On Uniqueness and Stability for Systems of Conservation Laws

CANDIDATE
Paola Goatin

SUPERVISOR
Prof. Alberto Bressan

Thesis submitted for the degree of *"Doctor Philosophiæ"*.

Academic Year 1999/2000

On Uniqueness and Stability for Systems of Conservation Laws

CANDIDATE
Paola Goatin

SUPERVISOR
Prof. Alberto Bressan

Thesis submitted for the degree of *"Doctor Philosophiæ"*.

Academic Year 1999/2000

Il presente lavoro costituisce la tesi presentata da Paola Goatin, sotto la direzione del Prof. Alberto Bressan, al fine di ottenere il diploma di *"Doctor Philosophiæ"* presso la Scuola Internazionale Superiore di Studi Avanzati, Classe di Matematica, Settore di Analisi Funzionale ed Applicazioni.

Ai sensi del Decreto del Ministro della Pubblica Istruzione n. 419 del 24 Aprile 1987, tale diploma di ricerca post-universitaria è equipollente al titolo di *"Dottore di Ricerca in Matematica"*.

Acknowledgments

I thank my supervisor and teacher Alberto Bressan for his expert guide through the world of conservation laws.

I thank Philippe LeFloch and Fabio Ancona for their precious advices and the fruitful collaborations.

I thank Martino Bardi for introducing me to mathematical research and for the constant encouragement.

I thank all the components of my research group for providing me with a pleasant and stimulating environment and for many helpful discussions.

I thank my colleagues and friends Ugo Boscain e Federico Becca for their support and their patience during these years. *I think this is the beginning of a beautiful friendship.*

It is impossible to list here all the people who contributed directly or indirectly to the accomplishment of this work. To all of them I express my gratitude. Only, I would give a special thank to my mum, my dad and my brother for the constant support that their love has given me. It's a pity that they are not going to read a word of what I have written here.

Trieste, September 5th, 2000.

Contents

Introduction

This thesis provides some new results concerning uniqueness and $\mathbf{L}^1$ stability of solutions to hyperbolic systems of conservation laws in one space dimension

$$u_t + f(u)_x = 0, \qquad x \in \mathbb{R}, \quad t \geq 0. \tag{1}$$

Here $u = u(t,x) \in \mathbb{R}^n$ is the vector of the conserved quantities, and the flux function $f : \Omega \to \mathbb{R}^n$ is a smooth vector field defined on a domain $\Omega \subset \mathbb{R}^n$. We shall assume that system (1) is strictly hyperbolic, i.e. that, for each $u \in \Omega$, the Jacobian matrix $Df(u)$ has n distinct eigenvalues $\lambda_1(u) < \cdots < \lambda_n(u)$, and that each characteristic family is either linearly degenerate or genuinely nonlinear in the sense of Lax. Systems of this type provide models of several nonlinear phenomena (for example in gas dynamics, traffic flow, elastodynamics) when dissipation effects, such as viscosity, are neglected.

It is well-known that, because of the nonlinear dependence of the characteristic speeds $\lambda_k(u)$ on the state variable u, solutions of the Cauchy problem for systems of conservation laws may develop discontinuities in finite time due to gradient catastrophe, no matter of the regularity of the initial data. Therefore, to achieve global existence results, it is essential to work within a class of discontinuous functions, interpreting the system of equations (1) in its distributional sense. Since, in general, weak solutions of the Cauchy problem for (1) are not unique, an entropy criterion for admissibility is usually added to rule out non-physical discontinuities.

The existence of global entropy weak solutions of the Cauchy problem for (1) with small total variation was first established by Glimm in 1965, in the fundamental paper [37], using a probabilistic algorithm. An alternative method for constructing solutions of the Cauchy problem, as limit of a sequence of piecewise constant approximate solutions defined by a front tracking algorithm, was introduced in the papers of Dafermos [27] for scalar equations and Di Perna [32] for 2×2 systems, then extended by Bressan [10] and Risebro [64] to general $n \times n$ systems. By generating more accurate approximations, this method provides additional insight on the behaviour of solutions. In the present work, we will take advantage of the properties of wave-front tracking approximations to estimate how solutions are affected when the initial data are changed, and to derive some regularity conditions on the solutions to (1) that guarantee the uniqueness of entropy admissible weak solutions of the corresponding Cauchy problem.

The well-posedness of the Cauchy problem for (1) has been studied by several authors

[55, 40, 56, 62], starting with the pioneering work of DiPerna [33]. However, apart from the scalar case [46, 69], a gap remained between the existence and the well-posedness theory since, while the global existence had been obtained within a space of **BV** functions, uniqueness and continuous dependence was established in all of these works only for solutions satisfying additional restrictive regularity conditions.

A completely different approach, the *semigroup approach*, introduced by Bressan in [11] and then developed with his collaborators in [15, 17, 18, 20, 22, 7], was the first to provide a general uniqueness theorem within the same class of **BV** functions where existence could be proven (see [14] for a resume on this activity). The basic idea involved in this approach consists in considering the problem of constructing a whole Lipschitz continuous semigroup of solutions to (1), compatible with the standard solutions of the Riemann problems, rather than comparing a single pair of solutions. More precisely, a *Standard Riemann Semigroup* generated by (1) is a map $S : \mathcal{D} \times [0, \infty[\mapsto \mathcal{D}$, defined on a closed domain $\mathcal{D} \subset \mathbf{L}^1$, with the properties:

(i) the domain $\mathcal{D}$ contains all the functions $\bar{u} \in \mathbf{L}^1$ with sufficiently small total variation;

(ii) for every $\bar{u} \in \mathcal{D}$, and all $t, s \geq 0$, one has

$$S_0\bar{u} = \bar{u}, \qquad S_t S_s \bar{u} = S_{t+s}\bar{u}\,; \tag{2}$$

(iii) there exists a Lipschitz constant L such that, for all $\bar{u}, \bar{v} \in \mathcal{D},\ \ t, s \geq 0$, one has

$$\big\| S_t\bar{u} - S_s\bar{v} \big\|_{\mathbf{L}^1} \leq L\big(|t-s| + \|\bar{u} - \bar{v}\|_{\mathbf{L}^1} \big)\,; \tag{3}$$

(iv) if $\bar{u} \in \mathcal{D}$ is piecewise constant, then for $t > 0$ sufficiently small the function $u(t, \cdot) = S_t\bar{u}$ coincides with the solution of (1) obtained by piecing together the standard entropic solutions of the Riemann problems determined by the jumps of $\bar{u}$.

It was shown in [11] that a Lipschitz continuous semigroup of solutions to (1) with the above properties is necessarily unique (up to the domain) and each trajectory $t \mapsto u(t, \cdot) = S_t\bar{u}$ turns out to be a weak, entropy-admissible solution of the corresponding Cauchy problem. Thus, once the existence of such a semigroup is established, a natural method is available to obtain the well-posedness theory. Namely, in order to prove the uniqueness of the solutions to a given Cauchy problem with initial data $\bar{u} = \bar{u}(x) \in \mathcal{D}$, it suffices to show that every entropy weak solution $u = u(t, x)$, that satisfies some mild assumption either on the local boundedness of the total variation or on the oscillation, actually coincides with the corresponding (unique) trajectory of the semigroup:

$$u(t, \cdot) = S_t \bar{u} \qquad \forall\, t \geq 0\,. \tag{4}$$

A regularity condition which implies the identity (4) was introduced by Bressan and LeFloch in [20]. This condition, called *Tame Variation*, controls the total variation of an entropy weak solution u along space-like segment in the t-x plain, and is clearly satisfied by solutions constructed by the Glimm scheme or by a front tracking algorithm.

In Chapter 1 we introduce a weaker regularity assumption, the so-called *Tame Oscillation*, which controls the local oscillation of u in a forward neighborhood of each point in the t-x plane, and is sufficient to guarantee uniqueness of entropy weak solutions. In turn, this result yields the uniqueness of weak solutions which satisfy a decay estimate on positive waves for genuinely nonlinear systems, thus extending a classical result proved by Oleinik [63] in the scalar case. In fact, this decay property implies both the entropy admissibility condition and the Tame Oscillation condition.

An alternative regularity assumption sufficient to guarantee uniqueness has been given by Bressan and Lewicka in [22]. This condition requires that the solution $u = u(t, x)$ has bounded variation along a suitable family of space-like curves. In all cases, the uniqueness is established within the same class of functions where an existence theorem can be proven.

In order to establish the $\mathbf{L}^1$ stability estimate (3) for a semigroup of solutions generated by (1), three different approaches were proposed in the past years.

1. *Comparison of solutions by homotopy and linearization.* To estimate the distance between two solutions u and v, one constructs a one parameter path $\gamma_t : \theta \mapsto u^\theta(t)$ of solutions joining u with v, and then study how the $\mathbf{L}^1$ length of γ_t varies in time. As long as all solutions u^θ remain sufficiently regular, the length of γ_t can be computed integrating a suitable defined weighted norm of a generalized tangent vector. By deriving careful a-priori estimates on the weighted norm of these tangent vectors, one provides a bound on the length of γ_t and hence on the distance $\|u(t) - v(t)\|_{\mathbf{L}^1}$. The $\mathbf{L}^1$ stability of entropy admissible $\mathbf{BV}$ solutions was obtained following this approach by Bressan and Colombo [15], in the case of systems of two equations, and by Bressan, Crasta and Piccoli [17], in the general

case of $n \times n$ systems with genuinely nonlinear or linearly degenerate characteristic fields.

2. *Construction of a Lyapunov type functional.* It is explicitly defined a functional $\Phi = \Phi(u, v)$ which is equivalent to the $\mathbf{L}^1$ distance and decreases along couples of solutions to the hyperbolic system (1). A key role in the stability analysis of this approach is played by a new *entropy functional* for genuinely nonlinear scalar equations introduced by Liu and Yang in [58]. The construction of a robust functional with the above properties for general $n \times n$ systems was established by Bressan, Liu and Yang in [23], and by Liu and Yang in [61].

3. *Haar's method of admissible averaging matrices.* Given any two entropy solutions u, v of (1), one considers an *averaging matrix* $A = A(u, v)$ satisfying the consistency property

$$A(u,v)(v-u) = f(v) - f(u), \qquad u, v \in \mathbb{R}^n. \tag{5}$$

The problem of proving the $\mathbf{L}^1$ stability estimate (3) is then (essentially) equivalent to showing the uniqueness and $\mathbf{L}^1$ stability for the linear hyperbolic system with discontinuous coefficient

$$\psi_t + (A\psi)_x = 0. \tag{6}$$

This approach, closely related to Holmgren's method (see [54] and the references therein), was first pursued by Hu and LeFloch in [41] to study the continuous dependence of solutions for systems of conservation laws.

The $\mathbf{L}^1$-stability results obtained in this thesis are obtained partly by using the omotopy approach (Chapters 2, 3), and partly following the Haar's method (Chapter 4).

In Chapter 2 we consider the problem of extending the construction of the Lipschitz continuous semigroup of solutions developed in [8, 15, 17, 23] to domains of $\mathbf{L}^\infty$ functions with possibly unbounded variation. We recall that, for scalar conservation laws, the uniqueness and continuous dependence of $\mathbf{L}^\infty$ solutions was established by a classical work of Kruzhkov [46]. On the other hand, in the case of $n \times n$ systems, the available stability results apply only to domains $\mathcal{D}$ of functions with uniformly bounded total variation. A counterexample by Bressan and Shen [24] of a 3×3 strictly hyperbolic, quasilinear system shows that, in general, the Cauchy problem with $\mathbf{L}^\infty$ data may not be well posed.

It is thus natural to ask in which cases the construction of a continuous semigroups of solutions can be pursued within domains of $\mathbf{L}^\infty$ functions. In Chapter 2 we show that this can be achieved in the case of Temple class systems, i.e. systems for which rarefaction and shock curves coincide [40, 66], under the assumption that all characteristic fields are genuinely nonlinear. Namely, for a large class of initial data $\bar{u} \in \mathbf{L}^\infty$, we prove the existence of solutions $t \to S_t\bar{u}$ to the corresponding Cauchy problem, obtained as limit of front-tracking approximations, which depend on $\bar{u}$ in a Lipschitz continuous way, w.r.t. the $\mathbf{L}^1$ norm. Using similar techniques to those developed in Chapter 1, we then show that each semigroup trajectory $u(t,\cdot) = S_t\bar{u}$ provides the unique weak solution of the Cauchy problem for (1), with initial data $\bar{u}$, which satisfies an entropy condition of Oleinik type, concerning the decay of positive waves.

The proof of the Lipschitz continuous dependence on the initial data of the trajectories of the semigroup is achieved by the homotopy type approach. As in [15, 8, 7], the basic idea consists in "differentiating" a family of front tracking approximate solutions w.r.t. a parameter which determines the locations of the jumps, and in providing a priori bound on the norm of the resulting "shift differential". The key stability estimate is obtained relying on two remarkable properties of genuinely nonlinear systems of Temple class:

a) By genuine nonlinearity and finite propagation speed, the total amount of waves in a solution $u(t,\cdot)$ which can be influenced by shifting a single wave-front of u at time $t = 0$ remains uniformly bounded for all $t > 0$.

b) For solutions of Temple class systems, the support of perturbations satisfies a special localization property.

In Chapter 3 we study the well-posedness of the initial-boundary value problem

$$\begin{aligned} &u_t + f(u)_x = 0, && t > 0,\ x > 0,\\ &u(0,x) = \bar{u}(x), && \\ &u(t,0) = \tilde{u}(t), && \end{aligned} \tag{7}$$

for the same class of Temple systems considered in Chapter 2. Here, following [34], the boundary condition is intended in the (weak) form

$$u(0+,t) \in \mathcal{V}(\tilde{u}(t)), \qquad t > 0, \tag{8}$$

where $\mathcal{V}(\tilde{u}(t))$ is a time-dependent set (the set of *admissible boundary values*) that is defined from the boundary data using the notion of Riemann problem, while $u(0+,t)$ represents the (weak) trace of u. As in the case of the Cauchy problem, also for the mixed initial-boundary value problem the existence and well-posedness theory has been established for scalar conservation laws within domains of $\mathbf{L}^\infty$ functions [45, 66, 49], while, in the case of $n \times n$ systems, the global existence and stability results available apply only to solutions with small total variation (see [1, 2, 3, 65] and references therein).

In order to extend these results to a class of $\mathbf{L}^\infty$ functions for $n \times n$ genuinely nonlinear Temple systems, we apply in Chapter 3 the same technique developed in Chapter 2 and construct a Lipschitz continuous semigroup of solutions to (7), whose trajectories satisfy a suitable entropy condition of Oleinik type.

Having in mind applications of Temple systems to study problems of oil reservoir simulation, multicomponent chromatography, as well as in models for traffic flows, in the second part of Chapter 3 we focus our attention on (7) from the point of view of control theory. Namely, following the same approach adopted by Ancona and Marson [5, 6] for scalar conservation laws, we fix a set $\mathcal{U} \subset \mathbf{L}^\infty$ of boundary data regarded as admissible controls, and, taking the initial data $\bar{u} \equiv 0$, we consider the set of attainable profiles at a fixed time T

$$\mathcal{A}(T,\mathcal{U}) \doteq \Big\{u(T,\cdot) : \ u \text{ is a solution to (7) with } \bar{u} \equiv 0 \text{ and } \tilde{u} \in \mathcal{U}\Big\},$$

and at a fixed point in space $\bar{x} > 0$

$$\mathcal{A}(\bar{x},\mathcal{U}) \doteq \Big\{u(\cdot,\bar{x}) : \ u \text{ is a solution to (7) with } \bar{u} \equiv 0 \text{ and } \tilde{u} \in \mathcal{U}\Big\}.$$

Motivated by applications to calculus of variations and problems of optimization we establish closure and compactness in the $\mathbf{L}^1$ topology of the attainable sets in connection with a class of $\mathbf{L}^\infty$ boundary controls.

In Chapter 4, following Hu and LeFloch [41], we investigate the $\mathbf{L}^1$ stability issue from the standpoint of Holmgren's and Haar's methods, and apply this technique to the scalar conservation law. Extending the result obtained in [41] for piecewise constant solutions, we establish sharp $\mathbf{L}^1$ continuous dependence estimates for general solutions of bounded variation. More precisely, we recover an estimate of the form

$$\|u(t) - v(t)\|_{w(t)} + \int_s^t M(\tau; u, v)\, d\tau \le \|u(s) - v(s)\|_{w(s)}, \quad 0 \le s \le t, \tag{9}$$

for any two entropy solutions of bounded variation u and v of (1), where $\|.\|_{w(t)}$ is a weighted norm equivalent to the standard $\mathbf{L}^1$ norm on the real line. In (9), the positive term $M(\tau; u, v)$ is determined explicitly evidentiating the contributions of all the discontinuities and the continuous parts of the solutions, and provides a sharp bound on the strict decrease of the $\mathbf{L}^1$ distance. Precisely, the term M turns out to be *cubic* in nature; a stronger *quadratic* decrease is produced where the solutions cross each other. Observe that, with a suitable choice of the definition of the wave strengths, the weighted norm reduces to the Liu-Yang's functional [58]. Our approach provides a new derivation and some generalization of this $\mathbf{L}^1$ functional. Note that the weight is far from being unique and we believe that this flexibility in choosing the weight may be helpful in certain applications.

Two different strategies are pursued. On one hand, we justify passing to the limit in an $\mathbf{L}^1$ estimate valid for piecewise constant wave-front tracking approximations. On the other hand, we use the technique of generalized characteristic and, following closely an approach by Dafermos [29], we derive the sharp $\mathbf{L}^1$ estimate directly from the equation. This approach can be extended to $n \times n$ systems with genuinely nonlinear characteristic fields ([39]).

Chapter 1

Oleinik type estimates and uniqueness for $n \times n$ conservation laws

1.1 Introduction to Chapter 1

Consider a scalar conservation law in one space dimension:

$$u_t + f(u)_x = 0. \tag{1.1}$$

If f is strictly convex, say $f''(u) \geq \kappa > 0$ for every u, a well known estimate of Oleinik [62, 67] states that

$$u(t,y) - u(t,x) \leq \frac{y-x}{\kappa t} \tag{1.2}$$

for all $t > 0$, $x < y$ and every entropy-admissible solution of (1.1). Conversely, if $u = u(t,x)$ is a weak solution satisfying (1.2), then u is entropy-admissible. In particular, given an initial condition

$$u(0,x) = \bar{u}(x), \tag{1.3}$$

the above decay estimate singles out a unique weak solution to the Cauchy problem, continuously depending on the initial data $\bar{u}$ in the $\mathbf{L}^1$ norm.

The aim of the present Chapter is to prove an analogous uniqueness theorem, valid for **BV** solutions of $n \times n$ hyperbolic systems. The following standard conditions [47, 67] will be assumed throughout:

♣ The function f is smooth, defined for u in a neighborhood Ω of the origin. The system (1.1) is strictly hyperbolic. Each characteristic field is either linearly degenerate or genuinely nonlinear.

Under these assumptions, it was proved in [15, 17] that there exists a family of entropy weak solutions to (1.1) continuously depending on the initial data. More precisely, there exists a closed domain $\mathcal{D} \subset \mathbf{L}^1(\mathbb{R};\ \mathbb{R}^n)$, constants $\eta_0, L > 0$, and a continuous semigroup $S : \mathcal{D} \times [0,\infty[\mapsto \mathcal{D}$ with the properties:

(i) Every function $\bar{u} \in \mathbf{L}^1$ with Tot.Var.$(\bar{u}) \leq \eta_0$ lies in $\mathcal{D}$.

(ii) For all $\bar{u}, \bar{v} \in \mathcal{D}$, $t, s \geq 0$ one has $\big\|S_t\bar{u} - S_s\bar{v}\big\|_{\mathbf{L}^1} \leq L\big(|t-s| + \|\bar{u}-\bar{v}\|_{\mathbf{L}^1}\big)$.

(iii) If $\bar{u} \in \mathcal{D}$ is piecewise constant, then for $t > 0$ sufficiently small the function $u(t,\cdot) = S_t\bar{u}$ coincides with the solution of (1.1), (1.3) obtained by piecing together the standard self-similar solutions of the corresponding Riemann problems.

(iv) Each trajectory $t \mapsto u(t,\cdot) = S_t\bar{u}$ is a weak, entropy-admissible solution of the corresponding Cauchy problem (1.1), (1.3).

(v) Every weak solution obtained as limit of Glimm or front tracking approximations coincides with the corresponding trajectory of the semigroup.

An alternative, much shorter proof of this same result was recently given in [23]. The positively invariant domain $\mathcal{D}$ has the form

$$\mathcal{D} = cl\Big\{u \in \mathbf{L}^1(I\!R; I\!R^n);\ \ u \text{ is piecewise constant},\quad V(u) + C_0 \cdot Q(u) < \delta_0\Big\}, \tag{1.4}$$

for some constants $C_0, \delta_0 > 0$. Here $V(u)$ and $Q(u)$ denote the total strength of waves and the wave interaction potential of u, while cl denotes closure in $\mathbf{L}^1$. On a given domain $\mathcal{D}$, the semigroup S with the above properties is unique.

Following [11], we say that a map S with the properties (i)–(iii) is a *Standard Riemann Semigroup* (SRS). See [12] for a general survey. These results provide a new method for proving the uniqueness of the solution u to a given Cauchy problem (1.1), (1.3). Namely, it now suffices to show that u coincides with the corresponding semigroup trajectory:

$$u(t,\cdot) = S_t\bar u \qquad \text{for all } t \geq 0. \tag{1.5}$$

In turn, a convenient way to prove (1.5) is to use the error estimate [13]

$$\big\|u(T) - S_T u(0)\big\|_{\mathbf{L}^1} \leq L \cdot \int_0^T \left\{ \liminf_{h\to 0+} \frac{\big\|u(t+h) - S_h u(t)\big\|_{\mathbf{L}^1}}{h} \right\} dt, \tag{1.6}$$

valid for every Lipschitz continuous map $u : [0,T] \mapsto \mathcal{D}$. By showing that the integrand on the right hand side of (1.6) vanishes for almost every time t, one can thus establish (1.5). This approach was adopted in [20], proving the uniqueness of the entropy weak solution $u = u(t,x)$ which satisfies an additional regularity assumption. This additional condition, called *Tame Variation*, controls the total variation of u along space-like segments in the t-x plane.

In the first part of this Chapter we show that the Tame Variation can be replaced by a weaker assumption, restricting the oscillation of u on a forward neighborhood of each given point. Observing that a weak solution of (1.1) is defined up to a set of measure zero in the t-x plane, for sake of definitness we shall henceforth consider its right continuous version, so that

$$u(t,x) = \lim_{y\to x+} u(t,y) \qquad \text{for all } \ (t,x). \tag{1.7}$$

This is meaningful since each map $x \mapsto u(t,x)$ has bounded variation. By $\lambda_1(u) < \cdots < \lambda_n(u)$ we denote the characteristic speeds, i.e. the eigenvalues of the Jacobian matrix $Df(u)$. Following [47, p. 555], we say that a shock is entropy admissible if the characteristics of the same genuinely nonlinear family impinge on the shock line from both sides.

For clarity, our main assumptions are listed below.

(A1) (Conservation Equations) The function $u = u(t,x)$ is a weak solution of the Cauchy problem (1.1), (1.3), taking values within the domain $\mathcal{D}$ of a Standard Riemann Semigroup S. More precisely, $u : [0,T] \mapsto \mathcal{D}$ is continuous w.r.t. the $\mathbf{L}^1$ distance. The initial condition (1.3) holds, together with

$$\iint \big(u\varphi_t + f(u)\varphi_x\big)\, dxdt = 0 \tag{1.8}$$

for every $\mathcal{C}^1$ function φ with compact support contained inside the open strip $]0,T[\times I\!R$.

(A2) (Entropy Condition) Let u have an approximate jump discontinuity at some point $(\tau,\xi) \in]0,T[\times I\!R$. More precisely, let there exist states $u^-, u^+ \in \Omega$ and a speed $\lambda \in I\!R$ such that, calling

$$U(t,x) \doteq \begin{cases} u^- & \text{if } x < \xi + \lambda(t-\tau), \\ u^+ & \text{if } x > \xi + \lambda(t-\tau), \end{cases} \tag{1.9}$$

there holds

$$\lim_{\rho\to 0+} \frac{1}{\rho^2} \int_{\tau-\rho}^{\tau+\rho} \int_{\xi-\rho}^{\xi+\rho} \Big|u(t,x) - U(t,x)\Big|\, dxdt = 0. \tag{1.10}$$

Then, for some $i \in \{1,\dots,n\}$, one has the entropy inequality:

$$\lambda_i(u^-) \geq \lambda \geq \lambda_i(u^+). \tag{1.11}$$

(A3) (Tame Oscillation Condition) There exists constants K and λ^* such that, at every point (τ,ξ), one has

$$\limsup_{t\to\tau+,\ x\to\xi} \big|u(t,x) - u(\tau,\xi)\big| \leq K \cdot \big|u(\tau,\xi+) - u(\tau,\xi-)\big| \tag{1.12}$$

and

$$\lim_{\substack{t\to\tau+,\ x\to\xi\pm \\ |x-\xi| > \lambda^*(t-\tau)}} u(t,x) = u(\tau,\xi\pm). \tag{1.13}$$

By $u(\tau, \xi\pm)$ we denote here the right and left limits of the map $x \mapsto u(\tau, x)$ at the point ξ. They certainly exist because this map has bounded variation. Observe that the Tame Oscillation consists of local $\mathbf{L}^\infty$ type estimates. These are much weaker than the global **BV** type estimates required by the Tame Variation assumption in [20]. Yet, **(A3)** suffices to guarantee uniqueness of entropy weak solutions:

Theorem 1.1.1 *Let the basic assumptions* **(♣)** *hold, so that the system (1.1) generates a Standard Riemann Semigroup* $S : \mathcal{D} \times [0, \infty[\mapsto \mathcal{D}$. *Then, for every* $\bar{u} \in \mathcal{D}$, $T > 0$, *the Cauchy problem (1.1), (1.3) has a unique weak solution* $u : [0, T] \mapsto \mathcal{D}$ *satisfying the assumptions* **(A1)**–**(A3)**. *Indeed, these conditions imply the identity (1.5).*

In the second part of this Chapter we show that, for genuinely nonlinear $n \times n$ systems, the Oleinik type estimates on the decay of positive waves imply both the Entropy Condition and the Tame Oscillation Condition stated above. This will provide an additional uniqueness theorem for weak solutions of the Cauchy problem (1.1), (1.3).

Decay estimates were proved in [57] for approximate solutions constructed by the Glimm scheme, and in [16] for exact solutions obtained as limit of front tracking approximations. A careful statement of these results requires some notations.

Let $A(u) = Df(u)$ be the Jacobian matrix of f at u. Smooth solutions of (1.1) thus satisfy the equivalent quasilinear system

$$u_t + A(u)u_x = 0\,. \tag{1.14}$$

Call $\lambda_1(u) < \cdots < \lambda_n(u)$ the eigenvalues of $A(u)$. Moreover, choose right and left eigenvectors $r_i(u)$, $l_i(u)$, $i = 1, \ldots, n$, normalized so that

$$|r_i| \equiv 1\,, \qquad \langle l_i,\ r_j\rangle = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j \end{cases} \tag{1.15}$$

for every $i \in \{1, \ldots, n\}$ and all u in the domain of f. The assumption on the genuine nonlinearity of the i-th characteristic family can be written as

$$\nabla\lambda_i \cdot r_i(u) \doteq \lim_{h\to 0} \frac{\lambda_i\big(u + hr_i(u)\big) - \lambda_i(u)}{h} \geq \kappa_i > 0\,. \tag{1.16}$$

For a given state $u \in I\!R^n$ and $i = 1, \ldots, n$, we denote by

$$\sigma \mapsto S_i(\sigma)(u), \qquad \sigma \mapsto R_i(\sigma)(u)$$

respectively the i-shock and the i-rarefaction curve through u, parametrized by arc-length. Moreover, we consider the composite curve

$$\Psi_i(\sigma)(u) \doteq \begin{cases} R_i(\sigma)(u) & \text{if } \sigma \geq 0, \\ S_i(\sigma)(u) & \text{if } \sigma < 0. \end{cases} \tag{1.17}$$

As in [16], the definition of the Glimm interaction functional can be extended to general **BV** functions. Let $u : I\!R \mapsto I\!R^n$ have bounded variation. Then the distributional derivative $\mu = Du$ is a vector measure. Let $x_1, x_2, \ldots$ be the points where u has a jump, say $\Delta u(x_\alpha) = u(x_\alpha+) - u(x_\alpha-)$. Call $\sigma_\alpha^1, \ldots, \sigma_\alpha^n$ the waves generated by the corresponding Riemann problem at x_α. Recalling (1.17), this means

$$u(x_\alpha+) = \Psi_n(\sigma_\alpha^n) \circ \cdots \circ \Psi_1(\sigma_\alpha^1)\big(u(x_\alpha-)\big)\,. \tag{1.18}$$

For $i = 1, \ldots, n$ we can now define μ^i as the signed measure such that, for every open interval J,

$$\mu^i(J) = \int_J l_i \cdot Du\,, \tag{1.19}$$

where

$$\begin{cases} l_i(x) = l_i\big(u(x)\big) & \text{if } u \text{ is continuous at } x, \\ l_i(x_\alpha) \cdot \Delta u(x_\alpha) = \sigma_\alpha^i & \text{if } u \text{ has a jump at } x_\alpha. \end{cases} \tag{1.20}$$

Observe that, for a point of jump as in (1.18), the definitions (1.19)-(1.20) simply mean $\mu^i\big(\{x_\alpha\}\big) = \sigma_\alpha^i$. In this case, (1.20) does not uniquely determine the value of $l_i(x_\alpha)$. However, since

$$\sigma_\alpha^i = l_i\big(u(x_\alpha)\big) \cdot \Delta u(x_\alpha) + \mathcal{O}(1) \cdot |\Delta u(x_\alpha)|^2\,, \tag{1.21}$$

where $\mathcal{O}$ is the Landau order symbol, we can choose the vector l_i so that

$$\Big| l_i(x_\alpha) - l_i\big(u(x_\alpha)\big) \Big| = \mathcal{O}(1) \cdot \big|\Delta u(x_\alpha)\big|\,. \tag{1.22}$$

Call μ^{i+}, μ^{i-} the positive and negative parts of the signed measure μ^i, so that

$$\mu^i = \mu^{i+} - \mu^{i-}, \qquad |\mu^i| = \mu^{i+} + \mu^{i-}. \tag{1.23}$$

The *total strength of waves* in u is defined as

$$V(u) \doteq \sum_{i=1}^n V_i(u), \qquad V_i(u) \doteq |\mu^i|(I\!R), \tag{1.24}$$

Let $\mathcal{N}$ be the set of those indices $i \in \{1, \ldots, n\}$ such that the i-th characteristic family is genuinely nonlinear. The *interaction potential* of waves in u is then defined as

$$Q(u) \doteq \sum_{1 \le i < j \le n} \big(|\mu^j| \times |\mu^i|\big)\big(\{(x,y);\quad x < y\}\big) + \sum_{i \in \mathcal{N}} \big(\mu^{i-} \times |\mu^i|\big)\big(\{(x,y);\quad x \neq y\}\big) . \tag{1.25}$$

The decay estimates in [16] can be stated as follows.

Proposition 1.1.2 *Let the i-th characteristic field be genuinely nonlinear. Then there exist constants $C_1, \kappa > 0$ such that, for every solution u with small total variation obtained as limit of wave-front tracking approximations, one has*

$$\mu_t^{i+}\big([a,b]\big) \le \frac{b-a}{\kappa(t-s)} + C_1 \cdot \big[Q(u(s)) - Q(u(t))\big]. \tag{1.26}$$

for every interval $[a,b]$ and all $t > s \ge 0$.

Here and in the sequel, by μ_t^{i+} we denote the measure of positive i-waves in $u(t,\cdot)$. Intuitively, (1.26) says that these positive waves can be split in two parts:

- The "old" waves, generated before time s, that have decayed throughout the interval $[s,t]$ due to genuine nonlinearity. Their density is $\mathcal{O}(1) \cdot (t-s)^{-1}$.
- The "new" waves, generated after time s. Their density can be arbitrarily large, but their total strength is controlled by the decrease in the interaction potential.

Our second main result provides a converse to Proposition 1.1.2, showing that the uniqueness of solutions to the Cauchy problem can also be derived from the following.

(A4) **(Decay Assumption)** There exist a constant $\kappa > 0$ and a nonincreasing function Z such that

$$\mu_t^{i+}\big([a,b]\big) \le \frac{b-a}{\kappa(t-s)} + \big[Z(s) - Z(t)\big]. \tag{1.27}$$

for every interval $[a,b]$ and all $i = 1, \ldots, n$, $t > s \ge 0$.

Theorem 1.1.3 *Let the system (1.1) be strictly hyperbolic, with each characteristic field genuinely nonlinear, so that it generates a Standard Riemann Semigroup S on a domain $\mathcal{D}$ as in (1.4), with $\delta_0 > 0$ sufficiently small. Then, for every $\bar{u} \in \mathcal{D}$, $T > 0$, the Cauchy problem (1.1), (1.3) has a unique solution $u : [0,T] \mapsto \mathcal{D}$ satisfying* **(A1)** *and* **(A4)** *for some constant κ and some nonincreasing function Z. Indeed, these conditions imply the identity (1.5).*

Several uniqueness results for entropy weak solutions to hyperbolic systems of conservation laws have appeared in the literature. For scalar conservation laws, the uniqueness and stability problem was completely solved by Kruzhkov [46]. In the case of systems, however, until recently all available theorems required additional regularity hypotheses on the solutions. In [63, 56], a uniqueness result is proved in the class of piecewise smooth functions, while in [33] it is shown that if a piecewise regular solution exists, then it is unique within a class of **BV** functions. The main result in [54] establishes the uniqueness of solutions under a restriction on the locations of centered rarefaction waves. For Temple class systems, stronger uniqueness theorems can be found in [30, 40]. Relying on the semigroup approach proposed in [11], the uniqueness theorem in [20] was the first one which could be applied within the same class of **BV** functions where an existence theorem is known. The present assumptions **(A3)** further weaken the regularity condition used in [20] and are clearly satisfied by the weak solutions obtained as limits of the Glimm scheme [37] or front tracking approximations [12]. As shown in [16], any limit of front tracking approximations satisfies the decay assumption **(A4)**, provided that all characteristic fields of the system (1.1) are genuinely nonlinear.

1.2 Preliminary results

In this section we collect some technical lemmas, for later use.

Lemma 1.2.1 *Let* $u : [0,T] \times \mathbb{R} \mapsto \mathbb{R}^n$ *be a function which satisfies the Conservation Equations* **(A1)***, say with* $\text{Tot.Var.}u(t,\cdot) \le M$ *for all* $t \in [0,T]$*. Then the map* $t \mapsto u(t,\cdot)$ *is Lipschitz continuous, i.e.*

$$\left\| u(\tau,\cdot) - u(\tau',\cdot) \right\|_{\mathbf{L}^1} \le L'|\tau - \tau'| \tag{1.28}$$

for some constant L' *and all* $\tau, \tau' \in [0,T]$.

