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# YANG-MILLS-HIGGS CONNECTIONS ON CALABI-YAU MANIFOLDS

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ABSTRACT. Let  $X$  be a compact connected Kähler–Einstein manifold with  $c_1(TX) \geq 0$ . If there is a semistable Higgs vector bundle  $(E, \theta)$  on  $X$  with  $\theta \neq 0$ , then we show that  $c_1(TX) = 0$ ; any  $X$  satisfying this condition is called a Calabi–Yau manifold, and it admits a Ricci–flat Kähler form [Ya]. Let  $(E, \theta)$  be a polystable Higgs vector bundle on a compact Ricci–flat Kähler manifold  $X$ . Let  $h$  be an Hermitian structure on  $E$  satisfying the Yang–Mills–Higgs equation for  $(E, \theta)$ . We prove that  $h$  also satisfies the Yang–Mills–Higgs equation for  $(E, 0)$ . A similar result is proved for Hermitian structures on principal Higgs bundles on  $X$  satisfying the Yang–Mills–Higgs equation.

## 1. INTRODUCTION

Let  $X$  be a compact connected Kähler–Einstein manifold with  $c_1(TX) \geq 0$ . A Higgs vector bundle on  $X$  is a holomorphic vector bundle  $E$  on  $X$  equipped with a holomorphic section  $\theta$  of  $\text{End}(E) \otimes \Omega_X$  such that  $\theta \wedge \theta = 0$ . The definition of semistable and polystable Higgs vector bundles is recalled in Section 2. We prove that if there is a semistable Higgs vector bundle  $(E, \theta)$  on  $X$  with  $\theta \neq 0$ , then  $c_1(TX) = 0$  (see Proposition 2.1).

Let  $X$  be a compact connected Calabi–Yau manifold, which means that  $X$  is a Kähler manifold with  $c_1(TX) = 0$ . Fix a Ricci–flat Kähler form on  $X$  [Ya]. Let  $(E, \theta)$  be a polystable Higgs vector bundle on  $X$ . Then there is a Hermitian structure on  $E$  that satisfies the Yang–Mills–Higgs equation for  $(E, \theta)$  (this equation is recalled in Section 2). Fix a Hermitian structure  $h$  on  $E$  satisfying the Yang–Mills–Higgs equation for  $(E, \theta)$ .

Our main theorem (Theorem 3.3) says that  $h$  also satisfies the Yang–Mills–Higgs equation for  $(E, 0)$ .

We give an example to show that if a Hermitian structure  $h_0$  on  $E$  satisfies the Yang–Mills–Higgs equation for  $(E, 0)$ , then  $h_0$  does not satisfy the Yang–Mills–Higgs equation for a general polystable Higgs vector bundle of the form  $(E, \theta)$  (see Remark 3.4). In

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Remark 3.5 we describe how a Yang–Mills–Higgs Hermitian structure for  $(E, \theta)$  can be constructed from a Yang–Mills–Higgs Hermitian structure for  $(E, 0)$ .

Theorem 3.3 extends to the more general context of principal  $G$ –bundles on  $X$  with a Higgs structure, where  $G$  is a connected reductive affine algebraic group defined over  $\mathbb{C}$ ; this is carried out in Section 4.

## 2. HIGGS FIELD ON A KÄHLER–EINSTEIN MANIFOLD

We recall that a Kähler metric is called *Kähler–Einstein* if its Ricci curvature is a constant real multiple of the Kähler form. Let  $X$  be a compact connected Kähler manifold admitting a Kähler–Einstein metric. We assume that  $c_1(TX) \geq 0$ ; this is equivalent to the condition that the above mentioned scalar factor is nonnegative. Fix a Kähler–Einstein form  $\omega$  on  $X$ . The cohomology class in  $H^2(X, \mathbb{R})$  given by  $\omega$  will be denoted by  $\tilde{\omega}$ .

Define the *degree* of a torsionfree coherent analytic sheaf  $F$  on  $X$  to be

$$\text{degree}(F) := (c_1(F) \cup \tilde{\omega}^{d-1}) \cap [X] \in \mathbb{R},$$

where  $d$  is the complex dimension of  $X$ . Throughout this paper, stability will be with respect to this definition of degree.

The holomorphic cotangent bundle of  $X$  will be denoted by  $\Omega_X$ . A *Higgs field* on a holomorphic vector bundle  $E$  on  $X$  is a holomorphic section  $\theta$  of  $\text{End}(E) \otimes \Omega_X = (E \otimes \Omega_X) \otimes E^*$  such that

$$\theta \wedge \theta = 0. \tag{2.1}$$

A *Higgs vector bundle* on  $X$  is a pair of the form  $(E, \theta)$ , where  $E$  is a holomorphic vector bundle on  $X$  and  $\theta$  is a Higgs field on  $E$ .

A Higgs vector bundle  $(E, \theta)$  is called *stable* (respectively, *semistable*) if for all nonzero coherent analytic subsheaves  $F \subset E$  with  $0 < \text{rank}(F) < \text{rank}(E)$  and  $\theta(F) \subseteq F \otimes \Omega_X$ , we have

$$\frac{\text{degree}(F)}{\text{rank}(F)} < \frac{\text{degree}(E)}{\text{rank}(E)} \quad (\text{respectively, } \frac{\text{degree}(F)}{\text{rank}(F)} \leq \frac{\text{degree}(E)}{\text{rank}(E)}).$$

A semistable Higgs vector bundle  $(E, \theta)$  is called *polystable* if it is a direct sum of stable Higgs vector bundles.

