Meromorphic differentials with imaginary periods on degenerating hyperelliptic curves

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Abstract

We provide a direct and explicit proof that imaginary (real) normalized differentials of the second kind with prescribed polar part do not develop additional singularities as the underlying hyperelliptic Riemann surface degenerates in an arbitrary way.

1 Introduction

Let $\mathcal{R}$ be a smooth compact Riemann surface of genus g, and choose a basis in the homology (a and b cycles). The main results of this paper is the continuity of certain normalized meromorphic differentials under degenerations of the Riemann surface $\mathcal{R}$. We start with the definition of the normalized differentials used in this paper.

Definition 1.1 *A meromorphic differential on a compact, smooth Riemann surface is called* **imaginary-normalized** *if all its periods are purely imaginary.*

This type of normalized differentials have been used in several context. First of all they are deeply intertwined with the theory of harmonic functions on Riemann surfaces. Indeed for differentials of the second kind[4] the real part of their antiderivative is a harmonic function globally defined on the given Riemann surface. Vice-versa, any harmonic non constant real function on $\mathcal{R}$ (or rather subsets thereof) yields upon complex differentiation an imaginary-normalized differential of the second kind [1]. Another appearance of normalized differentials was in the theory of Whitham modulation equations or nonlinear WKB method in [2]: twenty years there was an additional application in the more abstract theory of moduli spaces of pointed curves in [3].

Our specific interest to the subject arose in the relatively different context of asymptotic analysis of complex orthogonal polynomials, see [4]. It was then used in [5] to construct a "g-function", an integral part of the Deift–Zhou [6] steepest descent analysis of the asymptotics of non-hermitean complex orthogonal polynomials. In an analogous context it appears implicitly in the study of the semiclassical limit of the one-dimensional focusing nonlinear Schrödinger equation [7].

The goal of this paper is to study the behavior of an imaginary-normalized meromorphic differential η of the second kind under *arbitrary* degenerations of the underlying hyperelliptic Riemann surface $\mathcal{R}$. In Theorem 1.1 we prove that if the hyperelliptic curve $\mathcal{R}$ degenerates to a curve $\widetilde{\mathcal{R}}$ (also hyperelliptic) of lower genus (i.e. some or

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[4]This means that they can have poles but all residues vanish; the differentials of the first kind are the those holomorphic everywhere, and those of the third kind are all the other meromorphic differentials.

all branchpoints come together), then η continuously extends to an imaginary-normalized meromorphic differential on $\widetilde{\mathcal{R}}$ provided that none of the branchpoints of $\mathcal{R}$ tend to a singularity of η.

The above statement is not a priori obvious; for example, it is known that the first kind (holomorphic) differentials normalized so that they have vanishing A–cycles except one, acquire poles under the degeneration (see e.g. [8] p. 38).

A degeneration of a curve $\mathcal{R}$ is called simple (nodal) if no more than two branch points of $\mathcal{R}$ can tend to the same limit. For simple degenerations (nodes) of a hyperelliptic surface $\mathcal{R}$ the problem has been, in fact, addressed in [5, 7] (in different language and with different intents). For general compact Riemann surfaces this same problem was studied in [3]. The general structure of our proof goes along similar lines as in Theorem 5.1 of [3], however, our proof is valid for an arbitrary degeneration of the hyperelliptic surface. The key step in the proof in the present paper relies upon a certain bound (see the first paragraph in the proof our Theorem 1.1) on the boundary of a fixed disk around the pole of the differential. This bound is –in fact– the part that we tried to make absolutely explicit and self-contained, albeit just for the hyperelliptic case, and it constitutes in our opinion the main technical point of the proof (see Section 2). It seems (in our opinion) that this bound was not completely addressed in [3].

1.1 Setup and statement of results

Let $\widetilde{\eta}$ be a second kind meromorphic differential on a compact smooth Riemann surface $\mathcal{R}$ and let $A_{\widetilde{\eta}}, B_{\widetilde{\eta}}$ be the vectors of a, b periods of $\widetilde{\eta}$:

$$A_{j,\widetilde{\eta}} := \oint_{\mathbf{a}_j} \widetilde{\eta}, \quad B_{j,\widetilde{\eta}} := \oint_{\mathbf{b}_j} \widetilde{\eta}. \tag{1.1}$$

Let $\omega_1, \ldots, \omega_g$ be any basis of the first kind differentials. Denote the period matrices of the chosen basis as $\mathbb{A}, \mathbb{B}$, where

$$\oint_{\mathbf{a}_j} \omega_k = \mathbb{A}_{jk} \ , \quad \oint_{\mathbf{b}_j} \omega_k = \mathbb{B}_{jk}. \tag{1.2}$$

Proposition 1.1 *Given any meromorphic second kind differential $\widetilde{\eta}$ on a compact smooth Riemann surface $\mathcal{R}$, there exists a unique* imaginary-normalized *meromorphic differential η such that $\eta - \widetilde{\eta}$ is holomorphic. Such a differential can be expressed by the determinant formula*

$$\eta(p) = \det \left[\begin{array}{c|c|c} \mathbb{A} & \overline{\mathbb{A}} & A_{\widetilde{\eta}} + \overline{A}_{\widetilde{\eta}} \\ \hline \mathbb{B} & \overline{\mathbb{B}} & B_{\widetilde{\eta}} + \overline{B}_{\widetilde{\eta}} \\ \hline \omega_1(p) \ldots \omega_g(p) & 0 \ldots 0 & \widetilde{\eta}(p) \end{array} \right] \Big/ \det \left[\begin{array}{c|c} \mathbb{A} & \overline{\mathbb{A}} \\ \hline \mathbb{B} & \overline{\mathbb{B}} \end{array} \right], \tag{1.3}$$

where $p \in \mathcal{R}$.

Proof. The existence and uniqueness part of the argument below appears in [3] (they consider real-normalized differentials but the argument is the same). The uniqueness follows from the fact that two such differential must differ by a holomorphic differential with purely imaginary periods. By a standard theorem (Riemann Bilinear Relations; e.g. [1]), any such holomorphic differential vanishes identically. Let us now consider the existence. If $\eta = \widetilde{\eta} - \sum_{j=1}^{g} C_j \omega_j$ is an imaginary-normalized differential then the vector $C = (C_1, \ldots, C_g)^t$ solves the system

$$\begin{aligned} \mathbb{A}C + \overline{\mathbb{A}C} &= A_{\widetilde{\eta}} + \overline{A}_{\widetilde{\eta}} \\ \mathbb{B}C + \overline{\mathbb{B}C} &= B_{\widetilde{\eta}} + \overline{B}_{\widetilde{\eta}}. \end{aligned} \tag{1.4}$$

Using the elementary properties of determinants, one can easily see that $\eta = \eta(p)$ is given by (1.3). **Q.E.D.**

Remark 1.1 *Proposition 1.1 with η given by (1.3) can be extended to meromorphic differentials $\widetilde{\eta}$ of the third kind as long as all residues are real.*

Here and henceforth we specialize to a hyperelliptic curve $\mathcal{R}$ of genus $g \geq 0$ given by

$$w^2 = \prod_{j=1}^{2g+2} (z - \alpha_j) =: S(z; \vec{\alpha}), \tag{1.5}$$

where $\vec{\alpha} = (\alpha_1, \alpha_2, \cdots, \alpha_{2g+2})$ and $\alpha_j \neq \alpha_k$ for $j \neq k$, with standard basis of holomorphic differentials of the form

$$\omega_j := \frac{z^{j-1}\,\mathrm{d}z}{\sqrt{S(z)}}\ , \qquad j = 1, \ldots, g. \tag{1.6}$$

We start by considering a second kind differential η with only one singularity on $\mathcal{R}$. By a Möbius transformation we can assume that the singularity is at one of the points above $z = \infty$. Now, near its singularity, the differential has the expansion $\eta(z) = (2f(z) + \mathcal{O}(z^{-2}))\,\mathrm{d}z$, where $f(z)$ is a *polynomial* in z. Then, the differential $\widetilde{\eta}_{g,f} := \eta - f(z)\,\mathrm{d}z$ has singularities at both points above $z = \infty$ and

$$\widetilde{\eta}_{g,f}(z) = \left(\pm f(z) + \mathcal{O}(z^{-2})\right)\,\mathrm{d}z\ , z \to \infty, \tag{1.7}$$

where the two signs refer to the two sheets of the Riemann surface $\mathcal{R}$. Applied to the differential (1.7), Proposition 1.1 yields the following corollary.

Corollary 1.1 *For any Riemann surface (1.5) and for any polynomial $f(z)$ there exists a unique imaginary-normalized second kind differential*

$$\eta_{g,f}(z; \vec{\alpha}) = \frac{N(z; \vec{\alpha})\,\mathrm{d}z}{\sqrt{S(z; \vec{\alpha})}}, \tag{1.8}$$

such that

$$\frac{N(z; \vec{\alpha})}{\sqrt{S(z; \vec{\alpha})}} = \pm f(z) + O(z^{-2}) \qquad \text{as} \qquad z \to \infty, \tag{1.9}$$

with the $\pm$ referring to the two sheets of $\mathcal{R}$.

Notations in Corollary 1.1 emphasize the dependence of the polynomial N (of degree equal to $\deg f + g + 1$) on the branch points $\vec{\alpha}$ of $\mathcal{R}$, which is the main subject of the paper.