PROOF OF LEMMA 1.2.1. Fix any $\tau > \tau' > 0$ and construct a smooth approximation to the characteristic function of the interval $[\tau', \tau]$. For this purpose, take a smooth nondecreasing function $\alpha : \mathbb{R} \to [0,\ 1]$, such that

$$\alpha(x) = \begin{cases} 0 & \text{if } x \le -1, \\ 1 & \text{if } x \ge 1, \end{cases}$$

and define $\alpha_h(x) \doteq \alpha(x/h)$. As $h \to 0+$, α_h thus approaches the Heaviside function. Consider any smooth function $\psi = \psi(x)$ with compact support, and define

$$\varphi_h(t,x) = [\alpha_h(t-\tau') - \alpha_h(t-\tau)]\psi(x).$$

By the assumption on u, using φ_h in (1.8) and letting $h \to 0$, thanks to the $\mathbf{L}^1$ continuity of the function $t \mapsto u(t,\cdot)$ we obtain

$$\int \psi(x)[u(\tau,x) - u(\tau',x)]dx + \int_{\tau'}^{\tau}\int \psi_x(x) f(u)dxdt = 0.$$

It thus follows

$$\begin{aligned}\big\|u(\tau,\cdot) - u(\tau',\cdot)\big\|_{\mathbf{L}^1} &= \sup_{\psi\in\mathcal{C}_c^1,\ |\psi|\le 1} \int \psi(x)[u(\tau,x) - u(\tau',x)]dx \\ &\le \int_{\tau'}^{\tau} \text{Tot.Var.}\big\{f(u(t,\cdot))\big\}dt \\ &\le M \cdot Lip(f)\cdot |\tau - \tau'|,\end{aligned}$$

where by $Lip(f)$ we denote the Lipschitz constant of the function f on the domain $|u| \le M$. □

Lemma 1.2.2 *Let $w :]a,\ b[\mapsto I\!R^n$ be an integrable function such that, for some measure μ, one has*

$$\left|\int_{\zeta_1}^{\zeta_2} w(x)\,dx\right| \le \mu\big([\zeta_1,\ \zeta_2]\big), \qquad \textit{whenever } a < \zeta_1 < \zeta_2 < b. \tag{1.29}$$

Then

$$\int_a^b \big|w(x)\big|\,dx \le \mu\big(]a,\ b[\big). \tag{1.30}$$

Proof of Lemma 1.2.2. Observe that, in (1.29), one can replace the closed interval $[\zeta_1,\zeta_2]$ with an open one. Namely,

$$\left|\int_{\zeta_1}^{\zeta_2} w(x)\,dx\right| = \lim_{\varepsilon\to 0}\left|\int_{\zeta_1+\varepsilon}^{\zeta_2-\varepsilon} w(x)\,dx\right| \le \mu\big(]\zeta_1,\ \zeta_2[\big) \tag{1.31}$$

Next, fix any $\varepsilon > 0$. Then there exists a piecewise constant function v such that

$$\int_a^b \big|w(x) - v(x)\big|dx \le \varepsilon.$$

Calling $a = x_0 < \cdots < x_N = b$ the points of discontinuity of v, we compute

$$\begin{aligned}
\int_a^b \big|w(x)\big|dx &\leq \int_a^b \big|w(x) - v(x)\big|dx + \int_a^b \big|v(x)\big|dx \\
&\leq \varepsilon + \sum_j \int_{x_{j-1}}^{x_j} \big|v(x)\big|dx \\
&= \varepsilon + \sum_j \left| \int_{x_{j-1}}^{x_j} v(x)dx \right| \\
&\leq \varepsilon + \sum_j \left| \int_{x_{j-1}}^{x_j} w(x)dx \right| + \sum_j \left| \int_{x_{j-1}}^{x_j} \big(v(x) - w(x)\big)dx \right| \\
&\leq \varepsilon + \sum_j \mu\big(]x_{j-1},\ x_j[\big) + \int_a^b \big|v(x) - w(x)\big|dx \\
&\leq 2\varepsilon + \mu\big(]a,\ b[\big).
\end{aligned}$$

Since ε was arbitrary, this proves the lemma. □

Lemma 1.2.3 *Let $u : [0,T] \mapsto \mathcal{D}$ be Lipschitz continuous. At a given point (τ, ξ), let the conditions (1.9)-(1.10) hold, for some $u^-, u^+ \in \mathbb{R}^n$, $\lambda \in \mathbb{R}$. Then, for each $\lambda^* > 0$ one has*

$$\lim_{\rho\to 0+} \sup_{|h|\leq\rho} \frac{1}{\rho} \int_{\xi-\lambda^*\rho}^{\xi+\lambda^*\rho} \big|u(\tau + h,\ x) - U(\tau + h,\ x)\big|\ dx = 0. \tag{1.32}$$

Proof of Lemma 1.2.3. Assume, on the contrary, that there exists $\delta > 0$ and sequences $\rho_\nu \to 0$, $|h_\nu| < \rho_\nu$, such that

$$\int_{\xi-\lambda^*\rho_\nu}^{\xi+\lambda^*\rho_\nu} \big|u(\tau + h,\ x) - U(\tau + h,\ x)\big|\ dx > \delta\rho_\nu \tag{1.33}$$

for every $\nu \geq 1$. To fix the ideas, assume $h_\nu \geq 0$ for all ν, the other cases being entirely similar. By the Lipschitz continuity proved in (1.28), we have

$$\int_a^b \Big\{ \big|u(t,x) - U(t,x)\big| - \big|u(t',x) - U(t',x)\big| \Big\} dx \leq L^* |t - t'| \tag{1.34}$$

for some constant L^*, for every $t, t' \geq 0$ and every interval $[a, b]$. It is not restrictive to assume $L^* > 1/\delta$. The assumption (1.10) implies

$$\lim_{\rho\to 0+} \frac{1}{\rho^2} \int_{\tau-\rho}^{\tau+\rho} \int_{\xi-\lambda^*\rho}^{\xi+\lambda^*\rho} \Big|u(t,x) - U(t,x)\Big|\ dxdt = 0. \tag{1.35}$$

However, if (1.33) holds, by (1.34) it follows

$$\begin{aligned}
\int_{\tau-\rho_\nu}^{\tau+\rho_\nu} & \int_{\xi-\lambda^*\rho_\nu}^{\xi+\lambda^*\rho_\nu} \Big|u(t,x) - U(t,x)\Big|\, dxdt \\
&\geq \int_{\tau+h_\nu-(\delta/L^*)\rho_\nu}^{\tau+h_\nu} \int_{\xi-\lambda^*\rho_\nu}^{\xi+\lambda^*\rho_\nu} \Big\{ \big|u(\tau+h_\nu,\ x) \\
&\quad -U(\tau+h_\nu,\ x)\big| - L^*|\tau+h_\nu-t| \Big\}\, dxdt \\
&\geq \int_{\tau+h_\nu-(\delta/L^*)\rho_\nu}^{\tau+h_\nu} \delta\rho_\nu - L^*(\tau+h_\nu-t)\, dt \\
&= \delta(\delta/L^*)\rho_\nu^2
\end{aligned} \tag{1.36}$$

for all ν, in contradiction with (1.35). This proves the lemma. □

Corollary 1.2.4 *Under the same assumptions of Lemma 1.2.3, the states u^-, u^+ in (1.9)-(1.10) satisfy*

$$u^- = u(\tau, \xi-), \qquad\qquad u^+ = u(\tau, \xi+). \tag{1.37}$$

Indeed, taking $h = 0$ and letting $\rho \to 0+$ in (1.32) one obtains

$$\begin{aligned}
&\big|u\ \ (\tau,\xi+) - u^+\big| \\
&\leq \limsup_{\rho\to 0+} \frac{1}{\rho} \left(\int_\xi^{\xi+\rho} \big|u(\tau,\xi+) - u(\tau,x)\big|\, dx \ + \ \int_\xi^{\xi+\rho} \big|u(\tau,x) - u^+\big|\, dx \right) \\
&= 0.
\end{aligned}$$

The other identity is proved similarly.

Lemma 1.2.5 *Let u be a weak solution of (1.1). Let the conditions (1.9)-(1.10) hold at some point (τ, ξ). Then the states u^-, u^+ and the speed λ satisfy the Rankine-Hugoniot equations*

$$\lambda(u^+ - u^-) = f(u^+) - f(u^-). \tag{1.38}$$

Together with the entropy condition in **(A2)**, Lemma 1.2.5 shows that the states u^-, u^+ are connected either by an admissible shock or by a contact discontinuity, propagating with speed λ.

PROOF OF LEMMA 1.2.5. To prove the lemma, define the function

$$V(s,y) \doteq \begin{cases} u^- & \text{if } y < \lambda s, \\ u^+ & \text{if } y > \lambda s. \end{cases}$$

By (1.10) it follows

$$\lim_{\rho\to 0+} \iint_B \big|u(\tau+\rho s,\ \xi+\rho y) - V(s,y)\big|\ dyds = 0$$

for every bounded set $B \subset I\!R^2$. Moreover, (1.8) implies

$$\iint u(\tau+\rho s,\ \xi+\rho y)\phi_s(s,y) + f\big(u(\tau+\rho s,\ \xi+\rho y)\big)\phi_y(s,y)\ dyds = 0$$

for all $\phi \in \mathcal{C}^1_c$ and all $\rho > 0$. The two above relations imply

$$\iint V\phi_s + f(V)\phi_y\ dyds = 0,$$

showing that the function V itself is a solution of (1.1). Therefore the Rankine-Hugoniot equations (1.38) must hold. □

We conclude this section by recalling two local integral estimates that characterize the trajectories of a Standard Riemann Semigroup.

Two types of local approximate solutions for (1.1) will be considered. One is derived from the self-similar solution of a Riemann problem, the other is obtained by "freezing" the coefficients of the corresponding quasilinear hyperbolic system in a neighborhood of a given point.

Let $w : I\!R \mapsto I\!R^n$ be any **BV** function and fix any point $\xi \in I\!R$ where w has a jump. Call $\omega = \omega(t,x)$ the unique self-similar entropy solution of the Riemann problem

$$\omega_t + f(\omega)_x = 0, \qquad \omega(0,x) = \begin{cases} w(\xi-) & \text{if } x < 0, \\ w(\xi+) & \text{if } x > 0. \end{cases} \tag{1.39}$$

Let $\hat\lambda$ be an upper bound for the absolute values of all wave speeds. For $t \geq 0$, define

$$U^\sharp(t,x) \doteq \begin{cases} \omega(t,\ x-\xi) & \text{if } |x-\xi| \leq \hat\lambda t, \\ w(x) & \text{if } |x-\xi| > \hat\lambda t. \end{cases} \tag{1.40}$$

Observe that the function $t \mapsto U^\sharp(t,\cdot)$ is Lipschitz continuous w.r.t. the $\mathbf{L}^1$ distance, and approaches w as $t \to 0+$.

Next, call $\widetilde A \doteq Df\big(w(\xi)\big)$ the Jacobian matrix of f computed at the point $w(\xi)$. For $t \geq 0$, define $U^\flat(t,x)$ as the solution of the linear hyperbolic Cauchy problem with constant coefficients

$$U^\flat_t + \widetilde A U^\flat_x = 0, \qquad U^\flat(0) = w. \tag{1.41}$$

Lemma 1.2.6 *Let S be a Standard Riemann Semigroup generated by the system (1.1), with domain $\mathcal{D}$ as in (1.4). Then, there exists a constant C_1 such that the following holds. For every function $w \in \mathcal{D}$, every $\xi \in I\!R$ and $h, \rho > 0$, with the above definitions one has*

$$\frac{1}{h}\int_{\xi-\rho+h\hat{\lambda}}^{\xi+\rho-h\hat{\lambda}} \Big|(S_h w)(x) - U^\sharp(h,x)\Big|\,dx \\ \leq\ C_1 \cdot \textit{Tot.Var.}\big(w;\ ]\xi-\rho,\ \xi[\ \cup\ ]\xi,\ \xi+\rho[\big), \tag{1.42}$$

$$\frac{1}{h}\int_{\xi-\rho+h\hat{\lambda}}^{\xi+\rho-h\hat{\lambda}} \Big|(S_h w)(x) - U^\flat(h,x)\Big|\,dx \\ \leq C_1 \cdot \Big(\textit{Tot.Var.}\big(w;\ ]\xi-\rho,\ \xi+\rho[\big)\Big)^2. \tag{1.43}$$

For a proof, see [11, p. 217].

1.3 Proof of Theorem 1.1.1

As before, let $\hat{\lambda}$ be an upper bound for all wave speeds. Let u satisfy the hypotheses of Theorem 1.1.1. For every R, it suffices to show that $u(T,\cdot) = S_T\bar{u}$, restricted to the interval

$$J_T \doteq [-R+\hat{\lambda}T,\ R-\hat{\lambda}T], \tag{1.44}$$

for all $T > 0$. In turn, this can be deduced from the error estimate

$$\big\|u(T) - S_T\bar{u}\big\|_{\mathbf{L}^1(J_T)} \leq L \cdot \int_0^T \left\{\liminf_{h\to 0+} \frac{\big\|u(\tau+h) - S_h u(\tau)\big\|_{\mathbf{L}^1(J_{\tau+h})}}{h}\right\}\,d\tau, \tag{1.45}$$

if we prove that the integrand on the right hand side of (1.45) vanishes almost everywhere. Recalling Lemma 1.2.1 and the assumption $u(t,\cdot) \in \mathcal{D}$, we conclude that $u = u(t,x)$ is a **BV** function on the strip $[0,T] \times I\!R$, in the sense that the distributional derivatives $D_t u, D_x u$ are Radon measures [35]. By a well known structure theorem [35, 69], there exists a set $\widetilde{\mathcal{N}} \subset]0,T[\,\times I\!R$ of 1-dimensional Hausdorff measure zero such that, at every point $(\tau,\xi) \notin \widetilde{\mathcal{N}}$, u either is approximately continuous or has an approximate jump discontinuity. We claim that, at every such point of discontinuity, the jump cannot have a horizontal tangent. In other words, one can never have (1.10) in the case

$$U(t,x) = \begin{cases} u^- & \text{if} \quad t < \tau, \\ u^+ & \text{if} \quad t > \tau, \end{cases} \qquad u^+ \neq u^-. \tag{1.46}$$

Indeed, for every positive constant c (arbitrarily large), from (1.46), (1.10) and (1.28) we deduce

$$\begin{aligned} c|u^+ - u^-| &= \lim_{\rho\to 0+} \frac{1}{\rho^2} \int_\tau^{\tau+\rho} \int_\xi^{\xi+c\rho} |u^+ - u^-|\, dxdt \\ &\leq \limsup_{\rho\to 0+} \frac{1}{\rho^2} \int_\tau^{\tau+\rho} \int_\xi^{\xi+c\rho} \big|u(t,x) - u(2\tau - t, x)\big|\, dxdt \\ &\leq \lim_{\rho\to 0+} \frac{1}{\rho^2} \int_\tau^{\tau+\rho} L' \cdot 2(t-\tau)\, dt \\ &= L'. \end{aligned}$$

Hence $|u^+ - u^-| \leq L'/c$ for every c, against the second assumption in (1.46).

Taking the projection of $\widetilde{\mathcal{N}}$ on the t-axis, we conclude that there exists a set $\mathcal{N} \subset [0,T]$ of measure zero, containing the endpoints 0 and T, such that, at every point $(\tau,\xi) \in [0,T] \times I\!R$ with $\tau \notin \mathcal{N}$, the following property holds.

(P) Either u is approximately continuous at (τ,ξ), or it has an approximate jump discontinuity, in the sense that (1.9)-(1.10) hold for some states u^-, u^+ and some $\lambda \in I\!R$. In this second case, Corollary 1 implies $u^- = u(\tau,\xi-)$, $u^+ = u(\tau,\xi+)$. Moreover, by Lemma 1.2.5 the Rankine-Hugoniot equations (1.38) hold. Hence, by **(A2)** the entropy condition (1.11) holds as well.

Theorem 1.1.1 will be proved by establishing the basic relation

$$\liminf_{h\to 0+} \frac{1}{h} \int_{-R+\hat\lambda(\tau+h)}^{R-\hat\lambda(\tau+h)} \big|u(\tau+h,x) - \big(S_h u(\tau)\big)(x)\big| dx = 0 \qquad \text{for all } \tau \notin \mathcal{N}. \tag{1.47}$$

Let $\tau \in [0,T] \setminus \mathcal{N}$ and $\varepsilon > 0$ be given. For notational convenience, call μ the measure of total variation of $u(\tau,\cdot)$, so that

$$\mu\big(]a,\ b[\big) \doteq \text{Tot.Var.}\big\{u(\tau);\ ]a,\ b[\,\big\}. \tag{1.48}$$

Let $\xi'_1 < \ldots < \xi'_N$ be the points in $[-R,\ R]$ such that

$$\mu\big(\{\xi'_\alpha\}\big) \geq \varepsilon.$$

The boundedness of μ on any bounded interval implies that only finitely many such points exist. Since $\tau \notin \mathcal{N}$, for each $\alpha = 1,\ldots,N$ by the property **(P)** the two states

$u_\alpha^\pm \doteq u(\tau, \xi'_\alpha\pm)$ are connected by an entropy-admissible shock, travelling with some speed λ^α. Moreover, introducing the functions

$$U_\alpha(t,x) = \begin{cases} u_\alpha^- & \text{if } x < \xi'_\alpha + \lambda^\alpha(t-\tau), \\ u_\alpha^+ & \text{if } x > \xi'_\alpha + \lambda^\alpha(t-\tau), \end{cases} \tag{1.49}$$

one has

$$\lim_{\rho\to 0+} \frac{1}{\rho^2} \int_{\tau-\rho}^{\tau+\rho} \int_{\xi'_\alpha-\rho}^{\xi'_\alpha+\rho} \big|u(t,x) - U_\alpha(t,x)\big|\, dxdt = 0 \qquad (\alpha = 1,\ldots,N). \tag{1.50}$$

For any fixed $\lambda^* > 0$ (arbitrarily large), Lemma 1.2.3 implies

$$\lim_{h\to 0+} \frac{1}{h} \int_{\xi'_\alpha-\lambda^* h}^{\xi'_\alpha+\lambda^* h} \big|u(\tau+h,x) - U_\alpha(\tau+h,x)\big|\, dx = 0. \tag{1.51}$$

By (1.51) we can thus assume

$$\sum_{\alpha=1}^{N} \frac{1}{h} \int_{\xi'_\alpha-\lambda^* h}^{\xi'_\alpha+\lambda^* h} \Big|u(\tau+h,x) - U_\alpha(\tau+h,x)\Big|\, dx \le \varepsilon \tag{1.52}$$

for $h > 0$ sufficiently small. By (1.42) in Lemma 1.2.6 it follows that the semigroup trajectory satisfies an entirely similar estimate:

$$\begin{aligned} &\sum_{\alpha=1}^{N} \frac{1}{h} \int_{\xi'_\alpha-\lambda^* h}^{\xi'_\alpha+\lambda^* h} \Big|\big(S_h u(\tau)\big)(x) - U_\alpha(\tau+h,x)\Big|\, dx \\ &\quad \le C_1 \cdot \sum_{\alpha=1}^{N} \text{Tot.Var.}\Big\{u(\tau);\ ]\xi'_\alpha - 2\lambda^* h,\ \xi'_\alpha[\ \cup\ ]\xi'_\alpha,\ \xi'_\alpha + 2\lambda^* h[\Big\} \\ &\quad \le \varepsilon, \end{aligned} \tag{1.53}$$

for all $h > 0$ sufficiently small. Indeed, it is not restrictive to assume that λ^* is larger than all wave speeds. Combining (1.52) and (1.53) we deduce

$$\sum_{\alpha=1}^{N} \frac{1}{h} \int_{\xi'_\alpha-\lambda^* h}^{\xi'_\alpha+\lambda^* h} \Big|u(\tau+h,x) - \big(S_h u(\tau)\big)(x)\Big|\, dx \ \le\ 2\varepsilon \qquad \text{for all } h \in\,]0,\ h^*], \tag{1.54}$$

with $h^* > 0$ sufficiently small.

Next, we consider all the remaining points, where $u(\tau,\cdot)$ is either continuous or has a jump of strength $< \varepsilon$. Choose $\rho > 0$ such that

$$2\rho < \min_{\alpha=2,\cdots,N} |\xi'_\alpha - \xi'_{\alpha-1}|,$$

$$\mu\big(]a,\ b[\big) \leq \varepsilon \qquad \text{whenever} \quad b-a \leq 2\,\rho, \quad]a,b[\ \cap\ \{\xi'_1, \dots \xi'_N\} = \emptyset. \tag{1.55}$$

Such a ρ exists, because the total variation of $u(\tau)$ is bounded and $\mu\big(\{x\}\big) < \varepsilon$ whenever $x \notin \{\xi'_1, \dots \xi'_N\}$.

We can now select a finitely many points $\xi_1 < \dots < \xi_M$, such that the open intervals $I_\beta \doteq]\xi_\beta - \rho,\ \xi_\beta + \rho[$ satisfy:

(i) $\bigcup_{\beta=1}^M I_\beta \ \supseteq\ [-R,\ R] \setminus \{\xi'_1, \dots, \xi'_N\}$.

(ii) Every point x is contained in at most two distinct intervals I_β.

Observe that the closed intervals

$$I'_{\alpha,h} \doteq [\xi'_\alpha - h\lambda^*,\ \xi'_\alpha + h\lambda^*]$$

together with the open intervals

$$I_{\beta,h} \doteq]\xi_\beta - \rho + h\lambda^*,\ \xi_\beta + \rho - h\lambda^*[$$

still cover $[-R,\ R]$, for $h > 0$ sufficiently small.

We claim that, for every β, there exists $h_\beta > 0$ such that

$$\big|u(\tau+h, x) - u(\tau, \xi_\beta)\big| \leq (K+2)\varepsilon \qquad \text{for all } x \in I_{\beta,h}, \quad h \in [0, h_\beta]. \tag{1.56}$$

To see this, we apply (1.13) with ξ replaced by $\xi_\beta - \rho$. By (1.55), there exists $\delta^* > 0$ such that

$$\begin{aligned} \big|u(\tau+h,\ x) - u(\tau, \xi_\beta)\big| &\leq \Big|u(\tau+h,\ x) - u\big(\tau,\ (\xi_\beta - \rho)+\big)\Big| \\ &\quad + \Big|u\big(\tau,\ (\xi_\beta - \rho)+\big) - u(\tau, \xi_\beta)\Big| \\ &\leq \varepsilon + \varepsilon \end{aligned} \tag{1.57}$$

for all $h > 0$ and $x \in I_{\beta,h}$ with $x < \xi_\beta - \rho + \delta^*$. Similarly,

$$\big|u(\tau+h,\ x) - u(\tau, \xi_\beta)\big| \leq 2\varepsilon \tag{1.58}$$

for all $h > 0$ and $x \in I_{\beta,h}$ with $x > \xi_\beta + \rho - \delta^*$. For each point (τ, x) with $|x - \xi_\beta| \leq \rho - \delta^*$ we can now apply (1.12) and deduce

$$\big|u(\tau+h,\ y) - u(\tau, x)\big| \leq K\big|u(\tau, x+) - u(\tau, x-)\big| + \varepsilon \tag{1.59}$$

for $h > 0$ and $|y - x|$ sufficiently small. By (1.59) and (1.55), for some $\delta_x > 0$ there holds

$$\big|u(\tau + h,\ y) - u(\tau, \xi_\beta)\big| \leq \big|u(\tau + h,\ y) - u(\tau, x)\big| + \big|u(\tau, x) - u(\tau, \xi_\beta)\big| \\ \leq K\varepsilon + 2\varepsilon, \tag{1.60}$$

for $h \in [0, \delta_x]$, $|y - x| < \delta_x$. Covering the compact interval $[\xi_\beta - \rho + \delta^*,\ \xi_\beta + \rho - \delta^*]$ with finitely many subintervals $[x_\ell - \delta_\ell,\ x_\ell + \delta_\ell]$, $\ell = 1, \ldots, N$, and choosing $h_\beta \doteq \min\{\delta^*, \delta_1, \ldots, \delta_N\}$, we deduce (1.56).

Next, for each β, define $U^\flat$ to be the solution of the linear hyperbolic problem with constant coefficients

$$U_t^\flat + \widetilde{A} U_x^\flat = 0, \qquad U^\flat(\tau) = u(\tau), \tag{1.61}$$

where $\widetilde{A} \doteq Df\big(u(\tau, \xi_\beta)\big)$. Call $\tilde\lambda_1, \ldots \tilde\lambda_n$ the eigenvalues of $\widetilde{A}$ and let $\tilde l_i,\ \tilde r_i$ be the corresponding left and right eigenvectors, normalized as in (1.15). For every $i = 1, \ldots, n$ and every choice of $\zeta_1, \zeta_2 \in I_{\beta,h}$, we now estimate the quantity

$$E_i \doteq \int_{\zeta_1}^{\zeta_2} \big\langle \tilde l_i,\ u(\tau + h, x) - U^\flat(\tau + h, x)\big\rangle\, dx.$$

Observing that (1.61) implies

$$\big\langle \tilde l_i,\ U^\flat(\tau + h, x)\big\rangle = \big\langle \tilde l_i,\ U^\flat(\tau, x - h\tilde\lambda_i)\big\rangle = \big\langle \tilde l_i,\ u(\tau, x - h\tilde\lambda_i)\big\rangle,$$

and integrating (1.1) over the domain

$$D_i \doteq \Big\{(t, x);\quad t \in [\tau,\ \tau + h],\quad \zeta_1 + (t - \tau - h)\tilde\lambda_i \leq x \leq \zeta_2 + (t - \tau - h)\tilde\lambda_i\Big\},$$

we obtain

$$E_i = \int_\tau^{\tau+h} \Big\langle \tilde l_i,\ \big(f(u) - \tilde\lambda_i u\big)\big(t,\ \zeta_1 + (t - \tau - h)\tilde\lambda_i\big)\Big\rangle\, dt \\ - \int_\tau^{\tau+h} \Big\langle \tilde l_i,\ \big(f(u) - \tilde\lambda_i u\big)\big(t,\ \zeta_2 + (t - \tau - h)\tilde\lambda_i\big)\Big\rangle\, dt. \tag{1.62}$$

To estimate the quantity in (1.62), consider the states

$$u' \doteq u\big(t,\ \zeta_1 + (t - \tau - h)\tilde\lambda_i\big), \qquad u'' \doteq u\big(t,\ \zeta_2 + (t - \tau - h)\tilde\lambda_i\big), \qquad \tilde u \doteq u(\tau, \xi_\beta).$$

We then have

$$\Big\langle \tilde l_i,\ f(u'') - f(u') - \tilde\lambda_i(u'' - u')\Big\rangle = \Big\langle \tilde l_i,\ Df(\tilde u) \cdot (u'' - u') - \tilde\lambda_i(u'' - u')\Big\rangle \\ + \Big\langle \tilde l_i,\ A^*(u'' - u')\Big\rangle,$$

where the matrix A^* is defined by

$$A^* \doteq \int_0^1 \Big[Df\big(su'' + (1-s)u'\big) - Df(\tilde{u})\Big]\, ds.$$

Therefore,

$$\left|\Big\langle \tilde{l}_i,\ f(u'') - f(u') - \tilde{\lambda}_i(u''-u')\Big\rangle\right| \ \le\ C\cdot |u''-u'|\cdot\big(|u''-\tilde{u}| + |u'-\tilde{u}|\big), \tag{1.63}$$

for some constant C depending only on the second order derivatives of the function f. By (1.56),

$$|u''-\tilde{u}| + |u'-\tilde{u}| \le 2(K+2)\varepsilon. \tag{1.64}$$

Using (1.63)-(1.64) in (1.62) we deduce that, for some constant C_2,

$$\begin{aligned}
&\left|\int_{\zeta_1}^{\zeta_2} \big\langle \tilde{l}_i,\ u(\tau+h,x) - U^\flat(\tau+h,x)\big\rangle\, dx\right| \\
&\le C_2\ \varepsilon \int_\tau^{\tau+h} \text{Tot.Var.}\big\{u(t);\ J^i_{\zeta_1,\zeta_2}(t)\big\} dt,
\end{aligned} \tag{1.65}$$

where $J^i_{\zeta_1,\zeta_2}(t) \doteq\ ]\zeta_1 + (t-\tau-h)\tilde{\lambda}_i,\ \zeta_2 + (t-\tau-h)\tilde{\lambda}_i[$. Define the positive Radon measure $\mu_{i,h}$ by setting

$$\mu_{i,h}\big(]\zeta_1,\ \zeta_2[\big) \doteq \frac{1}{h}\int_\tau^{\tau+h} \text{Tot.Var.}\big\{u(t);\ J^i_{\zeta_1,\zeta_2}(t)\big\} dt,$$

for every open interval $]\zeta_1,\ \zeta_2[$. Thanks to (1.65), we can apply Lemma 1.2.2 and obtain

$$\begin{aligned}
\int_{\xi_\beta-\rho+\lambda^*h}^{\xi_\beta+\rho-\lambda^*h} & \left|\big\langle \tilde{l}_i,\ u(\tau+h,x) - U^\flat(\tau+h,x)\big\rangle\right| dx \\
& \le C_2\varepsilon h\cdot \mu_{i,h}\big(]\xi_\beta - \rho + \lambda^* h,\ \xi_\beta + \rho - \lambda^* h[\big) \\
& = C_2\varepsilon\cdot \int_\tau^{\tau+h} \text{Tot.Var.}\Big\{u(t);\ ]\xi_\beta - \rho + \lambda^*h + (t-\tau-h)\tilde{\lambda}_i, \\
& \qquad \xi_\beta + \rho - \lambda^*h + (t-\tau-h)\tilde{\lambda}_i\big[\Big\}\ dt \\
& \le C_2\varepsilon\cdot \int_0^h \text{Tot.Var.}\big\{u(\tau+h');\ I_{\beta,h'}\big\}\ dh'.
\end{aligned}$$

We are now ready to estimate

$$\begin{aligned}
\sum_{\beta=1}^M & \int_{\xi_\beta-\rho+\lambda^*h}^{\xi_\beta+\rho-\lambda^*h} \big|u(\tau+h,x) - U^\flat(\tau+h,x)\big|\ dx \\
& = \mathcal{O}(1)\cdot \sum_{\beta=1}^M \sum_{i=1}^n \int_{\xi_\beta-\rho+\lambda^*h}^{\xi_\beta+\rho-\lambda^*h} \left|\big\langle \tilde{l}_i,\ u(\tau+h,x) - U^\flat(\tau+h,x)\big\rangle\right| dx
\end{aligned}$$

$$= \mathcal{O}(1) \cdot \varepsilon \sum_{\beta=1}^{M} \int_{\tau}^{\tau+h} \text{Tot.Var.}\{u(t);\ ]\xi_\beta - \rho + \lambda^* t,\ \xi_\beta + \rho - \lambda^* t[\}dt$$
$$= \mathcal{O}(1) \cdot \varepsilon \int_{\tau}^{\tau+h} \text{Tot.Var.}\{u(t);\ I\!R\}dt$$
$$\leq C_3\ \varepsilon\ h, \tag{1.66}$$

for some constant C_3 and all $h > 0$ sufficiently small.

On the other hand, applying (1.43) with $w = u(\tau)$, by (1.55) we deduce

$$\frac{1}{h} \int_{\xi_\beta - \rho + \lambda^* h}^{\xi_\beta + \rho - \lambda^* h} \Big| \big(S_h u(\tau)\big)(x) - U^\flat(\tau + h, x) \Big|\ dx$$
$$\leq C_1 \Big(\text{Tot.Var.}\{u(\tau);\ ]\xi_\beta - \rho,\ \xi_\beta + \rho[\,\} \Big)^2$$
$$\leq C_1 \varepsilon \cdot \text{Tot.Var.}\{u(\tau);\ ]\xi_\beta - \rho,\ \xi_\beta + \rho[\,\}. \tag{1.67}$$

Indeed, it is not restrictive to assume λ^* larger than all wave speeds. The two inequalities (1.66) and (1.67) together yield

$$\sum_{\beta=1}^{M} \frac{1}{h} \int_{\xi_\beta - \rho + \lambda^* h}^{\xi_\beta + \rho - \lambda^* h} \Big| u(\tau + h, x) - \big(S_h u(\tau)\big)(x) \Big|\ dx$$
$$\leq \Big(C_3 + 2C_1 \cdot \text{Tot.Var.}\{u(\tau); I\!R\} \Big) \varepsilon.) \tag{1.68}$$

Since the intervals $I'_{\alpha,h}$, $I_{\beta,h}$ cover $[-R, R]$, from (1.52) and (1.68) we finally obtain

$$\frac{1}{h} \int_{-R}^{R} \Big| u(\tau + h, x) - \big(S_h u(\tau)\big)(x) \Big|\ dx \leq \Big(1 + C_3 + 2C_1\ \text{Tot.Var.}\{u(\tau); I\!R\} \Big) \varepsilon, \tag{1.69}$$

for every $h > 0$ sufficiently small. Since $\varepsilon > 0$ was arbitrary, this establishes (1.47). Using (1.47) in (1.45), Theorem 1.1.1 is proved.

1.4 Proof of Theorem 1.1.3

We first recall a result of real analysis [44, p. 320]. Let $Z : [0, T] \mapsto I\!R$ be a nonincreasing function. Then, for every $M > 0$ there holds

$$\text{meas} \left\{ \tau \in [0, T]\,;\ \left| \frac{Z(t) - Z(\tau)}{t - \tau} \right| > M \text{ for some } t \neq \tau \right\} \leq \frac{2\big[Z(0) - Z(T)\big]}{M}. \tag{1.70}$$

From (1.70) we deduce the existence of a set $\mathcal{N}$ of measure zero such that, for each $\tau \in [0, T] \setminus \mathcal{N}$, there exists a constant $M(\tau)$ satisfying

$$\big| Z(t) - Z(\tau) \big| \leq M(\tau) \big| t - \tau \big| \qquad \text{for all } t \in [0, T]. \tag{1.71}$$

Recalling the structure theorem for **BV** functions [35, 69], we can also assume that, for all $\tau \notin \mathcal{N}$ and all $\xi \in I\!R$, either u is approximately continuous or it has an approximate jump discontinuity at the point (τ, ξ).

From now on, we fix $\tau \notin \mathcal{N}$ and call $M = M(\tau)$ the corresponding constant in (1.71). We first prove that, at time τ, the Decay Assumption **(A4)** implies the Entropy Condition **(A2)**. Let ξ be a point of jump for the function $x \mapsto u(\tau, x)$, so that (1.9)-(1.10) and (1.38) hold. In particular, we have

$$u(\tau, \xi+) = \Psi_i(\sigma)\big(u(\tau, \xi-)\big)$$

for some $\sigma \in I\!R$ and some $i \in \{1, \ldots, n\}$. Applying (1.27) with $a = b = \xi$, we find

$$\sigma = \mu^i_\tau\big(\{\xi\}\big) \leq \big[Z(t) - Z(\tau)\big] \leq M(\tau - t) \tag{1.72}$$

for all $0 < t < \tau$. Letting $t \to \tau-$, we conclude $\sigma \leq 0$. Hence the entropy condition (1.11) holds.

It now remains to prove that the Tame Oscillation Condition **(A3)** also holds. Assuming that the constant $\delta_0 > 0$ in (1.4) is sufficiently small, we can choose a unit vector $\mathbf{e}$ and a constant $c_0 > 0$ such that

$$\frac{d}{d\sigma}\big\langle \mathbf{e},\ \Psi_i(\sigma)(u)\big\rangle \geq c_0 \tag{1.73}$$

for $|u| < \delta_0$, $|\sigma| < \delta_0$. Recalling that the maps $x \mapsto u(t, x)$ were assumed right continuous, from (1.73) it follows

$$\big\langle \mathbf{e},\ u(t, b) - u(t, a)\big\rangle \leq \sum_{i=1}^n \mu^{i+}_t\big(]a, b]\big) - c_0 \cdot \sum_{i=1}^n \mu^{i-}_t\big(]a, b]\big), \tag{1.74}$$

for all $a < b$ and $t \geq 0$. We claim that (1.12) holds with $K = 2/c_0$. Indeed, assume by contradiction that there exists a point ξ and sequences $h_\nu \to 0+$ and $x_\nu \to \xi$ such that

$$\big|u(\tau + h_\nu, x_\nu) - u(\tau, \xi)\big| > \frac{2}{c_0} \cdot \big|u(\tau, \xi+) - u(\tau, \xi-)\big|. \tag{1.75}$$

By Lemma 1.2.3 we can select sequences $a_\nu \to \xi-$ and $b_\nu \to \xi+$, with $a_\nu < x_\nu < b_\nu$, such that (by possibly taking a subsequence and relabelling)

$$u^-_\nu \doteq u(\tau + h_\nu, a_\nu) \to u(\tau, \xi-), \qquad u^+_\nu \doteq u(\tau + h_\nu, b_\nu) \to u(\tau, \xi+).$$

We can also assume that

$$\tilde{u}_\nu \doteq u(\tau + h_\nu, x_\nu) \to \tilde{u}$$

for some $\tilde{u}$. Set $\delta_\nu \doteq b_\nu - a_\nu$. Applying the decay estimate (1.27) to the interval $[\tau - \sqrt{\delta_\nu},\ \tau + h_\nu]$ we obtain

$$\begin{aligned}\mu^{i+}_{\tau+h_\nu}\big([a_\nu, b_\nu]\big) &\leq \frac{\delta_\nu}{\kappa(\sqrt{\delta_\nu} + h_\nu)} + \big[Z(\tau - \sqrt{\delta_\nu}) - Z(\tau + h_\nu)\big] \\ &\leq \frac{\sqrt{\delta_\nu}}{\kappa} + M(\sqrt{\delta_\nu} + h_\nu).\end{aligned} \tag{1.76}$$

Therefore,

$$\lim_{\nu\to\infty} \mu^{i+}_{\tau+h_\nu}\big([a_\nu, b_\nu]\big) = 0. \tag{1.77}$$

Using (1.74) and (1.77), and recalling that all functions $x \mapsto u(t,x)$ are right continuous, we now estimate

$$\begin{aligned}\big|u(\tau,\xi+) - u(\tau,\xi-)\big| &\geq \Big|\big\langle \mathbf{e},\ u(\tau,\xi+) - u(\tau,\xi-)\big\rangle\Big| \\ &= \lim_{\nu\to\infty} \Big|\langle \mathbf{e},\ u^+_\nu - u^-_\nu\rangle\Big| \\ &\geq c_0 \sum_{i=1}^n \liminf_{\nu\to\infty} \mu^{i-}_{\tau+h_\nu}\big(]a_\nu,\ b_\nu]\big).\end{aligned} \tag{1.78}$$

Moreover,

$$\begin{aligned}\big|u(\tau,\xi+) - \tilde{u}\big| &\leq \lim_{\nu\to\infty} \big|u^+_\nu - \tilde{u}_\nu\big| \\ &\leq \sum_{i=1}^n \liminf_{\nu\to\infty} \mu^{i-}_{\tau+h_\nu}\big(]x_\nu,\ b_\nu]\big).\end{aligned} \tag{1.79}$$

Together, (1.78) and (1.79) yield

$$\big|u(\tau,\xi+) - \tilde{u}\big| \leq \frac{1}{c_0}\big|u(\tau,\xi+) - u(\tau,\xi-)\big|, \tag{1.80}$$

which proves (1.12), since $u(\tau,\xi) = u(\tau,\xi+)$ by the convention (1.7).

Finally, we show that the decay assumption (1.27) also implies (1.13), where λ^* is any upper bound for all characteristic speeds. If the map $x \mapsto u(\tau,x)$ is continuous at $x = \xi$, then (1.13) is an easy consequence of (1.12). Now consider the other case, where u has an approximate jump discontinuity at (τ,ξ), so that (1.9)-(1.10) hold, for some u^-, u^+, λ satisfying (1.38). Assume by contradiction that exist sequences $h_\nu \to 0+$ and $x_\nu \to \xi+$, with $x_\nu - \xi > \lambda^* h_\nu$, such that

$$\lim_{\nu\to\infty} u(\tau + h_\nu,\ x_\nu) = \tilde{u} \neq u(\tau,\xi+). \tag{1.81}$$

Since $\lambda < \lambda^*$, by (1.32) there exist sequences $a_\nu, b_\nu \to \xi+$ with $\xi + \lambda h_\nu < a_\nu < x_\nu < b_\nu$ such that

$$u_\nu^- \doteq u(\tau + h_\nu,\ a_\nu) \to u(\tau, \xi+), \qquad u_\nu^+ \doteq u(\tau + h_\nu,\ b_\nu) \to u(\tau, \xi+).$$

Repeating the same arguments in (1.78)-(1.80), we conclude

$$\big|\tilde{u} - u(\tau, \xi+)\big| \le \frac{1}{c_0}\big|u(\tau, \xi+) - u(\tau, \xi+)\big| = 0.$$

This establishes the limit (1.13) in the sector where $x > \xi + \lambda^*(t - \tau)$. The case $x < \xi - \lambda^*(t - \tau)$ is entirely similar.

Thanks to the above analysis, for all $\tau \notin \mathcal{N}$ and $\xi \in I\!R$, the assumptions **(A2)**-**(A3)** hold. Therefore, we can repeat the arguments in Section 3 and establish (1.47). By (1.45), this implies $u(T) = S_T \bar{u}$ for all $T > 0$, proving Theorem 1.1.3.

1.5 Concluding remarks

We conclude with a few observations on the role of the regularity assumption **(A3)** toward the uniqueness of solutions of the Cauchy problem (1.1), (1.3). In our proof, this assumption is used at one single step, namely in the estimate (1.65) of the distance between a weak solution u and the solution $U^\flat$ of an approximating linear problem. While deriving this particular estimate, we are not relying on any entropy condition. Therefore we need an assumption which rules out the appearance of large oscillations immediately after time τ. For general $n \times n$ systems, it is not clear whether **(A3)** could actually be deduced from **(A1)**-**(A2)**, as in the scalar case.