Let  $\Lambda_\omega$  denote the adjoint of multiplication of differential forms on  $X$  by  $\omega$ . In particular,  $\Lambda_\omega$  sends a  $(p, q)$ –form on  $X$  to a  $(p-1, q-1)$ –form. Given a Higgs vector bundle  $(E, \theta)$  on  $X$ , the *Yang–Mills–Higgs* equation for the Hermitian structures  $h$  on  $E$  states that

$$\Lambda_\omega(\mathcal{K}_h + \theta \wedge \theta^*) = c\sqrt{-1} \cdot \text{Id}_E, \tag{2.2}$$

where  $\mathcal{K}_h \in C^\infty(X, \text{End}(E) \otimes \Omega_X^{1,1})$  is the curvature of the Chern connection on  $E$  for  $h$ , the adjoint  $\theta^*$  of  $\theta$  is with respect to  $h$ , and  $c$  is a constant scalar (it lies in  $\mathbb{R}$ ). A Hermitian structure on  $E$  is called Yang–Mills–Higgs for  $(E, \theta)$  if it satisfies the equation in (2.2).

**Proposition 2.1.** *If there is a semistable Higgs bundle  $(E, \theta)$  on  $X$  such that  $\theta \neq 0$ , then  $c_1(TX) = 0$ .*

*Proof.* The Higgs field  $\theta$  on  $E$  induces a Higgs field on  $\text{End}(E)$ , which we will denote by  $\widehat{\theta}$ . We recall that for any locally defined holomorphic sections  $s$  of  $\text{End}(E)$ ,

$$\widehat{\theta}(s) = [\theta, s].$$

Let

$$\theta' = \widehat{\theta} \otimes \text{Id}_{\Omega_X}. \quad (2.3)$$

This is a Higgs field for  $\text{End}(E) \otimes \Omega_X$ . We note that the integrability condition in (2.1) implies that  $\theta'(\theta) = 0$ .

Assume that  $(E, \theta)$  is semistable with  $\theta \neq 0$ , and also assume that  $c_1(TX) \neq 0$ . Since  $(X, \omega)$  is Kähler–Einstein with  $c_1(TX) \geq 0$ , the condition  $c_1(TX) \neq 0$  implies that the anti-canonical line bundle  $\bigwedge^d TX$  is positive, so  $X$  is a complex projective manifold. Also, the cohomology class of  $\omega$  is a positive multiple of the ample class  $c_1(TX)$ .

We shall use the fact that the tensor product of semistable Higgs bundles on a polarized complex projective manifold, with the induced Higgs field, is semistable [Si2, Cor. 3.8]. Thus,  $(\text{End}(E), \widehat{\theta})$  is semistable. Moreover, since  $\omega$  is Kähler–Einstein,  $\Omega_X$  is a polystable vector bundle, in particular it is semistable. Then  $(\Omega_X, 0)$  is a semistable Higgs bundle. As a result, the Higgs bundle  $(\text{End}(E) \otimes \Omega_X, \theta')$  is semistable.

The homomorphism

$$\mathcal{O}_X \longrightarrow \text{End}(E) \otimes \Omega_X, \quad f \longmapsto f\theta$$

defines a homomorphism of Higgs vector bundles

$$\varphi : (\mathcal{O}_X, 0) \longrightarrow (\text{End}(E) \otimes \Omega_X, \theta'). \quad (2.4)$$

As  $\theta \neq 0$ , the homomorphism  $\varphi$  in (2.4) is nonzero. Since  $(\text{End}(E) \otimes \Omega_X, \theta')$  is semistable, we have

$$0 = \frac{\text{degree}(\mathcal{O}_X)}{\text{rank}(\mathcal{O}_X)} = \frac{\text{degree}(\varphi(\mathcal{O}_X))}{\text{rank}(\varphi(\mathcal{O}_X))} \leq \frac{\text{degree}(\text{End}(E) \otimes \Omega_X)}{\text{rank}(\text{End}(E) \otimes \Omega_X)} = \frac{\text{degree}(\Omega_X)}{\text{rank}(\Omega_X)}; \quad (2.5)$$

the last equality follows from the fact that  $c_1(\text{End}(E)) = 0$ . Therefore,

$$\text{degree}(\Omega_X) \geq 0. \quad (2.6)$$

Recall that  $c_1(TX) \geq 0$  and  $X$  admits a Kähler–Einstein metric. So, (2.6) contradicts the assumption that  $c_1(TX) \neq 0$ . Therefore, we conclude that

$$c_1(TX) = 0. \quad (2.7)$$

Consequently,  $\omega$  is Ricci–flat, in particular,  $X$  is a Calabi–Yau manifold.  $\square$

A well-known theorem due to Simpson says that  $E$  admits an Hermitian structure that satisfies the Yang–Mills–Higgs equation for  $(E, \theta)$  if and only if  $(E, \theta)$  is polystable [Si1, Thm. 1] (see also [Si2]); when  $X$  is a compact Riemann surface and  $\text{rank}(E) = 2$ , this was first proved in [Hi].

The Chern connection on  $E$  for  $h$  will be denoted by  $\nabla^h$ . Let  $\widehat{\nabla}^h$  denote the connection on  $\text{End}(E) = E \otimes E^*$  induced by  $\nabla^h$ . The Levi–Civita connection on  $\Omega_X$  associated to  $\omega$  and the connection  $\widehat{\nabla}^h$  on  $\text{End}(E)$  together produce a connection on  $\text{End}(E) \otimes \Omega_X$ . This connection on  $\text{End}(E) \otimes \Omega_X$  will be denoted by  $\nabla^{\omega, h}$ .