Definition 1.2 *The differential $\eta_{g,f}(z; \vec{\alpha})$ from Corollary 1.1 will be called the imaginary-normalized differential on the Riemann surface (1.5) with principal part f.*

Let us define the set $\mathfrak{D}$ of "diagonals" as

$$\mathfrak{D} := \{\alpha_j = \alpha_k\ ,\ \ j \neq k\} \subset \mathbb{C}^{2g+2}. \tag{1.10}$$

We study the situation when $\vec{\alpha} \notin \mathfrak{D}$ is approaching $\vec{\rho} \in \mathfrak{D}$. Up to a permutation of components, the point $\vec{\rho}$ is of the form

$$\vec{\rho} = \Big(\overbrace{\rho_1, \ldots, \rho_1}^{\ell_1 \text{ times}}, \ldots, \overbrace{\rho_H, \ldots, \rho_H}^{\ell_H \text{ times}}, \rho_{H+1}, \rho_{H+2}, \ldots, \rho_m\Big) \ \text{ where } \ \sum_{j=1}^{H} \ell_j + (m - H) = 2g + 2, \ \ \ell_j \geq 2 \tag{1.11}$$

with $\rho_j \neq \rho_k,\ j \neq k$.

Definition 1.3 *A* **cluster point** *of the vector $\vec{\rho} \in \mathfrak{D}$ is any of the components ρ_j of $\vec{\rho}$, see (1.11), that appears repeated $\ell_j \geq 2$ times in $\vec{\rho}$. The* **multiplicity** *of the j-th cluster point of the vector $\vec{\rho}$ in (1.11) is the number ℓ_j, where $j = 1, \cdots, H$.*

As $\vec{\alpha} \to \vec{\rho}$ the Riemann surface $\mathcal{R}$, given by (1.5), becomes a singular hyperelliptic Riemann surface $\widehat{\mathcal{R}}$ and

$$S(z;\vec{\alpha}) \to \prod_{a=1}^{H}(z-\rho_a)^{2\kappa_a}\prod_{a=1}^{H}(z-\rho_a)^{\ell_a-2\kappa_a}\prod_{j=H+1}^{m}(z-\rho_j) = \prod_{a=1}^{H}(z-\rho_a)^{2\kappa_a}S_0(z,\vec{\beta}), \tag{1.12}$$

where $\kappa_a := \left\lfloor \frac{\ell_a}{2} \right\rfloor$ and $\vec{\beta}$ contains only one representative of each of the ρ_j's with odd multiplicity and all the non-cluster points (which are considered of multiplicity 1). The dimension of $\vec{\beta}$ is thus $2g_0 + 2$ with[5]

$$g_0 = g - \sum_{a=1}^{H} \kappa_a \ . \tag{1.13}$$

The hyperelliptic Riemann surface $\mathcal{R}_0$ given by

$$w^2 = S_0(z;\vec{\beta}) \tag{1.14}$$

is called the *desingularization* (or *normalization*) of the limiting curve $\widehat{\mathcal{R}}$. We prove the following continuity theorem.

Theorem 1.1 *Let $\mathcal{R}$ be the hyperelliptic Riemann surface (1.5) of genus g and $\mathcal{R}_0$, given by (1.14), be the desingularization of the singular Riemann surface $\widehat{\mathcal{R}}$ of genus g_0. Let $f(z)$ be a polynomial and $\eta_{g,f}(z;\vec{\alpha}), \eta_{g_0;f}(z;\vec{\beta})$ be the differentials in Corollary 1.1 on the hyperelliptic Riemann surfaces $\mathcal{R}, \mathcal{R}_0$, respectively. Then*

$$\lim_{\vec{\alpha}\to\vec{\rho}} \eta_{g,f}(z;\vec{\alpha}) = \eta_{g_0,f}(z;\vec{\beta}). \tag{1.15}$$

Equivalently, if $N(z;\vec{\alpha})$ and $N_0(z;\vec{\beta})$ are the polynomials in the numerators of $\eta_{g,f}(z;\vec{\alpha}), \eta_{g_0,f}(z;\vec{\beta})$, respectively (as in Corollary 1.1) then

$$\lim_{\vec{\alpha}\to\vec{\rho}} N(z;\vec{\alpha}) = \prod_{a=1}^{H}(z-\rho_a)^{\kappa_a} N_0(z;\vec{\beta}). \tag{1.16}$$

The continuity result of Theorem 1.1 can be extended to arbitrary imaginary-normalized differentials of the second kind. Indeed, any such differential η is the sum of several imaginary-normalized differentials of the second kind with only one pole (each at a different point). Let $\vec{f} = \{f_1, f_2, \cdots, f_n\}$ denote the principal parts of η at the poles $\vec{p} = \{p_1, p_2, \cdots, p_n\}$ with $p_j = (z_j, w_j)$ so that, e.g., $f_j(z) = P_j((z-z_j)^{-1})$, with P_j a given fixed polynomial. Since Möbius transformations preserve the periods, the Theorem 1.1 applies (with obvious modifications) to each of the η_{g,f_j}, and we obtain the following corollary.

[5] We allow complete degeneration so that $S_0 = 1$; formally this corresponds to genus $g_0 = -1$.

Corollary 1.2 *Let $\mathcal{R}$ and $\mathcal{R}_0$ be the hyperelliptic Riemann surfaces, considered in Theorem 1.1, and let $\eta_{g,\vec{f}}(z;\vec{\alpha})$ be an imaginary-normalized second kind differential on $\mathcal{R}$ with prescribed principal parts $f_j(z)$ at poles $p_j = (w_j, z_j)$. Then*

$$\lim_{\vec{\alpha}\to\vec{\rho}} \eta_{g,\vec{f}}(z;\vec{\alpha}) = \eta_{g_0,\vec{f}}(z;\vec{\beta}), \tag{1.17}$$

provided that none of the z–coordinates of the poles p_j of η coincides with one of the cluster points ρ_a. The limiting differential is the imaginary-normalized meromorphic differential on the desingularization of the limiting curve and with the same singular parts.

Proof of Theorem 1.1 Here we prove the theorem under the main technical assumption that the polynomial $N(z;\vec{\alpha})$ remains bounded as $\vec{\alpha} \to \vec{\rho}$ on a fixed circle $|z| = R$ with any $R > \max|\rho_j|$, or, which is the same, the coefficients of $N(z;\vec{\alpha})$ remain bounded in the limit. The *bulk of the paper*, namely, entire Section 2, *is devoted to the proof (Theorem 2.1) of this main assumption.*

Since $\eta = \eta_{g,f}$ is a second kind differential, the Riemann bilinear relations yield

$$\sum_{j=1}^{g}\left(\oint_{\mathbf{a}_j}\eta\oint_{\mathbf{b}_j}\overline{\eta} - \oint_{\mathbf{b}_j}\eta\oint_{\mathbf{a}_j}\overline{\eta}\right) = 4i\iint_{|z|<R}\frac{|N(z;\vec{\alpha})|^2\,\mathrm{d}^2z}{|S(z;\vec{\alpha})|} + 2\oint_{|z|=R}\frac{\overline{E}(z;\vec{\alpha})N(z;\vec{\alpha})\,\mathrm{d}z}{\sqrt{S(z;\vec{\alpha})}}\ , \tag{1.18}$$

where $E(z;\vec{\alpha}) = \int_{z_0}^{z}\frac{N(\zeta;\vec{\alpha})\,\mathrm{d}\zeta}{\sqrt{S(\zeta;\vec{\alpha})}}$ and d^2z denotes the Lebesgue area measure (here and below). Note that while $E(z;\vec{\alpha})$ depends on the basepoint of integration via an additive constant, the contour integral on $z = |R|$ in (1.18) is independent of this basepoint because Res $\eta|_{z=\infty} = 0$. Since all the periods of η are purely imaginary, the left hand side of (1.18) is zero. Thus, we obtain

$$4i\iint_{|z|<R}\frac{|N(z;\vec{\alpha})|^2\,\mathrm{d}^2z}{|S(z;\vec{\alpha})|} = -2\oint_{|z|=R}\frac{\overline{E}(z;\vec{\alpha})N(z;\vec{\alpha})\,\mathrm{d}z}{\sqrt{S(z;\vec{\alpha})}}. \tag{1.19}$$

By Theorem 2.1 the right hand side of (1.19) is **bounded** as $\vec{\alpha} \to \vec{\rho}$. So, the left hand side of (1.19) must remain bounded as $\vec{\alpha} \to \vec{\rho}$.

However, we do not know yet whether $N(z;\vec{\alpha})$ admits a limit as $\vec{\alpha} \to \vec{\rho}$. We shall thus now show that:

1. any limiting value of $N(z;\vec{\alpha})$ must be divisible by $\prod_{a=1}^{H}(z-\rho_a)^{\kappa_a}$;

2. all limiting values coincide with each other and hence the limit of $N(z;\vec{\alpha})$ exists.

Since $N(z;\vec{\alpha})$ is a polynomial in z, any of its limiting value is also a polynomial of degree not exceeding $\deg N$. Thus, we are considering the limiting values of the coefficients of the polynomial $N(z;\vec{\alpha})$. According to Theorem 2.1, the coefficients of $N(z;\vec{\alpha})$ remain bounded as $\vec{\alpha} \to \vec{\rho}$. Take any sequence $\{\vec{\alpha}_n\}_1^\infty$ such that $\vec{\alpha}_n \notin \mathfrak{D}$, $n \in \mathbb{N}$, and $\vec{\alpha}_n \to \vec{\rho}$. Then there is a subsequence $\{\vec{\alpha}'_n\}_1^\infty \subseteq \{\vec{\alpha}_n\}_1^\infty$, such that the limit $\lim_{n\to\infty} N(z;\vec{\alpha}'_n) = \widehat{N}(z;\vec{\rho})$ exists. Note also that, according to (1.12), $\lim_{\vec{\alpha}\to\vec{\rho}} S(z;\vec{\alpha}) = \prod_{a=1}^{H}(z-\rho_a)^{2\kappa_a}S_0(z;\vec{\beta})$ uniformly for $z \in \mathbb{D}(R)= \{|z| < R\}$.

