lockhart, 2017) and the Schur index of rank one H_{A_2} was given in (Cordova and Shao, 2016). To

[4]For $SU(3)$ and F_4, we are not aware how to compute the HL indices directly. Still, the Hilbert series are well-defined and computed in (Benvenuti, Hanany, and Mekareeya, 2010; Hanany, Mekareeya, and Razamat, 2013), which are in perfect agreement with our computation for elliptic genus from blowup equations.

[5]As in this thesis we only deal with the cases with three or four punctures, we also shorten $\mathcal{N}_{N,3}$ as $\mathcal{N}_N$ and $\mathcal{N}_{N,4}$ as $\mathcal{N}'_N$ in the latter sections, and same for those with hat.

compute the Hall-Littlewood indices and Schur indices of higher rank $H_{D_4,E_{6,7,8}}$ theories one will encounter certain subtle issues. Directly using the general formulas (7.2.6) and (7.2.11) fails to give correct results, because at a given order of v infinite number of Young diagrams λ contribute in. To cure such divergence, it was suggested in (Gaiotto and Razamat, 2012) that one reduce the flavor symmetry "one box at a time", that is to change one specific puncture by moving one box down in the associated Young diagram. The physical meaning of such operation is interpreted as coupling a free hypermultiplet to $H_G^{(k)}$ theory, which in our notation is just $\widetilde{H}_G^{(k)}$ theory. In the terminology of (Gaiotto and Razamat, 2012), $H_G^{(k)}$ are "bad" theories, while $\widetilde{H}_G^{(k)}$ are "good" theories. One can directly use (7.2.6) and (7.2.11) to compute the indices of $\widetilde{H}_G^{(k)}$, then divide by the index of a free hypermultiplet which is well defined, finally one will obtain the finite indices of $H_G^{(k)}$. Following this procedure, the Hall-Littlewood indices of rank two $H_{D_4,E_{6,7,8}}$ theories was computed in (Gaiotto and Razamat, 2012). Similarly, we computed the Schur indices of rank two and three $H_{D_4,E_{6,7,8}}$ theories which will be shown in details in later sections. For higher rank H_{A_2} we are not aware how to compute its Schur indices due to the irregular punctures of 6d construction. Although there exist no H_G^k theory for $G = F_4$, we suspect certain analogy can be constructed such that Hall-Littlewood indices still make sense as the Hilbert series of moduli space of k F_4 instantons, and the Schur indices can be associated with affine $\mathfrak{f}_4$ algebra. One support for such speculation is that the Hilbert series for arbitrary k F_4 instantons has been constructed from certain folding from E_6 (Cremonesi et al., 2014). Thus we sometimes informally denote the analogy as $H_{F_4}^{(k)}$ theories.

7.3 Rank one: Del Zotto-Lockhart's conjecture

In (Del Zotto and Lockhart, 2017), the authors found an intriguing structure of one string elliptic genera of 6d mininal $(1,0)$ SCFTs and a surprising relation between the elliptic genera and the supersymmetric indices of rank one H_G theories. Let us rephrase their conjecture here:

Conjecture 2 (Del Zotto-Lockhart)**.** For each $G \in \{A_2, D_4, F_4, E_{6,7,8}\}$, there exists a function $L_G^{(1)}(v, m_G, q_\tau) = \sum_{i,j=0}^{\infty} b_{i,j}^G q_\tau^i v^j$ such that

1. $b_{i,j}^G$ can be written as the sum of characters of irreducible representation of G with integral coefficients.
2. $L_G^{(1)}(v, m_G, 0)$ is the Hilbert series of the reduced moduli space of one G-instanton, i.e. the Hall-Littlewood index of the $H_G^{(1)}$ theory.
3. $L_G^{(1)}(q^{1/2}, m_G, q^2)$ is the Schur index of the $H_G^{(1)}$ theory.

4. The reduced one-string elliptic genus $\mathbb{E}_{h_G^{(1)}}(v)$ can be generated from $L_G^{(1)}(v)$ by the following formula in which the symmetry (5.4.24) is manifest:[6]

$$\begin{aligned}\mathbb{E}_{h_G^{(1)}}(v) =& v^{2h-1}q_\tau^{1/6}\sum_{n\geq 0}q_\tau^{2n}\Big[u^{4h}L_G(q_\tau^n v)-(-1)^{2h}u^{-4h}L_G(q_\tau^{n+1/2}/v)\\ &+(1+(-1)^{2h})q_\tau^{h+1/2}\left(u^2L_G(q_\tau^{n+1/2}v)-u^{-2}L_G(q_\tau^{n+1}/v)\right)\\ &+q_\tau^2\left((-1)^{2h}u^{4(1-h)}L_G(q_\tau^{n+1}v)-u^{-4(1-h)}L_G(q_\tau^{n+3/2}/v)\right)\Big]\end{aligned} \tag{7.3.1}$$

where $h=h_G^\vee/6$, $u=v/q_\tau^{1/4}$.

The conjectural formula (7.3.1) is quite intricate. Roughly speaking, it means the coefficient matrix of reduced one-string elliptic genus contains several "blocks", overlapping or non-overlapping, and each block contains infinite copies of the $L_G^{(1)}$ function. The number of blocks turns out to be 2 for $SU(3)$, 4 for F_4 and 6 for the other G. In the following we show the coefficient matrix of one-string elliptic genus of $SO(8)$ in a way consistent with our later higher rank discussion. The coefficient matrix of elliptic genus and the $L_G^{(1)}$ functions for other G can be found in (Del Zotto and Lockhart, 2017). Let us denote

$$\mathbb{E}_{h_{SO(8)}^{(1)}}(v,q_\tau,m_i=0)=v^5q_\tau^{-5/6}\sum_{i,j=0}^{\infty}c_{i,j}^{SO(8)}v^j(q_\tau v^{-4})^i. \tag{7.3.2}$$

Then we have Table 7.2 for the coefficients $c_{i,j}^{SO(8)}$ where each "block" is colored differently: the coefficients coming from the first term in the square bracket in (7.3.1) is colored red, the second black, the third blue, the forth orange, the fifth cyan and the last magenta. As we can see from the table, the reduced one-string elliptic indeed

i,j	0	2	4	6	8	10	12	14
0	1	28	300	1925	8918	32928	102816	282150
1	0	0	29	707	6999	42889	193102	699762
2	-1	0	0	$2\cdot 1$	$463+1$	9947	92391	544786
3	0	-28	-29	$-2\cdot 1$	$1-1$	$2\cdot 29$	$5280+29+2\cdot 28$	101850
4	0	0	-300	-707	$-463-1$	$-2\cdot 29$	$29-29$	$2\cdot 463+2\cdot 1$
5	0	0	0	-1925	-6999	-9947	$-5280-29-2\cdot 28$	$-2\cdot 463-2\cdot 1$
6	0	0	0	0	-8918	-42889	-92391	-101850
7	0	0	0	0	0	-32928	-193102	-544786
8	0	0	0	0	0	0	-102816	-699762
9	0	0	0	0	0	0	0	-282150

Table 7.2: Expansion coefficients $c_{i,j}^{SO(8)}$ for one $SO(8)$ instanton string.

depends on v^2. One can also see the symmetry (5.4.24) on the two sides of the ray with slop $-1/2$. Here the $L_{SO(8)}^{(1)}(v,q_\tau)$ function can be defined by all the red number in Table 7.2 with the red $+1$ and $+29$ moving out, as they come from $n=1$ term in

[6] Here the dependence on q_τ and Q_m are implied.

the summation. Thus we have

$$
\begin{aligned}
&L^{(1)}_{SO(8)}(v,q_\tau) = \\
&(1+28v^2+300v^4+1925v^6+8918v^8+32928v^{10}+102816v^{12}+282150v^{14}+\mathcal{O}(v^{16})) \\
&+(29+707v^2+6999v^4+42889v^6+193102v^8+699762v^{10}+\mathcal{O}(v^{12}))q_\tau \\
&+(463+9947v^2+92391v^4+544786v^6+\mathcal{O}(v^8))q_\tau^2 \\
&+(5280+101850v^2++\mathcal{O}(v^4))q_\tau^3+\mathcal{O}(q_\tau^4).
\end{aligned} \tag{7.3.3}
$$

Clearly, the first row in Table 7.2 gives the well-known Hilbert series for the reduced moduli space for one $SO(8)$ instanton, i.e. the Hall-Littlewood index for rank one $H_{SO(8)}$ theory:

$$
\begin{aligned}
L^{(1)}_{SO(8)}(v,0) &= \sum_{n=0}^{\infty} \chi^{SO(8)}_{n\theta} v^{2n} \\
&= 1+28v^2+300v^4+1925v^6+8918v^8+32928v^{10}+102816v^{12}+\mathcal{O}(v^{14}).
\end{aligned} \tag{7.3.4}
$$

Adding the red numbers from $L^{(1)}_{SO(8)}(v,q_\tau)$ in each column of Table 7.2 together, one expects to obtain the Schur index of rank one $H_{SO(8)}$ theory. Indeed, by making $v\to q^{1/2}$ to make contact with the literature, we obtain

$$
\begin{aligned}
L_{SO(8)}(q^{1/2},q^2) = &\,1+28q+329q^2+2632q^3+16380q^4+85764q^5 \\
&+393589q^6+1628548q^7+\mathcal{O}(q^8).
\end{aligned} \tag{7.3.5}
$$

Such series was actually already obtained by a lot of methods. For example, from the viewpoint of VOA/SCFT correspondece, it equals the vacuum character of affine Lie algebra $\mathfrak{so}(8)_{k=-2}$ (Beem et al., 2015b). From the nature that rank one $H_{SO(8)}$ theory is actually just $SU(2)$ gauge theory with $N_f=4$, the Schur index can be computed both from UV Lagrangian and IR wall-crossing formula (Cordova, Gaiotto, and Shao, 2016). See the Schur series from vacuum character up to q^{14} in the end of the appendix of (Cordova, Gaiotto, and Shao, 2016).

Such comparison between the reduced elliptic genus and Schur index for all other rank one H_G theory except $G=F_4$ has been done in (Del Zotto and Lockhart, 2017). In particular, all $L^{(1)}_G(v,q_\tau,m_G=0)$ functions are identified, and the conjectural formula (7.3.1) holds to substantial orders. Similarly, one can also couple a free hypermultiplet to establish the relation between original one-string elliptic genus $\mathbb{E}_{\tilde{h}^{(1)}_G}(v)$ and the Hall-Littlewood and Schur indices of $\widetilde{H}^{(k)}_G$ theory. Indeed, the Schur index of a 4d hypermultiplet is known to be (Cordova, Gaiotto, and Shao, 2016)

$$
\mathcal{I}^{\text{Schur}}_{h.m.} = \text{PE}\left[\frac{q^{1/2}}{1-q}(x+x^{-1})\right], \tag{7.3.6}
$$

which can also be obtained by taking limit $\mathbb{E}_{\text{c.m.}}(v,x,q_\tau)\to\mathbb{E}_{\text{cm}}(q^{1/2},x,q^2)$. The Hall-Littlewood index of a 4d hypermultiplet i.e. the Hilbert series of $\mathbb{C}^2$ is well-known to be

$$
\mathcal{I}^{\text{HL}}_{h.m.} = \frac{1}{(1-vx^{\pm1})}, \tag{7.3.7}
$$

which can also be obviously obtained by taking limit $\mathbb{E}_{\text{c.m.}}(v, x, q_\tau \to 0)$, with a factor $v q_\tau^{-1/6}$ absorbed into the overall factor of (5.4.1). This makes the whole story consistent.

In the viewpoint of pure 4d, this intriguing conjecture indicates there exists certain precise relation between the β-twisted partition function on $T^2 \times S^2$ and the partition function on $S^3 \times S^1$. We suspect the connection may be established by realizing one S^1 of T^2 as the Hopf fibration over S^2 to get $S^3 \times S^1$. To find the consequence of such realization one has to go into the details of localization.

7.4 Rank two

We would like to generalize Del Zotto-Lockhart's conjecture to the rank two cases, where there exist more flavour symmetry that is $SU(2)_x$ in H_G theories. To be precise, we want to find some functions $L_G^{(2)}(v, x, m_G, q_\tau) = \sum_{i,j=0}^{\infty} b_{i,j}^G q_\tau^i v^j$ such that

1. $b_{i,j}^G$ can be written as the sum of products between the character of irreducible representation of $SU(2)_x$ and the character of irreducible representation of G with integral coefficients.
2. $L_G^{(2)}(v, x, m_G, 0)$ is the Hilbert series of the reduced moduli space of two G-instanton, i.e. the Hall-Littlewood index of the $H_G^{(2)}$ theory.
3. $L_G^{(2)}(q^{1/2}, x, m_G, q^2)$ is the Schur index of the $H_G^{(2)}$ theory.
4. The reduced two-string elliptic genus $\mathbb{E}_{h_G^{(2)}}(v, x, m_G, q_\tau)$ can be generated from $L_G^{(2)}(v, x, m_G, q_\tau)$ and $L_G^{(1)}(v, x, m_G, q_\tau)$ functions.

It turns out the rank two cases are much more complicated than the rank one cases, one reason for which is that we can not rely on the additional symmetry (5.4.25). Although we have not achieved an exact formula to generate the two string elliptic genus, we successfully manage to identify the $L_G^{(2)}$ functions to substantial orders, which we will elaborate on later for each example. In fact, the leading and subleading q_τ order of $L_G^{(2)}(v, x, m_G, q_\tau)$ are just given by $g_{2,G}^{(0)}(v, x, m_G)$ in (5.4.6) and $g_{2,G}^{(1)}(v, x, m_G)$ in (5.4.7), while the subsubleading order is given by

$$(\chi_5 + (\chi_\theta + 2)\chi_3 + \chi_{\mathrm{Sym}^2\theta} + 2\chi_\theta + 3) + \Big((\chi_\theta + 1)\chi_4 + ((\chi_\theta + 1)^2 + (2\chi_\theta + 1))\chi_2\Big)v + \dots, \tag{7.4.1}$$

which differs from $g_{2,G}^{(2)}(v, x, m_G)$ in (5.4.8) by $1 + \chi_2 v + \dots$. Such difference is recognized as what we call "blue" series in contrast to the red $L_G^{(2)}$ functions. Indeed, the reason we also include $L_G^{(1)}$ in the last condition is that we observe a "blue" series appearing multiple times in the coefficient matrix of $\mathbb{E}_{h_G^{(2)}}$:

$$\begin{aligned} M_G^{(2),\text{blue}}(v, x) &= \sum_{n=0}^{\infty} v^n \sum_{i+2j=n+1} \chi_i \chi_{j\theta} = \frac{1}{(1 - vx)(1 - v/x)} g_{1,G}^{(0)}(v) \\ &= 1 + \chi_2 v + (\chi_3 + \chi_\theta)v^2 + (\chi_4 + \chi_\theta \chi_2)v^3 + (\chi_5 + \chi_\theta \chi_3 + \chi_{2\theta})v^4 \\ &\quad + (\chi_6 + \chi_\theta \chi_4 + \chi_{2\theta}\chi_2)v^5 + \dots . \end{aligned} \tag{7.4.2}$$

For example, the blue series always appear at q_τ order $h^\vee_G/3$ with leading v order $-2h^\vee_G/3$ (comparing to the leading q_τ order). The reason for such phenomenon is yet not clear to us.

On the other hand, from the technique of class $\mathcal{S}$ theory, we can compute the Schur index of $H^{(2)}_G$ theories for $G = D_4, E_{6,7,8}$. All of them are in agreement with our expectation from elliptic genera up to quite high orders. For example, from the $L^{(2)}_G$ functions, we are able to write down the following general formula for the Schur indices up to $q^{7/2}$:

$$\begin{aligned}\mathcal{I}^{\text{Schur}}_{H^{(2)}_G} &= \mathcal{I}^{\text{Schur}}_{\widetilde{H}^{(2)}_G}/\mathcal{I}^{\text{Schur}}_{h.m.} = 1 + (\chi_3+\chi_\theta)q + \chi_\theta\chi_2 q^{3/2} + \Big(\chi_5 + (\chi_\theta+1)\chi_3 + \chi_{\text{Sym}^2\theta} \\ &\quad + \chi_\theta + 1\Big)q^2 + \Big(\chi_\theta\chi_4 + (\chi_{2\theta}+\chi_{\text{Sym}^2\theta}+1)\chi_2\Big)q^{5/2} + \Big(\chi_7 + (\chi_\theta+1)\chi_5 \\ &\quad + (\chi_{2\theta}+\chi_{\text{Sym}^2\theta}+2\chi_\theta+3)\chi_3 + \chi_{\text{Sym}^3\theta} + (\chi_\theta+1)^2 - C_6(G)\Big)q^3 \\ &\quad + \Big(\chi_\theta\chi_6 + (\chi_{2\theta}+\chi_{\text{Alt}^2\theta}+2\chi_\theta+1)\chi_4 + (\chi_{3\theta}+2\chi_{2\theta}+(\chi_\theta+1)^2 \\ &\quad + \chi_{\text{Sym}^2\theta}+\chi_{\text{Alt}^2\theta}+B_2(G)+C_7(G))\chi_2\Big)q^{7/2} + \dots\,.\end{aligned} \tag{7.4.3}$$

In the following, we show the striking comparison between elliptic genus and indices at rank two for all symmetry group G.

$\mathbf{SU}(3)$

For $SU(3)$, let us denote the two-string elliptic genus as

$$\mathbb{E}_{h^{(2)}_{A_2}}(v,x,q_\tau,Q_m) = v^5 q_\tau^{-5/6}\sum_{i,j=0}^{\infty} c^{A_2}_{i,j}(x,Q_m)v^j(q_\tau v^{-4})^i. \tag{7.4.4}$$

Then we have the unrefined coefficients $c^{A_2}_{i,j}(x=1,Q_m=1)$ listed in Table 7.3. Keeping in mind that all such numbers can be refined to incorporate $SU(2)_x$, we show the unrefined coefficients just to make them look clearer. The red numbers give the definition of $L^{(2)}_G$ functions. In particular, they are in agreement with the universal expansion (5.4.6), (5.4.7) and (7.4.1). Note the red numbers in the first row agrees with the Hilbert series for reduced moduli space of two A_2 instantons in (Hanany, Mekareeya, and Razamat, 2013). The two red numbers in the $i=2$ rows are predicted from (7.4.1). Besides, the blue numbers agree with our proposal (7.4.2). Adding the red numbers in each column together, we expect to obtain a series that is equal to the Schur index of rank two H_{A_2} 4d SCFT.

The construction of $H^{(2)}_{A_2}$ theory from 6d involves irregular punctures. We are not aware how to directly compute its indices. We write our prediction from elliptic genus here: the Hall-Littlewood index of rank two H_{A_2} theory is

$$\begin{aligned}\mathcal{I}^{\text{HL}}_{H^{(2)}_{A_2}} &= 1 + (\chi_3+8)q + 8\chi_2 q^{3/2} + (\chi_5+8\chi_3+36)q^2 + (8\chi_4+55\chi_2)q^{5/2} \\ &\quad + (\chi_7+8\chi_5+63\chi_3+119)q^3 + (8\chi_6+55\chi_4+216\chi_2)q^{7/2} \\ &+(\chi_9+8\chi_7+63\chi_5+280\chi_3+322)q^4 + (8\chi_8+55\chi_6+280\chi_4+637\chi_2)q^{9/2} + \mathcal{O}(q^5)\,,\end{aligned} \tag{7.4.5}$$

i,j	-2	-1	0	1	2	3	4	5	6	7	8
0	0	0	1	0	11	16	65	142	335	700	1542
1	0	0	1	2	11	20	12+56	18+92	143+192	356+292	1091+517
2	-1	0	1	-2	2	0	51	150	473	1032	90+2225

i,j	9	10	11	12
0	2788	5350	9288	16184
1	2676+742	6387+1183	13476+1624	28204+2408
2	232+4024	8589	15552	30469

Table 7.3: Unrefined coefficients $c^{A_2}_{i,j}$ for the elliptic genus of two $SU(3)$ instanton strings.

which agrees with the Hilbert series of reduced moduli space of two $SU(3)$ instantons (Hanany, Mekareeya, and Razamat, 2013), and the Schur index of rank two H_{A_2} theory is

$$\begin{aligned}\mathcal{I}^{\text{Schur}}_{H^{(2)}_{A_2}} &= 1+(\chi_3+8)q+8\chi_2q^{3/2}+(\chi_5+9\chi_3+45)q^2+(8\chi_4+64\chi_2)q^{5/2}\\ &+(\chi_7+9\chi_5+82\chi_3+200)q^3+(8\chi_6+72\chi_4+360\chi_2)q^{7/2}\\ &+(\chi_9+9\chi_7+83\chi_5+479\chi_3+799)q^4+(8\chi_8+72\chi_6+496\chi_4+1608\chi_2)q^{9/2}+\mathcal{O}(q^5)\,.\end{aligned} \tag{7.4.6}$$

Taking $x=1$ in (7.4.6), we have the unrefined Schur index as

$$1+11q+16q^{3/2}+77q^2+160q^{5/2}+498q^3+1056q^{7/2}+2723q^4+5696q^{9/2}+\mathcal{O}(q^5)\,. \tag{7.4.7}$$

This is in complete agreement with results computed from chiral algebra (Beem et al., 2020), see also (Beem and Peelaers, 2020).[7]

SO(8)

The $H^{(2)}_{D_4}$ theory can be constructed by compactifying A_3 (2,0) 6d SCFT on a sphere with four square punctures $\{2^2\}$, i.e. 2222 theory, which is expected to be a $usp(4)$ gauge theory with four fundamental hypermultiplets and one anti-fundamental. On the other hand, the $\widetilde{H}^{(2)}_{D_4}$ theory can be constructed as a $222L$ theory, i.e. we replaces one $\{2^2\}$ puncture to $\{2,1^2\}$. In (Gaiotto and Razamat, 2012), the Hall-Littlewood indices of both $222L$ theory and $usp(4)+4f+1a$ theory was computed, which are in relation

$$\mathcal{I}_{222L}(v,x,m_i)=\frac{1}{1-vx^{\pm1}}\mathcal{I}_{usp(4)+4f+1a}(v,x,m_i)\,. \tag{7.4.8}$$

We expect and indeed checked to high orders

$$\mathcal{I}_{usp(4)+4f+1a}(v,x,m_i)=g^{(2)}_{0,D_4}(\tau,a,m_i). \tag{7.4.9}$$

For example, one can directly see the series coefficients in (A.12) of (Gaiotto and Razamat, 2012) agree with the q^0_τ entries in Table 7.4.

[7]We thank Beem and Rastelli for providing us their unpublished results on the unrefined Schur index of rank two $SU(3)$ theory.

The Schur index of $222L$ theory can be obtained in a similar manner. Following the general formula in (Gadde et al., 2013), we obtain

$$\mathcal{I}^{\text{Schur}}_{222L}(c,d,e;a,b) = \hat{\mathcal{N}}'_4\,\hat{\mathcal{K}}_1(c)\,\hat{\mathcal{K}}_1(d)\,\hat{\mathcal{K}}_1(e)\,\hat{\mathcal{K}}_2(a,b)\sum_\lambda \frac{\chi_\lambda(vb,v^{-1}b,b^{-1}a,b^{-1}a^{-1})}{\chi^2_\lambda(v^{-3},v^{-1},v,v^3)}$$
$$\times\chi_\lambda(vc,v^{-1}c,vc^{-1},v^{-1}c^{-1})\chi_\lambda(vd,v^{-1}d,vd^{-1},v^{-1}d^{-1})\chi_\lambda(ve,v^{-1}e,ve^{-1},v^{-1}e^{-1})\,, \tag{7.4.10}$$

where $(b_1 = b,\ b_2 = 1/b)$. The summation is over Young diagrams $\lambda = (\lambda_1,\lambda_2,\lambda_3,0)$. The $\hat{\mathcal{N}}$ and $\hat{\mathcal{K}}$ factors are given by

$$\begin{aligned}
\hat{\mathcal{N}}'_4 &= (v^2;v^2)^4\prod_{j=2}^{4}(v^{2j};v^2)^2\,,\\
\hat{\mathcal{K}}_1(b) &= \text{PE}\left[\frac{(v^2+v^4)(b^2+b^{-2}+2)}{1-v^2}\right]\,,\\
\hat{\mathcal{K}}_2(a,b) &= \text{PE}\left[\frac{3v^2+v^4+v^3b^{\pm2}a^{\pm1}+v^2a^{\pm2}}{1-v^2}\right]\,.
\end{aligned} \tag{7.4.11}$$

At last, one usually replaces $v\to q^{1/2}$ to make contact with literature. From the above formula, we computed the Schur index up to v^{20} as

$$\mathcal{I}^{\text{Schur}}_{222L} = 1+\chi_2 v+(2\chi_3+28)v^2+(2\chi_4+58\chi_2)v^3+(3\chi_5+87\chi_3+465)v^4+\dots\,. \tag{7.4.12}$$

Decoupling the free hypermultiplet, we obtain the Schur index of $H^{(2)}_{D_4}$ theory

$$\mathcal{I}^{\text{Schur}}_{H^{(2)}_{D_4}} = \mathcal{I}^{\text{Schur}}_{usp(4)+4f+1a} = \mathcal{I}^{\text{Schur}}_{222L}/\mathcal{I}^{\text{Schur}}_{h.m.} \tag{7.4.13}$$

up to q^{10}. The first 12 terms with full $SU(2)_x$ fugacity are

$$\begin{aligned}
\mathcal{I}^{\text{Schur}}_{H^{(2)}_{D_4}} &= 1+(\chi_3+28)q+28\chi_2q^{3/2}+(\chi_5+29\chi_3+435)q^2+(28\chi_4+707\chi_2)q^{5/2}\\
&+(\chi_7+29\chi_5+765\chi_3+4845)q^3+(28\chi_6+735\chi_4+9947\chi_2)q^{7/2}\\
&+(\chi_9+29\chi_7+766\chi_5+12337\chi_3+43353)q^4\\
&+(28\chi_8+735\chi_6+12607\chi_4+101878\chi_2)q^{9/2}\\
&+(\chi_{11}+29\chi_9+766\chi_7+12667\chi_5+141518\chi_3+330360)q^5\\
&+(28\chi_{10}+735\chi_8+12635\chi_6+155449\chi_4+845225\chi_2)q^{11/2}+\mathcal{O}(q^6)\,.
\end{aligned} \tag{7.4.14}$$

We can compare this with elliptic genus up to $q^{11/2}$. Let us denote the $SO(8)$ two-string elliptic genus as

$$\mathbb{E}_{h^{(2)}_{D_4}}(v,x,\tau,m_i=0) = v^{11}q_\tau^{-11/6}\sum_{i,j=0}^{\infty} c^{SO(8)}_{i,j}(x)v^j(q_\tau v^{-4})^i. \tag{7.4.15}$$

Then we have Table 7.4 for the coefficients $c^{SO(8)}_{i,j}(x=1)$. Here the red numbers are from the $L^{(2)}_{D_4}$ series. Add the red numbers in each column together, we expect to obtain a series that is equal to the Schur index of rank two H_{D_4} 4d SCFT. Indeed, we

i,j	0	1	2	3	4	5	6	7	8
0	1	0	31	56	495	1468	6269	19680	64768
1	0	0	0	0	32	58	1023	3322	19078
2	-1	-2	-31	-60	-389	-718	$-2972+2$	$-5226+2\cdot 2$	$560-16398+2\cdot 31+1$

i,j	9	10	11
0	187792	537021	1424526
1	69114	266799	886104
2	$1912-27570+2\cdot 60+2$	$20063-71670+2\cdot 389+31$	$83586-115770+2\cdot 718+60$

Table 7.4: Series coefficients $c_{i,j}^{SO(8)}$ for the elliptic genus of two $SO(8)$ instanton strings.

have

$$L_{D_4}^{(2)}(q^{1/2},x=1,m_{D_4}=0,q^2)=1+31q+56q^{3/2}+527q^2+1526q^{5/2}+7292q^3 \\ +23002q^{7/2}+84406q^8+258818q^{9/2}+823883q^5+2394216q^{11/2}+\dots\,. \tag{7.4.16}$$

On the other hand, by taking the unrefined limit $x=1$ in (7.4.13), we obtain the unrefined Schur series

$$\begin{aligned}&1+31q+56q^{3/2}+527q^2+1526q^{5/2}+7292q^3+23002q^{7/2}+84406q^8+258818q^{9/2}\\&+823883q^5+2394216q^{11/2}+6943434q^6+19082748q^{13/2}+51665849q^7\\&+134888730q^{15/2}+345764537q^8+862482876q^{17/2}+2112344321q^9\\&+5061362222q^{19/2}+11921262927q^{10}+\mathcal{O}(q^{21/2})\,.\end{aligned} \tag{7.4.17}$$

One can see the two series match perfectly up to $q^{11/2}$!

$\mathbf{F_4}$

Let us denote the two-string elliptic genus with gauge symmetry F_4 as

$$\mathbb{E}_{h_{F_4}^{(2)}}(v,x=1,\tau,m_i=0)=v^{17}q_\tau^{-17/6}\sum_{i,j=0}^{\infty}c_{i,j}^{F_4}v^j(q_\tau v^{-4})^i. \tag{7.4.18}$$

Then we have Table 7.5 for the unrefined coefficients $c_{i,j}^{F_4}$. The red numbers in the first row agrees with the Hilbert series for reduced moduli space of two F_4 instantons in (Hanany, Mekareeya, and Razamat, 2013). By summing over the red numbers in each column, we obtain certain analogy of Schur index of rank two H_G theory for F_4 up to $q^{11/2}$. The unrefined version is

$$\begin{aligned}&1+55q+104q^{3/2}+1595q^2+5072q^{5/2}+35226q^3+130240q^{7/2}+640886q^4\\&+2384608q^{9/2}+9769738q^5+34831256q^{11/2}+\mathcal{O}(q^6)\,.\end{aligned} \tag{7.4.19}$$

i,j	0	1	2	3	4	5	6	7	8	9
0	1	0	55	104	1539	4966	32091	119340	542109	1973088
1	0	0	0	0	56	106	3135	10900	97125	405480
2	0	0	0	0	0	0	0	0	1652+1	6040+2
3	1	2	55	108	1214	2320	15802	29284	143542−1	257800−2

i,j	10	11	12
0	7460100	25288640	84766812
1	2210027	9075756	38900537
2	99611+55	466860+108	3399668+1214
3	999970−55	1742140−108	5704242

Table 7.5: Series coefficients $c^{F_4}_{i,j}$ for the elliptic genus of two F_4 instanton strings.

This is in complete agreement with results computed from chiral algebra (Beem et al., 2020)![8]

$\mathbf{E_6}$

The $H^{(2)}_{E_6}$ theory can be constructed by compactifying A_5 (2,0) 6d SCFT on a sphere with three $\{2^3\}$ punctures, which is a "bad" theory. One can change one of the punctures to $\{2^2,1^2\}$ to add a decoupled hypermultiplet, i.e. the $\widetilde{H}^{(2)}_{E_6}$ theory. The Hall-Littlewood index of this theory was computed in (Gaiotto and Razamat, 2012). We expect and indeed checked

$$\mathcal{I}^{\mathrm{HL}}_{H^{(2)}_{E_6}}(v,x,m_{E_6}) = g^{(2)}_{0,E_6}(v,x,m_{E_6}). \tag{7.4.20}$$

The Schur index can be obtained in a similar manner. Following the general formula in (Gadde et al., 2013), we obtain

$$\begin{aligned}\mathcal{I}^{\mathrm{Schur}}_{\widetilde{H}^{(2)}_{E_6}} &= \hat{\mathcal{N}}_6\,\hat{\mathcal{K}}_1(a_1,a_2)\,\hat{\mathcal{K}}_1(a_3,a_4)\,\hat{\mathcal{K}}_2(a_5,a_6,x)\sum_\lambda \frac{\chi_\lambda(va_5,v^{-1}a_5,va_6,v^{-1}a_6,\frac{x}{a_5a_6},\frac{x^{-1}}{a_5a_6})}{\chi_\lambda(v^{-5},v^{-3},v^{-1},v^1,v^3,v^5)}\\ &\times\chi_\lambda(va_1,v^{-1}a_1,va_2,v^{-1}a_2,v\frac{1}{a_1a_2},v^{-1}\frac{1}{a_1a_2})\chi_\lambda(va_3,v^{-1}a_3,va_4,v^{-1}a_4,v\frac{1}{a_3a_4},v^{-1}\frac{1}{a_3a_4})\,.\end{aligned} \tag{7.4.21}$$

Here $\lambda=(\lambda_1,\cdots,\lambda_5,0)$ and ($b_3 \equiv \frac{1}{b_1b_2}$)

$$\hat{\mathcal{N}}_6 = (v^2;v^2)^3\prod_{j=2}^{6}(v^{2j};v^2)\,, \tag{7.4.22}$$

$$\begin{aligned}\hat{\mathcal{K}}_1(b_1,b_2) &= \prod_{\ell=1}^{2}\prod_{i,j=1}^{3}\mathrm{PE}\left[\frac{v^{2\ell}b_i/b_j}{1-v^2}\right],\\ \hat{\mathcal{K}}_2(b_1,b_2,x) &= \prod_{\ell=1}^{2}\prod_{i,j=1}^{2}\mathrm{PE}\left[\frac{v^{2\ell}b_i/b_j}{1-v^2}\right]\end{aligned} \tag{7.4.23}$$

[8] We thank Beem and Rastelli for providing us their unpublished results on the unrefined Schur index of rank two F_4 theory.

$$\times \text{PE}\left[\frac{2v^2+v^2x^{\pm 2}+v^3(b_1^2b_2x^{\pm 1})^{\pm 1}+v^3(b_1b_2^2x^{\pm 1})^{\pm 1}}{1-v^2}\right]. \tag{7.4.24}$$

At last, one need to replace $v \to q^{1/2}$. We computed the Schur index up to q^7:

$$\begin{aligned}
\mathcal{I}^{\text{Schur}}_{H^{(2)}_{E_6}} &= 1+(\chi_3+78)q+78\chi_2 q^{3/2}+(\chi_5+79\chi_3+3160)q^2+(78\chi_4+5512\chi_2)q^{5/2}\\
&+(\chi_7+79\chi_5+5670\chi_3+87751)q^3+(78\chi_6+5590\chi_4+201292\chi_2)q^{7/2}\\
&+(\chi_9+79\chi_7+5671\chi_5+248290\chi_3+1871196)q^4\\
&+(78\chi_8+5590\chi_6+250640\chi_4+5048654\chi_2)q^{9/2}\\
&+(\chi_{11}+79\chi_9+5671\chi_7+250400\chi_5+7248975\chi_3+32615793)q^5\\
&+(78\chi_{10}+5590\chi_8+250718\chi_6+7900243\chi_4+97665932\chi_2)q^{11/2}\\
&+(\chi_{13}+79\chi_{11}+5671\chi_9+250801\chi_7+7949911\chi_5+157280287\chi_3+483480405)q^6\\
&+(78\chi_{12}+5590\chi_{10}+250718\chi_8+7949591\chi_6+186447755\chi_4+1552411211\chi_2)q^{13/2}\\
&+(\chi_{15}+79\chi_{13}+5671\chi_{11}+250801\chi_9+7952421\chi_7+193661181\chi_5+2725694921\chi_3\\
&+6263699772)q^7+\ldots\,.
\end{aligned} \tag{7.4.25}$$

Note the leading terms up to $q^{7/2}$ agree with our general proposal (7.4.3).