**Proposition 2.2.** *Assume that the Hermitian structure  $h$  satisfies the Yang–Mills–Higgs equation in (2.2) for  $(E, \theta)$ . Then the section  $\theta$  of  $\text{End}(E) \otimes \Omega_X$  is flat (meaning covariantly constant) with respect to the connection  $\nabla^{\omega, h}$  constructed above.*

*Proof.* The Hermitian structure  $h$  on  $E$  produces an Hermitian structure on  $\text{End}(E)$ , which will be denoted by  $\widehat{h}$ . The connection  $\widehat{\nabla}^h$  on  $\text{End}(E)$  defined earlier is in fact the Chern connection for  $\widehat{h}$ . The Kähler form  $\omega$  and the Hermitian structure  $\widehat{h}$  together produce an Hermitian structure on  $\text{End}(E) \otimes \Omega_X$ . This Hermitian structure on  $\text{End}(E) \otimes \Omega_X$  will be denoted by  $h^\omega$ . We note that the connection  $\nabla^{\omega, h}$  in the statement of the proposition is the Chern connection for  $h^\omega$ .

Since  $\omega$  is Kähler–Einstein, the Hermitian structure on  $\Omega_X$  induced by  $\omega$  satisfies the Yang–Mills–Higgs equation for the Higgs vector bundle  $(\Omega_X, 0)$ . As  $h$  satisfies the Yang–Mills–Higgs equation for  $(E, \theta)$ , this implies that  $h^\omega$  satisfies the Yang–Mills–Higgs equation for the Higgs vector bundle  $(\text{End}(E) \otimes \Omega_X, \theta')$  constructed in (2.3). In particular, the Higgs vector bundle  $(\text{End}(E) \otimes \Omega_X, \theta')$  is polystable. The Proposition is obvious if  $\theta = 0$ . Assume that  $\theta \neq 0$ ; then  $\varphi$  defined in (2.4) is nonzero.

Since  $c_1(\Omega_X) = 0$ , the inequality in (2.5) is an equality. Now from [Si1, Prop. 3.3] it follows immediately that

- $\varphi(\mathcal{O}_X)$  in (2.4) is a subbundle of  $\text{End}(E)$ ,
- the orthogonal complement  $\varphi(\mathcal{O}_X)^\perp \subset \text{End}(E) \otimes \Omega_X$  of  $\varphi(\mathcal{O}_X)$  with respect to the Yang–Mills–Higgs Hermitian structure  $h^\omega$  is preserved by  $\theta'$ , and
- $(\varphi(\mathcal{O}_X)^\perp, \theta'|_{\varphi(\mathcal{O}_X)^\perp})$  is polystable with

$$\frac{\text{degree}(\varphi(\mathcal{O}_X)^\perp)}{\text{rank}(\varphi(\mathcal{O}_X)^\perp)} = \frac{\text{degree}(\text{End}(E) \otimes \Omega_X)}{\text{rank}(\text{End}(E) \otimes \Omega_X)} = 0.$$

We note that [Si1, Prop. 3.3] also says that the Hermitian structure on the image of  $\varphi$  induced by  $h^\omega$  satisfies the Yang–Mills–Higgs equation for the Higgs vector bundle  $(\varphi(\mathcal{O}_X), 0)$ . Since the above orthogonal complement  $\varphi(\mathcal{O}_X)^\perp \subset \text{End}(E) \otimes \Omega_X$  is a holomorphic subbundle,

- the connection  $\nabla^{\omega, h}$  preserves  $\varphi(\mathcal{O}_X)$ ,
- and the connection on  $\varphi(\mathcal{O}_X)$  obtained by restricting  $\nabla^{\omega, h}$  coincides with the Chern connection for the Hermitian structure  $h^\omega|_{\varphi(\mathcal{O}_X)}$ .

Also, recall that  $h^\omega|_{\varphi(\mathcal{O}_X)}$  satisfies the Yang–Mills–Higgs equation for the Higgs vector bundle  $(\varphi(\mathcal{O}_X), 0)$ . These together imply that all holomorphic sections of  $\varphi(\mathcal{O}_X)$  over  $X$  are flat with respect to the Yang–Mills–Higgs connection  $\nabla^{\omega, h}$  on  $\text{End}(E) \otimes \Omega_X$ . In particular, the section  $\theta$  is flat with respect to  $\nabla^{\omega, h}$ .  $\square$

**2.1. Decomposition of a Higgs field.** In view of Proposition 2.1, henceforth we assume that  $c_1(TX) = 0$ . Therefore, the Kähler–Einstein form  $\omega$  is Ricci–flat. For any point  $x \in X$ , the fiber of the vector bundle  $\Omega_X$  over  $x$  will be denoted by  $\Omega_{X,x}$ .

Let  $(E, \theta)$  be a polystable Higgs vector bundle on  $X$ . For any point  $x \in X$ , we have a homomorphism

$$\eta_x : T_x X \longrightarrow \text{End}(E_x), \quad \eta_x(v) = i_v(\theta(x)), \quad (2.8)$$

where  $i_v : \Omega_{X,x} \longrightarrow \mathbb{C}$ ,  $z \longmapsto z(v)$ , is the contraction of forms by the tangent vector  $v$ .