In [20], a stronger regularity assumption was used, namely

(A5) (Tame Variation Condition) There exists a constant C such that, for every horizontal segment Γ in the t-x plane and every space-like segment Γ' in the domain of dependency of Γ, one has

$$\text{Tot.Var.}\{u\,;\ \Gamma'\} \le C \cdot \text{Tot.Var.}\{u\,;\ \Gamma\}. \tag{1.82}$$

It is not difficult to show that **(A5)** implies **(A3)**. Indeed, fix any point (τ, ξ) and consider sequences $t_\nu \to \tau+$, $x_\nu \to \xi$. Let $\hat{\lambda}$ be an upper bound for all wave speeds and define

$$\rho_\nu \doteq |x_\nu - \xi| + \hat{\lambda}(t_\nu - \tau).$$

Clearly, $\rho_\nu \to 0$. Consider the horizontal segment Γ_ν with endpoints $(\tau,\ \xi-\rho_\nu)$, $(\tau,\ \xi+\rho_\nu)$, and the space-like segment Γ'_ν, with endpoints (t_ν, x_ν), $(\tau,\ \xi+\rho_\nu)$. Applying (1.82) we obtain

$$\begin{aligned}
\limsup_{\nu\to\infty} & \big|u(t_\nu, x_\nu) - u(\tau,\xi)\big| \\
&\leq \limsup_{\nu\to\infty} \Big\{ \big|u(t_\nu, x_\nu) - u(\tau, \xi+\rho_\nu)\big| + \big|u(\tau,\xi+\rho_\nu) - u(\tau,\xi)\big| \Big\} \\
&\leq (C+1)\cdot \limsup_{\nu\to\infty} \text{Tot.Var.}\big\{u(\tau)\,;\ \Gamma_\nu\big\} \\
&\leq (C+1)\cdot \big|u(\tau,\xi+) - u(\tau,\xi-)\big|.
\end{aligned}$$

This yields (1.12) with $K = C+1$.

To prove (1.13) with $\lambda^* = \hat\lambda$, assume $x_\nu \to \xi$, $t_\nu \to \tau+$, with $x_\nu > \xi + \hat\lambda(t_\nu - \tau)$. Calling Γ_ν the segment with endpoints $\big(\tau,\ x_\nu \pm \hat\lambda(t_\nu-\tau)\big)$ and Γ'_ν the segment with endpoints (t_ν, x_ν), $\big(\tau,\ x_\nu - \hat\lambda(t_\nu - \tau)\big)$, an application of (1.82) yields

$$\begin{aligned}
&\limsup_{\nu\to\infty} \big|u(t_\nu, x_\nu) - u(\tau,\xi+)\big| \\
&\leq \limsup_{\nu\to\infty} C\cdot \text{Tot.Var.}\big\{u(\tau)\,;\ \Gamma_\nu\big\} + \limsup_{\nu\to\infty} \big|u(\tau,\ x_\nu - \hat\lambda(t_\nu-\tau)) - u(\tau,\xi+)\big| \\
&= 0.
\end{aligned}$$

It is worth observing that Theorem 1.1.1 remains valid under assumptions somewhat weaker than **(A3)**. Namely, in (1.13) one can allow the positive number λ^* to depend arbitrarily on the point (τ,ξ). Moreover, instead of (1.12), one can assume that for almost every τ there exists a continuous function $K_\tau = K_\tau(\zeta)$ with $K_\tau(0) = 0$ such that

$$\limsup_{t\to\tau+,\ x\to\xi} \big|u(t,x) - u(\tau,\xi)\big| \leq K_\tau\Big(\big|u(\tau,\xi+) - u(\tau,\xi-)\big|\Big).$$

This extension requires only minor modifications of the original proof.

Chapter 2

Stability of $\mathbf{L}^\infty$ solutions of Temple class system

2.1 Introduction to Chapter 2

Consider the Cauchy problem for a strictly hyperbolic system of conservation laws in one space dimension

$$u_t + f(u)_x = 0, \tag{2.1}$$

$$u(0, x) = \bar{u}(x). \tag{2.2}$$

For initial data with small total variation, the global existence of weak solutions is well known [37]. More recently, various papers [8, 11, 15, 17, 18, 20, 22, 23, 59, 60] have established the uniqueness and Lipschitz continuous dependence on the initial data for entropy admissible **BV** solutions. We recall that, for a scalar conservation law, the classical work of Kruzhkov [46] proved the stability of solutions of (2.1) within a domain of $\mathbf{L}^\infty$ functions. On the other hand, in the case of $n \times n$ systems, the available stability results apply only to domains $\mathcal{D}$ of functions with uniformly bounded total variation.

It is thus natural to ask whether the Lipschitz continuous semigroups of solutions constructed in [8, 15, 17, 23] can be extended to domains of $\mathbf{L}^\infty$ functions with possibly unbounded variation. The present Chapter provides an answer to this question in the case of Temple class systems [40, 66]. Namely, we show that, if all characteristic fields are genuinely nonlinear, then for a wide class of initial data $\bar{u} \in \mathbf{L}^\infty$ the Cauchy problem (2.1)-(2.2) has a unique entropy weak solution, depending Lipschitz continuously on $\bar{u}$. Any two such solutions thus satisfy

$$\big\|u(t,\cdot) - v(t,\cdot)\big\|_{\mathbf{L}^1} \leq L \cdot \big\|u(0,\cdot) - v(0,\cdot)\big\|_{\mathbf{L}^1} \tag{2.3}$$

for some Lipschitz constant L independent of t. By a counterexample, we show that the assumption of genuine nonlinearity cannot be dropped. Indeed, for a particular 2×2 Temple class system with a linearly degenerate characteristic field, the $\mathbf{L}^\infty$ entropic solutions depend continuously but not Lipschitz continuously on the initial data.

The construction of the semigroup is achieved by the same technique as in [8, 15]. We consider a family of piecewise constant approximate solutions obtained by a front tracking algorithm [8, 10], which in this case represents a natural extension of [27]. The distance between two approximate solutions u, v is then estimated by constructing a path of solutions connecting u with v and keeping track of how the length of this path changes in time.

As in [8, 15], the heart of the matter is to provide a priori bounds on the norm of a shift-differential. Let a piecewise constant initial data $u(0, \cdot)$ be perturbed by shifting the location of one of its jumps. Call σ the strength of this jump and ξ its shift rate. At time $t > 0$, the corresponding solution $u(t,\cdot)$ will contain jumps, say of strength $\sigma_1, \ldots, \sigma_m$, shifting at rates $\xi_1, \ldots, \xi_m$. To prove stability, one has to establish the key estimate

$$\sum_{j=1}^{m} |\sigma_j \xi_j| \leq L \cdot |\sigma \xi| \tag{2.4}$$

with a constant L independent of t. In general, an estimate of the form (2.4) holds only within a class of solutions with uniformly small total variation. For genuinely nonlinear systems of Temple class, however, we show that (2.4) can be satisfied with a constant L independent of the total variation of u. This remarkable property is based on two special features of such systems:

1. By genuine nonlinearity and finite propagation speed, the total amount of waves in $u(t,\cdot)$ which can be influenced by shifting the wave-front σ remains uniformly bounded for all t. Indeed, call $\bar{x}$ the initial location of σ and let $\hat{\lambda}$ be an upper bound for the absolute values of all characteristic speeds. Then an infinitesimal shift of the position of σ can affect the values of $u(t,\cdot)$ only within the interval of dependency $I(t) \doteq [\bar{x} - \hat{\lambda}t,\ \bar{x}+\hat{\lambda}t]$. On the other hand, by the decay of positive waves due to genuine nonlinearity, the total amount of waves in $u(t,\cdot)$ contained in an interval $[a,b]$ can be estimated as $\mathcal{O}(1) \cdot \big[1 + (b-a)/t\big]$. In particular, the amount of waves contained in $I(t)$ is uniformly bounded.

2. For solutions of Temple class systems, the support of perturbations satisfies a special localization property. Namely, let u be a solution of (1.1) and fix an interval $[a,b]$. For $i = 1, \ldots, n$, call $t \mapsto x_i^a(t)$, $t \mapsto x_i^b(t)$ respectively the i-characteristics originating from the points a, b (for simplicity, in this informal discussion we are assuming that such characteristics are unique). Now consider a slightly perturbed solution v with $v(0,x) = u(0,x)$ for $x \notin [a,b]$. In general, at any time $t > 0$, the two solutions $u(t,\cdot)$, $v(t,\cdot)$ may have different values throughout the interval of dependency $I(t) \doteq \big[x_1^a(t),\ x_n^b(t)\big]$. However, introducing the integrated functions

$$U(t,x) \doteq \int_0^x u(t,y)\,dy, \qquad V(t,x) \doteq \int_0^x v(t,y)\,dy,$$

if $U(0,\cdot)$ and $V(0,\cdot)$ coincide outside $[a,b]$ then the solutions $u(t,\cdot)$, $v(t,\cdot)$ can be different only on a small neighborhood of the set

$$J(t) \doteq \bigcup_{i=1}^{n} \left[x_i^a(t),\ x_i^b(t) \right].$$

This property is closely related to the representation formula for a solution of the nonlinear equation $U_t + f(U_x) = 0$ in terms of the envelopes of n families of hyperplanes [66]. We shall repeatedly take advantage of this property in our calculations. Indeed, it will allow us to replace an initial data $\bar{u}$ with a new data $\bar{v}$ having uniformly bounded variation, without affecting the values of the solution $u(t, \cdot)$ in a neighborhood of a given point.

As soon as the basic estimate (2.4) is obtained, a standard argument yields the Lipschitz continuous dependence of front tracking approximations from the initial data. Since the Lipschitz constant is independent of the total variation, taking limits we obtain a semigroup of solutions defined on a domain of $\mathbf{L}^\infty$ functions. Using techniques developed in [18], we then prove that each semigroup trajectory $u(t, \cdot) = S_t\bar{u}$ provides the unique weak solution of the corresponding Cauchy problem (2.1)-(2.2) which satisfies a suitable "entropy condition" of Oleinik type [16, 18, 40].

Concerning the existence and uniqueness of solutions to Temple class systems with bounded initial data, some earlier results can be found in [30, 40, 66]. For some special 2×2 systems, the continuous dependence of solutions within a domain of $\mathbf{L}^\infty$ initial data has been analyzed in [9, 24].

The Chapter is organized as follows. Section 2.2 contains basic definitions and the statement of the main results. In Section 2.3 we describe the construction of front tracking approximate solutions, while Sections 2.4 and 2.5 contain the basic a priori estimates. A proof of the main theorems is given in Section 2.6, while the last section contains a simple example showing that the Lipschitz continuous dependence may fail if one of the characteristic fields is linearly degenerate.

2.2 Statement of the main results

Let (2.1) be a strictly hyperbolic system of conservation laws, where $f : \Omega \mapsto \mathbb{R}^n$ is a smooth vector field defined on some open set $\Omega \subseteq \mathbb{R}^n$. Call $A(u) \doteq Df(u)$ the Jacobian matrix of f and denote by $\lambda_1(u) < \cdots < \lambda_n(u)$ its eigenvalues. The right and left eigenvectors of $A(u)$ will be written as $r_i(u)$, $l_i(u)$ respectively, and normalized so that

$$\big|r_i(u)\big| = 1, \qquad\qquad l_i(u)\cdot r_j(u) = \begin{cases} 1 & \text{if} \quad i=j, \\ 0 & \text{if} \quad i\neq j. \end{cases} \tag{2.5}$$

We recall that the i-th field is *genuinely nonlinear* if, by a suitable orientation of the eigenvectors r_i, at every point $u\in\Omega$ one has $D\lambda_i\cdot r_i > 0$.

The system (2.1) is of *Temple class* if there exists a system of coordinates $w = (w_1,\ldots,w_n)$ consisting of Riemann invariants, and such that the level sets $\big\{u\in\Omega;\ w_i(u) = \text{constant}\big\}$ are hyperplanes [66]. For a Temple class system, the integral curve of the vector field r_i through a point u_0 is the straight line described by the $n-1$ equations

$$w_j(u) = w_j(u_0) \qquad\qquad j\neq i.$$

In particular, shock and rarefaction curves coincide [68].

We now consider a convex, compact set $E\subset\Omega$ having the form

$$E = \Big\{u\in\Omega;\quad w_i(u)\in[a_i,\ b_i] \qquad i=1,\ldots,n\Big\}, \tag{2.6}$$

and assume that, as u varies in E, a strengthened version of the strict hyperbolicity condition holds, namely

(SH) For any given $u_1,\ldots,u_n\in E$, the characteristic speeds at these points satisfy $\lambda_1(u_1) < \cdots < \lambda_n(u_n)$. Moreover, the eigenvectors $r_1(u_1),\ldots,r_n(u_n)$ are linearly independent.

Observe that the above assumption is certainly satisfied if the system is strictly hyperbolic and E is contained in a small neighborhood of a given point.

By a translation of coordinates, it is not restrictive to assume that $0\in E$ and $(w_1,\ldots,w_n)(0) = (0,\ldots,0)$. We now consider a positively invariant domain of $\mathbf{L}^\infty$ functions, with possibly unbounded variation:

$$\mathcal{D} \doteq \big\{u : I\!\!R \mapsto E;\quad u\in\mathbf{L}^1\big\}. \tag{2.7}$$

Our main result is concerned with the existence of a semigroup generated by the system (2.1) on the domain $\mathcal{D}$. Throughout the following, $w_i(t,x) \doteq w_i\big(u(t,x)\big)$ denotes the i-th Riemann coordinate of u.

Theorem 2.2.1 *Let the system (2.1) be of Temple class, let it satisfy the strict hyperbolicity condition* **(SH)** *and assume that all characteristic fields are genuinely nonlinear.*

Then there exist constants $L, \kappa > 0$ and a continuous semigroup $S : [0, +\infty[\times \mathcal{D} \to \mathcal{D}$ such that

$$\|S_t\bar{u} - S_t\bar{v}\|_{\mathbf{L}^1} \leq L \cdot \|\bar{u} - \bar{v}\|_{\mathbf{L}^1} \qquad \forall \bar{u}, \bar{v} \in \mathcal{D},\ t \geq 0. \tag{2.8}$$

Moreover, every trajectory $t \mapsto u(t, \cdot) = S_t\bar{u}$ is a weak solution of (2.1) and satisfies the entropy conditions

$$w_i(t, y) - w_i(t, x) \leq \frac{y - x}{\kappa t} \qquad \text{for all } \ x < y, \quad t > 0,\ i \in \{1, \ldots, n\}. \tag{2.9}$$

Our second result is concerned with the uniqueness of the semigroup. It shows that the Oleinik type estimates (2.9) on the decay of positive waves completely characterize the trajectories of the semigroup.

Theorem 2.2.2 *Let the assumptions of Theorem 2.2.1 hold. Let $u : [0, T] \mapsto \mathcal{D}$ be continuous as a map with values in $\mathbf{L}^1$, and provide a weak solution to the Cauchy problem (2.1)-(2.2). If the entropy conditions (2.9) hold, then u coincides with the corresponding semigroup trajectory, namely*

$$u(t, \cdot) = S_t\bar{u} \qquad \forall\ t \geq 0\,. \tag{2.10}$$

Remark 2.2.3 For any fixed M, the existence of a Lipschitz semigroup on the domain

$$\mathcal{D}_M \doteq \left\{ u : I\!R \mapsto E; \quad u \in \mathbf{L}^1, \quad \sum_i \text{Tot.Var.}\{w_i(u)\} \leq M \right\}. \tag{2.11}$$

was proved in [8]. For such solutions, the decay estimates (2.9) can be proved as in [16], while the argument in [18] would show that (2.9) implies (2.10). The key point in the present theorems is that the Lipschitz constant L in (2.8) and the constant κ in (2.8) are both independent of M. Hence the results remain valid for arbitrary solutions within the larger domain $\mathcal{D}$.

2.3 Front tracking approximations

We begin the proof of Theorem 2.2.1 by describing a front tracking algorithm [8, 10] which constructs piecewise constant approximate solutions of (2.1), continuously depending on the initial data.

Fix an integer $\nu \geq 1$ and consider the discrete set of points in E whose coordinates are integer multiples of $2^{-\nu}$:

$$E^\nu \doteq \{u \in E; \qquad w_i(u) \in 2^{-\nu}\mathbb{Z},\ i = 1, \ldots, n\}.$$

Moreover, consider the domain

$$\mathcal{D}^\nu \doteq \{u : \mathbb{R} \mapsto E^\nu; \quad u \in \mathbf{L}^1, \qquad u \text{ is piecewise constant}\}. \tag{2.12}$$

On $\mathcal{D}^\nu$ we now construct a semigroup S^ν whose trajectories are front tracking approximate solutions of (2.1). To this end, we first describe how to solve a Riemann problem with data $u^-, u^+ \in E^\nu$. In Riemann coordinates, assume that

$$u^- = (w_1^-, \ldots, w_n^-) \qquad u^+ = (w_1^+, \ldots, w_n^+).$$

Consider the intermediate states

$$\omega_0 = u^-, \quad \ldots, \quad \omega_i = (w_1^+, \ldots, w_i^+, w_{i+1}^-, \ldots, w_n^-), \quad \ldots, \quad \omega_n = u^+. \tag{2.13}$$

If $w_i^+ < w_i^-$, the solution will contain a single i-shock, connecting the states ω_{i-1}, ω_i and travelling with Rankine-Hugoniot speed $\lambda_i(\omega_{i-1}, \omega_i)$. Here and in the sequel, by $\lambda_i(u, u')$ we denote the i-th eigenvalue of the averaged matrix

$$A(u, u') \doteq \int_0^1 A\big(\theta u + (1-\theta)u'\big)\, d\theta. \tag{2.14}$$

If $w_i^+ > w_i^-$, the exact solution of the Riemann problem would contain a centered rarefaction wave. This is approximated by a rarefaction fan as follows. If $w_i^+ = w_i^- + p_i 2^{-\nu}$ we insert the states

$$\omega_{i,\ell} = (w_1^+, \ldots, w_i^- + 2^{-\nu}\ell, w_{i+1}^-, \ldots, w_n^-) \qquad\qquad \ell = 0, \ldots, p_i, \tag{2.15}$$

so that $\omega_{i,0} = \omega_{i-1},\ \omega_{i,p_i} = \omega_i$. Our front tracking solution will then contain p_i fronts of the i-th family, each connecting a couple of states $\omega_{i,\ell-1},\ \omega_{i,\ell}$ and travelling with speed $\lambda_i\big(\omega_{i,\ell-1},\ \omega_{i,\ell}\big)$.

For a given initial data $\bar u \in \mathcal{D}^\nu$, the approximate solution $u(t,\cdot) = S_t^\nu \bar u$ is now constructed as follows. At time $t = 0$ we solve each of the Riemann problems determined by the jumps in $\bar u$ according to the above procedure. This yields a piecewise constant function with finitely many fronts, travelling with constant speeds. The solution is then prolonged up to the first time where two or more of these fronts interact. At the interaction points, the new Riemann problems are again solved by the above procedure, etc...

As in [8], one checks that these front tracking approximations are well defined for all times $t \geq 0$. Indeed, the following properties hold.

- For each $i = 1, \ldots, n$, the map $t \mapsto \text{Tot.Var.}\{w_i(t,\cdot)\}$ is non-increasing.
- The total number of wave-fronts in $u(t,\cdot)$ does not increase in time.
- Each trajectory $t \mapsto u(t,\cdot) = S_t^\nu \bar{u}$ is a weak solution of (2.1) (because all fronts satisfy the Rankine-Hugoniot conditions), but may not be entropy-admissible (because of the presence of rarefaction fronts).

As $\nu \to \infty$, the domains $\mathcal{D}^\nu$ become dense in $\mathcal{D}$. We will thus define the semigroup S on $\mathcal{D}$ as a suitable limit of the flows S^ν. To ensure the existence of this limit, the key step is to prove the estimate

$$\lim_{\nu\to\infty} \left\| S_t^\nu \bar{u} - S_t^\nu \bar{v} \right\|_{\mathbf{L}^1} \leq L \cdot \|\bar{u} - \bar{v}\|_{\mathbf{L}^1} \tag{2.16}$$

for some constant L and any $\bar{u}, \bar{v} \in \mathcal{D}^\mu$, $\mu \geq 1$. For this purpose, given any two initial data $\bar{u}, \bar{v}$, we consider a continuous path $\gamma_0 : \theta \mapsto \bar{u}^\theta$, with $\gamma_0(0) = \bar{u}$, $\gamma_0(1) = \bar{v}$. More precisely, our path will be a *pseudopolygonal*, i. e. a finite concatenation of *elementary paths*, of the form

$$\theta \mapsto u^\theta := \sum_{\alpha=1}^{N} \omega_\alpha \cdot \chi_{]x^\theta_{\alpha-1},\, x^\theta_\alpha]}, \qquad x_\alpha^\theta = x_\alpha + \xi_\alpha \theta, \qquad \theta \in [a,b] \tag{2.17}$$

where χ_I is the characteristic function of the set I, $\omega_0, \ldots, \omega_N \in I\!R^n$ are constant states and ξ_α is the shift rate of the jump at x_α. In (2.17) it is assumed that $x_1^\theta < \ldots < x_N^\theta$ for $a < \theta < b$. If γ is an elementary path of the form (3.6), its $\mathbf{L}^1$ length is computed by

$$\|\gamma\|_{\mathbf{L}^1} = \int_a^b \sum_{\alpha=1}^{N} \left|\Delta u(x_\alpha)\right| \left|\frac{\partial x_\alpha^\theta}{\partial \theta}\right| d\theta = \sum_{\alpha=1}^{N} \left|\sigma_\alpha\right| \left|\xi_\alpha\right| (b-a), \tag{2.18}$$

where

$$\sigma_\alpha \doteq \Delta u(x_\alpha) = \omega_{\alpha+1} - \omega_\alpha. \tag{2.19}$$

Let $u^\theta(t,\cdot) = S_t^\nu \bar{u}^\theta$ be the corresponding solutions. Since the number of wave-fronts in these solutions is a-priori bounded and the locations of the interaction points in the

t-x plane are determined by a linear system of equations, it is clear that, at any time $\tau > 0$, the corresponding path $\gamma_\tau : \theta \mapsto u^\theta(\tau, \cdot)$ is still a pseudopolygonal. Moreover, there exist finitely many parameter values $0 = \theta_0 < \theta_1 < \ldots < \theta_m = 1$ such that the wave-front configuration of u^θ remains the same as θ ranges on each of the open intervals $I_j \doteq]\theta_{j-1}, \theta_j[$. In this case, the lengths of the paths γ_0 and γ_τ are measured by an expression of the form

$$\|\gamma\|_{\mathbf{L}^1} = \sum_{j=1}^{m} \int_{\theta_{j-1}}^{\theta_j} \sum_\alpha \left|\Delta u^\theta(x_\alpha^\theta)\right| \left|\frac{\partial x_\alpha^\theta}{\partial \theta}\right| d\theta. \tag{2.20}$$

By deriving a priori estimates on the integrand in (2.20), we will prove that $\|\gamma_\tau\|_{\mathbf{L}^1} \leq L\|\gamma_0\|_{\mathbf{L}^1}$, for some constant L independent of total variation.

2.4 Estimates on front tracking solutions

Throughout this section we fix $\nu \geq 1$ and consider a piecewise constant solution u constructed by the front tracking algorithm, so that $u(t, \cdot) = S_t^\nu \bar{u}$ for some $\bar{u} \in \mathcal{D}^\nu$. We then perturb this solution, shifting the locations x_α of the jumps at rates ξ_α. In other words, for θ suitably close to zero, the perturbation $u^\theta(t, \cdot)$ will be a function with jumps at the points $x_\alpha^\theta = x_\alpha + \theta\xi_\alpha$. As long as the wave-front configuration of the functions u, u^θ is the same, the shifts $\xi_\beta(t)$ are uniquely determined as linear functions of the shifts $\xi_\alpha(0)$. Some properties of these shift differentials are investigated in the next lemmas.

Lemma 2.4.1 *Consider a bounded, open region Γ in the t-x plane. Call σ_α, $\alpha = 1, \ldots, N$ the fronts entering Γ and let ξ_α be their shifts. Assume that the fronts leaving Γ, say σ_β, $\beta = 1, \ldots, N'$, are linearly independent. Then the shifts ξ_β are uniquely determined by the linear relation*

$$\sum_{\alpha=1}^{N} \xi_\alpha \sigma_\alpha = \sum_{\beta=1}^{N'} \xi_\beta \sigma_\beta\,. \tag{2.21}$$

Proof of Lemma 2.4.1. Let u, u^θ be the original and the perturbed solution. By possibly modifying the region Γ, we can assume that its boundary consists only of horizontal or vertical segments, and that all fronts of u cross the boundary $\partial\Gamma$ within segments parallel to the x-axis (fig. 2.1). We now integrate the conservation equation over the region Γ, using the divergence theorem. Observing that $u = u^\theta$ along all

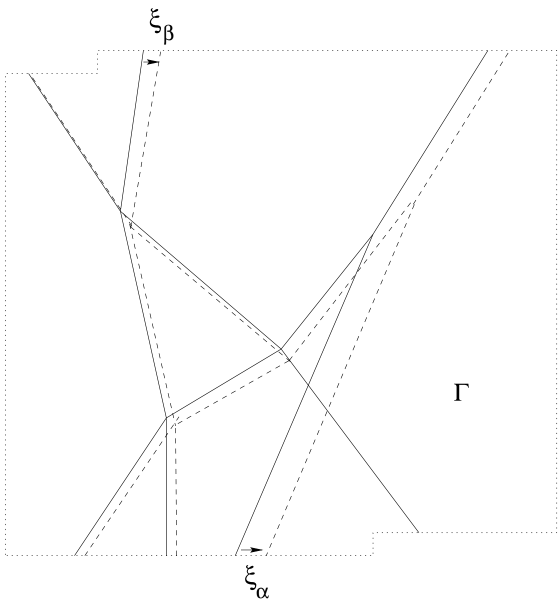

figure 2.1

vertical segments of the boundary and calling $\partial^-\Gamma$, $\partial^+\Gamma$ respectively the portions of the boundary containing incoming and outgoing fronts, we have

$$\begin{aligned} 0 &= \frac{1}{\theta}\int_\Gamma [u^\theta - u]_t + \big[f(u^\theta) - f(u)\big]_x \, dxdt \\ &= \frac{1}{\theta}\int_{\partial^+\Gamma} [u^\theta(t,x) - u(t,x)]dx - \frac{1}{\theta}\int_{\partial^-\Gamma} [u^\theta(t,x) - u(t,x)]dx \\ &= -\sum_{\beta=1}^{N'} \xi_\beta \sigma_\beta + \sum_{\alpha=1}^{N} \xi_\alpha \sigma_\alpha . \end{aligned}$$

Since the vectors $\sigma_1, \ldots, \sigma_{N'}$ are linearly independent, the coefficients ξ_β are uniquely determined by (2.21). □

Remark 2.4.2 According to Lemma 2.4.1, the shift rates of the outgoing fronts depend only on the shift rates of the incoming ones, and not on the order in which these wave-fronts interact inside Γ. More precisely, one can perform the following two operations, without changing the shift rates of the outgoing fronts:

(O1) Switch the order in which three fronts interact (fig. 2.2, fig. 2.3).

(O2) Invert the order of two fronts at time $t = 0$, provided that both fronts have zero shift rate (fig. 2.4).

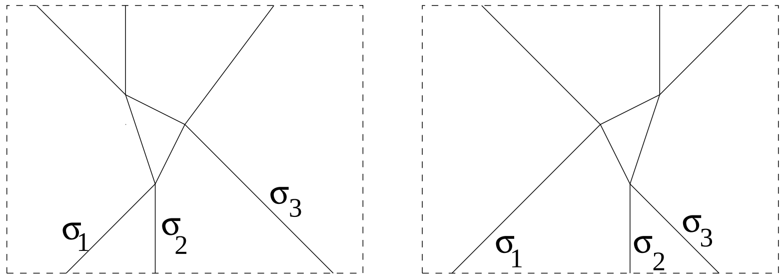

figure 2.2

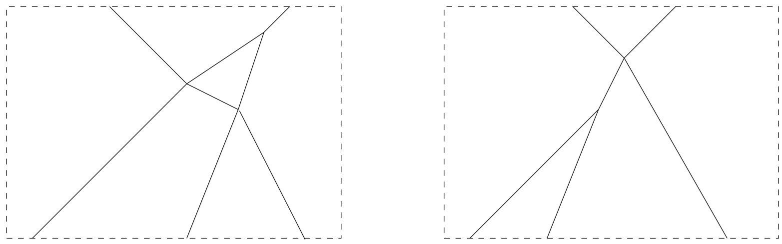

figure 2.3

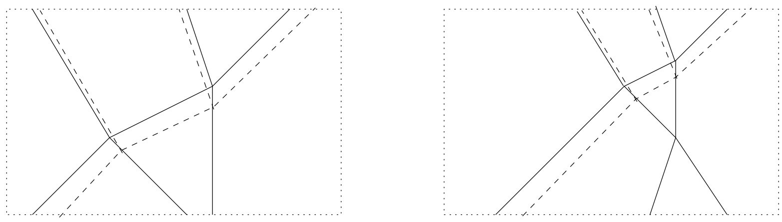

figure 2.4

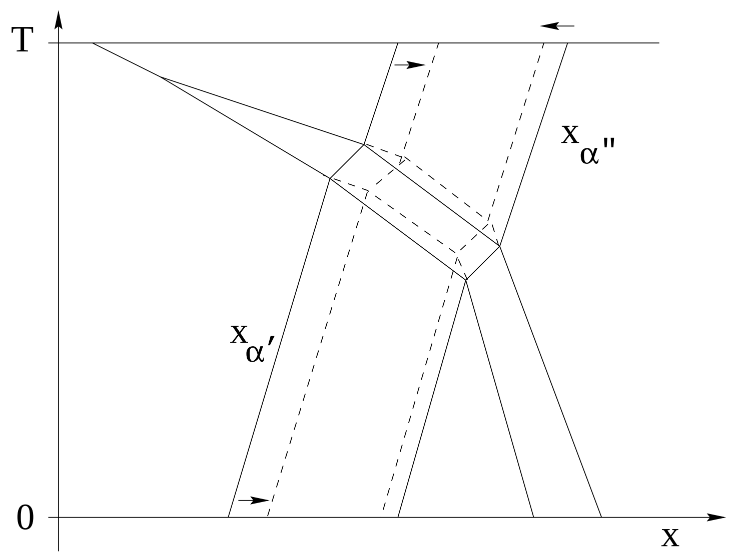

figure 2.5

This property will be repeatedly used in our future estimates. Indeed, in the calculation of a shift rate, we can suitably alter the order of wave interactions and thus reduce the problem to a case where the wave-front configuration is particularly simple.

Lemma 2.4.3 *Assume that a front tracking solution u contains two wave-fronts of the same k-th family, originating at distinct points and located at $x_{\alpha'}(t) \leq x_{\alpha''}(t)$, $t \in [0,T]$. Then it is possible to assign shift rates $\xi_\alpha(0)$ to all fronts in $u(0,\cdot)$ so that $\xi_{\alpha'}(0) = 1$ and moreover, in the corresponding solution u, all fronts $x_\beta(t)$ outside the strip $\Gamma \doteq \big\{(t,x);\ \ t \in [0,T],\ x_{\alpha'}(t) \leq x \leq x_{\alpha''}(t)\big\}$ have zero shift rate.*

In other words (fig. 2.5), the perturbation of the initial data can be chosen so that one particular front shifts at unit rate, but the corresponding solution remains unaffected outside the region Γ.

PROOF OF LEMMA 2.4.3. The shift rates $\xi_\alpha(0)$ will be chosen so that, for all t, x,

$$-\frac{d}{d\theta}\int_{-\infty}^{x} u^\theta(t,y)\ dy = \sum_{x_\alpha(t)<x} \xi_\alpha(t)\sigma_\alpha(t) = c(t,x)r_k\big(u(t,x)\big)\,, \tag{2.22}$$

where $c = c(t,x)$ is a scalar function identically equal to zero outside Γ. To achieve (2.22), we proceed by induction. We denote by $x_1(0) < \cdots < x_N(0)$ the locations of the jumps in $u(0,\cdot)$. Moreover, we consider the intermediate states and the jumps

$$u_\alpha \doteq u(0,x) \qquad x \in]x_\alpha,\ x_{\alpha+1}[\,, \qquad\qquad \sigma_\alpha \doteq u_\alpha - u_{\alpha-1}.$$

The initial shift rates are defined by setting

$$\xi_\alpha = 0 \qquad \text{if } \ 1 \le \alpha < \alpha', \qquad \xi_{\alpha'} = 1.$$

If $\alpha' < \alpha \le \alpha''$, assume by induction that $\xi_{\alpha-1}$ has already been defined, so that

$$\sum_{x_\beta < x_\alpha} \sigma_\beta \xi_\beta = c_{\alpha-1} r_k(u_{\alpha-1}). \tag{2.23}$$

for some constant $c_{\alpha-1}$. Two cases will be considered.

CASE 1. If the jump at x_α belongs to the k-th family, then the vectors $\sigma_\alpha, r_k(u_{\alpha-1}), r_k(u_\alpha)$ are all parallel, and any choice of the shift ξ_α will automatically satisfy

$$\sum_{x_\beta \le x_\alpha} \sigma_\beta \xi_\beta = c_\alpha r_k(u_\alpha). \tag{2.24}$$

CASE 2. If the jump at x_α belongs to the j-th family with $j \neq k$, since the system is of Temple class we have

$$\text{span}\,\big\{\sigma_\alpha,\ r_k(u_\alpha)\big\} = \text{span}\,\big\{\sigma_\alpha,\ r_k(u_{\alpha-1})\big\}.$$

Hence there exists a unique shift rate ξ_α and a constant c_α such that

$$c_\alpha r_k(u_\alpha) = \xi_\alpha \sigma_\alpha + c_{\alpha-1} r_k(u_{\alpha-1}). \tag{2.25}$$

Observe that, when $\alpha = \alpha''$, since the front belongs to the k-th family, we can choose the shift rate $\xi_{\alpha''}$ so that

$$\sum_{x_\beta \le x_{\alpha''}} \sigma_\beta \xi_\beta = 0. \tag{2.26}$$

We then define $\xi_\alpha = 0$ for all $\alpha > \alpha''$. This achieves (2.22) for all x, at time $t = 0$.

We claim that (2.22) remains valid at all later times. For this purpose, it is enough to study an arbitrary interaction between two wave-fronts, say occurring at a point $(\bar t, \bar x)$. We consider the case of two incoming fronts (fig. 2.6) of families $k_\alpha \neq k_\beta$, with $k_\alpha, k_\beta \neq k$. The other cases are similar, or easier. Call u_l, u_m, u_r respectively the left, middle and right states before interaction, and let u_m' be the middle state after interaction. We denote by $\sigma_\alpha, \sigma_\beta$ and ξ_α, ξ_β the jumps and shift rates of the incoming fronts, while $\sigma_\alpha', \sigma_\beta', \xi_\alpha', \xi_\beta'$ refer to the outgoing fronts. The inductive assumptions imply

$$\sum_{x_\gamma(\bar t) < \bar x} \sigma_\gamma(\bar t) \xi_\gamma(\bar t) = c_l r_k(u_l) \tag{2.27}$$

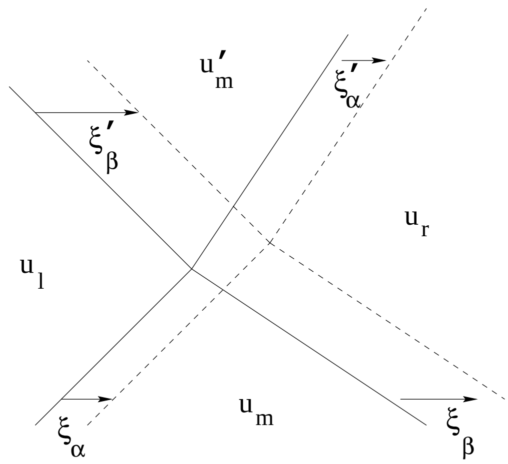

figure 2.6

for some constant c_l. Moreover

$$c_l r_k(u_l) + \xi_\alpha \sigma_\alpha = c_m r_k(u_m), \tag{2.28}$$

$$c_m r_k(u_m) + \xi_\beta \sigma_\beta = c_r r_k(u_r), \tag{2.29}$$

while the conservation equations yield

$$\xi'_\alpha \sigma'_\alpha + \xi'_\beta \sigma'_\beta = \xi_\alpha \sigma_\alpha + \xi_\beta \sigma_\beta. \tag{2.30}$$

From (2.27)–(2.30) it is clear that

$$\begin{aligned} \sum_{x_\gamma(\bar t)<\bar x} \sigma_\gamma(\bar t)\xi_\gamma(\bar t) + \xi'_\alpha \sigma'_\alpha + \xi'_\beta \sigma'_\beta &= c_l r_k(u_l) + \xi_\alpha \sigma_\alpha + \xi_\beta \sigma_\beta \\ &= c_r r_k(u_r). \end{aligned}$$

It remains to show that, for some constant c'_m, one has

$$c_l r_k(u_l) + \xi'_\beta \sigma'_\beta = c'_m r_k(u'_m). \tag{2.31}$$

This can be easily seen by writing

$$c_l r_k(u_l) + \xi'_\beta \sigma'_\beta = c_r r_k(u_r) - \xi'_\alpha \sigma'_\alpha \tag{2.32}$$

and observing that the vector on the left hand side belongs to span$\{r_k(u_r),\ r_{k_\beta}(u'_m)\}$, the vector on the right hand side lies in the plane span$\{r_k(u_l),\ r_{k_\alpha}(u'_m)\}$, and the intersection of these planes is the straight line spanned by $r_k(u'_m)$.

To check that all shifts vanish outside Γ, consider the case where the families of the incoming fronts are $k_\alpha \neq k$, $k_\beta = k$ and the front $t \mapsto x_\beta(t)$ marks the boundary of the region Γ. In this case, the inductive assumption implies $c_r = 0$. From the relation

$$c'_m r_k(u'_m) + \xi'_\alpha \sigma'_\alpha = c_r r_k(u_r) = 0,$$

by the linear independence of the vectors $r_k(u'_m)$, σ'_α it follows $\xi'_\alpha = 0$ as required. This completes the proof of Lemma 2.4.3. □

Lemma 2.4.4 *Let u be a front-tracking solution of (2.1), and consider two wave-fronts, say $t \mapsto x(t)$, $t \mapsto y(t)$, defined for $t \in [0,T]$. Then there exists a second front tracking solution $\tilde{u}$ with two fronts $\tilde{x}$, $\tilde{y}$ such that the following holds.*

(i) $\tilde{x}(0) = x(0)$, $\tilde{y}(0) = y(0)$, $\tilde{x}(T) = x(T)$, $\tilde{y}(T) = y(T)$.

(ii) $\tilde{u} = u$ *in a neighborhood of the points* $\big(0, x(0)\big)$, $\big(0, y(0)\big)$, $\big(T, x(T)\big)$, $\big(T, y(T)\big)$.