Suppose by contradiction that the above limit $\widehat{N}(z;\vec{\rho})$ is not divisible by $\prod_{a=1}^{H}(z-\rho_a)^{\kappa_a}$. Then the Lebesgue integral

$$\iint_{|z|<R}\frac{|\widehat{N}(z;\vec{\rho})|^2\,\mathrm{d}^2z}{|S_0(z;\vec{\beta})|\prod_{a=1}^{H}|z-\rho_a|^{2\kappa_a}} = +\infty. \tag{1.20}$$

Indeed then, near one of the points ρ_a where $\widehat{N}(z)$ is not divisible by $(z-\rho_a)^{\kappa_a}$, the integrand is of the form $\mathcal{O}(1)/|z-\rho_a|^\ell$, with $\ell \geq 2$ and thus divergent. But by Fatou's lemma

$$\iint_{|z|<R} \frac{|\widehat{N}(z;\vec{\rho})|^2}{|S_0(z;\vec{\beta})| \prod_{a=1}^H |z-\rho_a|^{2\kappa_a}} \leq \liminf_n \iint_{|z|<R} \frac{|N(z;\vec{\alpha}'_n)|^2 \, \mathrm{d}^2 z}{|S(z;\vec{\alpha}'_n)|} < \infty. \tag{1.21}$$

The obtained contradiction shows that

$$\widehat{N}(z;\vec{\rho}) = \prod_{a=1}^{H} (z-\rho_a)^{\kappa_a} \widehat{N}_0(z;\beta) \tag{1.22}$$

for some polynomial $\widehat{N}_0(z;\beta)$. Let us introduce

$$\widehat{\eta}(z;\vec{\rho}) := \frac{\widehat{N}(z;\vec{\rho})}{\sqrt{S(z;\vec{\rho})}} = \frac{\widehat{N}_0(z;\vec{\beta})}{\sqrt{S_0(z;\vec{\beta})}}. \tag{1.23}$$

According to (1.9),

$$\widehat{\eta}(z;\vec{\rho}) = \pm f(z) + \mathcal{O}(z^{-2}) \tag{1.24}$$

as $z \to \infty$ and, thus, $\widehat{\eta}(z;\vec{\rho})\,\mathrm{d}z$ is a second kind differential on the Riemann surface $\mathcal{R}_0$ given by $w^2 = S_0(z;\vec{\beta})$. Let us prove that $\widehat{\eta}(z;\vec{\rho})$ is an imaginary-normalized differential on $\mathcal{R}_0$. Consider loops $\gamma \subset \mathbb{C}$, such that in their interiors there are an even number of roots of $S_0(z)$; recall that the roots of $S_0(z)$ are all simple and in correspondence with each of the odd–multiplicity cluster points. We choose these γ so that their interior contains also small and mutually disjoint disks $\mathbb{D}_a$ around the corresponding cluster points ρ_a. In particular also the total number of α's in the region enclosed by γ is *even*. These loops span the homology of the Riemann surface $\mathcal{R}_0$ because there is only one branchpoint of S_0 at each ρ_a that has an *odd* multiplicity ℓ_a. They form closed contours on $\mathcal{R}_0$ and the corresponding periods are imaginary because they are limits of imaginary periods on $\mathcal{R}$, by construction. Thus the integrals $\oint_\gamma \widehat{\eta}(z;\vec{\rho})\,\mathrm{d}z$ are purely imaginary by the continuity argument. So, every limiting differential $\widehat{\eta}(z;\vec{\rho})$ is an imaginary-normalized differential on $\mathcal{R}_0$ satisfying (1.24). By Corollary 1.1, all such differentials coincide with $\eta_{g_0,f}(z;\vec{\beta})$. The proof of the theorem is complete. **Q.E.D.**

2 Proof of boundedness of the coefficients of $N(z,\vec{\alpha})$

We start by specializing the formula for $N(z,\vec{\alpha})$ in Corollary 1.1. Define the polynomial

$$Q(z;\vec{\alpha}) := \operatorname*{res}_{\zeta=\infty} \frac{f(\zeta)\sqrt{S(\zeta;\vec{\alpha})}}{z-\zeta}\,\mathrm{d}\zeta = \left(f(z)\sqrt{S(z;\vec{\alpha})}\right)_+ , \tag{2.1}$$

on $\mathbb{C}$, where the notation $()_+$ means the polynomial part and the branch of $\sqrt{S(z;\vec{\alpha})}$ is defined by $\sqrt{S(z;\vec{\alpha})} \to z^{g+1}$ as $z \to \infty$. In the following we shall occasionally omit the reference to the dependence on the α's.

Lemma 2.1 *The polynomial $Q(z;\vec{\alpha})$ is an entire function of $\vec{\alpha}$ and*

$$\frac{Q(z;\vec{\alpha})}{\sqrt{S(z;\vec{\alpha})}} - f(z) = \mathcal{O}(z^{-g-2}) \ , \quad |z| \to \infty. \tag{2.2}$$

As a consequence, the differential $\frac{Q(z;\vec{\alpha})}{\sqrt{S(z;\vec{\alpha})}}\,\mathrm{d}z$ has no residue at $z=\infty$.

Proof. Consider the sheet where $\sqrt{S(z)} \sim z^{g+1}$. Choose R, r_0 such that $R > r_0 > |\alpha_j|,\ j = 1, 2, \cdots, 2g+2$. Then for $r_0 < |z| < R$

$$\frac{Q(z)}{\sqrt{S(z)}} = \frac{1}{2i\pi\sqrt{S(z)}} \oint_{|\zeta|=R} \frac{f(\zeta)\sqrt{S(\zeta)}}{z-\zeta}\,\mathrm{d}\zeta = f(z) + \frac{1}{2i\pi\sqrt{S(z)}} \oint_{|\zeta|=r_0} \frac{f(\zeta)\sqrt{S(\zeta)}}{z-\zeta}\,\mathrm{d}\zeta, \tag{2.3}$$

so that the difference $Q(z)/\sqrt{S(z)} - f(z) = \mathcal{O}(z^{-g-2})$. It is also clear that $\sqrt{S(z;\vec\alpha)}$ restricted on a circle $|z| = R$ is a locally analytic function of the α's, and hence $Q(z;\vec\alpha)$ is also analytic in α's. Since R is arbitrarily large, Q is in fact entire in α's. The statement about the residue follows directly from (2.2). **Q.E.D.**

From Lemma 2.1 and Corollary 1.1 we have

$$\eta_{g,f} = \frac{Q(z;\vec\alpha) + \sum_{j=1}^{g} C_j(\vec\alpha) z^{j-1}}{\sqrt{S(z)}}\,\mathrm{d}z = \frac{N(z;\vec\alpha)\,\mathrm{d}z}{\sqrt{S(z)}}, \tag{2.4}$$

where, according to (1.3),

$$C_j(\vec\alpha) = \frac{1}{D(\vec\alpha)} \det\left[\begin{array}{c|c|c} \mathbb{A} & \overline{\mathbb{A}} & A_Q + \overline{A}_Q \\ \hline \mathbb{B} & \overline{\mathbb{B}} & B_Q + \overline{B}_Q \\ \hline e_j & 0 & 0 \end{array}\right],\quad N(z;\vec\alpha) = \frac{1}{D(\vec\alpha)} \det\left[\begin{array}{c|c|c} \mathbb{A} & \overline{\mathbb{A}} & A_Q + \overline{A}_Q \\ \hline \mathbb{B} & \overline{\mathbb{B}} & B_Q + \overline{B}_Q \\ \hline 1 \cdots z^{g-1} & 0 & Q(z) \end{array}\right] \tag{2.5}$$

$$\text{and}\qquad D(\vec\alpha) := \det\left[\begin{array}{c|c} \mathbb{A} & \overline{\mathbb{A}} \\ \hline \mathbb{B} & \overline{\mathbb{B}} \end{array}\right],\quad [A_Q]_j := \oint_{\mathbf{a}_j} \frac{Q(\zeta)\,\mathrm{d}\zeta}{\sqrt{S(\zeta)}},\quad [B_Q]_j := \oint_{\mathbf{b}_j} \frac{Q(\zeta)\,\mathrm{d}\zeta}{\sqrt{S(\zeta)}}. \tag{2.6}$$

Here e_j is the elementary row vector of size g.

Equation (2.4) does not imply that the polynomial $N(z;\vec\alpha)$ remains bounded for all possible bounded values of $\vec\alpha$'s. In particular, while $Q(z;\vec\alpha)$ has been shown to be an entire function of α's (hence bounded as $\vec\alpha \to \vec\rho$), the coefficients $C_j(\alpha)$ may become unbounded as $\vec\alpha \to \vec\rho$.

In the remaining part of the paper we will prove that all the coefficients $C_j(\vec\alpha)$ are bounded functions in a vicinity of any $\vec\rho \in \mathfrak{D} \subset \mathbb{C}^{2(g+1)}$. This proof requires estimates of both the lower bound of $D(\vec\alpha)$ (mainly in Section 2.1) and of the upper bound of the numerator of $C_j(\alpha)$. Some of these estimates are of their own interest as they connect with the theory of orthogonal polynomials with respect to the measure $\mathrm{d}^2 z / \prod |z-\alpha_j|$, see Section 2.2. We also need a formula for the determinant of the sum of matrices (based on the Laplace expansion) that is provided in Appendix A.

In the following Proposition 2.1 we recall some information about Riemann bilinear relations (see, for example, [1]).