Let us denote the two-string elliptic genus as

$$\mathbb{E}_{h^{(2)}_{E_6}}(v,x=1,\tau,m_i=0)=v^{23}q_\tau^{-23/6}\sum_{i,j=0}^{\infty}c^{E_6}_{i,j}v^j(q_\tau v^{-4})^i. \tag{7.4.26}$$

Then we have Table 7.6 for the coefficients $c^{E_6}_{i,j}$. Here the red numbers are from the

i,j	0	1	2	3	4	5	6	7	8	9
0	1	0	81	156	3320	11178	98440	401280	2344619	9785226
1	0	0	0	0	82	158	6723	24132	296879	1335694
2	0	0	0	0	0	0	0	0	3485+1	13112+2
3	0	0	0	0	0	0	0	0	−1	−2
4	−1	−2	−81	−160	−2669	−5178	−51445	−97712	−681945	−1266178

i,j	10	11	12	13
0	45870686	182872426	746229150	2782158570
1	9484963	44112702	236141466	1042037420
2	301488+81	1497516+160	14405643+2669	75613998+5178
3	−81	−160	102090−2669+83	563580−5178+322
4	−6819518 +2	−12372858 +2 · 2	−54611704 +2 · 81−83	−96850550 +2 · 160−322

i,j	14	15
0	10261780870	35695088906
1	4709271558	19202312882
2	486421964+51445	2415319754+97712
3	9603627−51445+7039	58071366−97712+24620
4	−365050846 + 2 · 2669−7039	−633251142+2 · 5178−24620

Table 7.6: Series coefficients $c^{E_6}_{i,j}$ for the unrefined elliptic genus of two E_6 instanton strings.

$L^{(2)}_{E_6}$ series. Add the red numbers in each column together, we expect to obtain a series that is equal to the Schur index of rank two H_{E_6} 4d SCFT. Indeed, we have

$$\begin{aligned} L^{(2)}_{E_6}(q^{1/2}, x=1, m_{E_6}=0, q^2) &= 1 + 81q + 156q^{3/2} + 3402q^2 + 11336q^{5/2} + 105163q^3 \\ &+ 425412q^{7/2} + 2644983q^4 + 11134032q^{9/2} + 55655137q^5 + 228482644q^{11/2} \\ &+ 996878349q^6 + 3900373568q^{13/2} + 15467078019q^7 + 57370792908q^{15/2} + \dots . \end{aligned} \tag{7.4.27}$$

On the other hand, taking $x = 1$ in (7.4.25), the unrefined Schur index is

$$\begin{aligned} &1 + 81q + 156q^{3/2} + 3402q^2 + 11336q^{5/2} + 105163q^3 + 425412q^{7/2} + 2644983q^4 \\ &+ 11134032q^{9/2} + 55655137q^5 + 228482644q^{11/2} + 996878349q^6 \\ &+ 3900373568q^{13/2} + 15467078019q^7 + \dots . \end{aligned} \tag{7.4.28}$$

We can see the two series match perfectly up to q^7!

$\mathbf{E_7}$

The $H^{(2)}_{E_7}$ theory can be constructed by compactifying A_7 (2,0) 6d SCFT on a sphere with one $\{4^2\}$ puncture and two $\{2^4\}$ punctures, which is a "bad" theory. One can change one of the $\{2^4\}$ punctures to $\{2^3, 1^2\}$ to add a decoupled hypermultiplet, i.e. the $\widetilde{H}^{(2)}_{E_7}$ theory. The Hall-Littlewood index of this theory was computed in (Gaiotto and Razamat, 2012). We find it agrees with our computation for $g^{(2)}_{0,E_7}(\tau, x, m_{E_7})$. The Schur index can be obtained in a similar manner. Following the general formula in (Gadde et al., 2013), we obtain

$$\begin{aligned} \mathcal{I}^{\text{Schur}}_{\widetilde{H}^{(2)}_{E_7}} &= \hat{\mathcal{N}}_8\, \hat{\mathcal{K}}_1(a_1,a_2,a_3)\, \hat{\mathcal{K}}_2(a_4,a_5,a_6,x)\, \hat{\mathcal{K}}_3(a_7) \\ &\times \sum_\lambda \frac{\chi_\lambda(v^3a_7, v^{-3}a_7, va_7, v^{-1}a_7, v^3a_7^{-1}, v^{-3}a_7^{-1}, va_7^{-1}, v^{-1}a_7^{-1})}{\chi_\lambda(v^{-7}, v^{-5}, v^{-3}, v^{-1}, v, v^3, v^5, v^7)} \\ &\times \chi_\lambda(va_1, v^{-1}a_1, va_2, v^{-1}a_2, va_3, v^{-1}a_3, v\frac{1}{a_1a_2a_3}, v^{-1}\frac{1}{a_1a_2a_3}) \\ &\times \chi_\lambda(va_4, v^{-1}a_4, va_5, v^{-1}a_5, va_6, v^{-1}a_6, \frac{x}{a_4a_5a_6}, \frac{x^{-1}}{a_4a_5a_6}) . \end{aligned} \tag{7.4.29}$$

Here

$$\hat{\mathcal{K}}_3(b) = \text{PE}\left[\frac{2(v^2+v^4+v^6+v^8)}{1-v^2}\right] \prod_{\ell=1}^{4} \text{PE}\left[\frac{v^{2\ell}b^{\pm 2}}{1-v^2}\right] , \tag{7.4.30}$$

$$\begin{aligned} \hat{\mathcal{K}}_2(b_1,b_2,b_3,x) &= \text{PE}\left[\frac{2v^2+v^2x^{\pm 2}}{1-v^2}\right] \left[\prod_{\ell=1}^{2}\prod_{i,j=1}^{3} \text{PE}\left[\frac{v^{2\ell}b_i/b_j}{1-v^2}\right]\right] \\ &\times \prod_{i=1}^{3}\left[\frac{v^3(x^{-1}\,b_i/b_4)^{\pm 1} + v^3(x\,b_i/b_4)^{\pm 1}}{1-v^2}\right] , \end{aligned} \tag{7.4.31}$$

$$\hat{\mathcal{K}}_1(b_1,b_2,b_3) = \prod_{\ell=1}^{2}\prod_{i,j=1}^{4} \text{PE}\left[\frac{v^{2\ell}b_i/b_j}{1-v^2}\right] . \tag{7.4.32}$$

At last, one need to replace $v \to q^{1/2}$. We computed the Schur index up to q^2 order.

After decoupling the free hypermultiplet, the Schur index of $H_{E_7}^{(2)}$ theory is given by

$$\begin{aligned}\mathcal{I}^{\text{Schur}}_{H^{(2)}_{E_7}} = \mathcal{I}^{\text{Schur}}_{\widetilde{H}^{(2)}_{E_7}} / \mathcal{I}^{\text{Schur}}_{h.m.} &= 1 + (\chi_3 + \mathbf{133})q + \mathbf{133}\chi_2 q^{3/2} \\ &+ (\chi_5 + (\mathbf{133}+1)\chi_3 + \text{Sym}^2\mathbf{133} + \mathbf{133} + 1)q^2 + \dots \,. \end{aligned} \tag{7.4.33}$$

Let us denote the two-string elliptic genus as

$$\mathbb{E}_{h^{(2)}_{E_7}}(v, x = 1, \tau, m_i = 0) = v^{35} q_\tau^{-35/6} \sum_{i,j=0}^{\infty} c^{E_7}_{i,j} v^j (q_\tau v^{-4})^i \,. \tag{7.4.34}$$

Then we have Table 7.7 for the coefficients $c^{E_7}_{i,j}$. Here the red numbers are from the

i,j	0	1	2	3	4	5	6	7	8	9
0	1	0	136	266	9315	32830	449050	2026080	17179899	84195608
1	0	0	0	0	137	268	18768	69544	1349005	6575250
2	0	0	0	0	0	0	0	0	9590+1	36982+2

Table 7.7: Series coefficients $c^{E_7}_{i,j}$ for the unrefined elliptic genus of two E_7 instanton.

$L^{(2)}_{E_7}$ series. Add the red numbers in each column together, we expect to obtain a series that is equal to the Schur index of rank two H_{E_7} 4d SCFT. Thus, we predict the unrefined Schur index as

$$\begin{aligned}&1 + 136q + 266q^{3/2} + 9452q^2 + 33098q^{5/2} + 467818q^3 + 2095624q^{7/2} + 18538494q^4 \\ &+ 90807840q^{9/2} + \mathcal{O}(q^5) \,.\end{aligned} \tag{7.4.35}$$

Indeed, taking $x = 1$ in (7.4.33), the unrefined Schur index is given by

$$1 + 136q + 266q^{3/2} + 9452q^2 + \mathcal{O}(q^{5/2}) \,. \tag{7.4.36}$$

We can see the two series match perfectly!

$\mathbf{E_8}$

The $H^{(2)}_{E_8}$ theory can be constructed by compactifying A_{11} (2,0) 6d SCFT on a sphere with three $\{6^2\}$, $\{4^3\}$ and $\{2^6\}$, which is a "bad" theory. One can change the $\{2^6\}$ puncture to $\{2^5, 1^2\}$ to add a decoupled free hypermultiplet, i.e. the $\widetilde{H}^{(2)}_{E_8}$ theory. Following the general formula in (Gadde et al., 2013), we obtain its Schur index as

$$
\begin{aligned}
&\mathcal{I}^{\text{Schur}}_{\widetilde{H}^{(2)}_{E_8}} = \hat{\mathcal{N}}_{12}\,\hat{\mathcal{K}}_1(a_1,a_2,a_3,a_4,a_5,x)\,\hat{\mathcal{K}}_2(a_6,a_7)\,\hat{\mathcal{K}}_3(a_8)\\
&\times\sum_{\lambda}\frac{\chi_\lambda(va_1,v^{-1}a_1,\ldots,va_5,v^{-1}a_5,\frac{x}{a_1\cdots a_5},\frac{x^{-1}}{a_1\cdots a_5})}{\chi_\lambda(v^{-11},v^{-9},\ldots,v^9,v^{11})}\\
&\times\chi_\lambda(v^3a_6,v^{-3}a_6,va_6,v^{-1}a_6,v^3a_7,v^{-3}a_7,va_7,v^{-1}a_7,v^3\frac{1}{a_6a_7},v^{-3}\frac{1}{a_6a_7},v\frac{1}{a_6a_7},v^{-1}\frac{1}{a_6a_7})\\
&\times\chi_\lambda(v^{-5}a_8,v^{-3}a_8,\ldots,v^3a_8,v^5a_8,v^{-5}a_8^{-1},v^{-3}a_8^{-1},\ldots,v^3a_8^{-1},v^5a_8^{-1})\,.
\end{aligned}
\tag{7.4.37}
$$

Here $\lambda=(\lambda_1,\cdots,\lambda_{11},0)$ and

$$
\begin{aligned}
\hat{\mathcal{K}}_3(b) &= \prod_{\ell=1}^{6}\mathrm{PE}\left[\frac{2v^{2\ell}+v^{2\ell}b^{\pm2}}{1-v^2}\right],\\
\hat{\mathcal{K}}_2(b_1,b_2) &= \prod_{\ell=1}^{4}\prod_{i,j=1}^{3}\mathrm{PE}\left[\frac{v^{2\ell}b_i/b_j}{1-v^2}\right],\\
\hat{\mathcal{K}}_1(c_1,c_2,c_3,c_4,c_5,x) &= \mathrm{PE}\left[\frac{2v^2+v^2\,x^{\pm2}}{1-v^2}\right]\left[\prod_{\ell=1}^{2}\prod_{i,j=1}^{5}\mathrm{PE}\left[\frac{v^{2\ell}c_i/c_j}{1-v^2}\right]\right]\\
&\quad\times\prod_{i=1}^{5}\mathrm{PE}\left[\frac{v^3(x^{-1}\,c_i/c_6)^{\pm1}+v^3(x\,c_i/c_6)^{\pm1}}{1-v^2}\right],
\end{aligned}
\tag{7.4.38}
$$

where $b_3\equiv\frac{1}{b_1b_2}$ and $c_6\equiv\frac{1}{c_1c_2c_3c_4c_5}$. At last, one need to replace $v\to q^{1/2}$. As the leading terms up to $q^{3/2}$ are contributed from rank one theory, the Schur index is given by

$$
\mathcal{I}^{\text{Schur}}_{\widetilde{H}^{(2)}_{E_8}} = 1+\chi_2q^{1/2}+(2\chi_3+\mathbf{248})q+(2\chi_4+2(\mathbf{248}+1)\chi_2)q^{3/2}+\ldots\,. \tag{7.4.39}
$$

After decoupling the free hypermultiplet, the Schur index of $H^{(2)}_{E_8}$ theory is

$$
\mathcal{I}^{\text{Schur}}_{H^{(2)}_{E_8}} = \mathcal{I}^{\text{Schur}}_{\widetilde{H}^{(2)}_{E_7}}/\mathcal{I}^{\text{Schur}}_{h.m.} = 1+(\chi_3+\mathbf{248})q+\mathbf{248}\chi_2q^{3/2}+\ldots\,. \tag{7.4.40}
$$

Let us denote the two-string elliptic genus as

$$
\mathbb{E}_{h^{(2)}_{E_8}}(v,x=1,\tau,m_i=0) = v^{59}q_\tau^{-59/6}\sum_{i,j=0}^{\infty}c^{E_8}_{i,j}v^j(q_\tau v^{-4})^i. \tag{7.4.41}
$$

Then we have Table 7.8 for the coefficients $c^{E_8}_{i,j}$. Here the red numbers are from the

i,j	0	1	2	3	4	5	6	7
0	1	0	251	496	31625	116248	2747875	13624000
1	0	0	0	0	252	498	63503	241742

Table 7.8: Series coefficients $c^{E_8}_{i,j}$ for the unrefined elliptic genus of two E_8 instanton strings.

$L^{(2)}_{E_8}$ series. Add the red numbers in each column together, we expect to obtain a series that is equal to the Schur index of rank two H_{E_8} 4d SCFT. Thus, we predict from the general formula (7.4.3) for the unrefined Schur index as

$$1+251q+496q^{3/2}+31877q^2+116746q^{5/2}+2811378q^3+13865742q^{7/2}+\mathcal{O}(q^4)\,. \tag{7.4.42}$$

Indeed, taking $x=1$ in (7.4.40), the unrefined Schur index is given by

$$1+251q+496q^{3/2}+\mathcal{O}(q^2)\,. \tag{7.4.43}$$

Indeed, the two series match perfectly!

7.5 Rank three and higher

We expect the Del Zotto-Lockhart's conjecture can be generalized to rank three and higher. From the universal leading expansion for three-string elliptic genus (5.4.9) and (5.4.10), we are able to predict the Schur index of rank three H_G SCFT up to order q^3:

$$\begin{aligned}\mathcal{I}^{\text{Schur}}_{H_G^{(3)}} =& 1+(\chi_3+\chi_\theta)q+(\chi_4+\chi_\theta\chi_2)q^{3/2}+\Big(\chi_5+(\chi_\theta+1)\chi_3+\chi_{\text{Sym}^2\theta}+\chi_\theta+2\Big)q^2\\&+\Big(\chi_6+(2\chi_\theta+2)\chi_4+2\chi_{\text{Sym}^2\theta}+\chi_\theta+1\Big)q^{5/2}\\&+\Big(2\chi_7+(3\chi_\theta+1)\chi_5+(\chi_{2\theta}+3\chi_{\text{Sym}^2\theta}+3\chi_\theta+5)\chi_3\\&\quad+\chi_{\text{Sym}^3\theta}+3\chi_{\text{Sym}^2\theta}+\chi_\theta+2\Big)q^3+\mathcal{O}(q^{7/2})\,.\end{aligned} \tag{7.5.1}$$

This is actually because (5.4.9) and (5.4.10) are also the definition of leading and subleading q_τ order of $L^{(3)}_G$ functions. Besides, we observe in the coefficient matrix of reduced three string elliptic genus, other than the $L^{(3)}_G$ function that appears as expected, the blue series also appears as in the rank two. The difference is that here the blue series are generated from the leading q_τ order of *two* string elliptic genus!

$$M^{(3),\text{blue}}_G(v,x)=\frac{1}{(1-vx)(1-v/x)}g^{(2)}_{0,G}(v,x). \tag{7.5.2}$$

Note $g^{(2)}_{0,G}(v,x)$ is also the leading q_τ order of $L^{(2)}_G$. In the following, we show the relation between reduced elliptic genus of three strings and the Schur index of $H^{(3)}_G$ theories for each G.

SU(3)

The formula for the elliptic genus of three $SU(3)$ string has been written down via Jeffrey-Kirwan residues in (Kim, Kim, and Park, 2016), using which we computed $\mathbb{E}_{h^{(3)}_{A_2}}$ up to q_τ^6 order. Denote

$$\mathbb{E}_{h^{(3)}_{A_2}}(v, x, \tau, m_i = 0) = v^8 q_\tau^{-4/3} \sum_{i,j=0}^{\infty} c_{i,j}^{SU(3)}(x) v^j (q_\tau v^{-4})^i . \tag{7.5.3}$$

Then the unrefined $L_G^{(3)}$ function is shown red in the coefficient matrix of $\mathbb{E}_{h^{(3)}_{A_2}}$ in Table 7.9. Note the red numbers are in agreement with our universal expansion (5.4.9)

i, j	0	1	2	3	4	5	6	7
0	1	0	11	20	90	218	698	1618
1	1	2	14	22	135+12	370+22	960+171	2250+502

Table 7.9: Coefficients $c_{i,j}^{SU(3)}$ for the unrefined elliptic genus of three $SU(3)$ instanton strings.

and (5.4.10), while the blue numbers are in agreement with our proposal (7.5.2).

The construction for rank three H_{A_2} theory from 6d involves certain irregular punctures as the rank two case. We are not aware how to compute its indices directly. We write down our prediction for the Schur index of rank three H_{A_2} theory here:

$$\begin{aligned} \mathcal{I}^{\text{Schur}}_{H^{(3)}_{A_2}} =& 1 + (\chi_3 + 8)q + (\chi_4 + 8\chi_2)q^{3/2} + (\chi_5 + 17\chi_3 + 46)q^2 \\ &+ (\chi_6 + 18\chi_4 + 81\chi_2)q^{5/2} + (2\chi_7 + 25\chi_5 + 164\chi_3 + 248)q^3 \\ &+ (\chi_8 + 27\chi_6 + 209\chi_4 + 557\chi_2)q^{7/2} + \mathcal{O}(q^4) . \end{aligned} \tag{7.5.4}$$

The unrefined limit is

$$\mathcal{I}^{\text{Schur}}_{H^{(3)}_{A_2}}(x = 1) = 1 + 11q + 20q^{3/2} + 102q^2 + 240q^{5/2} + 869q^3 + 2120q^{7/2} + \mathcal{O}(q^4) . \tag{7.5.5}$$

SO(8)

We can use class $\mathcal{S}$ theory technique to compute the HL and Schur index of rank three H_{D_4} 4d SCFT. The $H^{(3)}_{D_4}$ theory can be constructed by compactifying A_5 $(2,0)$ 6d SCFT on a sphere with four $\{3^2\}$ punctures, which is a "bad" theory. We need instead to consider $\widetilde{H}^{(3)}_{D_4}$ theory obtained from three $\{3^2\}$ punctures and one $\{3,2,1\}$

puncture. We compute the Schur index as

$$\mathcal{I}^{\text{Schur}}_{\widetilde{H}^{(3)}_{D_4}}(c_1,c_2,c_3;x,b) = \hat{\mathcal{N}}'_6\,\hat{\mathcal{K}}_1(c)\,\hat{\mathcal{K}}_1(d)\,\hat{\mathcal{K}}_1(e)\,\hat{\mathcal{K}}_2(x,b) \times \sum_{\lambda} \frac{\chi_\lambda(v^2b,b,v^{-2}b,vb^{-1}x,v^{-1}b^{-1}x,b^{-1}x^{-2})}{\chi^2_\lambda(v^{-5},v^{-3},v^{-1},v,v^3,v^5)} \times \prod_{i=1,2,3}\chi_\lambda(v^2c_i,c_i,v^{-2}c_i,v^2c_i^{-1},c_i^{-1},v^{-2}c_i^{-1})\,, \tag{7.5.6}$$

with ($b_1 = b,\ b_2 = 1/b$)

$$\hat{\mathcal{N}}'_6 = (v^2;v^2)^4\prod_{j=2}^{6}(v^{2j};v^2)^2\,, \tag{7.5.7}$$

$$\hat{\mathcal{K}}_1(b) = \mathrm{PE}\left[\frac{(v^2+v^4+v^6)(b^2+b^{-2}+2)}{1-v^2}\right], \tag{7.5.8}$$

$$\hat{\mathcal{K}}_2(a,b) = \mathrm{PE}\left[\frac{3v^2+2v^4+v^6+(v^3+v^5)(b^2a^{-1})^{\pm1}+v^3a^{\pm3}+v^4(b^2a^2)^{\pm1}}{1-v^2}\right]. \tag{7.5.9}$$

From the above formula, we compute the Schur index up to $q^{11/2}$. After decoupling the free hypermultiplet, we obtain

$$\begin{aligned}\mathcal{I}^{\text{Schur}}_{H^{(3)}_{D_4}} = \mathcal{I}^{\text{Schur}}_{\widetilde{H}^{(3)}_{D_4}}/\mathcal{I}^{\text{Schur}}_{h.m.} &= 1+(\chi_3+28)q+(\chi_4+28\chi_2)q^{3/2}+(\chi_5+57\chi_3+436)q^2\\ &+(\chi_6+58\chi_4+841\chi_2)q^{5/2}+(2\chi_7+85\chi_5+1607\chi_3+5308)q^3\\ &+(\chi_8+87\chi_6+2042\chi_4+14135\chi_2)q^{7/2}+(2\chi_9+115\chi_7+2806\chi_5+29042\chi_3\\ &+55871)q^4+(2\chi_{10}+115\chi_8+3242\chi_6+43166\chi_4+177896\chi_2)q^{9/2}\\ &+(2\chi_{11}+144\chi_9+4008\chi_7+60673\chi_5+392233\chi_3+527217)q^5\\ &+(2\chi_{12}+145\chi_{10}+4441\chi_8+75128\chi_6+649112\chi_4+1857119\chi_2)q^{11/2}+\mathcal{O}(q^6)\,.\end{aligned} \tag{7.5.10}$$

The unrefined limit is

$$\begin{aligned}\mathcal{I}^{\text{Schur}}_{H^{(3)}_{D_4}}(x=1) &= 1+31q+60q^{3/2}+612q^2+1920q^{5/2}+10568q^3+36968q^{7/2}\\ &+157850q^4+548848q^{9/2}+2036655q^5+6798456q^{11/2}+\mathcal{O}(q^6)\,.\end{aligned} \tag{7.5.11}$$

Let us denote the reduced three-string elliptic genus as

$$\mathbb{E}_{h^{(3)}_{D_4}}(v,x,\tau,m_i=0) = v^{17}q_\tau^{-17/6}\sum_{i,j=0}^{\infty}c^{SO(8)}_{i,j}v^j(q_\tau v^{-4})^i. \tag{7.5.12}$$

Then from (5.4.9) and (5.4.10), we expect to have Table 7.10 for the unrefined coefficients $c^{SO(8)}_{i,j}$. Here the red numbers are from the $L^{(3)}_{D_4}$ series. Add the red numbers in each column together, we expect to obtain a series that is equal to the Schur index of rank three H_{D_4} 4d SCFT. Indeed, we have

$$L^{(3)}_{D_4}(q^{1/2},x=1,Q_m=1,q^2) = 1+31q+60q^{3/2}+612q^2+1920q^{5/2}+10568q^3+\dots\,. \tag{7.5.13}$$

i, j	0	1	2	3	4	5	6
0	1	0	31	60	580	1858	9457
1	0	0	0	0	32	62	1111

Table 7.10: Expected coefficients $c_{i,j}^{SO(8)}$ for the unrefined elliptic genus of three $SO(8)$ instanton strings.

One can see the two series match perfectly up to q^3!

$\mathbf{E_6}$

The formula to compute the Hall-Littlewood index of rank three H_{E_6} SCFT has been written down in (Gaiotto and Razamat, 2012). Similarly, we compute the Schur index as

$$\mathcal{I}^{\text{Schur}}_{\widetilde{H}^{(3)}_{E_6}} = \hat{\mathcal{N}}_9\, \hat{\mathcal{K}}_1(a_1,a_2)\, \hat{\mathcal{K}}_1(a_3,a_4)\, \hat{\mathcal{K}}_2(a_5,a_6,x) \times$$
$$\sum_\lambda \frac{\chi_\lambda(v^2a_5, v^{-2}a_5, a_5, v^2a_6, v^{-2}a_6, a_6, v\frac{x}{a_5a_6}, v^{-1}\frac{x}{a_5a_6}, \frac{x^{-2}}{a_5a_6})}{\chi_\lambda(v^{-8},v^{-6},v^{-4},v^{-2},1,v^2,v^4,v^6,v^8)} \times$$
$$\times\, \chi_\lambda(v^2a_1, v^{-2}a_1, a_1, v^2a_2, v^{-2}a_2, a_2, v^2\frac{1}{a_1a_2}, v^{-2}\frac{1}{a_1a_2}, \frac{1}{a_1a_2})$$
$$\times\, \chi_\lambda(v^2a_3, v^{-2}a_3, a_3, v^2a_4, v^{-2}a_4, a_4, v^2\frac{1}{a_3a_4}, v^{-2}\frac{1}{a_3a_4}, \frac{1}{a_3a_4})\,.$$

with

$$\hat{\mathcal{K}}_1(b_1,b_2) = \text{PE}\Big[\sum_{i,j=1}^{3} \frac{(v^2+v^4+v^6)b_i/b_j}{1-v^2}\Big],$$
$$\hat{\mathcal{K}}_2(b_1,b_2,x) = \text{PE}\Big[\frac{1}{1-v^2}\Big((v^2+v^4+v^6)\Big(\sum_{i,j=1}^{2} b_i/b_j\Big) + 2v^2 + v^4 + v^3x^{\pm 3}$$
$$+ (b_1+b_2)((v^3+v^5)(b_3x)^{\pm 1} + v^4(b_3x^{-2})^{\pm 1})\Big)\Big],$$

where $b_1b_2b_3 = 1$. From the above formula, we computed the Schur index up to q^2. After decoupling the free hypermultiplet, we obtain

$$\begin{aligned}\mathcal{I}^{\text{Schur}}_{H^{(3)}_{E_6}} = \mathcal{I}^{\text{Schur}}_{\widetilde{H}^{(3)}_{E_6}}/\mathcal{I}^{\text{Schur}}_{h.m.} &= 1 + (\chi_3+78)q + (\chi_4+78\chi_2)q^{3/2} + (\chi_5+157\chi_3+3161)q^2 \\ &+ (\chi_6+158\chi_4+6241\chi_2)q^{5/2} + (2\chi_7+235\chi_5+11912\chi_3+91483)q^3 \\ &+ (\chi_8+237\chi_6+15072\chi_4+260821\chi_2)q^{7/2} + \mathcal{O}(q^4)\,.\end{aligned} \tag{7.5.14}$$

The unrefined limit is

$$\begin{aligned}\mathcal{I}^{\text{Schur}}_{H^{(3)}_{E_6}}(x=1) =& 1 + 81q + 160q^{3/2} + 3637q^2 + 13120q^{5/2} \\ &+ 128408q^3 + 583360q^{7/2} + \mathcal{O}(q^4)\,.\end{aligned} \tag{7.5.15}$$

On the other hand, the universal leading expansion (5.4.9) and (5.4.10) indicate the following Table 7.11 for the coefficients of reduced three-string elliptic genus for E_6. By adding the red numbers in each column together, one can indeed obtain the

i,j	0	1	2	3	4	5	6
0	1	0	81	160	3555	12958	121447
1	0	0	0	0	82	162	6961

Table 7.11: Coefficients $c_{i,j}^{E_6}$ for the unrefined elliptic genus of three E_6 instanton strings.

same unrefined Schur series as (7.5.15) up to q^3.

$\mathbf{F_4}$, $\mathbf{E_7}$, $\mathbf{E_8}$

The Schur indices with generic $SU(2)_x$ fugacity for rank three H_G theories can be predicted from (7.5.1) up to q^3 order. Let us just mark the unrefined series here:

$$\begin{aligned}
\mathcal{I}^{\text{Schur}}_{H^{(3)}_{F_4}} &= 1+55q+108q^{3/2}+1752q^2+6048q^{5/2}+45835q^3+\mathcal{O}(q^{7/2})\,,\\
\mathcal{I}^{\text{Schur}}_{H^{(3)}_{E_7}} &= 1+136q+270q^{3/2}+9852q^2+36990q^{5/2}+533401q^3+\mathcal{O}(q^{7/2})\,,\\
\mathcal{I}^{\text{Schur}}_{H^{(3)}_{E_8}} &= 1+251q+500q^{3/2}+32622q^2+126000q^{5/2}+3030748q^3+\mathcal{O}(q^{7/2})\,.
\end{aligned} \tag{7.5.16}$$

Note for F_4, we always mean the analogy for H_G theories.

In summary, we arrive at the final conjecture for arbitrary rank:

Conjecture 3. There exists an infinite series of functions with intergral expansion coefficients $L_G^{(n)}(v,x,m_G,q_\tau)=\sum_{i,j=0}^{\infty} b_{i,j}^{G,n} q_\tau^i v^j$, $n=1,2,\ldots$ such that

1. $b_{i,j}^{G,n}$ can be written as the sum of products between the character of irreducible representation of $SU(2)_x$ and the character of irreducible representation of G with integeral coefficients.
2. $L_G^{(n)}(v,x,m_G,0)$ is the Hilbert series of the reduced moduli space of n G-instanton, i.e. the Hall-Littlewood index of the $H_G^{(n)}$ theory.
3. $L_G^{(n)}(q^{1/2},x,m_G,q^2)$ is the Schur index of the $H_G^{(n)}$ theory.
4. The n-string elliptic genus $\mathbb{E}_{h_G^{(n)}}(v,x,m_G,q_\tau)$ can be generated from the first n L_G functions, i.e. $L_G^{(r)}(v,x,m_G,q_\tau)$, $r=1,2,\ldots,n$.

Chapter 8

Blowup Equations on $\mathbb{Z}_2$ Orbifold Spaces

This chapter is devoted to another kind of generalization of blowup equations of gauge theories, which directly inherits from Chapter 2. As mentioned in the introduction chapter, a natural generalization of Nakajima-Yoshioka's blowup equations on $\mathbb{C}^2$ is the blowup equations on ALE spaces $\mathbb{C}^2/\Gamma$, where the Nekrasov partition functions on the resolved spaces and orbifold spaces are connected. Indeed, lots of 4d blowup equations on $\mathbb{C}^2/\mathbb{Z}_2$ and even $\mathbb{C}^2/\mathbb{Z}_3$ have been found in (Bonelli, Maruyoshi, and Tanzini, 2011; Bonelli, Maruyoshi, and Tanzini, 2012a; Belavin et al., 2013; Ito, Maruyoshi, and Okuda, 2013; Bruzzo et al., 2016; Bruzzo, Sala, and Szabo, 2015). Here we focus on 5d blowup equations on $\mathbb{C}^2/\mathbb{Z}_2 \times S^1$ with $SU(N)$ gauge group, and use the new blowup equations we find to derive some conjectual functional equations for 5d Nekrasov partition functions inspired from the bilinear relations of Tau functions of some q-deformed isomonofromic systems (Bershtein and Shchechkin, 2017; Bershtein, Gavrylenko, and Marshakov, 2018; Bershtein, Gavrylenko, and Marshakov, 2019). To be precise, the conjectural functional equations of 5d $SU(2)$ partition function in (Bershtein and Shchechkin, 2017; Bershtein, Gavrylenko, and Marshakov, 2018) are inspired from q-Painleve III_3, see also (Bonelli, Grassi, and Tanzini, 2019). The conjectural functional equations of 5d $SU(N)$ partition function in (Bershtein, Gavrylenko, and Marshakov, 2019) are inspired from q-deformed perodic N-Toda systems.

The spirit of this chapter should be able to generalized to more than pure gauge theories. For example, it would be interesting to explore the relation between 5d $\mathbb{Z}_2$ blowup equations of $SU(2)$ $N_f = 4$ theory with the conjectural functional equations in (Jimbo, Nagoya, and Sakai, 2017) for q-Painleve VI. It would also be interesting to consider the $\mathbb{Z}_2$ type blowup equations for M-string theory, which should be related to the isomonodromic systems on one-punctured torus (Bonelli et al., 2020), and for E-string theory, which should be related to elliptic Painlevé equations (Mizoguchi and Yamada, 2002; Kels and Yamazaki, 2018). Presumably, $\mathbb{Z}_2$ type blowup equations for $SU(2)$ gauge theories with various number of fundamental matters if exist, should be connected to the bilinear relations of Tau functions of all Painlevé systems in Sakai's classification (Sakai, 2001). See a excellent review on such classification and bilinear relations in (Kajiwara, Noumi, and Yamada, 2017).

8.1 Nekrasov partition function on $\mathbb{C}^2/\mathbb{Z}_2$ orbifold

The orbifold partition functions on general ALE space were defined in (Kronheimer and Nakajima, 1990; Fucito, Morales, and Poghossian, 2004). In the case of A_1-ALE space, i.e. $\mathbb{C}^2/\mathbb{Z}_2$, the orbifold partition functions for gauge group $SU(2)$ were explicitly computed in (Belavin et al., 2013; Ito, 2012). It is easy to give a similar definition on $\mathbb{C}^2/\mathbb{Z}_2$ orbifold partition function in 5d. For gauge group $U(N)$, the 5d orbifold partition function depends the Chern number $c = 0, 1, 2, \ldots, N-1$. We follow (Ito, 2012) to give a definition on 5d $\mathbb{Z}_2$ orbifold $U(N)$ partition function.