(iii) Tot. Var.$\big\{\tilde{u}(0,\cdot)\big\} \leq C_0$, *for some constant C_0 depending only on the system (2.1) and on the set E.*

Proof of Lemma 2.4.4. To fix the ideas, assume $x(0) < y(0)$. The set $\mathbb{R}\setminus\big\{x(0),\ y(0)\big\}$ consists of three connected components which we call J_1, J_2, J_3. Similarly, the set $\mathbb{R} \setminus \big\{x(T),\ y(T)\big\}$ consists of three components J'_1, J'_2, J'_3. Assume that u contains two wave-fronts $t \mapsto z'(t)$, $t \mapsto z''(t)$, of the same k-th family, such that $z'(0)$ and $z''(0)$ lie in the same open set J_i and moreover $z'(T)$ and $z''(T)$ lie in the same J'_j. We now apply Lemma 2.4.3, with $x_{\alpha'} = z'$, $x_{\alpha''} = z''$. We claim that, if the k-waves at $z'(0)$ and $z''(0)$ have the same sign, we can choose the initial shift rates $\xi_\alpha(0)$ satisfying the additional conditions

(i) $\xi_{\alpha'}(0) = 1$, $\xi_{\alpha'}(t) \geq 0$ for all $t \in [0,T]$,

(ii) $\xi_\alpha(0) = 0$ if $\alpha' < \alpha < \alpha''$ and the front at x_α is of the k-th family,

(iii) $\xi_{\alpha''}(t) \leq 0$ for all $t \in [0,T]$.

To fix the ideas, let both waves be positive. Since the choice of $\xi_\alpha(0)$ is arbitrary for waves of the k-th family, it is clear that (ii) can be satisfied. To prove (i) and (iii), we first show that the constants $c(t,x)$ vanish outside the strip Γ bounded by z', z'' and are positive inside Γ. At time $t = 0$, by construction we have $c_{\alpha'} > 0$. Moreover, if

$c_{\alpha-1} > 0$ for some $\alpha' < \alpha < \alpha''$, then also $c_\alpha > 0$. This follows from (2.25) and the linear independence assumption **(SH)**. Similarly, at any interaction between two fronts in the interior of the strip Γ, from (2.31)-(2.32) we deduce that, if $c_l, c_m, c_r > 0$, then also $c'_m > 0$. By induction on all finitely many interaction points, we conclude $c > 0$ inside Γ.

Calling $\xi'(t)$, $\sigma'(t)$ respectively the shift rate of the jump at $z'(t)$ and its amplitude, from the relation

$$\xi'(t)\sigma'(t) = c(t, z'(t)+)r_k(u(t, z'(t)+)),$$

since σ' and r_k are parallel and have the same sign, recalling that $c > 0$ inside Γ we conclude $\xi'(t) > 0$. A similar argument yields that the shift of the front z'' satisfies $\xi''(t) < 0$.

By (i) and (iii), we can thus construct a one-parameter family of solutions u^θ shifting forward the front at $z'(\cdot)$ and shifting backwards the front at $z''(\cdot)$. The parameter θ can be raised up to a value $\bar{\theta}$ where two k-fronts in $u^\theta(0, \cdot)$ coincide. We thus obtain a second solution $u^{\bar{\theta}}$ which coincides with u outside the strip Γ bounded by the fronts $z'(\cdot)$ and $z''(\cdot)$, but with a smaller number of k-fronts. This construction can be repeated as long as the solution contains fronts of the same family and with the same sign, starting at different points within the same set J_i and ending within the same J'_j. In a finite number of steps, we obtain a new solution $\tilde{u}$ with the property that, for each $k = 1, \ldots, n$ and $i, j = 1, 2, 3$, there exists at most one point $\bar{x} \in J_i$ where a positive k-wave originates, terminating within J'_j, and similarly for negative k-waves. Of course, this implies that the total variation of $\tilde{u}$ is uniformly bounded, with a bound C_0 depending only on n and on the diameter of the set E. □

Next we seek an estimate on the amount of positive k-waves in a front tracking solution u at time $\tau > 0$, contained in a bounded interval $[a,\ b]$. As a preliminary, we prove

Lemma 2.4.5 *For a fixed $\nu \geq 1$, let $u(t, \cdot) = S_t^\nu \bar{u}$ be a front tracking solution of the system (2.1). For $t \in [0, T]$, let $x(t) \leq y(t)$ be the positions of two adjacent k-rarefaction fronts. Then for some constant $\kappa > 0$ one has*

$$y(\tau) - x(\tau) \geq \kappa\tau \cdot 2^{-\nu}. \tag{2.33}$$

PROOF OF LEMMA 2.4.5. By Lemma 2.4.4, by possibly replacing u by another solution $\tilde{u}$, it is not restrictive to assume that $\text{Tot.Var.}\{u(0,\cdot)\} \leq C_0$. Observe that, by construction, each k-rarefaction front has strength $\Delta w_k = 2^{-\nu}$. Let $k_\alpha \in \{1,\ldots,n\}$ be the family of the jump at x_α. Call $\sigma_\alpha \doteq \Delta u(t,x_\alpha) = u(t,x_\alpha+) - u(t,x_\alpha-)$ the jump at x_α and let $\|\sigma_\alpha\| \doteq |\Delta w_{k_\alpha}|$ the strength of the jump, measured in Riemann coordinates. Defining $m(t) \doteq y(t) - x(t)$, by the genuine nonlinearity of the k-th characteristic field we have an estimate of the form

$$\dot{m}(t) \geq c \cdot 2^{-\nu} - C_1 \cdot \sum_{x(t)<x_\alpha(t)<y(t)} \|\sigma_\alpha(t)\| \tag{2.34}$$

for some constants $C_1, c > 0$. As usual, we denote by σ_α the value of the jump at x_α, occurring in the k_α-th family. To estimate the contribution of the last term in (2.34), following [16] we introduce the function

$$\Phi(t) \doteq \sum_{k_\alpha \neq k} \phi_{k_\alpha}\big(t, x_\alpha(t)\big) \cdot \|\sigma_\alpha(t)\|$$

where

$$\phi_i(t,x) \doteq \begin{cases} 1 & \text{if } x < x(t) \\ \frac{y(t)-x}{m(t)} & \text{if } x \in [x(t),\ y(t)] \\ 0 & \text{if } x > y(t) \end{cases}$$

or

$$\phi_i(t,x) \doteq \begin{cases} 0 & \text{if } x < x(t) \\ \frac{x-x(t)}{m(t)} & \text{if } x \in [x(t),\ y(t)] \\ 1 & \text{if } x > y(t) \end{cases}$$

in the cases $i > k$ or $i < k$ respectively. Observe that the map $t \mapsto \Phi(t)$ is piecewise Lipschitz continuous, possibly with a finite number of downward jumps, occurring at interaction times. Because of the strict separation of the characteristic speeds, outside interactions we have

$$\begin{aligned} \dot{\Phi}(t) &= \sum_{k_\alpha \neq k} \|\sigma_\alpha\| \cdot \frac{d}{dt}\phi_{k_\alpha}\big(t, x_\alpha(t)\big) \\ &\leq - \sum_{x(t)<x_\alpha(t)<y(t)} \|\sigma_\alpha(t)\| \cdot \frac{C_2}{m(t)} \end{aligned} \tag{2.35}$$

for some constant $C_2 > 0$. Together, (2.34)-(2.35) yield

$$\dot{m}(t) - \frac{C_1}{C_2} \cdot \dot{\Phi}_i(t) m(t) \geq c \cdot 2^{-\nu}, \tag{2.36}$$

$$m(\tau) \geq e^{-(C_1/C_2)\Phi(0)} \cdot c\tau \cdot 2^{-\nu}. \tag{2.37}$$

The uniform bound on the total variation implies $\Phi(0) \leq C_0'$ for some constant C_0'. Hence the lemma holds with $\kappa = ce^{-C_0' C_1/C_2}$. □

Lemma 2.4.6 *Consider a front tracking solution $u(t,\cdot) = S_t^\nu \bar{u}$, with $\bar{u}$ containing N shock fronts of the k-th family. Then, for each $\tau > 0$ and every bounded interval $[a,\ b]$, one has*

$$Tot.Var.\{w_k(\tau,\cdot);\ [a,\ b]\} \leq \frac{2(b-a)}{\kappa\tau} + \|w_k\|_{L^\infty} + (N+1)2^{1-\nu}. \tag{2.38}$$

PROOF OF LEMMA 2.4.6. By Lemma 2.4.5, at time $\tau > 0$ any two adiacent k-rarefaction fronts of u are separated by a distance $\geq \kappa\tau \cdot 2^{-\nu}$. Therefore, the number of rarefaction fronts inside any interval $[a,b]$ is bounded by

$$1 + N + \frac{b-a}{\kappa\tau \cdot 2^{-\nu}}.$$

The positive variation of $w_k(\tau,\cdot)$ on $[a,b]$, i. e. the total amount of upward jumps, thus satisfies

$$\text{Pos.Var.}\{w_k(\tau,\cdot);\ [a,\ b]\} \leq (1+N)2^{-\nu} + \frac{b-a}{\kappa\tau}. \tag{2.39}$$

In turn, the total variation of $w_k(\tau,\cdot)$ on $[a,b]$ is bounded by $\|w_k\|_{\mathbf{L}^\infty}$ plus twice the positive variation of w_k. Hence (2.38) holds. □

2.5 Estimates on shift differentials

This section is devoted to the proof of the key estimate on shift differentials.

Lemma 2.5.1 *Let $u(t,\cdot) = S_t^\nu \bar{u}$ be a front tracking solution, with $\bar{u}$ containing N shocks. Assume that at time $t = 0$ one single front is shifted, say of the k-th family, located at $\bar{x}$ with shift rate $\bar{\xi}$ and amplitude $\bar{\sigma}$. Calling $\xi_\alpha(\tau)$, $\sigma_\alpha(\tau)$ the shift rates and the amplitudes of the fronts in $u(\tau,\cdot)$, for some constant C_3 depending only on the system (1.1) and on the domain E we then have*

$$\sum_\alpha \left|\xi_\alpha(\tau)\sigma_\alpha(\tau)\right| \leq C_3(1 + N2^{-\nu})|\bar{\xi}\bar{\sigma}|. \tag{2.40}$$

PROOF OF LEMMA 2.5.1. Consider one particular front, say x_{α^*}, of the j-th family, and call $\bar{y} \doteq x_{\alpha^*}(\tau)$ its terminal point. In the first part of the proof we shall establish the existence of constants C_4, C_5, depending only on the system (2.1) and on the domain E, such that the following properties hold.

(P1) If x_{α^*} is precisely the k-front starting at $\bar{x}$, then

$$\left|\xi_{\alpha^*}(\tau)\sigma_{\alpha^*}(\tau)\right| \leq C_5|\bar{\xi}\bar{\sigma}|. \tag{2.41}$$

(P2) If x_{α^*} is a j-front, with $j \neq k$, and the backward j-characteristics ending at $\bar{y}$ include fronts starting from both sides of $\bar{x}$, then (2.41) again holds.

(P3) If x_{α^*} is a j-front, and the j-fronts ending at $\bar{y}$ start either all at the left or all at the right of $\bar{x}$, one then has the sharper estimate

$$\left|\xi_{\alpha^*}(\tau)\right| \leq C_4|\bar{\xi}\bar{\sigma}|. \tag{2.42}$$

Toward a proof of **(P1)**–**(P3)** we observe that, besides the fronts starting at $\bar{x}$ and the ones ending at $\bar{y}$, one can single out four groups of waves:

(1) the waves starting on the left of $\bar{x}$ and ending on the left of $\bar{y}$;

(2) the waves starting on the right of $\bar{x}$ and ending on the right of $\bar{y}$;

(3) the waves starting on the right of $\bar{x}$ and ending on the left of $\bar{y}$;

(4) the waves starting on the left of $\bar{x}$ and ending on the right of $\bar{y}$.

According to Remark 2.4.2, in our computation of the shift rate $\xi_{\alpha^*}(\tau)$ of the front reaching $\bar{y}$, we can assume that the sets of waves in (1) and (2) are empty. Indeed, we can otherwise shift the locations of all these fronts of type (1) toward the left, until they all lie outside the domain of influence of the initial point $\bar{x}$. Similarly, fronts of type (2) can be shifted toward the right until they lie completely outside this domain of influence. Having achieved this simplification, consider the situation described in **(P1)**. We can construct a region of the form

$$\Gamma \doteq \{(t,x)\,;\quad t \in [0,\tau],\quad \gamma^-(t) < x < \gamma^+(t)\}$$

as in fig. 2.7, choosing Γ so that all k-fronts starting within $[\gamma^-(0),\ \gamma^+(0)]$ join together into the single k-front at $\bar{y}$. Again by Remark 2.4.2, we can assume that all fronts of type (4) originate from the interval $[\gamma^-(0),\ \bar{x}]$ and exit from Γ through the side γ^+. Similarly, we can assume that all fronts of type (3) originate from the interval $[\bar{x},\ \gamma^+(0)]$ and exit from Γ through the side γ^-.

Observe that the fronts of type (3) are those of families $i < k$, while the fronts of type (3) are those of families $i > k$. Applying Lemma 2.4.1 to the region Γ, we obtain

$$\bar{\xi}\bar{\sigma} = \sum_{\alpha \in C(\gamma^-)} \xi_\alpha \sigma_\alpha + \xi_{\alpha^*}(\tau)\sigma_{\alpha^*}(\tau) + \sum_{\alpha \in C(\gamma^+)} \xi_\alpha \sigma_\alpha\,. \tag{2.43}$$

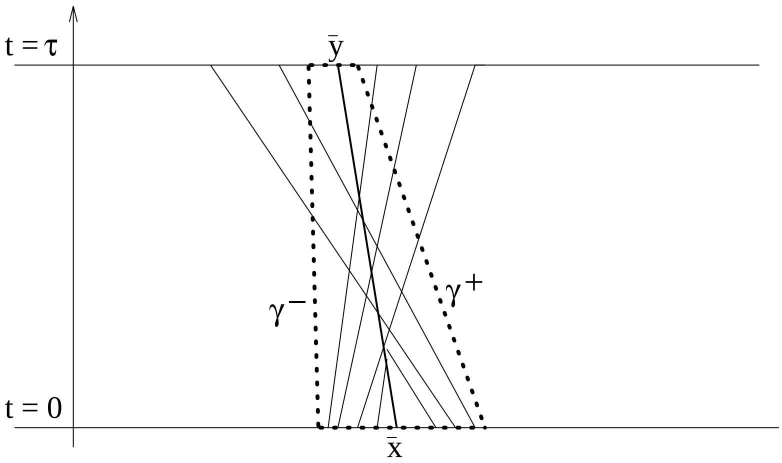

figure 2.7

In (2.43) the two summations refer to wave-fronts leaving Γ by crossing the boundary curves γ^- and γ^+ respectively. Calling $u^- = u(\tau, \bar{y}-)$ and $u^+ = u(\tau, \bar{y}+)$ the left and right states across the jump at $\bar{y}$, we now observe that the amplitude of each wave σ_α with $\alpha \in C(\gamma^-)$ satisfies

$$\sigma_\alpha \in \text{span}\big\{r_1(u^-), \ldots, r_{k-1}(u^-)\big\}. \tag{2.44}$$

Moreover, the amplitude of each wave σ_α with $\alpha \in C(\gamma^+)$ satisfies

$$\sigma_\alpha \in \text{span}\big\{r_{k+1}(u^-), \ldots, r_n(u^-)\big\}. \tag{2.45}$$

Since $\sigma_{\alpha^*}(\tau)$ is parallel to $r_k(u^+)$, from (2.43)–(2.45) and the assumption **(SH)** on the linear independence of the eigenvectors, the bound (2.41) follows.

We now establish **(P3)**, first in the case (fig. 2.8) where no j-wave ending at $\bar{y}$ crosses the k-wave starting at $\bar{x}$. Consider a curve γ running slightly to the right of the maximal backward j-front ending at $\bar{y}$. By Remark 2.4.2, after performing the operations **(O1)**-**(O2)** a number of times, we can consider an equivalent configuration with the following properties:

No front crosses γ from left to right. There exists some index $\ell \leq j$ such that only fronts of families $i < \ell$ can cross γ from right to left. Moreover, no i-front with $i < \ell$ terminates at time τ inside the half line $\big[\gamma(\tau),\ \infty\big[$, i. e. all these fronts cross γ before time τ.

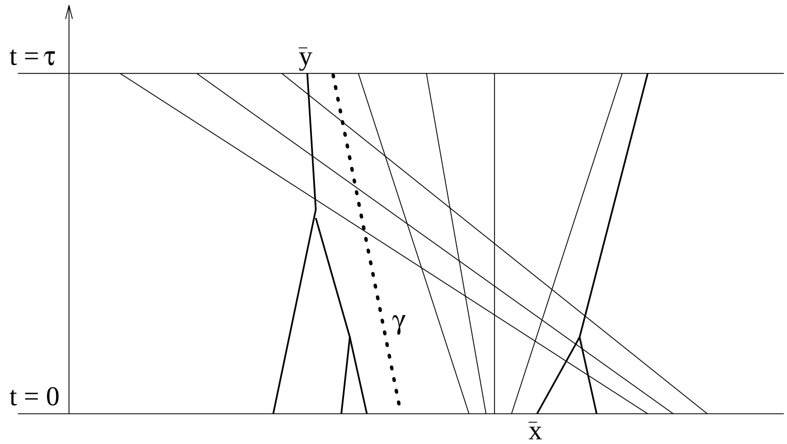

figure 2.8

Applying Lemma 2.4.1 to the region on the right of γ we obtain

$$\sum_{\alpha \in C(\gamma)} \xi_\alpha \sigma_\alpha + \sum_{x_\alpha(\tau) > \gamma(\tau)} \xi_\alpha(\tau)\sigma_\alpha(\tau) = \bar{\xi}\bar{\sigma}. \tag{2.46}$$

Here the first summation extends to all fronts crossing the curve γ. As before, call u^- and u^+ the left and right states across the jump at $\bar{y}$. Observing that the two sums on the right hand side of (2.46) are contained in

$$\mathrm{span}\{r_1(u^+), \ldots, r_{\ell-1}(u^+)\}, \qquad \mathrm{span}\{r_\ell(u^+), \ldots, r_n(u^+)\}, \tag{2.47}$$

using the assumption **(SH)** we conclude

$$\left| \sum_{\alpha \in C(\gamma)} \xi_\alpha \sigma_\alpha \right| \leq C_4 |\bar{\xi}\bar{\sigma}|. \tag{2.48}$$

We now apply again Lemma 2.4.1 to the region on the left of γ. Observing that the only incoming fronts which carry a nonzero shift rate are those crossing γ from right to left, and that the only outgoing j-front is the one ending at $\bar{y}$, we obtain

$$\sum_{x_\alpha(\tau) < \bar{y}} \xi_\alpha \sigma_\alpha + \xi_{\alpha^*}(\tau)\sigma_{\alpha^*}(\tau) = \sum_{\alpha \in C(\gamma)} \xi_\alpha \sigma_\alpha. \tag{2.49}$$

Recalling the normalization at (2.5), we observe that (2.49) implies

$$\left| \xi_{\alpha^*}(\tau)\sigma_{\alpha^*}(\tau) \right| = l_j(u^-) \cdot \sum_{\alpha \in C(\gamma)} \xi_\alpha \sigma_\alpha. \tag{2.50}$$

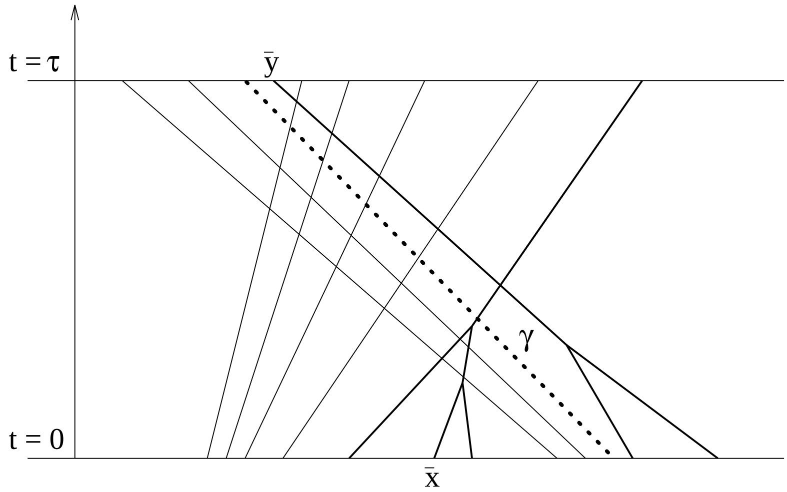

figure 2.9

On the other hand,

$$l_j(u^+) \cdot \sum_{\alpha \in C(\gamma)} \xi_\alpha \sigma_\alpha = 0. \tag{2.51}$$

Together, (2.50)-(2.51) imply

$$\left|\xi_{\alpha^*}(\tau)\sigma_{\alpha^*}(\tau)\right| \leq C'\left|l_j(u^+) - l_j(u^-)\right| \left|\sum_{\alpha \in C(\gamma)} \xi_\alpha \sigma_\alpha\right| \leq C_5\left|\sigma_{\alpha^*}(\tau)\right|\left|\bar{\xi}\bar{\sigma}\right|. \tag{2.52}$$

Hence, in this case, (2.42) holds.

We now prove **(P3)** in the case where $k > j$ and all j-waves running into $\bar{y}$ cross the k-wave starting from $\bar{x}$, as in figure 2.9. In this case, we construct a curve γ slightly to the left of the minimal backward j-front ending at $\bar{y}$. Observe that every wave-front crossing γ from left to right must be of a family $i > j$. Moreover, by Remark 2.4.2, we can assume that no wave crosses γ from right to left. Applying Lemma 2.4.1 to the region on the left of γ we obtain

$$\sum_{\alpha \in C(\gamma)} \xi_\alpha \sigma_\alpha = \bar{\xi}\bar{\sigma}. \tag{2.53}$$

We now apply again Lemma 2.4.1 to the region on the right of γ, observing that set of outgoing fronts, crossing the line $t = \tau$, contains the j-front at $\bar{y}$ plus other fronts on the right of $\bar{y}$, of families $i > j$. Together with (2.58), this yields

$$\xi_{\alpha^*}(\tau)\sigma_{\alpha^*}(\tau) + \sum_{x_\alpha(\tau) > \bar{y}} \xi_\alpha \sigma_\alpha = \sum_{\alpha \in C(\gamma)} \xi_\alpha(\tau)\sigma_\alpha(\tau) = \bar{\xi}\bar{\sigma}. \tag{2.54}$$

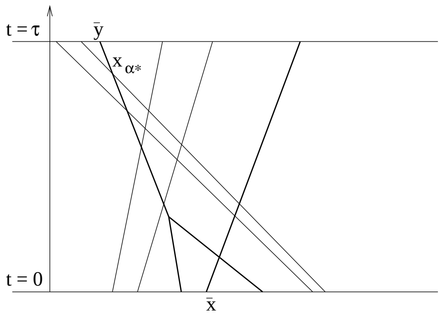

figure 2.10a

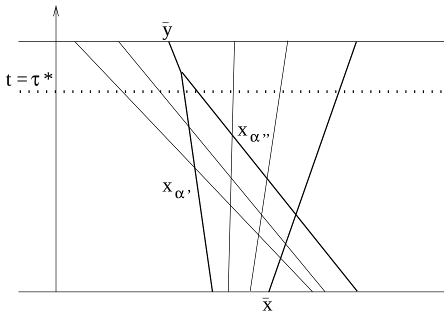

figure 2.10b

By (2.54),

$$\big|\xi_{\alpha^*}(\tau)\sigma_{\alpha^*}(\tau)\big| = l_j(u^+)\cdot \sum_{\alpha\in C(\gamma)} \xi_\alpha\sigma_\alpha. \tag{2.55}$$

Observing that

$$l_j(u^-)\cdot \sum_{\alpha\in C(\gamma)} \xi_\alpha\sigma_\alpha = 0,$$

we again have an estimate of the form (2.52), hence **(P3)** holds.

We finally consider the situation described in **(P2)**, where there exist j-waves starting both on the left and on the right of $\bar{x}$ which join into a single front ending at $\bar{y}$, (fig. 2.10a). According to Remark 2.4.2, we can simplify our computations by suitably changing the order of wave-front interactions. In particular, we can perform our estimate in terms of an equivalent wave-front configuration (fig. 2.10b), where the j-fronts ending at $\bar{y}$ and starting to the left of $\bar{x}$ join together in a single front $x_{\alpha'}$ at a time $\tau^* < \tau$, and all j-fronts ending at $\bar{y}$ and starting to the right of $\bar{x}$ join together in a single front $x_{\alpha''}$ at time τ^*. Applying **(P3)** to the fronts $x_{\alpha'}$, $x_{\alpha''}$, we obtain

$$\big|\xi_{\alpha'}(\tau^*)\big| \le C_5|\bar{\xi}\bar{\sigma}|, \qquad \big|\xi_{\alpha''}(\tau^*)\big| \le C_5|\bar{\xi}\bar{\sigma}|. \tag{2.56}$$

The final interaction of these two j-fronts determines a single front of strength $\sigma_{\alpha^*} = \sigma_{\alpha'} + \sigma_{\alpha''}$. Using Lemma 2.4.1, we compute

$$\begin{aligned} \big|\xi_{\alpha^*}(\tau)\sigma_{\alpha^*}(\tau)\big| &= \big|\xi_{\alpha'}(\tau^*)\sigma_{\alpha'}(\tau^*) + \xi_{\alpha''}(\tau^*)\sigma_{\alpha''}(\tau^*)\big| \\ &\le C'\cdot\Big(\big|\xi_{\alpha'}(\tau^*)\big| + \big|\xi_{\alpha''}(\tau^*)\big|\Big) \\ &\le C'C_5|\bar{\xi}\bar{\sigma}|. \end{aligned} \tag{2.57}$$

This establishes **(P2)**.

We now complete the proof of Lemma 2.5.1. Let $\hat{\lambda}$ be an upper bound for the absolute values of all characteristic speeds. If the only front shifted at time $t = 0$ is the one at $\bar{x}$, it follows that at a fixed time $\tau > 0$ the only fronts with nonzero shift rate can be the ones located inside the interval $[\bar{x} - \hat{\lambda}\tau,\ \bar{x} + \hat{\lambda}\tau]$. Recalling the estimate (2.38) on the total variation, using the properties **(P1)**–**(P3)** we thus have

$$\begin{aligned}\sum_\alpha \big|\xi_\alpha(\tau)\sigma_\alpha(\tau)\big| &\leq nC_4|\bar{\xi}\bar{\sigma}| + C_5|\bar{\xi}\bar{\sigma}| \cdot \text{Tot.Var.}\big\{u(\tau);\quad [\bar{x} - \hat{\lambda}\tau,\ \bar{x} + \hat{\lambda}\tau]\big\} \\ &\leq C_3(1 + N2^{-\nu})|\bar{\xi}\bar{\sigma}|,\end{aligned}$$

for a suitable constant C_3. □

Remark 2.5.2 For the front tracking solution u considered in Lemma 2.5.1, assume that all fronts in $u(0,\cdot)$ are shifted. More precisely, let $\xi_\alpha(0)$ be the shift rate of the front located at $x_\alpha(0)$, having amplitude $\sigma_\alpha(0)$. Call $\xi_\beta(\tau)$ the corresponding shift rate of the front of $u(\tau,\cdot)$ located at $x_\beta(\tau)$. Observing that the shift differential

$$\big(\xi_1(0), \dots \xi_N(0)\big) \mapsto \big(\xi_1(\tau), \dots \xi_{N'}(\tau)\big)$$

is a linear mapping, from (2.40) it easily follows

$$\sum_\beta \big|\xi_\beta(\tau)\sigma_\beta(\tau)\big| \leq C_3(1 + N2^{-\nu}) \cdot \sum_\alpha \big|\xi_\alpha(0)\sigma_\alpha(0)\big|. \tag{2.58}$$

2.6 Proof of the theorems

To construct the semigroup described in Theorem 2.2.1, we recall that, for every $M > 0$, by [8] there exists a Lipschitz semigroup S of solutions of (2.1) defined on the domain $\mathcal{D}_M$ defined at (2.11). Moreover, trajectories of this semigroup are the unique limits of front tracking approximations:

$$S_t\bar{u} = \lim_{\nu\to\infty} S_t^\nu \bar{u}_\nu$$

for every $\bar{u} \in \mathcal{D}_M$, $\bar{u}_\nu \in \mathcal{D}_M \cap \mathcal{D}^\nu$, $\bar{u}_\nu \to \bar{u}$ in $\mathbf{L}^1$. We need to show that the Lipschitz constant of this semigroup on $\mathcal{D}_M$ does not depend on M. For this purpose, consider any two piecewise constant initial data, say $\bar{u}, \bar{v} \in \mathcal{D}^\mu$. We can now construct a pseudopolygonal path $\gamma_0 : \theta \mapsto \bar{u}^\theta$ connecting $\bar{u}$ with $\bar{v}$ with the following properties. The $\mathbf{L}^1$ length of γ_0 satisfies

$$\|\gamma_0\|_{\mathbf{L}^1} \leq C_6\|\bar{u} - \bar{v}\|_{\mathbf{L}^1}\,. \tag{2.59}$$

Moreover, all functions $\bar{u}^\theta$ lie in $\mathcal{D}^\mu$ and have a uniformly bounded number of shocks, say $\leq N$.

Calling $\gamma_\tau : \theta \mapsto u^\theta \doteq S_\tau u^\theta$, we claim that the length of γ_τ remains a bounded multiple of the length of γ_0. Indeed, for every $\nu \geq \mu$, consider the path $\gamma_\tau^\nu : \theta \mapsto u^\theta \doteq S_\tau^\nu u^\theta$. Writing the length of this path in the form (2.20) and using Lemma 2.5.1 we obtain

$$\begin{aligned}\|\gamma_\tau^\nu\|_{\mathbf{L}^1} &= \sum_{j=1}^m \int_{\theta_{j-1}}^{\theta_j} \sum_\beta \left|\Delta u_\nu^\theta(\tau, x_\beta^\theta)\right| \left|\frac{\partial x_\beta^\theta(\tau)}{\partial\theta}\right| d\theta.\\ &\leq \sum_{j=1}^m \int_{\theta_{j-1}}^{\theta_j} C_3(1+N2^{-\nu}) \cdot \sum_\alpha \left|\Delta u_\nu^\theta(0, x_\alpha^\theta)\right| \left|\frac{\partial x_\alpha^\theta(0)}{\partial\theta}\right| d\theta.\\ &\leq C_3(1+N2^{-\nu}) \cdot \|\gamma_0^\nu\|_{\mathbf{L}^1}\,. \end{aligned} \tag{2.60}$$

Letting $\nu \to \infty$ we obtain (2.8) for all $\bar{u}, \bar{v} \in \mathcal{D}^*$. The semigroup S can thus be extended by continuity to the whole domain $\mathcal{D}$, preserving the property (2.8). Every semigroup trajectory is thus a limit of front tracking approximations, hence it provides a solution of the system (2.1).

Concerning the entropy condition (2.9), fix an interval $[a, b]$ and an initial condition $\bar{u}$. We can now approximate the solution $u(t, \cdot) = S_t\bar{u}$ with a sequence of front tracking solutions $u_\nu(t, \cdot) = S_t^\nu \bar{u}_\nu$, choosing initial data $\bar{u}_\nu$ having a number of shocks $N_\nu \leq \nu$. By (2.39), the total number of positive wave-fronts in $u_\nu(\tau, \cdot) = S_\tau^\nu \bar{u}_\nu$ on a given interval $[a, b]$ satisfies

$$\text{Pos.Var.}\big\{w_k^\nu(\tau, \cdot);\ [a,\ b]\big\} \leq (1+N_\nu)2^{-\nu} + \frac{b-a}{\kappa\tau}, \tag{2.61}$$

where $w_k^\nu \doteq w_k(u_\nu)$ is the k-th Riemann coordinate of u_ν. Letting $\nu \to \infty$ in (2.61) we obtain

$$\text{Pos.Var.}\big\{w_k\big(u(\tau, \cdot)\big);\ [a,\ b]\big\} \leq \frac{b-a}{\kappa\tau}. \tag{2.62}$$

Hence (2.9) holds. This completes the proof of Theorem 2.2.1.

Remark 2.6.1 In general the semigroup S_t constructed as above will not be Lipschitz continuous w. r. t. time. Indeed, the map $t \mapsto S_t\bar{u}$ from $[0, \infty[$ to $\mathbf{L}^1(\mathbb{R})$ may not be Lipschitz continuous at time $t = 0$ if $\bar{u}$ has unbounded total variation.

We now give a proof of Theorem 2.2.2. Fix any $R > 0$ and let $\hat{\lambda}$ be an upper bound for the absolute values of all characteristic speeds. Let u be a weak solution of (2.1) satifying the decay estimate (2.9). For every $\delta > 0$ the restriction of $u(t, \cdot)$ to the intervals $I(t) \doteq [-R + \hat{\lambda} t,\ R - \hat{\lambda} t]$, $t \in [\delta,\ R/\hat{\lambda}]$ has uniformly bounded variation. Therefore, the uniqueness theorem in [18] yields

$$u(t, \cdot) = S_{t-\delta} u(\delta) \qquad \text{restricted to } I(t). \tag{2.63}$$

$$\begin{aligned} \int_{I(t)} \big| u(t,x) - (S_t \bar{u})(x) \big|\, dx &= \int_{I(t)} \Big| \big(S_{t-\delta} u(\delta)\big)(x) - \big(S_{t-\delta} \circ S_\delta \bar{u}\big)(x) \Big|\, dx \\ &\leq L \cdot \big\| u(\delta, \cdot) - S_\delta \bar{u} \big\|_{\mathbf{L}^1} . \end{aligned} \tag{2.64}$$

Letting $\delta \to 0$ we conclude that the left hand side of (2.64) must be zero. Since R was arbitrary, this proves (2.10).

2.7 A counterexample

The following example shows that the Lipschitz continuous dependence on the initial data may not hold if one drops the assumption of genuine nonlinearity of all characteristic fields.

Consider the 2×2 system

$$\begin{aligned} (u_1)_t + \left(\frac{u_1}{1 + u_1 + u_2} \right)_x &= 0, \\ (u_2)_t + \left(\frac{u_2}{1 + u_1 + u_2} \right)_x &= 0, \end{aligned} \tag{2.65}$$

where $u_1,\ u_2 > 0$. Here, the first characteristic field is genuinely nonlinear, the second is linearly degenerate. In terms of the Riemann coordinates $w_1 \doteq u_1 + u_2$, $w_2 \doteq u_2/u_1$, this system takes the form

$$\begin{aligned} (w_1)_t + \left(\frac{w_1}{1 + w_1} \right)_x &= 0, \\ (w_2)_t + \frac{1}{1 + w_1} (w_2)_x &= 0. \end{aligned} \tag{2.66}$$

Fix some coordinate values $0 < w_1^- < w_1^+$, $0 < w_2^- < w_2^+$. We now consider an initial data $\bar{u}(x) = \big(\bar{w}_1(x),\ \bar{w}_2(x)\big)$ whose Riemann coordinates are defined as follows. Given three points $a < b < c$ on the real line, we take

$$\bar{w}_1(x) \doteq \begin{cases} w_1^- & \text{if } x < c, \\ \\ w_1^+ & \text{if } x > c. \end{cases}$$

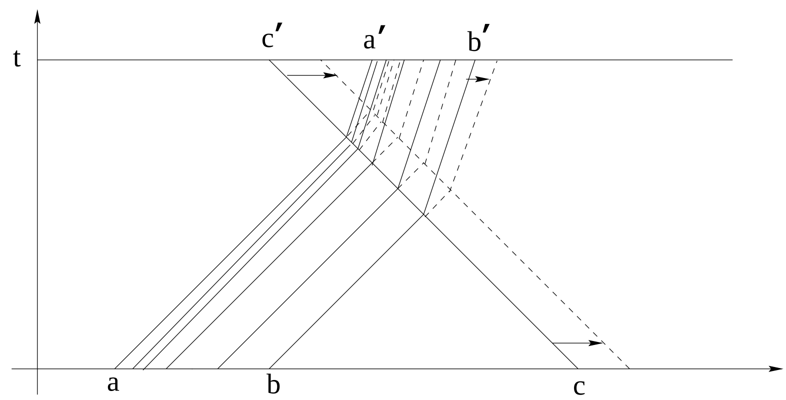

figure 2.11

Moreover we set

$$\bar{w}_2(x) \doteq w_2^- \qquad \text{if } \; x \notin [a, b]$$

and let $\bar{w}_2$ oscillate countably many times between the two values w_2^-, w_2^+ as x ranges inside $[a, b]$, so that its total variation is infinite.

The corresponding solution $u = u(t, x)$ will thus contain a 1-shock, travelling with speed λ_1, and countably many 2-contact discontinuities, travelling with speeds λ_2^-, λ_2^+ respectively before and after the interaction with the shock. An explicit computation yields

$$\lambda_1 = \frac{w_1^+/(1+w_1^+) - w_1^-/(1+w_1^-)}{w_1^+ - w_1^-}, \qquad \lambda_2^- = \frac{1}{1+w_1^-} > \lambda_2^+ = \frac{1}{1+w_1^+}. \tag{2.67}$$

Now consider a perturbed initial condition, obtained by shifting the 1-shock from c to $c+\theta$, and fix a time $t > 0$ large enough so that waves of different families will have crossed each other (fig. 2.11). Then $u(t, \cdot)$ contains a 1-shock located at a point $c' = c + \lambda_1 t$ and a family of 2-waves located within an interval $[a', b']$, while the corresponding perturbed solution $u^\theta(t, \cdot)$ will contain a 1-shock located at $c' + \theta$ and a family of 2-waves, located within the interval $[a' + \xi\theta, \; b' + \xi\theta]$. Here the shift rate is computed by

$$\xi = \frac{\lambda_2^- - \lambda_2^+}{\lambda_2^- - \lambda_1} \neq 0. \tag{2.68}$$

Observing that

$$\frac{1}{\theta} \cdot \big\| \bar{u}^\theta - \bar{u} \big\|_{\mathbf{L}^1} = \big| \Delta \bar{u}(c) \big|,$$

$$
\begin{aligned}
\limsup_{\theta\to 0} \ & \frac{1}{\theta}\int_{-\infty}^{\infty} \big|w_2^\theta(t,x) - w_2(t,x)\big| \ dx \\
& = \limsup_{\theta\to 0} \frac{1}{\theta}\int_{-\infty}^{\infty} \big|w_2(t,x-\xi\theta) - w_2(t,x)\big| \ dx \\
& = \xi \cdot \text{Tot.Var.}\big\{w_2(t,\cdot)\big\} = \infty,
\end{aligned}
$$

it is clear that the map $\bar{u} \mapsto S_t\bar{u}$ is not Lipschitz continuous w. r. t. the $\mathbf{L}^1$ distance.

