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# EXISTENCE RESULTS
# FOR SOME VARIATIONAL PROBLEMS
# IN SEGMENTATION THEORY

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TRIESTE

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Scuola Internazionale Superiore di Studi Avanzati

International School for Advanced Studies

"MAGISTER PHILOSOPHIAE" THESIS

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*To the dreams*

# CONTENTS

# CHAPTER 1

## INTRODUCTION

In this thesis we deal with the model case of some variational formulation of a segmentation problem.

Let $\Omega$ be a bounded domain of $\mathbf{R}^n$. Given a function $g \in L^\infty(\Omega)$, the corresponding segmentation problem consists in subdividing the domain $\Omega$ into appropriate regions, and in approximating, on each region, the function g by a smooth function. More precisely, we have to find a closed subset K of $\Omega$ and a function u defined on $\Omega \backslash K$, such that

(1) u is smooth on $\Omega \backslash K$,

(2) u is a good approximation of g on $\Omega \backslash K$,

(3) K is as "small" as possible.

The pair $(u, K)$ is called a segmentation of g. In a good segmentation the set K and the function u are chosen so that the approximation u eliminates the less relevant details of the datum g, but preserves some properties of its behaviour.

In the two dimensional case such a problem is suggested for instance by applications to computer vision. In this setting the function g, defined on a plain domain, represents an image: g is the grey level, i.e. g(x) measures the intensity of the light at x. One expects the function g to be discontinuous along the lines corresponding to sudden changes in the visible surfaces (e.g. the edges of the objects, surface markings, shadows, different colours). The image segmentation problem consists in finding a pair $(u, K)$ such that K is a set of curves decomposing the image into regions with relatively uniform intensity, while u is a smooth approximation of g on each region. The set K will be interpreted as the union of the lines which give the schematic description of the image.

For a general treatment of this subject we refer to A. Rosenfeld and A. C. Kak [23].

Similar situations arise in many other contexts, like the perception of speech, which requires segmenting time, the domain of the speech signal, into intervals during which a single phoneme is being pronounced, or radar data, in which g(x,y) represents the distance from a fixed point P in direction (x,y) to the nearest solid object.

The variational approach to the segmentation problem proposed at the beginning consists in minimizing an energy functional, depending on u and K, expressed by a sum of terms each measuring the deviation from one of the desired properties.

In order to confront the image segmentation problem D. Mumford and J. Shah [20], [21] developed this variational idea, proposing the following functional, defined for every closed subset K of $\bar{\Omega}$ ($\Omega$ is a plain domain) and for every function $u \in C^1(\Omega \setminus K)$:

$$(1.1) \qquad E(u,K) = \int_{\Omega \setminus K} |\nabla u|^2 \, dx \; + \; \int_{\Omega \setminus K} |u - g|^2 \, dx \; + \; \gamma \, \mathcal{H}^1(K) \, ,$$

where $\nabla u$ is the gradient of u, $\mathcal{H}^1$ denotes the one dimensional Hausdorff measure (see [14], 2.10.2), and $\gamma$ is a positive real number.

In the expression of E the first term requires u to vary smoothly on each connected component of $\Omega \setminus K$, while the second term forces u to be close to g. The last term, which penalises the total length of the discontinuity set K, is introduced in order to avoid a subdivision of the domain $\Omega$ into too many regions. Dropping any of these three terms inf E = 0 and in general we have no meaningful solutions.

For the proof of the existence of a minimizer for the functional E, conjectured in [21], see G. Dal Maso, J. M. Morel, and S. Solimini [10] and E. De Giorgi, M. Carriero, and A. Leaci [13] for the general case of the space dimension $n \geq 1$.

Given an open bounded subset $\Omega$ of $\mathbf{R}^n$ and a function $g \in L^\infty(\Omega)$, the n-dimensional version of functional (1.1), involving the (n-1)- dimensional Hausdorff measure $\mathcal{H}^{n-1}$, is defined by

$$(1.2) \qquad E(u,K) = \int_{\Omega \setminus K} |\nabla u|^2 \, dx \; + \; \int_{\Omega \setminus K} |u - g|^2 \, dx \; + \; \gamma \, \mathcal{H}^{n-1}(K),$$

3

for every closed subset K of $\bar{\Omega}$ and for every function $u \in C^1(\Omega \setminus K)$.

Both the proofs [10], [13] are based on the use of a new function space, named SBV($\Omega$) ( see [3] and [12] ), whose elements admit essential discontinuities along sets of codimension one. More precisely, a function $u \in L^1(\Omega)$ belongs to SBV($\Omega$) if and only if its distributional derivative Du is a vector measure which admits the Lebesgue decomposition

$$Du = (\nabla u)\, dx + (u^+ - u^-)\, v_u \, \mathcal{H}^{n-1}|_{S_u} \, ,$$

where $\nabla u$ in $L^1(\Omega, R^n)$ is the approximate differential of u, $S_u$ is the set of all jump points of u , $v_u$ is the unit normal to $S_u$, and $u^+$, $u^-$ are the approximate limits of u from both sides of $S_u$ (see Chapter 2 for the precise definitions).

In order to study the minimization problem corresponding to the functional (1.2) E. De Giorgi suggested the following weak formulation in SBV($\Omega$):

$$\min \{ E(u) : u \in SBV(\Omega) \},$$

where

(1.3) $\qquad E(u) = \int_\Omega |\nabla u|^2\, dx + \int_\Omega |u - g|^2\, dx + \gamma \, \mathcal{H}^{n-1}(S_u),$

( see [2], [3], [10], [12], [13] ).

By relying on a general compactness and semicontinuity theorem due to L. Ambrosio [2], it is easy to show that functional (1.3) achieves its minimum on SBV($\Omega$). The existence of a minimizer of the functional (1.2) is then proved by studying the regularity properties of the minimum points in SBV($\Omega$) ( see Chapter 3, Section 1 for more details ).

Further results about these problems can be found in [5], [11], [18], [19], [25].

The Mumford-Shah model, though quite simple, presents some limitations: it is unable to detect crease discontinuities ( i.e. points where the function is continuous, but the first derivative is discontinuous ) and presents the so called "gradient limit" effect. We try to explain what it means by a simple example modeled on the one-dimensional version of functional (1.2) ( $\mathcal{H}^{n-1}$ is reduced to the counting measure on R ). On the open interval

$I = (0,1)$ let us consider the ramp g, depending on a positive parameter m, defined by $g(t) = 0$ on $(0, \frac{1}{3})$, $g(t) = mt - \frac{m}{3}$ on $(\frac{1}{3}, \frac{2}{3})$ and $g(t) = \frac{m}{3}$ on $(\frac{2}{3}, 1)$. Let us examine the minimization problem corresponding to g. If the gradient m of the ramp is less than some threshold $g_l$, called "gradient limit", then the solution u is continuously differentiable on the whole interval $(0,1)$. On the contrary, if m exceeds $g_l$, at least one discontinuity appears in the reconstruction of the ramp (at $t = \frac{1}{2}$ if there is only one) and u might have even multiple breaks on the interval $(\frac{1}{3}, \frac{2}{3})$ if m is much greater than $g_l$ (see [6], 4.1.5). To sum up, from our point of view u is not a good approximation of g for m large enough, since their qualitative behaviours are too different. Moreover, some crease discontinuities seem meaningful in the shape of the datum and u never reconstructs them. In the two dimensional case it is easy to construct an analogous example, showing the same limitations for the functional (1.1) as a model in image reconstruction.

In order to overcome the deficiencies of such a model, A. Blake and A. Zisserman [6] suggest to modify functional (1.1), including the second order derivatives of u and a penalty for unit length of crease discontinuity.

Following the ideas of these authors, we propose the functionals, below defined for every pair of disjoint closed subsets $K_0$, $K_2$ of $\Omega$ and for every function $u \in C^0(\Omega \setminus K_0) \cap C^2(\Omega \setminus (K_0 \cup K_2))$ as

$$(1.4) \quad G(u, K_0, K_2) = \int_\Omega |\nabla^2 u|^2 \, dx + \int_\Omega |\nabla u|^2 \, dx + \int_\Omega |u - g|^2 \, dx +$$

$$+ \alpha \, \mathcal{H}^{n-1}(K_0) + \beta \, \mathcal{H}^{n-1}(K_2),$$

$$(1.5) \quad H(u, K_0, K_2) = \int_\Omega |\nabla^2 u|^2 \, dx + \int_\Omega |u - g|^2 \, dx + + \alpha \, \mathcal{H}^{n-1}(K_0) + \beta \, \mathcal{H}^{n-1}(K_2),$$

where $\alpha$ and $\beta$ are nonnegative real numbers, and $\nabla^2 u$ denotes the Hessian matrix of u.

Another possibility is to consider the square laplacian instead of the quadratic variation

$$(1.6) \quad \tilde{H}(u, K_0, K_2) = \int_\Omega |\Delta u|^2 \, dx + \int_\Omega |u - g|^2 \, dx + \alpha \, \mathcal{H}^{n-1}(K_0) + \beta \, \mathcal{H}^{n-1}(K_2),$$

where $\Delta u$ denotes the laplacian of u.

5

A. Blake and A. Zisserman deal only with the numerical aspects of these segmentation problems: making use of the finite element method they study the minimization problems associated to the functionals (1.4), (1.5), and (1.6), observing a better approximation of the datum in the cases of H and $\tilde{H}$. On the contrary, the functional G, including at the same time the first and the second order derivatives, has the worst of both models (1.2) and (1.5), the gradient limit of the first order energy as well as the additional computational load of the second order one.

It is easy to prove that functional (1.4) has a minimizer on a suitable space of functions of bounded variation, under the following restrictions on the values of the coefficients $\alpha$ and $\beta$:

(1.7)                         $0 < \beta \leq \alpha,$

( see Chapter 3, Section 2 ). The one suggested in this thesis is a possible weak formulation of the minimization problem corresponding to G, since no regularity theorems have been proved yet. The result is obtained by applying the compactness and semicontinuity theorems [2] and [4], due to L. Ambrosio, successively to a minimizing sequence ( $u_h$ ) and to the vector valued sequence ( $u_h$, $\nabla u_h$ ).

On the contrary, including the second order derivatives of the unknown function u, instead of the first, in the expression of the functional implies considerable mathematical difficulties. For instance, in dimension larger than one, it is not known whether it is possible to give a weak formulation of the minimization problems corresponding to the functionals (1.5) and (1.6), such as the one described above for the study of the functional (1.2).

As a first step in this direction, in the last part of this thesis we study the minimization problem associated to the functional (1.5) in dimension one, and we prove an existence theorem by applying the direct method of the calculus of variation.

More precisely, fixed the open interval I = (0,1) and given a function $g \in L^2(0,1)$, we consider the functional below defined for every pair of disjoint finite subsets $J_0$, $J_2$ of I and for every function $u \in C^0(I \setminus J_0) \cap C^2(I \setminus (J_0 \cup J_2))$ as

6

$$(1.8) \qquad F(u, J_0, J_2) = \int_0^1 |u''|^2 \, dt + \int_0^1 |u - g|^2 \, dt + \alpha \, \#(J_0) + \beta \, \#(J_2),$$

where $u''$ denotes the second derivative of u, $\alpha$ and $\beta$ are two fixed nonnegative real numbers, and # is the counting measure on **R**.

The existence of a minimum point is obtained by relying on the compactness and the lower semicontinuity of F with respect to the $L^1$- convergence.

Since the functional does not control the first derivative of u, except for the total number of its discontinuities, we expect that the functional F has a minimum point only under further restrictions, with respect to (1.7), on the values of the two coefficients $\alpha$ and $\beta$.

More precisely, we prove the compactness of the functional along sequences of functions on which F is bounded, provided that both the coefficients $\alpha$ and $\beta$ are strictly positive. This is a natural condition: in fact if we set $\alpha = 0$ or $\beta = 0$ the infimum of F is always zero and in general the associated minimization problem has no solutions ( see Examples 4.2.4, 4.2.5, and 4.4.3 ). The proof of the compactness theorem makes use of some interpolation inequalities for intermediate derivatives in order to apply an already known compactness result on appropriate subintervals of I. The limit function is then constructed by using a diagonal argument.