Let us first specify the charges of each parameters of $\epsilon_1, \epsilon_2, \vec{a} = \{a_1, a_2, \ldots, a_N\}$ under the orbifold $\mathbb{Z}_2$ action. It is easy to find under such action,

$$\epsilon_1 \to \epsilon_1 + \pi, \qquad \epsilon_2 \to \epsilon_2 - \pi, \tag{8.1.1}$$

which means ϵ_1, ϵ_2 have $\mathbb{Z}_2$ charge $+1$ and -1 respectively. On the other hand, Coulomb parameters a_m can be assigned with charge d_m, i.e.

$$a_m \to a_m + d_m \pi. \tag{8.1.2}$$

Here d_m could be 0 or 1. Every box (i, j) in the Young tableau in Nekrasov's formula (2.1.6) corresponds to a one-dimensional subspace in the localization with weight $a + (i-1)\epsilon_1 + (j-1)\epsilon_2$. Therefore the charge of this box is $d_m + (i-1) - (j-1)$. To define orbifold partition function we need to pick out those boxes with the same $\mathbb{Z}_2$ charge. Denote the number of the boxes with charge 0 and 1 by k_0 and k_1 respectively and $N_q = \#\{d_m = q\}$ where $q = 0, 1$. The c condition on the Chern class c of orbifold partition function and (N_q, k_0, k_1) is

$$c = N_1 - 2(k_1 - k_0). \tag{8.1.3}$$

Then the orbifold $U(N)$ partition function associated to Chern class c can be defined as

$$Z^{\diamond,c} = \sum_{c \text{ condition on } \vec{d}} Z^{\diamond,\vec{d}}. \tag{8.1.4}$$

Here $\vec{d} = \{d_1, d_2, \ldots, d_N\}$ and[1]

$$Z^{\diamond,\vec{d}}(\epsilon_1, \epsilon_2, a, \mathfrak{q}; \beta) = \sum_{\substack{\vec{Y} \\ k_1 - k_2 \text{ as required}}} \left[\frac{(\mathfrak{q}\beta^{2N})^{|\vec{Y}|/2}}{\prod_{\alpha,\beta=\{1,2,\ldots,N\}} n^{\vec{d},\vec{Y}}_{\alpha,\beta}(\epsilon_1, \epsilon_2, \vec{a}; \beta)} \right], \tag{8.1.5}$$

where $\vec{Y} = \{Y_1, Y_2, \ldots, Y_N\}$,

$$\begin{aligned} n^{\vec{d},\vec{Y}}_{\alpha,\beta}(\epsilon_1, \epsilon_2, \vec{a}; \beta) = & \prod_{s \in Y^{\vec{d}}_\alpha} \left(1 - e^{-\beta(-l_{Y_\beta}(s)\epsilon_1 + (a_{Y_\alpha}(s)+1)\epsilon_2 + a_\beta - a_\alpha)}\right) \\ & \times \prod_{t \in Y^{\vec{d}}_\beta} \left(1 - e^{-\beta((l_{Y_\alpha}(t)+1)\epsilon_1 - a_{Y_\beta}(t)\epsilon_2 + a_\beta - a_\alpha)}\right). \end{aligned}$$

[1]Note the definition of 5d instanton counting parameter $\mathfrak{q}$ here is slightly different from the one in Chapter 2 like equation (2.1.6). Here to make contact with the physics literature on bilinear relations, we do not add a factor like $e^{-N\beta(\epsilon_1+\epsilon_2)/2}$ with $\mathfrak{q}$.

with $Y_\alpha^{\vec{d}} = \{s \in Y_\alpha | -l_{Y_\beta}(s) + a_{Y_\alpha}(s) + 1 + d_\beta - d_\alpha \equiv 0 \pmod 2\}$ and $Y_\beta^{\vec{d}} = \{t \in Y_\beta | l_{Y_\alpha}(t) + 1 - a_{Y_\beta}(t) + d_\beta - d_\alpha \equiv 0 \pmod 2\}$.

Let us give an example for $G = U(5)$. The Chern class $c = 0,1,2,3,4,5$. For example, for $c = 0,1$, we have the following possibilities of $\vec{d}$ and (N_q, k_0, k_1):

N_1	N_0	(d_1,d_2,d_3,d_4,d_5)	$k_1 - k_2$	c
0	5	$(0,0,0,0,0)$	0	0
1	4	$(1,0,0,0,0)$ and permutations	0	1
2	3	$(1,1,0,0,0)$ and permutations	1	0
3	2	$(1,1,1,0,0)$ and permutations	1	1
4	1	$(1,1,1,1,0)$ and permutations	2	0
5	0	$(1,1,1,1,1)$	2	1

Using these constraints, we can directly compute $Z^{\diamond,0}$ and $Z^{\diamond,1}$. Similar for other Chern class c.

For the gauge group $SU(N)$, one just need to put constraints on the $U(N)$ fugacities $\vec{a}$ in the final step.

8.2 Nekrasov partition function on resolved $\widehat{\mathbb{C}^2/\mathbb{Z}_2}$ space

Let us denote $\mathbb{C}^2/\mathbb{Z}_2$ as X_2 and its blowup as $\widehat{X_2}$. The Nekrasov partition function on such resolved space can be defined and computed very similarly as in the $\widehat{\mathbb{C}^2}$ case in Chapter 2.2. The only difference lies in that now the weights of T^2 torus action on the two fixed points of $\widehat{X_2}$ become

$$(2\epsilon_1, \epsilon_2 - \epsilon_1) \quad \text{and} \quad (\epsilon_1 - \epsilon_2, 2\epsilon_2). \tag{8.2.1}$$

For example, the 4d $SU(2)$ partition function on $\widehat{X_2}$ with $(k,d) = (0,0)$ was given in (Bonelli, Maruyoshi, and Tanzini, 2011) as

$$\begin{aligned} Z_{\widehat{X_2}}(\epsilon_1,\epsilon_2,\vec{a};\mathfrak{q}) = \sum_{\vec{k}\in\mathbb{Z}/2} & \frac{\mathfrak{q}^{(\vec{k},\vec{k})}}{\prod_{\vec{\alpha}\in\Delta} l_{\vec{\alpha}}^{\diamondsuit,\vec{k}}(\epsilon_1,\epsilon_2,\vec{a})} \times \\ & Z(2\epsilon_1,\epsilon_2-\epsilon_1,\vec{a}+2\epsilon_1\vec{k};\mathfrak{q}) Z(\epsilon_1-\epsilon_2,2\epsilon_2,\vec{a}+2\epsilon_2\vec{k};\mathfrak{q}), \end{aligned} \tag{8.2.2}$$

where

$$l_{\vec{\alpha}}^{\diamondsuit,\vec{k}}(\epsilon_1,\epsilon_2,\vec{a}) = \begin{cases} \prod\limits_{\substack{i,j\geq 0, i+j\leq -2\langle\vec{k},\vec{\alpha}\rangle-2 \\ i+j\equiv -2\langle\vec{k},\vec{\alpha}\rangle-2 \pmod 2}} (-i\epsilon_1 - j\epsilon_2 + \langle\vec{a},\vec{\alpha}\rangle) & \text{if } \langle\vec{k},\vec{\alpha}\rangle < 0, \\ \prod\limits_{\substack{i,j\geq 0, i+j\leq 2\langle\vec{k},\vec{\alpha}\rangle-2 \\ i+j\equiv 2\langle\vec{k},\vec{\alpha}\rangle-2 \pmod 2}} ((i+1)\epsilon_1 + (j+1)\epsilon_2 + \langle\vec{a},\vec{\alpha}\rangle) & \text{if } \langle\vec{k},\vec{\alpha}\rangle > 0, \\ 1 & \text{otherwise.} \end{cases} \tag{8.2.3}$$

In 5d, we can similarly define the $SU(N)$ Nekrasov partition function on resolved $\widehat{X_2}$ as

$$Z_{k,d}(\epsilon_1,\epsilon_2,\vec{a};\mathfrak{q},\beta)=\sum_{\vec{k}\in-\frac{k}{2N}+\mathbb{Z}/2}\frac{(\mathfrak{q}\beta^{2N}e^{2d\beta(\epsilon_1+\epsilon_2)})^{(\vec{k},\vec{k})}e^{2d\beta(\vec{k},\vec{a})}}{\prod_{\vec{\alpha}\in\Delta}l^{\Diamond,\vec{k}}_{\vec{\alpha}}(\epsilon_1,\epsilon_2,\vec{a};\beta)}\times$$
$$Z(2\epsilon_1,\epsilon_2-\epsilon_1,\vec{a}+2\epsilon_1\vec{k};\mathfrak{q}e^{2d\beta\epsilon_1},\beta)Z(\epsilon_1-\epsilon_2,2\epsilon_2,\vec{a}+2\epsilon_2\vec{k};\mathfrak{q}e^{2d\beta\epsilon_2},\beta),\quad(8.2.4)$$

where

$$l^{\Diamond,\vec{k}}_{\vec{\alpha}}(\epsilon_1,\epsilon_2,\vec{a};\beta)=\begin{cases}\prod\limits_{\substack{i,j\geq0,i+j\leq-2\langle\vec{k},\vec{\alpha}\rangle-2\\ i+j\equiv-2\langle\vec{k},\vec{\alpha}\rangle-2\ (\mathrm{mod}\ 2)}}(1-e^{\beta(i\epsilon_1+j\epsilon_2-\langle\vec{a},\vec{\alpha}\rangle)}) & \text{if } \langle\vec{k},\vec{\alpha}\rangle<0,\\ \prod\limits_{\substack{i,j\geq0,i+j\leq2\langle\vec{k},\vec{\alpha}\rangle-2\\ i+j\equiv2\langle\vec{k},\vec{\alpha}\rangle-2\ (\mathrm{mod}\ 2)}}\left(1-e^{\beta(-(i+1)\epsilon_1-(j+1)\epsilon_2-\langle\vec{a},\vec{\alpha}\rangle)}\right) & \text{if } \langle\vec{k},\vec{\alpha}\rangle>0,\\ 1 & \text{otherwise.}\end{cases}\quad(8.2.5)$$

8.3 K-theoretic $\mathbb{Z}_2$ blowup equations

By explicit tests on the relation between 5d resolved and orbifold $SU(N)$ parition functions defined in the last two sections for $N=2,3,4,5,6$ to high orders of $\mathfrak{q}$, we find the following 5d $\mathbb{Z}_2$ blowup equations:

- $$Z_{k,0}=(\mathfrak{q}\beta^{2N})^{\frac{k(N-k)}{4N}}Z^{\diamond,k},\qquad k=0,1,2,\ldots,N-1.\quad(8.3.1)$$

- $$Z_{0,1}=Z^{\diamond,0}-(\mathfrak{q}\beta^{2N})^{1/2}e^{\beta(\epsilon_1+\epsilon_2)}Z^{\diamond,2}.\quad(8.3.2)$$
$$Z_{0,-1}=Z^{\diamond,0}-(\mathfrak{q}\beta^{2N})^{1/2}Z^{\diamond,2}.\quad(8.3.3)$$

- $$Z_{0,1/2}=Z_{0,-1/2}.\quad(8.3.4)$$

A few comments are in order. In the case of no insertion, i.e. $d=0$, the 5d $\mathbb{Z}_2$ blowup equations (8.3.1) resemble very much to Nakajima-Yoshioka's blowup equations (2.3.2). When there is nontrivial insertion, for example $d=1$, new phenomenon appears. The resolved partition function $Z_{0,1}$ involves the sum of orbifold partition functions at different c. In particular, equations (8.3.2) and (8.3.3) result in the following identitiy:

$$\frac{1}{1-e^{\beta(\epsilon_1+\epsilon_2)}}Z_{0,1}+\frac{1}{1-e^{-\beta(\epsilon_1+\epsilon_2)}}Z_{0,-1}=Z^{\diamond,0}\quad(8.3.5)$$

We also tested other k values with nonzero d, but unfortunely we did not find a good counterpart in the orbifold side.

In the case of $SU(2)$, the two independent orbifold partition functions $Z^{\diamond,0}$ and $Z^{\diamond,1}$ are also conveniently called even and odd partition functions, $Z^{\diamond,\mathrm{even}}$ and $Z^{\diamond,\mathrm{odd}}$. Due to the coincidence of $Z^{\diamond,0}=Z_{0,0}=Z^{\diamond,2}$, we obtain the following special cases of 5d $SU(2)$ $\mathbb{Z}_2$ blowup equations.

- $$Z_{0,0}(\epsilon_1,\epsilon_2,\vec{a},\mathfrak{q};\beta)=Z^{\diamond,\mathrm{even}}(\epsilon_1,\epsilon_2,\vec{a},\mathfrak{q};\beta).\quad(8.3.6)$$

- $$Z_{0,1}(\epsilon_1,\epsilon_2,\vec{a},\mathfrak{q};\beta)=\left(1-e^{\beta(\epsilon_1+\epsilon_2)}\mathfrak{q}^{1/2}\right)Z^{\diamond,\mathrm{even}}(\epsilon_1,\epsilon_2,\vec{a},\mathfrak{q};\beta). \tag{8.3.7}$$

- $$Z_{0,-1}(\epsilon_1,\epsilon_2,\vec{a};\mathfrak{q},\beta)=\left(1-\mathfrak{q}^{1/2}\right)Z^{\diamond,\mathrm{even}}(\epsilon_1,\epsilon_2,\vec{a},\mathfrak{q};\beta). \tag{8.3.8}$$

- $$Z_{1,0}(\epsilon_1,\epsilon_2,\vec{a};\mathfrak{q},\beta)=\mathfrak{q}^{1/8}Z^{\diamond,\mathrm{odd}}(\epsilon_1,\epsilon_2,\vec{a};\mathfrak{q},\beta). \tag{8.3.9}$$

Besides, we also checked the following relation only involving resolved partition function:

$$Z_{1,1/2}(\epsilon_1,\epsilon_2,\vec{a};\mathfrak{q},\beta)=Z_{1,-1/2}(\epsilon_1,\epsilon_2,\vec{a};\mathfrak{q},\beta). \tag{8.3.10}$$

These blowup equations can be used to derive some conjectural functional equations proposed in (Bershtein and Shchechkin, 2017; Bershtein, Gavrylenko, and Marshakov, 2018). Besides, we find if adding a factor $(-1)^{2k_1}$ in $Z_{1,0}$, one obtains an intriguing vanishing blowup equation

$$\begin{aligned}0=\sum_{\vec{k}\in\frac{1}{4}+\mathbb{Z}/2}&\frac{(-1)^{2k_1}(\mathfrak{q}\beta^4)^{(\vec{k},\vec{k})}}{\prod_{\vec{\alpha}\in\Delta}l_{\vec{\alpha}}^{\diamondsuit,\vec{k}}(\epsilon_1,\epsilon_2,\vec{a};\beta)}\times\\&Z(2\epsilon_1,\epsilon_2-\epsilon_1,\vec{a}+2\epsilon_1\vec{k};\mathfrak{q},\beta)Z(\epsilon_1-\epsilon_2,2\epsilon_2,\vec{a}+2\epsilon_2\vec{k};\mathfrak{q},\beta).\end{aligned} \tag{8.3.11}$$

In fact, this vanishing equation is nontrivial albeit the innocent look.

8.4 $\mathbb{Z}_2$ blowup equations and bilinear relations

8.4.1 Bershtein-Shchenkin's conjectures

We briefly review the results on the Tau functions and bilinear relations of q-Painlevé III_3 equation in (Bershtein and Shchechkin, 2017). The q-Painlevé III_3 equation was given in (Grammaticos and Ramani, 2016) as the following difference equation

$$G(qZ)G(q^{-1}Z)=\left(\frac{G(Z)-Z}{G(Z)-1}\right)^2, \tag{8.4.1}$$

It goes back to the differential Painlevé III_3 equation in the limit $q\to 1$. The Tau function $\mathcal{T}(u,s;q|Z)$ of the system is expected to satisfy the following bilinear relations

$$\begin{aligned}Z^{1/4}\mathcal{T}(u,s;q|qZ)\mathcal{T}(u,s;q|q^{-1}Z)&=\mathcal{T}(u,s;q|Z)^2+Z^{1/2}\mathcal{T}(uq,s;q|Z)\mathcal{T}(uq^{-1},s;q|Z),\\ \mathcal{T}(uq^2,s;q|Z)&=s^{-1}\mathcal{T}(u,s;q|Z),\end{aligned} \tag{8.4.2}$$

where u,s are some auxiliary parameters to define the system.

Remarkably, (Bershtein and Shchechkin, 2017) proposed an exact formula for the Tau function of q-Painlevé III_3 equation from the 5d $SU(2)$ Nekrasov partition function in the spirit of (Gamayun, Iorgov, and Lisovyy, 2013):

$$\mathcal{T}(u,s;q|Z)=\sum_{n\in\mathbb{Z}}s^nC(uq^{2n};q|Z)\frac{\mathcal{F}(uq^{2n};q^{-1},q|Z)}{(uq^{2n+1};q,q)_\infty(u^{-1}q^{-2n+1};q,q)_\infty} \tag{8.4.3}$$

where function $C(u;q|Z)$ satisfy equations

$$\frac{C(uq;q|Z)C(uq^{-1};q|Z)}{C(u;q|Z)^2} = -Z^{1/2} \tag{8.4.4}$$

$$\frac{C(uq;q|qZ)C(uq^{-1};q|q^{-1}Z)}{C(u;q|Z)^2} = -uZ^{1/4} \tag{8.4.5}$$

$$\frac{C(u;q|qZ)C(u;q|q^{-1}Z)}{C(u;q|Z)^2} = Z^{-1/4}, \tag{8.4.6}$$

and

$$(Z,t_1,t_2)_\infty := \exp\left(-\sum_{m=1}^{\infty}\frac{Z^m}{m(1-t_1^m)(1-t_2^m)}\right) = \mathrm{PE}\left(-\frac{Z}{(1-t_1)(1-t_2)}\right), \quad |Z|<1. \tag{8.4.7}$$

The Nekrasov partition function here is defined as

$$\mathcal{F}(u_1,u_2;q_1,q_2|Z) = \sum_{\lambda_1,\lambda_2} Z^{|\lambda_1|+|\lambda_2|}\frac{1}{\prod_{i,j=1}^2 N_{\lambda_i,\lambda_j}(u_i/u_j;q_1,q_2)}, \tag{8.4.8}$$

where

$$N_{\lambda,\mu}(u,q_1,q_2) = \prod_{s\in\lambda}(1-uq_2^{-a_\mu(s)-1}q_1^{\ell_\lambda(s)})\cdot\prod_{s\in\mu}(1-uq_2^{a_\lambda(s)}q_1^{-\ell_\mu(s)-1}). \tag{8.4.9}$$

Clearly, the bilinear relation (8.4.2) should put some constraints on the Nekrasov partition function. It turns out such constraints are related to the $\mathbb{C}^2/\mathbb{Z}_2$ blowup equations rather than $\mathbb{C}^2$ blowup equations. We state the conjecture in Appendix B of (Bershtein and Shchechkin, 2017) here. Assuming $|q_2|<1<|q_1|$, (Bershtein and Shchechkin, 2017) introduced the following notations:

$$\widehat{\mathcal{F}}_d(u,q_1,q_2|Z) = \sum_{2n\in\mathbb{Z}}\left(\frac{u^{2dn}(q_1q_2)^{4dn^2}Z^{2n^2}}{(uq_1^{4n-2},u^{-1}q_1^{-4n-2})^{(1)}_\infty(uq_1^{-1}q_2^{4n+1},u^{-1}q_1^{-1}q_2^{-4n+1})^{(2)}_\infty}\mathcal{F}_n^{(1)}(q_1^{2d}Z)\,\mathcal{F}_n^{(2)}(q_2^{2d}Z)\right), \tag{8.4.10}$$

where

$$\mathcal{F}_n^{(1)}(z) = \mathcal{F}(uq_1^{4n},q_1^2,q_1^{-1}q_2^1|z), \qquad \mathcal{F}_n^{(2)}(z) = \mathcal{F}(uq_2^{4n},q_1^1q_2^{-1},q_2^2|z),$$

$$(uq_1^{4n-2},u^{-1}q_1^{-4n-2})^{(1)}_\infty = (uq_1^{4n-2};q_1^{-2},q_1^{-1}q_2)_\infty(u^{-1}q_1^{-4n-2};q_1^{-2},q_1^{-1}q_2)_\infty,$$

$$(uq_1^{-1}q_2^{4n+1},u^{-1}q_1^{-1}q_2^{-4n+1})^{(2)}_\infty = (uq_1^{-1}q_2^{4n+1};q_1^{-1}q_2,q_2^2)_\infty(u^{-1}q_1^{-1}q_2^{-4n+1};q_1^{-1}q_2,q_2^2)_\infty.$$

They also introduced a "mod 2" version of Nekrasov partition function

$$\mathcal{F}_\Diamond(u,q_1,q_2|Z) = \mathcal{F}_{\Diamond,1-\mathrm{loop}}(u,q_1,q_2)\sum_{\lambda_1,\lambda_2} Z^{\frac{|\lambda_1|+|\lambda_2|}{2}}\frac{1}{\prod_{i,j=1}^2 N^{\Diamond}_{\lambda_i,\lambda_j}(q_1,q_2,u_i/u_j)}, \tag{8.4.11}$$

where

$$\mathcal{F}_{\diamondsuit,1-\mathrm{loop}}(u,q_1,q_2) =$$
$$\frac{1}{(uq_1^{-2};q_1^{-2},q_2^2)_\infty(uq_1^{-1}q_2;q_1^{-2},q_2^2)_\infty(u^{-1}q_1^{-2};q_1^{-2},q_2^2)_\infty(u^{-1}q_1^{-1}q_2;q_1^{-2},q_2^2)_\infty},$$
$$N^{\diamondsuit}_{\lambda,\mu}(q_1,q_2,u) = \prod_{s\in\lambda^\diamondsuit}(1-uq_2^{-a_\mu(s)-1}q_1^{\ell_\lambda(s)})\cdot\prod_{s\in\mu^\diamondsuit}(1-uq_2^{a_\lambda(s)}q_1^{-\ell_\mu(s)-1}). \tag{8.4.12}$$

with $\lambda^\diamondsuit = \{s\in\lambda|a_\mu(s)+\ell_\lambda(s)+1\equiv 0 \bmod 2\}$, $\mu^\diamondsuit = \{s\in\mu|a_\lambda(s)+\ell_\mu(s)+1\equiv 0 \bmod 2\}$.

Conjecture 4 (Bershtein-Shchenkin)**.**

$$(1-q_2q_1Z^{1/2})\mathcal{F}_\diamondsuit(u,q_1,q_2|Z) = \widehat{\mathcal{F}}_1(u,q_1,q_2|Z), \qquad \mathcal{F}_\diamondsuit(u,q_1,q_2|Z) = \widehat{\mathcal{F}}_0(u,q_1,q_2|Z). \tag{8.4.13}$$

Proof: Denote $L_n = (uq_1^{4n-2},u^{-1}q_1^{-4n-2})^{(1)}_\infty(uq_1^{-1}q_2^{4n+1},u^{-1}q_1^{-1}q_2^{-4n+1})^{(2)}_\infty$. We first show the poly-Pochammer symbols in (8.4.10) and (8.4.11) are equal when $n=0$, i.e. $L_0\mathcal{F}_{\diamondsuit,1-\mathrm{loop}} = 1$. In fact,

$$\begin{aligned}(uq_1^{-2};q_1^{-2},q_1^{-1}q_2)_\infty(uq_1^{-1}q_2;q_1^{-1}q_2,q_2^2)_\infty &= (uq_1^{-2};q_1^{-2},q_2^2)_\infty(uq_1^{-1}q_2;q_1^{-2},q_2^2)_\infty,\\ (u^{-1}q_1^{-2};q_1^{-2},q_1^{-1}q_2)_\infty(u^{-1}q_1^{-1}q_2;q_1^{-1}q_2,q_2^2)_\infty &= (u^{-1}q_1^{-2};q_1^{-2},q_2^2)_\infty(u^{-1}q_1^{-1}q_2;q_1^{-2},q_2^2)_\infty.\end{aligned} \tag{8.4.14}$$

These identities are easy to prove from formula (8.4.7). For example, in the first identity of (8.4.14), assuming $|u|<1$, we have

$$\begin{aligned}&(uq_1^{-2};q_1^{-2},q_1^{-1}q_2)_\infty(uq_1^{-1}q_2;q_1^{-1}q_2,q_2^2)_\infty\\ &=\mathrm{PE}\Big(-\frac{uq_1^{-2}}{(1-q_1^{-2})(1-q_1^{-1}q_2)}-\frac{uq_1^{-1}q_2}{(1-q_1^{-1}q_2)(1-q_2^2)}\Big)\\ &=\mathrm{PE}\Big(-\frac{uq_1^{-2}(1-q_2^2)+uq_1^{-1}q_2(1-q_1^{-2})}{(1-q_1^{-2})(1-q_1^{-1}q_2)(1-q_2^2)}\Big)\\ &=\mathrm{PE}\Big(-\frac{u(q_1^{-2}+q_1^{-1}q_2)}{(1-q_1^{-2})(1-q_2^2)}\Big)\end{aligned} \tag{8.4.15}$$

while

$$(uq_1^{-2};q_1^{-2},q_2^2)_\infty(uq_1^{-1}q_2;q_1^{-2},q_2^2)_\infty = \mathrm{PE}\Big(-\frac{u(q_1^{-2}+q_1^{-1}q_2)}{(1-q_1^{-2})(1-q_2^2)}\Big) \tag{8.4.16}$$

The identities (8.4.14) guarantee the zero order of (8.4.13).

Using the properties

$$\frac{(Z,t_1,t_2)_\infty}{(Zt_1,t_1,t_2)_\infty} = (Z,t_2)_\infty, \qquad \frac{(Z,t)_\infty}{(Zt,t)_\infty} = 1-Z, \tag{8.4.17}$$

and

$$\frac{(Z,t)_\infty}{(Zt^n,t)_\infty} = (1-Z)(1-Zt)\cdots(1-Zt^{n-1}) = \prod_{i=0}^{n-1}(1-Zt^i), \tag{8.4.18}$$

we find

$$
\begin{aligned}
\frac{L_{n+1/2}}{L_n} &= \frac{(uq_1^{4n};q_1^{-2},q_1^{-1}q_2)_\infty}{(uq_1^{4n-2};q_1^{-2},q_1^{-1}q_2)_\infty}\frac{(uq_1^{-1}q_2^{4n+3};q_1^{-1}q_2,q_2^2)_\infty}{(uq_1^{-1}q_2^{4n+1};q_1^{-1}q_2,q_2^2)_\infty}\\
&\quad\times\frac{(u^{-1}q_1^{-4n-4};q_1^{-2},q_1^{-1}q_2)_\infty}{(u^{-1}q_1^{-4n-2};q_1^{-2},q_1^{-1}q_2)_\infty}\frac{(u^{-1}q_1^{-1}q_2^{-4n-1};q_1^{-1}q_2,q_2^2)_\infty}{(u^{-1}q_1^{-1}q_2^{-4n+1};q_1^{-1}q_2,q_2^2)_\infty}\\
&=\frac{(uq_1^{4n};q_1^{-1}q_2)_\infty}{(uq_1^{-1}q_2^{4n+1};q_1^{-1}q_2)_\infty}\frac{(u^{-1}q_1^{-1}q_2^{-4n-1};q_1^{-1}q_2)_\infty}{(u^{-1}q_1^{-4n-2};q_1^{-1}q_2)_\infty}\\
&=\begin{cases}\prod_{i,j\geq0,\,i+j=4n-2}(1-uq_1^iq_2^j)(1-u^{-1}q_1^{-1-i}q_2^{-1-j}), & n>0,\\ \prod_{i,j\geq0,\,i+j=-4n-2}(1-uq_1^{-1-i}q_2^{-1-j})^{-1}(1-u^{-1}q_1^iq_2^j)^{-1}, & n<0.\end{cases}
\end{aligned}
\tag{8.4.19}
$$

Therefore,

$$
\frac{L_n}{L_0}=\begin{cases}\prod_{\substack{i,j\geq0,\,i+j\leq4n-2\\ i+j\equiv4n-2 \pmod 2}}(1-uq_1^iq_2^j)(1-u^{-1}q_1^{-1-i}q_2^{-1-j}), & n>0,\\ \prod_{\substack{i,j\geq0,\,i+j\leq-4n-2\\ i+j\equiv-4n-2 \pmod 2}}(1-uq_1^{-1-i}q_2^{-1-j})(1-u^{-1}q_1^iq_2^j), & n<0.\end{cases}
\tag{8.4.20}
$$

This is exactly the product $\prod_{\vec{\alpha}\in\Delta} l_{\vec{\alpha}}^{\lozenge,\vec{k}}(\epsilon_1,\epsilon_2,\vec{a};\boldsymbol{\beta}=1)$ with $u=e^{-2a}$ and $n=k_1$. Therefore, the Conjecture 4 is the result of $\mathbb{Z}_2$ blowup equations (8.3.6) and (8.3.7). This derivation is also recently obtained in (Shchechkin, 2020).

8.4.2 Bershtein-Gavrylenko-Marshakov's conjectures

The Tau function and bilinear relations of q-Painlevé III_3 equation in (Bershtein and Shchechkin, 2017) are related to the self-dual limit of Nekrasov paritition functio, i.e. $\epsilon_1+\epsilon_2\to0$ or $q_1q_2\to1$. To extend the full refined level that is generic ϵ_1,ϵ_2, (Bershtein, Gavrylenko, and Marshakov, 2018) introduced the *quantum q-Painlevé equations*. By studying the Tau functions and bilinear relations of quantum q-Painlevé III_3 equation, and expressing such Tau functions by Nekrasov partition function, they obtain more functional equations for 5d $SU(2)$ Nekrasov partition functions. Let us state their conjectures here. First, (Bershtein, Gavrylenko, and Marshakov, 2018) introduced the following notations.

$$
\begin{gathered}
\mathsf{F}(u;q_1,q_2|Z)=c_q(u|Z)\mathrm{F}(u;q_1,q_2|Z),\qquad \mathrm{F}(u;q_1,q_2|Z)=C_q(u;q_1,q_2)\mathcal{F}(u;q_1,q_2|Z),\\
\mathsf{F}^{(1)}(u|Z)=\mathsf{F}(u;q_1^2,q_1^{-1}q_2|Z),\quad \mathsf{F}^{(2)}(u|Z)=\mathsf{F}(u;q_1q_2^{-1},q_2^2|Z),\\
C_q(u;q_1,q_2)=(u;q_1,q_2)_\infty(u^{-1};q_1,q_2)_\infty,\quad c_q(u|Z)=\exp\left(\frac{-\log Z\,(\log u)^2}{4\log q_1\log q_2}\right).
\end{gathered}
\tag{8.4.21}
$$

Here $\mathcal{F}(u;q_1,q_2|Z)$ is still defined as in (8.4.8) with $u_1=u=-u_2$.

Conjecture 5 (Bershtein-Gavrylenko-Marshakov).

$$
\sum_{n\in\mathbb{Z}+\frac14}\left(Z^{2n^2}\mathrm{F}^{(1)}(uq_1^{4n}|Z)\mathrm{F}^{(2)}(uq_2^{4n}|Z)\right)=\sum_{n\in\mathbb{Z}-\frac14}\left(Z^{2n^2}\mathrm{F}^{(1)}(uq_1^{4n}|Z)\mathrm{F}^{(2)}(uq_2^{4n}|Z)\right),
\tag{8.4.22}
$$

$$\sum_{n\in\mathbb{Z}\pm\frac{1}{4}}\left(u^n(q_1q_2)^{2n^2}Z^{2n^2}\mathrm{F}^{(1)}(uq_1^{4n}|q_1Z)\mathrm{F}^{(2)}(uq_2^{4n}|q_2Z)\right)=$$

$$\sum_{n\in\mathbb{Z}\mp\frac{1}{4}}\left(u^{-n}(q_1q_2)^{-2n^2}Z^{2n^2}\mathrm{F}^{(1)}(uq_1^{4n}|q_1^{-1}Z)\mathrm{F}^{(2)}(uq_2^{4n}|q_2^{-1}Z)\right), \tag{8.4.23}$$

$$\sum_{2n\in\mathbb{Z}}\left(u^n(q_1q_2)^{2n^2}Z^{2n^2}\mathrm{F}^{(1)}(uq_1^{4n}|q_1Z)\,\mathrm{F}^{(2)}(uq_2^{4n}|q_2Z)\right)=$$

$$\sum_{2n\in\mathbb{Z}}\left(u^{-n}(q_1q_2)^{-2n^2}Z^{2n^2}\mathrm{F}^{(1)}(uq_1^{4n}|q_1^{-1}Z)\,\mathrm{F}^{(2)}(uq_2^{4n}|q_2^{-1}Z)\right) \tag{8.4.24}$$

$$\sum_{2n\in\mathbb{Z}}\left(u^{2n}(q_1q_2)^{4n^2}Z^{2n^2}\mathrm{F}^{(1)}(uq_1^{4n}|q_1^2Z)\,\mathrm{F}^{(2)}(uq_2^{4n}|q_2^2Z)\right)=$$

$$=(1-q_1q_2Z)\sum_{2n\in\mathbb{Z}}\left(Z^{2n^2}\mathrm{F}^{(1)}(uq_1^{4n}|Z)\,\mathrm{F}^{(2)}(uq_2^{4n}|Z)\right) \tag{8.4.25}$$

Proof: Using the formula

$$(Z;t_1^{-1},t_2)_\infty=(Zt_1;t_1,t_2)_\infty^{-1}, \tag{8.4.26}$$

we have

$$\begin{aligned}
C(uq_1^{4n};q_1^2,q_1^{-1}q_2)&=(uq_1^{4n};q_1^2,q_1^{-1}q_2)_\infty(u^{-1}q_1^{-4n};q_1^2,q_1^{-1}q_2)_\infty\\
&=(uq_1^{4n-2};q_1^2,q_1^{-1}q_2)_\infty^{-1}(u^{-1}q_1^{-4n-2};q_1^2,q_1^{-1}q_2)_\infty^{-1},\\
C(uq_2^{4n};q_1q_2^{-1},q_2^2)&=(uq_2^{4n};q_1q_2^{-1},q_2^2)_\infty(u^{-1}q_2^{-4n};q_1q_2^{-1},q_2^2)_\infty\\
&=(uq_1^{-1}q_2^{4n+1};q_1q_2^{-1},q_2^2)_\infty^{-1}(u^{-1}q_1^{-1}q_2^{-4n+1};q_1q_2^{-1},q_2^2)_\infty^{-1}.
\end{aligned} \tag{8.4.27}$$

This shows the two summations in the last relation (8.4.25) are exactly given by the $\widehat{\mathcal{F}}_d$ in (8.4.10). Due to our proof for conjecture 4, the last relation (8.4.25) holds.