**Lemma 2.3.** *For any two points  $x$  and  $y$  of  $X$ , there are isomorphisms*

$$\alpha : T_x X \longrightarrow T_y X \quad \text{and} \quad \beta : E_x \longrightarrow E_y$$

*such that  $\beta(\eta_x(v)(u)) = (\eta_y(\alpha(v)))(\beta(u))$  for all  $v \in T_x X$  and  $u \in E_x$ .*

*Proof.* Let  $h$  be an Hermitian structure on  $E$  satisfying the Yang–Mills–Higgs equation for  $(E, \theta)$ . As before, the Chern connection on  $E$  associated to  $h$  will be denoted by  $\nabla^h$ .

Fix a  $C^\infty$  path  $\gamma : [0, 1] \longrightarrow X$  such that  $\gamma(0) = x$  and  $\gamma(1) = y$ . Take  $\alpha$  to be the parallel transport of  $T_x X$  along  $\gamma$  for the Levi–Civita connection associated to  $\omega$ . Take  $\beta$  to be the parallel transport of  $E_x$  along  $\gamma$  for the above connection  $\nabla^h$ . Using Proposition 2.2 it is straightforward to deduce that

$$\beta(\eta_x(v)(u)) = (\eta_y(\alpha(v)))(\beta(u))$$

for all  $v \in T_x X$  and  $u \in E_x$ .  $\square$

From (2.1) it follows immediately that for any  $v_1, v_2 \in T_x X$ , we have

$$\eta_x(v_1) \circ \eta_x(v_2) = \eta_x(v_2) \circ \eta_x(v_1),$$

where  $\eta_x$  is constructed in (2.8). In view of this commutativity, there is a generalized eigenspace decomposition of  $E_x$  for  $\{\eta_x(v)\}_{v \in T_x X}$ . More precisely, we have distinct elements  $u_1^x, \dots, u_m^x \in \Omega_{X,x}$  and a decomposition

$$E_x = \bigoplus_{i=1}^m E_x^i \quad (2.9)$$

such that

- for all  $v \in T_x$  and  $1 \leq i \leq m$ ,

$$\eta_x(v)(E_x^i) \subseteq E_x^i, \quad (2.10)$$

- the endomorphism of  $E_x^i$

$$\eta_x(v)|_{E_x^i} - u_i^x(v) \cdot \text{Id}_{E_x^i} \quad (2.11)$$

is nilpotent.

Therefore, these elements  $\{u_i^x\}_{i=1}^m$  are the joint generalized eigenvalues of  $\{\eta_x(v)\}_{v \in T_x X}$ . Note however that there is no ordering of the elements  $\{u_i^x\}_{i=1}^m$ . From Lemma 2.3 it follows immediately that the integer  $m$  is independent of  $x$ .

Let  $Y'$  denote the space of all pairs of the form  $(x, \epsilon)$ , where  $x \in X$  and

$$\epsilon : \{1, \dots, m\} \longrightarrow \{u_i^x\}_{i=1}^m$$

is a bijection. Clearly,  $Y'$  is an étale Galois cover of  $X$  with the permutations of  $\{1, \dots, m\}$  as the Galois group. We note that  $Y'$  need not be connected. Fix a connected component  $Y \subset Y'$ . Let

$$\varpi : Y \longrightarrow X, \quad (x, \epsilon) \longmapsto x \quad (2.12)$$

be the projection. So  $\varpi$  is an étale Galois covering map.

For any  $y = (x, \epsilon) \in Y$ , and any  $i \in \{1, \dots, m\}$ , the element  $\epsilon(i) \in \{u_i^x\}_{i=1}^m$  will be denoted by  $\widehat{u}_i^{\varpi(y)}$ .

Therefore, from (2.9) we have a decomposition

$$\varpi^* E = \bigoplus_{i=1}^m F_i, \quad (2.13)$$

where the subspace  $(F_i)_y \subset (\varpi^* E)_y = E_{\varpi(y)}$ ,  $y \in Y$ , is the subspace of  $E_{\varpi(y)}$  which is the generalized simultaneous eigenspace of  $\{\eta_x(v)\}_{v \in T_{\varpi(y)} X}$  for the eigenvalue  $\widehat{u}_i^{\varpi(y)}(v)$  (the element  $\widehat{u}_i^{\varpi(y)}$  is defined above).

Clearly, (2.13) is a holomorphic decomposition of the holomorphic vector bundle  $\varpi^* E$ . Consider the Higgs field  $\varpi^* \theta \in H^0(Y, \text{End}(\varpi^* E) \otimes \Omega_Y)$  on  $\varpi^* E$ , where  $\Omega_Y = \varpi^* \Omega_X$  is the holomorphic cotangent bundle of  $Y$ . From (2.10) it follows immediately that

$$(\varpi^* \theta)(F_i) \subseteq F_i \otimes \Omega_Y. \quad (2.14)$$

Let

$$\theta_i := (\varpi^*\theta)|_{F_i} \quad (2.15)$$

be the Higgs field on  $F_i$  obtained by restricting  $\varpi^*\theta$ .