Moreover we study the lower semicontinuity of the functional F with respect to the $L^1$- convergence. It is easy to see that the first two terms are lower semicontinuous. On the contrary, the third and fourth terms are not semicontinuous separately ( see Examples 4.3.3 and 4.3.4 ). However, assuming the inequalities $\beta \leq \alpha \leq 2\beta$, we prove that the sum of the last two terms of F is lower semicontinuous with respect to the $L^1$- convergence along the sequences ($u_h$) on which the whole functional is bounded.

Besides, we show that such conditions on the coefficients $\alpha$ and $\beta$ are necessary to assure the lower semicontinuity of F ( again Examples 4.3.3 and 4.3.4 ).

The proof of the semicontinuity theorem is obtained by studying accurately the limits of the singular points of the sequence ($u_h$) converging to u.

Combining the compactness and the semicontinuity results, we prove that the minimization problem corresponding to the functional F has a solution if $0 < \beta \leq \alpha \leq 2\beta$. If one of these inequalities is not satisfied, then it is possible to construct functions g such that the associated functionals F have no minimizers (see Examples 4.4.3, 4.4.4, and 4.4.5), so we cannot assure the existence of a solution independently of the choice of the datum g.

This thesis is organized as follows: in Chapter 2 we recall some definitions and known results about the spaces $BV(\Omega)$, $SBV(\Omega)$, $GBV(\Omega)$, and $GSBV(\Omega)$ of functions of bounded variation, without giving proof; in Chapter 3 we expose the existence and regularity results concerning functional (1.2) and we prove the existence of a minimizer of functional (1.4) in a suitable generalized sense; finally Chapter 4 contains the main results on functional (1.8): in Section 4.2 we prove a compactness theorem, in Section 4.3 we study the lower semicontinuity of the functional, hence in Section 4.4 we obtain the existence of a minimizer.

The results of chapter 4 are contained in [9].

# CHAPTER 2

## SPACES OF FUNCTIONS OF BOUNDED VARIATION

In this chapter we introduce the space $BV(\Omega)$ of functions of bounded variation on a bounded open subset $\Omega$ of $R^n$. In Section 2.1 we list a series of definitions and well known results; in particular we specify the notions of approximate continuity, approximate differentiability, and jump set of a bounded variation function. In Section 2.2 and 2.3 we recall the definitions of some new spaces of functions of bounded variation introduced to study the segmentation problem described in the introduction.

## 2.1. The space $BV(\Omega)$ of the functions of bounded variation

Let us begin by listing the basic notations used in this thesis. Let $n \geq 1$ be an integer and $\Omega$ be a bounded open subset of $R^n$. We denote by $\mathcal{B}(\Omega)$ the $\sigma$-algebra of Borel subset of $\Omega$. By $\mathcal{L}^n(B)$ we denote the Lebesgue measure of a Lebesgue-measurable subset $B$ of $R^n$, and by $\mathcal{H}^{n-1}(B)$ the Hausdorff $(n-1)$-dimensional measure of any $B \in \mathcal{B}(\Omega)$ ( see [15], 2.10.2 ). Let $A, B \in \mathcal{B}(\Omega)$; when we write $A \subset\subset B$, we mean that the closure of $A$ is a compact subset of $\Omega$ containing in $B$. Let $k \geq 1$ be an integer; a Borel function $u : \Omega \to R^k$ is a vector $u = (u_1, .., u_k)$, where each $u_i$ is a Borel measurable function $: \Omega \to R$. In the sequel we identify the space of matrices with $k$ lines and $n$ columns with the vector space $R^{nk}$. Moreover for every $\varphi \in C_0(\Omega)$ we indicate by $\operatorname{supp} \varphi$ the support of $\varphi$.

Let $v : \mathcal{B}(\Omega) \to R^n$ be a vector valued Radon measure on $\Omega$ ( see [15], 2.2.5 ). We introduce the total variation of $v$ as the positive scalar measure $|v|$ defined by

(2.1.1)    $|\nu|(A) = \{ \sup\limits_{i\in N} \Sigma\ |\nu(A_i)| \ : \ A = \bigcup\limits_{i\in N} A_i \ , \ A_i \in \mathcal{B}(\Omega), \ A_i$ mutually disjoint $\}$

for every $A \in \mathcal{B}(\Omega)$ ( see [24], Chapter 6 ), where $|\cdot|$ denotes the euclidean norm in $\mathbf{R}^n$. We say that $\nu$ is a measure with finite total variation in $\Omega$ if $|\nu|(\Omega) < +\infty$.

**Definition 2.1.1.** ( see [24], 6.7 ) Let $\lambda : \mathcal{B}(\Omega) \to [0,+\infty)$, $\nu : \mathcal{B}(\Omega) \to \mathbf{R}^n$ be measures with finite total variation. We say that $\nu$ is *absolutely continuous* with respect to $\lambda$ and we write $\nu << \lambda$, if $\lambda(B) = 0$ implies that $\nu(B) = 0$ for every $B \in \mathcal{B}(\Omega)$.

If $\nu << \lambda$, then by the Radon-Nikodym theorem ( see [24], 6.10 ) there exists a function in $L^1(\Omega,\lambda)$, denoted by $\dfrac{\nu}{\lambda}$, such that $\nu(A) = \int\limits_A \dfrac{\nu}{\lambda}\ d\lambda$ for every $A \in \mathcal{B}(\Omega)$. This function is usually called the Radon-Nikodym derivative of $\nu$ with respect to $\lambda$.

**Definition 2.1.2.** ( see [17], Chapter 1 ) We say that $u \in L^1(\Omega)$ is a *function of bounded variation* in $\Omega$ if its distributional derivative $Du = (D_1 u, ..,D_n u)$ is a Radon measure of finite variation on $\Omega$.

For the general theory of functions of bounded variation we refer to [15], [16], [17], [27], [28], [29].

The space of the functions of bounded variation will be denoted by $BV(\Omega)$; moreover the space of all the functions belonging to $BV(\Omega')$ for every open set $\Omega' \subset\subset \Omega$, will be indicate by $BV_{loc}(\Omega)$. It is easy to verify that $W^{1,1}(\Omega) \subset BV(\Omega)$, where $W^{1,1}(\Omega)$ is the Sobolev space of the functions $u \in L^1(\Omega)$ with distributional derivative $Du \in L^1(\Omega,\mathbf{R}^n)$.

Let $u \in BV(\Omega)$. It is well known that for every open set $A \subseteq \Omega$ we have

$$\int\limits_A |Du| = \sup\ \{ \int\limits_\Omega u\ \text{div}\ g\ dx \ : \ g \in C_0^\infty(A,\mathbf{R}^n),\ |g| \leq 1 \ \}.$$

Endowed with the norm

10

$$\| u \|_{BV(\Omega)} = \| u \|_{L^1(\Omega)} + \int_\Omega |Du| \,,$$

$BV(\Omega)$ is a Banach space.

Let $k \geq 1$ be an integer and $u : \Omega \to R^k$ be a Borel function.

**Definition 2.1.3.** ( see [15], 2.9.9 ) We say that $x \in \Omega$ is a *Lebesgue point* of u if there exists $\tilde{u}(x) \in R^k$ such that

$$\lim_{r \to 0^+} r^{-n} \int_{B_r} |u(y) - \tilde{u}(x)| \, dy = 0 \,,$$

where $B_r(x)$ is the ball centered at x of radius r and $|\cdot|$ denotes the euclidean norm in $R^k$.

By $S_u$ we denote the *singular set* of u , defined as the set of all $x \in \Omega$ which are not Lebesgue points of u. If $u \in L^1(\Omega, R^k)$, then ( see [15], 2.9.9 ) $\mathcal{L}^n(S_u) = 0$ and $u = \tilde{u}$ a.e. on $\Omega \setminus S_u$. Note that $S_u$ is uniquely determined up to sets of Lebesgue measure zero.

If $u \in BV(\Omega)$, then the set $S_u$ is *countably (n-1)-rectifiable* in the sense of Federer ( see [15], 3.2.14 ), i.e.

$$(2.1.2) \qquad S_u \subseteq (\bigcup_{n=1}^{\infty} \Gamma_i) \cup N,$$

where $\Gamma_i$ are hypersurfaces of class $C^1$ and $\mathcal{H}^{n-1}(N) = 0$.

Moreover, for $\mathcal{H}^{n-1}$ a.e. $x \in S_u$ there exist two real numbers $u^-(x)$, $u^+(x)$ and a unit vector $v_u(x) \in R^n$ such that

$$(2.1.3) \qquad u^-(x) < u^+(x)$$

$$\lim_{r \to 0^+} r^{-n} \int_{B_r^+} |u(y) - u^+(x)| \, dy = 0 \,,$$

$$\lim_{r \to 0^+} r^{-n} \int_{B_r^-} |u(y) - u^-(x)| \, dy = 0 \,,$$

where $B_r^{\pm}(x) = B_r(x) \cap \{ y \in R^n : (y - x, \pm v_u(x)) > 0 \}$ (see [14], Theorem 4.5.9 ). It is

clear that $u^-(x)$, $u^+(x)$, $v_u(x)$ are uniquely determined by (2.1.3) and do not depend on the choice of $u$ in its equivalence class with respect to equality almost everywhere.

An equivalent definition of the triplet $(u^-(x), u^+(x), v_u(x))$ is the following.

Let $k \geq 1$ be an integer and $u : \Omega \to \mathbf{R}^k$ be a Borel function. Let $A \in \mathcal{B}(\Omega)$ and $x \in \Omega$.

**Definition 2.1.4.** ( see [15], 2.9.12 ) Let $C \in \mathcal{B}(\Omega)$. The *density* of the set C at the point x with respect to A is defined as

$$\lim_{r \to 0} \frac{|A \cap B_r(x) \cap C|}{|B_r(x) \cap C|} \, .$$

**Definition 2.1.5.** ( see [15], 2.9.12 ) We say that $\omega \in \mathbf{R}^k$ is the *approximate limit* of u at x with respect to the set A, and we write

$$\omega = \mathop{\mathrm{ap\,lim}}_{\substack{y \to x \\ y \in A}} u(y) \, ,$$

if and only if for every neighbourhood W of $\omega$ in $\mathbf{R}^k$, the set $\Omega \setminus u^{-1}(W)$ has density 0 at x with respect to A.

**Definition 2.1.6.** ( see [15], 2.9.12 ) We say that u is *approximately continuous* at x if and only if $\mathop{\mathrm{ap\,lim}}_{y \to x} u(y) = u(x)$.

In the scalar case, i.e. for a Borel function $u : \Omega \to [-\infty, +\infty]$, we can also define the *approximate upper and lower limits* $u^+, u^- : \Omega \to [-\infty, +\infty]$ as

$$(2.1.4) \quad u^+(x) = \inf \left\{ t \in [-\infty, +\infty] : \lim_{r \to 0+} \frac{|B_r(x) \cap \{y \in \Omega : u(y) > t\}|}{|B_r(x)|} = 0 \right\}$$

and

$$(2.1.5) \quad u^-(x) = \sup \left\{ t \in [-\infty, +\infty] : \lim_{r \to 0+} \frac{|B_r(x) \cap \{y \in \Omega : u(y) < t\}|}{|B_r(x)|} = 0 \right\}.$$

Then $u^+$ and $u^-$ are Borel functions ( see [15], 4.5.9 ).


**Proposition 2.1.7.** ( see [15], 2.9.13 ) *A Borel function* $u : \Omega \to R^k$ *is approximately continuous at almost every* $x \in \Omega$.


The approximately continuous representative of u will be indicated by $\tilde{u}$.

By N(u) we denote the set of points $x \in \Omega$, which are not approximate continuity points of u. This set is negligible with respect to $\mathcal{L}^n$ by Proposition 2.1.7. If $u \in L^1(\Omega, R^k)$, then N(u) differs from the singular set $S_u$ of u, defined above, up to a set of $\mathcal{H}^{n-1}$ measure zero.

In the scalar case $N(u) = \{ x \in \Omega : u^-(x) < u^+(x) \}$; a point $x \in N(u)$ is called a jump point of u, and N(u) is the jump set of u.

In the sequel, when it is possible we will identify the sets N(u) and $S_u$.