For the other three conjectural relations, the factors (8.4.27) are the same. Use our previous notation $L_n=C(uq_1^{4n};q_1^2,q_1^{-1}q_2)^{-1}C(uq_2^{4n};q_1q_2^{-1},q_2^2)^{-1}$. It is obvious that

$$L_{-n}(u)=L_n(u^{-1}). \tag{8.4.28}$$

Then the first conjectural relation is equivalent to (8.3.11). The second conjectural relation is equivalent to (8.3.10), while the third conjectural relation is actually equivalent to (8.3.4).

A more general conjecture for $SU(N)$ gauge theory with Chern-Simons level k was proposed in (Bershtein, Gavrylenko, and Marshakov, 2019). They define the total Nekrasov partition function as

$$Z^{N,k}(\vec{u};q_1,q_2|z)=Z_{\mathrm{cl}}^{N,k}(\vec{u};q_1,q_2|z)\cdot Z_{1-\mathrm{loop}}^{N}(\vec{u};q_1,q_2)\cdot Z_{\mathrm{inst}}^{N,k}(\vec{u};q_1,q_2|z)\,, \tag{8.4.29}$$

where

$$\begin{aligned}
Z_{\text{cl}}^{N,k}(\vec{u};q_1,q_2|z) &= \exp\left(\log z \frac{\sum(\log u_i)^2}{-2\log q_1 \log q_2} + k\frac{\sum(\log u_i)^3}{-6\log q_1\log q_2}\right),\\
Z_{1-\text{loop}}^{N}(\vec{u};q_1,q_2) &= \prod_{1\le i\neq j\le N}(u_i/u_j;q_1,q_2)_\infty\,,\\
Z_{\text{inst}}^{N,k}(\vec{u};q_1,q_2|z) &= \sum_{\vec{\lambda}}\frac{z^{|\vec{\lambda}|}\prod_{i=1}^N(\mathsf{T}_{\lambda^{(i)}}(u;q_1,q_2))^k}{\prod_{i,j=1}^N \mathsf{N}_{\lambda^{(i)},\lambda^{(j)}}(u_i/u_j;q_1,q_2)}\,,\\
\vec{\lambda} &= (\lambda^{(1)},\dots,\lambda^{(N)}),\ \ |\vec{\lambda}|=\sum|\lambda^{(i)}|,\ \ |\lambda|=\sum\lambda_j,\\
\mathsf{N}_{\lambda,\mu}(u,q_1,q_2) &= \prod_{s\in\lambda}(1-uq_2^{-a_\mu(s)-1}q_1^{\ell_\lambda(s)})\cdot\prod_{s\in\mu}(1-uq_2^{a_\lambda(s)}q_1^{-\ell_\mu(s)-1})\,,\\
\mathsf{T}_\lambda(u;q_1,q_2) &= u^{|\lambda|}q_1^{\frac{1}{2}(\|\lambda'\|-|\lambda'|)}q_2^{\frac{1}{2}(\|\lambda\|-|\lambda|)} = \prod_{(i,j)\in\lambda}uq_1^{i-1}q_2^{j-1},\quad \|\lambda\|=\sum\lambda_j^2\,.
\end{aligned}\tag{8.4.30}$$

Define the Fourier transformed Nekrasov functions by

$$\mathcal{T}_j^{N,k}(\vec{u},\vec{s};q|z) = \sum_{\vec{\Lambda}\in Q_{N-1}+\omega_j} s^\Lambda Z^{N,k}(\vec{u}q^{\vec{\Lambda}};q^{-1},q|z),\qquad j\in\mathbb{Z}/N\mathbb{Z}\,.\tag{8.4.31}$$

Conjecture 6. The functions (8.4.31) satisfy the bilinear relations

$$\mathcal{T}_j^{N,k}(qz)\mathcal{T}_j^{N,k}(q^{-1}z) = \mathcal{T}_j^{N,k}(z)^2 - z^{1/N}\mathcal{T}_{j+1}^{N,k}(q^{k/N}z)\mathcal{T}_{j-1}^{N,k}(q^{-k/N}z)\,.\tag{8.4.32}$$

Proof: Let us focus on the $k=0$ and $j=0$ case. Clearly, by a completely parallel proof as in the previous $SU(2)$ case,

$$\mathcal{T}_0^{N,0}(qz)\mathcal{T}_0^{N,0}(q^{-1}z) = \mathcal{T}_0^{N,0}(z)^2 - z^{1/N}\mathcal{T}_1^{N,0}(z)\mathcal{T}_{-1}^{N,0}(z)\tag{8.4.33}$$

is the direct consequence of combining equations (8.3.1) and (8.3.2) when taking the self-dual limit $\epsilon_1+\epsilon_2=0$.

For $j\neq 0$, it seems the functional equations of Nekrasov partition function given in Conjectrue 6 can not be derived from $\mathbb{Z}_2$ blowup equations. It would be interesting to further explore the origin of those functional equations.

Chapter 9

Summary and Future Directions

In this thesis, we begin from the K-theoretic blowup equations in (Nakajima and Yoshioka, 2005b; Gottsche, Nakajima, and Yoshioka, 2009a; Nakajima and Yoshioka, 2011) and study two kinds of generalizations, one is for the refined topological string theory on arbitrary local Calabi-Yau threefolds inspired by geometric engineering, the other is the elliptic blowup equations for arbitrary 6d $(1,0)$ SCFTs in the atomic classification (Heckman et al., 2015). The major results can be summarized into following three equations:

- blowup equations for refined topological strings (4.0.2),
- elliptic blowup equations for rank one 6d SCFTs (5.2.1),
- elliptic blowup equations for arbitrary rank 6d SCFTs (6.1.5).

The second and third equations are the natural elliptic lift of K-theoretic blowup equations for 5d $\mathcal{N}=1$ (quiver) gauge theories, which are also the first equation specializing to local elliptic Calabi-Yau threefolds.

The study on blowup equations always involves two steps: first one establishes the validity of blowup equations by demonstrating that the partition functions or equivalently the elliptic genera satisfy these equations; in the second step one has to develop efficient procedures to solve the blowup equations. The first step has been very successful for all kinds of theories including refined topological strings on lots of toric Calabi-Yau threefolds including resolved conifold, local $\mathbb{P}^2$, local $\mathbb{P}^1\times\mathbb{P}^1$, resolved $\mathbb{C}^3/\mathbb{Z}_5$, $X_{N,m}$ geometries and tons of local elliptic Calabi-Yau – in general non-toric – associated to all 6d $(1,0)$ SCFTs in the atomic classification, for instance local half K3 associated to E-string theory. In the elliptic cases, we are able to write the blowup equations in gauge language, use the quantities of 6d SCFTs and obtain elegant elliptic blowup equations. In particular, we studied extensively all the rank one theories which are labeled by an integer n, a gauge symmetry G, a flavor symmetry F and matter in representation $\mathfrak{R}$ in Chapter 5. We further studied in Chapter 6 the higher-rank 6d SCFTs and give the gluing rules which make it easy to write down all admissible blowup equations for any 6d $(1,0)$ SCFTs in the atomic classification. In particular, we explicitly present the elliptic blowup equations for E–, M–string chains, three higher rank non-Higgsable clusters, ADE chains of -2 curves, conformal matter theories and the blown-ups of $(-n)$-curves. We checked the blowup equations for lots of Calabi-Yau geometries by known techniques such as refined topological vertex, holomorphic anomaly equations, and lots of 6d theories using the elliptic genera from 2d quiver gauge theories and tested lots of leading degree theta identities from vanishing blowup equations.

For the second step, we have developed four efficient techniques to extract information from blowup equations, which are the ϵ_1, ϵ_2 expansion, refined BPS expansions, recursion formulas and Weyl orbit expansions. Each method typically requires different inputs. The first two methods are for general local Calabi threefolds, while the last two methods are specially for those with gauge theories correspondences such as the 5d $\mathcal{N} = 1$ gauge theories and 6d $(1,0)$ SCFTs. We found the unity blowup equations are much more constraining than vanishing blowup equations, and for all the examples we have studied, as long as there exist unity blowup equation, the refined instanton partition function can be fully determined.

We divide all 6d $(1,0)$ SCFTs theories into three classes.

- class **A**: rank one with $n \geq 3$, without unpaired half hypermultiplet,
- class **B**: rank one with $n = 1,2$ or higher rank, without unpaired half hypermultiplet,
- class **C**: with unpaired half hypermultiplet.

We find that for classes **A** and **B**, there always exist unity blowup equations and possibly also vanishing blowup equations, while for class **C**, there only exist vanishing blowup equations. This has the following implications for the solvability of the elliptic genera from the blowup equations: For class **A**, we obtain a recursion formula that determines the elliptic genera completely, i.e. for arbitrary numbers of strings from the unity blowup equations, which is the ideal situation. For class **B**, we can solve the elliptic genera and the refined BPS invariants order by order from the Weyl orbit expansion, the refined BPS expansion or the ϵ_1, ϵ_2 expansion. For class **C**, we do not have a universal description how to solve elliptic genera from vanishing blowup equations.[1] Fortunately, for the rank one theories which are the most interesting, classes **A** and **B** make up the most of them, while class **C** only contains the remaining 12 theories.

Using the elliptic blowup equations and the solving technique we developed, we explicitly compute the one and two-string elliptic genera for lots of rank-one theories in class **A** and **B**, which recover all previous partial results from refined topological string, modular bootstrap, Hilbert series from monopole formulas, 5d partition functions, 2d quiver gauge theories and the β-twisted partition function of $\mathcal{N} = 2$ superconformal H_G theories. Most of our results on elliptic genera are new and our methods work for arbitrary number of strings and arbitrary gauge and flavor fugacities. The elliptic genera we solved out from blowup equations could be useful in many aspects. For example, they would help to identify the 2d quiver description of the 6d minimal SCFT with exceptional gauge symmetry, see some attempts for $G = E_7$ in (Kim et al., 2018). They also serve as the calibration to determine modular ansatz for higher-string elliptic genus and the web of topological vertex for the associated non-toric Calabi-Yau threefolds.

We also studied many important properties of blowup equations, such as the modularity associated to the monodromy group of mirror curves. In particular, we have shown how the blowup equations for refined topological strings are nontrivially consistent with refined holomorphic/modular anomaly equations. We also proposed the non-holomorphic version and non-perturbative version of blowup equations. For elliptic blowup equations, things become more rigid. We have proved a

[1] Numerical study on $n = 7, G = E_7$ theory, a typical class **C** theory shows that around half of the refined BPS invariants can be determined from the seven vanishing blowup equations.

stronger version of modularity under $SL(2,\mathbb{Z})$, which is a strong support that they hold for arbitrary number of strings. This is closely related to the anomaly cancellation, which is new 6d phenomenon compared to 5d.

There are still many open questions concerning blowup equations. We list some in the following.

- The most important and imminent question perhaps is how to rigorously prove the generalized blowup equations for local Calabi-Yau threefolds. Since the refined BPS invariants for non-compact Calabi-Yau threefolds have been rigorously defined (Choi, Katz, and Klemm, 2014; Nekrasov and Okounkov, 2014; Maulik and Toda, 2016), these(4.0.2) functional equations for the partition functions are indeed well-formulated mathematical conjectures. The question actually contains two parts:

  1, how to prove the refined partition function satisfies blowup equations?
  2, how to prove the refined partition function can be fully determined by blowup equations?

  The proof of Göttsche-Nakajima-Yoshioka's K-theoretic blowup equations (Nakajima and Yoshioka, 2005b; Gottsche, Nakajima, and Yoshioka, 2009a; Nakajima and Yoshioka, 2011) relies deeply on the structure of gauge theories, which may not be exactly suitable for general Calabi-Yau setting, as the latter does not necessarily engineer a gauge theory, for instance local $\mathbb{P}^2$. Although lots of toric Calabi-Yau threefolds can be realized as $X_{N,m}$ geometries and their reductions thus their blowup equations can be seen proved via gauge theories, it would be desirable to give a direct proof based on the Calabi-Yau setting, even just for local $\mathbb{P}^2$. There are some recent interesting developments on the algebraic derivation of Nakajima-Yoshioka's blowup equations using the so called Urod algebra, see for example (Creutzig, 2020). It would be interesting to see whether such algebra approach can be extended to arbitrary toric Calabi-Yau cases. Another idea comes from the recent work (Bousseau et al., 2020) where the traditional topological string partition function is related to NS free energy by relative Gromov-Witten theory. This resembles the behavior of limits of blowup equations as we already seen in Chapter 4.1.2. It is interesting to study whether the surprising relation in (Bousseau et al., 2020) indeed takes root in blowup equations and whether their approach can be used to derive the blowup equations for local $\mathbb{P}^2$. Besides, for local elliptic Calabi-Yau threefolds that engineer 6d $(1,0)$ SCFTs, which are in general non-toric, we give the elliptic form of blowup equations in the gauge theory language. To prove them, one first need a rigorous definition of the elliptic genera of BPS strings of 2d $(0,4)$ SCFTs, which in general is not known so far to our knowledge.

  About the second part of the question, we have shown for the class **A** 6d SCFTs, there exist recursion formulas such that the elliptic genera can be solved exactly. This resembles the Corollary 2.8 of (Nakajima and Yoshioka, 2005b) where 5d Nekrasov partition function can be solved recursively with respect to instanton number. However, we found blowup equations can do more than this. Even for class **B** 6d SCFTs for which there is no recursion formula, the unity blowup equations can still determine elliptic genera in certain ansatz. It would be interesting to further explore to what extent blowup equations can determine partition function when there is no explicit recursion formulas. More generally, one can ask how to rigorously prove blowup equations can solve all BPS invariants in the refined BPS expansion.

- How blowup equations work for open refined topological string theory? Consider in M-theory, one M5-brane wraps the $\mathbb{C}_{z_1} \times S^1$ subspace of 5d Ω background and the Lagrangian 3-cycle of a local Calabi-Yau. The refined partition function of such configuration is called the ϵ_1-brane partition function $\Psi_1(t, x, \epsilon_1, \epsilon_2)$ which contains the contribution from both open and closed M2-branes, see for example (Aganagic et al., 2012). Here x is the open moduli which gives the position of the brane, or equivalently measures the size of the boundary of worldsheet Riemann surface. We expect such brane partition function Ψ_1 to satisfy blowup equations like

$$\Lambda(t, \epsilon_1, \epsilon_2, r) = \sum_{N \in \mathbb{Z}^g} (-)^{|N|} \frac{\Psi_1(t + \epsilon_1 R, x, \epsilon_1, \epsilon_2 - \epsilon_1)\, Z(t + \epsilon_2 R, \epsilon_1 - \epsilon_2, \epsilon_2)}{\Psi_1(t, x, \epsilon_1, \epsilon_2)}. \tag{9.0.1}$$

  This should serve as a generalization of Nekrasov's blowup equations for 4D $\mathcal{N} = 2$ gauge theory with surface defect on the z_1 plane (Nekrasov, 2020). We leave these open blowup equations and their relation with τ functions of isomonodromic systems for future study. One can even consider to incorporate knots and links. The BPS invariants of a knot or link are encoded in the open topological string partition function (Ooguri and Vafa, 2000; Labastida and Marino, 2001b; Labastida and Marino, 2001a; Labastida, Marino, and Vafa, 2000). It is interesting to consider whether blowup equations exist for such circumstances and whether they are able to determine the knot invariants.

- The formalism of blowup equations in (4.0.2) for refined topological string theory is suitable for the region near large radius point in the complex moduli space of local Calabi-Yau. It is interesting to consider do and how blowup equations work near conifold points and orbifold points in the moduli space. In section 7 of (Huang, Sun, and Wang, 2018), some preliminary case study shows that a very similar form of blowup equations as in (4.0.2) can also exist near conifold points and orbifold points. This may not be so surprising since in Chapter 4.4, we have shown the form of blowup equations is nontrivially consistent with refined holomoprhic anomaly equations, and the latter indeed apply to the generic points in the moduli space. It would be interesting to systematically study the hehavior of blowup equations near conifold points and orbifold points. In particular, the region near conifold point correspondes to the strong coupling region in supersymmetric gauge theories, and contact term equations were indeed found in strong coupling long time ago in (Edelstein and Mas, 1999). This also gives an inspiration that the form of blowup equations may hold for a generic point in the moduli space.

- How blowup equations work for little string theories? These are some 6d non-local theories without gravity and can be regarded as the natural further extention of 6d SCFTs. The full classification was achieved in (Bhardwaj et al., 2016). Some typical examples include the affine extension of the ADE chains of (-2)-curves studied in Chapter 6.4. From the viewpoint of geometry, they are engineered by local Calabi-Yau threefolds with a double elliptic fibration.

The partition function of lots of little string theories can be computed by refined topological vertex and 2d quiver gauge theories, see for example (Kim, Kim, and Lee, 2016; Kim and Lee, 2017; Bastian et al., 2018). It would be interesting to see how to modify the current form of elliptic blowup equations to make it work for little strings.

- Can blowup equations work for 6d SCFTs with "frozen singularity"? These 6d SCFTs (Tachikawa, 2016; Bhardwaj et al., 2018) are not covered in the atomic classification, but with new ingredient called $O7_+$-planes. It would be interesting to see whether the partition functions of such SCFTs also satisfy certain blowup equations.

- Can blowup equations work for topological strings on compact Calabi-Yau threefolds? Although it is commonly believed that only local Calabi-Yau threefolds exhibit refined formulation, recently in (Huang, Katz, and Klemm, 2015; Huang, Katz, and Klemm, 2020) it was observed that certain compact elliptic Calabi-Yau threefolds like elliptic $\mathbb{P}^2, \mathbb{F}_n$ may also have a well-defined refinement. See also the recent refined results for $T^6/(\mathbb{Z}_2 \times \mathbb{Z}_2)$ in (Hayashi et al., 2019c). It is interesting to see if the refined partition function of those compact Calabi-Yau satisfies some blowup equations as well and if the blowup equations can determine all the GV invariants. If so, it would be a fascinating progress since no known method can determine the all-genus all-degree invariants of such compact Calabi-Yau threefolds.

- Are the 6d SCFTs with unpaired half-hypermultiplets really impossible to solve by blowup equations? As we see in the main text, the reason the elliptic genera of these 6d SCFTs can not be fully determined is that there exist no unity but only vanishing blowup equations. One possible remedy draws inspiration from the massless E-string theory, which corresponds to a naturally realized elliptic non-compact Calabi-Yau 3-fold with two Kähler parameters. Although the theory itself has only one vanishing blowup equation and is therefore not solvable, once one of the eight possible mass parameters is turned on, there are enough unity blowup equations to allow for a complete solution of the theory. This example suggests that in some cases one may be able to recover the necessary unity blowup equations after deforming the theory with additional parameters. Other possible remedy may be to combine the vanishing blowup equations and modular ansatz together.

- Can one give a direct derivation on the universal one-string elliptic genus formula (5.3.4), even just for the pure gauge cases? A possible proof may be obtained by using the Kac-Weyl character formulas and following the 5d derivation in the Appendix A of (Keller et al., 2012).

- In Chapter 4 we see the compatibility formulas between the two quantization schemes of mirror curves are the NS limit of vanishing blowup equations. However, in Chapter 5 we encounter some 6d gauge theories for which there is no vanishing blowup equations, such as the 6d $(1,0)$ pure gauge E_8 theory. This raises intriguing question that how the NS quantization conditions and GHM conjecture work for 6d quantum curves. There may be new ingredients coming into play.

Furthermore, in Chapter 7 we studied a surprising conjectural relation (Del Zotto et al., 2018) between the elliptic genera of pure gauge 6d $(1,0)$ SCFTs and the Schur indices of 4d $\mathcal{N}=2$ H_G SCFTs, and generalized it from one string elliptic genera to higher strings. In particular, we explicitly compute the Schur indices for lots of rank-two and three H_G SCFTs, which could be useful for further study on SCFT/VOA correspondence (Beem et al., 2020). For theories with matter, it was identified in (Del Zotto and Lockhart, 2018) that the worldsheet $(0,4)$ theories also correspond to some 4d $\mathcal{N}=2$ SCFTs but with some $(0,4)$ surface defects. The Schur indices of such configurations have rarely been studied, see some preliminary results in (Pan and Peelaers, 2018). It is interesting to see if the Schur indices of such 4d SCFTs with $(0,4)$ defects are also related to the elliptic genera of 6d $(1,0)$ SCFTs with matter.

In Chapter 8, we studied the K-theoretic blowup equations on $\mathbb{C}^2/\mathbb{Z}_2$ and used them to derive some bilinear relations of the τ functions of q-Toda systems. As we have seen, not all bilinear relations are from $\mathbb{Z}_2$ type blowup equations but some other interesting functional equations of Nekrasov partition function. It would be interesting to further explore the origin of those functional equations. Besides, it is also interesting to further study the (K-theoretic) blowup equations on general ALE spaces. The instanton partition function on such spaces for all classical gauge groups was recently defined in (Nakajima, 2018). For $U(N)$ gauge group, the orbifold partition function was explicitly computed in for example (Fucito, Morales, and Poghossian, 2004), while the resolved partition function was computed in for example (Bonelli et al., 2013). Some interesting blowup equations on A_{p-1}-ALE spaces with $p \geq 3$ were already found in (Ito, Maruyoshi, and Okuda, 2013). One can even consider whether the blowup equations for refined topological strings can be extended to $\mathbb{Z}_2$ type or general ALE spaces. In such cases, the orbifold partition function persumably will pick out refined BPS invariants by some generalized B-field condition. See also an interesting study on holomorphic anomaly equations for ALE spaces in (Krefl and Shih, 2013). Following the consistency condition we give in Chapter 4.4.4, one may be able to extend the relation between blowup equations and holomorphic anomaly equations to general ALE spaces.

Appendix A

Lie Algebraic Conventions

We use the same Lie algebraic conventions as in (Gu et al., 2020b). Given a simple Lie algebra $\mathfrak{g}$ of rank r, there are four important r-dimensional lattices: the root and coroot lattices $Q, Q^\vee$, as well as the weight and coweight lattices $P, P^\vee$. They are related to each other by

$$Q^\vee \subset P^\vee \subset \mathfrak{h}_\mathbb{C} \, , \tag{A.0.1}$$

$$Q \subset P \subset \mathfrak{h}_\mathbb{C}^* \, , \tag{A.0.2}$$

where $\mathfrak{h}_\mathbb{C}, \mathfrak{h}_\mathbb{C}^* \cong \mathbb{C}^r$ denote the complexified Cartan subalgebra and its dual equipped with the natural pairing

$$\langle \bullet, \bullet \rangle : \mathfrak{h}_\mathbb{C}^* \times \mathfrak{h}_\mathbb{C} \to \mathbb{C} \, . \tag{A.0.3}$$

The root and coroot lattices $Q, Q^\vee$ are spanned by the simple roots α_i and the simple coroots $\alpha_j^\vee$, whose pairings are entries of the Cartan matrix A

$$\langle \alpha_i, \alpha_j^\vee \rangle = A_{ij}. \tag{A.0.4}$$

The weight and coweight lattices $P, P^\vee$ are spanned by the fundamental weights ω_i and the fundamental coweights $\omega_i^\vee$, defined through

$$\langle \alpha_i, \omega_j^\vee \rangle = \langle \omega_i, \alpha_j^\vee \rangle = \delta_{ij} \, , \tag{A.0.5}$$

in other words, they are the duals of the coroot and the root lattices respectively. Every weight vector ω can be represented by the coefficients λ_i in its decomposition in terms of the fundamental weights, which are called the Dynkin labels

$$\omega = \sum_i \lambda_i \omega_i. \tag{A.0.6}$$

A weight vector is said to be *dominant* if all of its Dynkin labels are non-negative integers. Likewise, we can represent a coweight vector $\omega^\vee$ by the coefficients $\lambda_i^\vee$ in its decomposition in terms of the fundamental coweights

$$\omega^\vee = \sum_i \lambda_i^\vee \omega_i^\vee. \tag{A.0.7}$$

We will also call $\lambda_i^\vee$ the Dynkin labels of the coweight $\omega^\vee$ and say the coweight vector is dominant if all $\lambda_i^\vee$ are non-negative. Dominant (co)weight vectors can be used to label Weyl orbits as each Weyl orbit of (co)weight vectors has one and only one dominant element.

We define the Weyl invariant bilinear form $(\bullet,\bullet)$ on $\mathfrak{h}_{\mathbb{C}}$ by

$$(k,\ell) := \frac{1}{2h^\vee_{\mathfrak{g}}} \sum_{\alpha\in\Delta} \langle\alpha,k\rangle\langle\alpha,\ell\rangle, \quad k,\ell\in\mathfrak{h}_{\mathbb{C}}, \tag{A.0.8}$$

where $h^\vee_{\mathfrak{g}}$ is the dual Coxeter number of $\mathfrak{g}$. It has the nice property that the norm $||k||^2 = (k,k)$ of any coroot is an even integer, and in particular the norm of the shortest non-zero coroot $\theta^\vee$ is two. Note that the dual Coxeter number $h^\vee_{\mathfrak{g}}$ can be interpreted as the Dynkin index of the adjoint representation $\mathfrak{adj}$, while for an arbitrary representation R its Dynkin index ind_R is defined by (Di Francesco, Mathieu, and Senechal, 1997)

$$\mathrm{Tr}_R(\mathcal{R}(J^a)\mathcal{R}(J^b)) = 2\,\mathrm{ind}_R\,\delta_{ab}, \tag{A.0.9}$$

where $\mathcal{R}(J^a)$ is the matrix representation of the generator J^a of $\mathfrak{g}$. Consequently the bilinear form (A.0.8) can be expressed in terms of any representation R of $\mathfrak{g}$ through

$$(k,\ell) = \frac{1}{2\,\mathrm{ind}_R} \sum_{\omega\in R} \langle\omega,k\rangle\langle\omega,\ell\rangle, \quad k,\ell\in\mathfrak{h}_{\mathbb{C}}, \tag{A.0.10}$$

where we have used the same symbol R for the weight space of the representation.

The bilinear form $(\bullet,\bullet)$ is symmetric and non-degenerate. It then defines an isomorphism from $\mathfrak{h}_{\mathbb{C}}$ to $\mathfrak{h}^*_{\mathbb{C}}$ by

$$\begin{aligned} \varphi : \mathfrak{h}_{\mathbb{C}} &\xrightarrow{\sim} \mathfrak{h}^*_{\mathbb{C}} \\ k &\mapsto \varphi(k) = (k,\bullet); \end{aligned} \tag{A.0.11}$$

in other words, we have

$$\langle\varphi(k),\ell\rangle = (k,\ell), \quad \forall \ell\in\mathfrak{h}_{\mathbb{C}}. \tag{A.0.12}$$

The isomorphism then induces a Weyl invariant bilinear form on $\mathfrak{h}^*_{\mathbb{C}}$

$$(\omega,\eta) = \langle\omega,\varphi^{-1}(\eta)\rangle = (\varphi^{-1}(\omega),\varphi^{-1}(\eta)), \quad \omega,\eta\in\mathfrak{h}^*_{\mathbb{C}}. \tag{A.0.13}$$

Concretely we have

$$\varphi(\alpha_i^\vee) = \frac{||\alpha_i^\vee||^2}{2}\alpha_i, \quad \varphi(\omega_i^\vee) = \frac{||\alpha_i^\vee||^2}{2}\omega_i. \tag{A.0.14}$$

It is easy to see that the Dynkin labels $\lambda_i^\vee$ of a coweight $\omega^\vee$ and the Dynkin labels λ_i of its isomorphic weight vector $\omega = \varphi(\omega^\vee)$ are related by

$$\lambda_i = \lambda_i^\vee \frac{||\alpha_i^\vee||^2}{2}. \tag{A.0.15}$$

We list below the norms of simple coroots of simple Lie algebras used in the main text.

- $A_n, D_n, E_{6,7,8}$: These are simply laced Lie algebras and all the simple coroots have norm 2.

- $B_n (n\geq 2)$:
$$||\alpha_i^\vee||^2 = 2, \ \ i=1,\ldots,n-1, \quad ||\alpha_n^\vee||^2 = 4. \tag{A.0.16}$$

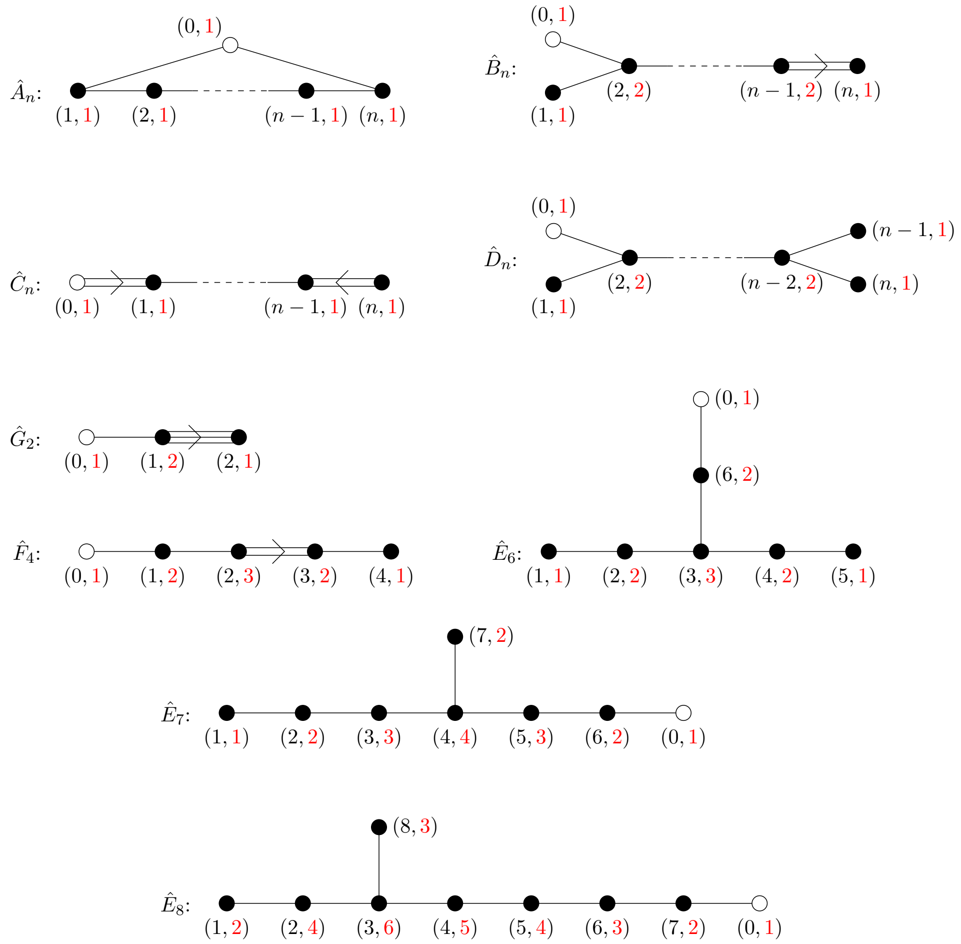


FIGURE A.1: Affine Dynkin diagrams associated to simple Lie algebras. The i-th node with comark m_i is labeled by the pair (i, m_i) where m_i is colored in red. In each diagram, the white node is the affine node, and the black nodes are nodes of simple Lie algebra. The arrows point from short coroots to long coroots. We follow the same node order and same representation names as in the `LieART` package of `Mathematica` (Feger and Kephart, 2015; Feger, Kephart, and Saskowski, 2019).

- $C_n (n \geq 2)$:

$$||\alpha_i^\vee||^2 = 4, \quad i = 1, \ldots, n-1, \quad ||\alpha_n^\vee||^2 = 2. \tag{A.0.17}$$

- G_2:

$$||\alpha_1^\vee||^2 = 2, \quad ||\alpha_2^\vee||^2 = 6. \tag{A.0.18}$$

- F_4:

$$||\alpha_1^\vee||^2 = ||\alpha_2^\vee||^2 = 2, \quad ||\alpha_3^\vee||^2 = ||\alpha_4^\vee||^2 = 4. \tag{A.0.19}$$

We give in Fig. A.1 the affine Dynkin diagrams of simple Lie algebras and the ordering of nodes used in the thesis.

In the main text, to lighten notation we use $\cdot$ to represent both the pairing $\langle \bullet, \bullet \rangle$ and the bilinear form $(\bullet, \bullet)$.