Chapter 3

$\mathbf{L}^1$ stability for Temple class systems with $\mathbf{L}^\infty$ initial and boundary data

3.1 Introduction to Chapter 3

We consider the initial-boundary value problem for a strictly hyperbolic system of conservation laws in one space dimension,

$$u_t + f(u)_x = 0, \tag{3.1}$$

$$u(0,x) = \bar{u}(x), \tag{3.2}$$

$$u(t,0) = \tilde{u}(t), \tag{3.3}$$

on the domain $\Omega = \{(t,x) \in \mathbb{R}^2 : t \geq 0, x \geq 0\}$. Here $f : U \mapsto \mathbb{R}^n$ is a smooth vector field defined on some open set $U \subseteq \mathbb{R}^n$. The problem (3.1)-(3.3) is usually not well-posed when the boundary data are required to be assumed in the (strong) sense (3.3), even when (3.1) is a linear system (see [45]). In fact, different notions of the boundary condition have been considered in the literature, see [1, 65] for definition and references. Here, following [34, 42], we will deal with the (weak) form

$$f\big(u(0+,t)\big) \in f\big(\mathcal{V}(\tilde{u}(t))\big), \qquad \text{for a.e.} \quad t > 0, \tag{3.4}$$

where $\mathcal{V}(\tilde{u}(t)) \subset U$ is a time-dipendent set (the set of *admissible boundary values*) that is defined from the boundary data using the notion of Riemann problem, while $f(u(0+,t))$ represents the (weak) trace of $f(u)$ at the boundary $x = 0$.

Most of the study on the boundary condition for (3.1) has been restricted to the scalar equation. We recall that, for scalar conservation laws, the existence and continuous dependence on the initial and boundary data of global solutions to the mixed problem (3.1)-(3.3) was proved within domains of $\mathbf{L}^\infty$ functions [45, 49, 66]. On the other hand, in the case of $n \times n$ systems global existence and stability of entropy weak solutions has been established only for data with small total variation (see [1, 2, 3, 65] and references therein).

Here, having in mind to study the initial-boundary value problem from the point of view of control theory (where it is natural to regard the boundary data as varying into a prescribed set of $\mathbf{L}^\infty$ controls with possibly unbounded variation), we extend the existence and the stability results in [43, 49] to the case of Temple class systems with genuinely nonlinear characteristic fields.

We rely on the stability result obtained in Chapter 2 for the Cauchy problem for Temple class systems with $\mathbf{L}^\infty$ initial data, and we apply the same technique to construct a Lipshitz continuous semigroup of solutions to (3.1)-(3.3), whose trajectories satisfy a suitable entropy condition of Oleinik type.

In the case of scalar equations, the problem with integrable (possibly unbounded) boundary data has been studied from the point of view of control by [5, 6], relying on an explicit representation formula for the solution derived by LeFloch in [49]. A contractive property was also established. On the same line, we fix a set $\mathcal{U} \subset \mathbf{L}^\infty(\mathbb{R}^+)$ of boundary data regarded as admissible controls, and, taking the initial data $\bar{u} \equiv 0$, we consider the set of attainable profiles at a fixed time T

$$\mathcal{A}(T,\mathcal{U}) \doteq \Big\{ u(T,\cdot)\,;\ u \text{ is a solution to (3.1)-(3.3) with } \bar{u} \equiv 0 \text{ and } \tilde{u} \in \mathcal{U} \Big\},$$

and at a fixed point in space $\bar{x} > 0$

$$\mathcal{A}(\bar{x},\mathcal{U}) \doteq \Big\{ u(\cdot,\bar{x})\,;\ u \text{ is a solution to (3.1)-(3.3) with } \bar{u} \equiv 0 \text{ and } \tilde{u} \in \mathcal{U} \Big\}.$$

Motivated by applications to calculus of variations and problems of optimization we establish closure and compactness in the $\mathbf{L}^1$ topology of the attainable sets in connection with a class of $\mathbf{L}^\infty$ boundary controls.

3.2 Preliminaries and statement of the main results

Let $f : U \mapsto \mathbb{R}^n$ be the flux function of the strictly hyperbolic system (3.1), and denote by $\lambda_1(u) < \cdots < \lambda_n(u)$ the eigenvalues of the Jacobian matrix $Df(u)$. Throughout the paper we shall assume that, for some fixed index $p \in \{1, \ldots, n\}$, there holds

$$\lambda_{n-p}(u) < 0 < \lambda_{n-p+1}(u) \qquad \text{for every} \quad u \in U. \tag{3.5}$$

Moreover, by possibly considering a sufficiently small restriction of the domain U, we may assume that f is invertible and that all characteristic speeds have a uniform upper bound

$$\lambda_i(u) \leq \widehat{\lambda} \qquad \forall\, u \in U.$$

Choose right and left eigenvectors $r_i(u)$, $l_i(u)$, $i = 1, \ldots, n$, of $Df(u)$ normalized so that

$$\big|r_i(u)\big| = 1, \qquad\qquad l_i(u) \cdot r_j(u) = \begin{cases} 1 \text{ if } i = j, \\ 0 \text{ if } i \neq j. \end{cases} \tag{3.6}$$

We assume that each i-th characteristic field r_i is *genuinely nonlinear* in the sense of Lax, i.e. that, by choosing a suitable orientation of the eigenvectors $r_i(u)$, at every point

$u \in \Omega$ one has $D\lambda_i \cdot r_i(u) > 0$. Moreover, the system (3.1) is of Temple class according with the following

Definition 3.2.1 *A system of conservation laws is of Temple class if there exists a system of coordinates $w = (w_1, \ldots, w_n)$ consisting of Riemann invariants, and such that the level sets $\{u \in \Omega;\ w_i(u) = constant\}$ are hyperplanes (see [66]).*

For a Temple class system, the integral curve of the vector field r_i through a point u_0 is the straight line described by the $n-1$ linear equations

$$w_j(u) = w_j(u_0) \qquad\qquad j \neq i. \tag{3.7}$$

In particular, shock and rarefaction curves coincide. Throughout the paper, we will often write $w_i(t,x) \doteq w_i\big(u(t,x)\big)$ to denote the i-th Riemann coordinate of a solution $u = u(t,x)$ to (3.1).

We introduce next a boundary condition that is formulated in terms of the weak trace of $f(u)$ at the the boundary $x = 0$, and is related to the notion of Riemann problem in the same spirit of [9]. To this purpose, letting $u(t,x) = W(\xi = x/t;\ u_L, u_R)$, $u_L,\ u_R \in U$, denote the self-similar solution of the Riemann problem for (3.1) with initial data

$$u(0,x) = \begin{cases} u_L \text{ if } x < 0, \\ \\ u_R \text{ if } x > 0, \end{cases}$$

for any given boundary state $\tilde{u} \in U$, we define the set of *admissible states at the boundary*

$$\mathcal{V}(\tilde{u}) := \big\{W(0+;\ \tilde{u}, u_R)\ ;\ u_R \in U\big\}. \tag{3.8}$$

Definition 3.2.2 *A function $u : \Omega \mapsto \mathbb{R}^n$ is an entropy weak solution of (3.1)-(3.3) if*

(i) u is a weak solution to (3.1) and satisfies the initial condition (3.2) in the sense that, for every $\mathcal{C}^1$ function ϕ with compact support contained in the set $\{(t,x) \in \mathbb{R}^2;\ x > 0\}$, there holds

$$\int_0^{+\infty}\int_0^{+\infty} \big(u(t,x)\cdot\phi_t(t,x) + f(u(t,x))\cdot\phi_x(t,x)\big)dx\ dt + \int_0^{+\infty} \bar{u}(x)\cdot\phi(0,x)dx = 0.$$

(ii) the flux $f(u)$ admits a weak trace at the boundary $x = 0$, i.e. there exists a measurable function $\Psi : \mathbb{R}^+ \mapsto \mathbb{R}^n$ such that

$$\lim_{x \to 0+} \int_0^t f(u(s,x))\ ds = \int_0^t \Psi(s)\ ds \qquad t \geq 0, \tag{3.9}$$

and the boundary condition (3.3) is satisfied in the following sense

$$\Psi(t) \in f\big(\mathcal{V}(\tilde{u}(t))\big) \qquad \text{for a.e.} \quad t \geq 0. \tag{3.10}$$

(iii) u satisfies the following entropy conditions. For any $0 < x < y$, and $t > 0$, there holds

$$w_i(t,y) - w_i(t,x) \leq \frac{y-x}{\kappa t} \qquad \text{if} \quad i \in \{1, \ldots, n-p\}, \tag{3.11}$$

$$w_i(t,y) - w_i(t,x) \leq \frac{C}{\kappa} \log \left\{ \left[\frac{(\hat{\lambda} t + y) - |\hat{\lambda} t - y|}{2x} - 1 \right]^+ + 1 \right\} + \frac{y-x}{\kappa t} \qquad \text{if} \quad i \in \{n-p+1, \ldots, n\}. \tag{3.12}$$

for some constant $C > 0$ depending only on the system (3.1) (here $[a]^+ \doteq \max\{a, 0\}$ denotes the positive part of $a \in \mathbb{R}$).

Remark 3.2.3 The set of admissible flux values at the boundary $f\big(\mathcal{V}(\tilde{u})\big)$ can be expressed in Riemann coordinates as

$$f\big(\mathcal{V}(\tilde{u})\big) = \Big\{ f(u)\ ;\ w_j(u) = w_j(\tilde{u}) \quad \forall\ \ j = n-p+1, \ldots, n \Big\}. \tag{3.13}$$

Hence, by the invertibiliy of the smooth map f, the above boundary condition (3.10) is equivalent to the existence of a measurable map $\phi : \mathbb{R}^+ \mapsto \mathbb{R}^n$ such that

$$f(\phi(t)) = \Psi(t) \qquad \text{for a.e.} \quad t \geq 0, \tag{3.14}$$

and that satisfies the set of equalities

$$w_j\big(\phi(t)\big) = w_j\big(\tilde{u}(t)\big) \qquad j = n-p+1, \ldots, n. \tag{3.15}$$

This means that the boundary condition (3.10) guarantees that, at almost every time t the solution to the Riemann problem for (3.1), having initial data with left state $u^L = \tilde{u}(t)$ and right state $u^R = \phi(t)$ contains only waves with negative speeds and, in particular, its restriction to the region Ω takes costant value $\phi(t)$.

We now consider a convex, compact set $K \subset U$ having the form

$$K = \Big\{ u \in U; \quad w_i(u) \in [a_i,\ b_i] \qquad i = 1, \dots, n \Big\}, \tag{3.16}$$

and assume that, as u varies in K, a strengthened version of the strict hyperbolicity condition holds, namely

(SH) For any given $u_1, \dots, u_n \in K$, the characteristic speeds at these points satisfy $\lambda_1(u_1) < \dots < \lambda_n(u_n)$. Moreover, the eigenvectors $r_1(u_1), \dots, r_n(u_n)$ are linearly independent.

Observe that the above assumption is certainly satisfied if the system is strictly hyperbolic and K is contained in a small neighborhood of a given point. By a translation of coordinates, it is not restrictive to assume that $0 \in K$ and $(w_1, \dots, w_n)(0) = (0, \dots, 0)$.

Due to the presence of the boundary data, the flow map $u(0, \cdot) \mapsto u(t, \cdot)$ is not time homogeneus. To recast the problem in a semigroup framework, it is convenient to incorporate the boundary data $\tilde{u}$ in the domain of the semigroup. More precisely, consider the positively invariant domain of $\mathbf{L}^\infty$ functions, with possibly unbounded variations,

$$\mathcal{D} \doteq \Big\{ \mathbf{p} = (\bar{u}, \tilde{u}) \ ; \ \bar{u},\ \tilde{u} \in \mathbf{L}^1(\mathbb{R}^+, K) \Big\}. \tag{3.17}$$

We define

$$\text{Tot.Var.}(\mathbf{p}) \doteq \text{Tot.Var.}(\bar{u}) + \text{Tot.Var.}(\tilde{u}) \tag{3.18}$$

and we introduce the distance

$$d(\mathbf{p}_1, \mathbf{p}_2) \doteq \|\bar{u}_1 - \bar{u}_2\|_{\mathbf{L}^1} + \|\tilde{u}_1 - \tilde{u}_2\|_{\mathbf{L}^1}. \tag{3.19}$$

With the above notation, we construct a semigroup S acting on $\mathcal{D}$, in the sense that

$$\begin{array}{rl} S : \mathbb{R}^+ \times \mathcal{D} \mapsto & \mathcal{D} \\ t \quad , \ \mathbf{p} \mapsto & S_t \mathbf{p}, \end{array} \tag{3.20}$$

where, if $\mathbf{p} = (\bar{u}, \tilde{u})$, $S_t \mathbf{p} = \big(E_t \mathbf{p},\, \mathcal{T}_t \tilde{u}\big)$. Here $\mathcal{T}_t$ is the translation operator, i.e. $(\mathcal{T}_t \tilde{u})(s) \doteq \tilde{u}(t+s)$, while the evolution operator $E : \mathbb{R}^+ \times \mathcal{D} \mapsto \mathbf{L}^1(\mathbb{R}^+, K)$ is such that $E_t \mathbf{p} = u(t, \cdot)$, u being the solution to (3.1)-(3.3).

Our main result is concerned with the existence of a semigroup generated by the system (3.1) on the domain $\mathcal{D}$.

Theorem 3.2.4 *Let (3.1) be a system of Temple class with all characteristic fields genuinely nonlinear, and assume that the strict hyperbolicity condition* (**SH**) *holds. Then, there exist a continuous semigroup S of the form (3.20) and some constant $C > 0$ depending only the system (3.1) so that, for every fixed $T > 0$, $\delta > 0$, and for all $(\bar{u}, \bar{v}), (\tilde{u}, \tilde{v}) \in \mathcal{D}$, letting $L \doteq L(\delta, T) = C(1 + \log(T/\delta))$, one has*

$$\left\| E_t(\bar{u}, \tilde{u}) - E_t(\bar{v}, \tilde{v}) \right\|_{\mathbf{L}^1([\delta, +\infty[)} \leq L \cdot \left(\|\bar{u} - \bar{v}\|_{\mathbf{L}^1} + \|f(\tilde{u}) - f(\tilde{v})\|_{\mathbf{L}^1} \right) \tag{3.21}$$

for all $t \in [0,\ T]$. Moreover, the map $t \mapsto E_t\mathbf{p}$ yields an entropy weak solution (in the sense of Definition 3.2.2) to the initial-boundary value problem (3.1)-(3.3).

Remark 3.2.5 As in the case of the Cauchy problem [19], the map $t \mapsto E_t\mathbf{p}$ constructed as above may not be Lipschitz continuous at time $t = 0$ w.r.t. the $\mathbf{L}^1$ distance if $\mathbf{p}$ has unbounded total variation. Moreover, the evolution operator $\mathbf{p} \mapsto E_t\mathbf{p}$ will not be, in general, Lipschitz continuous w.r.t. the topology of $\mathbf{L}^1(\mathbb{R}^+, K)$.

Theorem 3.2.6 *Let (3.1) be a system of Temple class satisfying the same assumptions as in Theorem 3.2.4. Let $u = u(t,x)$ be an entropy weak solution to the mixed problem (3.1)-(3.3) on the region Ω. Assume that the maps $t \to u(t,\cdot)$, $x \to u(\cdot,x)$, are continuous on the domains $\{t \in \mathbb{R}\ :\ t \geq 0\}$, $\{x \in \mathbb{R}\ :\ x \geq 0\}$, w.r.t. the $\mathbf{L}^1$ topology, and that the map $(t,x) \to \big(u(t,\cdot),\, u(\cdot,x)\big)$ takes values in the domain $\mathcal{D}$ defined in (3.17). Then, u coincides with the corresponding semigroup trajectory, namely*

$$u(t,\cdot) = E_t(\bar{u}, \tilde{u}) \qquad \forall \quad t \geq 0. \tag{3.22}$$

3.3 Outline of the proof

We describe here the basic steps in the proof of Theorem 3.2.4. The technical estimates involved in the proof will be then worked out in the remaining sections. As in [19] we shall first construct a sequence of flow maps S^ν whose trajectories are front tracking approximate solutions [8, 10] of (3.1) in the region Ω, depending Lipschitz continuously on the initial and boundary data. Next, for any fixed $\mathcal{M} > 0$, we shall prove the convergence of such a sequence of flow maps to a Lipschitz continuous semigroup of solutions, defined on the domain

$$\mathcal{D}_{\mathcal{M}} \doteq \Big\{ \mathbf{p} \in \mathcal{D};\ \text{Tot.Var.}\{\mathbf{p}\} \leq \mathcal{M} \Big\}. \tag{3.23}$$

Finally, we will show that the Lipschitz constant of this semigroup is indeed independent on the bound on the total variation $\mathcal{M}$.

We describe now a front-tacking algorithm which represents a natural extension of [18]. Fix an integer $\nu \geq 1$ and consider the discrete set of points in K whose coordinates are integer multiples of $2^{-\nu}$:

$$K^\nu \doteq \Big\{ u \in K\,;\ w_i(u) \in 2^{-\nu}\mathbb{Z}, \quad i = 1, \ldots, n \Big\}.$$

Moreover, consider the domain

$$\mathcal{D}^\nu \doteq \Big\{ \mathbf{p} = (u, u') : \mathbb{R}^+ \mapsto K^\nu \times K^\nu; \qquad u,\, u' \in \mathbf{L}^1,\ u,\, u' \text{ are piecewise constant} \Big\}. \tag{3.24}$$

On $\mathcal{D}^\nu$ we now construct a semigroup S^ν whose trajectories are front tracking approximate solutions of (3.1). To this end, we first describe how to solve a Riemann problem with data $u^-, u^+ \in K^\nu$. In Riemann coordinates, assume that

$$w(u^-) \doteq w^- = (w_1^-, \ldots, w_n^-) \qquad w(u^+) \doteq w^+ = (w_1^+, \ldots, w_n^+).$$

Consider the intermediate states

$$\omega_0 = u^-, \qquad \ldots, \qquad \omega_i = u(w_1^+, \ldots, w_i^+, w_{i+1}^-, \ldots, w_n^-), \qquad \ldots, \qquad \omega_n = u^+. \tag{3.25}$$

If $w_i^+ < w_i^-$, the solution will contain a single i-shock, connecting the states ω_{i-1}, ω_i and travelling with Rankine-Hugoniot speed $\lambda_i(\omega_{i-1}, \omega_i)$. Here and in the sequel, by $\lambda_i(u, u')$ we denote the i-th eigenvalue of the averaged matrix

$$A(u, u') \doteq \int_0^1 Df\big(\theta u + (1-\theta)u'\big)\, d\theta. \tag{3.26}$$

If $w_i^+ > w_i^-$, the exact solution of the Riemann problem would contain a centered rarefaction wave. This is approximated by a rarefaction fan as follows. If $w_i^+ = w_i^- + p_i 2^{-\nu}$ we insert the states

$$\omega_{i,\ell} = (w_1^+, \ldots, w_i^- + 2^{-\nu}\ell, w_{i+1}^-, \ldots, w_n^-) \qquad \ell = 0, \ldots, p_i, \tag{3.27}$$

so that $\omega_{i,0} = \omega_{i-1},\ \omega_{i,p_i} = \omega_i$. Our front tracking solution will then contain p_i fronts of the i-th family, each connecting a couple of states $\omega_{i,\ell-1},\ \omega_{i,\ell}$ and travelling with speed $\lambda_i\big(\omega_{i,\ell-1},\ \omega_{i,\ell}\big)$.

For a given data $\mathbf{p} = (\bar{u}, \tilde{u}) \in \mathcal{D}^\nu$, the approximate solution $u(t, \cdot) \doteq E_t^\nu \mathbf{p}$ is now constructed as follows. At time $t = 0$ we solve each of the Riemann problems determined by the jumps in $\bar{u}$ according to the above procedure. At the origin, construct the solution to the Riemann problem with data $u_L = \tilde{u}(0+)$, $u_R = \bar{u}(0+)$, and take its restriction to the domain Ω, for small times. This yields a piecewise constant function with finitely many fronts, travelling with constant speeds. The solution is then prolonged up to the first time where one of the following situations occurs:

a) two or more fronts interact;

b) one or more discontinuities hit the boundary;

c) the boundary condition $\tilde{u}$ has a jump.

Observe that it is not restrictive to assume that only one discontinuity hits the boundary and only one of the previous situations can occour at any given time. At the interaction points, the new Riemann problems are again solved by the above procedure, etc... Here, by interaction point, with a slight abuse of notations, we mean a point where one of the events a), b) or c) takes place. Note that in case b) no new wave appears, i.e. waves exiting the boundary are produced only by the jumps of $\tilde{u}$.

As in [8] and Chapter 2, one checks that these front tracking approximations are well defined for all times $t \geq 0$. Indeed, the following properties hold.

- For each $i = 1, \ldots, n$, the total variation of $u(t, \cdot)$, measured w.r.t. the Riemann coordinates, coincides with the total strength of waves in $w_i(t, \cdot)$ and is non-increasing in time.

- The number of wave-fronts in $u(t, \cdot)$ is non-increasing at each interaction. Hence, the total number of wave-fronts in $u(t, \cdot)$ remains finite.

It is now possible to define a ν-approximate semigroup $S^\nu : \mathbb{R}^+ \times \mathcal{D}^\nu \mapsto \mathcal{D}^\nu$ as in (3.20) by setting, for any $\mathbf{p} = (\bar{u}, \tilde{u})$, $S_t^\nu \mathbf{p} = \big(E_t^\nu \mathbf{p}, \mathcal{T}_t \tilde{u}\big)$. The uniqueness of the definition of the approximate solution guarantees that E_t^ν, and hence S_t^ν, satisfy the standard semigroup properties, i.e.

$$S_0^\nu = \text{Identity}, \qquad\qquad S_t^\nu \circ S_s^\nu = S_{t+s}^\nu.$$

Each trajectory $t \mapsto E_t^\nu \mathbf{p}$ is a weak solution of (3.1) (because all fronts satify the Rankine-Hugoniot conditions), but may not be entropy-admissible (because of the presence of rarefaction fronts).

We next proceed towards an estimate of the Lipschitz constant for E^ν following the same technique adopted in [18]. The basic idea to estimate the distance between two approximate solutions u, v, consists in constructing a continuous path of solutions u^θ connecting u, v, and then study how the length of the path $\theta \to u^\theta(t,\cdot)$ varies in time. In particular, given any two couples of initial and boundary data $\mathbf{p}_1 = (\bar{u}_1,\, \tilde{u}_1)$, $\mathbf{p}_2 = (\bar{u}_2,\, \tilde{u}_2)$ in $\mathcal{D}^\nu$, we consider a suitable class of continuous paths (pseudopolygonals) that connect $\mathbf{fp}_1 \doteq (\bar{u}_1,\, f(\tilde{u}_1))$ with $\mathbf{fp}_2 \doteq (\bar{u}_2,\, f(\tilde{u}_2))$ by merely shifting the space and time positions of the jumps in $\bar{u}_1$, $\bar{u}_2$, and in $f(\tilde{u}_1)$, $f(\tilde{u}_2)$, respectively. More precisely, a *pseudopolygonal* with values in

$$\mathcal{FD}^\nu \doteq \big\{\mathbf{fp} = (u, f(u'));\quad (u,u') \in \mathcal{D}^\nu\big\},$$

is a finite concatenation of *elementary paths* $\gamma : \theta \mapsto \big(\bar{u}^\theta,\, f(\tilde{u}^\theta)\big)$ of the form

$$\begin{aligned} &\bar{u}^\theta(x) = \textstyle\sum_{\alpha=1}^{n} \overline{\omega}_\alpha \cdot \chi_{]x^\theta_{\alpha-1},\, x^\theta_\alpha]}(x), \quad x^\theta_\alpha = x_\alpha + \xi_\alpha\theta,\, x \geq 0, \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \theta \in [a,b], \\ &f(\tilde{u}^\theta(t)) = \textstyle\sum_{\beta=1}^{\widetilde{n}} f(\widetilde{\omega}_\beta) \cdot \chi_{]t^\theta_{\beta-1},\, t^\theta_\beta]}(t), \;\; t^\theta_\beta = t_\beta + \widetilde{\xi}_\beta\theta, \;\; t \geq 0, \end{aligned} \tag{3.28}$$

with $x^\theta_{\alpha-1} < x^\theta_\alpha$, $t^\theta_{\alpha-1} < t^\theta_\alpha$, for all $\theta \in [a,\, b]$ and $\alpha = 1,\dots,n$, $\beta = 1,\dots,\widetilde{n}$. Here, χ_I is the characteristic function of the interval I, $\overline{\omega}_\alpha$, $\widetilde{\omega}_\beta \in K^\nu$ are constant states and ξ_α, $\widetilde{\xi}_\beta$ are, respectively, the (space) shift rate of the jump in $\bar{u}^\theta$ at x_α, and the (time) shift rate of the jump in $f(\tilde{u}^\theta)$ at t_β. A simple example of pseudopolygonal joining two couples of initial data and boundary flux $\mathbf{fp}_1 = (\bar{u}_1,\; f(\tilde{u}_1))$, $\mathbf{fp}_2 = (\bar{u}_2,\; f(\tilde{u}_2))$, is given by

$$\theta \mapsto \big(\bar{u}_1 \cdot \chi_{[0,\theta[} + \bar{u}_2 \cdot \chi_{]\theta,+\infty[}\,,\;\; f(\tilde{u}_1) \cdot \chi_{[0,\theta[} + f(\tilde{u}_2) \cdot \chi_{]\theta,+\infty[}\big).$$

The $\mathbf{L}^1$ length of an elementary path γ of the form (3.28), is then computed by

$$\begin{aligned} \|\gamma\|_{\mathbf{L}^1} &= \int_a^b \left\{ \sum_{\alpha=1}^{N} \left|\Delta\bar{u}^\theta(x_\alpha)\right| \left|\frac{\partial x^\theta_\alpha}{\partial\theta}\right| + \sum_{\beta=1}^{\widetilde{N}} \left|\widetilde{\Delta}\tilde{u}^\theta(t_\beta)\right| \left|\frac{\partial t^\theta_\beta}{\partial\theta}\right| \right\} d\theta \\ &= \left\{ \sum_{\alpha=1}^{N} \left|\sigma_\alpha\right| |\xi_\alpha| + \sum_{\beta=1}^{\widetilde{N}} \left|\widetilde{\sigma}_\beta\right| |\widetilde{\xi}_\beta| \right\} (b-a), \end{aligned} \tag{3.29}$$

where

$$\sigma_\alpha \doteq \Delta\bar{u}^\theta(x_\alpha) = \overline{\omega}_{\alpha+1} - \overline{\omega}_\alpha, \qquad\qquad \widetilde{\sigma}_\beta \doteq \widetilde{\Delta}\tilde{u}^\theta(t_\beta) = f(\widetilde{\omega}_{\beta+1}) - f(\widetilde{\omega}_\beta). \tag{3.30}$$

If we consider a pseudopolygonal $\gamma_0 : \theta \mapsto \big(\bar{u}^\theta,\, f(\tilde{u}^\theta)\big)$, $\theta \in [0,\, 1]$, with values in $\mathcal{FD}^\nu$, and let $u^\theta_\nu(t,\cdot) = E^\nu_t(\bar{u}^\theta,\, \tilde{u}^\theta)$ be the corresponding solution, since the number of wave-fronts in these solutions is a-priori bounded and the locations of the interaction points in the t-x plane are determined by a linear system of equations, it follows that, at any time $t > 0$, the path

$$\gamma_t^\nu : \theta \mapsto \big(u_\nu^\theta(t, \cdot),\ \mathcal{T}_t f(\tilde{u}^\theta)\big) \qquad \theta \in [0,\, 1], \tag{3.31}$$

is still a pseudopolygonal. Moreover, there exist finitely many parameter values $0 = \theta_0 < \theta_1 < \ldots < \theta_m = 1$ such that the wave-front configuration of u_ν^θ remains the same as θ ranges on each of the open intervals $I_j \doteq]\theta_{j-1}, \theta_j[$. In this case, the length of the path γ_t^ν is measured by an expression of the form

$$\|\gamma_t^\nu\|_{\mathbf{L}^1} = \sum_{j=1}^m \int_{\theta_{j-1}}^{\theta_j} \sum_\alpha \big|\Delta u_\nu^\theta(t,\, x_\alpha^\theta)\big| \left|\frac{\partial x_\alpha^\theta}{\partial\theta}\right| d\theta + \sum_{j=1}^m \int_{\theta_{j-1}}^{\theta_j} \sum_{\{\beta\,:\, t_\beta^\theta > t\}} \big|\widetilde{\Delta}\tilde{u}^\theta(t_\beta^\theta)\big| \left|\frac{\partial t_\beta^\theta}{\partial\theta}\right| d\theta\,. \tag{3.32}$$

The second term of the sum in (3.32) is clearly uniformly bounded in time. Thus, to estimate the $\mathbf{L}^1$ distance between two approximate solutions $E_t^\nu \mathbf{p}_1$, $E_t^\nu \mathbf{p}_2$, it will be sufficient to provide an a-priori bound on the integrand of the first term in (3.32). In particular, we shall first fix $\mathcal{M} > 0$, and show that, for any given T, $\delta > 0$, there exists some constant $L_\mathcal{M} = L_\mathcal{M}(T,\, \delta)$ such that, letting $\pi_1((\bar{u},\ f(\tilde{u}))) = \bar{u}$ denote the canonical projection on the first component of any couple $\mathbf{fp} = (\bar{u},\ f(\tilde{u})) \in \mathcal{FD}^\nu$, there holds

$$\|\pi_1 \circ \gamma_t^\nu\|_{\mathbf{L}^1([\delta, +\infty[)} \le L_\mathcal{M} \cdot \|\gamma_0\|_{\mathbf{L}^1} \qquad\qquad t \in [0,\, T]\,, \tag{3.33}$$

for every pseudopolygonal $\gamma_0 : [0,\, 1] \to \mathcal{FD}^\nu$ joining two couples of initial data and boundary flux in $\{\mathbf{fp};\ \ \mathbf{p} \in D_\mathcal{M} \cap \mathcal{D}^\nu\}$. Introducing the seminorm

$$\big\|(\bar{u}, f(\tilde{u}))\big\|_\delta \doteq \big\|\bar{u}\big\|_{\mathbf{L}^1([\delta, +\infty[)} + \big\|f(\tilde{u})\big\|_{\mathbf{L}^1}, \tag{3.34}$$

and observing that the $\mathbf{L}^1$ lengths of the paths γ_0^ν, γ_t^ν, satisfy

$$\|\gamma_0^\nu\|_{\mathbf{L}^1} \le C_0 \cdot d(\mathbf{fp}_1,\, \mathbf{fp}_2),$$

$$\big\|E_t^\nu \mathbf{p}_1 - E_t^\nu \mathbf{p}_2\big\|_{\mathbf{L}^1([\delta, +\infty[)} \le C_0 \cdot \|\gamma_t^\nu\|_\delta, \tag{3.35}$$

for some constant $C_0 > 0$, we deduce from (3.33) a uniform Lipschitz estimates for the flow maps E^ν of the type

$$\big\|E_t^\nu \mathbf{p}_1 - E_t^\nu \mathbf{p}_2\big\|_{\mathbf{L}^1([\delta, +\infty[)} \le L'_\mathcal{M} \cdot d(\mathbf{fp}_1,\, \mathbf{fp}_2), \qquad t \in [0,\, T]\,, \tag{3.36}$$

for some other constant $L'_\mathcal{M} = L'_\mathcal{M}(T,\, \delta)$, and any $\mathbf{p}_1, \mathbf{p}_2 \in D_\mathcal{M} \cap \mathcal{D}^\nu$. As $\nu \to \infty$, the Lipshitz constant for E^ν remains uniformly bounded, while the domain $D_\mathcal{M} \cap \mathcal{D}^\nu$

become dense in $\mathcal{D}_{\mathcal{M}}$. In the limit, a continuous flow map E is obtained, defined on the domain $\mathcal{D}_M$, and satisfying the estimate (3.21).

To extend the flow map E to the whole domain $\mathcal{D}$ preserving the property (3.21), by similar arguments as above we will prove the estimate

$$\left\|E_t\mathbf{p}_1 - E_t\mathbf{p}_2\right\|_{\mathbf{L}^1([\delta,+\infty[)} \leq L'' \cdot d(\mathbf{fp}_1,\, \mathbf{fp}_2), \qquad t \in [0,\, T]\,, \tag{3.37}$$

for some constant $L'' = L''(T,\, \delta)$ independent on the total variation, and for any $\mathbf{p}_1, \mathbf{p}_2 \in \mathcal{D}^\mu$, $\mu \geq 1$. In the last part of the proof of Theorem 3.2.4, following the same technique adopted in the above analysis we establish a Lipshitz continuous dependence in space of the flow map $f(E)$ of the type

$$\left\|f\left(E_{(\cdot)}\mathbf{p}_1(x)\right) - f\left(E_{(\cdot)}\mathbf{p}_2(x)\right)\right\|_{\mathbf{L}^1([\tau_1,\,\tau_2])} \leq L''' \cdot d(\mathbf{fp}_1,\, \mathbf{fp}_2), \qquad x \in [0, \overline{\lambda}\,\tau_1]\,, \tag{3.38}$$

where $\overline{\lambda} \doteq \left(\inf_u \lambda_{n-p+1}(u)\right)$, and $L''' = L'''(\tau_1,\, \tau_2)$ is some constant depending on $\tau_2 > \tau_1 > 0$. Relying on this property, we prove the existence of the trace of $f(E_t\mathbf{p}(x))$ at $x = 0$. for any $\mathbf{p} \in \mathcal{D}$. Finally, we show that $E_t\mathbf{p}$ fulfills the boundary condition (3.15) and satisfies the entropy conditions (3.11)-(3.12).

3.4 Preliminary results

Throughout this section we fix $\nu \geq 1$ and consider a piecewise constant solution u constructed by the front tracking algorithm, so that

$$S_t^\nu(\bar{u}, \tilde{u}) = (E_t^\nu(\bar{u}, \tilde{u}), \mathcal{T}_t\tilde{u}) = (u(t, \cdot), \mathcal{T}_t\tilde{u}) \qquad \text{for some } (\bar{u}, \tilde{u}) \in \mathcal{D}^\nu.$$

We then perturb this solution, shifting the (space) locations x_α of the jumps in $\bar{u}$ at rates ξ_α, and the (time) locations t_α of the jumps in $\tilde{u}$ at rates $\tilde{\xi}_\alpha$ (fig. 3.1). In other words, for θ suitably close to zero, the perturbation $u^\theta(t, \cdot)$ will be a function with jumps at the points $x_\beta^\theta = x_\beta + \theta\xi_\beta$. In the same way, $u^\theta(\cdot, x)$ will jump at points $t_\beta^\theta = t_\beta - \theta\tilde{\xi}_\beta$ with $\tilde{\xi}_\beta = \xi_\beta / \lambda_{k_\beta}$, where λ_{k_β} is the slope of the discontinuity. As long as the wave-front configuration of the functions u, u^θ is the same, the *space-shifts* $\xi_\beta(t)$ and the *time-shifts* $\tilde{\xi}_\beta(x)$ are uniquely determined as linear functions of the shifts $\xi_\alpha,\ \tilde{\xi}_\alpha$.

Remark 3.4.1 We denote by $\sigma_\alpha(t) = u(t, x_\alpha+) - u(t, x_\alpha-)$ the strength of the jump of u at $(t, x_\alpha(t))$, and by $\tilde{\sigma}_\alpha(x) = f(u(t_\alpha+, x)) - f(u(t_\alpha-, x))$ the jump of the flux along

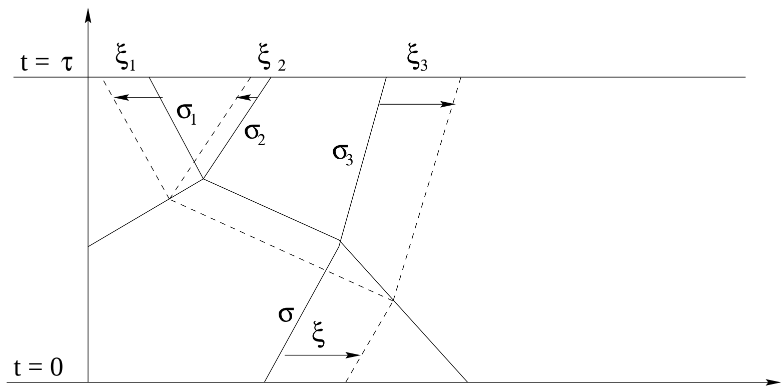

figure 3.1

the vertical direction. Since approximate solutions are indeed weak solutions, by the Rankine-Hugoniot equations we have the identity

$$\tilde{\xi}_\alpha \tilde{\sigma}_\alpha = \frac{\xi_\alpha}{\lambda_{k_\alpha}} \lambda_{k_\alpha} \sigma_\alpha = \xi_\alpha \sigma_\alpha .$$

In the following we will use both notations, depending on which one is more convenient.

We collect here some properties of these shift differentials (see [18]).

Lemma 3.4.2 *Consider a bounded, open region Γ in Ω. Call σ_α, $\alpha = 1, \dots, M$ the fronts entering Γ and let ξ_α be their shifts. Assume that the fronts leaving Γ, say σ_β, $\beta = 1, \dots, M'$, are linearly independent. Then the shifts ξ_β are uniquely determined by the linear relation*

$$\sum_{\alpha=1}^{M} \xi_\alpha \sigma_\alpha = \sum_{\beta=1}^{M'} \xi_\beta \sigma_\beta \, . \tag{3.39}$$

Remark 3.4.3 According to Lemma 3.4.2 the shift rates of the outgoing fronts depend only on the shift rates of the incoming ones, and not on the order in which these wavefronts interact inside Γ. More precisely, one can perform the following two operations, without changing the shift rates of the outgoing fronts:

(O1) Switch the order in which three fronts interact (fig. 3.2).

(O2) Invert the order of two fronts at $t = 0$ or at $x = 0$, provided that both fronts have zero shift rate (fig. 3.3).

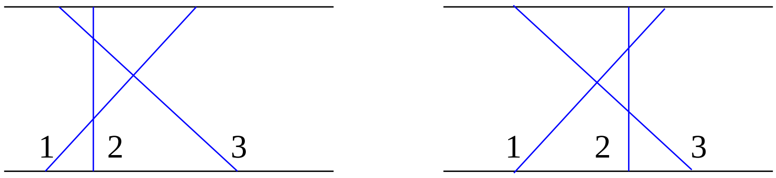

figure 3.2

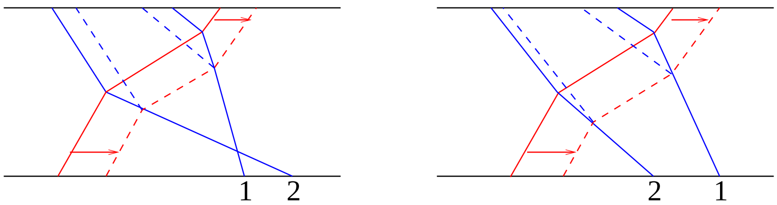

figure 3.3

This property will be repeatedly used in our future estimates. Indeed, in the computation of a shift rate, we can suitably alter the order of wave interactions and thus reduce the problem to a case where the wave-front configuration is particularly simple.