Proposition 2.1 *Let* $\omega(z;\vec\alpha) = \frac{h(z)}{\sqrt{S(z;\vec\alpha)}}\mathrm{d}z$ *and* $\eta(z;\vec\alpha) = \frac{q(z)\,\mathrm{d}z}{\sqrt{S(z;\vec\alpha)}}$ *be second kind meromorphic differentials on the Riemann surface* $\mathcal{R}$ *given by (1.5). If all the branchpoints of* $\mathcal{R}$ *are within the disk* $|z| = R_0 < R$ *and* $\mathbf{a}_j, \mathbf{b}_j$, $j = 1 \ldots, g$, *is a canonical basis of cycles then*

$$\sum_{j=1}^{g} \left(\oint_{\mathbf{a}_j} \eta \oint_{\mathbf{b}_j} \omega - \oint_{\mathbf{b}_j} \eta \oint_{\mathbf{a}_j} \omega \right) = 2 \oint_{|z|=R} E(z;\vec\alpha) \frac{h(z)}{\sqrt{S(z;\vec\alpha)}}\,\mathrm{d}z, \tag{2.7}$$

$$\sum_{j=1}^{g} \left(\oint_{\mathbf{a}_j} \eta\ \overline{\oint_{\mathbf{b}_j} \omega} - \oint_{\mathbf{b}_j} \eta\ \overline{\oint_{\mathbf{a}_j} \omega} \right) = 4i \iint_{|z|<R} \frac{q(z)\overline{h(z)}\,\mathrm{d}^2 z}{|S(z;\vec\alpha)|} + 2 \oint_{|z|=R} \frac{E(z;\vec\alpha)\overline{h(z)}}{\sqrt{\overline{S(z;\vec\alpha)}}}\,\overline{\mathrm{d}z}, \tag{2.8}$$

where the Abelian integral

$$E(z;\vec\alpha) := \int_{z_0}^{z} \eta(\zeta) = \int_{z_0}^{z} \frac{q(\zeta)\,\mathrm{d}\zeta}{\sqrt{S(\zeta;\vec\alpha)}} \tag{2.9}$$

is analytic for $|z| \geq R$*. Here* $\mathrm{d}^2 z = \frac{i}{2} dz \wedge \overline{dz}$*.*

It is clear that the integral in (2.7) is a bounded and analytic function of the branchpoints α's [6], provided that they lie in the disk $|z| < R_0$. Similarly, the second integral in (2.8) is a bounded and smooth (it depends on $\overline{\alpha}_j$'s) function of α's. These integrals will be referred to as the "boundary terms". However, the remaining area-integral ("bulk term") in (2.8) is generically divergent when two or more α's collide, because in this case the integrand ceases to be $L^1_{loc}(\mathrm{d}^2 z)$; we are precisely going to analyze these divergences in this section. For a given $\vec\rho \in \mathfrak{D}$ define the distance

$$r = \min\left\{1; \frac{|\rho_i - \rho_j|}{10},\ 1 \leq i < j \leq m\right\}, \tag{2.10}$$

where ρ_i, ρ_j are distinct components of the vector $\vec\rho$ given by (1.11), and the partition

$$\mathbb{D}_j := \{|z - \rho_j| < r\}\ , \quad j = 1 \ldots H\ , \qquad \mathbb{D}_0 := \{z \in \mathbb{C} :\ |z| \leq R\} \setminus \bigcup_{j=1}^{H} \mathbb{D}_j \tag{2.11}$$

of the disk $\mathbb{D}(R) = \{z \in \mathbb{C} :\ |z| \leq R\}$. Here and henceforth, without loss of generality, we assume $R > 1$.

Notation 1 *Given a vector* $\vec z \in \mathbb{C}^g$ *and a multi-index* $\vec m = (m_1, \ldots, m_H) \in \mathbb{N}^H$ *such that* $|\vec m| \leq g$ *(with* $|\vec m| = \sum_j m_j$*) we shall adopt the following notation for the subvectors:* $\vec z^{(1)}_{m_1} = (z_1, \ldots, z_{m_1})$*,* $\vec z^{(2)}_{m_2} = (z_{m_1+1}, \ldots, z_{m_1+m_2})$ *etc. In the case* $|\vec m| < g$ *we also define* $\vec z^{(0)}_{m_0} = (z_{g-|\vec m|+1}, \ldots, z_g)$ *with* $m_0 := g - |\vec m|$*.*

Notation 2 *For a vector* $\vec\alpha \in \mathbb{C}^{2g+2}$ *in a neighborhood of* $\vec\rho$ *given by (1.11) we shall adopt the following notation:* $\vec\alpha^{(1)}_{\ell_1} := (\alpha_1, \ldots, \alpha_{\ell_1}) \in \mathbb{D}_1^{\ell_1}$*,* $\vec\alpha^{(2)}_{\ell_2} = (\alpha_{\ell_1+1}, \ldots, \alpha_{\ell_1+\ell_2})$ *etc.*

Lemma 2.2 *If* $\mathcal{R}$ *is a Riemann surface given by (1.5) then*

$$D(\vec\alpha) = (4i)^g I_g(\infty;\vec\alpha), \qquad \text{where} \qquad I_g(R;\vec\alpha) := \frac{1}{g!} \iint_{\mathbb{D}(R)^g} \prod_{1\leq k<j\leq g} |z_j - z_k|^2 \prod_{j=1}^{g} \frac{\mathrm{d}^2 z_j}{|S(z_j;\vec\alpha)|}\ . \tag{2.12}$$

Proof. The Riemann bilinear relations for holomorphic differentials and $R \to \infty$ can be written in terms of the period matrices $\mathbb{A}, \mathbb{B}$ as follows

$$\mathbb{A}^t \mathbb{B} = \mathbb{B}^t \mathbb{A}\ , \tag{2.13}$$

so that the matrix $M := \mathbb{A}^t\mathbb{B}$ is *symmetric*. Then

$$D(\vec\alpha) = \det\left[\begin{array}{c|c} \mathbb{A} & \overline{\mathbb{A}} \\ \hline \mathbb{B} & \overline{\mathbb{B}} \end{array}\right] = \frac{1}{\det\mathbb{A}\det\mathbb{B}} \det\left[\begin{array}{c|c} \mathbb{B}^t\mathbb{A} & \mathbb{B}^t\overline{\mathbb{A}} \\ \hline \mathbb{A}^t\mathbb{B} & \mathbb{A}^t\overline{\mathbb{B}} \end{array}\right] = \frac{1}{\det\mathbb{A}\det\mathbb{B}} \det\left[\begin{array}{c|c} M & \mathbb{B}^t\overline{\mathbb{A}} \\ \hline M & \mathbb{A}^t\overline{\mathbb{B}} \end{array}\right] = \tag{2.14}$$

$$= \frac{1}{\det\mathbb{A}\det\mathbb{B}} \det\left[\begin{array}{c|c} M & \mathbb{B}^t\overline{\mathbb{A}} \\ \hline 0 & \mathbb{A}^t\overline{\mathbb{B}} - \mathbb{B}^t\overline{\mathbb{A}} \end{array}\right] = \frac{\det M}{\det\mathbb{A}\det\mathbb{B}} \det\left[\mathbb{A}^t\overline{\mathbb{B}} - \mathbb{B}^t\overline{\mathbb{A}}\right] = \det\left[\mathbb{A}^t\overline{\mathbb{B}} - \mathbb{B}^t\overline{\mathbb{A}}\right]. \tag{2.15}$$

[6] In fact they can be extended to entire functions by enlarging R. We shall not need to use this extension.

The entries of this last matrix are (recall that $\omega_j(z) = \frac{z^{j-1}\,\mathrm{d}z}{\sqrt{S(z)}}$)

$$\left[\mathbb{A}^t\overline{\mathbb{B}} - \mathbb{B}^t\overline{\mathbb{A}}\right]_{j,k} = \sum_{\ell=1}^{g}\left(\oint_{\mathbf{a}_\ell}\omega_j\overline{\oint_{\mathbf{b}_\ell}\omega_k} - \oint_{\mathbf{b}_\ell}\omega_j\overline{\oint_{\mathbf{a}_\ell}\omega_k}\right) = \iint_{\mathcal{R}}\omega_j\wedge\overline{\omega_k} = 4i\iint_{\mathbb{C}}\frac{z^{j-1}\overline{z}^{k-1}\,\mathrm{d}^2 z}{|S(z)|}. \tag{2.16}$$

We then use Andreief identity [9] (see App. B) that allows to express the determinant of the matrix of integrals (2.16) as a multiple integral of a determinant as follows

$$g!\det\left[4i\iint_{\mathbb{C}}\frac{z^{j-1}\overline{z}^{k-1}\,\mathrm{d}^2 z}{|S(z)|}\right] = (4i)^g\iint_{\mathbb{C}^g}\prod_{\ell=1}^{g}\frac{\mathrm{d}^2 z_\ell}{|S(z_\ell)|}\det[z_j^{k-1}]_{j,k}\det[\overline{z}_j^{k-1}]_{j,k} = (4i)^g\iint_{\mathbb{C}^g}\prod_{\ell=1}^{g}\frac{\mathrm{d}^2 z_\ell}{|S(z_\ell)|}\prod_{k<j}|z_k - z_j|^2 \tag{2.17}$$

Q.E.D.

2.1 Cluster expansion for the denominator

We invite the reader to refer to the notation of Definition 1.3. and in Notation 1 and 2.

Lemma 2.3 *Let the multi-index $\vec{m} \in \mathbb{N}^H$ be such that $|\vec{m}| = g$. Suppose that $\vec{z}^{(1)}_{m_1} \in \mathbb{D}_1^{m_1}$, $\vec{z}^{(2)}_{m_2} \in \mathbb{D}_2^{m_2}, \cdots, \vec{z}^{(H)}_{m_H} \in \mathbb{D}_H^{m_H}$. Then*

$$|\Delta_g(\vec{z})| = \prod_{1\le k<j\le g}|z_j - z_k|^2 \ge r^{(g+1)g^3}\prod_{a=1}^{H}|\Delta_{m_a}(\vec{z}^{(a)}_{m_a})|\ , \tag{2.18}$$

where $\Delta_{m_a}(\vec{z}^{(a)}_{m_a})$ denotes the Vandermonde of the subset of the $m_a \ge 1$ variables that fall within the disk $\mathbb{D}_a$ (if $m_a = 0$ we understand the corresponding term as 1) .