Appendix B

Useful Identities

Here we collect some definitions and identities which are useful in the main text. Jacobi theta functions with characteristics are defined as

$$\begin{aligned}
\theta_1^{[a]}(\tau,z) &= -\,\mathrm{i}\sum_{k\in\mathbb{Z}}(-1)^{k+a}q_\tau^{(k+1/2+a)^2/2}Q_z^{k+1/2+a},\\
\theta_2^{[a]}(\tau,z) &= \sum_{k\in\mathbb{Z}}q_\tau^{(k+1/2+a)^2/2}Q_z^{k+1/2+a},\\
\theta_3^{[a]}(\tau,z) &= \sum_{k\in\mathbb{Z}}q_\tau^{(k+a)^2/2}Q_z^{k+a},\\
\theta_4^{[a]}(\tau,z) &= \sum_{k\in\mathbb{Z}}(-1)^{k+a}q_\tau^{(k+a)^2/2}Q_z^{k+a},
\end{aligned} \tag{B.0.1}$$

The plethystic exponent operation PE is defined as

$$\mathrm{PE}[f(x)] = \exp\left[\sum_{n=1}^{\infty}\frac{1}{n}f(x^n)\right]. \tag{B.0.2}$$

Using the triple product formula of Jacobi theta tunction θ_1

$$\theta_1(\tau,z) = \mathrm{i}q_\tau^{\frac{1}{12}}Q_z^{-\frac{1}{2}}\eta(\tau)\prod_{n=1}^{\infty}\left(1-Q_zq_\tau^{n-1}\right)\left(1-\frac{q_\tau^n}{Q_z}\right), \tag{B.0.3}$$

we can simplify the following plethystic exponentials which often appear in the evaluation of multiplet contributions to the one-loop partition function

$$\mathrm{PE}\left[\frac{Q}{1-q_\tau}\right] = \prod_{n=0}^{\infty}\frac{1}{1-Qq_\tau^n}, \tag{B.0.4}$$

and

$$\mathrm{PE}\left[\left(Q_z+\frac{q_\tau}{Q_z}\right)\left(\frac{1}{1-q_\tau}\right)\right] = \frac{\mathrm{i}q_\tau^{\frac{1}{12}}Q_z^{-\frac{1}{2}}\eta(\tau)}{\theta_1(\tau,z)}. \tag{B.0.5}$$

In the following, we would like to present some elementary but useful formulas when dealing with blowup equations. Denote

$$f_{(j_L,j_R)}(q_1,q_2) = \frac{\chi_{j_L}(q_L)\chi_{j_R}(q_R)}{\left(q_1^{1/2}-q_1^{-1/2}\right)\left(q_2^{1/2}-q_2^{-1/2}\right)}, \qquad \chi_j(q) = \frac{q^{2j+1}-q^{-2j-1}}{q-q^{-1}}, \tag{B.0.6}$$

which is the spin-related prefactor in the contribution to the one-loop partition function of a multiplet with spin (j_L, j_R). It satisfies the relations

$$f_{(j_L,j_R)}(q_1^{-1}, q_2^{-1}) = f_{(j_L,j_R)}(q_1, q_2) = f_{(j_L,j_R)}(q_2, q_1) \,, \tag{B.0.7}$$

$$f_{(j_L,j_R)}(q_1^{-1}, q_2) = f_{(j_L,j_R)}(q_1, q_2^{-1}) = f_{(j_R,j_L)}(q_1, q_2) \,. \tag{B.0.8}$$

In the blowup equation this prefactor contributes by

$$Bl_{(j_L,j_R,R)}(q_1, q_2) = f_{(j_L,j_R)}(q_1, q_2/q_1)q_1^R + f_{(j_L,j_R)}(q_1/q_2, q_2)q_2^R - f_{(j_L,j_R)}(q_1, q_2), \quad 2R \in \mathbb{Z}. \tag{B.0.9}$$

The B field condition translates to the condition

$$2j_L + 2j_R + 1 \equiv 2R \; (\mathrm{mod}\, 2) \,. \tag{B.0.10}$$

It is easy to find that under this condition the apparent denominator of $Bl_{(j_L,j_R,R)}(q_1, q_2)$ can always be factored out so that

$$Bl_{(j_L,j_R,R)}(q_1, q_2) = \text{finite series in } q_1, q_2 \,. \tag{B.0.11}$$

We call (B.0.11) *fundamental identities*. Note that since

$$Bl_{(j_L,j_R,-R)}(q_1, q_2) = Bl_{(j_L,j_R,R)}(q_1^{-1}, q_2^{-1}) \,, \tag{B.0.12}$$

we only need to consider the cases with $R \geq 0$.

In the following, we present some frequently used instances of the fundamental identities for small spins.

- For $(j_L, j_R) = (0, 0)$, R should be half integers. Then

$$Bl_{(0,0,R)}(q_1, q_2) = - \sum_{\substack{m,n\geq 0 \\ m+n\leq R-3/2}} q_1^{m+1/2} q_2^{n+1/2} \,, \quad R \geq 1/2 \,. \tag{B.0.13}$$

- For $(j_L, j_R) = (1/2, 0)$, R should be integers. Then

$$Bl_{(1/2,0,R)}(q_1, q_2) = \begin{cases} - \sum\limits_{\substack{m,n\geq 0 \\ 1\leq m+n\leq R}} q_1^m q_2^n - \sum\limits_{\substack{m,n\geq 0 \\ m+n\leq R-3}} q_1^{m+1} q_2^{n+1}, & R \geq 1 \,, \\ -1 \,, & R = 0 \,. \end{cases} \tag{B.0.14}$$

- For $(j_L, j_R) = (0, 1/2)$, R should be integers. Then

$$Bl_{(0,1/2,R)}(q_1, q_2) = - \sum_{\substack{m,n\geq 0 \\ m+n\leq R-1}} q_1^m q_2^n - \sum_{\substack{m,n\geq 0 \\ m+n\leq R-2}} q_1^{m+1} q_2^{n+1} \,, \quad R \geq 0 \,. \tag{B.0.15}$$

As we have seen in the main text, the contribution of vector multiplets can always be factorized as products of

$$T_V(z, R) = \mathrm{PE}\left[- \left(Bl_{(0,1/2,R)}(q_1, q_2) Q_z + Bl_{(0,1/2,-R)}(q_1, q_2) \frac{q_\tau}{Q_z} \right) \left(\frac{1}{1 - q_\tau} \right) \right] . \tag{B.0.16}$$

Here $R \in \mathbb{Z}$. Using (B.0.15) and (B.0.5) and assuming $R \geq 0$, it can be written as

$$T_V(z,R) = \prod_{\substack{m,n\geq 0\\ m+n\leq R-1}} \frac{\mathrm{i} q_\tau^{1/12}\eta\left(Q_z q_1^m q_2^n\right)^{-1/2}}{\theta_1(z+m\epsilon_1+n\epsilon_2)} \prod_{\substack{m,n\geq 0\\ m+n\leq R-2}} \frac{\mathrm{i} q_\tau^{1/12}\eta\left(Q_z q_1^{m+1} q_2^{n+1}\right)^{-1/2}}{\theta_1(z+(m+1)\epsilon_1+(n+1)\epsilon_2)}$$
$$= \left(\mathrm{i} q_\tau^{1/12} Q_z^{-1/2}\right)^{R^2} (q_1q_2)^{-\frac{(R-1)R(R+1)}{6}} \breve{\theta}_R(z)\,, \qquad \text{(B.0.17)}$$

where

$$\breve{\theta}_V(z,R) = \prod_{\substack{m,n\geq 0\\ m+n\leq R-1}} \frac{\eta}{\theta_1(z+m\epsilon_1+n\epsilon_2)} \prod_{\substack{m,n\geq 0\\ m+n\leq R-2}} \frac{\eta}{\theta_1(z+(m+1)\epsilon_1+(n+1)\epsilon_2)}, \quad R\in\mathbb{Z}. \qquad \text{(B.0.18)}$$

In the case of $R < 0$ we can use the above expression for $-R$ with $\epsilon_{1,2}$ replaced by $-\epsilon_{1,2}$ or equivalently $q_{1,2}$ replaced by $1/q_{1,2}$. In both cases, $\breve{\theta}_V(z,R)$ is a multivariate Jacobi form of weight zero and index quadratic form

$$\mathrm{Ind}^V_{\breve{\theta}}(z,R) = -\frac{R^2z^2}{2} - \frac{(R-1)R(R+1)}{3}z(\epsilon_1+\epsilon_2) - \frac{(R-1)R^2(R+1)}{12}(\epsilon_1^2+\epsilon_1\epsilon_2+\epsilon_2^2). \qquad \text{(B.0.19)}$$

Similarly, the contribution of hyper multiplets can always be factorized as products of

$$T_H(z,R) = \mathrm{PE}\left[\left(Bl_{(0,0,R)}(q_1,q_2)Q_z + Bl_{(0,0,-R)}(q_1,q_2)\frac{q_\tau}{Q_z}\right)\left(\frac{1}{1-q_\tau}\right)\right]. \qquad \text{(B.0.20)}$$

Here $R \in \mathbb{Z}+1/2$. Using (8.3.6) and (B.0.5) and assuming $R \geq 0$, it can be written as

$$T_H(z,R) = \prod_{\substack{m,n\geq 0\\ m+n\leq R-3/2}} \frac{\theta_1(z+(m+1/2)\epsilon_1+(n+1/2)\epsilon_2)}{\mathrm{i} q_\tau^{1/12}\left(Q_z q_1^{m+1/2} q_2^{n+1/2}\right)^{-1/2}\eta} \qquad \text{(B.0.21)}$$
$$= \left(\frac{Q_z^{1/2}}{\mathrm{i} q_\tau^{1/12}}\right)^{\frac{(R^2-1/4)}{2}} (q_1q_2)^{\frac{(R-1/2)R(R+1/2)}{12}} \breve{\theta}_H(z,R), \qquad \text{(B.0.22)}$$

where

$$\breve{\theta}_H(z,R) := \prod_{\substack{m,n\geq 0\\ m+n\leq R-3/2}} \frac{\theta_1(z+s(m+1/2)\epsilon_1+s(n+1/2)\epsilon_2)}{\eta}\,, \quad R\in\frac{1}{2}+\mathbb{Z}, \qquad \text{(B.0.23)}$$

In the case of $R < 0$ we can use the above expression for $-R$ with $\epsilon_{1,2}$ replaced by $-\epsilon_{1,2}$ or equivalently $q_{1,2}$ replaced by $1/q_{1,2}$. In both cases, $\breve{\theta}_H(z,R)$ is a multivariate Jacobi form of weight zero and index quadratic form

$$\mathrm{Ind}^H_{\breve{\theta}}(z,R) = \frac{(R+1/2)(R-1/2)}{4}z^2 + \frac{R(R-1/2)(R+1/2)}{6}z(\epsilon_1+\epsilon_2)$$
$$+ \frac{(R-1/2)(R+1/2)(R^2-3/4)}{24}(\epsilon_1^2+\epsilon_2^2) + \frac{(R-1/2)(R+1/2)(R^2+3/4)}{24}\epsilon_1\epsilon_2. \qquad \text{(B.0.24)}$$

At last, we collect some useful formulas for Chapter 8.

$$\frac{(q^{x\pm n}q;q,q)_\infty}{(q^x q;q,q)_\infty^2} = \begin{cases} \prod_{\substack{i,j\geq 0,\, i+j\leq 2n-2 \\ i+j\equiv 2n-2 \pmod 2}} (1-q^x q^{\frac{j-i}{2}}), & n>0, \\ \prod_{\substack{i,j\geq 0,\, i+j\leq -2n-2 \\ i+j\equiv -2n-2 \pmod 2}} (1-q^x q^{\frac{j-i}{2}}), & n<0. \end{cases} \tag{B.0.25}$$

$$\frac{(q^{x\pm(n+1/2)}q;q,q)_\infty}{(q^{x\pm 1/2}q;q,q)_\infty} = \begin{cases} \prod_{\substack{i,j\geq 0,\, i+j\leq 2n-1 \\ i+j\equiv 2n-1 \pmod 2}} (1-q^x q^{\frac{j-i}{2}}), & n>0, \\ \prod_{\substack{i,j\geq 0,\, i+j\leq -2n-1 \\ i+j\equiv -2n-1 \pmod 2}} (1-q^x q^{\frac{j-i}{2}}), & n<-1. \end{cases} \tag{B.0.26}$$

Appendix C

Functional Equations for Theta Functions of Even Unimodular Lattices

The unity blowup equations for one E-string elliptic genus (5.5.12) give a set of interesting functional equations for E_8 theta function. Here we prove E_8 theta function is the unique solution for such equations up to a free function of τ. This statement can be generalized to the theta function associated to any positive definite even unimodular lattice that is generated by roots. The generalization and proof were shown to us by Don Zagier.

Proposition 3. Denote $\mathfrak{H}$ as the Poincare upper half plane. Let Λ be a positive definite even unimodular lattice that is generated by its roots, and let f be a holomorphic function on $\mathfrak{H} \times \Lambda_{\mathbb{C}}$ satisfying the functional equation

$$\begin{aligned} \theta_1(\epsilon_2)\theta_1(\alpha\cdot m+\epsilon_2)f(\tau,m+\epsilon_1\alpha)-\theta_1(\epsilon_1)\theta_1(\alpha\cdot m+\epsilon_1)f(\tau,m+\epsilon_2\alpha) \\ =\theta_1(\epsilon_2-\epsilon_1)\theta_1(\alpha\cdot m+\epsilon_1+\epsilon_2)f(\tau,m), \end{aligned} \tag{C.0.1}$$

for all roots α of Λ and all $\epsilon_1,\epsilon_2\in\mathbb{C}$. Then f is a multiple (depending only on τ) of the theta series

$$\theta_\Lambda(\tau,m)=\sum_{w\in\Lambda}e^{2\pi \mathrm{i}(w\cdot w/2+m\cdot w)}. \tag{C.0.2}$$

Proof: Fix τ and also a root α and a vector $m_0\in\Lambda$ with $m_0\cdot\alpha=0$, and set $F(\tau,\lambda)=f(m_0+\lambda\alpha)$, $\lambda\in\mathbb{C}$. Using $\alpha\cdot\alpha=2$ and setting $h_1=\lambda+\epsilon_1$, $h_2=\lambda+\epsilon_2$, $h_3=\lambda$, we can write (C.0.1) in a symmetric form,

$$\sum_{i\,(\mathrm{mod}\,3)}\theta_1(h_{i+1}-h_{i-1})\theta_1(h_{i+1}+h_{i-1})F(h_i)=0,\quad (\text{any }\{h_i\}_{i\,(\mathrm{mod}\,3)}\in\mathbb{C}^3). \tag{C.0.3}$$

Here the τ dependence is implicit. Changing h_1 to h_1+1 and $h_1+\tau$ with h_2 and h_3 fixed, we find $F(h+1)=F(h)$ and $F(h+\tau)=q^{-1}\xi^{-2}F(h)$, where $q=e^{2\pi \mathrm{i}\tau}$, $\xi=e^{2\pi \mathrm{i}h}$. Thus,

$$f(m+\alpha)=f(m),\qquad f(m+\alpha\tau)=e^{-2\pi \mathrm{i}(\tau+\alpha\cdot m)}f(m). \tag{C.0.4}$$

Since Λ is even unimodular and generated by all roots α, the first equation of (C.0.4) implies that we can write $f(\tau,m)$ as Fourier expansion $\sum_{w\in\Lambda}c_w(q)q^{w\cdot w/2}e^{2\pi \mathrm{i}m\cdot w}$ for some coefficients $c_w(q)$. The second equation of (C.0.4) implies that $c_{w+\alpha}(q)=c_w(q)$ for all w and α, so $c_w(q)=c_0(q)$ and $f(\tau,m)=c_0(q)\theta_\Lambda(\tau,m)$.

Appendix D

Elliptic Genera

We make a list for the computational results in Chapter 5 and 6.

- Elliptic genera

  Although our computation on the elliptic genera of rank one 6d $(1,0)$ SCFTs in general contain all gauge and flavor fugacities, for some theories we only present the results with all fugacities turned off. For most theories especially the exceptional theories, we not only show the elliptic genera with fugacities turned off, but also the v expansion with gauge and flavor fugacities turned on.

  Class **A**

  - $n=3$, $G=\mathfrak{su}(3)$, $\mathbb{E}_1$ (5.5.54), $\mathbb{E}_2$ (5.5.55)
    $G=\mathfrak{so}(7)$, $\mathbb{E}_1$ (5.5.52), $\mathbb{E}_2$ (5.5.53)
    $G=\mathfrak{so}(8)$, $\mathbb{E}_1$ (D.0.24), $\mathbb{E}_2$ (D.0.26)
    $G=\mathfrak{so}(9)$, $\mathbb{E}_1$ (D.0.27), $\mathbb{E}_2$ (D.0.28)
    $G=\mathfrak{so}(10)$, $\mathbb{E}_1$ (D.0.30), $\mathbb{E}_2$ (D.0.32)
    $G=G_2$, $\mathbb{E}_1$ (5.5.70), $\mathbb{E}_2$ (5.5.73)
    $G=F_4$, $\mathbb{E}_1$ (5.5.91), $\mathbb{E}_2$ (5.5.93)
    $G=E_6$, $\mathbb{E}_1$ (5.5.111), $\mathbb{E}_2$ (D.0.35)
  - $n=4$, $G=\mathfrak{so}(8)$, $\mathbb{E}_1$ (5.5.62), $\mathbb{E}_2$ (5.5.63)
    $G=\mathfrak{so}(9)$, $\mathbb{E}_1$ (5.5.64), $\mathbb{E}_2$ (5.5.65)
    $G=\mathfrak{so}(10)$, $\mathbb{E}_1$ (5.5.66), $\mathbb{E}_2$ (5.5.68)
    $G=F_4$, $\mathbb{E}_1$ (5.5.88), $\mathbb{E}_2$ (5.5.90)
    $G=E_6$, $\mathbb{E}_1$ (5.5.108), $\mathbb{E}_2$ (D.0.36)
    $G=E_7$, $\mathbb{E}_1$ (5.5.128)
  - $n=5$, $G=F_4$, $\mathbb{E}_1$ (5.5.85), $\mathbb{E}_2$ (5.5.87)
    $G=E_6$, $\mathbb{E}_1$ (5.5.105), $\mathbb{E}_2$ (D.0.39)
  - $n=6$, $G=E_6$, $\mathbb{E}_1$ (5.5.104), $\mathbb{E}_2$ (D.0.42)
    $G=E_7$, $\mathbb{E}_1$ (5.5.125)
  - $n=8$, $G=E_7$, $\mathbb{E}_1$ (5.5.124), $\mathbb{E}_2$ (D.0.44)
  - $n=12$, $G=E_8$, $\mathbb{E}_1$ (5.5.133), $\mathbb{E}_2$ (5.5.135)

  Class **B**

  - $n=1$, $G=\mathfrak{su}(3)$, $\mathbb{E}_1$ (5.5.35)
    $G=\mathfrak{su}(4)$, $\mathbb{E}_1$ (D.0.1)
    $G=\mathfrak{so}(7)$, $\mathbb{E}_1$ (D.0.3)

$G = \mathfrak{so}(8)$, $\mathbb{E}_1$ (D.0.5)
$G = \mathfrak{so}(9)$, $\mathbb{E}_1$ (D.0.7)
$G = G_2$, $\mathbb{E}_1$ (5.5.81)
$G = F_4$, $\mathbb{E}_1$ (5.5.98)
$G = E_6$, $\mathbb{E}_1$ (5.5.118)

- $n = 2$, $G = \mathfrak{so}(9)$, $\mathbb{E}_1$ (D.0.10)
  $G = \mathfrak{so}(10)$, $\mathbb{E}_1$ (D.0.12)
  $G = \mathfrak{so}(11)$, $\mathbb{E}_1$ (D.0.15)
  $G = \mathfrak{so}(12)_a$, $\mathbb{E}_1$ (D.0.20)
  $G = G_2$, $\mathbb{E}_1$ (5.5.75)
  $G = F_4$, $\mathbb{E}_1$ (5.5.94)
  $G = E_6$, $\mathbb{E}_1$ (5.5.114)
  $G = E_7$, $\mathbb{E}_1$ (5.5.130)

- Exact v expansion formulas for 5d one-instanton Hilbert series

  For a lot of rank-one theories with matters, the exact formulas for the v expansion of 5d one-instanton partition function have been proposed in (Del Zotto and Lockhart, 2018) and (Kim et al., 2019). Benefited from the results of blowup equations, we further obtain the exact formulas for the following new theories

  - $n = 1$, $G = \mathfrak{su}(3)$ (5.5.36), $\mathfrak{su}(4)$ (D.0.2), $\mathfrak{so}(7)$ (D.0.4), $\mathfrak{so}(8)$ (D.0.6), $\mathfrak{so}(9)$ (D.0.8)
    G_2 (5.5.84), F_4 (D.0.9), E_6 (5.5.121)
  - $n = 2$, $G = \mathfrak{so}(9)$ (D.0.11), $\mathfrak{so}(10)$ (D.0.13), $\mathfrak{so}(11)$ (D.0.17), $\mathfrak{so}(12)_a$ (D.0.22), E_6 (5.5.117), E_7 (D.0.23)
  - $n = 3$, $G = \mathfrak{so}(12)$ (D.0.33), E_6 (D.0.34)
  - $n = 4$, $G = E_6$ (5.5.110), E_7 (D.0.38)

- Modular ansatz

  Among the ten theories whose modular ansatz for reduced one-string elliptic genus were not fixed in (Del Zotto and Lockhart, 2018), five of them listed below belong to class **A** or **B**. Benefitting from blowup equations, we are able to determine their modular ansatz. See results in the `Mathematica` file `ModularAnsatzAppendix.nb` on the website.

  - $n = 1$, $G = E_6$
  - $n = 2$, $G = \mathfrak{so}(11)$, E_6, E_7
  - $n = 4$, $G = E_7$

- Vanishing theta identities

  We checked the leading degree identities for all the vanishing blowup equations in Table 5.7, 5.8 and 5.9 up to $\mathcal{O}(q^{20})$. We write down the explicit form of the vanishing identities for the following theories:

  - $n = 1$, $G = \mathfrak{su}(3)$ (5.2.25, 5.2.26)
  - $n = 1$, $G = \mathfrak{su}(N)$ (5.5.28, 5.5.29, 5.5.30, 5.5.31)
  - $n = 1$, $G = \mathfrak{sp}(N)$ (5.5.22, 5.5.23)

- $n = 2, G = \mathfrak{su}(N)$ (5.5.39, 5.5.40, 5.5.41, 5.5.42)
- $n = 3, G = \mathfrak{so}(7)$ (5.5.49, 5.5.50)
- $n = 4, G = \mathfrak{so}(8+N)$ (5.5.59, 5.5.60)
- $n = 1, 2, \ldots, 6, G = E_6$ (5.5.102, 5.5.103)
- $n = 1, 2, \ldots, 8, G = E_7$ (5.5.122, 5.5.123)
- NHC 3, 2 (6.3.6)
- NHC 3, 2, 2 (6.3.21)
- NHC 2, 3, 2 (6.3.32, 6.3.35)
- D_4 quiver of -2 curves (6.4.4, 6.4.5)
- (E_6, E_6) conformal matter (6.5.5, 6.5.6, 6.5.7)
- (E_7, E_7) conformal matter (6.5.10)
- blown-up of $-n$ curve with $n = 2, 3, 4, 5, 7, 9, 10, 11$ (6.6.2, 6.6.3)

In the following we record more results on the one-string and two-string elliptic genera for certain rank one theories which we obtain from blowup equations. Note all "..." in the polynomial of v means palindromic. More detailed results can be found on the website.

$\mathbf{n = 1}$, $\mathbf{G} = \mathfrak{su}(\mathbf{4})$, $\mathbf{F} = \mathfrak{su}(\mathbf{12})_{\mathfrak{a}} \times \mathfrak{su}(\mathbf{2})_{\mathfrak{b}}$

Using the Weyl orbit expansion, we turn on a subgroup $\mathfrak{su}(2) \times \mathfrak{su}(2)$ of the full flavor to compute the elliptic genus. We obtain the reduced one-string elliptic genus as

$$\mathbb{E}_{h^{(1)}_{1,\mathfrak{su}(4)}}(q_\tau, v, m_{\mathfrak{su}(4)} = 0, m_F = 0) = q_\tau^{-1/3} + q_\tau^{2/3} v^{-2} \sum_{n=0}^{\infty} q_\tau^n \frac{P_n(v)}{(1+v)^6}, \tag{D.0.1}$$

where

$$P_0(v) = 15 - 18v - 261v^2 - 72v^3 + 2934v^4 + 10676v^5 + \cdots + 15v^{10}.$$

This agrees with the modular ansatz in (Del Zotto and Lockhart, 2018). Using the result with flavor fugacities turned on, we obtain the following exact v expansion formula for the subleading q order coefficient, which contains the 5d one-instanton Nekrasov partition function:

$$\begin{aligned}
&\chi^{\mathfrak{su}(4)}_{(1,0,1)} v^{-2} - (\chi^{\mathfrak{su}(4)}_{(1,0,0)} \chi^F_{(00000000001)_a} + c.c.) v^{-1} - \chi^{\mathfrak{su}(4)}_{(0,1,0)} \chi^F_{(1)_b} v^{-1} + \chi^F_{(10000000001)_a \oplus (2)_b} \\
&+ \chi^{\mathfrak{su}(4)}_{(1,0,1)} + 1 + \chi^F_{(1)_b} (\chi^F_{(01000000000)_a} v - \chi^{\mathfrak{su}(4)}_{(1,0,0)} \chi^F_{(10000000000)_a} v^2 + \chi^{\mathfrak{su}(4)}_{(2,0,0)} v^3 + c.c.) \\
&+ \chi^{\mathfrak{su}(4)}_{(0,1,0)} \chi^F_{(1)_b} v - \chi^{\mathfrak{su}(4)}_{(1,0,1)} v^2 + (\chi^F_{(00010000000)_a} v^2 - \chi^{\mathfrak{su}(4)}_{(1,0,0)} \chi^F_{(00100000000)_a} v^3 \\
&+ \chi^{\mathfrak{su}(4)}_{(2,0,0)} \chi^F_{(01000000000)_a} v^4 - \chi^{\mathfrak{su}(4)}_{(3,0,0)} \chi^F_{(10000000000)_a} v^5 + \chi^{\mathfrak{su}(4)}_{(4,0,0)} v^6 + c.c.) \\
+ &\sum_{n=0}^{\infty} \Big[\chi^F_{(1)_b} \Big(\chi^{\mathfrak{su}(4)}_{(n,0,n)} \chi^F_{(00001000000)_a} v^{3+2n} + (-\chi^{\mathfrak{su}(4)}_{(n+1,0,n)} \chi^F_{(00001000000)_a} v^{4+2n} \\
&+ \chi^{\mathfrak{su}(4)}_{(n+2,0,n)} \chi^F_{(00010000000)_a} v^{5+2n} - \chi^{\mathfrak{su}(4)}_{(n+3,0,n)} \chi^F_{(00100000000)_a} v^{6+2n}
\end{aligned}$$

$$
\begin{aligned}
&+\chi^{\mathfrak{su}(4)}_{(n+4,0,n)}\chi^F_{(01000000000)_a}v^{7+2n}-\chi^{\mathfrak{su}(4)}_{(n+5,0,n)}\chi^F_{(10000000000)_a}v^{8+2n}+\chi^{\mathfrak{su}(4)}_{(n+6,0,n)}v^{9+2n}+c.c.)\Big)\\
&+\Big(-\chi^{\mathfrak{su}(4)}_{(n,1,n)}\chi^F_{(00000100000)_a}v^{4+2n}+(\chi^{\mathfrak{su}(4)}_{(n+1,1,n)}\chi^F_{(00001000000)_a}v^{5+2n}\\
&-\chi^{\mathfrak{su}(4)}_{(n+2,1,n)}\chi^F_{(00010000000)_a}v^{6+2n}+\chi^{\mathfrak{su}(4)}_{(n+3,1,n)}\chi^F_{(00100000000)_a}v^{7+2n}\\
&-\chi^{\mathfrak{su}(4)}_{(n+4,1,n)}\chi^F_{(01000000000)_a}v^{8+2n}+\chi^{\mathfrak{su}(4)}_{(n+5,1,n)}\chi^F_{(10000000000)_a}v^{9+2n}\\
&-\chi^{\mathfrak{su}(4)}_{(n+6,1,n)}v^{10+2n}+c.c.)\Big)\Big].
\end{aligned}
\tag{D.0.2}
$$

We have checked this agrees with the localization formula (5.5.27) from 2d quiver gauge theory.

$\mathbf{n=1}$, $\mathbf{G}=\mathfrak{so}(7)$, $\mathbf{F}=\mathfrak{sp}(2)_a\times\mathfrak{sp}(6)_b$

Using the Weyl orbit expansion, we turn on a diagonal subgroup $\mathfrak{sp}(1)\times\mathfrak{sp}(1)$ of the flavor group to compute the elliptic genus. We obtain the reduced one-string elliptic genus as

$$
\mathbb{E}_{h^{(1)}_{1,\mathfrak{so}(7)}}(q_\tau,v)=q_\tau^{-1/3}+q_\tau^{2/3}v^{-2}\sum_{n=0}^{\infty}q_\tau^n\frac{P_n(v)}{(1+v)^8},
\tag{D.0.3}
$$

where

$$
\begin{aligned}
P_0(v)=&\,21+44v-294v^2-1156v^3+475v^4+13400v^5+38508v^6\\
&+13400v^7+\cdots+21v^{12}.
\end{aligned}
$$

This agrees with the modular ansatz in (Del Zotto and Lockhart, 2018). Using the result with flavor fugacities turned on, we obtain the following exact v expansion formula for the subleading q order coefficient, which contains the 5d one-instanton Nekrasov partition function:

$$
\begin{aligned}
&\chi^{\mathfrak{so}(7)}_{(010)}v^{-2}-(\chi^{\mathfrak{so}(7)}_{(100)}\chi^F_{(10)_a}+\chi^{\mathfrak{so}(7)}_{(001)}\chi^F_{(100000)_b})v^{-1}+(\chi^{\mathfrak{so}(7)}_{(002)}+\chi^F_{(20)_a}+\chi^F_{(200000)_b}+1)\\
&+\chi^F_{(10)_a\otimes(010000)_b}v+(\chi^F_{(000100)_b}+\chi^F_{(01)_a\otimes(010000)_b}+\chi^{\mathfrak{so}(7)}_{(100)}\chi^F_{(01)_a})v^2\\
&+(\chi^F_{(10)_a\otimes(000100)_b}-\chi^{\mathfrak{so}(7)}_{(001)}\chi^F_{(01)_a\otimes(100000)_b}-\chi^{\mathfrak{so}(7)}_{(100)}\chi^F_{(100000)_b})v^3\\
&-(\chi^{\mathfrak{so}(7)}_{(100)}\chi^F_{(000100)_b}+\chi^{\mathfrak{so}(7)}_{(001)}\chi^F_{(10)_a\otimes(001000)_b}+\chi^{\mathfrak{so}(7)}_{(010)}\chi^F_{(010000)_b}-\chi^{\mathfrak{so}(7)}_{(002)}\chi^F_{(01)_a})v^4\\
&+(\chi^{\mathfrak{so}(7)}_{(101)}\chi^F_{(001000)_b}+\chi^{\mathfrak{so}(7)}_{(002)}\chi^F_{(10)_a\otimes(010000)_b}+\chi^{\mathfrak{so}(7)}_{(011)}\chi^F_{(100000)_b})v^5\\
&-(\chi^{\mathfrak{so}(7)}_{(102)}\chi^F_{(010000)_b}+\chi^{\mathfrak{so}(7)}_{(003)}\chi^F_{(10)_a\otimes(100000)_b}+\chi^{\mathfrak{so}(7)}_{(012)})v^6\\
&+(\chi^{\mathfrak{so}(7)}_{(103)}\chi^F_{(100000)_b}+\chi^{\mathfrak{so}(7)}_{(004)}\chi^F_{(10)_a})v^7-\chi^{\mathfrak{so}(7)}_{(104)}v^8+\\
\sum_{n=0}^{\infty}&\Big[\chi^{\mathfrak{so}(7)}_{(0n0)}\chi^F_{(01)_a\otimes(000001)_b}v^{4+2n}-(\chi^{\mathfrak{so}(7)}_{(1n0)}\chi^F_{(10)_a\otimes(000001)_b}+\chi^{\mathfrak{so}(7)}_{(0n1)}\chi^F_{(01)_a\otimes(000010)_b})v^{5+2n}\\
&+(\chi^{\mathfrak{so}(7)}_{(2n0)}\chi^F_{(000001)_b}+\chi^{\mathfrak{so}(7)}_{(1n1)}\chi^F_{(10)_a\otimes(000010)_b}+\chi^{\mathfrak{so}(7)}_{(0n2)}\chi^F_{(01)_a\otimes(000100)_b})v^{6+2n}\\
&-(\chi^{\mathfrak{so}(7)}_{(2n1)}\chi^F_{(000010)_b}+\chi^{\mathfrak{so}(7)}_{(1n2)}\chi^F_{(10)_a\otimes(000100)_b}+\chi^{\mathfrak{so}(7)}_{(0n3)}\chi^F_{(01)_a\otimes(001000)_b})v^{7+2n}
\end{aligned}
$$

$$
\begin{aligned}
&+(\chi^{\mathfrak{so}(7)}_{(2n2)}\chi^{F}_{(000100)_b}+\chi^{\mathfrak{so}(7)}_{(1n3)}\chi^{F}_{(10)_a\otimes(001000)_b}+\chi^{\mathfrak{so}(7)}_{(0n4)}\chi^{F}_{(01)_a\otimes(010000)_b})v^{8+2n}\\
&-(\chi^{\mathfrak{so}(7)}_{(2n3)}\chi^{F}_{(001000)_b}+\chi^{\mathfrak{so}(7)}_{(1n4)}\chi^{F}_{(10)_a\otimes(010000)_b}+\chi^{\mathfrak{so}(7)}_{(0n5)}\chi^{F}_{(01)_a\otimes(100000)_b})v^{9+2n}\\
&+(\chi^{\mathfrak{so}(7)}_{(2n4)}\chi^{F}_{(010000)_b}+\chi^{\mathfrak{so}(7)}_{(1n5)}\chi^{F}_{(10)_a\otimes(100000)_b}+\chi^{\mathfrak{so}(7)}_{(0n6)}\chi^{F}_{(01)_a})v^{10+2n}\\
&-(\chi^{\mathfrak{so}(7)}_{(2n5)}\chi^{F}_{(100000)_b}+\chi^{\mathfrak{so}(7)}_{(1n6)}\chi^{F}_{(10)_a})v^{11+2n}+\chi^{\mathfrak{so}(7)}_{(2n6)}v^{12+2n}\Big].
\end{aligned}
\tag{D.0.4}
$$

After turning off all gauge and flavor fugacities, this goes back to the rational function of v by Weyl dimension formulas.