Equip  $Y$  with the pulled back Kähler form  $\varpi^*\omega$ . Consider the Hermitian structure  $\varpi^*h$  on  $\varpi^*E$ , where  $h$ , as before, is a Hermitian structure on  $E$  satisfying the Yang–Mills–Higgs equation for  $(E, \theta)$ . It is straightforward to check that  $\varpi^*h$  satisfies the Yang–Mills–Higgs equation for  $(\varpi^*E, \varpi^*\theta)$ . In particular,  $(\varpi^*E, \varpi^*\theta)$  is polystable. The restriction of  $\varpi^*h$  to the subbundle  $F_i$  in (2.13) will be denoted by  $h_i$ . Since

$$(\varpi^*E, \varpi^*\theta) = \bigoplus_{i=1}^m (F_i, \theta_i),$$

where  $\theta_i$  is constructed in (2.15), and  $\varpi^*h$  satisfies the Yang–Mills–Higgs equation for  $(\varpi^*E, \varpi^*\theta)$ , it follows that  $h_i$  satisfies the Yang–Mills–Higgs equation for  $(F_i, \theta_i)$  [Si1, p. 878, Theorem 1]. Consequently,  $(F_i, \theta_i)$  is polystable. We note that the polystability of  $(F_i, \theta_i)$  also follows from the fact that  $(F_i, \theta_i)$  is a direct summand of the polystable Higgs vector bundle  $(\varpi^*E, \varpi^*\theta)$ .

Let

$$\mathrm{tr}(\theta_i) \in H^0(Y, \Omega_Y) \quad (2.16)$$

be the trace of  $\theta_i$ . Let  $r_i$  be the rank of the vector bundle  $F_i$ . Define

$$\tilde{\theta}_i := \theta_i - \frac{1}{r_i} \mathrm{Id}_{F_i} \otimes \mathrm{tr}(\theta_i) \in H^0(Y, \mathrm{End}(F_i) \otimes \Omega_Y). \quad (2.17)$$

We note that  $\tilde{\theta}_i$  is also a Higgs field on  $F_i$ .

**Corollary 2.4.** *The section  $\theta_i \in H^0(Y, \mathrm{End}(F_i) \otimes \Omega_Y)$  in (2.15) is flat with respect to the connection on  $\mathrm{End}(F_i) \otimes \Omega_Y$  constructed from  $h_i$  and  $\varpi^*\omega$ . Similarly,  $\tilde{\theta}_i$  in (2.17) is flat with respect to this connection on  $\mathrm{End}(F_i) \otimes \Omega_Y$ .*

*Proof.* We noted earlier that  $h_i$  satisfies the Yang–Mills–Higgs equation for  $(F_i, \theta_i)$ . From this it follows that  $h_i$  also satisfies the Yang–Mills–Higgs equation for  $(F_i, \tilde{\theta}_i)$ . Therefore, substitutions of  $(F_i, \theta_i, h_i)$  and  $(F_i, \tilde{\theta}_i, h_i)$  in place of  $(E, \theta, h)$  in Proposition 2.2 yield the result.  $\square$

**Proposition 2.5.** *The Higgs field  $\tilde{\theta}_i$  on  $F_i$  in (2.17) vanishes identically.*

*Proof.* Since the endomorphism in (2.11) is nilpotent, it follows that

$$\tilde{\theta}_i(y)(v) \in \mathrm{End}(\varpi^*E_y) = \varpi^*\mathrm{End}(E_y) = \mathrm{End}(E_{\varpi(y)})$$

is nilpotent for all  $y \in Y$  and  $v \in T_y Y$ . Consider the homomorphism

$$\tilde{\theta}_i : F_i \longrightarrow F_i \otimes \Omega_Y, \quad z \longmapsto \tilde{\theta}_i(y)(z) \quad \forall z \in (F_i)_y. \quad (2.18)$$

Let

$$\mathcal{V}_i := \text{kernel}(\tilde{\theta}_i) \subset F_i \quad (2.19)$$

be the kernel of it. From Corollary 2.4 it follows that the subsheaf  $\mathcal{V}_i \subset F_i$  is a subbundle. We also note that  $\mathcal{V}_i$  is of positive rank.

Let

$$\tilde{\theta}_i^f = \tilde{\theta}_i \otimes \text{Id}_{\Omega_Y}$$

be the Higgs field on  $F_i \otimes \Omega_Y$ . Since  $\varpi^*\omega$  is Kähler–Einstein, and  $h_i$  satisfies the Yang–Mills–Higgs equation for  $(F_i, \tilde{\theta}_i)$ , the Hermitian structure on  $F_i \otimes \Omega_Y$  induced by the combination of  $h_i$  and  $\varpi^*\omega$  satisfies the Yang–Mills–Higgs equation for  $(F_i \otimes \Omega_Y, \tilde{\theta}_i^f)$ . In particular,  $(F_i \otimes \Omega_Y, \tilde{\theta}_i^f)$  is polystable.