Clearly, from Definition 2.1.6 it follows that $x \in N(u)$, where $u = (u_1, .. , u_k) : \Omega \to R^k$ is a Borel function, if and only if there exists an index $i \in \{ 1, .. , k \}$ such that x is a jump point of $u_i$ , i.e.

$$(2.1.6) \qquad N(u) = \bigcup_{i=1}^{k} N(u_i).$$

Let $x \in \Omega$ and $u \in BV(\Omega)$. There are the following possibilities: either the approximate limit of the function u at x exists and therefore, changing the value of this function at the point x we obtain u approximately continuous at x, or we can find one and only one (n-1)-dimensional hyperplane such that at this point there exist the approximate limits with respect to each of the half-spaces separated by the hyperplane. The normal to this hyperplane will be denoted by $\nu_u$ and concide with the unit vector defined in (2.1.3) up to sign.

The points where both the two conditions above are not satisfied consist in a set negligible with respect to $\mathcal{H}^{n-1}$.


13

**Definition 2.1.8.** Let $u : \Omega \to \mathbf{R}^k$ be a Borel function. Let $x \in \Omega \backslash S_u$ such that $\tilde{u}(x) \neq \infty$. We say that a linear mapping $L \in \mathbf{R}^{nk}$ is the *approximate differential* of u at x if

$$\operatorname*{ap\,lim}_{y \to x} \frac{|u(y) - \tilde{u}(x) - <L, y - x>|}{|y - x|} = 0.$$

If the approximate differential of u at x exists it is unique, and we will denote it by $\nabla u(x)$.

We observe that the function u is approximately differentiable at x if and only if all the k components $u_i$ are approximately differentiable at x. Moreover the set

$$\{ x \in \Omega \backslash S_u : \tilde{u}(x) \neq \infty \text{ and exists } \nabla u(x) \}$$

belongs to $\mathcal{B}(\Omega)$ and $\nabla u$ is a Borel function from this set into $\mathbf{R}^{nk}$.

**Proposition 2.1.9.** ( see [8] ) *Let* $u \in BV(\Omega, \mathbf{R}^k)$. *Then* u *is approximately differentiable almost everywhere in* $\Omega$.

**Proposition 2.1.10.** ( see [27], 4.5 ) *Let* $u \in BV(\Omega)$. *Let us consider the total variation of* Du, *defined in* (2.1.1). *Then* $|Du|(C) = 0$ *for every* $C \in \mathcal{B}(\Omega)$ *with* $\mathcal{H}^{n-1}(C) = 0$.

**Proposition 2.1.11.** *Let* $u \in BV(\Omega)$. *Then the distributional derivative* Du *is a measure which admits the decomposition*

$$Du = (Du)_a + (Du)_s ,$$

*where* $(Du)_a$ *is absolutely continuous and* $(Du)_s$ *is singular with respect to* $\mathcal{L}^n$. *In addition the approximate differential* $\nabla u \in L^1(\Omega, \mathbf{R}^n)$ *and*

$$(Du)_a(B) = \int_B \nabla u \, dx \qquad \text{for every } B \in \mathcal{B}(\Omega).$$

Notice that the approximate differential $\nabla u$ is the Radon-Nikodym derivative of $(Du)_a$ with respect to the Lebesgue measure.

## 2.2. The space $SBV(\Omega)$ of the special functions of bounded variation

In order to define the space $SBV(\Omega)$ we require a deeper analysis of the distributional derivative Du of a function of bounded variation.

**Proposition 2.2.1.** *Let* $u \in BV(\Omega)$. *Then the distributional derivative Du can be decomposed into three mutually singular measures:*

$$(2.2.1) \qquad Du = \nabla u \, \mathcal{L}^n_{|\Omega} + Ju + Cu.$$

*The first term in the right hand side of (2.2.1) corresponds to the absolutely continuous part with respect to* $\mathcal{L}^n$, *accordingly to Proposition 2.1.11. The second term corresponds to the "jump" part of the derivative, and is related to* $S_u$, $u^+$, $u^-$ *by*

$$(2.2.2) \qquad Ju(B) = \int_{B \cap S_u} (u^+ - u^-) \, \nu_u \, d\mathcal{H}^{n-1} \quad \textit{for every } B \in \mathcal{B}(\Omega).$$

*The measure Cu is a bounded Radon measure on* $\Omega$ *with values in* $\mathbf{R}^n$ *such that*

$$(2.2.3) \qquad |Cu|(B) = 0 \qquad \textit{whenever} \quad \mathcal{H}^{n-1}(B) < +\infty.$$

*Moreover the following implications hold*

$$u \in W^{1,1}(\Omega) \Leftrightarrow \mathcal{H}^{n-1}(S_u) = 0, Cu = 0;$$

$$u \in W^{1,p}(\Omega) \Rightarrow \mathcal{H}^{n-p}(S_u) = 0, Cu = 0.$$

For the proof of (2.2.2) see for instance [27], 4.4; for (2.2.3) see [2], Section 3; for the last two implications see [15], [29].

The measure Cu is "diffuse" and it may have support on sets which have Hausdorff dimension between (n-1) and n. Recalling the well-known Cantor-Vitali function, E. De Giorgi and L. Ambrosio [12] called Cu the Cantor part of the derivative Du; for this function, in fact, we have $\nabla u = 0$ almost everywhere, $S_u = \varnothing$, and $Du = Cu$.

Let us consider the functional (1.3); since it controls only the n-dimensional and the (n-1)-dimensional part of the derivative it does not seem natural to consider as admissible solutions functions like Cantor-Vitali's. Therefore we need to construct a subspace of $BV(\Omega)$ in order to avoid pathological functions and to gain compactness of the sublevels of the functional.

**Definition 2.2.2.** (see [2],[12] ) We say that a function $u \in BV(\Omega)$ is a *special function of bounded variation* if $Cu \equiv 0$.

The space of all special functions of bounded variation in $\Omega$ is denoted by $SBV(\Omega)$.

For the properties of a function $u \in SBV(\Omega)$ we refer to [2], [3], [4], [22].

Of course we have the inclusions $W^{1,1}(\Omega) \subset SBV(\Omega) \subset BV(\Omega)$, and u in $SBV(\Omega)$ belongs to $W^{1,1}(\Omega)$ if and only if $\mathcal{H}^{n-1}(S_u) = 0$.

Endowed with the BV norm, $SBV(\Omega)$ is a Banach space. Clearly a function $u \in SBV(\Omega)$ satisfies all the properties of a bounded variation function.

## 2.3 The spaces $GBV(\Omega)$, $GSBV(\Omega)$ of the generalized functions of bounded variation

In many cases the compactness of the sublevels of functionals like (1.3) may fail (see for instance [4], Example 5.3) and an enlargement of $SBV(\Omega)$ is needed.

**Definition 2.3.1.** Let $u : \Omega \to R^k$ be a borel function. We say that u is a *generalized function of bounded variation* in $\Omega$ if

$$\Phi(u) \in BV_{loc}(\Omega) \qquad \text{for any } \Phi \in C^1(\Omega,R^k) \text{ with } supp(\nabla\Phi) \subset\subset R^k.$$

We denote by $GBV(\Omega,R^k)$ such class of functions. The class of functions $GSBV(\Omega,R^k)$ is defined similarly, by requiring $\Phi(u) \in SBV_{loc}(\Omega)$.

For the properties of these spaces we refer to [2], [3], [12], [22].

In the case $k = 1$ it can be easily seen that

$$u \in GBV(\Omega) \iff (u \wedge N) \vee (-N) \in BV_{loc}(\Omega) \quad \text{for any } N \in \mathbf{N}.$$

A similar equivalence is true in $GSBV(\Omega)$. In the case $k > 1$, the space of test functions $\Phi$ can be taken equal to $C_0^1(\Omega, \mathbf{R}^k)$.

If $u \in L^\infty(\Omega, \mathbf{R}^k)$, then

$$(2.3.1) \qquad\qquad u \in BV_{loc}(\Omega, \mathbf{R}^k) \iff u \in GBV(\Omega, \mathbf{R}^k)$$

and the corresponding equivalence holds for $GSBV(\Omega, \mathbf{R}^k)$.

The generalized functions of bounded variation inherit many properties of the ordinary functions of bounded variation. In particular, given a function $u \in GBV(\Omega, \mathbf{R}^k)$, we can define the singular set $S_u$ as above ( see Definition 2.1.3 ), and $S_u$ is countably $(n-1)$-rectifiable in the same sense of (2.1.2). In addition, for $\mathcal{H}^{n-1}$ a.e. $x \in S_u$ there exists a unit vector $\nu_u \in \mathbf{R}^n$ so that there exist the approximate limits with respect to each of the half-spaces separated by the hyperplane orthogonal to $\nu_u$.

If $u \in BV(\Omega)$ then (2.1.3) implies that $u^-(x)$, $u^+(x) \in \mathbf{R}$ for $\mathcal{H}^{n-1}$ a.e. $x \in S_u$. On the contrary, for a function $u \in GBV(\Omega)$ it may happen that

$$\mathcal{H}^{n-1}(\{\, x \in S_u : u^+(x) = \infty \text{ or } u^-(x) = \infty \,\}) > 0.$$

For a function $u \in GBV(\Omega, \mathbf{R}^k)$ Proposition 2.1.9 continues to hold ( see [4], Proposition 1.4 ). Nevertheless the approximate differential $\nabla u$ may not belong to $L^1(\Omega, \mathbf{R}^{nk})$ ( consider for instance $n = 1$, $k = 1$, $\Omega = (-\frac{\pi}{2}, \frac{\pi}{2})$, and $u(x) = \operatorname{tg} x$ ).

# CHAPTER 3

## EXISTENCE RESULTS FOR SOME FUNCTIONALS
## WITH FIRST ORDER DERIVATIVES

In this chapter, which is principally based on [2], [4], [10], [13], we describe the existence and regularity results proved in literature in order to study functional (1.2). In Section 3.1 we explain how the existence of a minimizer had been proved. The method, proposed by E. De Giorgi, is a typical application of the classical direct method of the calculus of variations. Finally, Section 3.2 is devoted to an application of the results of the previous one. We prove the existence, in a generalized sense, of a minimizer of functional (1.4), whose leading term contains both the first and the second order derivatives of the unknown function.

## 3.1. An existence result for the Mumford-Shah minimization problem

Minimization problems for functionals like (1.2) are typical examples of a larger class of variational problems, called free discontinuity problems (see [11]), arising from mathematical physics. Such problems occur for instance in the mathematical theory of liquid cristals (see [14], [26]) and in a lot of interesting situations where the functional to be minimized is the sum of a surface energy and a volume energy.

For problems of this kind E. De Giorgi and his school have proposed a unified approach (see [2], [3], [10], [12], [13]) based on the use of the space $GSBV(\Omega)$ of the generalized special functions of bounded variation (see Chapter 2, Section 3).

The general method consists in the following steps:

- weak formulation of the minimization problem in the space GSBV($\Omega$);

- proof of the existence of a minimum point in GSBV($\Omega$);

- study of the regularity properties of the solutions, such as the smoothness of the discontinuity set $S_u$ and the differentiability of the solution u on its approximate continuity set $\Omega \setminus S_u$.

Let $\Omega$ be a bounded open subset of $\mathbf{R}^n$. Given a function $g \in L^2(\Omega)$, let us consider the following functional, defined for every closed subset K of $\bar{\Omega}$ and for every function $u \in C^1(\Omega \setminus K)$ as

$$(3.1.1) \qquad E(u, K) = \int_{\Omega \setminus K} |\nabla u|^2 \, dx + \int_{\Omega \setminus K} |u - g|^2 \, dx + \gamma \, \mathcal{H}^{n-1}(K),$$

where $\nabla u$ is the gradient of u, $\mathcal{H}^{n-1}$ denotes the (n-1)-dimensional Hausdorff measure, and $\gamma$ is a positive real number.

As pointed out in the introduction, E. De Giorgi suggested the following weak formulation in GSBV($\Omega$)

$$\min \{ E(u) : u \in \text{GSBV}(\Omega) \}, \quad \text{where}$$

$$(3.1.2) \qquad E(u) = \int_{\Omega} |\nabla u|^2 \, dx + \int_{\Omega} |u - g|^2 \, dx + \alpha \, \mathcal{H}^{n-1}(S_u).$$

The existence of a minimizer of functional (3.1.2) is proved by relying on a general compactness and semicontinuity theorem of L. Ambrosio (see [2] and [4]), adapted below to our problem.