$\mathbf{n=1}$, $\mathbf{G}=\mathfrak{so}(8)$, $\mathbf{F}=\mathfrak{sp}(3)_a\times\mathfrak{sp}(3)_b\times\mathfrak{sp}(3)_c$

Using the Weyl orbit expansion, we turn on a subgroup $\mathfrak{sp}(1)\times\mathfrak{sp}(1)\times\mathfrak{sp}(1)$ of the flavor group to compute the elliptic genus. We obtain the reduced one-string elliptic genus as

$$
\mathbb{E}_{h^{(1)}_{1,\mathfrak{so}(8)}}(q_\tau,v)=q_\tau^{-1/3}+q_\tau^{2/3}v^{-2}\sum_{n=0}^{\infty}q_\tau^{n}\frac{P_n(v)}{(1+v)^{10}},
\tag{D.0.5}
$$

where

$$
\begin{aligned}
P_0(v)=&\,4(7+34v-22v^2-496v^3-1128v^4+1326v^5+14327v^6+35392v^7\\
&+14327v^8+\cdots+7v^{14}).
\end{aligned}
$$

This agrees with the modular ansatz in (Del Zotto and Lockhart, 2018). Using the result with flavor fugacities turned on, we find the following exact formula for the subleading q order coefficient, which contains the 5d one-instanton Nekrasov partition function:

$$
\begin{aligned}
&\chi^{F}_{(010)_a\otimes(010)_b\otimes(010)_c}v^4-(\chi^{G}_{(1000)}\chi^{F}_{(100)_a\otimes(010)_b\otimes(010)_c}+\text{tri.})v^5+(\chi^{F}_{(100)_a\otimes(100)_b\otimes(001)_c}+\text{tri.})v^3\\
&-(\chi^{G}_{(1000)}\chi^{F}_{(100)_b\otimes(001)_c}+\text{tri.})v^4+(\chi^{F}_{(010)_b\otimes(010)_c}+\text{tri.})v^2+(\chi^{G}_{(2000)}\chi^{F}_{(010)_b\otimes(010)_c}+\text{tri.})v^6\\
&+(\chi^{G}_{(0011)}\chi^{F}_{(010)_a\otimes(100)_b\otimes(100)_c}+\text{tri.})v^6+(v-\chi^{G}_{(0100)}v^5-\chi^{G}_{(1011)}v^7)\chi^{F}_{(100)_a\otimes(100)_b\otimes(100)_c}\\
&-(\chi^{G}_{(2010)}\chi^{F}_{(100)_b\otimes(010)_c}+\text{tri.})v^7+(\chi^{G}_{(1000)}\chi^{F}_{(001)_a}+\text{tri.})v^3+(\chi^{G}_{(0011)}\chi^{F}_{(001)_a}+\text{tri.})v^5\\
&-(\chi^{G}_{(0100)}\chi^{F}_{(010)_a}+\text{tri.})v^4+(\chi^{G}_{(0022)}\chi^{F}_{(010)_a}+\text{tri.})v^8+(\chi^{G}_{(1100)}\chi^{F}_{(100)_b\otimes(100)_c}+\text{tri.})v^6\\
&+(\chi^{G}_{(2011)}\chi^{F}_{(100)_b\otimes(100)_c}+\text{tri.})v^8-(\chi^{G}_{(1000)}\chi^{F}_{(100)_a}+\text{tri.})v^{-1}-(\chi^{G}_{(0111)}\chi^{F}_{(100)_a}+\text{tri.})v^7\\
&-(\chi^{G}_{(1022)}\chi^{F}_{(100)_a}+\text{tri.})v^9+\chi^{G}_{(0100)}v^{-2}+\chi^{G}_{(0100)}+\chi^{F}_{(200)_a\oplus(200)_b\oplus(200)_c}+1\\
&+\chi^{G}_{(0200)}v^6+\chi^{G}_{(1111)}v^8+\chi^{G}_{(2022)}v^{10}+\\
&\sum_{n=0}^{\infty}\Big[\chi^{G}_{(0n00)}\chi^{F}_{(001)_a\otimes(001)_b\otimes(001)_c}v^{5+2n}-(\chi^{G}_{(1n00)}\chi^{F}_{(010)_a\otimes(001)_b\otimes(001)_c}+\text{tri.})v^{6+2n}\\
&+(\chi^{G}_{(1n10)}\chi^{F}_{(010)_a\otimes(010)_b\otimes(001)_c}+\text{tri.})v^{7+2n}-\chi^{G}_{(1n11)}\chi^{F}_{(010)_a\otimes(010)_b\otimes(010)_c}v^{8+2n}\\
&-(\chi^{G}_{(2n10)}\chi^{F}_{(100)_a\otimes(010)_b\otimes(001)_c}+\text{tri.})v^{8+2n}-(\chi^{G}_{(3n00)}\chi^{F}_{(001)_b\otimes(001)_c}+\text{tri.})v^{8+2n}\\
&+(\chi^{G}_{(2n11)}\chi^{F}_{(100)_a\otimes(010)_b\otimes(010)_c}+\text{tri.})v^{9+2n}+(\chi^{G}_{(3n10)}\chi^{F}_{(010)_b\otimes(001)_c}+\text{tri.})v^{9+2n}
\end{aligned}
$$

$$
\begin{aligned}
&+(\chi^G_{(2n20)}\chi^F_{(100)_a\otimes(100)_b\otimes(001)_c}+\text{tri.})v^{9+2n}-(\chi^G_{(3n20)}\chi^F_{(100)_b\otimes(001)_c}+\text{tri.})v^{10+2n}\\
&-(\chi^G_{(3n11)}\chi^F_{(010)_b\otimes(010)_c}+\text{tri.})v^{10+2n}-(\chi^G_{(1n22)}\chi^F_{(010)_a\otimes(100)_b\otimes(100)_c}+\text{tri.})v^{10+2n}\\
&+\chi^G_{(2n22)}\chi^F_{(100)_a\otimes(100)_b\otimes(100)_c}v^{11+2n}+(\chi^G_{(3n21)}\chi^F_{(100)_b\otimes(010)_c}+\text{tri.})v^{11+2n}\\
&+(\chi^G_{(0n33)}\chi^F_{(001)_a}+\text{tri.})v^{11+2n}-(\chi^G_{(1n33)}\chi^F_{(010)_a}+\text{tri.})v^{12+2n}\\
&-(\chi^G_{(3n22)}\chi^F_{(100)_b\otimes(100)_c}+\text{tri.})v^{12+2n}+(\chi^G_{(2n33)}\chi^F_{(100)_a}+\text{tri.})v^{13+2n}-\chi^G_{(3n33)}v^{14+2n}\Big].
\end{aligned}
\tag{D.0.6}
$$

Here "tri." means the two or five more terms implied by triality of both $\mathfrak{so}(8)$ and the three $\mathfrak{sp}(3)$ flavor groups together. We represent the v expansion terms both inside and outside the infinite summation in a descending order of the flavor representations. By Weyl dimension formulas of $\mathfrak{so}(8)$ and $\mathfrak{sp}(3)$, the above exact formula goes back to the rational function of v after turning off the gauge and flavor fugacities.

$\mathbf{n=1}$, $\mathbf{G=\mathfrak{so}(9)}$, $\mathbf{F=\mathfrak{sp}(4)_a\times\mathfrak{sp}(3)_b}$

Using the Weyl orbit expansion, we turn on the subgroup $\mathfrak{sp}(1)\times\mathfrak{sp}(1)$ of the flavor group to compute the elliptic genus. We obtain the reduced one-string elliptic genus as

$$
\mathbb{E}_{h^{(1)}_{1,\mathfrak{so}(9)}}(q_\tau,v)=q_\tau^{-1/3}+q_\tau^{2/3}v^{-2}\sum_{n=0}^{\infty}q_\tau^n\frac{P_n(v)}{(1+v)^{12}},
\tag{D.0.7}
$$

where

$$
\begin{aligned}
P_0(v)=2(&18+132v+227v^2-936v^3-5226v^4-7904v^5+17037v^6\\
&+118788v^7+263632v^8+118788v^9+\cdots+18v^{16}).
\end{aligned}
$$

This agrees with the modular ansatz in (Del Zotto and Lockhart, 2018). Using the result with flavor fugacities turned on, we obtain the following exact formula for the subleading q order coefficient, which contains the 5d one-instanton Nekrasov partition function:

$$
\begin{aligned}
&\chi^{\mathfrak{so}(7)}_{(0100)}\chi^F_{(1000)_a}v^{-2}-(\chi^{\mathfrak{so}(7)}_{(1000)}\chi^F_{(1000)_a}+\chi^{\mathfrak{so}(7)}_{(0001)}\chi^F_{(002)_b})v^{-1}+\chi^{\mathfrak{so}(7)}_{(0100)}+\chi^F_{(2000)_a}+\chi^F_{(200)_b}\\
&+1+\chi^F_{(1000)_a\otimes(200)_b}v+(\chi^F_{(020)_b}+\chi^F_{(0100)_a\otimes(010)_b})v^2\\
&+(\chi^F_{(1000)_a\otimes(101)_b}+\chi^F_{(0010)_a\otimes(010)_b}+\chi^{\mathfrak{so}(7)}_{(0001)}\chi^F_{(001)_b})v^3+\dots\\
+&\sum_{n=0}^{\infty}\Big[\chi^{\mathfrak{so}(7)}_{(0n00)}\chi^F_{(0001)_a\otimes(002)_b}v^{6+2n}-(\chi^{\mathfrak{so}(7)}_{(0n01)}\chi^F_{(0001)_a\otimes(011)_b}+\chi^{\mathfrak{so}(7)}_{(1n00)}\chi^F_{(0010)_a\otimes(002)_b})v^{7+2n}\\
&+(\chi^{\mathfrak{so}(7)}_{(0n02)}\chi^F_{(0001)_a\otimes(101)_b}+\chi^{\mathfrak{so}(7)}_{(0n10)}\chi^F_{(0001)_a\otimes(020)_b}+\chi^{\mathfrak{so}(7)}_{(1n01)}\chi^F_{(0010)_a\otimes(011)_b}\\
&+\chi^{\mathfrak{so}(7)}_{(2n00)}\chi^F_{(0100)_a\otimes(002)_b})v^{8+2n}-(\chi^{\mathfrak{so}(7)}_{(0n03)}\chi^F_{(0001)_a\otimes(001)_b}+\chi^{\mathfrak{so}(7)}_{(0n11)}\chi^F_{(0001)_a\otimes(110)_b}\\
&+\chi^{\mathfrak{so}(7)}_{(1n02)}\chi^F_{(0010)_a\otimes(101)_b}+\chi^{\mathfrak{so}(7)}_{(1n10)}\chi^F_{(0010)_a\otimes(020)_b}+\chi^{\mathfrak{so}(7)}_{(2n01)}\chi^F_{(0100)_a\otimes(011)_b}\\
&+\chi^{\mathfrak{so}(7)}_{(3n00)}\chi^F_{(1000)_a\otimes(002)_b})v^{9+2n}+(\chi^{\mathfrak{so}(7)}_{(0n20)}\chi^F_{(0001)_a\otimes(200)_b}+\chi^{\mathfrak{so}(7)}_{(0n12)}\chi^F_{(0001)_a\otimes(010)_b}\\
&+\chi^{\mathfrak{so}(7)}_{(1n03)}\chi^F_{(0010)_a\otimes(001)_b}+\chi^{\mathfrak{so}(7)}_{(1n11)}\chi^F_{(0010)_a\otimes(110)_b}+\chi^{\mathfrak{so}(7)}_{(2n02)}\chi^F_{(0100)_a\otimes(101)_b}
\end{aligned}
$$

$$
\begin{aligned}
&+\chi^{\mathfrak{so}(7)}_{(2n10)}\chi^{F}_{(0100)_a\otimes(020)_b}+\chi^{\mathfrak{so}(7)}_{(3n01)}\chi^{F}_{(1000)_a\otimes(011)_b}+\chi^{\mathfrak{so}(7)}_{(4n00)}\chi^{F}_{(002)_b})v^{10+2n}\\
&-(\chi^{\mathfrak{so}(7)}_{(0n21)}\chi^{F}_{(0001)_a\otimes(100)_b}+\chi^{\mathfrak{so}(7)}_{(1n20)}\chi^{F}_{(0010)_a\otimes(200)_b}+\chi^{\mathfrak{so}(7)}_{(1n12)}\chi^{F}_{(0010)_a\otimes(010)_b}\\
&+\chi^{\mathfrak{so}(7)}_{(2n03)}\chi^{F}_{(0100)_a\otimes(001)_b}+\chi^{\mathfrak{so}(7)}_{(2n11)}\chi^{F}_{(0100)_a\otimes(110)_b}+\chi^{\mathfrak{so}(7)}_{(3n02)}\chi^{F}_{(1000)_a\otimes(101)_b}\\
&+\chi^{\mathfrak{so}(7)}_{(3n10)}\chi^{F}_{(1000)_a\otimes(020)_b}+\chi^{\mathfrak{so}(7)}_{(4n01)}\chi^{F}_{(011)_b})v^{11+2n}+(\chi^{\mathfrak{so}(7)}_{(0n30)}\chi^{F}_{(0001)_a}\\
&+\chi^{\mathfrak{so}(7)}_{(1n21)}\chi^{F}_{(0010)_a\otimes(100)_b}+\chi^{\mathfrak{so}(7)}_{(2n20)}\chi^{F}_{(0100)_a\otimes(200)_b}+\chi^{\mathfrak{so}(7)}_{(2n12)}\chi^{F}_{(0100)_a\otimes(010)_b}\\
&+\chi^{\mathfrak{so}(7)}_{(3n03)}\chi^{F}_{(1000)_a\otimes(001)_b}+\chi^{\mathfrak{so}(7)}_{(3n11)}\chi^{F}_{(1000)_a\otimes(110)_b}+\chi^{\mathfrak{so}(7)}_{(4n02)}\chi^{F}_{(101)_b}+\chi^{\mathfrak{so}(7)}_{(4n10)}\chi^{F}_{(020)_b})v^{12+2n}\\
&-(\chi^{\mathfrak{so}(7)}_{(1n30)}\chi^{F}_{(0010)_a}+\chi^{\mathfrak{so}(7)}_{(2n21)}\chi^{F}_{(0100)_a\otimes(100)_b}+\chi^{\mathfrak{so}(7)}_{(3n20)}\chi^{F}_{(1000)_a\otimes(200)_b}\\
&+\chi^{\mathfrak{so}(7)}_{(3n12)}\chi^{F}_{(1000)_a\otimes(010)_b}+\chi^{\mathfrak{so}(7)}_{(4n03)}\chi^{F}_{(001)_b}+\chi^{\mathfrak{so}(7)}_{(4n11)}\chi^{F}_{(110)_b})v^{13+2n}\\
&+(\chi^{\mathfrak{so}(7)}_{(2n30)}\chi^{F}_{(0100)_a}+\chi^{\mathfrak{so}(7)}_{(3n21)}\chi^{F}_{(1000)_a\otimes(100)_b}+\chi^{\mathfrak{so}(7)}_{(4n20)}\chi^{F}_{(200)_b}+\chi^{\mathfrak{so}(7)}_{(4n12)}\chi^{F}_{(010)_b})v^{14+2n}\\
&-(\chi^{\mathfrak{so}(7)}_{(3n30)}\chi^{F}_{(1000)_a}+\chi^{\mathfrak{so}(7)}_{(4n21)}\chi^{F}_{(100)_b})v^{15+2n}+\chi^{\mathfrak{so}(7)}_{(4n30)}v^{16+2n}\Big].
\end{aligned}
\tag{D.0.8}
$$

The sporadic terms outside the infinite summations are too long to present, thus here we only present those in a few leading orders.

$\mathbf{n = 1}$, $\mathbf{G = F_4}$, $\mathbf{F = \mathfrak{sp}(4)}$

Using v expansion method, we turn on all flavor $\mathfrak{sp}(4)$ fugacities to compute the reduced one-string elliptic genus. The 5d one-instanton Nekrasov partition function is contained in the subleading q order, for which we find the following exact formula

$$
\begin{aligned}
&\chi^{\mathfrak{sp}(4)}_{(0030)}v^7+\chi^{\mathfrak{sp}(4)}_{(0201)}v^6-\chi^{F_4}_{(0001)}\chi^{\mathfrak{sp}(4)}_{(0120)}v^8+\chi^{F_4}_{(0002)}\chi^{\mathfrak{sp}(4)}_{(1020)}v^9+\chi^{\mathfrak{sp}(4)}_{(1101)}(v^5-\chi^{F_4}_{(0001)}v^7)\\
&+\chi^{F_4}_{(0010)}\chi^{\mathfrak{sp}(4)}_{(0210)}v^9+\chi^{\mathfrak{sp}(4)}_{(0101)}(-\chi^{F_4}_{(0001)}v^6+\chi^{F_4}_{(0002)}v^8)+\chi^{\mathfrak{sp}(4)}_{(0020)}(v^4-\chi^{F_4}_{(0003)}v^{10})\\
&+\chi^{\mathfrak{sp}(4)}_{(2001)}(v^4+\chi^{F_4}_{(0010)}v^8)-\chi^{F_4}_{(0011)}\chi^{\mathfrak{sp}(4)}_{(1110)}v^{10}-\chi^{\mathfrak{sp}(4)}_{(0300)}(\chi^{F_4}_{(1000)}v^8+\chi^{F_4}_{(0100)}v^{10})\\
&-\chi^{F_4}_{(0011)}\chi^{\mathfrak{sp}(4)}_{(1001)}v^9+\chi^{\mathfrak{sp}(4)}_{(0110)}(v^3-\chi^{F_4}_{(0012)}v^{11})+\chi^{\mathfrak{sp}(4)}_{(2010)}(-\chi^{F_4}_{(1000)}v^7+\chi^{F_4}_{(0020)}v^{11})\\
&+\chi^{\mathfrak{sp}(4)}_{(1200)}(\chi^{F_4}_{(1001)}v^9+\chi^{F_4}_{(0101)}v^{11})+\chi^{\mathfrak{sp}(4)}_{(0001)}(\chi^{F_4}_{(0001)}v^4+\chi^{F_4}_{(0100)}v^8+\chi^{F_4}_{(0020)}v^{10})\\
&-\chi^{\mathfrak{sp}(4)}_{(1010)}(\chi^{F_4}_{(1000)}v^6-\chi^{F_4}_{(1001)}v^8+\chi^{F_4}_{(0021)}v^{12})+\chi^{\mathfrak{sp}(4)}_{(0200)}(v^2-\chi^{F_4}_{(1002)}v^{10}-\chi^{F_4}_{(0102)}v^{12})\\
&-\chi^{\mathfrak{sp}(4)}_{(2100)}(\chi^{F_4}_{(1010)}v^{10}+\chi^{F_4}_{(0110)}v^{12})-\chi^{\mathfrak{sp}(4)}_{(0010)}(\chi^{F_4}_{(1000)}v^5+\chi^{F_4}_{(1010)}v^9-\chi^{F_4}_{(0030)}v^{13})\\
&+\chi^{\mathfrak{sp}(4)}_{(1100)}(\chi^{F_4}_{(1011)}v^{11}+\chi^{F_4}_{(0111)}v^{13})+\chi^{\mathfrak{sp}(4)}_{(3000)}(v+\chi^{F_4}_{(2000)}v^9+\chi^{F_4}_{(1100)}v^{11}+\chi^{F_4}_{(0200)}v^{13})\\
&+\chi^{\mathfrak{sp}(4)}_{(0100)}(\chi^{F_4}_{(2000)}v^8-\chi^{F_4}_{(1020)}v^{12}-\chi^{F_4}_{(0120)}v^{14})+\chi^{\mathfrak{sp}(4)}_{(2000)}(1-\chi^{F_4}_{(2001)}v^{10}-\chi^{F_4}_{(1101)}v^{12}\\
&-\chi^{F_4}_{(0201)}v^{14})+\chi^{\mathfrak{sp}(4)}_{(1000)}(-\chi^{F_4}_{(0001)}v^{-1}+\chi^{F_4}_{(2010)}v^{11}+\chi^{F_4}_{(1110)}v^{13}+\chi^{F_4}_{(0210)}v^{15})\\
&+(\chi^{F_4}_{(1000)}v^{-2}+\chi^{F_4}_{(1000)}+1-\chi^{F_4}_{(3000)}v^{10}-\chi^{F_4}_{(2100)}v^{12}-\chi^{F_4}_{(1200)}v^{14}-\chi^{F_4}_{(0300)}v^{16})
\end{aligned}
$$

$$
\sum_{n=0}^{\infty}\Big[\chi^{F_4}_{(n000)}\chi^{\mathfrak{sp}(4)}_{(0003)}v^{8+2n}-\chi^{F_4}_{(n001)}\chi^{\mathfrak{sp}(4)}_{(0012)}v^{9+2n}+(\chi^{F_4}_{(n010)}\chi^{\mathfrak{sp}(4)}_{(0021)}+\chi^{F_4}_{(n002)}\chi^{\mathfrak{sp}(4)}_{(0102)})v^{10+2n}
$$

$$
\begin{aligned}
&- (\chi^{F_4}_{(n100)}\chi^{\mathfrak{sp}(4)}_{(0030)} + \chi^{F_4}_{(n011)}\chi^{\mathfrak{sp}(4)}_{(0111)} + \chi^{F_4}_{(n003)}\chi^{\mathfrak{sp}(4)}_{(1002)})v^{11+2n} \\
&+ (\chi^{F_4}_{(n012)}\chi^{\mathfrak{sp}(4)}_{(1011)} + \chi^{F_4}_{(n020)}\chi^{\mathfrak{sp}(4)}_{(0201)} + \chi^{F_4}_{(n101)}\chi^{\mathfrak{sp}(4)}_{(0120)} + \chi^{F_4}_{(n004)}\chi^{\mathfrak{sp}(4)}_{(0002)})v^{12+2n} \\
&- (\chi^{F_4}_{(n102)}\chi^{\mathfrak{sp}(4)}_{(1020)} + \chi^{F_4}_{(n013)}\chi^{\mathfrak{sp}(4)}_{(0011)} + \chi^{F_4}_{(n021)}\chi^{\mathfrak{sp}(4)}_{(1101)} + \chi^{F_4}_{(n110)}\chi^{\mathfrak{sp}(4)}_{(0210)})v^{13+2n} \\
&+ (\chi^{F_4}_{(n022)}\chi^{\mathfrak{sp}(4)}_{(0101)} + \chi^{F_4}_{(n103)}\chi^{\mathfrak{sp}(4)}_{(0020)} + \chi^{F_4}_{(n030)}\chi^{\mathfrak{sp}(4)}_{(2001)} + \chi^{F_4}_{(n111)}\chi^{\mathfrak{sp}(4)}_{(1110)} \\
&+ \chi^{F_4}_{(n200)}\chi^{\mathfrak{sp}(4)}_{(0300)})v^{14+2n} - (\chi^{F_4}_{(n031)}\chi^{\mathfrak{sp}(4)}_{(1001)} + \chi^{F_4}_{(n112)}\chi^{\mathfrak{sp}(4)}_{(0110)} + \chi^{F_4}_{(n120)}\chi^{\mathfrak{sp}(4)}_{(2010)} \\
&+ \chi^{F_4}_{(n201)}\chi^{\mathfrak{sp}(4)}_{(1200)})v^{15+2n} + (\chi^{F_4}_{(n040)}\chi^{\mathfrak{sp}(4)}_{(0001)} + \chi^{F_4}_{(n121)}\chi^{\mathfrak{sp}(4)}_{(1010)} + \chi^{F_4}_{(n202)}\chi^{\mathfrak{sp}(4)}_{(0200)} \\
&+ \chi^{F_4}_{(n210)}\chi^{\mathfrak{sp}(4)}_{(2100)})v^{16+2n} - (\chi^{F_4}_{(n130)}\chi^{\mathfrak{sp}(4)}_{(0010)} + \chi^{F_4}_{(n211)}\chi^{\mathfrak{sp}(4)}_{(1100)} + \chi^{F_4}_{(n300)}\chi^{\mathfrak{sp}(4)}_{(3000)})v^{17+2n} \\
&+ (\chi^{F_4}_{(n220)}\chi^{\mathfrak{sp}(4)}_{(0100)} + \chi^{F_4}_{(n301)}\chi^{\mathfrak{sp}(4)}_{(2000)})v^{18+2n} - \chi^{F_4}_{(n310)}\chi^{\mathfrak{sp}(4)}_{(1000)}v^{19+2n} + \chi^{F_4}_{(n400)}v^{20+2n}\Big].
\end{aligned} \tag{D.0.9}
$$

Turning off all F_4 and $\mathfrak{sp}(4)$ fugacities, the above exact formula reduces to the rational function of v in (5.5.98) by Weyl dimension formulas.

$\mathbf{n = 2,\ G = \mathfrak{so}(9),\ F = \mathfrak{sp}(3)_a \times \mathfrak{sp}(2)_b}$

Using the Weyl orbit expansion, we turn on the subgroup $\mathfrak{sp}(1) \times \mathfrak{sp}(1)$ of the flavor group to compute the elliptic genus. We obtain the reduced one-string elliptic genus as

$$
\mathbb{E}_{h^{(1)}_{2,\mathfrak{so}(9)}}(q_\tau, v) = q_\tau^{1/6} v^{-1} \sum_{n=0}^{\infty} q_\tau^n \frac{P_n(v)}{(1+v)^{12}}, \tag{D.0.10}
$$

where

$$
\begin{aligned}
P_0(v) = (1-v)^2(&1 + 14v + 93v^2 + 392v^3 + 1181v^4 + 2658v^5 \\
&+ 4106v^6 + 2658v^7 + \cdots + v^{12}).
\end{aligned}
$$

This agrees with the modular ansatz in (Del Zotto and Lockhart, 2018). Using the result with flavor fugacities turned on, we obtain the following exact v expansion formula for the leading q order coefficient, which contains the reduced 5d one-instanton Nekrasov partition function:

$$
\begin{aligned}
&- \chi^F_{(001)_a}v^4 - \chi^F_{(010)_a}(\chi^F_{(20)_b}v^5 - \chi^{\mathfrak{so}(9)}_{(0001)}\chi^F_{(10)_b}v^6 + \chi^{\mathfrak{so}(9)}_{(0010)}v^7) + \chi^F_{(100)_a}(-\chi^F_{(01)_b}v^4 \\
&+ \chi^{\mathfrak{so}(9)}_{(1000)}\chi^F_{(20)_b}v^6 - \chi^{\mathfrak{so}(9)}_{(1001)}\chi^F_{(10)_b}v^7 + \chi^{\mathfrak{so}(9)}_{(1010)}v^8 + \chi^{\mathfrak{so}(9)}_{(0100)}v^6) + \chi^F_{(01)_b}(-v^3 + \chi^{\mathfrak{so}(9)}_{(1000)}v^5) \\
&- \chi^{\mathfrak{so}(9)}_{(2000)}\chi^F_{(20)_b}v^7 + \chi^{\mathfrak{so}(9)}_{(2001)}\chi^F_{(10)_b}v^8 + v^{-1} - \chi^{\mathfrak{so}(9)}_{(1100)}v^7 - \chi^{\mathfrak{so}(9)}_{(2010)}v^9 + \\
&\sum_{n=0}^{\infty}\Big[\chi^F_{(001)_a}\Big(-\chi^{\mathfrak{so}(9)}_{(0n00)}\chi^F_{(02)_b}v^{6+2n} + \chi^{\mathfrak{so}(9)}_{(0n01)}\chi^F_{(11)_b}v^{7+2n} - (\chi^{\mathfrak{so}(9)}_{(0n02)}\chi^F_{(01)_b} \\
&+ \chi^{\mathfrak{so}(9)}_{(0n10)}\chi^F_{(20)_b})v^{8+2n} + \chi^{\mathfrak{so}(9)}_{(0n11)}\chi^F_{(10)_b}v^{9+2n} - \chi^{\mathfrak{so}(9)}_{(0n20)}v^{10+2n}\Big) \\
&+ \chi^F_{(010)_a}\Big(\chi^{\mathfrak{so}(9)}_{(1n00)}\chi^F_{(02)_b}v^{7+2n} - \chi^{\mathfrak{so}(9)}_{(1n01)}\chi^F_{(11)_b}v^{8+2n} + (\chi^{\mathfrak{so}(9)}_{(1n02)}\chi^F_{(01)_b} \\
&+ \chi^{\mathfrak{so}(9)}_{(1n10)}\chi^F_{(20)_b})v^{9+2n} - \chi^{\mathfrak{so}(9)}_{(1n11)}\chi^F_{(10)_b}v^{10+2n} + \chi^{\mathfrak{so}(9)}_{(1n20)}v^{11+2n}\Big)
\end{aligned}
$$

$$+\chi^F_{(100)_a}\Big(-\chi^{\mathfrak{so}(9)}_{(2n00)}\chi^F_{(02)_b}v^{8+2n}+\chi^{\mathfrak{so}(9)}_{(2n01)}\chi^F_{(11)_b}v^{9+2n}-(\chi^{\mathfrak{so}(9)}_{(2n02)}\chi^F_{(01)_b}$$
$$+\chi^{\mathfrak{so}(9)}_{(2n10)}\chi^F_{(20)_b})v^{10+2n}+\chi^{\mathfrak{so}(9)}_{(2n11)}\chi^F_{(10)_b}v^{11+2n}-\chi^{\mathfrak{so}(9)}_{(2n20)}v^{12+2n}\Big)$$
$$+\Big(\chi^{\mathfrak{so}(9)}_{(3n00)}\chi^F_{(02)_b}v^{9+2n}-\chi^{\mathfrak{so}(9)}_{(3n01)}\chi^F_{(11)_b}v^{10+2n}+(\chi^{\mathfrak{so}(9)}_{(3n02)}\chi^F_{(01)_b}+\chi^{\mathfrak{so}(9)}_{(3n10)}\chi^F_{(20)_b})v^{11+2n}$$
$$-\chi^{\mathfrak{so}(9)}_{(3n11)}\chi^F_{(10)_b}v^{12+2n}+\chi^{\mathfrak{so}(9)}_{(3n20)}v^{13+2n}\Big)\Big]. \tag{D.0.11}$$

A few leading terms in the v expansion has been determined in (H.20) of (Del Zotto and Lockhart, 2018).

$\mathbf{n=2}$, $\mathbf{G=\mathfrak{so}(10)}$, $\mathbf{F=\mathfrak{sp}(4)_a\times\mathfrak{su}(2)_b\times\mathfrak{u}(1)_c}$

Using the Weyl orbit expansion, we turn on the subgroup $\mathfrak{sp}(1)\times\mathfrak{su}(2)\times\mathfrak{u}(1)$ of the flavor group to compute the elliptic genus. We obtain the reduced one-string elliptic genus as

$$\mathbb{E}_{h^{(1)}_{2,\mathfrak{so}(10)}}(q_\tau,v)=q_\tau^{1/6}v^{-1}\sum_{n=0}^{\infty}q_\tau^n\frac{P_n(v)}{(1+v)^{14}}, \tag{D.0.12}$$

where

$$P_0(v)=(1-v)^2(1+16v+122v^2+592v^3+2060v^4+5472v^5+11287v^6+16496v^7$$
$$+11287v^8+5472v^9+2060v^{10}+592v^{11}+122v^{12}+16v^{13}+v^{14}).$$

This agrees with the modular ansatz in (Del Zotto and Lockhart, 2018). Using the result with flavor fugacities turned on, we obtain the following exact v expansion formula for the leading q order coefficient, which contains the reduced 5d one-instanton Nekrasov partition function:

$$v^{-1}-\chi^F_{(2)_b}v^3-\chi^F_{(1000)_a\otimes((2)_c\oplus(-2)_c)}v^4+(\chi^{\mathfrak{so}(10)}_{(10000)}\chi^F_{(2)_c\oplus(-2)_c}-\chi^F_{(0100)_a\otimes(2)_b}$$
$$-\chi^F_{(0001)_a})v^5+(\chi^F_{(0010)_a\otimes((2)_c\oplus(-2)_c)}-\chi^{\mathfrak{so}(10)}_{(10000)}\chi^F_{(1000)_a})\chi^F_{(2)_b}v^6+\cdots+$$
$$\sum_{n=0}^{\infty}\Big[(\chi^{\mathfrak{u}(1)}_{(-4)\oplus(4)}+\chi^{\mathfrak{su}(2)}_{(4)})\Big(-\chi^{\mathfrak{so}(10)}_{(0n000)}\chi^{\mathfrak{sp}(4)}_{(0001)}v^{7+2n}+\chi^{\mathfrak{so}(10)}_{(1n000)}\chi^{\mathfrak{sp}(4)}_{(0010)}v^{8+2n}$$
$$-\chi^{\mathfrak{so}(10)}_{(2n000)}\chi^{\mathfrak{sp}(4)}_{(0100)}v^{9+2n}+\chi^{\mathfrak{so}(10)}_{(3n000)}\chi^{\mathfrak{sp}(4)}_{(1000)}v^{10+2n}-\chi^{\mathfrak{so}(10)}_{(4n000)}v^{11+2n}\Big)+\Big((\chi^{\mathfrak{u}(1)}_{(-3)}\chi^{\mathfrak{su}(2)}_{(1)}$$
$$+\chi^{\mathfrak{u}(1)}_{(1)}\chi^{\mathfrak{su}(2)}_{(3)})(\chi^{\mathfrak{so}(10)}_{(0n001)}\chi^{\mathfrak{sp}(4)}_{(0001)}v^{8+2n}-\chi^{\mathfrak{so}(10)}_{(1n001)}\chi^{\mathfrak{sp}(4)}_{(0010)}v^{9+2n}+\chi^{\mathfrak{so}(10)}_{(2n001)}\chi^{\mathfrak{sp}(4)}_{(0100)}v^{10+2n}$$
$$-\chi^{\mathfrak{so}(10)}_{(3n001)}\chi^{\mathfrak{sp}(4)}_{(1000)}v^{11+2n}+\chi^{\mathfrak{so}(10)}_{(4n001)}v^{12+2n})+c.c.\Big)$$
$$+\chi^{\mathfrak{u}(1)}_{(-2)\oplus(2)}\chi^{\mathfrak{su}(2)}_{(2)}\Big(-\chi^{\mathfrak{so}(10)}_{(0n100)}\chi^{\mathfrak{sp}(4)}_{(0001)}v^{9+2n}+\chi^{\mathfrak{so}(10)}_{(1n100)}\chi^{\mathfrak{sp}(4)}_{(0010)}v^{10+2n}$$
$$-\chi^{\mathfrak{so}(10)}_{(2n100)}\chi^{\mathfrak{sp}(4)}_{(0100)}v^{11+2n}+\chi^{\mathfrak{so}(10)}_{(3n100)}\chi^{\mathfrak{sp}(4)}_{(1000)}v^{12+2n}-\chi^{\mathfrak{so}(10)}_{(4n100)}v^{13+2n}\Big)$$
$$-\Big(\chi^{\mathfrak{u}(1)}_{(2)}(\chi^{\mathfrak{so}(10)}_{(0n020)}\chi^{\mathfrak{sp}(4)}_{(0001)}v^{9+2n}-\chi^{\mathfrak{so}(10)}_{(1n020)}\chi^{\mathfrak{sp}(4)}_{(0010)}v^{10+2n}+\chi^{\mathfrak{so}(10)}_{(2n020)}\chi^{\mathfrak{sp}(4)}_{(0100)}v^{11+2n}$$
$$-\chi^{\mathfrak{so}(10)}_{(3n020)}\chi^{\mathfrak{sp}(4)}_{(1000)}v^{12+2n}+\chi^{\mathfrak{so}(10)}_{(4n020)}v^{13+2n})+c.c.\Big)$$
$$+\Big(\chi^{\mathfrak{u}(1)}_{(-1)}\chi^{\mathfrak{su}(2)}_{(1)}(\chi^{\mathfrak{so}(10)}_{(0n101)}\chi^{\mathfrak{sp}(4)}_{(0001)}v^{10+2n}-\chi^{\mathfrak{so}(10)}_{(1n101)}\chi^{\mathfrak{sp}(4)}_{(0010)}v^{11+2n}+\chi^{\mathfrak{so}(10)}_{(2n101)}\chi^{\mathfrak{sp}(4)}_{(0100)}v^{12+2n}$$

$$
\begin{aligned}
&-\chi^{\mathfrak{so}(10)}_{(3n101)}\chi^{\mathfrak{sp}(4)}_{(1000)}v^{13+2n}+\chi^{\mathfrak{so}(10)}_{(4n101)}v^{14+2n})+c.c.\Big)-\chi^{\mathfrak{su}(2)}_{(2)}(\chi^{\mathfrak{so}(10)}_{(0n011)}\chi^{\mathfrak{sp}(4)}_{(0001)}v^{9+2n}\\
&-\chi^{\mathfrak{so}(10)}_{(1n011)}\chi^{\mathfrak{sp}(4)}_{(0010)}v^{10+2n}+\chi^{\mathfrak{so}(10)}_{(2n011)}\chi^{\mathfrak{sp}(4)}_{(0100)}v^{11+2n}-\chi^{\mathfrak{so}(10)}_{(3n011)}\chi^{\mathfrak{sp}(4)}_{(1000)}v^{12+2n}\\
&+\chi^{\mathfrak{so}(10)}_{(4n011)}v^{13+2n})+\Big(-\chi^{\mathfrak{so}(10)}_{(0n200)}\chi^{\mathfrak{sp}(4)}_{(0001)}v^{11+2n}+\chi^{\mathfrak{so}(10)}_{(1n200)}\chi^{\mathfrak{sp}(4)}_{(0010)}v^{12+2n}\\
&-\chi^{\mathfrak{so}(10)}_{(2n200)}\chi^{\mathfrak{sp}(4)}_{(0100)}v^{13+2n}+\chi^{\mathfrak{so}(10)}_{(3n200)}\chi^{\mathfrak{sp}(4)}_{(1000)}v^{14+2n}-\chi^{\mathfrak{so}(10)}_{(4n200)}v^{15+2n}\Big)\Big]. \qquad \text{(D.0.13)}
\end{aligned}
$$

The sporadic terms outside the infinite summation are too long to present, thus here we only show some in leading orders. In general they can be recovered from the terms inside the infinite summation. Note the complex conjugate *c.c.* interchanges the Dynkin labels of spinor and conjugate spinor representations of gauge $\mathfrak{so}(10)$ and reverses the charge of $\mathfrak{u}(1)$ flavor simultaneously. We also checked this expression from 5d blowup equations. A few leading terms in the v expansion has been determined in (H.21) of (Del Zotto and Lockhart, 2018).