Lemma 3.4.4 *Assume that a front tracking solution u contains two wave-fronts of the same k-th family, originating at distinct points and located at $x_{\alpha'}(t) \leq x_{\alpha''}(t)$, $t \in [t_0,\ t_1]$, $0 \leq t_0 < t_1 \leq T$. Then it is possible to assign shift rates ξ_α to all fronts in $\bar{u}$ and $\tilde{u}$ so that $\xi_{\alpha'} = 1$ and moreover, in the corresponding solution u, all fronts $x_\beta(t)$ outside the strip $\Gamma \doteq \big\{(t,x) \in \Omega;\quad t \in [t_0'',\ t_1'],\ x_{\alpha'}(t) \leq x \leq x_{\alpha''}(t)\big\}$ have zero shift rate.*

In other words, the perturbation of the initial data can be chosen so that one particular front shifts at unit rate, but the corresponding solution remains unaffected outside the region Γ (fig. 3.4). For a proof of the lemma, proceed as in Chapter 2, noting that it is not restrictive to assume, if necessary, that there are no waves originating from the origin. In fact, if $\tilde{w}(0+) = (\tilde{w}_1, \ldots, \tilde{w}_n)$ and $\bar{w}(0+) = (\bar{w}_1, \ldots, \bar{w}_n)$ are the data at the origin, we can shift the jump on the t-axe by adding the state $w^* = \tilde{w}(0+) = (\tilde{w}_1, \ldots, \tilde{w}_{n-p}, \bar{w}_{n-p+1}, \ldots, \bar{w}_n)$.

Lemma 3.4.5 *Let u be a front-tracking solution of (3.1)-(3.3), and consider two wave-fronts, say $t \mapsto x(t)$, $t \mapsto y(t)$, defined for t in some subinterval of $[0,T]$. Call A_I, B_I, A_F, B_F the starting and the ending points respectively (they can be both on the x and on*

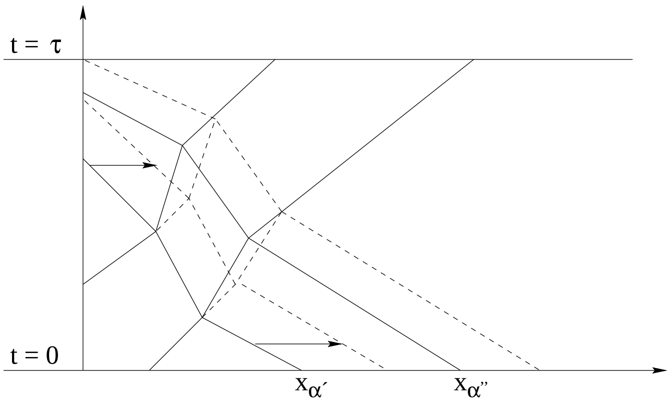

figure 3.4

the t axe). Then there exists a second front tracking solution $v = E^\nu \mathbf{p}$ *with two fronts* $\tilde{x}$, $\tilde{y}$ *such that the following holds.*

(i) $\tilde{A}_I = A_I$, $\tilde{B}_I = B_I$, $\tilde{A}_F = A_F$, $\tilde{B}_F = B_F$.

(ii) $v = u$ *in a neighborhood of these points.*

(iii) Tot. Var.$\{\mathbf{p}(0)\} \leq C_0$, *for some constant* C_0 *depending only on the system* (1.1) *and on the set* K.

PROOF OF LEMMA 3.4.5. To fix the ideas, assume $A_I = (t_I,\ x_I)$, $B_I = (s_I,\ y_I)$ with $x_I \leq y_I$. The set $\{0\} \times I\!R^+ \cup [0,\ T] \times \{0\} \cup \{T\} \times I\!R^+ \setminus \{A_I,\ B_I,\ A_F,\ B_F\}$ consists of five connected components which we call J_i, $i = 1, \ldots, 5$. Assume that u contains two wave-fronts $t \mapsto z'(t)$, $t \mapsto z''(t)$, of the same k-th family, such that z'_I and z''_I lie in the same open set J_i and moreover z'_F and z''_F lie in the same J_j. We can then proceed as in [8], obtaining a new solution v with the property that, for each $k = 1, \ldots, n$ and $i, j \in \{1, \ldots, 5\}$, there exists at most one point $\bar{x} \in J_i$ where a positive k-wave originates, terminating within J_j, and similarly for negative k-waves. Of course, this implies that the total variation of $\tilde{u}$ is uniformly bounded, with a bound C_0 depending only on n and on the diameter of the set K. □

Due to genuine nonlinearity, the amount of positive waves in $u(t, \cdot)$ contained in a bounded interval $[a,\ b]$ dacays in time. We have the following result:

Lemma 3.4.6 *Consider a front tracking solution* $u(t, \cdot) = E^\nu_t(\bar{u}, \tilde{u})$, *with* $\bar{u}$, $\tilde{u}$ *containing together at most* M *shock fronts of the* k*-th family. Then, for each* $\tau > 0$ *and every* *bounded interval* $[a,\ b]$*, with* $0 < a \le b$*, one has*

$$\textit{Tot. Var.}\big\{w_k(\tau,\cdot);\ [a,\ b]\big\} \le \frac{2(b-a)}{\kappa\tau} + \|w_k\|_{L^\infty} + (M+1)2^{1-\nu}\,, \tag{3.40}$$

for $k = 1, \ldots, n-p$*,*

$$\begin{aligned}\textit{Tot. Var.}\big\{w_k(\tau,\cdot);\ [a,\ b]\big\} \le \frac{2C_1}{\kappa}\log\left\{\left[\frac{(\hat\lambda\tau+b)-|\hat\lambda\tau-b|}{2a}-1\right]^+ +1\right\}\\ +2\frac{(b-a)}{\kappa\tau} + \|w_k\|_{L^\infty} + (M+1)2^{1-\nu}\,,\end{aligned} \tag{3.41}$$

for $k = n-p+1, \ldots, n$*, for some constant* C_1 *depending only on the system.*

Proof of Lemma 3.4.6. We give the proof of the statement for $k \in \{n-p+1, \ldots, n\}$, the other case being easier. By Lemma 2.4.5 in Chapter 2 at time $\tau > 0$ any two adiacent k-rarefaction fronts of u are separated by a distance $\ge \kappa(\tau - t_0)\cdot 2^{-\nu}$, where $t_0 \ge 0$ is the beginning time of the rarefaction fronts. Hence the distance between rarefaction fronts exiting the boundary grows at least linearly with the distance from the t axe. Therefore, the number of rarefaction fronts starting from the boundary and crossing any interval $[a,b]$ with $0 < a < b \le \hat\lambda\tau$ is bounded by

$$1 + M + \frac{C_1}{\kappa\cdot 2^{-\nu}}\log\left(\frac{b}{a}\right)$$

for some constant C_1 depending on the system. The positive variation of $w_k(\tau,\cdot)$ on $[a,b]$, i. e. the total amount of upward jumps, thus satisfies

$$\text{Pos.Var.}\big\{w_k(\tau,\cdot);\ [a,\ b]\big\} \le (1+M)2^{-\nu} + \frac{C_1}{\kappa}\log\left(\frac{b}{a}\right). \tag{3.42}$$

On the contrary, if the usual decay estimate holds for the waves starting form $t = 0$:

$$\text{Pos.Var.}\big\{w_k(\tau,\cdot);\ [a,\ b]\big\} \le (1+M)2^{-\nu} + \frac{b-a}{\kappa\tau}\,. \tag{3.43}$$

In turn, the total variation of $w_k(\tau,\cdot)$ on a generic interval $[a,b]$ containing the point $\hat\lambda\tau$ is bounded by $\|w_k\|_{\mathbf{L}^\infty}$ plus twice the positive variation of w_k. Hence (3.40) holds. □

3.5 Estimates on shift differentials

In this section we recover the key estimate on shift differentials. We use the same techique as in Chapter 2, and we refer to it for the details.

Lemma 3.5.1 *Let $u(t,\cdot) = E_t^\nu(\bar u, \tilde u)$ be a front tracking solution, with $\bar u$, $\tilde u$ containing together M shocks. Assume that at time $t = 0$ (respectively on the boundary at $x = 0$) the fronts located at x_β (t_β respectively) are shifted with shift rate ξ_β ($\tilde\xi_\beta = \xi_\beta/\lambda_{k_\beta}$ respectively) and amplitude σ_β (or $\tilde\sigma_\beta$). Then there exists a constant C_3 depending only on the system (3.1) and on the domain K, such that for any $\delta > 0$, calling $\xi_\alpha(\tau)$, $\sigma_\alpha(\tau)$ the shift rates and the amplitudes of the fronts in $u(\tau,\cdot)$, $\tau \in [0,\ T]$, we have*

$$\sum_{x_\alpha(\tau)>\delta} \left|\xi_\alpha(\tau)\sigma_\alpha(\tau)\right| \le C_3(1+M2^{-\nu})(1+\log(T/\delta))\cdot\left(\sum_\beta \left|\xi_\beta\sigma_\beta\right| + \sum_\beta \left|\tilde\xi_\beta\tilde\sigma_\beta\right|\right). \quad (3.44)$$

Proof of Lemma 3.5.1. Assume first that only one front $\bar\sigma$ starting at $\bar x$ (or $\bar t$) is shifted, say of the k-th family, with shift rate $\bar\xi$. Consider one particular front, say x_{α^*}, of the j-th family, and call $\bar y \doteq x_{\alpha^*}(\tau)$ its terminal point. There are constants C_4, C_5, depending only on the system (3.1) and on the domain K, such that the following properties hold.

(P1) If x_{α^*} is precisely the k-front starting at $\bar x$ ($\bar t$ respectively), then

$$\left|\xi_{\alpha^*}(\tau)\sigma_{\alpha^*}(\tau)\right| \le C_5|\bar\xi\bar\sigma|.. \quad (3.45)$$

(P2) If x_{α^*} is a j-front, with $j \ne k$, and the backward j-characteristics ending at $\bar y$ include fronts starting from both sides of $\bar x$ ($\bar t$ respectively), then (3.45) again holds.

(P3) If x_{α^*} is a j-front, and the j-fronts ending at $\bar y$ start all at the same side of $\bar x$ ($\bar t$ respectively), one then has the sharper estimate

$$\left|\xi_{\alpha^*}(\tau)\right| \le C_4|\bar\xi\bar\sigma|. \quad (3.46)$$

For a detailed proof of **(P1)**–**(P3)** we refer to Chapter 2. The same techique holds, with minor changes. Here we limit ourselves to show the proof of **(P3)**.

Observe that, besides the fronts starting at $\bar x$ (or $\bar t$) and the ones ending at $\bar y$, one can single out four groups of waves:

(1) the waves starting on the left of $\bar x$ (respectively, after $\bar t$) and ending on the left of $\bar y$;

(2) the waves starting on the right of $\bar x$ (respectively, before $\bar t$) and ending on the right of $\bar y$;

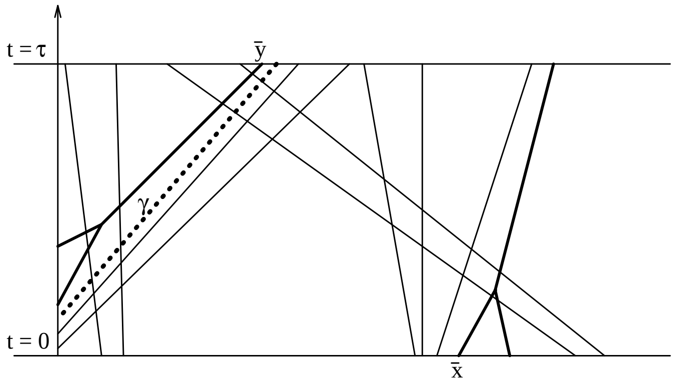

figure 3.5

(3) the waves starting on the right of $\bar{x}$ (respectively, before $\bar{t}$) and ending on the left of $\bar{y}$;

(4) the waves starting on the left of $\bar{x}$ (respectively, after $\bar{t}$) and ending on the right of $\bar{y}$.

According to Remark 3.4.3, in our computation of the shift rate $\xi_{\alpha^*}(\tau)$ of the front reaching $\bar{y}$, it is not restrictive not to take in account the sets of waves in (1) and (2). Indeed, we can otherwise shift the locations of all these fronts of type (1) towards the left, until they all lie outside the domain influenced by the shift at $\bar{x}$. Similarly, fronts of type (2) can be shifted toward the right until they lie completely outside this domain of influence.

Let's consider the case (fig. 3.5) where no j-wave ending at $\bar{y}$ crosses the k-wave starting at $\bar{x}$ (or $\bar{t}$). Consider a curve γ running slightly to the right of the minimal backward j-front ending at $\bar{y}$. By Lemmas 3.4.2, 3.4.4, after performing the operations **(O1)-(O2)** a number of times, we can consider an equivalent configuration with the following properties:

No front crosses γ from left to right. There exists some index $\ell \leq j$ such that only fronts of families $i < \ell$ can cross γ from right to left, and we can assume that the waves of type (1) have zero shift rate at every time in the interval $[0,\ \tau]$.

Applying Remark 3.4.3 to the region on the right of γ we obtain

$$\sum_{\alpha \in C(\gamma)} \xi_\alpha \sigma_\alpha + \sum_{x_\alpha(\tau) > \gamma(\tau)} \xi_\alpha(\tau)\sigma_\alpha(\tau) = \bar{\xi}\bar{\sigma}. \tag{3.47}$$

Here the first summation extends to all fronts crossing the curve γ with non-zero shift rate. Call u^- and u^+ the left and right states across the jump at $\bar{y}$. Observing that the two sums on the right hand side of (3.47) are contained in

$$\text{span}\{r_1(u^+), \dots, r_{\ell-1}(u^+)\}, \qquad \text{span}\{r_\ell(u^+), \dots, r_n(u^+)\}, \tag{3.48}$$

using the assumption **(SH)** on the linear independence of the eigenvectors, we conclude

$$\left| \sum_{\alpha \in C(\gamma)} \xi_\alpha \sigma_\alpha \right| \leq C_4 |\bar{\xi}\bar{\sigma}|. \tag{3.49}$$

We now apply again Remark 3.4.3 to the region on the left of γ. Observing that the only incoming fronts which carry a nonzero shift rate are those crossing γ from right to left, and that the only outgoing shifted j-front is the one ending at $\bar{y}$, we obtain

$$\sum_{x_\alpha(\tau) < \bar{y}} \xi_\alpha \sigma_\alpha + \xi_{\alpha^*}(\tau)\sigma_{\alpha^*}(\tau) = \sum_{\alpha \in C(\gamma)} \xi_\alpha \sigma_\alpha. \tag{3.50}$$

Recalling the normalization at (3.6), we observe that (3.50) implies

$$\left|\xi_{\alpha^*}(\tau)\sigma_{\alpha^*}(\tau)\right| = l_j(u^-) \cdot \sum_{\alpha \in C(\gamma)} \xi_\alpha \sigma_\alpha. \tag{3.51}$$

On the other hand,

$$l_j(u^+) \cdot \sum_{\alpha \in C(\gamma)} \xi_\alpha \sigma_\alpha = 0. \tag{3.52}$$

Together, (3.51) and (3.52) imply

$$\left|\xi_{\alpha^*}(\tau)\sigma_{\alpha^*}(\tau)\right| \leq C' \left|l_j(u^+) - l_j(u^-)\right| \left| \sum_{\alpha \in C(\gamma)} \xi_\alpha \sigma_\alpha \right| \leq C_5 \left|\sigma_{\alpha^*}(\tau)\right| |\bar{\xi}\bar{\sigma}|. \tag{3.53}$$

Hence, in this case, (3.46) holds.

We now prove **(P3)** in the case where $k > j$ and all j-waves running into $\bar{y}$ cross the k-wave starting from $\bar{t}$, as in figure 3.6. In this case, we construct a curve γ slightly to the left of the maximal backward j-front ending at $\bar{y}$. Observe that every wave-front crossing γ from left to right must be of a family $i > j$. Moreover, we can assume that no wave crosses γ from right to left. Applying Remark 3.4.3 to the region on the left of γ we obtain

$$\sum_{\alpha \in C(\gamma)} \xi_\alpha \sigma_\alpha + \sum_{x_\alpha(\tau) < \gamma(\tau)} \xi_\alpha(\tau)\sigma_\alpha(\tau) = \bar{\xi}\bar{\sigma}. \tag{3.54}$$

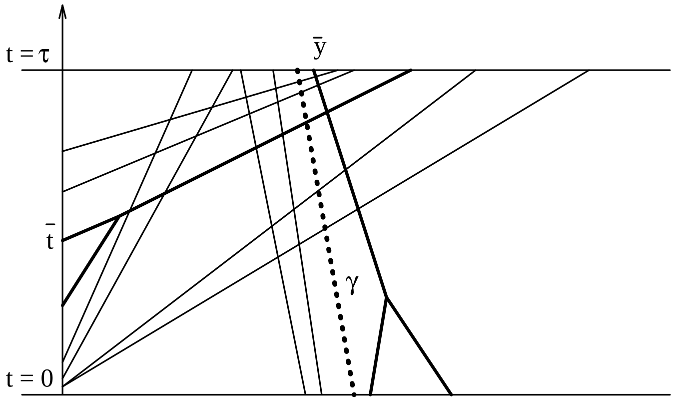

figure 3.6

Since the waves crossing γ must belong to different families from the ones ending inside the interval $[0,\ \gamma(\tau)]$ (recall that interacting waves of the same family produce a single wave front), (3.54) implies

$$\left| \sum_{\alpha \in C(\gamma)} \xi_\alpha \sigma_\alpha \right| \leq C_5 |\bar{\xi} \bar{\sigma}|. \tag{3.55}$$

We now apply again Remark 3.4.3 to the region on the right of γ, observing that the set of outgoing fronts, crossing the line $t = \tau$, contains the j-front at $\bar{y}$ plus other fronts on the right of $\bar{y}$, of families $i > j$. This yields

$$\xi_{\alpha^*}(\tau)\sigma_{\alpha^*}(\tau) + \sum_{x_\alpha(\tau) > \bar{y}} \xi_\alpha(\tau)\sigma_\alpha(\tau) = \sum_{\alpha \in C(\gamma)} \xi_\alpha \sigma_\alpha. \tag{3.56}$$

By (3.56),

$$\big| \xi_{\alpha^*}(\tau)\sigma_{\alpha^*}(\tau) \big| = l_j(u^+) \cdot \sum_{\alpha \in C(\gamma)} \xi_\alpha \sigma_\alpha. \tag{3.57}$$

Observing that

$$l_j(u^-) \cdot \sum_{\alpha \in C(\gamma)} \xi_\alpha \sigma_\alpha = 0,$$

we again have an estimate of the form (3.53), hence **(P3)** holds. The other cases are similar or easier.

We now complete the proof of Lemma 3.5.1. Let $\hat{\lambda}$ be an upper bound for the absolute values of all characteristic speeds. If the only front shifted is located at time $t = \bar{t}$ at $x = \bar{x}$, it follows that at a fixed time $\tau > 0$ the only fronts with nonzero shift rate can be the ones located inside the interval $[x_0,\ \bar{x}+\hat{\lambda}(\tau-\bar{t})]$, where $x_0 = \max\{0,\ \bar{x}-\hat{\lambda}(\tau-\bar{t})\}$. Recalling the estimate (3.40)-(3.41) on the total variation, using the properties **(P1)**–**(P3)** we thus have

$$\sum_{x_\alpha(\tau)>\delta} \left|\xi_\alpha(\tau)\sigma_\alpha(\tau)\right| \leq nC_5|\bar{\xi}\bar{\sigma}| + C_4|\bar{\xi}\bar{\sigma}| \cdot \text{Tot.Var.}\{u(\tau)\,;\quad [\delta,\ \bar{x}+\hat{\lambda}(\tau-\bar{t})]\}$$
$$\leq C_3(1+M2^{-\nu})(1+\log(\tau/\delta))|\xi\sigma|$$
$$\leq C_3(1+M2^{-\nu})(1+\log(T/\delta))|\bar{\xi}\bar{\sigma}|, \tag{3.58}$$

for a suitable constant C_3.

Assume now that all fronts in $\bar{u}$, $\tilde{u}$ are shifted. More precisely, let $\xi_\alpha(0)$ be the shift rate of the front located at $x_\alpha(0)$ ($t_\alpha(0)$ respectively), having amplitude $\sigma_\alpha(0)$. Call $\xi_\beta(\tau)$ the corresponding shift rate of the front of $u(\tau,\cdot)$ located at $x_\beta(\tau)$. Observing that the shift differential

$$\big(\xi_1(0),\dots\xi_M(0)\big) \mapsto \big(\xi_1(\tau),\dots\xi_{M'}(\tau)\big)$$

is a linear mapping, (3.44) easily follows from (3.58). □

In order to prove that the constructed semigroup of solutions satisfies the boundary condition, we will make use of the following result of continuous dependence along vertical segments.

Lemma 3.5.2 *Let $u(t,x) = E_t^\nu(\bar{u},\tilde{u})(x)$ be a front tracking solution containing at most M shocks. Assume that at $x=0$ the fronts located at t_β (whose amplitude will be misured by the jump $\tilde{\sigma}_\beta$ of the flux) are shifted with time-shift rate $\tilde{\xi}_\beta$. Then there exists some constant C_6 independent on the total variation such that, for any $t_2 > t_1 > 0$, denoting with $\tilde{\xi}_\alpha(x)$, $\tilde{\sigma}_\alpha(x)$ the shift and the amplitude of the fronts in u along the segment $\{(t,x);\ t\in[t_1,\ t_2]\}$, $0 < x < \big(\inf_u \lambda_{n-p+1}(u)\big)t_1$, we have*

$$\sum_{t_\alpha(x)\in[t_1,\ t_2]} \left|\tilde{\xi}_\alpha(x)\tilde{\sigma}_\alpha(x)\right| \leq C_6(1+M2^{-\nu})(1+\log(t_2/t_1)) \cdot \sum_{t_\beta} \left|\tilde{\xi}_\beta\tilde{\sigma}_\beta\right|, \tag{3.59}$$

Proof of Lemma 3.5.2. We give here a sketch of the proof, which is similar to the one of Lemma 3.5.1. The estimates **(P1)**–**(P3)** can be recovered here with minor modifications. As an example, we illustrate the case in which the particular front to be estimated at $x>0$ belongs to a family $j\in\{1,\dots.n-p\}$, so it started on the x axe (fig. 3.7). Call $\bar{s}$ the time in which this front crosses the line $\{(t,x);\ t\in[t_1,\ t_2]\}$.

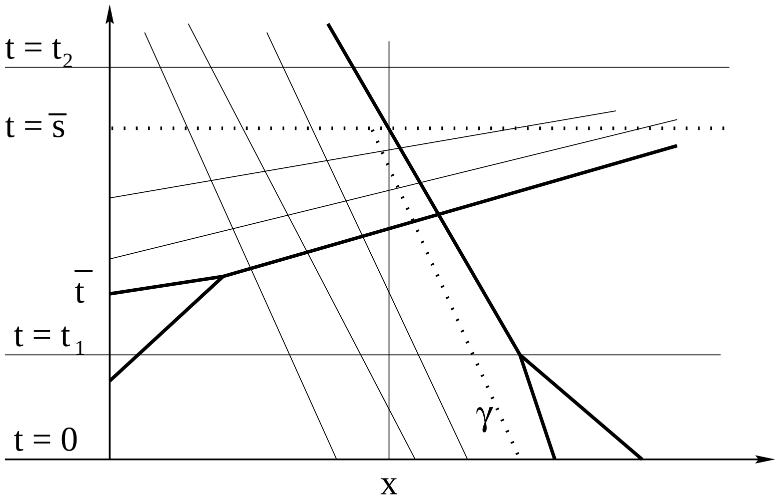

figure 3.7

Assume first that only one front $\tilde{\sigma}(0)$, starting at $t = \bar{t}$, is shifted, say of family $k \in \{n-p+1, \dots, n\}$. After performing the usual semplifications, we can reduce to the situation illustrated in figure 3.7. We consider the straight line $t = \bar{s}$ and a curve γ running slightly to the left of the maximal backward j-front passing through $(\bar{s}, x)$. Applying the divergence theorem to the region on the left of γ we obtain

$$\tilde{\xi}\tilde{\sigma}(0) = \sum_{\alpha \in C(\gamma)} \xi_\alpha \sigma_\alpha + \sum_{x_\alpha(\bar{s}) < \gamma(\bar{s})} \xi_\alpha(\bar{s}) \sigma_\alpha(\bar{s}). \tag{3.60}$$

By linear independence of the vectors on the left hand side of (3.60) we have

$$\left| \sum_{\alpha \in C(\gamma)} \xi_\alpha \sigma_\alpha \right| \leq C \big| \tilde{\xi}\tilde{\sigma}(0) \big|. \tag{3.61}$$

Then we consider the region on the right of γ, where we compute

$$\sum_{\alpha \in C(\gamma)} \xi_\alpha \sigma_\alpha = \xi\sigma(x) + \sum_{x_\alpha(\bar{s}) > x} \xi_\alpha(\bar{s}) \sigma_\alpha(\bar{s}). \tag{3.62}$$

From (3.62), by (3.6) and (3.61) we obtain

$$\begin{aligned} \big|\tilde{\xi}\tilde{\sigma}(\bar{s})\big| &= \big|\xi\sigma(x)\big| = \left| l_j(u^+) \cdot \sum_{\alpha \in C(\gamma)} \xi_\alpha \sigma_\alpha \right| \leq C \big| l_j(u^+) - l_j(u^-) \big| \left| \sum_{\alpha \in C(\gamma)} \xi_\alpha \sigma_\alpha \right| \\ &\leq C \big|\sigma(x)\big| \big|\tilde{\xi}\tilde{\sigma}(0)\big|, \end{aligned}$$

and we recover **(P3)**. Here u^- and u^+ denote as usual the left and right states across the jump at $(\bar{s}, x)$.

To get the final estimates, we make use of estimates on the total variation analogous to (3.40)-(3.41). More precisely, for any $b > a$ and $0 < x < \big(\inf_u \lambda_{n-p+1}(u)\big)a$, one gets

$$\text{Tot.Var.}\big\{w_k(\cdot, x);\ [a,\ b]\big\} \leq \frac{2C_1}{\kappa} \log\left(\frac{b}{a}\right) + \|w_k\|_{L^\infty} + (M+1)2^{1-\nu}\,, \tag{3.63}$$

for $k = 1, \ldots, n-p$,

$$\text{Tot.Var.}\big\{w_k(\cdot, x);\ [a,\ b]\big\} \leq \frac{2(b-a)}{\kappa x} + \|w_k\|_{L^\infty} + (M+1)2^{1-\nu}\,, \tag{3.64}$$

for $k = n-p+1, \ldots, n$, the proof being enterely analogous to the one of Lemma 3.4.6.
□

3.6 Proof of the theorems

1. Existence of the semigroup on domains of BV functions. In order to construct the semigroup described in Theorem 3.2.4, we shall first define a Lipschitz continuous flow map E on every domain

$$\mathcal{D}_{\mathcal{M}} \doteq \Big\{\mathbf{p} \in \mathcal{D};\quad \text{Tot.Var.}\{\mathbf{p}\} \leq \mathcal{M}\Big\} \qquad \mathcal{M} > 0,$$

obtained as a limit of the approximate flow maps E^ν constructed in Section 3.3 on the domains $\mathcal{D}^\nu$. To this end, consider any two piecewise constant couples of initial and boundary data, say $\mathbf{p}_1,\ \mathbf{p}_2 \in \mathcal{D}_{\mathcal{M}} \cap \mathcal{D}^\nu$, and construct a pseudopolygonal path $\gamma_0^\nu : \theta \mapsto \mathbf{fp}^\theta = (\bar{u}^\theta, f(\tilde{u}^\theta))$ connecting $\mathbf{fp}_1$ with $\mathbf{fp}_2$ as described in Section 3.3. All functions $(\bar{u}^\theta, \tilde{u}^\theta)$ lie in $\mathcal{D}_{\mathcal{M}} \cap \mathcal{D}^\nu$ and have a uniformly bounded number of shocks, say $\leq M$. Call $u_\nu^\theta(t, \cdot) = E_t^\nu(\bar{u}^\theta,\ \tilde{u}^\theta)$ the corresponding solution and consider the path $\gamma_t^\nu : \theta \mapsto \big(u_\nu^\theta(t, \cdot),\ \mathcal{T}_t f(\tilde{u}^\theta)\big)$. Writing the length of this path in the form (3.32), and using Lemma 3.5.1, for any fixed T, $\delta > 0$, and for every $t \in [0,\ T]$, we obtain the estimate

$$\begin{aligned}
\|\pi_1 \circ \gamma_t^\nu\|_{\mathbf{L}^1([\delta,+\infty[)} &= \sum_{j=1}^m \int_{\theta_{j-1}}^{\theta_j} \sum_{\{\alpha\ :\ x_\alpha^\theta > \delta\}} \big|\Delta u_\nu^\theta(t, x_\alpha^\theta)\big| \left|\frac{\partial x_\alpha^\theta(t)}{\partial \theta}\right| d\theta \\
&\leq \sum_{j=1}^m \int_{\theta_{j-1}}^{\theta_j} C_3(1 + M2^{-\nu})(1 + \log(T/\delta)) \\
&\qquad \left(\sum_\beta \big|\Delta u_\nu^\theta(0, x_\beta^\theta)\big| \left|\frac{\partial x_\beta^\theta(0)}{\partial \theta}\right| + \sum_{\beta'} \big|\widetilde{\Delta} u_\nu^\theta(t_{\beta'}^\theta, 0)\big| \left|\frac{\partial t_{\beta'}^\theta(0)}{\partial \theta}\right|\right) d\theta \\
&\leq C_3(1 + M2^{-\nu})(1 + \log(T/\delta)) \cdot \|\gamma_0^\nu\|_{\mathbf{L}^1}\,.
\end{aligned} \tag{3.65}$$

Observing that any function in $\mathcal{D}_{\mathcal{M}} \cap \mathcal{D}^\nu$ has at most $2^\nu \mathcal{M}$ jumps, from (3.65) we derive (3.33) with $L_{\mathcal{M}} = C_3(1+\mathcal{M})(1+\log(T/\delta))$, which, in turn, because of (3.35), clearly implies (3.36).

Once we have established the uniform Lipshitz continuity of the maps E_t^ν (and hence of the approximate semigroup S_t^ν) on the domains $\mathcal{D}_{\mathcal{M}} \cap \mathcal{D}^\nu$, since the union $\cup_{\nu \geq 1} \mathcal{D}_{\mathcal{M}} \cap \mathcal{D}^\nu$ is dense in $\mathcal{D}_{\mathcal{M}}$ we will define the map E on $\mathcal{D}_{\mathcal{M}}$ as the limit

$$E_t(\mathbf{p}) \doteq \mathbf{L}^1 - \lim_{\nu \to \infty} E_t^\nu(\mathbf{p}^\nu) \qquad \mathbf{p}^\nu \in \mathcal{D}_{\mathcal{M}} \cap \mathcal{D}^\nu,\ \mathbf{p}^\nu \to \mathbf{p} \text{ in } \mathbf{L}^1. \tag{3.66}$$

In order to prove that the assignment (3.66) yields a well-defined map, since any sequence $E_t^\nu \mathbf{p}^\nu$ is uniformly bounded in $\mathbf{L}^\infty$, it is sufficient to show that, for every given $\mathbf{p} \in \mathcal{D}_{\mathcal{M}}$, and for any $\delta > 0$, if $\mathbf{p}^\nu \in \mathcal{D}_{\mathcal{M}} \cap \mathcal{D}^\nu$ is any sequence that converges to $\mathbf{p}$ in $\mathbf{L}^1$, then the sequence $E_t^\nu \mathbf{p}^\nu \restriction_{[\delta, +\infty[}$, is Cauchy in $\mathbf{L}^1$. Indeed, for any $\mu > \nu$, using (3.35) (possibly with a different constant $L'_{\mathcal{M}}$), we obtain

$$\begin{aligned}\left\|E_t^\mu \mathbf{p}^\mu - E_t^\nu \mathbf{p}^\nu\right\|_{\mathbf{L}^1([\delta,+\infty[)} &\leq \left\|E_t^\mu \mathbf{p}^\mu - E_t^\mu \mathbf{p}^\nu\right\|_{\mathbf{L}^1([\delta,+\infty[)} + \left\|E_t^\mu \mathbf{p}^\nu - E_t^\nu \mathbf{p}^\nu\right\|_{\mathbf{L}^1([\delta,+\infty[)} \\ &\leq L'_{\mathcal{M}} \cdot d(\mathbf{p}^\mu, \mathbf{p}^\nu) + \left\|S_t^\mu \mathbf{p}^\nu - S_t^\nu \mathbf{p}^\nu\right\|_\delta,\end{aligned} \tag{3.67}$$

where $\|\cdot\|_\delta$ denotes the seminorm introduced in (3.34). To estimate the second term in (3.67), we shall use the same type of error estimate established in [14, Theorem 2.9] for the distance between a Lipschitz continuous map and the trajectory of a Lipshitz continuous semigroup which can be restated as follows.

Lemma 3.6.1 *Let $\mathcal{D} \subset B$, $\mathcal{D}' \subset B'$ be subsets of two Banach spaces $(B, \|\cdot\|_B)$, $(B', \|\cdot\|_{B'})$. Let $S : \mathcal{D} \times [0, T] \mapsto \mathcal{D}$ be a continuous semigroup, $\Gamma : [0, T] \mapsto \mathcal{D}$ be a continuous map, and let $\rho : \mathcal{D} \mapsto \mathcal{D}'$ be a map such that*

$$\begin{aligned}\left\|\rho(S_t \mathbf{p}_1) - \rho(S_s \mathbf{p}_2)\right\|_{B'} &\leq L \cdot \left\{\left\|\mathbf{p}_1 - \mathbf{p}_2\right\|_B + |t-s|\right\}, \\ \left\|\rho(\Gamma(t)) - \rho(\Gamma(s))\right\|_{B'} &\leq L \cdot |t-s|,\end{aligned} \tag{3.68}$$

for some constant $L > 0$. Then, for any $\tau \in [0, T]$, one has the estimate

$$\left\|\rho(\Gamma(\tau)) - \rho(S_\tau \Gamma(0))\right\|_{B'} \leq L \cdot \int_0^\tau \left\{\liminf_{h \to 0+} \frac{\left\|\rho(\Gamma(t+h)) - \rho(S_h \Gamma(t))\right\|_{B'}}{h}\right\} dt. \tag{3.69}$$

Set

$$\mathcal{D} \doteq \mathcal{D}_{\mathcal{M}} \cap \mathcal{D}^\mu, \qquad \mathcal{D}' \doteq \left\{(\bar{u} \restriction_{[\delta,+\infty[}, \tilde{u})\ ;\ \mathbf{p} = (\bar{u}, \tilde{u}) \in \mathcal{D}\right\},$$

and let $\rho \doteq \rho_\delta : \mathcal{D} \to \mathcal{D}'$ be the restriction map $\mathbf{p} = (\bar{u}, \tilde{u}) \mapsto \rho(\mathbf{p}) = \big(\bar{u} \restriction_{[\delta, +\infty[}, \tilde{u}\big)$. Observe that, if we let $\Gamma : [0, T] \to \mathcal{D}_{\mathcal{M}} \cap \mathcal{D}^\nu \subset \mathcal{D}_{\mathcal{M}} \cap \mathcal{D}^\mu$ be the map $\Gamma(t) = S_t^\nu \mathbf{p}^\nu$, and $S \doteq S^\mu$, the Lipschitz continuity of the maps S^ν, S^μ, and the uniform bound on the total variation of the trajectories $t \mapsto E_t^\nu \mathbf{p}$, $\mathbf{p} \in \mathcal{D}_{\mathcal{M}} \cap \mathcal{D}^\nu$, clearly imply the estimates (3.68). Thus, we may apply Lemma 3.6.1 and, from (3.67), (3.69), we derive

$$\left\|E_t^\mu \mathbf{p}^\mu - E_t^\nu \mathbf{p}^\nu\right\|_{\mathbf{L}^1([\delta,+\infty[)} \le L'_{\mathcal{M}} \cdot d(\mathbf{p}^\mu, \mathbf{p}^\nu) + L \cdot \int_0^t \left\{ \liminf_{h \to 0+} \frac{\left\|S_{s+h}^\nu \mathbf{p}^\nu - S_h^\mu S_s^\nu \mathbf{p}^\nu\right\|_{\mathbf{L}^1}}{h} \right\} ds. \tag{3.70}$$

With the same arguments in [8], letting $\mathbf{q} \doteq S_s^\nu \mathbf{p}^\nu$, we can now estimate the integrand in (3.70) by

$$\frac{1}{h}\left\|S_h^\nu \mathbf{q} - S_h^\mu \mathbf{q}\right\|_{\mathbf{L}^1} = \frac{1}{h}\left\|E_h^\nu \mathbf{q} - E_h^\mu \mathbf{q}\right\|_{\mathbf{L}^1} \le C_7 \cdot 2^{-\nu} \mathcal{M}, \tag{3.71}$$

for some constant $C_7 > 0$. Hence, (3.70) together with (3.71) yields

$$\left\|E_t^\mu \mathbf{p}^\mu - E_t^\nu \mathbf{p}^\nu\right\|_{\mathbf{L}^1([\delta,+\infty[)} \le L'_{\mathcal{M}} \cdot d(\mathbf{p}^\mu, \mathbf{p}^\nu) + LC_7 \mathcal{M} \cdot 2^{-\nu} t, \tag{3.72}$$

which clearly shows that $E_t^\nu \mathbf{p}^\nu \restriction_{[\delta,+\infty[}$ is a Cauchy sequence in the $\mathbf{L}^1$ norm and that his limit does not depend on the choice of the sequence $\mathbf{p}^\nu$. Thus, the map in (3.66) is well-defined on every domain $\mathcal{D}_{\mathcal{M}}$.