**Theorem 3.1.1.** *Let $\gamma$ be a positive real number and $k \geq 1$ be an integer. Given a function $g \in L^2(\Omega, \mathbf{R}^k)$, let $(u_h)_{h \in \mathbf{N}}$ be a sequence of functions in the space GSBV($\Omega, \mathbf{R}^k$) such that*

$$(3.1.3) \qquad \int_{\Omega} |\nabla u_h|^2 \, dx + \int_{\Omega} |u_h - g|^2 \, dx + \gamma \, \mathcal{H}^{n-1}(S_{u_h}) \leq C$$

*for some finite constant $C > 0$. Then there exists a subsequence $(u_{h_j})_{j \in \mathbf{N}}$ converging*

19

*almost everywhere in* $\Omega$ *to a function* $u \in GSBV(\Omega, R^k)$. *In addition the approximate differentials* $\nabla u_{h_j}$ *converge to* $\nabla u$ *weakly in* $L^1(\Omega, R^{nk})$ *and*

$$\mathcal{H}^{n-1}(S_u) \leq \lim_{j \to \infty} \inf \ \mathcal{H}^{n-1}(S_{u_{h_j}}) .$$

The proof of this theorem ( see [2], Theorem 2.1, and [4], Theorem 2.2 and 3.5 ) is obtained by extending the one dimensional result, which is easy to prove, to the general case of the space dimension $n \geq 1$, using the slicing properties of functions of bounded variation ( see [2], Theorem 3.3 ).

**Remark 3.1.2.** Hypothesis (3.1.3) implies that the sequence $(u_h)$ is bounded in $L^2(\Omega, R^k)$. In fact, for every $h \in N$ we have

$$\int_\Omega |u_h|^2 \, dx \ \leq \ 2 \ ( \int_\Omega |u_h - g|^2 \, dx \ + \ \int_\Omega |g|^2 \, dx ) \ \leq \ 2 \ ( C + \int_\Omega |g|^2 \, dx ) = M.$$

**Remark 3.1.3.** Indeed, the subsequence $(u_{h_j})$ found in Theorem 3.1.1 converges strongly in $L^p(\Omega, R^k)$ for every $1 \leq p < 2$. This follows from the fact that $(u_{h_j})$ is at the same time bounded in $L^2(\Omega, R^k)$ (see Remark 3.1.2) and convergent to u almost everywhere in $\Omega$. In fact, of these two conditions the former implies that the sequence $(|u_{h_j}|^p)$ is equi-integrable on $\Omega$, hence we conclude by the dominated convergence theorem. Nevertheless, in general we do not have strong convergence in $L^2(\Omega, R^k)$, because the sequence $(|u_{h_j}|^2)$ is not equi-integrable (in the case $n = 1$, $k = 1$, and $\Omega = (0,1)$ consider for instance the sequence $(u_h)$ defined by $u_h(t) = \sqrt{h}$ if $t \in (0, \frac{1}{h})$, $u_h(t) = 0$ if $t \in [\frac{1}{h}, 1)$, which satisfies (3.1.3) for any g).

**Remark 3.1.4.** Let us prove that, corresponding to the subsequence $(u_{h_j})$ found in Theorem 3.1.1, the approximate differentials $\nabla u_{h_j}$ converge to $\nabla u$ weakly in $L^2(\Omega, R^{nk})$. By hypothesis (3.1.3) the sequence $(\nabla u_{h_j})$ is bounded in $L^2(\Omega, R^{nk})$. Therefore, by the

weak compactness of a bounded sequence in a reflexive Banach space, from every subsequence of ($\nabla u_{h_j}$) we can extract a further subsequence convergent weakly in $L^2(\Omega, \mathbb{R}^{nk})$ to some function f. By the uniqueness of the weak limit $f = \nabla u$, hence the whole subsequence ($\nabla u_{h_j}$) converges to $\nabla u$ weakly in $L^2(\Omega, \mathbb{R}^{nk})$, by the Urysohn property.

By applying Theorem 3.1.1 (for $k = 1$) to a minimizing sequence for the functional (3.1.2) we prove the existence of at least one minimizer in the space GSBV($\Omega$).

**Corollary 3.1.5.** *Given a function* $g \in L^2(\Omega)$ *and a positive real number* $\gamma$, *let us consider the functional* E *defined by* (3.1.2) *on* GSBV($\Omega$). *Then the minimization problem*

(3.1.6)
$$\min \{ E(u) : u \in GSBV(\Omega) \}$$

*has a solution* .

Proof — Let ($u_h$) be a minimizing sequence for the problem (3.1.6). The Compactness Theorem 3.1.1 provides a subsequence ($u_{h_j}$) converging a.e. in $\Omega$ to some function $u \in$ GSBV($\Omega$). Moreover, from the Fatou's Lemma and the weak lower semicontinuity of the $L^2$ norm, it follows that (see Remark 3.1.4)

$$\int_\Omega |u - g|^2\, dx \ \leq\ \lim_{j \to \infty} \inf \int_\Omega |u_{h_j} - g|^2\, dx \quad \text{and} \quad \int_\Omega |\nabla u|^2\, dx \ \leq\ \lim_{j \to \infty} \inf \int_\Omega |\nabla u_{h_j}|^2\, dx \ .$$

Therefore the functional (3.1.2) is lower semicontinuous along the sequences of functions converging a.e. in $\Omega$ on which it is bounded. In particular functional (3.1.2) is lower semicontinuous along the subsequence ($u_{h_j}$), that is

$$E(u) \ \leq\ \lim_j E(u_{h_j}) \ = \ \inf \{ E(u) : u \in GSBV(\Omega) \}.$$

Hence u is a minimizer of the functional E and this concludes the proof.            ◆

**Remark 3.1.6.** If the function $g \in L^\infty(\Omega)$, then it is easy to prove ( see [3], Section 3 ) that the solution u of the minimization problem (3.1.6) belongs to SBV($\Omega$) $\cap$ $L^\infty(\Omega)$ and

$\| u \|_\infty \leq \| g \|_\infty$ ( the special functions of bounded variation are discussed in Chapter 2, Section 2 ).

In [13] E. De Giorgi, M. Carriero, and A. Leaci proved that functional (3.1.1) attains its minimum for every choice of the function $g \in L^\infty(\Omega)$. Moreover the minimum values of (3.1.1) and (3.1.2) are equal and are achieved at (essentially) the same minimum points, in the following sense (see Remark 3.1.6 ):

(a) if $u \in SBV(\Omega)$ is a minimum point of (3.1.2), then $(\tilde{u}, S_{\tilde{u}})$, where $\tilde{u}$ is the approximately continuous representative of u, is a minimum point of (3.1.1) and $\mathcal{H}^{n-1}(\overline{S}_u \setminus S_u) = 0$ ;

(b) if (u,K) is a minimum point of (3.1.1) then u (arbitrarily extended to $K \cap \Omega$ ) is a minimum point of (3.1.2) on $SBV(\Omega)$; moreover $\overline{S}_u \subseteq K$ and $\mathcal{H}^{n-1}(K \setminus S_u) = 0$ .

The proof of these results relies on a Poincaré - Wirtinger inequality for functions in $SBV(\Omega)$ and on regularization tecniques developed for the study of minimal oriented boundaries.

Limited to the dimension two, another proof of the same result is due to G. Dal Maso, J.M. Morel, and S. Solimini [10] and is based on completely different ideas and techniques, such as some interesting estimates on the singular set of a function $u \in SBV(\Omega)$.

Without assuming the function g to be essentially bounded, A. Leaci recently proved, in the two dimensional case, that the minimization problem corresponding to the functional (3.1.1) has a solution if $g \in L^r(\Omega)$ with r > 4. In the general case of the space dimension $n \geq 1$, there is a conjecture, due to E. De Giorgi, which says that we have the existence of a minimizer of functional (3.1.1) if $g \in L^r(\Omega)$ with r > 2n.

## 3.2. Some functionals with first and second order derivatives

In the introduction we have proposed the following functional, including at the same time the first and second order derivatives, defined for every pair of disjoint closed subsets $K_0$, $K_2$ of $\Omega$ and for every function $u \in C^0(\Omega \setminus K_0) \cap C^2(\Omega \setminus (K_0 \cup K_2))$, as

$$G(u, K_0, K_2) = \int_\Omega |\overset{2}{\nabla}u|^2 \, dx + \int_\Omega |\nabla u|^2 \, dx + \int_\Omega |u - g|^2 \, dx +$$

$$+ \alpha \, \mathcal{H}^{n-1}(K_0) + \beta \, \mathcal{H}^{n-1}(K_2),$$

where $\overset{2}{\nabla}u$ denotes the Hessian matrix of $u$.

As a weak formulation in a generalized sense, we suggest the following functional $G$, well defined on the space $X = \{ u \in GSBV(\Omega) : \nabla u \in GSBV(\Omega, \mathbf{R}^n) \}$, as

(3.2.1) $$G(u) = \int_\Omega |\overset{2}{\nabla}u|^2 \, dx + \int_\Omega |\nabla u|^2 \, dx + \int_\Omega |u - g|^2 \, dx +$$

$$+ \alpha \, \mathcal{H}^{n-1}(S_u) + \beta \, \mathcal{H}^{n-1}(S_{\nabla u} \setminus S_u),$$

where $\nabla u = (\nabla_1 u, .., \nabla_n u)$ is the approximate differential of $u$, $\overset{2}{\nabla}u$ in $\mathbf{R}^{n^2}$ is the matrix of the second approximate differentials ( $(\overset{2}{\nabla}u)_{ij}$ is the absolutely continuous part of $D_j(\nabla_i u)$ ), $S_u$ is the singular set of $u$, and $S_{\nabla u} = \bigcup_{i=1}^{n} S_{\nabla_i u}$ is the singular set of $\nabla u$ ( see Chapter 2 for the precise definitions ).

In this section we prove an existence theorem for the minimization problem corresponding to functional (3.2.1) on the space $X$ by applying Theorem 3.1.1 successively to a minimizing sequence ($u_h$) and to the vector valued sequence ($u_h, \nabla u_h$).

To do this, first me must rewrite functional (3.2.1) in order to obtain a more tractable term containing $S_{\nabla u_h}$.

Let us observe that $S_u \cup S_{\nabla u} = (S_{\nabla u} \setminus S_u) \cup S_u$ with $(S_{\nabla u} \setminus S_u) \cap S_u = \varnothing$; hence

$$\mathcal{H}^{n-1}(S_u \cup S_{\nabla u}) = \mathcal{H}^{n-1}(S_{\nabla u} \setminus S_u) + \mathcal{H}^{n-1}(S_u).$$

Therefore

$$\alpha \, \mathcal{H}^{n-1}(S_u) + \beta \, \mathcal{H}^{n-1}(S_{\nabla u} \setminus S_u) = \alpha \, \mathcal{H}^{n-1}(S_u) + \beta \, \mathcal{H}^{n-1}(S_u \cup S_{\nabla u}) - \beta \, \mathcal{H}^{n-1}(S_u) =$$

$$= (\alpha - \beta)\, \mathcal{H}^{n-1}(S_u) + \beta\, \mathcal{H}^{n-1}(S_u \cup S_{\nabla u}) = a\, \mathcal{H}^{n-1}(S_u) + b\, \mathcal{H}^{n-1}(S_u \cup S_{\nabla u})$$

where we set $a = \alpha - \beta$ and $b = \beta$.

Functional (3.2.1) can be rewrite as

$$(3.2.2) \qquad G(u) = \int_\Omega |\nabla^2 u|^2\, dx + \int_\Omega |\nabla u|^2\, dx + \int_\Omega |u - g|^2\, dx +$$

$$+ a\, \mathcal{H}^{n-1}(S_u) + b\, \mathcal{H}^{n-1}(S_u \cup S_{\nabla u}),$$

We can expect to have a solution of the minimization problem associated with functional (3.2.2) only if $a \geq 0$ and $b > 0$; in fact if $b = 0$ we get $\inf \{\, G(u) : u \in GSBV(\Omega)\, \} = 0$ and in general we have no solutions.

Now we are able to prove the following theorem.