$\mathbf{n=2}$, $\mathbf{G=\mathfrak{so}(11)}$, $\mathbf{F=\mathfrak{sp}(5)_a\times\mathfrak{so}(2)_b}$

There are 128 unity blowup equations in total. Let us regard the flavor subgroup $\mathfrak{so}(2)$ as $\mathfrak{u}(1)$. The r fields $\lambda_{\mathfrak{sp}(5)}$ takes value in $\mathcal{O}^{\mathfrak{sp}(5)}_{[00001]}$, while $\lambda_{\mathfrak{u}(1)}=\pm1/2$. Using the Weyl orbit expansion method, we turn on a subgroup $\mathfrak{sp}(1)\times\mathfrak{u}(1)$ of the flavor and compute the one-string elliptic genus to $\mathcal{O}(q_\tau^2)$. For example, with gauge and flavor fugacities turned off we obtain the reduced one-string elliptic genus as[1]

$$
\mathbb{E}_{h^{(1)}_{2,\mathfrak{so}(11)}}(q_\tau,v)=q_\tau^{1/6}\sum_{n=0}^{\infty}q_\tau^n\frac{(1-v)^2P_n(v)}{v(1+v)^{16}}, \qquad \text{(D.0.15)}
$$

where

$$
\begin{aligned}
P_0(v)=&\,1+18v+155v^2+852v^3+3367v^4+10208v^5+24624v^6+47390v^7\\
&+66362v^8+\cdots+v^{16},\\
P_1(v)=&\,v^{-2}(55+816v+5505v^2+21936v^3+55038v^4+79650v^5+18864v^6\\
&-193544v^7-427293v^8-245690v^9+410958v^{10}-\cdots+55v^{20}). \qquad \text{(D.0.16)}
\end{aligned}
$$

Turning on all gauge and flavor fugacities, we find the following exact formula for the leading q_τ order of reduced one-string elliptic genus, i.e. the Hilbert series:

$$
\begin{aligned}
\chi^F_{(-2)_b\oplus(2)_b}\Big(&-v^3-\chi^F_{(00100)_a}v^6+(\chi^G_{(10000)}\chi^F_{(01000)_a}-\chi^F_{(00010)_a})v^7\\
&+(\chi^G_{(10000)}\chi^F_{(00100)_a}-\chi^G_{(20000)}\chi^F_{(10000)_a})v^8+(\chi^G_{(30000)}-\chi^G_{(20000)}\chi^F_{(01000)_a})v^9\\
&+\chi^G_{(30000)}\chi^F_{(10000)_a}v^{10}-\chi^G_{(40000)}v^{11}\Big)
\end{aligned}
$$

[1] In (Del Zotto and Lockhart, 2018), the modular ansatz for the reduced one-string elliptic genus of this theory is determined up to two unfixed parameters. Using our result from blowup equations, we are able to determine their two unfixed parameters as

$$
a_1=\frac{16291}{1283918464548864},\ a_2=\frac{9983}{7703510787293184}. \qquad \text{(D.0.14)}
$$

$$
\begin{aligned}
&\chi^F_{(-1)_b\oplus(1)_b}\Big(\chi^G_{(00001)}\chi^F_{(00010)_a}v^8-\chi^G_{(10001)}\chi^F_{(00100)_a}v^9+\chi^G_{(20001)}\chi^F_{(01000)_a}v^{10}\\
&\qquad\qquad -\chi^G_{(30001)}\chi^F_{(10000)_a}v^{11}+\chi^G_{(40001)}v^{12}\Big)\\
&+\Big(v^{-1}-\chi^F_{(10000)_a}v^4+(\chi^G_{(10000)}-\chi^F_{(01000)_a})v^5+(\chi^G_{(10000)}\cdot\chi^F_{(10000)_a}-\chi^F_{(00001)_a})v^6\\
&\quad -(\chi^G_{(20000)}+\chi^F_{(00010)_a})v^7+\chi^G_{(10000)\oplus(01000)}\chi^F_{(00100)_a}v^8\\
&\quad -(\chi^G_{(00100)}\chi^F_{(00010)_a}+\chi^G_{(20000)\oplus(11000)}\chi^F_{(01000)_a})v^9\\
&\quad +(\chi^G_{(10100)}\chi^F_{(00100)_a}+\chi^G_{(30000)\oplus(21000)}\chi^F_{(10000)_a})v^{10}\\
&\quad -(\chi^G_{(20100)}\chi^F_{(01000)_a}+\chi^G_{(40000)\oplus(31000)})v^{11}+\chi^G_{(30100)}\chi^F_{(10000)_a}v^{12}-\chi^G_{(40100)}v^{13}\Big)\\
&+\sum_{n=0}^{\infty}\Big[\chi^F_{(-4)_b\oplus(4)_b}\Big(-v^{8+2n}\chi^G_{(0n000)}\chi^F_{(00001)_a}+v^{9+2n}\chi^G_{(1n000)}\chi^F_{(00010)_a}-v^{10+2n}\chi^G_{(2n000)}\chi^F_{(00100)_a}\\
&\quad +v^{11+2n}\chi^G_{(3n000)}\chi^F_{(01000)_a}-v^{12+2n}\chi^G_{(4n000)}\chi^F_{(10000)_a}+v^{13+2n}\chi^G_{(5n000)}\Big)\\
&+\chi^F_{(-3)_b\oplus(3)_b}\Big(v^{9+2n}\chi^G_{(0n001)}\chi^F_{(00001)_a}-v^{10+2n}\chi^G_{(1n001)}\chi^F_{(00010)_a}+v^{11+2n}\chi^G_{(2n001)}\chi^F_{(00100)_a}\\
&\quad -v^{12+2n}\chi^G_{(3n001)}\chi^F_{(01000)_a}+v^{13+2n}\chi^G_{(4n001)}\chi^F_{(10000)_a}-v^{14+2n}\chi^G_{(5n001)}\Big)\\
&+\chi^F_{(-2)_b\oplus(2)_b}\Big(-v^{10+2n}(\chi^G_{(0n100)}+\chi^G_{(0n010)})\chi^F_{(00001)_a}+v^{11+2n}(\chi^G_{(1n100)}+\chi^G_{(1n010)})\chi^F_{(00010)_a}\\
&\quad -v^{12+2n}(\chi^G_{(2n100)}+\chi^G_{(2n010)})\chi^F_{(00100)_a}+v^{13+2n}(\chi^G_{(3n100)}+\chi^G_{(3n010)})\chi^F_{(01000)_a}\\
&\quad -v^{14+2n}(\chi^G_{(4n100)}+\chi^G_{(4n010)})\chi^F_{(10000)_a}+v^{15+2n}(\chi^G_{(5n100)}+\chi^G_{(5n010)})\Big)\\
&+\chi^F_{(-1)_b\oplus(1)_b}\Big(v^{9+2n}(\chi^G_{(0n001)}+v^2\chi^G_{(0n101)})\chi^F_{(00001)_a}-v^{10+2n}(\chi^G_{(1n001)}+v^2\chi^G_{(1n101)})\chi^F_{(00010)_a}\\
&\quad +v^{11+2n}(\chi^G_{(2n001)}+v^2\chi^G_{(2n101)})\chi^F_{(00100)_a}-v^{12+2n}(\chi^G_{(3n001)}+v^2\chi^G_{(3n101)})\chi^F_{(01000)_a}\\
&\quad +v^{13+2n}(\chi^G_{(4n001)}+v^2\chi^G_{(4n101)})\chi^F_{(10000)_a}-v^{14+2n}(\chi^G_{(5n001)}+v^2\chi^G_{(5n101)})\Big)\\
&+\Big(-v^{8+2n}(\chi^G_{(0n000)}+v^2(\chi^G_{(0n100)}+\chi^G_{(0n002)})+v^4\chi^G_{(0n200)})\chi^F_{(00001)_a}\\
&\quad +v^{9+2n}(\chi^G_{(1n000)}+v^2(\chi^G_{(1n100)}+\chi^G_{(1n002)})+v^4\chi^G_{(1n200)})\chi^F_{(00010)_a}\\
&\quad -v^{10+2n}(\chi^G_{(2n000)}+v^2(\chi^G_{(2n100)}+\chi^G_{(2n002)})+v^4\chi^G_{(2n200)})\chi^F_{(00100)_a}\\
&\quad +v^{11+2n}(\chi^G_{(3n000)}+v^2(\chi^G_{(3n100)}+\chi^G_{(3n002)})+v^4\chi^G_{(3n200)})\chi^F_{(01000)_a}\\
&\quad -v^{12+2n}(\chi^G_{(4n000)}+v^2(\chi^G_{(4n100)}+\chi^G_{(4n002)})+v^4\chi^G_{(4n200)})\chi^F_{(10000)_a}\\
&\quad +v^{13+2n}(\chi^G_{(5n000)}+v^2(\chi^G_{(5n100)}+\chi^G_{(5n002)})+v^4\chi^G_{(5n200)})\Big)\Big].
\end{aligned}
\tag{D.0.17}
$$

The subleading q_τ order of reduced one-string elliptic genus is

$$
\begin{aligned}
&\mathbf{55}v^{-3}-(\mathbf{11}\cdot\chi^F_{(10000)_a}+\mathbf{32}\cdot\chi^F_{(-1)_b\oplus(1)_b})v^{-2}+(\mathbf{55}+\chi^F_{(20000)_a}+2)v^{-1}\\
&+\chi^F_{(10000)_a\otimes((-1)_b\oplus(1)_b\oplus(0)_b)}+(\chi^F_{(01000)_a\otimes((-2)_b\oplus(2)_b)}+\chi^F_{(-4)_b\oplus(4)_b}+1)v\\
&+(\chi^F_{(00100)_a}+\mathbf{32}\cdot\chi^F_{(-1)_b\oplus(1)_b})v^2+\mathcal{O}(v^3)
\end{aligned}
\tag{D.0.18}
$$

Let us further denote

$$\mathbb{E}_{h^{(1)}_{2,\mathfrak{so}(11)}}(q_\tau, v) = q_\tau^{1/6} v^{-1} \sum_{i,j} c_{i,j} v^j (q_\tau/v^2)^i. \tag{D.0.19}$$

Then we have the following table D.1 for the coefficients c_{ij}. Note the red numbers in the first column are just the dimensions of representations $k\theta$ of $\mathfrak{so}(11)$ where θ is the adjoint representation. The blue numbers in the second column are given by $-10\dim(\chi^{\mathfrak{so}(11)}_{[1n000]}) - 2\dim(\chi^{\mathfrak{so}(11)}_{[0n001]})$ with $n = i-1$, consistent with the fact that the matter is in representation $(\mathbf{11}, \mathbf{10}^a) \oplus (\mathbf{32}, \mathbf{2}^b)$. The orange number 112 in the third column is given by $\dim(\mathfrak{so}(11)) + \dim(\mathfrak{sp}(5)\times\mathfrak{u}(1)) + 1 = 55 + 55 + 1 + 1 = 112$. These are the constraints given in (Del Zotto and Lockhart, 2018) by analyzing the spectral flow to NSR elliptic genus, which our result satisfies perfectly.

i,j	0	1	2	3	4	5	6	7	8	9
0	1	0	0	0	-2	-10	-33	-242	408	18544
1	55	-174	112	30	91	174	-150	-686	-651	-33420
2	1144	-7106	17037	-17196	2998	330	6602	15822	-16128	-16234

Table D.1: Series coefficients $c_{i,j}$ for the one-string elliptic genus of $\mathfrak{n} = 2$ $\mathfrak{so}(11)$ model.

$\mathbf{n = 2}$, $\mathbf{G = \mathfrak{so}(12)_a}$, $\mathbf{F = \mathfrak{sp}(6)_a \times \mathfrak{so}(2)_b}$

This is a chiral theory in the sense that the spinor and conjugate spinor representations of $\mathfrak{so}(12)$ are not on an equal footing. The chirality comes from the matter representation $(\mathbf{32_s}, \mathbf{2}_b)$. This is reflected in the vanishing r fields in Table 5.6 and also the exact v expansion formula below (D.0.22). Using the Weyl orbit expansion, we turn on the subgroup $\mathfrak{sp}(1)\times\mathfrak{u}(1)$ of the flavor group to compute the elliptic genus. We obtain the reduced one-string elliptic genus as

$$\mathbb{E}_{h^{(1)}_{2,\mathfrak{so}(12)}}(q_\tau, v) = q_\tau^{1/6} v^{-1} \sum_{n=0}^{\infty} q_\tau^n \frac{P_n(v)}{(1+v)^{18}}, \tag{D.0.20}$$

where

$$\begin{aligned} P_0(v) = (1-v)^2(&1 + 20v + 192v^2 + 1180v^3 + 5226v^4 + 17804v^5 + 48575v^6 \\ &+ 108512v^7 + 197370v^8 + 267144v^9 + 197370v^{10} + \cdots + v^{18}). \end{aligned} \tag{D.0.21}$$

This agrees with the modular ansatz in (Del Zotto and Lockhart, 2018). Using the result with flavor fugacities turned on, we obtain the following exact v expansion formula for the leading q order coefficient, which contains the reduced 5d one-instanton Nekrasov partition function:

$$\begin{aligned} &v^{-1} - \chi^F_{(2)_b\oplus(-2)_b} v^3 - \chi^F_{(010000)_a} v^5 + \chi^{\mathfrak{so}(12)}_{(100000)} \chi^F_{(100000)_a} v^6 - (\chi^G_{(200000)} \chi^F_{(01)_b} \\ &+ \chi^F_{(000100)_a\otimes((2)_b\oplus(-2)_b)} + \chi^F_{(000001)_a}) v^7 + \chi^{\mathfrak{so}(12)}_{(100000)} \chi^F_{(001000)_a\otimes((2)_b\oplus(-2)_b)} v^8 + \cdots + \\ &\sum_{n=0}^{\infty} \Big[\chi^{\mathfrak{u}(1)}_{(-4)\oplus(4)} \Big(-\chi^{\mathfrak{so}(12)}_{(0n0000)} \chi^{\mathfrak{sp}(6)}_{(000001)} v^{9+2n} + \chi^{\mathfrak{so}(12)}_{(1n0000)} \chi^{\mathfrak{sp}(6)}_{(000010)} v^{10+2n} - \chi^{\mathfrak{so}(12)}_{(2n0000)} \chi^{\mathfrak{sp}(6)}_{(000100)} v^{11+2n} \end{aligned}$$

$$
\begin{aligned}
&+\chi^{\mathfrak{so}(12)}_{(3n0000)}\chi^{\mathfrak{sp}(6)}_{(001000)}v^{12+2n}-\chi^{\mathfrak{so}(12)}_{(4n0000)}\chi^{\mathfrak{sp}(6)}_{(010000)}v^{13+2n}+\chi^{\mathfrak{so}(12)}_{(5n0000)}\chi^{\mathfrak{sp}(6)}_{(100000)}v^{14+2n}\\
&-\chi^{\mathfrak{so}(12)}_{(6n0000)}v^{15+2n}\Big)+\chi^{\mathfrak{u}(1)}_{(-3)\oplus(3)}\Big(\chi^{\mathfrak{so}(12)}_{(0n0001)}\chi^{\mathfrak{sp}(6)}_{(000001)}v^{10+2n}-\chi^{\mathfrak{so}(12)}_{(1n0001)}\chi^{\mathfrak{sp}(6)}_{(000010)}v^{11+2n}\\
&+\chi^{\mathfrak{so}(12)}_{(2n0001)}\chi^{\mathfrak{sp}(6)}_{(000100)}v^{12+2n}-\chi^{\mathfrak{so}(12)}_{(3n0001)}\chi^{\mathfrak{sp}(6)}_{(001000)}v^{13+2n}+\chi^{\mathfrak{so}(12)}_{(4n0001)}\chi^{\mathfrak{sp}(6)}_{(010000)}v^{14+2n}\\
&-\chi^{\mathfrak{so}(12)}_{(5n0001)}\chi^{\mathfrak{sp}(6)}_{(100000)}v^{15+2n}+\chi^{\mathfrak{so}(12)}_{(6n0001)}v^{16+2n}\Big)-\chi^{\mathfrak{u}(1)}_{(-2)\oplus(2)}\Big(\chi^{\mathfrak{so}(12)}_{(0n0100)}\chi^{\mathfrak{sp}(6)}_{(000001)}v^{11+2n}\\
&-\chi^{\mathfrak{so}(12)}_{(1n0100)}\chi^{\mathfrak{sp}(6)}_{(000010)}v^{12+2n}+\chi^{\mathfrak{so}(12)}_{(2n0100)}\chi^{\mathfrak{sp}(6)}_{(000100)}v^{13+2n}-\chi^{\mathfrak{so}(12)}_{(3n0100)}\chi^{\mathfrak{sp}(6)}_{(001000)}v^{14+2n}\\
&+\chi^{\mathfrak{so}(12)}_{(4n0100)}\chi^{\mathfrak{sp}(6)}_{(010000)}v^{15+2n}-\chi^{\mathfrak{so}(12)}_{(5n0100)}\chi^{\mathfrak{sp}(6)}_{(100000)}v^{16+2n}+\chi^{\mathfrak{so}(12)}_{(6n0100)}v^{17+2n}\Big)\\
&+\chi^{\mathfrak{u}(1)}_{(-1)\oplus(1)}\Big(\chi^{\mathfrak{so}(12)}_{(0n1010)}\chi^{\mathfrak{sp}(6)}_{(000001)}v^{12+2n}-\chi^{\mathfrak{so}(12)}_{(1n1010)}\chi^{\mathfrak{sp}(6)}_{(000010)}v^{13+2n}+\chi^{\mathfrak{so}(12)}_{(2n1010)}\chi^{\mathfrak{sp}(6)}_{(000100)}v^{14+2n}\\
&-\chi^{\mathfrak{so}(12)}_{(3n1010)}\chi^{\mathfrak{sp}(6)}_{(001000)}v^{15+2n}+\chi^{\mathfrak{so}(12)}_{(4n1010)}\chi^{\mathfrak{sp}(6)}_{(010000)}v^{16+2n}-\chi^{\mathfrak{so}(12)}_{(5n1010)}\chi^{\mathfrak{sp}(6)}_{(100000)}v^{17+2n}\\
&+\chi^{\mathfrak{so}(12)}_{(6n1010)}v^{18+2n}\Big)-\Big((\chi^{\mathfrak{so}(12)}_{(0n0020)}+\chi^{\mathfrak{so}(12)}_{(0n2000)}v^2)\chi^{\mathfrak{sp}(6)}_{(000001)}v^{11+2n}\\
&-(\chi^{\mathfrak{so}(12)}_{(1n0020)}+\chi^{\mathfrak{so}(12)}_{(1n2000)}v^2)\chi^{\mathfrak{sp}(6)}_{(000010)}v^{12+2n}+(\chi^{\mathfrak{so}(12)}_{(2n0020)}+\chi^{\mathfrak{so}(12)}_{(2n2000)}v^2)\chi^{\mathfrak{sp}(6)}_{(000100)}v^{13+2n}\\
&-(\chi^{\mathfrak{so}(12)}_{(3n0020)}+\chi^{\mathfrak{so}(12)}_{(3n2000)}v^2)\chi^{\mathfrak{sp}(6)}_{(001000)}v^{14+2n}+(\chi^{\mathfrak{so}(12)}_{(4n0020)}+\chi^{\mathfrak{so}(12)}_{(4n2000)}v^2)\chi^{\mathfrak{sp}(6)}_{(010000)}v^{15+2n}\\
&-(\chi^{\mathfrak{so}(12)}_{(5n0020)}+\chi^{\mathfrak{so}(12)}_{(5n2000)}v^2)\chi^{\mathfrak{sp}(6)}_{(100000)}v^{16+2n}+(\chi^{\mathfrak{so}(12)}_{(6n0020)}+\chi^{\mathfrak{so}(12)}_{(6n2000)}v^2)v^{17+2n}\Big)\Big].
\end{aligned}
\tag{D.0.22}
$$

The sporadic terms outside the infinite summation are too long to present, thus here we only show some in leading orders. In general they can be recovered from the terms inside the infinite summation. We also checked this expression from 5d blowup equations. A few leading terms in the v expansion has been determined in (H.22) of (Del Zotto and Lockhart, 2018).

$\mathbf{n=2}$, $\mathbf{G=E_7}$, $\mathbf{F=\mathfrak{so}(6)}$

Let us regard the flavor group as $\mathfrak{su}(4)$ to present the elliptic genus. We use both the v expansion method and the recursion formula from 5d blowup equations to compute the leading q order of the reduced one-string elliptic genus, and find the following exact formula:

$$
\begin{aligned}
&-\chi^{su(4)}_{(8,0,4)\oplus(4,0,8)}v^{15}-\chi^{su(4)}_{(7,0,5)\oplus(5,0,7)}\chi^{E_7}_{(n000001)}v^{18}+\chi^{su(4)}_{(6,0,6)}(\chi^{E_7}_{(1000000)}v^{17}+\chi^{E_7}_{(0100000)}v^{19})\\
&+\chi^{su(4)}_{(7,1,3)\oplus(3,1,7)}\chi^{E_7}_{(0000010)}v^{16}+\chi^{su(4)}_{(6,1,4)\oplus(4,1,6)}\chi^{E_7}_{(0000011)}v^{19}\\
&-\chi^{su(4)}_{(5,1,5)}(\chi^{E_7}_{(1000010)}v^{18}+\chi^{E_7}_{(0100010)}v^{20})+\dots\\
+\sum_{n=0}^{\infty}\Big[&-\chi^{su(4)}_{(12,0,0)\oplus(0,0,12)}\chi^{E_7}_{(n000000)}v^{17+2n}+\chi^{su(4)}_{(11,0,1)\oplus(1,0,11)}\chi^{E_7}_{(n000010)}v^{18+2n}\\
&-\chi^{su(4)}_{(10,0,2)\oplus(2,0,10)}\chi^{E_7}_{(n000100)}v^{19+2n}+\chi^{su(4)}_{(9,0,3)\oplus(3,0,9)}\chi^{E_7}_{(n001000)}v^{20+2n}\\
&-\chi^{su(4)}_{(8,0,4)\oplus(4,0,8)}\chi^{E_7}_{(n010000)}v^{21+2n}+\chi^{su(4)}_{(7,0,5)\oplus(5,0,7)}\chi^{E_7}_{(n100001)}v^{22+2n}\\
&-\chi^{su(4)}_{(6,0,6)}(\chi^{E_7}_{(n000002)}v^{21+2n}+\chi^{E_7}_{(n200000)}v^{23+n})-\chi^{su(4)}_{(10,1,0)\oplus(0,1,10)}\chi^{E_7}_{(n000020)}v^{19+2n}
\end{aligned}
$$

$$
\begin{aligned}
&+\chi^{su(4)}_{(9,1,1)\oplus(1,1,9)}\chi^{E_7}_{(n000110)}v^{20+2n}-\chi^{su(4)}_{(8,1,2)\oplus(2,1,8)}\chi^{E_7}_{(n001010)}v^{21+2n}\\
&+\chi^{su(4)}_{(7,1,3)\oplus(3,1,7)}\chi^{E_7}_{(n010010)}v^{22+2n}-\chi^{su(4)}_{(6,1,4)\oplus(4,1,6)}\chi^{E_7}_{(n100011)}v^{23+2n}\\
&+\chi^{su(4)}_{(5,1,5)}(\chi^{E_7}_{(n000012)}v^{22+2n}+\chi^{E_7}_{(n200010)}v^{24+n})+\dots\Big].
\end{aligned}
\tag{D.0.23}
$$

The full dependence on flavor representations are too long to present. Here we only show the terms involving the largest representations of $\mathfrak{su}(4)$ with Dynkin label (b_1, b_2, b_3) satisfying $b_1 + 2b_2 + b_3 = 12$ and $b_2 = 0, 1$.

$\mathbf{n = 3}$, $\mathbf{G = \mathfrak{so}(8)}$, $\mathbf{F = \mathfrak{sp}(1)_a \times \mathfrak{sp}(1)_b \times \mathfrak{sp}(1)_c}$

Denote the reduced one-string elliptic genus with all gauge and flavor fugacities turned off as

$$
\mathbb{E}_{h^{(1)}_{3,\mathfrak{so}(8)}}(q_\tau, v) = q_\tau^{-1/3} v^4 \sum_{n=0}^{\infty} q_\tau^n \frac{P_n(v)}{(1-v)^4(1+v)^{10}}. \tag{D.0.24}
$$

From the recursion formula from blowup equations, we obtain

$$
\begin{aligned}
P_0(v) &= 1 + 14v - 37v^2 + 68v^3 - 37v^4 + 14v^5 + v^6,\\
P_1(v) &= v^{-6}(1 + 6v + 11v^2 - 4v^3 - 41v^4 - 50v^5 + 43v^6\\
&\qquad + 564v^7 - 1310v^8 + 1752v^9 - \dots + v^{18}).
\end{aligned}
$$

These agree with the modular ansatz in (Del Zotto and Lockhart, 2018). With all gauge and flavor fugacities turned on, we reobtain the exact formula for the leading q order of the reduced one-string elliptic genus in (Del Zotto and Lockhart, 2018) and (Kim et al., 2019) as

$$
\begin{aligned}
v^4 + \sum_{n=0}^{\infty} \Big[&\chi^{\mathfrak{so}(8)}_{(0n00)}\chi^{F}_{(1)_a\otimes(1)_b\otimes(1)_c} v^{5+2n} - \Big(\chi^{\mathfrak{so}(8)}_{(1n00)}\chi^{F}_{(1)_b\otimes(1)_c} + \text{tri.}\Big) v^{6+2n}\\
&+ \Big(\chi^{\mathfrak{so}(8)}_{(1n10)}\chi^{F}_{(1)_c} + \text{tri.}\Big) v^{7+2n} - \chi^{\mathfrak{so}(8)}_{(1n11)} v^{8+2n}\Big].
\end{aligned}
\tag{D.0.25}
$$

We also obtain the subleading q order as

$$
v^{-2} - 2v^2 + (\chi^F_{(2)_a\oplus(2)_b\oplus(2)_c} + 1 + \chi^{\mathfrak{so}(8)}_{(0100)})v^4 + \chi^F_{(1)_a\otimes(1)_b\otimes(1)_c}(\chi^{\mathfrak{so}(8)}_{(0100)} + 4)v^5 + \mathcal{O}(v^6).
$$

Denote the reduced two-string elliptic genus with all gauge and flavor fugacities turned off as

$$
\mathbb{E}_{h^{(2)}_{3,\mathfrak{so}(8)}}(q_\tau, v) = q_\tau^{-5/6} v^9 \sum_{n=0}^{\infty} q_\tau^n \frac{P^{(2)}_n(v)}{(1-v)^{10}(1+v)^{10}(1+v+v^2)^{11}}. \tag{D.0.26}
$$

We obtain

$$
\begin{aligned}
P^{(2)}_0(v) =& 1 + 19v + 94v^2 + 77v^3 + 31v^4 + 592v^5 + 1681v^6 + 1395v^7 + 942v^8 + 3775v^9\\
&+ 7249v^{10} + 5434v^{11} + 3008v^{12} + \dots + v^{24},\\
P^{(2)}_1(v) =& v^{-6}(1 + 24v + 152v^2 + 541v^3 + 1377v^4 + 2582v^5 + 3949v^6 + 5335v^7\\
&+ 9170v^8 + 13009v^9 + 6362v^{10} - 5437v^{11} + 23841v^{12} + 92713v^{13} + 134067v^{14}
\end{aligned}
$$

$$+169449v^{15}+309565v^{16}+451272v^{17}+425964v^{18}+359168v^{19}+\cdots+v^{38}).$$

$\mathbf{n=3}$, $\mathbf{G}=\mathfrak{so}(\mathbf{9})$, $\mathbf{F}=\mathfrak{sp}(\mathbf{2})\times\mathfrak{sp}(\mathbf{1})$

Denote the reduced one-string elliptic genus with all gauge and flavor fugacities turned off as

$$\mathbb{E}_{h^{(1)}_{3,\mathfrak{so}(9)}}(q_\tau,v)=q_\tau^{-1/3}v^5\sum_{n=0}^{\infty}q_\tau^n\frac{P_n(v)}{(1-v)^4(1+v)^{12}}. \tag{D.0.27}$$

We obtain

$$\begin{aligned}P_0(v)=&-2(2+19v-62v^2+106v^3-62v^4+19v^5+2v^6),\\ P_1(v)=&-v^{-7}(1+8v+24v^2+24v^3-37v^4-132v^5-144v^6+180v^7\\ &+2004v^8-5264v^9+7056v^{10}-\cdots+v^{20}).\end{aligned}$$

Denote the reduced two-string elliptic genus with all gauge and flavor fugacities turned off as

$$\mathbb{E}_{h^{(2)}_{3,\mathfrak{so}(9)}}(q_\tau,v)=q_\tau^{-5/6}v^{11}\sum_{n=0}^{\infty}q_\tau^n\frac{P_n^{(2)}(v)}{(1-v)^{10}(1+v)^{12}(1+v+v^2)^{13}}. \tag{D.0.28}$$

We obtain

$$\begin{aligned}P_0^{(2)}(v)=&10+174v+707v^2+851v^3-109v^4+1860v^5+11190v^6+16610v^7+6728v^8\\ &+7008v^9+43183v^{10}+70861v^{11}+45001v^{12}+18164v^{13}+\cdots+10v^{26},\\ P_1^{(2)}(v)=&v^{-7}(4+74v+398v^2+1414v^3+3488v^4+6697v^5+9871v^6+12142v^7\\ &+18585v^8+43069v^9+55702v^{10}-10441v^{11}-73597v^{12}+105935v^{13}\\ &+359120v^{14}+239627v^{15}+114575v^{16}+750264v^{17}+1400325v^{18}\\ &+990699v^{19}+470338v^{20}+\cdots+4v^{40}).\end{aligned} \tag{D.0.29}$$

$\mathbf{n=3}$, $\mathbf{G}=\mathfrak{so}(\mathbf{10})$, $\mathbf{F}=\mathfrak{sp}(\mathbf{3})\times\mathfrak{u}(\mathbf{1})$

Denote the reduced one-string elliptic genus with all gauge and flavor fugacities turned off as

$$\mathbb{E}_{h^{(1)}_{3,\mathfrak{so}(10)}}(q_\tau,v)=q_\tau^{-1/3}v^6\sum_{n=0}^{\infty}q_\tau^n\frac{P_n(v)}{(1-v)^4(1+v)^{14}}. \tag{D.0.30}$$

We obtain

$$\begin{aligned}P_0(v)=&2(7+54v-210v^2+344v^3-210v^4+54v^5+7v^6),\\ P_1(v)=&v^{-8}(1+10v+41v^2+80v^3+35v^4-178v^5-419v^6-428v^7\\ &+676v^8+7284v^9-20742v^{10}+28016v^{11}-\cdots+v^{22}).\end{aligned} \tag{D.0.31}$$

Denote the reduced two-string elliptic genus with all gauge and flavor fugacities turned off as

$$\mathbb{E}_{h^{(2)}_{3,\mathfrak{so}(10)}}(q_\tau,v)=q_\tau^{-5/6}v^{13}\sum_{n=0}^{\infty}q_\tau^n\frac{P_n^{(2)}(v)}{(1-v)^{10}(1+v)^{18}(1+v+v^2)^{15}},$$

we obtain

$$
\begin{aligned}
P_0^{(2)}(v) &= 2(45+932v+6264v^2+21096v^3+37801v^4+32448v^5+31299v^6+178325v^7\\
&\quad +549579v^8+838987v^9+682443v^{10}+561148v^{11}+1511348v^{12}+3259788v^{13}\\
&\quad +3952706v^{14}+2932464v^{15}+2106794v^{16}+\cdots+45v^{32}),\\
P_1^{(2)}(v) &= 2v^{-8}(7+159v+1412v^2+7693v^3+29780v^4+87899v^5+205494v^6\\
&\quad +388084v^7+597939v^8+790211v^9+1104282v^{10}+1974138v^{11}+3342747v^{12}\\
&\quad +3399917v^{13}+355771v^{14}-2250673v^{15}+2821724v^{16}+13232633v^{17}\\
&\quad +15679593v^{18}+11581039v^{19}+25206981v^{20}+61068134v^{21}+81796560v^{22}\\
&\quad +66229422v^{23}+50700908v^{24}+\cdots+7v^{48}).
\end{aligned}
\tag{D.0.32}
$$