Proof. Since all the variables belong to the various $\mathbb{D}_1, \ldots, \mathbb{D}_H$ and the pairwise distance is no less than $8r$, each term $|z_j - z_k|^2$ in $|\Delta(\vec{z})|$ with the variables in different disks is no less than $(8r)^2$. Counting those pairs we get $\sum_{j<k} m_j m_k$. Thus their contribution is a factor with the lower bound $(8r)^{2\sum_{j<k} m_j m_k}$. Removing the 8 and using $r < 1$, we can simplify the lower bound by $r^{H(H-1)g^2}$. Here we used the fact that there are at most $H = g+1$ clusters. **Q.E.D.**

Lemma 2.4 *Let $\epsilon \in (0, \frac{r}{2})$ and let $\vec{\alpha}$ be such that all the components of subvectors $\vec{\alpha}^{(n)}_{\ell_n}$ are ϵ-close to ρ_n, $n = 1, \cdots, H$. If $\vec{z}$ is as in Lemma 2.3 then*

$$\prod_{j=1}^{g}\frac{1}{|S(z_j)|} \ge R^{-2g(g+1)}\prod_{a=1}^{H}\frac{1}{|S_a(\vec{z}^{(a)}_{m_a})|}, \qquad \text{where} \quad S_a(\vec{z}^{(a)}_{m_a}) := \prod_{z\in\vec{z}^{(a)}_{m_a}}\prod_{\alpha\in\vec{\alpha}^{(a)}_{\ell_a}}(z-\alpha) \tag{2.19}$$

and the notation $\alpha \in \vec{\alpha}^{(a)}_{\ell_a}$ means that the dummy variable α runs over all the components of $\vec{\alpha}^{(a)}_{\ell_a}$ and the notation $z \in \vec{z}^{(a)}_{m_a}$ means that z runs over all the components of $\vec{z}^{(a)}_{m_a}$.

Proof. If $z \in \mathbb{D}_a$ then

$$|S(z)| \le R^{2g+2-\ell_a}\prod_{\alpha\in\vec{\alpha}^{(a)}_{\ell_a}}|z-\alpha| \le R^{2g+2}\prod_{\alpha\in\vec{\alpha}^{(a)}_{\ell_a}}|z-\alpha|. \tag{2.20}$$

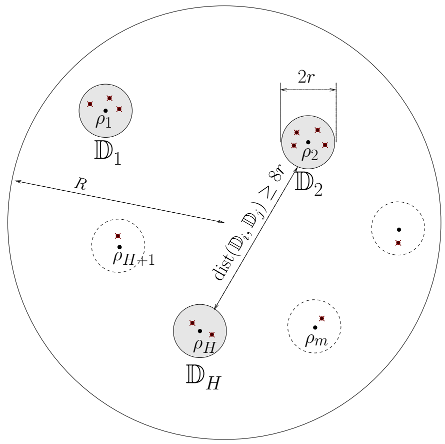

Figure 1: Typical arrangement of cluster disks $\mathbb{D}_a$ (shaded) and cluster points ρ_a with indicated the locations of α_j's (crosses). The α's in each disk tend to its center.

Multiplying over the points $\vec{z}^{(a)}_{m_a} \in \mathbb{D}_a^{m_a}$ and then over the clusters, we obtain

$$\prod_{j=1}^{g} \frac{1}{|S(z_j)|} \geq K_2 \prod_{a=1}^{H} \frac{1}{|S_a(\vec{z}^{(a)}_{m_a})|}, \qquad \text{where} \quad K_2 := R^{-(2g+2)|\vec{m}|} \geq R^{-2g(g+1)} \ . \tag{2.21}$$

Q.E.D.

Proposition 2.2 *Let*

$$I_{m_a,a}(r;\vec{\alpha}^{(a)}_{\ell_a}) := \frac{1}{m_a!} \iint_{\mathbb{D}_a^{m_a}} \frac{|\Delta_{m_a}|^2}{|S_a(\vec{z}^{(a)}_{m_a})|} \,\mathrm{d}^2 \vec{z}^{(a)}_{m_a}, \quad \text{where} \quad d^2\vec{z}^{(a)}_{m_a} := \prod_{s=1}^{m_a} \mathrm{d}^2 z_s. \tag{2.22}$$

Then the integrals $I_g(R;\vec{\alpha})$ defined in (2.12) satisfy

$$I_g(\infty;\vec{\alpha}) \geq I_g(R;\vec{\alpha}) \geq \frac{r^{(g+1)g^3}}{R^{2g(g+1)}} \sum_{\substack{\vec{m}\in\mathbb{N}^H \\ |\vec{m}|=g}} \prod_{a=1}^{H} I_{m_a,a}(r;\vec{\alpha}^{(a)}_{\ell_a}). \tag{2.23}$$

Proof. The first inequality (2.23) is obvious, so we concentrate on the second. Using Notation 1 and (2.11), denote (throughout, $\mathbb{N} = \{0,1,2,\ldots\}$)

$$\mathbb{D}_{\widehat{m}} := \mathbb{D}_0^{m_0} \times \mathbb{D}_1^{m_1} \times \cdots \times \mathbb{D}_H^{m_H} \ , \qquad \mathbb{D}_{\vec{m}} := \mathbb{D}_1^{m_1} \times \cdots \times \mathbb{D}_H^{m_H} \qquad \binom{g}{\vec{m}} := \frac{g!}{m_0! m_1! \cdots m_H!}. \tag{2.24}$$

where $\widehat{m} = (m_0, m_1, \cdots, m_H)$. Then we have the identity

$$g! I_g(R) = \sum_{\widehat{m}\in\mathbb{N}^{H+1}:\ |\widehat{m}|=g} \binom{g}{\vec{m}} \iint_{\mathbb{D}_{\widehat{m}}} |\Delta_g|^2 \prod_{j=1}^{g} \frac{\mathrm{d}^2 z_j}{|S(z_j)|}. \tag{2.25}$$

Each term in the sum is a positive number. Keeping only the terms with $m_0 = 0$ and using Lemmas 2.3, Lemma 2.4, we obtain

$$
\begin{aligned}
g! I_g(R) &\geq \sum_{\vec{m}\in\mathbb{N}^H:\ |\vec{m}|=g} \binom{g}{\vec{m}} \iint_{\mathbb{D}_{\widehat{m}}} |\Delta_g|^2 \prod_{j=1}^{g} \frac{\mathrm{d}^2 z_j}{|S(z_j)|} \geq r^{(g+1)g^3} \sum_{\vec{m}\in\mathbb{N}^H:\ |\vec{m}|=g} \binom{g}{\vec{m}} \iint_{\mathbb{D}_{\vec{m}}} \prod_{a=1}^{H} |\Delta_{m_a}|^2 \prod_{j=1}^{g} \frac{\mathrm{d}^2 z_j}{|S(z_j)|} \\
&\geq r^{(g+1)g^3} R^{-2g(g+1)} \sum_{\vec{m}\in\mathbb{N}^H:\ |\vec{m}|=g} \binom{g}{\vec{m}} \prod_{a=1}^{H} \iint_{\mathbb{D}_a^{m_a}} |\Delta_{m_a}|^2 \frac{\mathrm{d}^2 \vec{z}_{m_a}}{|S_a(\vec{z}_{m_a})|} \\
&= g! r^{(g+1)g^3} R^{-2g(g+1)} \sum_{\vec{m}\in\mathbb{N}^H:\ |\vec{m}|=g} \prod_{a=1}^{H} I_{m_a,a} \geq g! r^{(g+1)g^3} R^{-2g(g+1)} \sum_{\vec{m}\in\mathbb{N}^H:\ |\vec{m}|=g} \prod_{a=1}^{H} I_{m_a,a}.
\end{aligned}
\tag{2.26}
$$

Q.E.D.

2.2 Lower and upper bounds

In this section we will establish certain bounds for each of the $I_{m_a,a}$ in the cluster expansion of Proposition 2.2. To simplify our analysis we consider a cluster of $\ell_a \geq 2$ points in a disk centered at the origin (instead of ρ_a). Instead of using the notation from $I_{m_a,a}(r; \vec{\alpha}^{(a)}_{\ell_a})$ we shall use the generic notation

$$
J_n(r; \alpha_1, \ldots, \alpha_\ell) := \frac{1}{n!} \iint_{\mathbb{D}(r)^n} |\Delta_n|^2 \prod_{s=1}^{n} \frac{\mathrm{d}^2 z_j}{|T(z_j)|}, \quad \Delta_n = \prod_{1\leq i<j\leq n} |z_i - z_j|, \quad T(z) := \prod_{j=1}^{\ell} (z - \alpha_j). \tag{2.27}
$$

Here n plays the role of m_a, $(\alpha_1, \ldots, \alpha_\ell)$ plays the role of $\vec{\alpha}^{(a)}_{\ell_a}$ and $\prod_{j=1}^n T(z_j)$ plays the role of $S_a(\vec{z}^{(a)}_{m_a})$ in (2.22) (up to translation such that $\rho_a = 0$).

We shall first recall some general fact about these integrals and their relation with orthogonal polynomials.

Consider the monic holomorphic orthogonal polynomials (OPs) over $\mathbb{D}(r)$ with weight $\frac{\mathrm{d}^2 z}{|T(z)|}$), i.e, consider the sequence $P_n(z) = z^n + \ldots$ satisfying

$$
\int_{\mathbb{D}(r)} P_n(z) \overline{P_m(z)} \frac{\mathrm{d}^2 z}{|T(z)|} = h_n \delta_{mn}, \qquad m, n \in \mathbb{N} \cup \{0\}, \tag{2.28}
$$

where $h_n \geq 0$ and δ_{mn} is the Kronecker delta.