**Theorem 3.2.1.** *Let* a,b *two real numbers such that* $a \geq 0$, $b > 0$. *Let* $(u_h)$ *be a sequence of functions in the space* X *such that*

$$(3.2.3) \qquad \int_\Omega |\nabla^2 u_h|^2\, dx + \int_\Omega |\nabla u_h|^2\, dx + \int_\Omega |u_h - g|^2\, dx +$$

$$+ a\, \mathcal{H}^{n-1}(S_{u_h}) + b\, \mathcal{H}^{n-1}(S_{u_h} \cup S_{\nabla u_h}) \leq C,$$

*for some finite constant* $C > 0$. *Then there exists a subsequence* $(u_{h_j})$ *such that*

(i) $(u_{h_j}, \nabla u_{h_j})$ *converges strongly in* $L^1(\Omega, \mathbb{R}^{n+1})$ *to* $(u, \nabla u)$ *with* $u \in X \cap L^2(\Omega)$,

   $\nabla u \in L^2(\Omega, \mathbb{R}^n)$,

(ii) $(\nabla^2 u_{h_j})$ *converges weakly in* $L^2(\Omega, \mathbb{R}^{n^2})$ *to* $\nabla^2 u$,

(iii) $\mathcal{H}^{n-1}(S_u) \leq \displaystyle\liminf_{h \to \infty} \mathcal{H}^{n-1}(S_{u_h})$,

(iv) $\mathcal{H}^{n-1}(S_u \cup S_{\nabla u}) \leq \displaystyle\liminf_{j \to \infty} \mathcal{H}^{n-1}(S_{u_{h_j}} \cup S_{\nabla u_{h_j}})$.

Proof. By hypothesis (3.2.3) we obtain in particular that

$$\int_\Omega |\nabla u_h|^2\, dx + \int_\Omega |u_h - g|^2\, dx + a\, \mathcal{H}^{n-1}(S_{u_h}) \leq C.$$

Therefore we can apply Theorem 3.1.1 (for $k = 1$) to the sequence $(u_h)$ of functions in

24

the space GSBV($\Omega$) in order to extract a subsequence, which we continue to label with ($u_h$),

such that ( see Remark 3.1.3 and Remark 3.1.4 )

(3.2.4)   ($u_h$) converges strongly in $L^1(\Omega)$ to some function $u \in$ GSBV($\Omega$) $\cap L^2(\Omega)$,

$\quad\quad\quad$ ($\nabla u_h$) converges to $\nabla u$ weakly in $L^2(\Omega,\mathbf{R}^n)$,

$$\mathcal{H}^{n-1}(S_u) \leq \lim_{h \to \infty} \inf \; \mathcal{H}^{n-1}(S_{u_h}).$$

As a second step, let us consider the vector valued sequence ($z_h$) of functions in the space GSBV($\Omega,\mathbf{R}^{n+1}$), defined by $z_h = (u_h, \nabla u_h)$. By the definition of the singular set of a vector valued sequence ( see (2.1.6) ) we get $S_{z_h} = S_{u_h} \cup S_{\nabla u_h}$.

Again by hypothesis (3.2.3) we have that

$$\int_\Omega |\nabla \overset{2}{u}_h|^2 \, dx + \int_\Omega |\nabla u_h|^2 \, dx + \int_\Omega |u_h - g|^2 \, dx + b \; \mathcal{H}^{n-1}(S_{u_h} \cup S_{\nabla u_h}) \leq C,$$

which implies (see Remark 3.1.2) that

$$\int_\Omega (|\nabla \overset{2}{u}_h|^2 + |\nabla u_h|^2) \, dx + \int_\Omega |\nabla u_h|^2 \, dx + \int_\Omega |u_h|^2 \, dx + \mathcal{H}^{n-1}(S_{u_h} \cup S_{\nabla u_h}) \leq K,$$

for some finite constant $K > 0$. This inequality can be rewrite in term of $z_h$ as

$$\int_\Omega |\nabla z_h|^2 \, dx + \int_\Omega |z_h|^2 \, dx + \mathcal{H}^{n-1}(S_{z_h}) \leq K.$$

Therefore, we are in a position to apply Theorem 3.1.1 (with $k = n + 1$ and $g \equiv 0$) to the sequence ($z_h$) in the space GSBV($\Omega,\mathbf{R}^{n+1}$). From this theorem there exists a subsequence of ($z_h$), which we still denote by ($z_h$), converging strongly in $L^1(\Omega,\mathbf{R}^{n+1})$ to some function $z \in$ GSBV($\Omega,\mathbf{R}^{n+1}$) ( see Remark 3.1.3 ). Recalling (3.2.4), this means that ($u_h, \nabla u_h$) converges strongly in $L^1(\Omega,\mathbf{R}^{n+1})$ to ($u, \nabla u$). In addition

$\quad\quad\quad$ ($\nabla z_h$) converges to $\nabla z$ weakly in $L^2(\Omega,\mathbf{R}^{n(n+1)})$,

which implies that ($\overset{2}{\nabla}u_h$) converges to $\overset{2}{\nabla}u$ weakly in $L^2(\Omega,\mathbf{R}^{n^2})$. Finally, we get also

$$\mathcal{H}^{n-1}(S_z) \leq \lim_{h \to \infty} \inf \; \mathcal{H}^{n-1}(S_{z_h}), \text{ that is}$$

$$\mathcal{H}^{n-1}(S_u \cup S_{\nabla u}) \leq \lim_{h \to \infty} \inf \; \mathcal{H}^{n-1}(S_{u_h} \cup S_{\nabla u_h}),$$

which concludes the proof. $\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad$ ◆

To sum up, using the same arguments as in the Proof of Corollary 3.1.5, we obtain that

(a) functional (3.2.2) has a minimizer if $a \geq 0$ and $b > 0$;

(b) functional (3.2.1) has a minimum point if $0 < \beta \leq \alpha$.

# CHAPTER 4

## ONE DIMENSIONAL EXISTENCE RESULT

## FOR A NEW FUNCTIONAL

## WITH SECOND ORDER DERIVATIVES

In this chapter we study the minimization problem corresponding to functional (1.8) and we prove an existence theorem by applying the direct method of the calculus of variations.

The existence of a minimum point is obtained by relying on the compactness and the lower semicontinuity of the functional with respect to the $L^1$- convergence. More precisely, in Section 4.2 we prove the compactness of the functional along sequences of functions on which F is bounded, while in Section 4.3 we prove a semicontinuity theorem under some restrictions on the values of the two coefficients $\alpha$ and $\beta$. Finally, in Section 4.4 we obtain the existence result and we present some examples showing that, failing the conditions on $\alpha$ and $\beta$, we cannot assure the existence of a minimizer independently of the choice of the datum g.

All the results of this chapter can be found in [9].

## 4.1. Preliminary results and formulation of the minimization problem

Let $u : (a,b) \rightarrow R$ , $- \infty < a < b < + \infty$ , be a function in the space $L^2(a,b)$. We indicate by u' and u" the first and the second derivatives of u in the sense of distributions.

By $H^2(a,b)$ we denote the Sobolev space of the functions $u \in L^2(a,b)$ such that both u' and u" are representable by a square-integrable function. Let us fix $u \in H^2(a,b)$. By taking

the appropriate representative of u in its equivalence class with respect to equality almost everywhere, both u and u' are absolutely continuous, hence bounded, on the closed interval [a,b] (see for instance [7], VIII.2). In the definition of $H^2(a,b)$ we can require the square-integrability only for the function u and for its second derivative u", by means of the following interpolation inequality.

**Proposition 4.1.1** _ Let $u \in L^2(a,b)$, *and suppose that* $u" \in L^2(a,b)$. *Then* $u \in H^2(a,b)$ *and*

$$\int_a^b |u'(t)|^2 \, dt \ \leq \ R \ ( \int_a^b |u(t)|^2 \, dt \ + \ \int_a^b |u"(t)|^2 \, dt \ ),$$

*with* R *depending only on the length of the interval* (a,b) *in the following way*

$$R(a,b) = 2 \cdot 9^2 \ \max \{ \ (b-a)^2 , (b-a)^{-2} \ \}.$$

Proof _ It is enough to adapt the proof of Lemma 4.10 of [1].

We begin by showing that $u' \in AC([a,b])$. To this purpose, let us define on the interval (a,b) the function $v(x) = \int_a^x u"(t) \, dt$, in such a way to get $v \in AC([a,b])$ and $v'(x) = u"(x)$ for almost every $x \in (a,b)$. Then the distribution ( v − u' ) has the derivative identically zero, which means that $u' = v + c$ for some constant c and gives $u' \in AC([a,b])$. Note that we have also proved that $u \in H^2(a,b)$.

Now, suppose for the moment that a = 0 and b = 1 and fix $0 < \xi < \frac{1}{3}$ and $\frac{2}{3} < \eta < 1$.

Since u is continuous on the closed interval $[\xi,\eta]$ and derivable on the open interval $(\xi,\eta)$, by the Lagrange's theorem there exists $\lambda \in (\xi,\eta)$ such that

$$| u'(\lambda) | = |\frac{u (\eta) - u (\xi)}{\eta - \xi} | \leq 3 | u (\xi) | + 3 | u (\eta) |.$$

Using the absolute continuity of u', it follows that for any $s \in (0,1)$

$$| u'(x) | = | u'(\lambda) + \int_\lambda^x u"(t) \, dt | \leq 3 | u (\xi) | + 3 | u (\eta) | + \int_0^1 | u"(t) | \, dt.$$

28

Integrating the above inequality with respect to $\xi$ over $(0,\frac{1}{3})$ and with respect to $\eta$ over $(\frac{2}{3},1)$, we obtain

$$\frac{1}{9}\mid u'(x)\mid \leq \int_0^{1/3}\mid u(\xi)\mid d\xi + \int_{2/3}^1\mid u(\eta)\mid d\eta +\frac{1}{9}\int_0^1\mid u''(t)\mid dt \leq \int_0^1\mid u(t)\mid dt +\frac{1}{9}\int_0^1\mid u''(t)\mid dt.$$

Therefore, by Holder's inequality, we get

$$\mid u'(x)\mid^2 \leq 2\cdot 9^2 \int_0^1\mid u(t)\mid^2 dt + 2\int_0^1\mid u''(t)\mid^2 dt \qquad \text{for every } x \in (0,1),$$

which allows us to conclude that

$$\int_0^1\mid u'(t)\mid^2 dt \leq 2\cdot 9^2 \left( \int_0^1\mid u(t)\mid^2 dt + \int_0^1\mid u''(t)\mid^2 dt \right).$$

Finally, by a change of variable, we obtain

$$\int_a^b\mid u'(t)\mid^2 dt \leq 2\cdot 9^2 (b-a)^{-2} \int_a^b\mid u(t)\mid^2 dt + 2\cdot 9^2 (b-a)^2 \int_a^b\mid u''(t)\mid^2 dt,$$

which is our thesis, if we set $R(a,b) = 2\cdot 9^2 \max\{ (b-a)^2, (b-a)^{-2} \}$. $\qquad\qquad\blacklozenge$

We point out a well known compactness result concerning sequences of functions in the space $H^2(a,b)$ ( see for instance [7], ch. VIII, Théorème VIII.7 ).

**Proposition 4.1.2** _ _Let_ $(u_h)_{h\in N}$ _be a bounded sequence in_ $H^2(a,b)$. _Then there exist a subsequence_ $(u_{h_k})_{k\in N}$ _and a function_ $u \in H^2(a,b)$, _such that_

(i) $u_{h_k} \to u$ _strongly in_ $H^1(a,b)$, _weakly in_ $H^2(a,b)$, _and uniformly on_ (a,b),

(ii) $u'_{h_k} \to u'$ _uniformly on_ (a,b).

Proof _ By hypothesis, the sequence $(u_h)$ is bounded in $H^1(a,b)$. Therefore, by the compactness of bounded sequences in a reflexive Banach space, we can pass to a subsequence $u_{h_k}$ converging weakly in $H^1(a,b)$ to some function u. Moreover, a weakly convergent sequence in $H^1(a,b)$ converges strongly in $L^2(a,b)$, which implies that, up to a

subsequence, we can suppose that $u_{h_k}$ converges to u almost everywhere. By the same argument, applied to the sequence $u'_{h_k}$ ( also bounded in $H^1$ ), we get a function $v \in H^1(a,b)$ such that, passing if necessary to another subsequence, $u'_{h_k} \to v$ strongly in $L^2(a,b)$, almost everywhere on (a,b) and weakly in $H^1(a,b)$. By the uniqueness of the weak limit, v coincides with u' and this concludes the proof. ♦

By $\mathcal{H}^2(a,b)$ we denote the space of the functions u in $L^2(a,b)$ such that it is possible to find a partition of the interval (a,b) by means of a finite number of points $x^0 = a < x^1 < .. < x^k < b = x^{k+1}$ $(k \in N)$, such that the restriction of u to the subinterval $(x^i, x^{i+1})$ belongs to $H^2(x^i, x^{i+1})$, for every $i = 0, .., k$.