Note that

$$\frac{\text{degree}(F_i \otimes \Omega_Y)}{\text{rank}(F_i \otimes \Omega_Y)} = \frac{\text{degree}(F_i)}{\text{rank}(F_i)} + \frac{\text{degree}(\Omega_Y)}{\text{rank}(\Omega_Y)} = \frac{\text{degree}(F_i)}{\text{rank}(F_i)}; \quad (2.20)$$

the last equality follows from the fact that  $c_1(\Omega_Y) = 0$ . The homomorphism  $\tilde{\theta}_i$  in (2.18) is compatible with the Higgs fields  $\tilde{\theta}_i$  and  $\tilde{\theta}_i^f$  on  $F_i$  and  $F_i \otimes \Omega_Y$  respectively, meaning  $\tilde{\theta}_i \circ \tilde{\theta}_i^f = \tilde{\theta}_i^f \circ \tilde{\theta}_i$ . From the definition of  $\mathcal{V}_i$  in (2.19) it follows immediately that  $\tilde{\theta}_i|_{\mathcal{V}_i} = 0$ . Hence  $(\mathcal{V}_i, 0)$  is a Higgs subbundle of  $(F_i, \tilde{\theta}_i)$ . Since both  $(F_i, \tilde{\theta}_i)$  and  $(F_i \otimes \Omega_Y, \tilde{\theta}_i^f)$  are semistable of same slope (see (2.20)), we conclude that  $(\mathcal{V}_i, 0)$  is a Higgs subbundle of  $(F_i, \tilde{\theta}_i)$  of same slope (same as that of  $F_i$ ). Now, as  $(F_i, \tilde{\theta}_i)$  is polystable, the Higgs subbundle  $(\mathcal{V}_i, 0)$  of same slope has a direct summand.

Let  $(W_i, \theta_i^c) \subset (F_i, \tilde{\theta}_i)$  be a direct summand of  $(\mathcal{V}_i, 0)$ . If  $W_i = 0$ , then the proof is complete. So assume that  $W_i \neq 0$ .

Substituting  $(W_i, \theta_i^c)$  in place of  $(F_i, \tilde{\theta}_i)$  in the above argument and iterating the argument, we conclude that  $\tilde{\theta}_i = 0$ .  $\square$

**Corollary 2.6.** *Let  $X$  be a compact 1-connected Calabi–Yau manifold. If  $(E, \theta)$  is a polystable Higgs vector bundle on  $X$ , then  $\theta = 0$ .*

*Proof.* Since  $X$  is simply connected, it follows that  $\varpi$  in (2.12) is an isomorphism. We have  $H^0(X, \Omega_X) = 0$ , because  $b_1(X) = 0$  and  $\dim H^0(X, \Omega_X) = b_1(X)/2$ . Therefore,  $\text{tr}(\theta_i)$  in (2.16) vanishes identically, and hence  $\tilde{\theta}_i$  in (2.17) is  $\theta_i$  itself. Now Proposition 2.5 completes the proof.  $\square$

## 3. INDEPENDENCE OF YANG–MILLS–HIGGS HERMITIAN STRUCTURE

As before,  $X$  is a compact connected Kähler manifold with  $c_1(TX) = 0$ , and  $\omega$  is a Ricci-flat Kähler form on  $X$ . Let  $(E, \theta)$  be a polystable Higgs vector bundle on  $X$ . Let  $h$  be a Hermitian structure on  $E$  satisfying the Yang–Mills–Higgs equation for  $(E, \theta)$ . We will continue to use the set-up of Section 2.

**Lemma 3.1.** *The decomposition in (2.13) is orthogonal with respect to the pulled back Hermitian structure  $\varpi^*h$  on  $\varpi^*E$ .*

*Proof.* The decomposition in (2.13) gives a decomposition of the Higgs vector bundle  $(\varpi^*E, \varpi^*\theta)$

$$(\varpi^*E, \varpi^*\theta) = \bigoplus_{i=1}^m (F_i, \theta_i),$$

where  $\theta_i$  are constructed in (2.15). Recall that  $(\varpi^*E, \varpi^*\theta)$  and all  $(F_i, \theta_i)$  are polystable. If  $\tilde{h}_i$ ,  $1 \leq i \leq m$ , is a Hermitian structure on  $F_i$  satisfying the Yang–Mills–Higgs equation for  $(F_i, \theta_i)$ , then the Hermitian structure  $\bigoplus_{i=1}^m \tilde{h}_i$  on  $\varpi^*E$ , constructed using the decomposition in (2.13), clearly satisfies the Yang–Mills–Higgs equation for  $(\varpi^*E, \varpi^*\theta)$ .

Any two Hermitian structures on  $\varpi^*E$  that satisfy the Yang–Mills–Higgs equation for  $(\varpi^*E, \varpi^*\theta)$ , differ by a holomorphic automorphism of the Higgs vector bundle  $(\varpi^*E, \varpi^*\theta)$  [Si1, p. 878, Theorem 1]. In particular, there is a holomorphic automorphism

$$T : \varpi^*E \longrightarrow \varpi^*E$$

such that  $(T \otimes \text{Id}_{\Omega_Y}) \circ (\varpi^*\theta) = (\varpi^*\theta) \circ T$ , and

$$\bigoplus_{i=1}^m \tilde{h}_i(a, b) = \varpi^*h(T(a), T(b)). \quad (3.1)$$

Therefore, the lemma follows once it is shown that any holomorphic automorphism of the Higgs vector bundle  $(\varpi^*E, \varpi^*\theta)$  preserves the decomposition in (2.13). Note that the decomposition in (2.13) is orthogonal for the above Hermitian structure  $\bigoplus_{i=1}^m \tilde{h}_i$  on  $\varpi^*E$ . If the above automorphism  $T$  preserves the decomposition in (2.13), then from (3.1) it follows immediately that the decomposition in (2.13) is orthogonal with respect to  $\varpi^*h$ .