Proposition 2.3 *The norms h_n of the monic orthogonal polynomials (2.28) satisfy*

$$
h_n = \frac{J_{n+1}(r; \alpha_1, \ldots \alpha_\ell)}{J_n(r; \alpha_1, \ldots \alpha_\ell)}. \tag{2.29}
$$

Proof. This is a well known fact that we review for the reader's convenience [10]. The monic OPs are explicitly expressed as

$$
P_n := \frac{1}{\det[\mu_{ij}]_{0\leq i,j\leq n-1}} \det \begin{bmatrix} \mu_{00} & \cdots & \mu_{0n-1} & 1 \\ \mu_{10} & \cdots & \mu_{1n-1} & z \\ \vdots & & & \vdots \\ \mu_{00} & \cdots & \mu_{nn-1} & z^n \end{bmatrix}, \qquad \text{where} \qquad \mu_{ij} := \int_{\mathbb{D}(r)} z^i \overline{z}^j \frac{\mathrm{d}^2 z}{|T(z)|}. \tag{2.30}
$$

The determinant in the denominator is equal to $J_n(r)$ due to the Andreief's identity ([9]). Multiplying $P_n(z)$ by $\overline{z}^j$ one obtains 0 if $j \leq n-1$ and (2.29) for $j = n$. **Q.E.D**

We recall here that the square norm of the n-th monic OP h_n can also be represented through the variational formula ([11])

$$h_n = \inf_{z_1,\ldots,z_n \in \mathbb{D}(r)} \iint_{\mathbb{D}(r)} \prod_{j=1}^{n} |z - z_j|^2 \frac{\mathrm{d}^2 z}{|T(z)|}. \tag{2.31}$$

Proposition 2.4 *The integrals $J_n(r) = J_n(r; \alpha_1, \ldots, \alpha_\ell)$ satisfy*

$$J_n(r) \geq J_{n-1}(r) \int_0^r \frac{2\pi s^{2n-1}\,\mathrm{d}s}{(s + |\vec{\alpha}|)^\ell} \geq J_{n-1}(r) \int_0^r \frac{2\pi s^{2n-1}\,\mathrm{d}s}{(s+r)^\ell} \geq J_{n-1}(r) \frac{\pi}{n(2r)^\ell} r^{2n}, \tag{2.32}$$

where $|\vec{\alpha}| = \max_{j=1,\cdots,\ell} |\alpha_j|$, so that $\forall m \geq n$:

$$J_m(r) \geq J_n(r) \frac{(n!)\pi^{m-n} r^{(m+n+1)(m-n)}}{(m!)(2r)^{\ell(m-n)}}. \tag{2.33}$$

Proof. The formula (2.33) follows simply from (2.32) by induction on $m - n$. Since $J_n = h_{n-1} J_{n-1}$ it suffices to get a lower-bound for h_{n-1}. Consider the integral

$$F(z_1, \ldots, z_{n-1}) := \int_{\mathbb{D}(r)} \prod_{j=1}^{n-1} |z - z_j|^2 \frac{\mathrm{d}^2 z}{|T(z)|} \tag{2.34}$$

in (2.31). Since $|T(z)| \leq (|z| + |\vec{\alpha}|)^\ell$, we have

$$F(z_1, \ldots, z_{n-1}) \geq \int_{\mathbb{D}(r)} \prod_{j=1}^{n-1} |z - z_j|^2 \frac{\mathrm{d}^2 z}{(|z| + |\vec{\alpha}|)^\ell}. \tag{2.35}$$

We take the infimum over all configurations of z_j's; the right hand side will then be the minimal square-norm of monic polynomials of degree k in $L^2\left(\mathbb{D}(r), \frac{\mathrm{d}^2 z}{(|z|+|\vec{\alpha}|)^\ell}\right)$. It is well known that the minimum is achieved on the corresponding orthogonal polynomial. In this case, since the measure is invariant by rotations, the orthogonal polynomial is simply the monomial z^{n-1} and its square norm is precisely the right hand side of (2.32). Thus

$$J_n(r) \geq J_{n-1}(r) \inf_{z_1,\ldots,z_{n-1}} F(z_1, \ldots, z_{n-1}) \geq J_{n-1}(r) \int_{s<r} \frac{2\pi s^{2n-1}\,\mathrm{d}s}{(s + |\vec{\alpha}|)^\ell}. \tag{2.36}$$

The second and third inequalities in (2.32) are obvious since $|\vec{\alpha}| < r$. **Q.E.D.**

Lemma 2.5 *Let $\mathbb{A}_k, \mathbb{B}_k$ denote matrices $\mathbb{A}, \mathbb{B}$, where the k-th column is replaced by $A_Q + \overline{A_Q}$ in $\mathbb{A}$ or by $B_Q + \overline{B_Q}$ respectively, see (2.6). If e_k denotes the k-th standard row vector then*

$$(-1)^{k-1} \det \left[\begin{array}{c|c|c} \mathbb{A} & \overline{\mathbb{A}} & A_Q + \overline{A}_Q \\ \hline \mathbb{B} & \overline{\mathbb{B}} & B_Q + \overline{B}_Q \\ \hline e_k & 0 & 0 \end{array} \right] = \det \left[\overline{\mathbb{B}}^t \mathbb{A}_k - \overline{\mathbb{A}}^t \mathbb{B}_k \right]. \tag{2.37}$$

Proof. Using (2.13), we have

$$(-1)^{k-1}\det\left[\begin{array}{c|c|c}\mathbb{A} & \overline{\mathbb{A}} & A_{\mathbf{Q}}+\overline{A}_{\mathbf{Q}}\\ \hline \mathbb{B} & \overline{\mathbb{B}} & B_{\mathbf{Q}}+\overline{B}_{\mathbf{Q}}\\ \hline e_k & 0 & 0\end{array}\right] = \det\left[\begin{array}{c|c}\mathbb{A}_k & \overline{\mathbb{A}}\\ \hline \mathbb{B}_k & \overline{\mathbb{B}}\end{array}\right] = \frac{\det\left[\begin{array}{c|c}\overline{\mathbb{B}}^t\mathbb{A}_k & \overline{\mathbb{B}}^t\overline{\mathbb{A}}\\ \hline \overline{\mathbb{A}}^t\mathbb{B}_k & \overline{\mathbb{A}}^t\overline{\mathbb{B}}\end{array}\right]}{\det[\overline{\mathbb{A}}^t\overline{\mathbb{B}}]} =$$
$$= \frac{\det\left[\begin{array}{c|c}\overline{\mathbb{B}}^t\mathbb{A}_k-\overline{\mathbb{A}}^t\mathbb{B}_k & 0\\ \hline \overline{\mathbb{A}}^t\mathbb{B}_k & \overline{\mathbb{A}}^t\overline{\mathbb{B}}\end{array}\right]}{\det[\overline{\mathbb{A}}^t\overline{\mathbb{B}}]} = \det\left[\overline{\mathbb{B}}^t\mathbb{A}_k-\overline{\mathbb{A}}^t\mathbb{B}_k\right]. \tag{2.38}$$

Q.E.D.

Lemma 2.6 *Let*

$$M_{ij}^{(a)} := \iint_{\mathbb{D}_a}\frac{\overline{z^{i-1}}\,\widetilde{\psi}_j(z)\,\mathrm{d}^2z}{|S(z)|}\ ,\quad a=1,\dots,H, \tag{2.39}$$

$$M_{ij}^{(0)} := (4i)\iint_{\mathbb{D}_0}\frac{\overline{z^{i-1}}\,\widetilde{\psi}_j(z)\,\mathrm{d}^2z}{|S(z)|}+4\oint_{\partial\mathbb{D}(R)}\overline{\left(\int_{z_0}^{z}\frac{\zeta^{j-1}\,\mathrm{d}\zeta}{\sqrt{S(\zeta)}}\right)}\Re\left(\frac{\widetilde{\psi_i(z)\,\mathrm{d}z}}{\sqrt{S(z)}}\right), \tag{2.40}$$

where $\widetilde{\psi}_j(z)=z^{j-1}$ for $j\neq k$ and $\widetilde{\psi}_k(z)=Q(z)$. Then the matrix $M^{(0)}$ is continuous in $\vec{\alpha}$ in a polydisk of radius r around $\vec{\rho}$ and

$$\left[\overline{\mathbb{B}}^t\mathbb{A}_k-\overline{\mathbb{A}}^t\mathbb{B}_k\right] = M^{(0)}+M^{(1)}+\ldots+M^{(H)}. \tag{2.41}$$