2. Extension of the semigroup to domains of $\mathbf{L}^\infty$ functions. To ensure the existence of the map E on the whole domain $\mathcal{D}$ of functions of possibly unbounded variation, we will prove now the estimate (3.37) for some Lipschitz constant L'' independent on the total variation. To this purpose, consider any two couples $\mathbf{p}_1$, $\mathbf{p}_2 \in \mathcal{D}^\mu$, and construct as above a pseudopolygonal path $\gamma_0 : \theta \mapsto \mathbf{fp}^\theta = (\bar{u}^\theta, f(\tilde{u}^\theta))$ taking values in $\mathcal{F}\mathcal{D}^\mu$, that connects $\mathbf{fp}_1$ with $\mathbf{fp}_2$ and has the following property. All functions $(\bar{u}^\theta, \tilde{u}^\theta)$ have a uniformly bounded number of jumps and hence lie in some domain $\mathcal{D}_{\mathcal{M}}$, $\mathcal{M} > 0$. Then, calling $u_\nu^\theta(t, \cdot) = E_t^\nu(\bar{u}^\theta, \tilde{u}^\theta)$ the corresponding ν-approximate solution, since by (3.66) we have

$$E_t(\bar{u}^\theta, \tilde{u}^\theta) = \lim_{\nu \to \infty} E_t^\nu(\bar{u}^\theta, \tilde{u}^\theta) = \lim_{\nu \to \infty} u_\nu^\theta(t, \cdot), \tag{3.73}$$

in order to establish (3.37) we will show that the length of the path $\gamma_t^\nu : \theta \mapsto \big(u_\nu^\theta(t, \cdot), \mathcal{T}_t f(\tilde{u}^\theta)\big)$ remains a bounded multiple of the length of γ_0, independent on ν. Indeed, for any fixed T, $\delta > 0$, and for every $\nu \ge \mu$, letting M be a uniform bound on the number of shocks in $(\bar{u}^\theta, \tilde{u}^\theta)$, and using Lemma 3.5.1, we obtain by the same arguments in (3.65) the estimate

$$\|\pi_1 \circ \gamma_t^\nu\|_{\mathbf{L}^1([\delta,+\infty[)} \le C_3(1 + M2^{-\nu})(1 + \log(T/\delta)) \cdot \|\gamma_0\|_{\mathbf{L}^1}$$

which, in turn, because of (3.35), implies

$$\left\|E_t^\nu \mathbf{p}_1 - E_t^\nu \mathbf{p}_2\right\|_{\mathbf{L}^1([\delta,+\infty[)} \leq C_8(1+M2^{-\nu})(1+\log(T/\delta)) \cdot d(\mathbf{fp}_1,\, \mathbf{fp}_2) \tag{3.74}$$

for some other constant $C_8 > 0$. Letting $\nu \to \infty$ in (3.74), because of (3.73) we obtain (3.37) for all $\mathbf{p}_1,\ \mathbf{p}_2 \in \mathcal{D}^\mu$. Since the domains $\mathcal{D}^\mu,\ \mu \geq 1$, are dense in $\mathcal{D}$, relying on (3.37) we can now extend the map E by continuity to the whole domain $\mathcal{D}$ setting

$$E_t(\mathbf{p}) \doteq \mathbf{L}^1 - \lim_{\mu\to\infty} E_t(\mathbf{p}^\mu) \qquad\qquad \mathbf{p}^\mu \in \mathcal{D}^\mu,\ \mathbf{p}^\mu \to \mathbf{p} \text{ in } \mathbf{L}^1\,. \tag{3.75}$$

Clearly, the map in (3.75) preserves the property (3.37), proving (3.21). Moreover, any trajectory $t \mapsto E_t(\mathbf{p})$, being the limit of front tracking approximations, provides by standard arguments [10, 14] a weak solution to problem (3.1)-(3.2).

3. Lipschitz continuity in space. Towards a proof of the existence of the trace of $f(u(t,x)) \doteq f\big(E_t\mathbf{p}(x)\big)$ at the boundary $x = 0$, we shall first establish the Lipshitz continuous dipendence in space (3.38) for the map $f(E)$. Fix $\tau_2 > \tau_1 > 0$, and observe that, because of (3.66), for every given $\mathbf{p} \in \mathcal{D}_\mathcal{M}$, the sequence $E^\nu_{(\cdot)}\mathbf{p}$ converges to $E_{(\cdot)}\mathbf{p}$ in $\mathbf{L}^1\big([0,\,\tau_2] \times I\!R^+;\, K\big)$. Hence, relying also on the continuity of the maps $x \mapsto E^\nu_{(\cdot)}\mathbf{p},\ x \mapsto E_{(\cdot)}\mathbf{p}$, we deduce that

$$f\big(E_{(\cdot)}(\mathbf{p})(x)\big)\restriction_{[\tau_1,\,\tau_2]} = \mathbf{L}^1 - \lim_{\nu\to\infty} f\big(E^\nu_{(\cdot)}(\mathbf{p})(x)\big)\restriction_{[\tau_1,\,\tau_2]} \qquad \text{for all } x. \tag{3.76}$$

Therefore we may proceed as in the proof of (3.37) to establish the estimate (3.38) for any given pair of couples $\mathbf{p}_1,\ \mathbf{p}_2 \in \mathcal{D}^\mu$. We construct a pseudopolygonal path $\gamma_0 : \theta \mapsto \mathbf{fp}^\theta = (\bar u^\theta, f(\tilde u^\theta))$ taking values in $\mathcal{FD}^\mu$, and connecting $\mathbf{fp}_1$ with $\mathbf{fp}_2$, so that all functions $(\bar u^\theta, \tilde u^\theta)$ have a uniformly bounded number of shocks $\leq M$, and lie in some domain $\mathcal{D}_\mathcal{M},\ \mathcal{M} > 0$. Then, for every $\nu \geq \mu$, calling $u^\theta_\nu(t,\cdot) \doteq E^\nu_t(\bar u^\theta,\, \tilde u^\theta)$ the corresponding ν-approximate solution, we consider the pseudopolygonal path

$$\gamma^\nu_x : \theta \mapsto \big(\mathcal{T}_x \bar u^\theta,\, f\big(u^\theta_\nu(\cdot,\, x)\big)\big) \tag{3.77}$$

with values in $\mathcal{FD}^\nu$. Letting $\pi_2((\bar u,\, f(\tilde u))) = f(\tilde u)$ denote the canonical projection on the second component of $(\bar u,\, f(\tilde u))$, and using Lemma 3.5.2, for every $0 < x < \big(\inf_u \lambda_{n-p+1}(u)\big)\tau_1$ we compute as in (3.65)

$$\|\pi_2 \circ \gamma^\nu_x\|_{\mathbf{L}^1([\tau_1,\tau_2])} = \sum_{j=1}^m \int_{\theta_{j-1}}^{\theta_j} \sum_{\{\alpha\,:\, t^\theta_\alpha \in[\tau_1,\tau_2]\}} \big|\widetilde\Delta u^\theta_\nu(t^\theta_\alpha, x)\big| \left|\frac{\partial t^\theta_\alpha(x)}{\partial\theta}\right| d\theta$$

$$\leq \sum_{j=1}^{m} \int_{\theta_{j-1}}^{\theta_j} C_6(1+M2^{-\nu})(1+\log(\tau_2/\tau_1))$$
$$\left(\sum_{\beta} \left|\Delta u_\nu^\theta(0, x_\beta^\theta)\right| \left| \frac{\partial x_\beta^\theta(0)}{\partial \theta} \right| + \sum_{\beta'} \left|\widetilde{\Delta} u_\nu^\theta(t_{\beta'}^\theta, 0)\right| \left| \frac{\partial t_{\beta'}^\theta(0)}{\partial \theta} \right| \right) d\theta$$
$$\leq C_6(1+M2^{-\nu})(1+\log(\tau_2/\tau_1)) \cdot \|\gamma_0\|_{\mathbf{L}^1}. \tag{3.78}$$

Introducing the seminorm

$$\left\|(\bar{u}, f(\tilde{u}))\right\|_{[\tau_1,\tau_2]} \doteq \left\|\bar{u}\right\|_{\mathbf{L}^1} + \left\|f(\tilde{u})\right\|_{\mathbf{L}^1([\tau_1,\tau_1])},$$

and observing that the $\mathbf{L}^1$ length of the path γ_x^ν satisfies

$$\left\|f\left(E_{(\cdot)}^\nu \mathbf{p}_1(x)\right) - f\left(E_{(\cdot)}^\nu \mathbf{p}_2(x)\right)\right\|_{\mathbf{L}^1([\tau_1,\tau_1])} \leq C_9 \cdot \|\gamma_x^\nu\|_{[\tau_1,\tau_2]}, \tag{3.79}$$

for some constant $C_9 > 0$, we deduce from (3.78) the estimate

$$\left\|f\left(E_{(\cdot)}^\nu \mathbf{p}_1(x)\right) - f\left(E_{(\cdot)}^\nu \mathbf{p}_2(x)\right)_{\mathbf{L}^1([\tau_1,\tau_1])} \leq C_{10}(1+M2^{-\nu})(1+\log(\tau_2/\tau_1)) \cdot d(\mathbf{fp}_1, \mathbf{fp}_2) \tag{3.80}$$

for some other constant $C_{10} > 0$. Letting $\nu \to \infty$ in (3.79), thanks to (3.76) we obtain (3.38) for all $\mathbf{p}_1,\ \mathbf{p}_2 \in \mathcal{D}^\mu$. By continuity, and relying on the density of the domains $\mathcal{D}^\mu,\ \mu \geq 1$ in $\mathcal{D}$, we then extend the estimate (3.38) to any pair $\mathbf{p}_1,\ \mathbf{p}_2$ in $\mathcal{D}$.

4. Boundary conditions. Let $u(t,x) \doteq S_t\mathbf{p}(x)$, $\mathbf{p} = (\bar{u},\ \tilde{u}) \in D$, be the weak solution constructed above, and consider a sequence $\mathbf{p}^\nu = (\bar{u}^\nu,\ \tilde{u}^\nu) \in \mathcal{D}^\nu$ converging to $\mathbf{p}$ in $\mathbf{L}^1$. Call $u^\nu(t,x) \doteq S_t\mathbf{p}^\nu(x)$ the corresponding solution. Since every $\mathbf{p}^\nu$ lies in some domain $D_{\mathcal{M}_\nu}$, all functions $u^\nu(t,x)$ have bounded total variation and satisfy the boundary condition (3.15)

$$\lim_{x\to 0+} w_j\big(u^\nu(t,x)\big) = w_j\big(\tilde{u}^\nu(t)\big) \qquad \text{for a.e.} \quad t \geq 0, \quad j = n-p+1,\dots,n. \tag{3.81}$$

Now, fix $\tau_2 > \tau_1 > 0$. By (3.38) and because of the invertibility property of the flux function f, let $C_{11} > 0$ be some constant such that

$$\left\|w_j\big(u^\nu(\cdot,x)\big) - w_j\big(u(\cdot,x)\big)\right\|_{\mathbf{L}^1([\tau_1,\ \tau_2])} \leq C_{11} \cdot d(\mathbf{p}^\nu,\ \mathbf{p}). \tag{3.82}$$

Then, (3.81), (3.82) together imply that, for any $j = n-p+1,\dots,n$, the functions $w_j\big(u(\cdot,x)\big)$, $w_j\big(f(u(\cdot,x))\big)$ have a strong limit as $x \to 0$ and

$$\lim_{x\to 0+} \int_{\tau_1}^{\tau_2} \left|w_j\big(u(t,x)\big) - w_j\big(\tilde{u}(t)\big)\right| dt = 0, \tag{3.83}$$

thus showing that $u(t,x)$ fulfills the boundary condition (3.15). On the other hand, by the decay of the total variation, also $w_j\big(f(u(\cdot,x))\big)$, $j=1,\dots,n-p$, have a strong limit as $x\to 0$, which completes the proof of the existence of the trace of $f(u(t,x))$ at $x=0$, according with Definition 3.2.2.

5. Uniqueness. Concerning the entropy conditions (3.11)-(3.12), fix an interval $[a,b]$ and a couple of initial-boundary conditions $(\bar u,\tilde u)$. We can now approximate the solution $u(t,\cdot)=E_t(\bar u,\tilde u)$ with a sequence of front tracking solutions $u^\nu(t,\cdot)=E_t^\nu(\bar u^\nu,\tilde u^\nu)$, choosing initial data $\bar u^\nu$, $\tilde u^\nu$ having a number of shocks $N_\nu\le\nu$. By (3.40)-(3.41), the total number of positive wave-fronts in $u^\nu(\tau,\cdot)=E_\tau^\nu(\bar u^\nu,\tilde u^\nu)$ on a given interval $[a,b]$ satisfies

$$\text{Pos.Var.}\big\{w_k^\nu(\tau,\cdot);\ [a,\ b]\big\}\le\frac{(b-a)}{\kappa\tau}+(N_\nu+1)2^{1-\nu}, \tag{3.84}$$

for $k=1,\dots,n-p$,

$$\text{Pos.Var.}\big\{w_k^\nu(\tau,\cdot);\ [a,\ b]\big\}\le\frac{C_2}{\kappa}\log\left\{\left[\frac{(\hat\lambda\tau+b)-|\hat\lambda\tau-b|}{2a}-1\right]^+ +1\right\} +\frac{(b-a)}{\kappa\tau}+(N_\nu+1)2^{1-\nu}, \tag{3.85}$$

for $k=n-p+1,\dots,n$, where $w_k^\nu\doteq w_k(u^\nu)$ is the k-th Riemann coordinate of u^ν. Letting $\nu\to\infty$ in (3.73)-(3.74), by the lower semicontinuity of the total variation we obtain

$$\text{Pos.Var.}\big\{w_k(\tau,\cdot);\ [a,\ b]\big\}\le\frac{(b-a)}{\kappa\tau}, \tag{3.86}$$

for $k=1,\dots,n-p$,

$$\text{Pos.Var.}\big\{w_k(\tau,\cdot);\ [a,\ b]\big\}\le\frac{C_2}{\kappa}\log\left\{\left[\frac{(\hat\lambda\tau+b)-|\hat\lambda\tau-b|}{2a}-1\right]^+ +1\right\}+\frac{(b-a)}{\kappa\tau}, \tag{3.87}$$

for $k=n-p+1,\dots,n$. Hence (3.11)-(3.12) holds. This completes the proof of Theorem 3.2.4.

Regarding Theorem 3.2.6, let us fix any $R>0$ and let $\hat\lambda$ be an upper bound for the absolute value of all characteristic speeds. Let u be a weak entropy solution according to Definition 3.2.2 (and continuous in x=0, t=0). For every $\delta>0$ the restriction of $u(t,\cdot)$ to the intervals $I(t)\doteq[\delta,R-\hat\lambda t]$, $t\in[\delta,R/\hat\lambda]$ has uniformly bounded total variation. Therefore, the uniqueness theorem in Chapter 1, Theorem 1.1.3 yields

$$u(t,\cdot)=E_{t-\delta}\big(u(\delta,\delta+\cdot),u(\delta+\cdot,\delta)\big)\qquad\text{restricted to } I(t),$$

$$
\begin{aligned}
&\int_{I(t)} \left|u(t,x) - E_t(\bar{u},\tilde{u})(x)\right| dx \\
&= \int_{I(t)} \left|E_{t-\delta}\big(u(\delta,\delta+\cdot),u(\delta+\cdot,\delta)\big)(x) - E_{t-\delta}\big(E_\delta(\bar{u},\tilde{u})(\delta+\cdot),E_{\delta+\cdot}(\bar{u},\tilde{u})(\delta)\big)(x)\right| dx \\
&\le L\cdot \left\|u(\delta,\cdot) - E_\delta\big(\bar{u},\tilde{u}\big)\right\|_{\mathbf{L}^1} + \left\|f\big(u(\delta+\cdot,\delta)\big) - f\big(E_{\delta+\cdot}(\bar{u},\tilde{u})(\delta)\big)\right\|_{\mathbf{L}^1}. \qquad (3.88)
\end{aligned}
$$

Letting $\delta \to 0$ we conclude that the left-hand side of (3.75) must be zero. Since R was arbitrary, this proves Theorem 3.2.6.

3.7 Properties of the attainable sets

Following [5, 6] we focus the attention on the mixed initial-boundary value problem

$$
\begin{aligned}
&u_t + f(u)_x = 0, && && (3.89)\\
&u(0,x) = 0, && t,x>0, && (3.90)\\
&u(t,0) = \tilde{u}(t), && && (3.91)
\end{aligned}
$$

from the point of view of control theory, regarding the boundary data $\tilde{u}$ as a control. We shall be concerned with the basic properties of the attainable sets for (3.89)-(3.91)

$$
\mathcal{A}(T,\mathcal{U}) \doteq \Big\{u(T,\cdot):\ u \text{ is a weak entropic solution to (3.89)-(3.91) with } \tilde{u}\in\mathcal{U}\Big\},
$$

$$
\mathcal{A}(\bar{x},\mathcal{U}) \doteq \Big\{u(\cdot,\bar{x}):\ u \text{ is a weak entropic solution to (3.89)-(3.91) with } \tilde{u}\in\mathcal{U}\Big\},
$$

which consist of all profiles that can be attained at a fixed time $T > 0$ and at a fixed point $\bar{x} > 0$ by solutions of (3.89)-(3.91) with boundary data that varies inside a given class $\mathcal{U} \subseteq \mathbf{L}^\infty(I\!R^+)$ of admissible boundary controls. In particular, we establish the compactness of these sets.

Theorem 3.7.1 *Consider a set K of the form (3.16) and define*

$$
\mathcal{U} \doteq \Big\{\tilde{u}\in K;\ w_j(f(\tilde{u})) \in [\alpha_j,\ \beta_j],\ j = n-p+1,\dots,n\Big\}
$$

for some $-\infty < \alpha_j \le \beta_j < +\infty$. Then $\mathcal{A}(T,\mathcal{U})$, $T>0$, and $\mathcal{A}(\bar{x},\mathcal{U})$, $\bar{x}>0$, are compact subsets of $\mathbf{L}^1(I\!R^+)$ and $\mathbf{L}^1{}_{loc}(I\!R^+)$ respectively.

PROOF OF THEOREM 3.7.1. We will show the proof for the set $\mathcal{A}(T,\mathcal{U})$. For the set $\mathcal{A}(\bar{x},\mathcal{U})$ the procedure is enterely similar. Let $\{u^\nu\}_{\nu\in\mathcal{N}}$ be a sequence of weak entropic solutions of (3.89)-(3.91) and $\{\tilde{u}^\nu\}_{\nu\in\mathcal{N}} \subset \mathcal{U}$ the corresponding boundary data.

Denote by w_i^ν the Riemann coordinates of u^ν. Being K bounded and by finite propagation speed there exist $C,\ \alpha > 0$ such that

$$|w_i^\nu(t,x)| \leq \begin{cases} C, & \text{if } x < \alpha, \\ 0, & \text{if } x \geq \alpha. \end{cases} \qquad \forall t \in [0,\ T], \qquad \forall \nu \in \mathcal{N}. \tag{3.92}$$

Hence $\{w_i^\nu(T,\cdot)\}_{\nu\in\mathcal{N}}$, $\{w_i^\nu(\cdot,\cdot)\}_{\nu\in\mathcal{N}}$ are weak* relatively compact in $\mathbf{L}^\infty(I\!R^+)$, $\mathbf{L}^\infty(\Omega)$ respectively. Thus, up to a subsequence, we have

$$w_i^\nu(T,\cdot) \stackrel{*}{\rightharpoonup} w_i \qquad \text{in } \mathbf{L}^\infty(I\!R^+), \tag{3.93}$$

$$w_i^\nu \stackrel{*}{\rightharpoonup} v_i \qquad \text{in } \mathbf{L}^\infty(\Omega), \tag{3.94}$$

for some functions $w \in \mathbf{L}^\infty(I\!R^+, I\!R^n)$, $v \in \mathbf{L}^\infty(\Omega, I\!R^n)$. We shall prove that $u \doteq u(w) \in \mathcal{A}(T,\mathcal{U})$ and that there exists a subsequence of $\{w_i^\nu(T,\cdot)\}_{\nu\in\mathcal{N}}$ converging to w_i in $\mathbf{L}^1(I\!R^+)$. The entropy conditions (3.11)-(3.12) guarantee that, for every $\delta > 0$, there exists C_δ such that

$$\text{Tot.Var.}\{w_i^\nu(t,\cdot);[\delta,+\infty)\} \leq C_\delta \qquad \forall t \in [0,\ T], \qquad \forall \nu \in \mathcal{N}.$$

Moreover there exists a constant $L = L(\delta,T)$ such that

$$\int_\delta^{+\infty} |w_i^\nu(t,x) - w_i^\nu(s,x)|\, dx \leq L|t-s| \qquad \forall t,s \in [0,\ T], \qquad \forall \nu \in \mathcal{N}.$$

By Helly's Theorem for any δ fixed there exists a subsequence $\{w_i^{\nu_j}(t,\cdot)\}_{j\in\mathcal{N}}$ which converges to some function $v_{\delta,i}(t,\cdot)$ in $\mathbf{L}^1{}_{loc}([\delta,+\infty))$. But (3.94) implies that such a function must coincide with v_i and hence, by using (3.92), for every $t \in [0,\ T]$ the original sequence $\{w_i^\nu(t,\cdot)\}_{\nu\in\mathcal{N}}$ converges to $v_i(t,\cdot)$ in $\mathbf{L}^1(I\!R^+)$. In particular, from the convergence of $\{w_i^\nu(T,\cdot)\}_{\nu\in\mathcal{N}}$ to $v_i(T,\cdot)$ and (3.93) it follows that $v_i(T,\cdot) = w_i$.

By the $\mathbf{L}^1$ convergence, $u = u(v)$ is a weak solution to (3.89) and satisfies the initial condition in divergence form, as required by (i) in Definition 3.2.2. Also, by the lower semicontinuity of the total variation, the decay estimates (3.11)-(3.12) are also satisfied. Thus to complete the proof it remains to show that u is a solution of (3.89)-(3.91) corresponding to a boundary data $\tilde{u} \in \mathcal{U}$.

By compactness, the sequence $\{\Psi^\nu\}_{j\in\mathcal{N}}$ of the traces of the functions $f(u^\nu)$ will admit a weak limit Ψ. Since by hypotheses $w_j(\Psi^\nu) \in [\alpha_j,\ \beta_j]$ and the set $\big\{g \in I\!R^n;\ w_j(g) \in [\alpha_j,\ \beta_j],\ j = n-p+1,\dots,n\big\}$ is convex, it follows that $w_j(\Psi) \in [\alpha_j,\ \beta_j]$, $j = n-p+1,\dots,n$. Hence, since f is invertible, $\tilde{u} \doteq f^{-1}(\Psi) \in \mathcal{U}$. If we now pass to the limit as $\nu \to \infty$ in

$$\int_0^{+\infty}\int_0^{+\infty} \big(u^\nu(t,x)\cdot\phi_t(t,x) + f(u^\nu(t,x))\cdot\phi_x(t,x)\big)dx\ dt + \int_0^T \Psi^\nu(t)\cdot\phi(t,0)dt = 0\,,$$

with $\phi \in \mathcal{C}^1$ with compact support contained in the set $\{(t,x) \in I\!R^2;\ t > 0\}$, we get

$$\int_0^{+\infty}\int_0^{+\infty} \big(u(t,x)\cdot\phi_t(t,x) + f(u(t,x))\cdot\phi_x(t,x)\big)dx\ dt + \int_0^T \Psi(t)\cdot\phi(t,0)dt = 0\,. \quad (3.95)$$

From (3.95), by applying the divergence theorem on small rectangles approximating the boundary, one easily gets

$$\lim_{x\to 0+}\int_0^\tau f(u(s,x))\ ds = \int_0^\tau \Psi(s)\ ds \qquad \tau \geq 0. \quad (3.96)$$

This concludes the proof. □

Chapter 4

Sharp $\mathbf{L}^1$ stability estimates for hyperbolic conservation laws

4.1 Introduction to Chapter 4

We are interested in the continuous dependence of entropy solutions to hyperbolic conservation laws

$$u_t + f(u)_x = 0, \qquad u(x,t) \in I\!R,\ x \in I\!R,\ t > 0, \tag{4.1}$$

where the flux $f : I\!R \to I\!R$ is a smooth and convex function. After works by Liu and Yang [58] and Dafermos [29], we aim at deriving sharp $\mathbf{L}^1$ estimates of the form

$$\|u^{II}(t) - u^I(t)\|_{w(t)} + \int_s^t M(\tau; u^I, u^{II})\, d\tau \le \|u^{II}(s) - u^I(s)\|_{w(s)}, \quad 0 \le s \le t, \tag{4.2}$$

for any two entropy solutions of bounded variation u^I and u^{II} of (4.1), where $\|.\|_{w(t)}$ is a weighted $\mathbf{L}^1$ norm equivalent to the standard $\mathbf{L}^1$ norm on the real line. In (4.2), the positive term $M(\tau; u^I, u^{II})$ is intended to provide a sharp bound on the strict decrease of the $\mathbf{L}^1$ norm. The estimate with $w \equiv 1$ and $M \equiv 0$ is of course well-known.

Recall that the fundamental issue of the uniqueness and continuous dependence for hyperbolic systems of conservation laws was initiated by Bressan and his collaborators (see [14, 17] and the references therein). A major contribution came from Liu and Yang [58, 60] who introduced a decreasing $\mathbf{L}^1$ functional ensuring (4.2). This research culminated in papers by Bressan, Liu and Yang [23] and Hu and LeFloch [41], and Liu and Yang [61].

In the present paper, we restrict attention to scalar conservation laws and, following Hu and LeFloch [41], we investigate the stability issue from the standpoint of Holmgren's and Haar's methods ([54] and the references therein). The problem under consideration is (essentially) equivalent to showing the uniqueness and $\mathbf{L}^1$ stability for the following hyperbolic equation with discontinuous coefficient:

$$\psi_t + \big(a\,\psi\big)_x = 0, \qquad \psi(x,t) \in I\!R,\ x \in I\!R,\ t > 0. \tag{4.3}$$

That is, for solutions with bounded variation we aim at deriving an estimate like

$$\|\psi(t)\|_{w(t)} + \int_s^t \tilde{M}(\tau; a, \psi)\, d\tau \le \|\psi(s)\|_{w(s)}, \qquad 0 \le s \le t. \tag{4.4}$$

For the application to (4.1) one should define a by

$$a = a(u^I, u^{II}) = \frac{f(u^{II}) - f(u^I)}{u^{II} - u^I}. \tag{4.5}$$

One may also consider the equation (4.3) for more general coefficients a.

Recall that the existence and uniqueness of solutions to the Cauchy problem associated with (4.3) was established in LeFloch [50] in the class of bounded measures, under the assumption $a_x \leq E$ for some constant E. The latter holds when a is given by (4.5) (at least when u^I and u^{II} contain no rarefaction center on the line $t = 0$ which holds "generically"). See also Crasta and LeFloch [25, 26] for further existence results.

It must be observed that we restrict attention here to more regular solutions, having bounded total variation, as this is natural in view of the application to the conservation law (4.1). In this direction, recall that an $\mathbf{L}^1$ stability result like (4.4) was established by Hu and LeFloch (see [41], Section 5, and our Theorem 4.2.2 below) in the class of piecewise Lipschitz continuous solutions, with $\tilde{M} \equiv 0$ however. This uniqueness and stability result was achieved under the assumption that the coefficient a does not contain any rarefaction shock (see Section 4.2 below for the definition). In [41] the authors made the following essential observation:

> The linearized equation (4.3)-(4.5) based on two entropy solutions of (4.1) does not exhibit rarefaction shocks. (4.6)

(This is also true for systems of conservation laws, as far as solutions with small amplitude are concerned.) One of our aims here is to extend the $\mathbf{L}^1$ stability result for (4.3) in [41] to arbitrary solutions of bounded variation.

The present result relies also heavily on the contribution by Liu and Yang [58] who, for approximate solutions constructed by the Glimm scheme, discovered a weighted norm having a sharp decay of the form (4.2). Subsequently, the Liu-Yang's functional was extended by Dafermos ([29], Chapter 11) to arbitrary functions of bounded variation (**BV**) and, using the notion of generalized characteristics, Dafermos derived precisely an estimate of the form (4.2) valid for **BV** solutions.

The aim of this Chapter is to provide a new derivation and some generalization of this $\mathbf{L}^1$ functional. Toward the derivation of bounds like (4.2) or (4.4) we make the following preliminary observations:

(1) The geometrical properties of the propagating discontinuities in a (Lax, fast or slow undercompressive, rarefaction shocks, according to the terminology in [41]) play an essential role. It turns out that the (jump of the) weight $w(x, t)$ should be assigned precisely on each *undercompressive discontinuity*. On the other hand, Lax discontinuities are very stable and do not require weight, while (in exact entropy soutions) rarefaction shocks do not arise, according to (4.6).

(2) Certain (invariance) properties on the coefficient a are necessary to define the weight globally in space; see (4.15)-(4.16) in Section 4.2.

(3) The weight however is far from being unique and we believe that this flexibility in choosing the weight may be helpful in certain applications.

The content of this Chapter is as follows.

In Section 4.2, we consider piecewise constant solutions of (4.3) and introduce a class of weighted norms satisfying a sharp bound of the form (4.4). See Theorem 4.2.3 below. All undercompressive and Lax discontinuities contribute to the decrease of the $\mathbf{L}^1$ norm. For the sake of comparison, we also consider the $\mathbf{L}^1$ norm without weight; see Theorem 4.2.2.

In Section 4.3, we point out that the setting of Section 4.2 covers the case of the conservation law (4.1). Passing to the limit in wave front tracking approximations, in Theorem 4.3.5 we arrive to the sharp bound (1.2) for general **BV** solutions. The proof is based on fine convergence properties established earlier by Bressan and LeFloch [21] and on a technique of stability of nonconservative products developed by DalMaso, LeFloch, and Murat [31] and LeFloch and Liu [53].

Next, in Sections 4.4 and 4.5 we return to the equation (4.3) studied in Section 4.2 but, now, we deal with general **BV** solutions. We follow closely ideas developed by Dafermos [28, 29] for solutions of (4.1), and extend them to the linear equation (4.3). Using generalized characteristics we establish first a maximum principle in Theorem 4.4.5. Finally, in Theorem 4.5.1 using the technique of generalized characteristics, we establish the sharp $\mathbf{L}^1$ stability property (4.4) directly, for general **BV** solutions of (4.3). The result applies in particular to the conservation law (4.1) and allows us to recover (4.2).

Throughout the paper, we always assume that all functions of bounded variation under consideration are normalized to be defined everywhere as right-continuous functions.

4.2 Decreasing norms for piecewise constant solutions

Given a piecewise constant function $a : I\!R \times I\!R_+ \to I\!R$, let us consider the linear hyperbolic equation

$$\psi_t + \big(a\,\psi\big)_x = 0, \qquad \psi(x,t) \in I\!R, \tag{4.7}$$

and restrict attention to piecewise constant solutions. By definition, the function a admits a set of jump points $\mathcal{J}(a)$, consisting of finitely many straightlines defined on open time intervals, together with a finite set of interaction points $\mathcal{I}(a)$, consisting of the end points of the lines in $\mathcal{J}(a)$. The function a is constant in each connected component of the complement $\mathcal{C}(a)$ of $\mathcal{I}(a) \cup \mathcal{J}(a)$. At a point $(x,t) \in \mathcal{J}(a)$ we denote by $\lambda^a = \lambda^a(x,t)$ the speed of the discontinuity and $a_\pm = a_\pm(x,t) = a(x\pm,t)$ the left- and right-hand traces. It is tacitly assumed that the discontinuity speeds λ^a remain uniformly bounded. Finally the function is normalized to be right-continuous. A similar notation is used for the function ψ.

The geometrical property of the coefficient a play a central role for the analysis of (4.7), so we recall the following terminology [41]:

Definition 4.2.1 *A point $(x,t) \in \mathcal{J}(a)$ is called a Lax discontinuity iff*

$$a_-(x,t) > \lambda^a(x,t) > a_+(x,t),$$

a slow undercompressive discontinuity iff

$$\lambda^a(x,t) \leq \min\big(a_-(x,t), a_+(x,t)\big),$$

a fast undercompressive discontinuity iff

$$\lambda^a(x,t) \geq \max\big(a_-(x,t), a_+(x,t)\big),$$

and a rarefaction-shock discontinuity iff

$$a_-(x,t) < \lambda^a(x,t) < a_+(x,t).$$

For each $t > 0$, we denote by $\mathcal{L}(a), \mathcal{S}(a), \mathcal{F}(a)$, and $\mathcal{R}(a)$ the set of points $(x,t) \in \mathcal{J}(a)$ corresponding to Lax, slow undercompressive, fast undercompressive, and rarefaction-shock discontinuities, respectively.

Theorem 4.2.2 *Consider a piecewise constant speed $a = a(x,t)$. Let ψ be any piecewise constant solution of (4.7). Then we have for all $0 \leq s \leq t$*

$$\begin{aligned}
&\|\psi(t)\|_{\mathbf{L}^1} + \int_s^t \sum_{(x,\tau)\in\mathcal{L}(a)} 2\left(a_-(x,\tau) - \lambda^a(x,\tau)\right) |\psi_-(x,\tau)|\, d\tau \\
&= \|\psi(s)\|_{\mathbf{L}^1} + \int_s^t \sum_{(x,\tau)\in\mathcal{R}(a)} 2\left(\lambda^a(x,\tau) - a_-(x,\tau)\right) |\psi_-(x,\tau)|\, d\tau. \qquad (4.8)
\end{aligned}$$

In (4.8), the left-hand traces are chosen for definiteness only. Indeed it will be noticed in the proof below that for all $(x, \tau) \in \mathcal{L}(a) \cup \mathcal{R}(a)$

$$\big(\lambda^a(x,\tau) - a_-(x,\tau)\big)\,|\psi_-(x,\tau)| = -\big(\lambda^a(x,\tau) - a_+(x,\tau)\big)\,|\psi_+(x,\tau)|$$

Observe that the Lax discontinuities contribute to the decrease of the $\mathbf{L}^1$ norm, while the rarefaction-shocks increase it. On the other hand, the undercompressive discontinuities do not modify the $\mathbf{L}^1$ norm. When a contains no rarefaction shocks (this is the case when (4.7) is a linearized equation derived from entropy solutions of a conservation law, as discovered in Hu and LeFloch [41]), Theorem 5.1 yields

$$\|\psi(t)\|_{\mathbf{L}^1} \le \|\psi(s)\|_{\mathbf{L}^1}, \qquad 0 \le s \le t, \tag{4.9}$$

where we neglected the favorable contribution of the Lax discontinuities appearing in the left-hand side of (4.8). In particular, (4.9) implies that the Cauchy problem for (4.7) admits a unique solution (in the class of piecewise constant functions at this stage), provided a has no rarefaction-shock discontinuities.

On the other hand, it is clear that the sign of the function ψ is important for the sake of deriving the $\mathbf{L}^1$ stability of the solutions ψ of (4.7). For instance, if ψ has a constant sign for all (x, t), then (4.9) holds as an *equality*

$$\|\psi(t)\|_{\mathbf{L}^1} = \|\psi(s)\|_{\mathbf{L}^1}, \qquad 0 \le s \le t,$$

which implies that the Cauchy problem for (4.7) admits at most one solution ψ of a given sign.

PROOF OF THEOREM 4.2.2. Denote by $\mathbf{P}(E)$ the projection of a subset E of the (x, t)-plane on the t-axis. By definition, any piecewise Lipschitz continuous solution ψ is also Lipschitz continuous in time with values in $\mathbf{L}^1(I\!R)$. So, it is enough to derive (4.8) for all $t \notin E := \mathbf{P}\big(\mathcal{I}(a) \cup \mathcal{I}(\psi)\big)$. The latter is just a finite set. The following is valid in each open interval I such that $I \cap E = \emptyset$.

We denote by $x_j(t)$ for $t \in I$ and $j = 1, \cdots, m$ the discontinuity lines where the function $\psi(., t)$ changes sign, with the convention that

$$(-1)^j\,\psi(x,t) \ge 0 \qquad \text{for } x \in [x_j(t), x_{j+1}(t)]. \tag{4.10}$$

Set $\psi_j^{\pm}(t) = \psi_\pm(x_j(t), t)$, $\lambda_j(t) = \lambda^a(x_j(t), t)$, etc. Then by using that ψ solves (4.7) we find (for all t in the interval I)

$$\frac{d}{dt} \int_{I\!R} |\psi(x,t)|\,dx$$

$$
\begin{aligned}
&= \frac{d}{dt}\sum_{j=1}^{m}(-1)^j \int_{x_j(t)}^{x_{j+1}(t)} \psi(x,t)\,dx \\
&= \sum_{j=1}^{m}(-1)^j \left(\int_{x_j(t)}^{x_{j+1}(t)} \partial_t\psi(x,t)\,dx + \lambda_{j+1}(t)\,\psi_{j+1}^-(t) - \lambda_j(t)\,\psi_j^+(t)\right) \\
&= \sum_{j=1}^{m}(-1)^j \left(\int_{x_j(t)}^{x_{j+1}(t)} -\partial_x(a(x,t)\,\psi(x,t))\,dx + \lambda_{j+1}(t)\,\psi_{j+1}^-(t) - \lambda_j(t)\,\psi_j^+(t)\right) \\
&= \sum_{j=1}^{m}(-1)^j \Big((a_j^+(t) - \lambda_j(t))\,\psi_j^+(t) + (a_j^-(t) - \lambda_j(t))\,\psi_j^-(t)\Big).
\end{aligned}
$$

The Rankine-Hugoniot relation associated with (4.7) reads

$$(a_j^+(t) - \lambda_j(t))\,\psi_j^+(t) = (a_j^-(t) - \lambda_j(t))\,\psi_j^-(t), \tag{4.11}$$

therefore by (4.10)

$$\frac{d}{dt}\int_{I\!R} |\psi(x,t)|\,dx = 2\sum_{j=1}^{m} \pm(a_j^\pm(t) - \lambda_j(t))\,|\psi_j^\pm(t)|. \tag{4.12}$$

Consider each point $x_j(t)$ successively. If $x_j(t)$ is a Lax discontinuity, then $a_j^-(t) > \lambda_j(t) > a_+(t)$ and both coefficients $\pm(a_j^\pm(t) - \lambda_j(t))$ are negative. If $x_j(t)$ is a rarefaction-shock discontinuity, then $a_j^-(t) < \lambda_j(t) < a_+(t)$ and the coefficients $\pm(a_j^\pm(t) - \lambda_j(t))$ are positive. These two cases lead us to the two sums in (4.8). Indeed one just needs to observe the following: if (x,τ) correspond to a Lax or rarefaction-shock discontinuity of the speed a, but ψ does not change sign at (x,τ) (so it is not counted in (4.12)), then actually by the Rankine-Hugoniot relation (see (4.11)) we conclude easily that

$$\psi_-(x,\tau) = \psi_+(x,\tau) = 0,$$

and so it does not matter to include the point (x,τ) in the sums (4.8).