Let us fix $u \in \mathcal{H}^2(a,b)$; by the properties of the functions in the space $H^2$, u is absolutely continuous and bounded on each subinterval $(x^i, x^{i+1})$. Moreover u is continuously differentiable on each $(x^i, x^{i+1})$, in the classical sense that there exists the limit of the difference quotient. The function so constructed will be denoted by $\dot{u}$ and we will refer to it as pointwise derivative. On each subinterval $\dot{u}$ is absolutely continuous; in particular there exist the right and the left limits $\dot{u}(x^i + 0)$ and $\dot{u}(x^i - 0)$ of $\dot{u}$ at every point $x^i$ of the subdivision and they are finite.

By the derivability properties of the absolutely continuous functions, starting from $\dot{u}$ it is also possible to define almost everywhere the second pointwise derivative of u, denoted by $\ddot{u}$, in the same sense explained above. Moreover $\ddot{u}$ coincides with the distributional derivative u" almost everywhere on each subinterval $(x^i, x^{i+1})$, hence $\ddot{u}$ is square integrable on the whole interval (a,b).

Let us fix an index $i \in \{ 1,..,k \}$; for the corresponding point $x^i$ in the subdivision we give the following definitions.

We say that $x^i$ is a jump point of u if $u(x^i + 0) \neq u(x^i - 0)$ and we denote by $S_u$ the set of jump points of u.

We say that $x^i$ is a crease point of u if $x^i$ is a point of continuity of u, but

$\dot{u}(x^i + 0) \neq \dot{u}(x^i - 0)$. Hence the set of crease points of u is $S_{\dot{u}} \setminus S_u$.

Notice that if a point $x^i$ of the subdivision is neither a jump nor a crease point of u, then $u \in H^2(x^{i-1}, x^{i+1})$ and $x^i$ can be removed from the subdivision.

For every open subset A of (a,b) and for every $u \in \mathcal{H}^2(a,b)$, we denote by $J_0(u,A)$ the cardinality of the set $S_u \cap A$ and by $J_1(u,A)$ the cardinality of the set $(S_{\dot{u}} \setminus S_u) \cap A$. In the particular case A = (a,b), we write $J_0(u)$ and $J_1(u)$ for the total number of jump and crease points of u, respectively.

Finally, we are in a position to formulate our minimization problem.

Given a function $g \in L^2(0,1)$ and two nonnegative real numbers $\alpha$ and $\beta$, let us consider the following functional

(4.1.1) $\qquad F(u) = \int_0^1 |\ddot{u}|^2 \, dt \; + \; \int_0^1 |u - g|^2 \, dt \; + \; \alpha \, J_0(u) \; + \; \beta \, J_1(u)$,

which is well defined for $u \in \mathcal{H}^2(0,1)$.

In this chapter we study the minimization problem

(4.1.2) $\qquad \min \; \{ F(u) : u \in \mathcal{H}^2(0,1) \}$

and we prove the existence of at least one minimizer.

We conclude this section mentioning the connection between the space $\mathcal{H}^2(0,1)$ and the space of the special functions of bounded variation on (0,1) ( see Chapter 2, Section 2 ). It is easy to see that $\mathcal{H}^2(0,1)$ is contained in the space $X = \{ u \in SBV(0,1) : \nabla u \in SBV(0,1) \}$, where $\nabla u$ is the approximate differential of u ( see Definition 2.1.8 ); more precisely the first and second pointwise derivatives $\dot{u}$ and $\ddot{u}$ coincide a.e. on (0,1) with the approximate differentials $\nabla u$ and $\nabla^2 u$. In addition the extension of the functional F to the space X is well defined and the minimization problem (4.1.2) is equivalent to

$\qquad \min \; \{ F(u) : u \in SBV(0,1), \nabla u \in SBV(0,1) \}$,

in the sense that the minimum values are the same and they are attained at the same minimum points.

## 4.2. Compactness Theorem

In this section we prove a compactness theorem for sequences of functions in the space $\mathcal{H}^2(0,1)$ along which the functional F, defined in (4.1.1), is bounded. The purpose is to use this result, together with the semicontinuity of F (discussed in the next section), to apply the direct method of the calculus of variations in order to prove the existence of at least one solution of the minimization problem (4.1.2).

**Theorem 4.2.1** (Compactness) — *Let* $(u_h)_{h \in N}$ *be a sequence of functions in the space* $\mathcal{H}^2(0,1)$ *such that*

$$(4.2.1) \qquad \int_0^1 | \ddot{u}_h |^2 \, dt \; + \; \int_0^1 | u_h - g |^2 \, dt \; + \; \alpha \, J_0(u_h) \; + \; \beta \, J_1(u_h) \; \leq \; C$$

*for some finite constant* $C > 0$. *If both the coefficients* $\alpha$ *and* $\beta$ *are strictly positive, then there exists a subsequence* $(u_{h_k})_{k \in N}$ *converging strongly in* $L^1(0,1)$ *to a function* $u \in \mathcal{H}^2(0,1)$. *In addition* $(\dot{u}_{h_k})$ *converges to* $\dot{u}$ *a.e. and* $(\ddot{u}_{h_k})$ *converges to* $\ddot{u}$ *weakly in* $L^2(0,1)$.

**Remark 4.2.2** — Indeed, using the same arguments of Remark 3.1.3, we can prove taht the subsequence $(u_{h_k})$ found in the Compactness Theorem 4.2.1 converges strongly in $L^p(0,1)$ for every $1 \leq p < 2$.

We premise to the proof some examples showing that Theorem 4.2.1 does not hold when one of the coefficients $\alpha$ and $\beta$ is zero. The following example show that we can construct simple sequences of functions satisfying (4.2.1), but not compact in $L^1(0,1)$.

**Example 4.2.3** – Let us fix $\alpha = 0$, $\beta \geq 0$, and a function $g \in L^2(0,1)$. The sequence

$$u_h(t) = \begin{cases} +1, & t \in \left( \dfrac{k}{h}, \dfrac{2k+1}{2h} \right), & k = 0, \ldots, h-1, \\[3mm] -1, & t \in \left( \dfrac{2k+1}{2h}, \dfrac{k+1}{h} \right), & k = 0, \ldots, h-1. \end{cases}$$

is not compact in $L^1(0,1)$, but satisfies (4.2.1): in fact $F(u_h) \leq 2 \left( 2 + \int_0^1 |g|^2 \, dt \right)$ for every h. In the case $\alpha > 0$, $\beta = 0$, we obtain the same conclusions by considering the sequence

$$u_h(t) = \begin{cases} 4ht - (4k+1), & t \in \left( \dfrac{k}{h}, \dfrac{2k+1}{2h} \right), & k = 0, \ldots, h-1, \\[3mm] -4ht + (4k+3), & t \in \left( \dfrac{2k+1}{2h}, \dfrac{k+1}{h} \right), & k = 0, \ldots, h-1. \end{cases}$$

Nevertheless, even if the sequence we are considering is compact in $L^2(0,1)$, we cannot assure that the limit function belongs to the space $\mathcal{H}^2(0,1)$.

**Example 4.2.4** – Let us fix $\alpha = 0$, $\beta \geq 0$, and $g \in L^2(0,1) \setminus \mathcal{H}^2(0,1)$. It is well known that there exists a sequence of step functions $(u_h)$ in the space $\mathcal{H}^2(0,1)$ such that $\| u_h - g \|_{L^2} < \dfrac{1}{h}$ for every h. Since for every $h \in \mathbb{N}$ we get $\ddot{u}_h = 0$, $J_1(u_h) = 0$, and $\int_0^1 |u_h - g|^2 \, dt \leq 1$, the sequence $(u_h)$ satisfies the hypotheses of Theorem 4.2.1 (with $\alpha = 0$ and $\beta \geq 0$), but every subsequence converges strongly in $L^2(0,1)$ to the function g which does not belong to $\mathcal{H}^2(0,1)$.

**Example 4.2.5** – Let us consider $\alpha > 0$, $\beta = 0$, and a function g as in the previous example. By the density of the piecewise affine functions in $L^2(0,1)$ we can construct a sequence $(u_h)$ in the space $\mathcal{H}^2(0,1)$, converging to g strongly in $L^2(0,1)$, such that for every $h \in \mathbb{N}$ we have $\ddot{u}_h = 0$, $J_0(u_h) = 0$, and $\int_0^1 |u_h - g|^2 \, dt \leq 1$. Therefore the

33

sequence $(u_h)$ satisfies the hypotheses of Theorem 4.2.1 (with $\alpha > 0$ and $\beta = 0$), but every subsequence of $(u_h)$ converges strongly in $L^2(0,1)$ to the function $g$ which does not belong to $\mathcal{H}^2(0,1)$.

Proof of Theorem 4.2.1 _ By Remark 3.1.2 the sequence $(u_h)$ is bounded in $L^2(0,1)$. More precisely for every $h \in N$ we have

(4.2.2) $\quad \int_0^1 |u_h|^2\, dt \;\leq\; 2\,(C + \int_0^1 |g|^2\, dt\,) = M.$

By hypothesis (4.2.1), we deduce also that $J_0(u_h) + J_1(u_h) \leq \dfrac{C}{\alpha} + \dfrac{C}{\beta}$, hence for every $h \in N$ there exists a finite number of points $0 < x_h^1 < .. < x_h^{m_h} < 1$ such that

$S_{u_h} \cup S_{\dot u_h} = \{\, x_h^1, .. , x_h^{m_h} \,\}$ with $m_h \leq \dfrac{C}{\alpha} + \dfrac{C}{\beta}$. Up to a subsequence we can assume that $m_h = m$ for every $h \in N$. Let us consider now the following $(m + 2)$ sequences

$$(x_h^0),\quad (x_h^1),\quad .. \quad,\quad (x_h^m),\quad (x_h^{m+1}),$$

where we set $x_h^0 = 0$, $x_h^{m+1} = 1$ for every $h$. By the compactness of the interval I, we can find m points $0 \leq x^1 \leq .. \leq x^m \leq 1$ such that, passing, if necessary, to a subsequence with respect to h, $x_h^i \to x^i$ for every $i = 1, .. , m$. Therefore there exist $(\bar m + 2)$ distinct limit points $(\bar m \leq m)$ $x^0 = 0 < x^1 < .. < x^{\bar m} < 1 = x^{\bar m+1}$ and a partition $(I_r)_{r=0,..,\bar m+1}$ of the set $\{0, 1, .. , m+1\}$ such that the following two conditions are satisfied:

(a) $i \in I_r, j \in I_s, r < s \quad\Rightarrow\quad i < j;$

(b) $x_h^i \to x^r \qquad \forall\, i \in I_r,\ \forall\, r = 0, .. , \bar m+1.$

Let us consider now the subsequence of $(u_h)$ corresponding to the increasing sequence of indices determined in the previous construction.

The idea of the proof consists in using the interpolation inequality of Proposition 4.1.1 in order to apply the compactness result of Proposition 4.1.2 on appropriate subintervals of $(0,1)$.

To this purpose, let us fix an index $r \in \{0, 1, .., \bar{m}\}$ and consider the corresponding subinterval $(x^r, x^{r+1})$. Let $\sigma$ be the greatest index in $I_r : (x_h^\sigma)$ is the greatest sequence converging to $x^r$, while $(x_h^{\sigma+1})$ is the least sequence converging to $x^{r+1}$. As $x^r < x^{r+1}$, for every $\varepsilon < (x^{r+1} - x^r)/2$ there exists $h_\varepsilon \in N$ sufficiently large such that $x_h^\sigma < x^r + \varepsilon$ and $x_h^{\sigma+1} > x^{r+1} - \varepsilon$ for every $h \geq h_\varepsilon$, which means that $[x^r + \varepsilon, x^{r+1} - \varepsilon]$ does not contain any jump or crease point of $u_h$ for every $h \geq h_\varepsilon$.