Proof. Using Riemann bilinear relations, we obtain

$$\left[\overline{\mathbb{B}}^t\mathbb{A}_k-\overline{\mathbb{A}}^t\mathbb{B}_k\right]_{ij} = (4i)\iint_{\mathbb{D}(R)}\frac{\overline{z^{i-1}}\,\widetilde{\psi}_j(z)\,\mathrm{d}^2z}{|S(z)|}+2\oint_{\partial\mathbb{D}(R)}\overline{\left(\int_{z_0}^{z}\frac{\zeta^{i-1}\,\mathrm{d}\zeta}{\sqrt{S(\zeta)}}\right)}\frac{\widetilde{\psi}_j(z)\,\mathrm{d}z}{\sqrt{S(z)}}+$$
$$\overline{2\oint_{\partial\mathbb{D}(R)}\left(\int_{z_0}^{z}\frac{\zeta^{i-1}\,\mathrm{d}\zeta}{\sqrt{S(\zeta)}}\right)\frac{\widetilde{\psi}_j(z)\,\mathrm{d}z}{\sqrt{S(z)}}} = (4i)\iint_{\mathbb{D}(R)}\frac{\overline{z^{i-1}}\,\widetilde{\psi}_j(z)\,\mathrm{d}^2z}{|S(z)|}+4\oint_{\partial\mathbb{D}(R)}\overline{\left(\int_{z_0}^{z}\frac{\zeta^{i-1}\,\mathrm{d}\zeta}{\sqrt{S(\zeta)}}\right)}\Re\left(\frac{\widetilde{\psi_j(z)\,\mathrm{d}z}}{\sqrt{S(z)}}\right), \tag{2.42}$$

where the factor 2 in front of the line integrals is due to the fact that there are 2 copies of $\partial\mathbb{D}(R)$ on the two sheets of the Riemann surface[7]. It should be noted that the basepoint z_0 of the integration is irrelevant because changing it yields a term proportional to $\oint_{|z|=R}\frac{\overline{z^{i-1}\,\mathrm{d}z}}{\sqrt{S(z)}}$. This term is zero because the holomorphic differentials have no residue at infinity. We choose z_0 so that $|z_0|>R$. Then the integrals in (2.42) are uniformly bounded for all α's from $\mathbb{D}(R)$. Separate the contribution coming from the circles $\mathbb{D}_a$ near the cluster points (2.11) in the area-integrals in (2.42), we obtain

$$\left[\overline{\mathbb{B}}^t\mathbb{A}_k-\overline{\mathbb{A}}^t\mathbb{B}_k\right]_{ij} = (4i)\left(\iint_{\mathbb{D}_1}+\iint_{\mathbb{D}_2}+\ldots+\iint_{\mathbb{D}_H}+\iint_{\mathbb{D}_0}\right)\frac{\overline{z^{i-1}}\,\widetilde{\psi}_j(z)\,\mathrm{d}^2z}{|S(z)|}+$$
$$4\oint_{\partial\mathbb{D}(R)}\overline{\left(\int_{z_0}^{z}\frac{\zeta^{i-1}\,\mathrm{d}\zeta}{\sqrt{S(\zeta)}}\right)}\Re\left(\frac{\widetilde{\psi_j(z)\,\mathrm{d}z}}{\sqrt{S(z)}}\right) = M_{ij}^{(1)}+M_{ij}^{(2)}+\ldots+M_{ij}^{(H)}+M_{ij}^{(0)}\ . \tag{2.43}$$

This proves (2.41). The matrix $M^{(0)}$ is continuous as $\vec{\alpha}\to\vec{\rho}$ because the "bulk" term is integrated only in the region where no α's coalesce and the cluster points in $\vec{\rho}$ are bounded away from $\partial\mathbb{D}(R)$. Similarly, the entries $M_{j,i}^{(a)}$ are bounded functions of all the α's that do not belong to the a-th cluster. **Q.E.D.**

[7] The last term in (2.42) is nonzero only for the column $j=k$ when $\widetilde{\psi}_k=Q$.

Lemma 2.7 *In the notation of Proposition A.2 (in particular, $[m] = \{1, \dots, m\}$) we have*

$$\det\left[\overline{\mathbb{B}}^t\mathbb{A}_k - \overline{\mathbb{A}}^t\mathbb{B}_k\right] = \det\left[\sum_{a=0}^{H} M^{(a)}\right] = \sum_{\substack{\vec m\in\mathbb{N}^{H+1}\\ |\vec m|=g}} \sum_{\vec{\mathcal{I}}\in\binom{[g]}{\vec m}} \sum_{\vec{\mathcal{J}}\in\binom{[g]}{\vec m}} (-1)^{C(\vec{\mathcal{J}},\vec{\mathcal{I}})} \prod_{a=0}^{H} M^{(a)}_{\mathcal{I}_a,\mathcal{J}_a}, \tag{2.44}$$

where

$$M^{(a)}_{\mathcal{I}_a\mathcal{J}_a} = \frac{(4i)^{m_a}}{m_a!} \iint_{\mathbb{D}_a^{m_a}} \det[\widetilde\psi_j(z_s)]_{\mathcal{J}_a,[m_a]} \overline{\det[z_s^{i-1}]_{\mathcal{I}_a,[m_a]}} \prod_{i=1}^{m_a} \frac{\mathrm{d}^2 z_i}{|S(z_i)|}\ , \quad m_a = |\mathcal{I}_a| = |\mathcal{J}_a|. \tag{2.45}$$

Proof. Equation (2.44) with $M^{(a)}_{\mathcal{I}_a\mathcal{J}_a} := \det\left[M^{(a)}_{ij}\right]_{i\in\mathcal{I}_a, j\in\mathcal{J}_a}$ follows from Lemma 2.6 and Proposition A.2. Equation (2.45) follows from Andreief's identity. **Q.E.D.**

Lemma 2.8 *The determinant of each of the minors (2.45) satisfy*

$$\left|M^{(a)}_{\mathcal{I}_a\mathcal{J}_a}\right| \le \frac{4^{m_a}\sup_{\vec\zeta\in\mathbb{D}_a^{m_a}}\left|G_{\mathcal{I}_a}(\vec\zeta)\widetilde G_{\mathcal{J}_a}(\vec\zeta)\right|}{m_a! R^{2g+2}} I_{m_a,a}(r;\vec\alpha^{(a)}_{\ell_a}), \tag{2.46}$$

where $I_{m_a,a}$ is defined in (2.22),

$$\widetilde G_{\mathcal{J}_a}(z_1,\dots,z_{m_a}) := \frac{\det[\widetilde\psi_j(z_s)]_{\mathcal{J}_a,[m_a]}}{\Delta_{m_a}}\ , \quad G_{\mathcal{I}_a}(z_1,\dots,z_{m_a}) := \frac{\det[\psi_i(z_s)]_{\mathcal{I}_a,[m_a]}}{\Delta_{m_a}}. \tag{2.47}$$

and the constant in front of $I_{m_a,a}(r;\vec\alpha^{(a)}_{\ell_a})$ in (2.46) is uniformly bounded as $\vec\alpha\to\vec\rho$.

Proof. From (2.45) we see that their absolute values can be bounded as

$$\begin{aligned}
|M^{(a)}_{\mathcal{I}_a\mathcal{J}_a}| &\le \frac{4^{m_a}}{m_a!}\iint_{\mathbb{D}_a^{m_a}} \left|\det[\widetilde\psi_j(z_s)]_{\mathcal{J}_a,[m_a]}\right| \left|\det[\psi_i(z_s)]_{\mathcal{I}_a,[m_a]}\right| \prod_{i=1}^{m_a}\frac{\mathrm{d}^2 z_i}{|S(z_i)|} = \\
&= \frac{4^{m_a}}{m_a!}\iint_{\mathbb{D}_a^{m_a}} \left|\frac{\det[\widetilde\psi_j(z_s)]_{\mathcal{J}_a,[m_a]}}{\Delta_{m_a}}\right| \left|\frac{\det[\psi_i(z_s)]_{\mathcal{I}_a,[m_a]}}{\Delta_{m_a}}\right| |\Delta_{m_a}|^2 \prod_{i=1}^{m_a}\frac{\mathrm{d}^2 z_i}{|S(z_i)|} \le \\
&\overset{\text{Lemma 2.4}}{\le} \frac{4^{m_a}}{m_a! R^{2g+2}}\iint_{\mathbb{D}_a^{m_a}} \left|\frac{\det[\widetilde\psi_j(z_s)]_{\mathcal{J}_a,[m_a]}}{\Delta_{m_a}}\right| \left|\frac{\det[\psi_i(z_s)]_{\mathcal{I}_a,[m_a]}}{\Delta_{m_a}}\right| |\Delta_{m_a}|^2 \prod_{i=1}^{m_a}\frac{\mathrm{d}^2 z_i}{|S_a(z_i)|} = \\
&= \frac{4^{m_a}}{m_a! R^{2g+2}}\iint_{\mathbb{D}_a^{m_a}} \left|\widetilde G_{\mathcal{J}_a}(\vec z) G_{\mathcal{I}_a}(\vec z)\right| |\Delta_{m_a}|^2 \prod_{i=1}^{m_a}\frac{\mathrm{d}^2 z_i}{|S_a(z_i)|},
\end{aligned} \tag{2.48}$$

where $S_a(z) := \prod_{\alpha\in\vec\alpha^{(a)}_{\ell_a}}(z-\alpha)$. In general, the expressions (2.47) are symmetric polynomials in the variables $z_1,\dots,z_{m_1}$. In fact $G_{\mathcal{I}_a}$ are *independent of* $\vec\alpha$ (by our assumption on ψ_j) and $\widetilde G_{\mathcal{J}_a}$ depend on $\vec\alpha$ only through $Q(z;\vec\alpha)$, and thus analytically by Lemma 2.1. In either cases these two functions are uniformly bounded in the polydisk $\mathbb{D}_a^{m_a}$ as $\vec\alpha\to\vec\rho$. The proof of the lemma is completed. **Q.E.D.**

We now prove the boundedness of the polynomial $N(z;\vec\alpha)$ near any point $\vec\rho\in\mathfrak{D}$. This does not imply continuity (or the existence of the limit) of N but is an important step in proving the existence of such a limit. For this reason it is convenient to separate its proof from the continuity Theorem 1.1.