From the construction of the decomposition in (2.13) it follows that the  $m$  sections

$$\frac{1}{r_1} \text{tr}(\theta_1), \dots, \frac{1}{r_m} \text{tr}(\theta_m) \in H^0(Y, \Omega_Y)$$

in (2.16) and (2.17) are distinct; as mentioned just before (2.9), the elements  $\{u_i^x\}_{i=1}^m$  are all distinct. Indeed, (2.13) is the generalized eigenspace decomposition for  $\varpi^*\theta$ , and  $\frac{1}{r_1} \text{tr}(\theta_1), \dots, \frac{1}{r_m} \text{tr}(\theta_m)$  are the eigenvalues. It now follows that any automorphism of the Higgs vector bundle  $(\varpi^*E, \varpi^*\theta)$  preserves the decomposition in (2.13). As observed earlier, this completes the proof.  $\square$

**Lemma 3.2.** *The section*

$$\theta \wedge \theta^* \in C^\infty(X, \text{End}(E) \otimes \Omega_X^{1,1})$$

(see (2.2)) *vanishes identically.*

*Proof.* Consider  $\theta_i$  defined in (2.15). From Proposition 2.5 it follows immediately that

$$\tilde{\theta}_i \wedge \tilde{\theta}_i^* = 0. \quad (3.2)$$

Since the decomposition in (2.13) is orthogonal by Lemma 3.1, from (3.2) and (2.17) we conclude that

$$(\varpi^* \theta) \wedge (\varpi^* \theta^*) = 0.$$

This implies that  $\theta \wedge \theta^* = 0$ . □

**Theorem 3.3.** *Let  $(E, \theta)$  be a polystable Higgs vector bundle on  $X$  equipped with a Yang–Mills–Higgs structure  $h$ . Then  $h$  also satisfies the Yang–Mills–Higgs equation for the Higgs vector bundle  $(E, 0)$ .*

*Proof.* In view of Lemma 3.2, this follows immediately from (2.2). □

**Remark 3.4.** It should be clarified that the converse of Theorem 3.3 is not valid. In other words, if  $h$  is an Hermitian structure on  $E$  satisfying the Yang–Mills–Higgs equation for  $(E, 0)$ , then  $h$  need not satisfy the Yang–Mills–Higgs equation for  $(E, \theta)$ . The reason for it is that the automorphism group of  $(E, 0)$  is in general bigger than the automorphism group of  $(E, \theta)$ . To give an example, take  $X$  to be a complex elliptic curve equipped with a flat metric. Take  $E$  to be the trivial vector bundle  $\mathcal{O}_X^{\oplus 2}$  on  $X$  of rank two. Let  $\theta$  be the Higgs field on  $\mathcal{O}_X^{\oplus 2}$  given by the matrix

$$A := \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix};$$

fixing a trivialization of  $\Omega_X$ , we identify the Higgs fields on  $\mathcal{O}_X^{\oplus 2}$  with the  $2 \times 2$  complex matrices. This Higgs vector bundle  $(E, \theta)$  is polystable because the matrix  $A$  is semisimple. The Hermitian structure on  $\mathcal{O}_X^{\oplus 2}$  given by the standard inner product on  $\mathbb{C}^2$  satisfies the Yang–Mills–Higgs equation for  $(E, 0)$ , but this Hermitian structure does not satisfy Yang–Mills–Higgs equation for  $(E, \theta)$  (because  $AA^* \neq A^*A$ ).

**Remark 3.5.** Let  $(E, \theta)$  be a polystable Higgs vector bundle on  $X$ . From Theorem 3.3 we know that the Higgs vector bundle  $(E, 0)$  is polystable. Fix a Hermitian structure  $h_0$  on  $E$  satisfying the Yang–Mills–Higgs equation for  $(E, 0)$ . Any other Hermitian structure on  $E$  that satisfies the Yang–Mills–Higgs equation for  $(E, 0)$  differs from  $h_0$  by a holomorphic automorphism of  $E$ . Take a holomorphic automorphism  $T$  of  $E$  such that the Hermitian structure  $h := T^*h_0$  on  $E$  has the following property:

$$\theta \wedge \theta^{*h} = 0,$$

where  $\theta^{*h}$  is the adjoint of  $\theta$  constructed using  $h$ . From Lemma 3.2 and Theorem 3.3 it follows that such an automorphism  $T$  exists. The above Hermitian structure  $h$  satisfies the Yang–Mills–Higgs equation for  $(E, \theta)$ .

#### 4. POLYSTABLE PRINCIPAL HIGGS $G$ -BUNDLES

Let  $G$  be a connected reductive affine algebraic group defined over  $\mathbb{C}$ . The Lie algebra of  $G$  will be denoted by  $\mathfrak{g}$ . As before,  $X$  is a compact connected Kähler manifold equipped with a Ricci–flat Kähler form  $\omega$ . Let  $E_G \rightarrow X$  be a holomorphic principal  $G$ -bundle. Its adjoint vector bundle  $E_G \times^G \mathfrak{g}$  will be denoted by  $\text{ad}(E_G)$ . A Higgs field on  $E_G$  is a holomorphic section

$$\theta \in H^0(X, \text{ad}(E_G) \otimes \Omega_X)$$

such that the section  $\theta \wedge \theta$  of  $\text{ad}(E_G) \otimes \Omega_X^2$  vanishes identically. A Higgs  $G$ -bundle on  $X$  is a pair of the form  $(E_G, \theta)$ , where  $E_G$  is a holomorphic principal  $G$ -bundle on  $X$ , and  $\theta$  is a Higgs field on  $E_G$ .