By the definition of $\mathcal{H}^2(0,1)$, we obtain that $u_h \in H^2(x^r + \varepsilon, x^{r+1} - \varepsilon)$ for every $h \geq h_\varepsilon$. Moreover, by the interpolation inequality of Proposition 4.1.1 and the boundedness of the sequence $(u_h)$ in $L^2(0,1)$ (see (4.2.2)), for every $h \geq h_\varepsilon$ we have the estimate

$$\int_{x^r + \varepsilon}^{x^{r+1} - \varepsilon} |u_h'|^2 \, dt \leq R \left( \int_{x^r + \varepsilon}^{x^{r+1} - \varepsilon} |u_h''|^2 \, dt + \int_{x^r + \varepsilon}^{x^{r+1} - \varepsilon} |u_h|^2 \, dt \right) \leq R(C + M),$$

with $R = R(x^r + \varepsilon, x^{r+1} - \varepsilon)$.

This is equivalent to say that $(u_h)$ is bounded in $H^2(x^r + \varepsilon, x^{r+1} - \varepsilon)$.

Finally, we are in a position to apply the compactness result of Proposition 4.1.2 to the sequence $(u_h)$ in order to find a subsequence such that

(i) $u_h \to u$ strongly in $H^1(x^r + \varepsilon, x^{r+1} - \varepsilon)$ and uniformly on $(x^r + \varepsilon, x^{r+1} - \varepsilon)$,

(ii) $\dot{u}_h \to u'$ uniformly on $(x^r + \varepsilon, x^{r+1} - \varepsilon)$,

(iii) $\ddot{u}_h \to u''$ weakly in $L^2(x^r + \varepsilon, x^{r+1} - \varepsilon)$,

for some function $u \in H^2(x^r + \varepsilon, x^{r+1} - \varepsilon)$.

Besides, the weak lower semicontinuity of the $L^2$ norm implies that

$$(4.2.3) \qquad \int_{x^r + \varepsilon}^{x^{r+1} - \varepsilon} |u''|^2 \, dt \leq \liminf_{h \to \infty} \int_{x^r + \varepsilon}^{x^{r+1} - \varepsilon} |\ddot{u}_h|^2 \, dt \ .$$

Since for every integer $j > 2/(x^{r+1} - x^r)$ we can construct a new subsequence of $(u_h)$ satisfying conditions (i), (ii), and (iii) with $\varepsilon = \dfrac{1}{j}$, using a diagonal argument we get a subsequence such that

35

(i)' $u_h \to u$ strongly in $H^1_{loc}( x^r , x^{r+1} )$ and pointwise on $( x^r, x^{r+1} )$,

(ii)' $\dot{u}_h \to \dot{u}$ pointwise on $( x^r, x^{r+1} )$,

(iii)' $\ddot{u}_h \to \ddot{u}$ weakly in $L^2(K)$ for every compact subset K of $( x^r, x^{r+1} )$,

for some function $u \in H^2_{loc}( x^r , x^{r+1} )$.

Since it is easy to see that the subsequence $(u_h)$ so found converges to u strongly in $L^1( x^r, x^{r+1} )$ ( the arguments are the same as in Remark 3.1.3 ), to complete the first part of the proof it remains to prove that $u \in H^2( x^r, x^{r+1} )$.

Since $u_h \to u$ pointwise on $( x^r, x^{r+1} )$ and $\int_{x^r}^{x^{r+1}} |u_h|^2 \, dt \le M$ for every h, from the Fatou's corollary it follows that $u \in L^2( x^r, x^{r+1} )$. Moreover, inequality (4.2.3) implies that

$$\int_{x^r+\varepsilon}^{x^{r+1}-\varepsilon} |u''|^2 \, dt \le \liminf_{h \to \infty} \int_{x^r}^{x^{r+1}} |\ddot{u}_h|^2 \, dt \le C \quad \text{for every } \varepsilon \text{ positive sufficiently small,}$$

which gives $\int_{x^r}^{x^{r+1}} |u''|^2 \, dt \le C$ as $\varepsilon \to 0$. Since $u'' \in L^2(x^r, x^{r+1})$, we conclude that the function u belongs to the space $H^2(x^r, x^{r+1})$ by applying Proposition 4.1.1.

Finally, since the sequence $(\ddot{u}_h)$ is bounded in $L^2(x^r, x^{r+1})$ and condition (iii)' holds, we can pass to a subsequence of $(u_h)$ such that $(\ddot{u}_h)$ converges to $\ddot{u}$ weakly in $L^2( x^r, x^{r+1} )$, using the weak compactness of bounded sequences in a reflexive Banach space. Since we can repeat the whole construction on every subinterval and the number of the subintervals is finite, the proof is complete. ♦

## 4.3. Semicontinuity Theorem

In this section we study the lower semicontinuity of the functional F, defined in (4.1.1), with respect to the $L^1$- convergence. The main difficulty lies in the fact that the last two terms,

which penalise jump and crease points of u, are not lower semicontinuous separately. Nevertheless, the following theorem shows that the sum of the last two terms of the functional is lower semicontinuous, along the sequences on which F is bounded, under some restrictions on the values of the coefficients $\alpha$ and $\beta$.

**Theorem 4.3.1** (Semicontinuity) _ *Let* $(u_h)_{h \in N}$ *be a sequence of functions in the space* $\mathcal{H}^2(0,1)$ *such that*

$$(4.3.1) \qquad \int_0^1 |\ddot{u}_h|^2 \, dt + \int_0^1 |u_h - g|^2 \, dt + \alpha J_0(u_h) + \beta J_1(u_h) \leq C$$

*for some finite constant* $C > 0$. *Assume also that*

    *(i)* $0 < \beta \leq \alpha \leq 2\beta$,

    *(ii)* $(u_h)$ *converges strongly in* $L^1(0,1)$ *to some function* $u \in \mathcal{H}^2(0,1)$.

*Then*

$$(4.3.2) \qquad \alpha J_0(u) + \beta J_1(u) \leq \liminf_{h \to \infty} (\alpha J_0(u_h) + \beta J_1(u_h)).$$

**Remark 4.3.2** _ In the statement of Theorem 4.3.1 it is not necessary to require that $u \in \mathcal{H}^2(0,1)$, since, under assumption (4.3.1), it follows directly from the Compactness Theorem 4.2.1 that the strong limit in $L^1(0,1)$ of a sequence of functions in the space $\mathcal{H}^2(0,1)$ belongs to the same space. Moreover, under the hypotheses of Theorem 4.3.1, we have that the second pointwise derivatives $\ddot{u}_h$ converge weakly in $L^2(0,1)$ to $\ddot{u}$.

In order to explain better the relations on the coefficients $\alpha$ and $\beta$, we present here some examples showing why the semicontinuity inequality (4.3.2) is not satisfied, failing these hypotheses.

**Example 4.3.3** _ Let us fix $\alpha > 2\beta \geq 0$. The idea is to construct a sequence of functions

with two crease points, which become a jump point in the limit function. Since the penalty for a jump point is larger than the one for two crease points, inequality (4.3.2) might not be satisfied. Let us consider the sequence

$$u_h(t) = \begin{cases} 0, & \text{if } t \in (0, \frac{1}{2}), \\ h\,M\,(t - \frac{1}{2}), & \text{if } t \in [\frac{1}{2}, \frac{1}{2} + \frac{1}{h}], \\ M, & \text{if } t \in (\frac{1}{2} + \frac{1}{h}, 1), \end{cases} \qquad (h \geq 3, M > 0)$$

which converges strongly in $L^2(0,1)$ to the function $u \in \mathcal{H}^2(0,1)$, defined by $u(t) = 0$ on $(0, \frac{1}{2}]$, $u(t) = M$ on $(\frac{1}{2}, 1)$. Clearly, for every h we have $J_0(u_h) = 0$ and $J_1(u_h) = 2$, whereas for the limit function u it results $J_0(u) = 1$ and $J_1(u) = 0$. This shows that the sum of the last two terms of the functional F is not semicontinuous along $(u_h)$.

**Example 4.3.4** — Suppose now that the penalty for a jump point is less than the one for a crease point: $0 \leq \alpha < \beta$. In this case, if we consider the sequence

$$u_h(t) = \begin{cases} 0, & \text{if } t \in (0, \frac{1}{2}], \\ M\,(t - \frac{1}{2}) + \frac{1}{h}, & \text{if } t \in (\frac{1}{2}, 1), \end{cases} \qquad (M > 0)$$

we conclude as in the previous example that the semicontinuity inequality (4.3.2) does not hold. In fact $J_0(u_h) = 1$ and $J_1(u_h) = 0$ for every h, while for the limit function u, that is $u(t) = 0$ on $(0, \frac{1}{2}]$ and $u(t) = Mt - \frac{M}{2}$ on $(\frac{1}{2}, 1)$, we have $J_0(u) = 0$ and $J_1(u) = 1$.

**Remark 4.3.5** — Examples 4.3.3 and 4.3.4 show also that $J_0$ and $J_1$ are not lower semicontinuous if they are considered separately.

The following lemma is a first step in the proof of Theorem 4.3.1; in fact it is introduced in order to solve the main difficulty we meet with, that is to prove that a sequence of crease points cannot originate a jump point in the limit function. More generally, Lemma 4.3.6 concerns some properties of the limits of the singular points of the sequence $(u_h)$ converging

to u.

**Lemma 4.3.6** — *Let* $(u_h)_{h \in \mathbb{N}}$ *be a sequence of functions in the space* $\mathcal{H}^2(0,1)$ *and let* (a,b) *be a subinterval of* (0,1). *Let us consider a sequence* $(x_h)$ *of points in* (a,b) *converging to some point* $x \in$ (a,b). *Assume also that*

*(i)* $(\int_0^1 |\ddot{u}_h|^2\, dt + \int_0^1 |u_h|^2\, dt)$ *is bounded by a constant C independent of* h,

*(ii)* $u_h$ *has no jump points in* (a,b) *and no crease points in* (a,b)\{$x_h$},

*(iii)* $(u_h)$ *converges strongly in* $L^1(0,1)$ *to some function* $u \in \mathcal{H}^2(0,1)$.

*Then* $u \in H^1(a,b) \cap H^2((a,b)\setminus\{x\})$, *which means that* u *has no jump points on* (a,b) *and* x *is the unique possible crease point on* (a,b).

*In addition if* $u_h \in H^2(a,b)$ *for* h *sufficiently large, then* $u \in H^2(a,b)$.

<u>Proof</u> — It is easy to see that if $u_h \in H^2(a,b)$ for h large enough, then $u \in H^2(a,b)$. In fact by hypothesis (i) and Proposition 4.1.1 the sequence $(u_h)$ is bounded in $H^2(a,b)$, hence $u \in H^2(a,b)$ by Proposition 4.1.2 and hypothesis (iii).

Let us suppose now that $x_h$ is a crease point of $u_h$ frequently with respect to h. By hypothesis, for every h the functions $u_h$ and $\dot{u}_h$ belong to $AC(a,x_h) \cap AC(x_h,b)$ and every $x_h$ is not a jump point of $u_h$. It is well known that these conditions allow us to say that $\dot{u}_h$ is the first derivative of $u_h$ in the sense of distributions on the whole interval (a,b).

Let us show that this sequence of derivatives is uniformly bounded in $L^2(a,b)$.

As $u_h \in H^2(a,x_h) \cap H^2(x_h,b)$, by Proposition 4.1.1 we get the estimate

$$\int_a^b |\dot{u}_h|^2\, dt \; \leq \; R_h^1 \left( \int_a^{x_h} |\ddot{u}_h|^2\, dt \; + \; \int_a^{x_h} |u_h|^2\, dt \right) + R_h^2 \left( \int_{x_h}^b |\ddot{u}_h|^2\, dt + \int_{x_h}^b |u_h|^2\, dt \right)$$

with $R_h^1$ and $R_h^2$ depending only on the length of $(a, x_h)$ and $(x_h, b)$ respectively. Since $(x_h)$ converges to the point x in the interior of the interval (a,b), $R_h^1$ and $R_h^2$ converge to some finite constants $R^1$ and $R^2$ (see Proposition 4.1.1). By hypothesis (i) our claim is proved.

We conclude that $u \in H^1(a,b)$ as strong limit in $L^1(0,1)$ of a sequence bounded in $H^1(a,b)$.