Theorem 2.1 *The coefficients of the polynomial $N(z,\alpha)$ in (2.4) remain bounded as $\vec\alpha\to\vec\rho\in\mathfrak{D}$.*

Proof. According to (2.4) and Lemma 2.1, it is sufficient to prove that all the coefficients $C_j(\vec{\alpha})$ are bounded as $\vec{\alpha} \to \vec{\rho}$. From (2.5) we see that $C_j(\vec{\alpha})$ is the ratio of (2.37) and $D(\vec{\alpha})$. Then, according to (2.44),

$$\left|\det\left[\overline{\mathbb{B}}^t \mathbb{A}_k - \overline{\mathbb{A}}^t \mathbb{B}_k\right]\right| \leq \sum_{\substack{\vec{m}\in\mathbb{N}^{H+1}\\ |\vec{m}|=g}} \sum_{\vec{\mathcal{I}}\in\binom{[g]}{\vec{m}}} \sum_{\vec{\mathcal{J}}\in\binom{[g]}{\vec{m}}} \prod_{a=0}^{H} |M^{(a)}_{\mathcal{I}_a,\mathcal{J}_a}|. \tag{2.49}$$

Using Lemma 2.6 (the matrix $M^{(0)}$ and all its minors are bounded) and (2.46), we obtain

$$\sum_{\vec{\mathcal{I}}\in\binom{[g]}{\vec{m}}} \sum_{\vec{\mathcal{J}}\in\binom{[g]}{\vec{m}}} \prod_{a=0}^{H} |M^{(a)}_{\mathcal{I}_a,\mathcal{J}_a}| \leq K_1 \sum_{\substack{\vec{m}\in\mathbb{N}^{H}\\ |\vec{m}|\leq g}} \sum_{\vec{\mathcal{I}}\in\binom{[g]}{\vec{m}}} \sum_{\vec{\mathcal{J}}\in\binom{[g]}{\vec{m}}} \prod_{a=1}^{H} |M^{(a)}_{\mathcal{I}_a,\mathcal{J}_a}| \leq K_2 \sum_{\substack{\vec{m}\in\mathbb{N}^{H}\\ |\vec{m}|\leq g}} \prod_{a=1}^{H} I_{m_a,a} \tag{2.50}$$

for some constants K_1, K_2 that do not depend on $\vec{\alpha}$ in a small polydisk around $\vec{\rho}$. On the other hand, the modulus of the denominator $|D(\vec{\alpha})| = 4^g I_g(\infty;\vec{\alpha})$ is bounded from below by Proposition 2.2. and thus

$$|C_j(\vec{\alpha})| \leq K_3 \frac{\sum_{\substack{\vec{m}\in\mathbb{N}^{H}\\ |\vec{m}|\leq g}} \prod_{a=1}^{H} I_{m_a,a}}{\sum_{\substack{\vec{n}\in\mathbb{N}^{H}\\ |\vec{n}|= g}} \prod_{a=1}^{H} I_{n_a,a}} \tag{2.51}$$

for some constant K_3 that do not depend on $\vec{\alpha}$ in a small polydisk around $\vec{\rho}$. For each term $\vec{m}$ in the numerator of (2.51) there is a term $\vec{n}$ in the denominator with $n_a \geq m_a$. According to Proposition 2.4, each ratio $\frac{I_{m_a,a}}{I_{n_a,a}}$, $m_a \leq n_a$, is uniformly bounded as $\vec{\alpha} \to \vec{\rho}$. Thus (2.51) is bounded and the theorem is proved. **Q.E.D.**

A Generalized Laplace expansion

Let $\binom{[n]}{\ell}$ be the set of subsets of $[n] = \{1, 2, \ldots, n\}$ consisting of ℓ (distinct) elements in ascending order $\mathcal{I} = \{1 \leq i_1 < i_2 < \ldots < i_\ell \leq n\} \in \binom{[n]}{\ell}$. For $\mathcal{I} \in \binom{[n]}{\ell}$ we denote with $\mathcal{I}'$ the (ordered) complementary set in $[n]$. (Note that the cardinality of $\binom{[n]}{\ell}$ is $\binom{n}{\ell}$, which should explain the notation.)

Lemma A.1 *Let M, T be two square matrices of size n. Then*

$$\det(M+T) = \det M + \sum_{\ell=1}^{n} \sum_{\mathcal{I}\in\binom{[n]}{\ell}} \det[(M,T)_{\mathcal{I}}], \tag{A.1}$$

where the symbol $(M,T)_{\mathcal{I}}$ means the matrix obtained by substituting the columns $\{i_1, \ldots, i_k\} = \mathcal{I}$ of M with the corresponding columns of T.

The proof follows from the multilinearity of the determinant with respect to the columns. Now each of those determinants can be expanded using the general Laplace expansion:

Lemma A.2 (Laplace expansion formula) *For each $\mathcal{I} \in \binom{[n]}{\ell}$ fixed, we have*

$$\det A = \sum_{\mathcal{J}\in\binom{[n]}{\ell}} (-1)^{|\mathcal{I}|+|\mathcal{J}|} \det A_{\mathcal{J},\mathcal{I}} \det A_{\mathcal{J}',\mathcal{I}'}, \tag{A.2}$$

where $|\mathcal{I}| = i_1 + i_2 + \ldots i_\ell$ and $A_{\mathcal{I},\mathcal{J}}$ denotes the submatrix indexed by $i \in \mathcal{I}, j \in \mathcal{J}$.

Combining the two lemmas we have:

Proposition A.1 *Let M, T be two square matrices of size n. Then*

$$\det[M+T] = \det M + \sum_{\ell=1}^{n} \sum_{\mathcal{I}\in\binom{[n]}{\ell}} \sum_{\mathcal{J}\in\binom{[n]}{\ell}} (-1)^{|\mathcal{I}|+|\mathcal{J}|} \det[M_{\mathcal{J},\mathcal{I}}] \det[T_{\mathcal{J}',\mathcal{I}'}]. \tag{A.3}$$

Repeated application of Proposition A.1 leads to:

Proposition A.2 *Let $M_0, M_1, \ldots, M_H$ be square matrices of size n and $\vec{\ell} = (\ell_0, \ldots, \ell_H) \in \mathbb{N}^{H+1}$ such that $|\vec{\ell}| := \sum \ell_j = n$. Denote by $\binom{[n]}{\vec{\ell}}$ the set of all partitions $\vec{\mathcal{I}} = (\mathcal{I}_0, \ldots, \mathcal{I}_H)$ of $\{1, \ldots, n\}$ of cardinality $\ell_0, \ldots, \ell_H$ respectively (each of which ordered in ascending order)*[8].

Then

$$\det\left[\sum_{a=0}^{H} M_a\right] = \sum_{\substack{\vec{\ell}\in\mathbb{N}^{H+1}\\ |\vec{\ell}|=n}} \sum_{\vec{\mathcal{I}}\in\binom{[n]}{\vec{\ell}}} \sum_{\vec{\mathcal{J}}\in\binom{[n]}{\vec{\ell}}} (-1)^{C(\vec{\mathcal{J}},\vec{\mathcal{I}})} \prod_{a=0}^{H} \det[(M_a)_{\mathcal{I}_a,\mathcal{J}_a}], \tag{A.4}$$

where $C(\vec{\mathcal{J}},\vec{\mathcal{I}})$ is an integer that is inconsequential to our purposes.

B Andréief's identity

Let $(X,\, \mathrm{d}\mu)$ be a measure space and $\psi_1, \ldots, \psi_n$, $\phi_1, \ldots, \phi_n$ be measurable functions. The identity [9] is the following algebraic identity, as long as all the integrals involved make sense:

$$\iint_{X^n} \det\left[\psi_j(x_k)\right]_{j,k} \det\left[\phi_j(x_k)\right]_{j,k} \prod_{j=1}^{n} \mathrm{d}\mu(x_j) = n! \det\left[\int_X \phi_j(x)\psi_k(x)\,\mathrm{d}\mu(x)\right]_{j,k} \tag{B.1}$$

References

[1] H. M. Farkas and I. Kra. *Riemann surfaces*, volume 71 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1992.

[2] I. M. Krichever. The averaging method for two-dimensional "integrable" equations. *Funktsional. Anal. i Prilozhen.*, 22(3):37–52, 96, 1988.

[3] S. Grushevsky and I. Krichever. The universal Whitham hierarchy and the geometry of the moduli space of pointed Riemann surfaces. In *Surveys in differential geometry. Vol. XIV. Geometry of Riemann surfaces and their moduli spaces*, volume 14 of *Surv. Differ. Geom.*, pages 111–129. Int. Press, Somerville, MA, 2009.

[4] M. Bertola and M. Y. Mo. Commuting difference operators, spinor bundles and the asymptotics of orthogonal polynomials with respect to varying complex weights. *Adv. Math.*, 220(1):154–218, 2009.

[8] We note that the cardinality of $\binom{[n]}{\vec{\ell}}$ is $\binom{n}{\ell_0,\ldots,\ell_H} = \frac{n!}{\ell_0!\cdots\ell_H!}$, which –again– should make clear the choice of notation.

[5] M. Bertola. Boutroux curves with external field: equilibrium measures without a variational problem. *Anal. Math. Phys.*, 1(2-3):167–211, 2011.

[6] P. Deift and X. Zhou. A steepest descent method for oscillatory Riemann-Hilbert problems. *Bull. Amer. Math. Soc. (N.S.)*, 26(1):119–123, 1992.

[7] A. Tovbis and S. Venakides. Nonlinear steepest descent asymptotics for semiclassical limit of integrable systems: continuation in the parameter space. *Comm. Math. Phys.*, 295(1):139–160, 2010.

[8] J. D. Fay. *Theta functions on Riemann surfaces*. Lecture Notes in Mathematics, Vol. 352. Springer-Verlag, Berlin, 1973.

[9] C. Andréief. Note sur une relation entre les intégrales définies des produits des fonctions. *Mém. de la Soc. Sci., Bordeaux*, 3(2):1–14, 1883.

[10] G. Szegö. *Orthogonal polynomials*. American Mathematical Society, Providence, R.I., fourth edition, 1975. American Mathematical Society, Colloquium Publications, Vol. XXIII.

[11] E. B. Saff and V. Totik. *Logarithmic potentials with external fields*, volume 316 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1997. Appendix B by Thomas Bloom.