$\mathbf{n=3,\ G=\mathfrak{so}(12),\ F=\mathfrak{sp}(5)}$

This theory belongs to class **C** which only has vanishing blowup equations. The leading q order of reduced one-string elliptic genus, i.e. the reduced 5d one-instanton partition function was partially determined in (Del Zotto and Lockhart, 2018). Using the vanishing blowup equations, we are able to fix it as

$$
\begin{aligned}
&v^8\chi^{\mathfrak{sp}(5)}_{(00010)}-v^9\chi^{\mathfrak{so}(12)}_{(100000)}\chi^{\mathfrak{sp}(5)}_{(00100)}+v^{10}\chi^{\mathfrak{so}(12)}_{(200000)}\chi^{\mathfrak{sp}(5)}_{(01000)}-v^{11}\chi^{\mathfrak{so}(12)}_{(300000)}\chi^{\mathfrak{sp}(5)}_{(10000)}+v^{12}\chi^{\mathfrak{so}(12)}_{(400000)}\\
&+\sum_{n=0}^{\infty}\Big[-v^{10+2n}\chi^{\mathfrak{so}(12)}_{(0n0010)}\chi^{\mathfrak{sp}(5)}_{(00001)}+v^{11+2n}(\chi^{\mathfrak{so}(12)}_{(1n0010)}\chi^{\mathfrak{sp}(5)}_{(00010)}+\chi^{\mathfrak{so}(12)}_{(0n1000)}\chi^{\mathfrak{sp}(5)}_{(00001)})\\
&-v^{12+2n}(\chi^{\mathfrak{so}(12)}_{(2n0010)}\chi^{\mathfrak{sp}(5)}_{(00100)}+\chi^{\mathfrak{so}(12)}_{(1n1000)}\chi^{\mathfrak{sp}(5)}_{(00010)})+v^{13+2n}(\chi^{\mathfrak{so}(12)}_{(3n0010)}\chi^{\mathfrak{sp}(5)}_{(01000)}+\chi^{\mathfrak{so}(12)}_{(2n1000)}\chi^{\mathfrak{sp}(5)}_{(00100)})\\
&-v^{14+2n}(\chi^{\mathfrak{so}(12)}_{(4n0010)}\chi^{\mathfrak{sp}(5)}_{(10000)}+\chi^{\mathfrak{so}(12)}_{(3n1000)}\chi^{\mathfrak{sp}(5)}_{(01000)})+v^{15+2n}(\chi^{\mathfrak{so}(12)}_{(5n0010)}+\chi^{\mathfrak{so}(12)}_{(4n1000)}\chi^{\mathfrak{sp}(5)}_{(10000)})\\
&-v^{16+2n}\chi^{\mathfrak{so}(12)}_{(5n1000)}\Big].
\end{aligned}
\tag{D.0.33}
$$

$\mathbf{n=3,\ G=E_6,\ F=\mathfrak{su}(3)_a\times\mathfrak{u}(1)_b}$

From the recursion formula, we obtain the following exact formula for the leading q order of reduced one-string elliptic genus, i.e. the reduced 5d one-instanton partition function:

$$
\begin{aligned}
&\Big(v^{10}\chi^F_{(03)_a\oplus(6)_b}-v^{11}\chi^{E_6}_{(100000)}\chi^F_{(12)_a\oplus(5)_b}+v^{12}\chi^{E_6}_{(010000)}\chi^F_{(21)_a\oplus(4)_b}+v^{12}\chi^{E_6}_{(200000)}\chi^F_{(02)_a\oplus(4)_b}\\
&+v^9\chi^F_{(06)_a\oplus(3)_b}-v^{11}\chi^{E_6}_{(000001)}\chi^F_{(30)_a\oplus(3)_b}-v^{13}\chi^{E_6}_{(001000)}\chi^F_{(30)_a\oplus(3)_b}-v^{11}\chi^{E_6}_{(110000)}\chi^F_{(11)_a\oplus(3)_b}\\
&+\chi^F_{(3)_b}v^7+\chi^{E_6}_{(000100)}\chi^F_{(31)_a\oplus(2)_b}v^{12}-\chi^{E_6}_{(100000)}\chi^F_{(12)_a\oplus(2)_b}v^{10}+\chi^{E_6}_{(100001)}\chi^F_{(20)_a\oplus(2)_b}v^{12}\\
&+\chi^{E_6}_{(101000)}\chi^F_{(20)_a\oplus(2)_b}v^{14}+\chi^{E_6}_{(020000)}\chi^F_{(01)_a\oplus(2)_b}v^{14}-\chi^{E_6}_{(000010)}\chi^F_{(32)_a\oplus(1)_b}v^{11}\\
&-\chi^{E_6}_{(100100)}\chi^F_{(21)_a\oplus(1)_b}v^{13}+\chi^{E_6}_{(010000)}\chi^F_{(02)_a\oplus(1)_b}v^{11}-\chi^{E_6}_{(010001)}\chi^F_{(10)_a\oplus(1)_b}v^{13}\\
&-\chi^{E_6}_{(011000)}\chi^F_{(10)_a\oplus(1)_b}v^{15}+v^8\chi^F_{(03)_a}+c.c.\Big)+\chi^F_{(33)_a}v^{10}+\chi^{E_6}_{(101010)}\chi^F_{(22)_a}v^{12}\\
&-\chi^{E_6}_{(000001)}\chi^F_{(11)_a}v^{10}+\chi^{E_6}_{(010100)}\chi^F_{(11)_a}v^{14}+\chi^{E_6}_{(000002)}v^{12}+\chi^{E_6}_{(001001)}v^{14}+\chi^{E_6}_{(002000)}v^{16}
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{n=0}^{\infty}\Big[v^{11+2n}\chi^{E_6}_{(00000n)}\chi^{F}_{(9)_b\oplus(-9)_b}-v^{12+2n}(\chi^{E_6}_{(10000n)}\chi^{F}_{(01)_a\oplus(8)_b}+c.c.)\\
&\quad+v^{13+2n}(\chi^{E_6}_{(01000n)}\chi^{F}_{(02)_a\oplus(7)_b}+\chi^{E_6}_{(20000n)}\chi^{F}_{(10)_a\oplus(7)_b}+c.c.)\\
&\quad-v^{14+2n}(\chi^{E_6}_{(00100n)}\chi^{F}_{(03)_a\oplus(6)_b}+\chi^{E_6}_{(11000n)}\chi^{F}_{(11)_a\oplus(6)_b}+\chi^{E_6}_{(30000n)}\chi^{F}_{(00)_a\oplus(6)_b}+c.c.)\\
&\quad+v^{13+2n}(\chi^{E_6}_{(00010n)}\chi^{F}_{(04)_a\oplus(5)_b}+c.c.)\\
&\quad+v^{15+2n}(\chi^{E_6}_{(10100n)}\chi^{F}_{(12)_a\oplus(5)_b}+\chi^{E_6}_{(02000n)}\chi^{F}_{(20)_a\oplus(5)_b}+\chi^{E_6}_{(21000n)}\chi^{F}_{(01)_a\oplus(5)_b}+c.c.)\\
&\quad-v^{12+2n}(\chi^{E_6}_{(00001n)}\chi^{F}_{(05)_a\oplus(4)_b}+c.c.)-v^{14+2n}(\chi^{E_6}_{(10010n)}\chi^{F}_{(13)_a\oplus(4)_b}+c.c.)\\
&\quad-v^{16+2n}(\chi^{E_6}_{(01100n)}\chi^{F}_{(21)_a\oplus(4)_b}+\chi^{E_6}_{(20100n)}\chi^{F}_{(02)_a\oplus(4)_b}+\chi^{E_6}_{(12000n)}\chi^{F}_{(10)_a\oplus(4)_b}+c.c.)\\
&\quad+v^{11+2n}(\chi^{E_6}_{(00000n)}\chi^{F}_{(06)_a\oplus(3)_b}+c.c.)+v^{13+2n}(\chi^{E_6}_{(10001n)}\chi^{F}_{(14)_a\oplus(3)_b}+c.c.)\\
&\quad+v^{15+2n}(\chi^{E_6}_{(01010n)}\chi^{F}_{(22)_a\oplus(3)_b}+\chi^{E_6}_{(20010n)}\chi^{F}_{(03)_a\oplus(3)_b}+c.c.)\\
&\quad+v^{17+2n}(\chi^{E_6}_{(00200n)}\chi^{F}_{(30)_a\oplus(3)_b}+\chi^{E_6}_{(11100n)}\chi^{F}_{(11)_a\oplus(3)_b}+\chi^{E_6}_{(03000n)}\chi^{F}_{(3)_b}+c.c.)\\
&\quad-v^{12+2n}(\chi^{E_6}_{(10000n)}\chi^{F}_{(15)_a\oplus(2)_b}+c.c.)\\
&\quad-v^{14+2n}(\chi^{E_6}_{(01001n)}\chi^{F}_{(23)_a\oplus(2)_b}+\chi^{E_6}_{(20001n)}\chi^{F}_{(04)_a\oplus(2)_b}+c.c.)\\
&\quad-v^{16+2n}(\chi^{E_6}_{(00110n)}\chi^{F}_{(31)_a\oplus(2)_b}+\chi^{E_6}_{(11010n)}\chi^{F}_{(12)_a\oplus(2)_b}+c.c.)\\
&\quad-v^{18+2n}(\chi^{E_6}_{(10200n)}\chi^{F}_{(20)_a\oplus(2)_b}+\chi^{E_6}_{(02100n)}\chi^{F}_{(01)_a\oplus(2)_b}+c.c.)\\
&\quad+v^{13+2n}(\chi^{E_6}_{(01000n)}\chi^{F}_{(24)_a\oplus(1)_b}+\chi^{E_6}_{(20000n)}\chi^{F}_{(05)_a\oplus(1)_b}+c.c.)\\
&\quad+v^{15+2n}(\chi^{E_6}_{(00101n)}\chi^{F}_{(32)_a\oplus(1)_b}+\chi^{E_6}_{(11001n)}\chi^{F}_{(13)_a\oplus(1)_b}+\chi^{E_6}_{(00020n)}\chi^{F}_{(40)_a\oplus(1)_b}+c.c.)\\
&\quad+v^{17+2n}(\chi^{E_6}_{(10110n)}\chi^{F}_{(21)_a\oplus(1)_b}+\chi^{E_6}_{(02010n)}\chi^{F}_{(02)_a\oplus(1)_b}+c.c.)\\
&\quad+v^{19+2n}\chi^{E_6}_{(01200n)}(\chi^{F}_{(10)_a\oplus(1)_b}+c.c.)\\
&\quad-v^{14+2n}\chi^{E_6}_{(00100n)}\chi^{F}_{(33)_a}-v^{14+2n}(\chi^{E_6}_{(11000n)}\chi^{F}_{(14)_a}+c.c.)\\
&\quad-v^{16+2n}\chi^{E_6}_{(10101n)}\chi^{F}_{(22)_a}-v^{16+2n}(\chi^{E_6}_{(02001n)}\chi^{F}_{(03)_a}+c.c.)\\
&\quad-v^{18+2n}\chi^{E_6}_{(01110n)}\chi^{F}_{(11)_a}-v^{20+2n}\chi^{E_6}_{(00300n)}\Big]
\end{aligned}
\tag{D.0.34}
$$

After turning off all E_6 gauge fugacities, the above exact formula reduces to the result (A.17) of (Kim et al., 2019) by Weyl dimension formula of representations of E_6. Further turning off all flavor fugacities, one obtains the rational function of v in (5.5.111).

Denote the reduced two-string elliptic genus with all gauge and flavor fugacities turned off as

$$\mathbb{E}_{h^{(2)}_{3,E_6}}(q_\tau, v) = q_\tau^{-5/6} v^{15} \sum_{n=0}^{\infty} q_\tau^n \frac{P_n^{(2)}(v)}{(1-v)^{10}(1+v)^{26}(1+v+v^2)^{23}},$$

we obtain

$$P_0^{(2)}(v) = 3 + 159v + 4245v^2 + 72622v^3 + 863819v^4 + 7446591v^5 + 47902516v^6$$

$$+ 235241313v^7 + 896085222v^8 + 2671738023v^9 + 6257280290v^{10} + 11565342413v^{11}$$
$$+ 17441014579v^{12} + 24757146408v^{13} + 43167107703v^{14} + 92340625269v^{15}$$
$$+ 184446978968v^{16} + 297014465909v^{17} + 380602273913v^{18} + 427769333206v^{19}$$
$$+ 533426305310v^{20} + 825794587232v^{21} + 1287690035763v^{22} + 1693325870657v^{23}$$
$$+ 1815742557209v^{24} + 1695462175970v^{25} + 1602451245554v^{26} + \cdots + 3v^{52}. \tag{D.0.35}$$

$\mathbf{n = 4}$, $\mathbf{G = E_6}$, $\mathbf{F = \mathfrak{su}(2) \times \mathfrak{u}(1)}$

Denote the reduced two-string elliptic genus with all gauge and flavor fugacities turned off as

$$\mathbb{E}_{h^{(2)}_{4,E_6}}(q_\tau, v) = q_\tau^{-11/6} v^{19} \sum_{n=0}^{\infty} q_\tau^n \frac{P_n^{(2)}(v)}{(1-v)^{22}(1+v)^{28}(1+v+v^2)^{23}}, \tag{D.0.36}$$

we obtain

$$P_0^{(2)}(v) = 6 + 200v + 2632v^2 + 17758v^3 + 75489v^4 + 243367v^5 + 760467v^6 + 2577888v^7$$
$$+ 8317316v^8 + 23236506v^9 + 58513940v^{10} + 143767140v^{11} + 347390848v^{12}$$
$$+ 786032254v^{13} + 1633105895v^{14} + 3195818881v^{15} + 6041990014v^{16}$$
$$+ 10959026237v^{17} + 18715741117v^{18} + 30093383834v^{19} + 46262367433v^{20}$$
$$+ 68471264635v^{21} + 96730928747v^{22} + 129436722092v^{23} + 164888050451v^{24}$$
$$+ 201811431341v^{25} + 237209409984v^{26} + 265667738531v^{27}$$
$$+ 282914996487v^{28} + 288440699594v^{29} + \cdots + 6v^{58},$$
$$P_1^{(2)}(v) = -v^{-2}(8 + 262v + 2954v^2 + 6882v^3 - 125701v^4 - 1314279v^5 - 6621327v^6$$
$$- 23006770v^7 - 69417453v^8 - 213977845v^9 - 651520698v^{10} - 1773023963v^{11}$$
$$- 4276376371v^{12} - 9730496854v^{13} - 21781260461v^{14} - 46890358519v^{15}$$
$$- 94029008670v^{16} - 176640724111v^{17} - 318761640562v^{18} - 556066823089v^{19}$$
$$- 924340036971v^{20} - 1452988495522v^{21} - 2179171428592v^{22} - 3147790892042v^{23}$$
$$- 4365630688208v^{24} - 5770440288994v^{25} - 7276423650370v^{26}$$
$$- 8812083976234v^{27} - 10262845252021v^{28} - 11435602558269v^{29}$$
$$- 12163726096281v^{30} - 12402928893114v^{31} + \cdots + 20v^{62}). \tag{D.0.37}$$

$\mathbf{n = 4}$, $\mathbf{G = E_7}$, $\mathbf{F = \mathfrak{so}(4)}$

Regarding the flavor symmetry F as $\mathfrak{su}(2) \times \mathfrak{su}(2)$, we obtain the following exact formula for the leading q order of reduced one-string elliptic genus, i.e. the reduced 5d one-instanton partition function:

$$- \chi^F_{(8,4)\oplus(4,8)} v^{15} - \chi^F_{(7,5)\oplus(5,7)} \chi^{E_7}_{(0000001)} v^{18} + \chi^F_{(6,6)} (\chi^{E_7}_{(1000000)} v^{17} + \chi^{E_7}_{(0100000)} v^{19})$$
$$+ \chi^F_{(7,3)\oplus(3,7)} \chi^{E_7}_{(0000010)} v^{16} + \chi^F_{(6,4)\oplus(4,6)} \chi^{E_7}_{(0000011)} v^{19} - \chi^F_{(5,5)} (\chi^{E_7}_{(1000010)} v^{18}$$

$$
\begin{aligned}
&+\chi^{E_7}_{(0100010)}v^{20}) - \chi^{F}_{(6,2)\oplus(2,6)}\chi^{E_7}_{(0000100)}v^{17} - \chi^{F}_{(5,3)\oplus(3,5)}\chi^{E_7}_{(0000101)}v^{20}\\
&+\chi^{F}_{(4,4)}(-v^{13}+\chi^{E_7}_{(10000100)}v^{19}+\chi^{E_7}_{(0100100)}v^{21}) - \chi^{F}_{(6,0)\oplus(0,6)}v^{13}\\
&+\chi^{F}_{(5,1)\oplus(1,5)}\chi^{E_7}_{(0001000)}v^{18}+\chi^{F}_{(4,2)\oplus(2,4)}(\chi^{E_7}_{(1000000)}v^{15}+\chi^{E_7}_{(0001001)}v^{21})\\
&-\chi^{F}_{(3,3)}(\chi^{E_7}_{(0000001)}v^{16}+\chi^{E_7}_{(1001000)}v^{20}+\chi^{E_7}_{(0101000)}v^{22}) - \chi^{F}_{(4,0)\oplus(0,4)}(\chi^{E_7}_{(0100000)}v^{17}\\
&+\chi^{E_7}_{(0010000)}v^{19}) + \chi^{F}_{(3,1)\oplus(1,3)}(\chi^{E_7}_{(0000101)}v^{18}-\chi^{E_7}_{(0010001)}v^{22}) + \chi^{F}_{(2,2)}(-\chi^{E_7}_{(2000000)}v^{17}\\
&-\chi^{E_7}_{(0000002)}v^{19}+\chi^{E_7}_{(1010000)}v^{21}+\chi^{E_7}_{(0110000)}v^{23}) + \chi^{F}_{(2,0)\oplus(0,2)}(\chi^{E_7}_{(1000002)}v^{21}\\
&+\chi^{E_7}_{(0100002)}v^{23}) - \chi^{F}_{(1,1)}(\chi^{E_7}_{(2000001)}v^{20}+(\chi^{E_7}_{(1100001)}+\chi^{E_7}_{(0000003)})v^{22}+\chi^{E_7}_{(0200001)}v^{24})\\
&-v^{11}+\chi^{E_7}_{(3000000)}v^{19}+\chi^{E_7}_{(2100000)}v^{21}+\chi^{E_7}_{(1200000)}v^{23}+\chi^{E_7}_{(0300000)}v^{25}+\\
\sum_{n=0}^{\infty}\Big[&-\chi^{F}_{(12,0)\oplus(0,12)}\chi^{E_7}_{(n000000)}v^{17+2n}+\chi^{F}_{(11,1)\oplus(1,11)}\chi^{E_7}_{(n000010)}v^{18+2n}\\
&-\chi^{F}_{(10,2)\oplus(2,10)}\chi^{E_7}_{(n000100)}v^{19+2n}+\chi^{F}_{(9,3)\oplus(3,9)}\chi^{E_7}_{(n001000)}v^{20+2n}\\
&-\chi^{F}_{(8,4)\oplus(4,8)}\chi^{E_7}_{(n010000)}v^{21+2n}+\chi^{F}_{(7,5)\oplus(5,7)}\chi^{E_7}_{(n100001)}v^{22+2n}\\
&-\chi^{F}_{(6,6)}v^{21+2n}(\chi^{E_7}_{(n000002)}+v^2\chi^{E_7}_{(n200000)}) - \chi^{F}_{(10,0)\oplus(0,10)}\chi^{E_7}_{(n000020)}v^{19+2n}\\
&+\chi^{F}_{(9,1)\oplus(1,9)}\chi^{E_7}_{(n000110)}v^{20+2n}-\chi^{F}_{(8,2)\oplus(2,8)}\chi^{E_7}_{(n001010)}v^{21+2n}\\
&+\chi^{F}_{(7,3)\oplus(3,7)}\chi^{E_7}_{(n010010)}v^{22+2n}-\chi^{F}_{(6,4)\oplus(4,6)}\chi^{E_7}_{(n10011)}v^{23+2n}\\
&+\chi^{F}_{(5,5)}v^{22+2n}(\chi^{E_7}_{(n000012)}+v^2\chi^{E_7}_{(n200010)}) - \chi^{F}_{(8,0)\oplus(0,8)}\chi^{E_7}_{(n000200)}v^{21+2n}\\
&+\chi^{F}_{(7,1)\oplus(1,7)}\chi^{E_7}_{(n001100)}v^{22+2n}-\chi^{F}_{(6,2)\oplus(2,6)}\chi^{E_7}_{(n010100)}v^{23+2n}\\
&+\chi^{F}_{(5,3)\oplus(3,5)}\chi^{E_7}_{(n100101)}v^{24+2n}-\chi^{F}_{(4,4)}v^{23+2n}(\chi^{E_7}_{(n000102)}+v^2\chi^{E_7}_{(n200100)})\\
&-\chi^{F}_{(6,0)\oplus(0,6)}\chi^{E_7}_{(n002000)}v^{23+2n}+\chi^{F}_{(5,1)\oplus(1,5)}\chi^{E_7}_{(n011000)}v^{24+2n}\\
&-\chi^{F}_{(4,2)\oplus(2,4)}\chi^{E_7}_{(n101001)}v^{25+2n}+\chi^{F}_{(3,3)}v^{24+2n}(\chi^{E_7}_{(n001002)}+v^2\chi^{E_7}_{(n201000)})\\
&-\chi^{F}_{(4,0)\oplus(0,4)}\chi^{E_7}_{(n020000)}v^{25+2n}+\chi^{F}_{(3,1)\oplus(1,3)}\chi^{E_7}_{(n110001)}v^{26+2n}\\
&-\chi^{F}_{(2,2)}v^{25+2n}(\chi^{E_7}_{(n010002)}+v^2\chi^{E_7}_{(n210000)}) - \chi^{F}_{(2,0)\oplus(0,2)}\chi^{E_7}_{(n200002)}v^{27+2n}\\
&+\chi^{F}_{(1,1)}v^{26+2n}(\chi^{E_7}_{(n100003)}+v^2\chi^{E_7}_{(n300001)}) - (\chi^{E_7}_{(n000004)}+v^4\chi^{E_7}_{(n400000)})v^{25+2n}\Big].
\end{aligned}
\tag{D.0.38}
$$

After turning off all E_7 gauge fugacities, the above exact formula reduces to the result (A.20) of (Kim et al., 2019) by Weyl dimension formula of representations of E_7. Further turning off all flavor fugacities, one obtains the rational function of v in (5.5.128).

$\mathbf{n = 5, G = E_6, F = \mathfrak{u}(1)}$

Denote the reduced two-string elliptic genus with all gauge and flavor fugacities turned off as

$$\mathbb{E}_{h^{(2)}_{5,E_6}}(q_\tau, v) = q_\tau^{-17/6} v^{21} \sum_{n=0}^{\infty} q_\tau^n \frac{P_n^{(2)}(v)}{(1-v)^{34}(1+v)^{30}(1+v+v^2)^{23}}, \tag{D.0.39}$$

we obtain

$$\begin{aligned} P_0^{(2)}(v) = {} & 1 + 21v + 153v^2 + 904v^3 + 5116v^4 + 25914v^5 + 116029v^6 + 477409v^7 \\ & + 1823569v^8 + 6443864v^9 + 21148972v^{10} + 64945868v^{11} + 187225307v^{12} \\ & + 507470579v^{13} + 1296690701v^{14} + 3132384316v^{15} + 7167102255v^{16} \\ & + 15555191149v^{17} + 32075501088v^{18} + 62937552731v^{19} + 117653600727v^{20} \\ & + 209750655294v^{21} + 356983566607v^{22} + 580561108791v^{23} + 902887841711v^{24} \\ & + 1343669144748v^{25} + 1914685757018v^{26} + 2613923784990v^{27} + 3420367203355v^{28} \\ & + 4291402109101v^{29} + 5164404456225v^{30} + 5962900573462v^{31} + 6606847822339v^{32} \\ & + 7025662161955v^{33} + 7170987830896v^{34} + \cdots + v^{68}, \end{aligned} \tag{D.0.40}$$

and

$$\begin{aligned} P_1^{(2)}(v) = {} & 84 + 1870v + 15150v^2 + 92382v^3 + 509942v^4 + 2529414v^5 + 11170010v^6 \\ & + 45018822v^7 + 167914134v^8 + 580737756v^9 + 1867913107v^{10} + 5619089721v^{11} \\ & + 15872495069v^{12} + 42199602702v^{13} + 105848677375v^{14} + 251124006621v^{15} \\ & + 564703393888v^{16} + 1205575234175v^{17} + 2447284329306v^{18} + 4730834408879v^{19} \\ & + 8719854968064v^{20} + 15341684421093v^{21} + 25790951006163v^{22} \\ & + 41466404452278v^{23} + 63813198389587v^{24} + 94061792487301v^{25} \\ & + 132885858904299v^{26} + 180032677369322v^{27} + 234011514454012v^{28} \\ & + 291950610885280v^{29} + 349716381424128v^{30} + 402326438406440v^{31} \\ & + 444618538975344v^{32} + 472069443334672v^{33} + 481585928612732v^{34} + \ldots + 2v^{68}). \end{aligned} \tag{D.0.41}$$

$\mathbf{n = 6, G = E_6}$

Denote the reduced two-string elliptic genus with all gauge and flavor fugacities turned off as

$$\mathbb{E}_{h^{(2)}_{6,E_6}}(q_\tau, v) = q_\tau^{-23/6} v^{23} \sum_{n=0}^{\infty} q_\tau^n \frac{P_n^{(2)}(v)}{(1-v)^{46}(1+v)^{32}(1+v+v^2)^{23}}, \tag{D.0.42}$$

we obtain

$$P_0^{(2)}(v) = 1 + 9v + 94v^2 + 739v^3 + 5121v^4 + 31432v^5 + 173895v^6 + 874485v^7$$

$$+ 4036298v^8 + 17200367v^9 + 68039474v^{10} + 250943933v^{11} + 866242068v^{12}$$
$$+ 2807705547v^{13} + 8569454706v^{14} + 24690503239v^{15} + 67304396959v^{16}$$
$$+ 173919980352v^{17} + 426790882149v^{18} + 996158535441v^{19}$$
$$+ 2214670938701v^{20} + 4695878015170v^{21} + 9507297417908v^{22}$$
$$+ 18398716114730v^{23} + 34066083855696v^{24} + 60399840583490v^{25}$$
$$+ 102628223553496v^{26} + 167232472484542v^{27} + 261500117384417v^{28}$$
$$+ 392614934492341v^{29} + 566271723784347v^{30} + 784947220008032v^{31}$$
$$+ 1046126546231772v^{32} + 1340924322289616v^{33} + 1653587141756229v^{34}$$
$$+ 1962268356880815v^{35} + 2241216639463322v^{36} + 2464163123099051v^{37}$$
$$+ 2608327634962043v^{38} + 2658213934310966v^{39} + \cdots + v^{78}\,.$$

$$P_1^{(2)}(v) = (1+v^2)(82 + 896v + 9129v^2 + 73825v^3 + 515477v^4 + 3176394v^5 + 17567385v^6$$
$$+ 88082527v^7 + 404122599v^8 + 1707996910v^9 + 6687039606v^{10} + 24365673656v^{11}$$
$$+ 82957003626v^{12} + 264812209428v^{13} + 794925309293v^{14} + 2249848989493v^{15}$$
$$+ 6017588149603v^{16} + 15241390482586v^{17} + 36623148751459v^{18}$$
$$+ 83623554563863v^{19} + 181712020504595v^{20} + 376267731853770v^{21}$$
$$+ 743340720549339v^{22} + 1402570753853399v^{23} + 2530053857442778v^{24}$$
$$+ 4367001323365453v^{25} + 7218179887542376v^{26} + 11433257908228549v^{27}$$
$$+ 17365401325615558v^{28} + 25305594210396759v^{29} + 35398201343930359v^{30}$$
$$+ 47551931562200552v^{31} + 61367940071565626v^{32} + 76109936363599780v^{33}$$
$$+ 90737018750916024v^{34} + 104007721490984500v^{35} + 114645634265369518v^{36}$$
$$+ 121537998101131452v^{37} + 123925354694394472v^{38} + \cdots + v^{76}). \qquad \text{(D.0.43)}$$

$\mathbf{n = 8,\ G = E_7}$

Denote the reduced two-string elliptic genus with all gauge and flavor fugacities turned off as

$$\mathbb{E}_{h^{(2)}_{8,E_7}}(q_\tau, v) = q_\tau^{-35/6} v^{35} \sum_{n=0}^{\infty} q_\tau^n \frac{P_n^{(2)}(v)}{(1-v)^{70}(1+v)^{52}(1+v+v^2)^{35}}, \qquad \text{(D.0.44)}$$

we obtain

$$P_0^{(2)}(v) = 1 + 17v + 237v^2 + 2628v^3 + 25193v^4 + 213819v^5 + 1638666v^6 + 11476871v^7$$
$$+ 74152233v^8 + 445070980v^9 + 2495671432v^{10} + 13133928036v^{11} + 65121712327v^{12}$$
$$+ 305215505275v^{13} + 1356033968529v^{14} + 5725284334978v^{15}$$
$$+ 23021851542594v^{16} + 88338636956104v^{17} + 324035139906700v^{18}$$
$$+ 1138031848052668v^{19} + 3832341391241046v^{20} + 12390621413785440v^{21}$$
$$+ 38509222288582663v^{22} + 115175603408208175v^{23} + 331836472263902521v^{24}$$

$$
\begin{aligned}
&+ 921861932483495244v^{25} + 2471530433876763846v^{26} + 6399961693050532054v^{27}\\
&+ 16018745367471142680v^{28} + 38781560068496818142v^{29}\\
&+ 90876821066275028695v^{30} + 206242719899419463791v^{31}\\
&+ 453576963793872584712v^{32} + 967171231109021529977v^{33}\\
&+ 2000571291562232590513v^{34} + 4016126507767354504238v^{35}\\
&+ 7828073649219480743672v^{36} + 14820947289312246349740v^{37}\\
&+ 27267076918737091016348v^{38} + 48764087264312469202730v^{39}\\
&+ 84802326792798968389732v^{40} + 143449590902653729399624v^{41}\\
&+ 236104043071240448693797v^{42} + 378216261606533139497461v^{43}\\
&+ 589822792928957883073617v^{44} + 895677339869346647226824v^{45}\\
&+ 1324728639658651633703727v^{46} + 1908697079658876873038411v^{47}\\
&+ 2679565476854052143878502v^{48} + 3665936157860425562998541v^{49}\\
&+ 4888414479465062757831170v^{50} + 6354435158683924634396271v^{51}\\
&+ 8053206553397859455383003v^{52} + 9951646269406905770095206v^{53}\\
&+ 11992251412402642586454948v^{54} + 14093734406768042617860546v^{55}\\
&+ 16154939755233169917249815v^{56} + 18062065264884658609927825v^{57}\\
&+ 19698620890606501833935055v^{58} + 20956986683280640928389866v^{59}\\
&+ 21750009714684524653667914v^{60} + 22020920210850484561094012v^{61}\\
&+ 21750009714684524653667914v^{62} + \cdots + v^{122},
\end{aligned}
$$

$$
\begin{aligned}
P_1^{(2)}(v) = (1+v^2)(&137 + 2597v + 37024v^2 + 419921v^3 + 4077137v^4 + 34901534v^5\\
&+ 268811177v^6 + 1887255497v^7 + 12196657853v^8 + 73094300214v^9\\
&+ 408614442098v^{10} + 2140990474296v^{11} + 10556715862964v^{12}\\
&+ 49151597538306v^{13} + 216730904533865v^{14} + 907396069059573v^{15}\\
&+ 3615374924636545v^{16} + 13736293007916068v^{17} + 49857926256318138v^{18}\\
&+ 173164585658174276v^{19} + 576354715341079126v^{20} + 1840835604225541174v^{21}\\
&+ 5649018624246617909v^{22} + 16674709176092326437v^{23}\\
&+ 47394303096706259811v^{24} + 129836595656234291790v^{25}\\
&+ 343131359707453293583v^{26} + 875542039936399623515v^{27}\\
&+ 2158650362542725175948v^{28} + 5146221002718346735055v^{29}\\
&+ 11870970192394860758359v^{30} + 26512271515436823962474v^{31}\\
&+ 57361948999125457686102v^{32} + 120296712068566252009120v^{33}\\
&+ 244657061843538883914723v^{34} + 482772800850075541856889v^{35}\\
&+ 924703269912581018952608v^{36} + 1719956874446161848789295v^{37}\\
&+ 3107832107172475492688890v^{38} + 5457334614794588632799143v^{39}
\end{aligned}
$$

$$
\begin{aligned}
&+ 931610676445282459379765 7v^{40} + 1546525603920295879405168 8v^{41} \\
&+ 2497337038629592138075376 1v^{42} + 3923867303317889155831426 5v^{43} \\
&+ 6000394964418188328799655 4v^{44} + 8932580015338243438894976 3v^{45} \\
&+ 12947949050019944994043004 0v^{46} + 18278478543142076574300894 5v^{47} \\
&+ 25134772058123468295152899 1v^{48} + 33672832751060509737806050 8v^{49} \\
&+ 43956329432025573814053592 7v^{50} + 55919211942925973730359828 3v^{51} \\
&+ 69335079157480812429836155 9v^{52} + 83800323143387317123464523 8v^{53} \\
&+ 98737340920649748997601827 0v^{54} + 113421839674116178345697890 8v^{55} \\
&+ 127034640763471507177412334 4v^{56} + 138733982435600988303225175 8v^{57} \\
&+ 147740010601105979478429370 0v^{58} + 153419930108312978687877083 0v^{59} \\
&+ 155361026270205442540731032 0v^{60} + \cdots + v^{120}).
\end{aligned} \tag{D.0.45}
$$

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