It remains to prove that $u \in H^2((a,b)\setminus\{x\})$. As $u$ has no jump points on $(a,b)$ and belongs to $\mathcal{H}^2(0,1)$, it is enough to show that $u$ has no crease points on $(a,b)\setminus\{x\}$.

To this purpose let us fix a point $y \in (a,b)\setminus\{x\}$ and an open neighbourhood $U$ of $y$ contained in $(a,b)\setminus\{x\}$. Since $u_h \in H^2(U)$ for $h$ sufficiently large, by arguing as in the first step of the proof we obtain that $u \in H^2(U)$, which proves our claim. ♦


Proof of Theorem 4.3.1 — By the definition of $\mathcal{H}^2(0,1)$, corresponding to $u$ there exists a finite number of points $x^0 = 0 < x^1 < .. < x^k < 1 = x^{k+1}$ such that $u \in H^2(x^i, x^{i+1})$ for every $i = 0, .., k$, and $S_u \cup S_{\dot{u}} = \{ x^1, .., x^k \}$. Let us fix $x^i \in S_u \cup S_{\dot{u}}$ and a neighbourhood $U^i$ of $x^i$: then $( S_{u_h} \cup S_{\dot{u}_h} ) \cap U^i \neq \emptyset$ for $h$ large enough. In fact, if there existed a subsequence $( u_{h_k} )$ such that $u_{h_k} \in H^2(U^i)$ for every $k$, we should obtain that $x^i$ is neither a jump point nor a crease point of $u$ by Lemma 4.3.6.

Now, for every $i = 1, .., k$, let us consider a neighbourhood $U^i$ of $x^i$ so that each $U^i$ is contained in $(0,1)$ and $U^i \cap U^j = \emptyset$ if $i \neq j$.

By the properties of the limit inferior, to prove inequality (4.3.2) it is enough to show that

$$(4.3.3) \quad \alpha\, J_0(u, U^i) + \beta\, J_1(u, U^i) \leq \liminf_{h \to \infty} ( \alpha\, J_0(u_h, U^i) + \beta\, J_1(u_h, U^i) ),$$

for each one of these neighbourhoods ( the definitions of $J_0(u, U^i)$ and $J_1(u, U^i)$ can be found in the first section ).

To this purpose, let us fix $i \in \{ 1, .., k \}$. For the corresponding $U^i$, we distinguish three cases: 1) we have $J_0(u_h, U^i) + J_1(u_h, U^i) > 1$ for $h$ sufficiently large;

2) we have $J_0(u_h, U^i) + J_1(u_h, U^i) = 1$ for $h$ sufficiently large;

3) both the conditions $J_0(u_h, U^i) + J_1(u_h, U^i) > 1$ and $J_0(u_h, U^i) + J_1(u_h, U^i) = 1$ hold frequently with respect to $h$.

In the first case, since $( S_u \cup S_{\dot{u}} ) \cap U^i = \{ x^i \}$ and $\beta \leq \alpha \leq 2\beta$, we have

$$\alpha \, J_0(u, U^i) + \beta \, J_1(u, U^i) \leq \alpha \leq 2\beta \leq \alpha \, J_0(u_h, U^i) + \beta \, J_1(u_h, U^i)$$

for h large enough. This means that inequality (4.3.3) is satisfied independently of the fact that $x^i$ is a jump or a crease point of u.

In the second case, let us denote by $x^i_h$ the unique point in $(S_{u_h} \cup S_{\dot{u}_h}) \cap U^i$.

If $x^i$ is a crease point of u, again semicontinuity inequality (4.3.3) holds because of the relation $\beta \leq \alpha$, independently of the kind of discontinuity which $u_h$ presents in $x^i_h$.

In the case where $x^i$ is a jump point of u, let us observe that $(x^i_h)$ converges to $x^i$.

Otherwise, arguing by contradiction as above, we obtain that $x^i$ is neither a jump nor a crease point of u. In addition $x^i_h$ are jump points of $u_h$ for h sufficiently large. In fact if there existed a subsequence $(u_{h_k})$ such that $x^i_{h_k}$ were a crease point of $u_{h_k}$ for every k, then $x^i$ should not be a jump point of the limit function u by Lemma 4.3.6. Therefore, we can conclude that inequality (4.3.3) is satisfied also in this case.

In the third case it is enough to split the sequence $(u_h)$ into two subsequences satisfying conditions $J_0(u_h, U^i) + J_1(u_h, U^i) > 1$ and $J_0(u_h, U^i) + J_1(u_h, U^i) = 1$ respectively, and to apply the above arguments separately to each subsequence.

As we have shown inequality (4.3.3) in all the cases considered, the proof is complete. ◆

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**Remark 4.3.7** — Under the hypotheses of Theorem 4.3.1, from the weak lower semicontinuity of the $L^2$ norm, it follows that (see Remark 4.3.2)

$$\int_0^1 |u - g|^2 \, dt \leq \liminf_{h \to \infty} \int_0^1 |u_h - g|^2 \, dt \quad \text{and} \quad \int_0^1 |\ddot{u}|^2 \, dt \leq \liminf_{h \to \infty} \int_0^1 |\ddot{u}_h|^2 \, dt.$$

Therefore we observe that, for every choice of the coefficients $\alpha$ and $\beta$ such that $0 \leq \beta \leq \alpha \leq 2\beta$, F is lower semicontinuous along the sequences of functions converging strongly in $L^1(0,1)$ on which F is bounded.

## 4.4. Existence Theorem

In this section we combine the results of the previous two sections in order to prove that the functional F has at least one minimum point on the space $\mathcal{H}^2(0,1)$.

**Theorem 4.4.1** (Existence) $-$ *Given a function* $g \in L^2(0,1)$ *and two nonnegative real numbers* $\alpha$ *and* $\beta$, *let us consider the functional* F *defined by* (4.1.1) *on* $\mathcal{H}^2(0,1)$. *If*

$$(4.4.1) \qquad\qquad 0 < \beta \leq \alpha \leq 2\beta,$$

*then the minimization problem*

$$(4.4.2) \qquad\qquad \min \ \{ \ F(u) : \ u \in \mathcal{H}^2(0,1) \ \}$$

*has a solution* .

Proof $-$ The proof is the same as in Corollary 3.1.5. ♦

**Remark 4.4.2** $-$ Every solution u of the minimization problem (4.4.2) is piecewise smooth and precisely $u \in C^3((0,1)\backslash(S_u \cup S_{\dot u}))$. Indeed, by the definition of the space $\mathcal{H}^2(0,1)$, corresponding to u there exists a finite number of points $x^0 = 0 < x^1 < .. < x^k < 1 = x^{k+1}$ such that $u \in H^2(x^i, x^{i+1})$ for every $i = 0, .., k$, and $S_u \cup S_{\dot u} = \{ x^1, .., x^k \}$. As a consequence of the Euler equation u satisfies $u^{IV} + u = g$ in the sense of distributions on each subinterval $(x^i, x^{i+1})$. As $u \in H^2(x^i, x^{i+1})$ for every i and $g \in L^2(0,1)$, from this equation we deduce that the fourth derivative $u^{IV} \in L^2(x^i, x^{i+1})$ for every i, which implies that $u'''$ is absolutely continuous on each subinterval $(x^i, x^{i+1})$.

When the coefficients $\alpha$ and $\beta$ do not satisfy relations (4.4.1), not only the direct method of the calculus of variations do not apply, but we cannot assure the existence of a solution for the minimization problem (4.4.2) independently of the choice of the datum g. In the

following examples we construct explicitely functions g such that the corresponding functionals have no minimizers.


**Example 4.4.3** _ If $\alpha = 0$ or $\beta = 0$, using the same arguments of Examples 4.2.4 and 4.2.5, for every $g \in L^2(0,1) \backslash \mathcal{H}^2(0,1)$ we get inf $\{ F(v) : v \in \mathcal{H}^2(0,1) \} = 0$, but the infimum is never achieved.

Notice that, being $g \in \mathcal{H}^2(0,1)$, it is clearly the unique minimum point.


**Example 4.4.4** _ Let us suppose $\alpha > 2\beta > 0$ and consider the piecewise constant function $g_M$, depending on a positive parameter M, defined by $g_M(t) = 0$ if $t \in (0, \frac{1}{2}]$, $g_M(t) = M$ if $t \in (\frac{1}{2}, 1)$. Corresponding to the datum $g_M$, the functional defined in (4.1.1) will be denoted by $F_M$, to emphasize the dependence on M. Therefore, for every $v \in \mathcal{H}^2(0,1)$ we

have $\qquad F_M(v) = \int_0^1 |\ddot{v}|^2 dt + \int_0^1 |v - g_M|^2 dt + \alpha J_0(v) + \beta J_1(v)$ .

The idea is to determine M sufficiently large so that the minimization problem (4.4.2) for $F_M$ has no solutions. To this purpose, let us observe that the sequence ($u_h$) of Example 4.3.3 converges to $g_M$ strongly in $L^2(0,1)$ and $F_M(u_h) = 2\beta + \frac{M^2}{3h}$ for every h.

Now, since $F_M(g_M) = \alpha > 2\beta$, it remains to choose M in order to get

(4.4.3)    $F_M(v) > \alpha$ for every function $v \in \mathcal{H}^2(0,1)$ such that $J_0(v) = 0$ and $J_1(v) \leq 1$,

( for the sake of brevity, in the sequel we will denote by $H_{0,1}$ this subset of $\mathcal{H}^2(0,1)$ ).

In fact, if this condition is satisfied, then ($u_h$) is a minimizing sequence for $F_M$, inf $\{ F_M(v) : v \in \mathcal{H}^2(0,1) \} = 2\beta$, but there are no minimizers.

By means of easy calculations, we get

(4.4.4)    inf $\{ F_M(v) : v \in H_{0,1} \} \geq$ inf $\{ \int_0^1 |\ddot{v}|^2 dt + \int_0^1 |v - g_M|^2 dt : v \in H_{0,1} \} =$

$= M^2$ inf $\{ \int_0^1 |\ddot{v}|^2 dt + \int_0^1 |v - g_1|^2 dt : v \in H_{0,1} \}$.

Clearly, if

(4.4.5) $\quad \inf \{ \int_0^1 |\dot{v}|^2 \, dt \; + \; \int_0^1 |v - g_1|^2 \, dt \; : v \in H_{0,1} \} > 0,$

our claim is proved, since inequalities (4.4.4) imply that $\inf \{ F_M(v) : v \in H_{0,1} \}$ tends to $+\infty$ as $M \to +\infty$.

Finally, the infimum in (4.4.5) is a minimum because the functional

$$\overline{F}_1(v) = \int_0^1 |\dot{v}|^2 \, dt \; + \; \int_0^1 |v - g_1|^2 \, dt$$

is compact and lower semicontinuous on $H_{0,1}$ with respect to the $L^1$- convergence ( see Compactness Theorem 4.2.1, Remark 4.3.7, and Lemma 4.3.6). Moreover the minimum value is strictly positive since it is attained at a function different from g ( $g \notin H_{0,1}$ ).

**Example 4.4.5** — Let us fix $0 < \alpha < \beta$. Given a constant $M > 0$, let us define the function $g_M \in \mathcal{H}^2(0,1)$ by $g_M(t) = 0$ if $t \in (0, \frac{1}{2}]$ and $g_M(t) = Mt - \frac{M}{2}$ if $t \in (\frac{1}{2}, 1)$. Again, the functional defined by (4.1.1) associated to $g_M$ will be denoted by $F_M$. The constant M acts as a parameter and plays the same rôle as in the previous example: for M sufficiently large the minimization problem (4.4.2) for $F_M$ has no solutions.

To prove this fact, let us observe that the sequence $(u_h)$ of Example 4.3.4 converges to $g_M$ strongly in $L^2(0,1)$ and $F_M(u_h) = \alpha + \frac{1}{2h^2}$ for every h.

Moreover, it is easier than in the previous example to determine the threshold $M_0$ for the parameter M so that, for every $M > M_0$, $F_M(v) > F_M(g_M) = \beta > \alpha$ on $H^2(0,1)$. Therefore, if $M > M_0$, we can conclude that $(u_h)$ is a minimizing sequence, $\inf \{ F_M(v) : v \in \mathcal{H}^2(0,1) \} = \alpha$, but there are no minimum points.

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