Fix a maximal compact subgroup

$$K_G \subset G.$$

A Hermitian structure on a holomorphic principal  $G$ -bundle  $E_G$  on  $X$  is a  $C^\infty$  reduction of structure group of  $E_G$

$$E_{K_G} \subset E_G$$

to the subgroup  $K_G$ . There is a unique  $C^\infty$  connection  $\nabla$  on the principal  $K_G$ -bundle  $E_{K_G}$  such that the connection on  $E_G$  induced by  $\nabla$  is compatible with the holomorphic structure of  $E_G$  [At, p. 191–192, Proposition 5]. Using the decomposition  $\mathfrak{g} = \text{Lie}(K) \oplus \mathfrak{p}$ , given any Higgs field  $\theta$  on  $E_G$ , we have

$$\theta^* \in C^\infty(X; \text{ad}(E_G) \otimes \Omega_X^{0,1}).$$

Let  $(E_G, \theta)$  be a Higgs  $G$ -bundle on  $X$ . The center of the Lie algebra  $\mathfrak{g}$  will be denoted by  $z(\mathfrak{g})$ . Since the adjoint action of  $G$  on  $z(\mathfrak{g})$  is trivial, we have an injective homomorphism

$$\psi : X \times z(\mathfrak{g}) \hookrightarrow \text{ad}(E_G) \tag{4.1}$$

from the trivial vector bundle with fiber  $z(\mathfrak{g})$ . This homomorphism  $\psi$  produces an injective homomorphism

$$\widehat{\psi} : z(\mathfrak{g}) \hookrightarrow H^0(X, \text{ad}(E_G)).$$

A Hermitian structure  $E_{K_G} \subset E_G$  is said to satisfy the Yang–Mills–Higgs equation for  $(E_G, \theta)$  if there is an element  $c \in z(\mathfrak{g})$  such that

$$\Lambda_\omega(\mathcal{K}(\nabla) + \theta \wedge \theta^*) = \widehat{\psi}(c),$$

where  $\mathcal{K}(\nabla)$  is the curvature of the connection  $\nabla$  associated to the reduction  $E_{K_G}$ , and  $\theta^*$  is defined above.

It is known that  $(E_G, \theta)$  admits a Yang–Mills–Higgs Hermitian structure if and only if  $(E_G, \theta)$  is polystable [Si2], [BS, p. 554, Theorem 4.6]. (See [BS] for the definition of a polystable Higgs  $G$ -bundle.)

**Lemma 4.1.** *Let  $(E_G, \theta)$  be a Higgs  $G$ -bundle on  $X$  equipped with an Hermitian structure  $E_{K_G} \subset E_G$  that satisfies the Yang–Mills–Higgs equation for  $(E_G, \theta)$ . Then*

$$\theta \wedge \theta^* = 0.$$

*Proof.* This follows by applying Lemma 3.2 to the Higgs vector bundle associated to  $(E_G, \theta)$  for the adjoint action of  $G$  on  $\mathfrak{g}$ . Consider the adjoint Higgs vector bundle  $(\text{ad}(E_G), \text{ad}(\theta))$ . The reduction  $E_{K_G}$  produces a Hermitian structure on the vector bundle  $\text{ad}(E_G)$  that satisfies the Yang–Mills–Higgs equation for  $(\text{ad}(E_G), \text{ad}(\theta))$ . Now Lemma 3.2 says that

$$\text{ad}(\theta) \wedge \text{ad}(\theta)^* = 0.$$

This immediately implies that the  $C^\infty$  section  $\theta \wedge \theta^*$  of  $\text{ad}(E_G) \otimes \Omega_X^{1,1}$  is actually a section of  $\psi(z(\mathfrak{g})) \otimes \Omega_X^{1,1}$ , where  $\psi$  is the homomorphism in (4.1).

Take any holomorphic character  $\chi : G \rightarrow \mathbb{C}^*$ . Let

$$L^\chi := E_G \times^\chi \mathbb{C} \rightarrow X$$

be the holomorphic line bundle associated to  $E_G$  for  $\chi$ . The Higgs field  $\theta$  defines a Higgs field on  $L^\chi$  using the homomorphism of Lie algebras

$$d\chi : \mathfrak{g} \rightarrow \mathbb{C} \tag{4.2}$$

associated to  $\chi$ ; this Higgs field on  $L^\chi$  will be denoted by  $\theta^\chi$ . Since  $L^\chi$  is a line bundle, we have  $\theta^\chi \wedge (\theta^\chi)^* = 0$  (Lemma 3.2 is not needed for this). As  $\theta \wedge \theta^*$  is a section of  $\psi(z(\mathfrak{g})) \otimes \Omega_X^{1,1}$ , from this it can be deduced that  $\theta \wedge \theta^* = 0$ . Indeed, given any nonzero element  $v \in z(\mathfrak{g})$ , there is a holomorphic character

$$\chi : G \rightarrow \mathbb{C}^*$$

such that  $d\chi(v) \neq 0$  (defined in (4.2)). □

**Theorem 4.2.** *Let  $(E_G, \theta)$  be a polystable Higgs  $G$ -bundle on  $X$ , and let  $E_{K_G} \subset E_G$  be an Hermitian structure that satisfies the Yang–Mills–Higgs equation for  $(E_G, \theta)$ . Then the Hermitian structure  $E_{K_G} \subset E_G$  also satisfies the Yang–Mills–Higgs equation for  $(E_G, 0)$ .*

*Proof.* In view of the Yang–Mills–Higgs equation for  $(E_G, \theta)$ , this follows immediately from Lemma 4.1. □

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