Suppose next that $x_j(t)$ is an undercompressive discontinuity. Then the two sides of (4.11) have different sign, therefore

$$(a_j^+(t) - \lambda_j(t))\,\psi_j^+(t) = (a_j^-(t) - \lambda_j(t))\,\psi_j^-(t) = 0,$$

and the corresponding term in (4.12) vanishes. □

Our objective now is to derive an improved version of Theorem 4.2.2, based on a weighted $\mathbf{L}^1$ norm adapted to the equation (4.7). For piecewise constant functions, we set

$$\|\psi(t)\|_{w(t)} := \int_{I\!R} |\psi(x,t)|\, w(x,t)\, dx, \tag{4.13}$$

where $w = w(x,t) > 0$ is a piecewise constant and uniformly bounded function. We determine this function based on the following constrain on its jumps, at each discontinuity of the speed a,

$$w_+(x,t) - w_-(x,t) = \begin{cases} \leq 0 & \text{if } (x,t) \in \mathcal{S}(a), \\ \\ \geq 0 & \text{if } (x,t) \in \mathcal{F}(a). \end{cases} \tag{4.14}$$

The weight is chosen so that the left-hand trace of a slow undercompressive discontinuity and the right-hand trace of a fast one are weighted more. This is consistent with the immediate observation that the terms $\big(\lambda_j(t) - a_j^-(t)\big)\, |\psi_j^-(t)|$ and $\big(a_j^+(t) - \lambda_j(t)\big)\, |\psi_j^+(t)|$ have a favorable (negative) sign for slow and fast undercompressive discontinuities, respectively. On the other hand, the jumps of w at Lax or rarefaction-shock discontinuities will remain unconstrained. This choice is motivated by the two observations:

(i) Lax shocks already provide us with a good contribution in (4.8),

(ii) rarefaction shocks are the source of instability and non-uniqueness and cannot be "fixed up".

The constraint in (4.14) is different for slow and for fast undercompressive discontinuities. To actually exhibit a (uniformly bounded) weight satisfying (4.14), we put a restriction on how the nature of the discontinuities change in time as wave interactions take place. (An incoming wave may be a slow undercompressive one and become a fast one after the interaction, etc. A different constrain is placed before and after the interaction.)

Precisely, we suppose that, to the speed $a = a(x,t)$, we can associate on one hand a function $\kappa : I\!R \times I\!R_+ \to I\!R$ having bounded total variation and such that $\mathcal{J}(\kappa) \subset \mathcal{J}(a)$ and $\mathcal{I}(\kappa) \subset \mathcal{I}(a)$, and on the other hand a partition of the discontinuities

$$\mathcal{J}(a) = \mathcal{J}^I(a) \cup \mathcal{J}^{II}(a), \tag{4.15}$$

so that, for each $(x,t) \in \mathcal{J}(a)$, the limits $\kappa_\pm = \kappa_\pm(x,t)$ determine if the wave is slow or fast on its left or right side, as follows:

$$\text{sgn}\, \big(a_\pm(x,t) - \lambda(x,t)\big) = \begin{cases} \text{sgn}\, \kappa_\mp & \text{if } (x,t) \in \mathcal{J}^I(a), \\ -\text{sgn}\, \kappa_\mp & \text{if } (x,t) \in \mathcal{J}^{II}(a). \end{cases} \tag{4.16}$$

Here we use sgn $(y) = -1, 0, 1$ iff $y < 0, y = 0, y > 0$, respectively. Therefore a discontinuity $(x,t) \in \mathcal{J}^I(a)$ (for instance) is

$$\begin{aligned}
&\text{a Lax one iff } \kappa_- < 0 \text{ and } \kappa_+ > 0,\\
&\text{a slow undercompressive one iff } \kappa_- \geq 0 \text{ and } \kappa_+ \geq 0,\\
&\text{a fast undercompressive one iff } \kappa_- \leq 0 \text{ and } \kappa_+ \leq 0,\\
&\text{a rarefaction-shock iff } \kappa_- > 0 \text{ and } \kappa_+ < 0.
\end{aligned}$$

Furthermore, to measure the strength of the jumps, we introduce a piecewise constant function, $b = b(x,t)$, having the same jump points as the function a. For instance, we could assume that there exist constants $C_1, C_2 > 0$ such that at each discontinuity of a

$$C_1 \, |a_+(y,t) - a_-(y,t)| \leq |b_+(y,t) - b_-(y,t)| \leq C_2 \, |a_+(y,t) - a_-(y,t)|. \tag{4.17}$$

However, strictly speaking, this condition will not be used, in the present section at least.

Based on the functions κ and b and for t except wave interaction times, we can set

$$\begin{aligned}
V^I(x,t) &= \sum_{\substack{(y,t)\in\mathcal{J}^I(a),\\ y<x}} |b_+(y,t) - b_-(y,t)|,\\
V^{II}(x,t) &= \sum_{\substack{(y,t)\in\mathcal{J}^{II}(a),\\ y<x}} |b_+(y,t) - b_-(y,t)|,
\end{aligned} \tag{4.18}$$

so that the total variation of $b(t)$ on the interval $(-\infty, x)$ decomposes into

$$TV_{-\infty}^x(b(t)) = V^I(x,t) + V^{II}(x,t). \tag{4.19}$$

Fix some parameter $m \geq 0$. Consider now the weight-function defined for each $(x,t) \in \mathcal{C}(a)$ by

$$w(x,t) = \begin{cases} m + V^I(\infty,t) - V^I(x,t) + V^{II}(x,t) & \text{if } \kappa(x,t) > 0,\\[2ex] m + V^I(x,t) + V^{II}(\infty,t) - V^{II}(x,t) & \text{if } \kappa(x,t) \leq 0. \end{cases} \tag{4.20}$$

It is immediate to see that indeed (4.14) holds and that with (4.17)

$$m \leq w(x,t) \leq m + TV(b(t)) \leq m + C_2 \, TV(a(t)), \qquad x \in I\!R. \tag{4.21}$$

Note also that the weight depends on b and a, but not on the solution.

Theorem 4.2.3 *Consider a piecewise constant speed $a = a(x,t)$ admitting a decomposition (4.15)-(4.16) and satisfying the total variation estimate (4.21). Consider the weight function $w = w(x,t)$ defined by (4.19). Let ψ be any piecewise constant solution of the linear hyperbolic equation (4.7). Then the weighted norm (4.13) satisfies for all $0 \le s \le t$*

$$\begin{aligned}
&\|\psi(t)\|_{w(t)} \\
&+ \int_s^t \sum_{(x,\tau)\in\mathcal{L}(a)} \Big(2\,m + TV(b) - |b_+(x,\tau) - b_-(x,\tau)|\Big)\,\big|a_-(x,\tau) - \lambda(x,\tau)\big|\,|\psi_-(x,\tau)|\,d\tau \\
&+ \int_s^t \sum_{(x,\tau)\in\mathcal{S}(a)\cup\mathcal{F}(a)} |b_+(x,\tau) - b_-(x,\tau)|\,\big|a_-(x,\tau) - \lambda(x,\tau)\big|\,|\psi_-(x,\tau)|\,d\tau \\
&= \|\psi(s)\|_{w(s)} + \int_s^t \sum_{(x,\tau)\in\mathcal{R}(a)} \Big(2\,m + TV(b)\Big)\,\big|a_-(x,\tau) - \lambda(x,\tau)\big|\,|\psi_-(x,\tau)|\,d\tau \\
&\qquad + \int_s^t \sum_{(x,\tau)\in\mathcal{R}(a)} |b_+(x,\tau) - b_-(x,\tau)|\,\big|a_-(x,\tau) - \lambda(x,\tau)\big|\,|\psi_-(x,\tau)|\,d\tau. \qquad (4.22)
\end{aligned}$$

The statement (4.22) is sharper than (4.8), as *all* discontinuities contribute now to the decrease of the weighted $\mathbf{L}^1$ norm. Note that as $m \to \infty$, we recover exactly (4.8) from (4.22).

Proof of Theorem 4.2.3. We proceed similarly as in the proof of Theorem 4.2.2. However, $x_j(t)$ for $t \in I$ (some open interval avoiding the interaction points in a or ψ) denote now all the jump points in either a or ψ. We obtain as before the identity

$$\begin{aligned}
&\frac{d}{dt}\int_{I\!R} |\psi(x,t)|\,w(x,t)\,dx \\
&= \sum_{j=1}^{m} \Big((\lambda_j(t) - a_j^-(t))\,|\psi_j^-(t)|\,w_j^-(t) + (a_j^+(t) - \lambda_j(t))\,|\psi_j^+(t)|\,w_j^+(t)\Big) \\
&= \sum_{j=1}^{m} \Big(\text{sgn}\,\big(\lambda_j(t) - a_j^-(t)\big)\,w_j^-(t) \\
&\qquad\qquad + \text{sgn}\,\big(a_j^+(t) - \lambda_j(t)\big)\,w_j^+(t)\Big)\,\big|\lambda_j(t) - a_j^-(t)\big|\,|\psi_j^-(t)|\,, \qquad (4.23)
\end{aligned}$$

where we used the Rankine-Hugoniot relation (4.11).

If $x_j(t)$ is a Lax discontinuity in $\mathcal{J}^I(a)$, then by (4.17) we have $\kappa_- < 0$ and $\kappa_+ > 0$. So by (4.20) we find

$$w_j^- = m + V^I(x_j(t)-) + V^{II}(\infty) - V^{II}(x_j(t)-),$$

$$w_j^+ = m + V^I(\infty) - V^I(x_j(t)+) + V^{II}(x_j(t)+),$$

and so

$$\begin{aligned}\operatorname{sgn}\,\big(\lambda_j(t) - a_j^-(t)\big)\, w_j^-(t) \;+\; \operatorname{sgn}\,\big(a_j^+(t) - \lambda_j(t)\big)\, w_j^+(t)& \\ = -w_j^-(t) - w_j^+(t)& \\ = -2\,m - TV(b) + |b_j^+(t) - b_j^-(t)|.& \qquad (4.24)\end{aligned}$$

If $x_j(t)$ is a rarefaction-shock discontinuity in $\mathcal{J}^I(a)$, then by (4.17) we have $\kappa_- > 0$ and $\kappa_+ < 0$. By (4.19) we find

$$\begin{aligned} w_j^- &= m + V^I(\infty) - V^I(x_j(t)-) + V^{II}(x_j(t)-),\\ w_j^+ &= m + V^I(x_j(t)+) + V^{II}(\infty) - V^{II}(x_j(t)+),\end{aligned}$$

and so

$$\begin{aligned}\operatorname{sgn}\,\big(\lambda_j(t) - a_j^-(t)\big)\, w_j^-(t) \;+\; \operatorname{sgn}\,\big(a_j^+(t) - \lambda_j(t)\big)\, w_j^+(t)& \\ = w_j^-(t) + w_j^+(t)& \\ = 2\,m + TV(b) + |b_j^+(t) - b_j^-(t)|.& \qquad (4.25)\end{aligned}$$

If $x_j(t)$ is a fast undercompressive discontinuity in $\mathcal{J}^I(a)$, then by (4.17) we have $\kappa_- \le 0$ and $\kappa_+ \le 0$. By (4.19) we find

$$\begin{aligned}&\operatorname{sgn}\,\big(\lambda_j(t) - a_j^-(t)\big)\, w_j^-(t) + \operatorname{sgn}\,\big(a_j^+(t) - \lambda_j(t)\big)\, w_j^+(t)\\ &= w_j^-(t) - w_j^+(t)\\ &= m + V^I(x_j(t)-) + V^{II}(\infty) - V^{II}(x_j(t)-)\\ &-m - V^I(x_j(t)+) - V^{II}(\infty) + V^{II}(x_j(t)+)\\ &= -|b_j^+(t) - b_j^-(t)|. \qquad (4.26)\end{aligned}$$

Similarly for slow undercompressive discontinuities in $\mathcal{J}^I(a)$ we obtain

$$\operatorname{sgn}\,\big(\lambda_j(t) - a_j^-(t)\big)\, w_j^-(t) + \operatorname{sgn}\,\big(a_j^+(t) - \lambda_j(t)\big)\, w_j^+(t) = -|b_j^+(t) - b_j^-(t)|. \qquad (4.27)$$

Using (4.24)-(4.27) in (4.23) we conclude that

$$\begin{aligned}&\|\psi(t)\|_{w(t)}\\ &+\int_s^t \sum_{(x,\tau)\in\mathcal{L}(a)} \Big(2\,m + TV(b) - |b_+(x,\tau) - b_-(x,\tau)|\Big)\big|a_-(x,\tau) - \lambda(x,\tau)\big|\,|\psi_-(x,\tau)|\,d\tau\end{aligned}$$

$$
\begin{aligned}
&+\int_s^t \sum_{(x,\tau)\in\mathcal{S}(a)\cup\mathcal{F}(a)} |b_+(x,\tau)-b_-(x,\tau)|\,\big|a_-(x,\tau)-\lambda(x,\tau)\big|\,|\psi_-(x,\tau)|\,d\tau\\
&=\|\psi(s)\|_{w(s)}\\
&+\int_s^t \sum_{(x,\tau)\in\mathcal{R}(a)} \Big(2\,m+TV(b)+|b_+(x,\tau)-b_-(x,\tau)|\Big)\big|a_-(x,\tau)-\lambda(x,\tau)\big|\,|\psi_-(x,\tau)|\,d\tau,
\end{aligned}
$$

which is equivalent to (4.22). □

Using that $\mathcal{R}(a)$ is included in the set of points where ψ changes sign, it is easy to deduce from (4.22) that:

Corollary 4.2.4 *Under the assumptions and notations in Theorem 4.2.3, we have for all* $0 \le s \le t$

$$
\begin{aligned}
\|\psi(t)\|_{w(t)} \le\ & \|\psi(s)\|_{w(s)}\\
&+\Big(2\,m+TV(b)\Big)\sup_{\substack{(x,\tau)\in\mathcal{R}(a)\\ s\le\tau\le t}}\big|b_+(x,\tau)-b_-(x,\tau)\big|\int_s^t TV(\psi(\tau))\,d\tau
\end{aligned}
$$

and, in particular, letting $m\to\infty$

$$
\|\psi(t)\|_{\mathbf{L}^1} \le \|\psi(s)\|_{\mathbf{L}^1} + 2\sup_{\substack{(x,\tau)\in\mathcal{R}(a)\\ s\le\tau\le t}}\big|b_+(x,\tau)-b_-(x,\tau)\big|\int_s^t TV(\psi(\tau))\,d\tau. \tag{4.28}
$$

Finally, in view of Corollary 4.2.4, in case the function a contains no rarefaction shocks, we deduce that

$$
\|\psi(t)\|_{w(t)} \le \|\psi(s)\|_{w(s)}, \qquad 0\le s\le t.
$$

Observe that this result is achieved, based on a weight that depends on an arbitrary function, b, and on the sole assumption that a decomposition (4.15)-(4.16) of the jumps of a is available. However, our result in this section covers only piecewise constant solutions. We will see in Section 4.5 that a stronger structure assumption on the coefficients a is necessary to handle general solutions of bounded variation.

4.3 Sharp $\mathbf{L}^1$ estimate for hyperbolic conservation laws

In this section, we apply Theorem 4.2.3 to the case that a is the averaging coefficient (4.4) based on two entropy solutions of (4.1). First, we check that the assumptions required in Section 4.2 on the coefficient a do hold in this situation. Therefore Theorem 4.2.3 applies to the piecewise constant solutions defined by the wave-front tracking (also called polygonal approximation) algorithm proposed by Dafermos in [27]. Next, we observe that, with a suitable choice of the definition of the wave strengths, the weighted norm in Section 4.2 reduces to Liu-Yang's functional. Finally we rigorously justify the passage to the limit in the estimate of Theorem 4.2.3 when the number of wave fronts tends to infinity and exact entropy solutions of (4.1) are recovered.

Consider the nonlinear scalar conservation law:

$$u_t + f(u)_x = 0, \qquad u(x,t) \in I\!R, \tag{4.29}$$

where the flux $f : I\!R \to I\!R$ is a smooth function. Let u^I and u^{II} be two bounded entropy solutions of (4.29) having bounded total variation. Given $h > 0$ let us approximate the data $u^I(0)$ and $u^{II}(0)$ by piecewise constant functions $u^{I,h}(0)$, $u^{II,h}(0)$, having finitely many jumps and such that as $h \to 0$

$$u^{I,h}(0) \to u^I(0), \quad u^{II,h}(0) \to u^{II}(0) \qquad \text{in the } L^1 \text{ norm}, \tag{4.30}$$

$$TV(u^{I,h}(0)) \to TV(u^I(0)), \quad TV(u^{II,h}(0)) \to TV(u^{II}(0)). \tag{4.31}$$

Applying Dafermos' scheme [27], we can construct corresponding, piecewise constant, approximate solutions $u^{I,h}$ and $u^{II,h}$ having finitely many jump lines and for $t \geq s \geq 0$ and $p \in [1,\infty]$

$$\|u^{I,h}(t)\|_{\mathbf{L}^p} \leq \|u^{I,h}(s)\|_{\mathbf{L}^p}, \qquad \|u^{II,h}(t)\|_{\mathbf{L}^p} \leq \|u^{II,h}(s)\|_{\mathbf{L}^p}, \tag{4.32}$$

and for all $-\infty \leq A + M\,(t-s) \leq B - M\,(t-s)$

$$\begin{aligned} &TV_{A+M\,(t-s)}^{B-M\,(t-s)}\big(u^{I,h}(t)\big) \leq TV_A^B\big(u^{I,h}(s)\big), \\ &TV_{A+M\,(t-s)}^{B-M\,(t-s)}\big(u^{II,h}(t)\big) \leq TV_A^B\big(u^{II,h}(s)\big). \end{aligned} \tag{4.33}$$

More precisely, the functions $u^{I,h}$ and $u^{II,h}$ are exact solutions of (4.29) satisfying therefore the Rankine-Hugoniot relation at every jump. They contain two kinds of jump discontinuities: *Lax shocks* satisfy the so-called Oleinik entropy inequalities, while *rarefaction jumps* do not, but have small strength, that is

$$|u^{I,h}(x+,t) - u^{I,h}(x-,t)| \leq h, \qquad |u^{II,h}(x+,t) - u^{II,h}(x-,t)| \leq h. \tag{4.34}$$

Furthermore, for a subsequence $h \to 0$ at least, we have for each time $t \geq 0$

$$u^{I,h}(t) \to u^I(t), \quad u^{II,h}(t) \to u^{II}(t) \qquad \text{in the } \mathbf{L}^1 \text{ norm.}$$

To study the $\mathbf{L}^1$ distance between these approximate solutions, we set

$$\psi := u^{II,h} - u^{I,h},$$

which is one solution of the linear hyperbolic equation

$$\psi_t + \left(a^h\, \psi\right)_x = 0, \qquad a^h(x,t) := \frac{f(u^{II,h}(x,t)) - f(u^{I,h}(x,t))}{u^{II,h}(x,t) - u^{I,h}(x,t)}. \tag{4.35}$$

First of all, based on Theorem 4.2.2 and (4.33)-(4.34), we obtain immediately:

Theorem 4.3.1 *The approximate solutions $u^{I,h}$ and $u^{II,h}$ satisfy the following $\mathbf{L}^1$ stability estimate for all $0 \leq s \leq t$*

$$\begin{aligned} &\|u^{II,h}(t) - u^{I,h}(t)\|_{\mathbf{L}^1} \\ &\quad + \int_s^t \sum_{(x,\tau)\in\mathcal{L}(a)} 2\left(a^h(x-,\tau) - \lambda^{a^h}(x,\tau)\right) |u^{II,h}(x-,\tau) - u^{I,h}(x-,\tau)|\, d\tau \\ &\leq \|u^{II,h}(s) - u^{I,h}(s)\|_{\mathbf{L}^1} \\ &\quad + 2h\,(t-s)\,\|f''\|_\infty \Big(TV(u^{I,h}(0)) + TV(u^{II,h}(0))\Big). \end{aligned} \tag{4.36}$$

From the functions u^I and u^{II} we define the function a as in (4.35). Recall that the wave front tracking scheme converge locally uniformly (see the proof of Theorem 4.3.5 below for a the definition), so that the $\mathbf{BV}$ solutions u^I and u^{II} are endowed with additional regularity properties. Consider for instance the function u^I. In particular, for all but countably many times t and for each x, either x is a point of continuity of u^I in the classical sense (say $(x,t) \in \mathcal{C}(u^I)$) or else it is a point of jump in the classical sense (say $(x,t) \in \mathcal{J}(u^I)$) and, to the discontinuity, one can also associate a shock speed, denoted by $\lambda^I(x,t)$.

From the properties shared by u^I and u^{II}, one deduces immediately a similar property for the coefficient a. Excluding countably many times at most, at each point of jump of a we can define the propagation speed $\lambda^a(x,t)$ of the discontinuity located at the point (x,t). Namely, we have

$$\lambda^a(x,t) = \begin{cases} \lambda^I(x,t) & \text{if } (x,t) \in \mathcal{J}(u^I), \\ \lambda^{II}(x,t) & \text{if } (x,t) \in \mathcal{J}(u^{II}). \end{cases}$$

In the limit $h \to 0$ we deduce from (4.36) that:

Corollary 4.3.2 *For all* $0 \le s \le t$ *we have*

$$\begin{aligned}
&\|u^{II}(t) - u^I(t)\|_{\mathbf{L}^1} \\
&\qquad + \int_s^t \sum_{(x,\tau)\in\mathcal{L}(a)} 2\left(a(x-,\tau) - \lambda^a(x,\tau)\right) |u^{II}(x-,\tau) - u^I(x-,\tau)|\, d\tau \\
&\le \|u^{II}(s) - u^I(s)\|_{\mathbf{L}^1}. \qquad (4.37)
\end{aligned}$$

We omit the proof of Corollary 4.3.2 as (4.37) is a consequence of a stronger estimate proven in Theorem 4.3.5 below (by taking $m \to \infty$ in (4.45)). Note that (4.37) is a stronger statement than the standard $\mathbf{L}^1$ contraction estimate

$$\|u^{II}(t) - u^I(t)\|_{\mathbf{L}^1} \le \|u^{II}(s) - u^I(s)\|_{\mathbf{L}^1}.$$

PROOF OF THEOREM 4.3.1. We apply the estimate (4.8) with ψ replaced with $u^{II,h} - u^{I,h}$. We just need to observe (see [41]) that all the rarefaction-shock discontinuities in a^h are due to rarefaction fronts in $u^{I,h}$ or in $u^{II,h}$, which have small strength according to (4.34). In other words we have

$$\begin{aligned}
&\int_s^t \sum_{(x,\tau)\in\mathcal{R}(a)} 2\left(\lambda^a(x,\tau) - a_-(x,\tau)\right) |\psi_-(x,\tau)|\, d\tau \\
&\qquad \le \sup_{\substack{(x,\tau)\in\mathcal{R}(a)\\ s\le\tau\le t}} 2\left|a_+(x,\tau) - a_-(x,\tau)\right| \int_s^t TV(\psi(\tau))\, d\tau \\
&\qquad \le 2\,\|f''\|_\infty\, h \int_s^t TV(\psi(\tau))\, d\tau \\
&\qquad \le 2\,\|f''\|_\infty\, h\,(t-s)\left(TV(u^{I,h}(0)) + TV(u^{II,h}(0))\right).
\end{aligned}$$

This establishes (4.36). □

We now want to apply Theorem 4.2.3 and control a weighted norm of $u^{II,h} - u^{I,h}$. In this direction, our main observation is:

Lemma 4.3.3 *When the function* f *is strictly convex, the coefficient* a^h *satisfies all of the assumptions (4.15)-(4.16).*

PROOF OF LEMMA 4.3.3. The function a^h is piecewise constant, and we can associate to this function an obvious decomposition of the form (4.15). To establish (4.16), consider for instance a jump point $(x,t) \in \mathcal{J}(u^{I,h}) \cap \mathcal{C}(u^{II,h})$, together with its left- and right-hand traces u_-^I and u_+^I. Since $u^{I,h}$ is a solution of (4.29), the corresponding speed $\lambda = \lambda(x,t)$ satisfies the Rankine-Hugoniot relation:

$$-\lambda\left(u_+^I - u_-^I\right) + f(u_+^I) - f(u_-^I) = 0.$$

Thus the term in the left-hand side of (4.16) takes the form

$$\begin{aligned} a_\pm(x,t) - \lambda(x,t) &= \frac{f(u^{II}) - f(u_\pm^I)}{u^{II} - u_\pm^I} - \frac{f(u_+^I) - f(u_-^I)}{u_+^I - u_-^I} \\ &= \int_0^1 \Big(f'\big(\theta\, u^{II} + (1-\theta)\, u_\pm^I\big) - f'\big(\theta\, u_\mp^I + (1-\theta)\, u_\pm^I\big)\Big)\, d\theta. \end{aligned}$$

Thus we obtain

$$\begin{aligned} & a_\pm(x,t) - \lambda(x,t) = \mu\left(u^{II} - u_\mp^I\right), \\ & \mu := \int_0^1\int_0^1 f''\Big(\rho\left(\theta\, u^{II} + (1-\theta)\, u_\pm^I\right) + (1-\rho)\left(\theta\, u_\mp^I + (1-\theta)\, u_\pm^I\right)\Big)\, \theta\, d\theta d\rho. \end{aligned} \tag{4.38}$$

Since f is strictly convex, the coefficient is bounded away from zero. In view of (4.38), if we now choose $\kappa(x,t) := u^{II,h} - u^{I,h}$, the desired property (4.16) holds true. □

Next, we define the weight $w^h = w^h(x,t)$ associated with the function a^h, by the formula (4.20) in which we specify

$$\kappa^h(x,t) := u^{II,h} - u^{I,h}. \tag{4.39}$$

It follows immediately from Theorem 4.2.3 that:

Theorem 4.3.4 *Suppose that the function f is strictly convex. The approximate solutions constructed by Dafermos scheme satisfy the $\mathbf{L}^1$ stability estimate for all $0 \le s \le t$*

$$\begin{aligned} & \|u^{II,h}(t) - u^{I,h}(t)\|_{w^h(t)} \\ & + \int_s^t \sum_{(x,\tau)\in\mathcal{L}(a^h)} \Big(2\, m + TV(b^h) - |b^h(x+,\tau) - b^h(x-,\tau)|\Big) \\ & \qquad \big|a^h(x-,\tau) - \lambda^h(x,\tau)\big|\, |u^{II,h}(x-,\tau) - u^{I,h}(x-,\tau)|\, d\tau \\ & + \int_s^t \sum_{(x,\tau)\in\mathcal{S}(a^h)\cup\mathcal{F}(a^h)} |b^h(x+,\tau) - b^h(x-,\tau)| \\ & \qquad \big|a^h(x-,\tau) - \lambda^h(x,\tau)\big|\, |u^{II,h}(x-,\tau) - u^{I,h}(x-,\tau)|\, d\tau \end{aligned}$$

$$
\begin{aligned}
&= \|u^{II,h}(s) - u^{I,h}(s)\|_{w^h(s)} \\
&+ \int_s^t \sum_{(x,\tau)\in\mathcal{R}(a^h)} \Big(2\,m + TV(b^h) + |b^h(x+,\tau) - b^h(x-,\tau)|\Big) \\
&\qquad \big|a^h(x-,\tau) - \lambda^h(x,\tau)\big|\,|u^{II,h}(x-,\tau) - u^{I,h}(x-,\tau)|\,d\tau,
\end{aligned}
\tag{4.40}
$$

where a^h is the averaging coefficient defined in (4.35) and $\lambda^h(x,\tau)$ represents the speed of the discontinuity located at $(x,\tau) \in \mathcal{J}(a^h)$.

We emphasize that (4.40) is an *equality* in which the contribution to the $\mathbf{L}^1$ norm of each type of wave appears clearly. The coefficient a^h exhibits three types of waves: the Lax and undercompressive discontinuities in a^h contribute to the decay of the $\mathbf{L}^1$ weighted distance. The statement (4.40) quantifies sharply this effect. On the other hand, the rarefaction-shocks appearing in the right-hand side of (4.40) increase the $\mathbf{L}^1$ norm.

In the rest of this section, we assume that the function $b = b^h$ is chosen to be specifically

$$
b^h(x+,t) - b^h(x-,t) = \begin{cases} u^{I,h}(x+,t) - u^{I,h}(x-,t) & \text{if } (x,t) \in \mathcal{J}(u^{I,h}), \\ u^{II,h}(x+,t) - u^{II,h}(x-,t) & \text{if } (x,t) \in \mathcal{J}(u^{II,h}), \end{cases}
\tag{4.41}
$$

but a more general definition is possible.

Our next purpose is to pass to the limit ($h \to 0$) in the statement established in Theorem 4.3.4 for piecewise constant approximate solutions. We recover here a result derived by Dafermos [29] via a different approach. Recall the notation $\mathcal{C}(u^I)$, $\mathcal{S}(u^I)$, etc introduced earlier. Denote by $\mathcal{I}(u^I)$ the countable set of interactions times. Let $V^I(t)$ be the total variation function associated with $u^I(t)$. Based on the functions $V^I(t)$ and $V^{II}(t)$, we then define the weight w as in (4.20) but with (4.18) replaced by the total variation functions of $u^I(t)$ and $u^{II}(t)$, with $\kappa := u^{II} - u^I$ and

$$
b(x+,t) - b(x-,t) = \begin{cases} u^{I}(x+,t) - u^{I}(x-,t) & \text{if } (x,t) \in \mathcal{J}(u^{I}), \\ u^{II}(x+,t) - u^{II}(x-,t) & \text{if } (x,t) \in \mathcal{J}(u^{II}). \end{cases}
\tag{4.42}
$$

Furthermore, to any functions of bounded variation u, v, w in the space variable x (the time variable being fixed) we associate the measure on $\mathbb{R}$

$$
\mu = \big(a(u,v) - f'(u)\big)\,(v-u)\,dw
$$

understood as the nonconservative product in the sense of Dal Maso, LeFloch and Murat [31] and characterized by the following two conditions:

(1) If B is a Borel set included in the set of continuity points of w

$$\mu(B) = \int_B \big(a(u,v) - f'(u)\big)\,(v-u)\,dw, \tag{4.43}$$

where the integral is defined in a classical sense;

(2) If x is a point of jump of w, then

$$\begin{aligned}\mu(\{x\}) = \frac{1}{2}\Big(&\big(a(u_+,v_+) - a(u_-,u_+)\big)\,(v_+ - u_+)\\ &+\big(a(u_-,v_-) - a(u_-,u_+)\big)\,(v_- - u_-)\Big)\,|w_+ - w_-|\end{aligned} \tag{4.44}$$

with $u_\pm = u(x\pm)$, etc.

Note that, if $u = u^I$ and $v = u^{II}$, the two terms $\big(a(u_\pm,v_\pm) - a(u_-,u_+)\big)\,(v_\pm - u_\pm)$ in fact coincide.

Theorem 4.3.5 *Let the function f be strictly convex and let u^I and u^{II} be two entropy solutions of bounded variation of the conservation law* (1.1). *For all $0 \le s \le t$ we have*

$$\begin{aligned}&\|u^{II}(t) - u^I(t)\|_{w(t)}\\ &+\int_s^t \sum_{(x,\tau)\in\mathcal{L}(a)\cap\mathcal{J}(u^I)} q\,\big|a\big(u_-^I,u_-^{II}\big) - a\big(u_+^I,u_-^I\big)\big|\,|u^{II}-u^I|\,d\tau\\ &+\int_s^t \sum_{(x,\tau)\in\mathcal{L}(a)\cap\mathcal{J}(u^{II})} q\,\big|a\big(u_-^I,u_-^{II}\big) - a\big(u_+^{II},u_-^{II}\big)\big|\,|u^{II}-u^I|\,d\tau\\ &+\int_s^t\int_{\mathbb{R}} \big(a(u^I,u^{II}) - f'(u^I)\big)\,(u^{II}-u^I)\,dV^I\,d\tau\\ &+\int_s^t\int_{\mathbb{R}} \big(a(u^I,u^{II}) - f'(u^{II})\big)\,(u^I-u^{II})\,dV^{II}\,d\tau\\ &\le \|u^{II}(s) - u^I(s)\|_{w(s)}.\end{aligned} \tag{4.45}$$

where $q = q(\tau) = 2\,m + TV(u^I(\tau)) + TV(u^{II}(\tau))$.

Observe that the terms in integrals in (4.45) globally contribute to the decrease of weighted norm, as is better seen rewriting the formula as follows (V_c^I and V_c^{II} being the continuous parts of the measures V^I and V^{II}):

$$\begin{aligned}&\|u^{II}(t) - u^I(t)\|_{w(t)}\\ &+\int_s^t \sum_{(x,\tau)\in\mathcal{L}(a)\cap\mathcal{J}(u^I)} \big(q - |u_+^I - u_-^I|\big)\,\big|a\big(u_-^I,u_+^{II}\big) - a\big(u_+^I,u_-^I\big)\big|\,|u^{II}-u^I|\,d\tau\end{aligned}$$

$$
\begin{aligned}
&+\int_s^t \sum_{(x,\tau)\in\mathcal{L}(a)\cap\mathcal{J}(u^{II})} \big(q-|u_+^{II}-u_-^{II}|\big)\,\big|a\big(u_-^I,u_-^{II}\big)-a\big(u_+^{II},u_-^{II}\big)\big|\,|u^{II}-u^I|\,d\tau \\
&+\int_s^t \sum_{(x,\tau)\in\big(\mathcal{S}(a)\cup\mathcal{F}(a)\big)\cap\mathcal{J}(u^I)} \big|a\big(u_-^I,u_-^{II}\big)-a\big(u_+^I,u_-^I\big)\big|\,|u^{II}-u^I|\,|u_+^I-u_-^I|d\tau \\
&+\int_s^t \sum_{(x,\tau)\in\big(\mathcal{S}(a)\cup\mathcal{F}(a)\big)\cap\mathcal{J}(u^{II})} \big|a\big(u_-^I,u_-^{II}\big)-a\big(u_+^{II},u_-^{II}\big)\big|\,|u^{II}-u^I|\,|u_+^{II}-u_-^{II}|d\tau \\
&+\int_s^t\int_{I\!R}\big|a(u^I,u^{II})-f'(u^I)\big|\,|u^{II}-u^I|\,dV_c^I\,d\tau \\
&+\int_s^t\int_{I\!R}\big|a(u^I,u^{II})-f'(u^{II})\big|\,|u^I-u^{II}|\,dV_c^{II}\,d\tau \\
&\le \|u^{II}(s)-u^I(s)\|_{w(s)}.
\end{aligned}
\tag{4.46}
$$

The following estimate is a direct consequence of the definition (4.43)-(4.44):

Lemma 4.3.6 *There exists a constant $C>0$ such that for all functions of bounded variation $u,\tilde u,v,\tilde v,w$ defined on some interval $[\alpha,\beta]$*

$$
\begin{aligned}
&\left|\int_\alpha^\beta \big(a(u,v)-f'(u)\big)\,(v-u)\,dw-\int_\alpha^\beta \big(a(\tilde u,\tilde v)-f'(\tilde u)\big)\,(\tilde v-\tilde u)\,dw\right| \\
&\le C\,\big(\|\tilde u-u\|_{L^\infty(\alpha,\beta)}+\|\tilde v-v\|_{L^\infty(\alpha,\beta)}\big)\,TV_{[\alpha,\beta]}(w).
\end{aligned}
\tag{4.47}
$$

PROOF OF THEOREM 4.3.5.

Step 1 : Preliminaries.

For each $t\ge 0$, the functions $V^{I,h}(t)$ and $V^{II,h}(t)$ associated with the wave front tracking approximations $u^{I,h}(t)$ and $u^{II,h}(t)$ are of uniformly bounded variation as $h\to 0$. The measures $dV^{I,h}$ and $dV^{II,h}$ are also Lipschitz continuous in time (with constant independent of h) for the weak convergence, except at interaction points. On the other hand, interaction times in the limiting solutions are at most countable. Therefore, extracting subsequences if necessary, the measures $dV^{I,h}$ and $dV^{II,h}$ converge to some limiting (non-negative) measures, say:

$$
dV^{I,h}(t)\to d\bar V^I(t) \qquad dV^{II,h}(t)\to d\bar V^{II}(t). \tag{4.48}
$$

By lower semi-continuity, we have at each time t

$$
dV^I(t)\le d\bar V^I(t), \qquad dV^{II}(t)\le d\bar V^{II}(t), \tag{4.49}
$$

and, in particular, at each (x,t)

$$V^I(x,t) \le \bar V^I(x,t), \qquad V^{II}(x,t) \le \bar V^{II}(x,t). \tag{4.50}$$

$$\begin{aligned} &V^I(+\infty,t) - V^I(x,t) \le \bar V^I(+\infty,t) - \bar V^I(x,t),\\ &V^{II}(+\infty,t) - V^{II}(x,t) \le \bar V^{II}(+\infty,t) - \bar V^{II}(x,t). \end{aligned} \tag{4.51}$$

Based on the functions $\bar V^I(t)$ and $\bar V^{II}(t)$, on the coefficient $\kappa := u^{II} - u^I$ and on the function in (4.42), we can define a weight denoted by $\bar w$, along the same lines as in (4.20). We will show that the left-hand side of (4.45) is bounded above by

$$\begin{aligned} &\|u^{II}(t) - u^I(t)\|_{\bar w(t)}\\ &+ \int_s^t \sum_{(x,\tau)\in\mathcal{L}(a)\cap\mathcal{J}(u^I)} \bar q \left|a\big(u_-^I, u_-^{II}\big) - a\big(u_+^I, u_-^I\big)\right| |u^{II} - u^I|\, d\tau\\ &+ \int_s^t \sum_{(x,\tau)\in\mathcal{L}(a)\cap\mathcal{J}(u^{II})} \bar q \left|a\big(u_-^I, u_-^{II}\big) - a\big(u_+^{II}, u_-^{II}\big)\right| |u^{II} - u^I|\, d\tau\\ &+ \int_s^t \int_{I\!R} \big(a\big(u^I, u^{II}\big) - f'(u^I)\big)\, (u^{II} - u^I)\, d\bar V^I(y,\tau) d\tau\\ &+ \int_s^t \int_{I\!R} \big(a\big(u^I, u^{II}\big) - f'(u^{II})\big)\, (u^I - u^{II})\, d\bar V^{II}(y,\tau) d\tau \end{aligned} \tag{4.52}$$

where $\bar q := 2\,m + \bar V^I(+\infty) + \bar V^{II}(+\infty)$, and that (4.52) coincides with the desired upper bound $\|u^{II}(s) - u^I(s)\|_{w(s)}$. The former statement is postponed to Step 5 below and we focus now on the latter.

Fix some $t \ge s \ge 0$ and rewrite (4.40) in the equivalent form

$$\begin{aligned} &\|u^{II,h}(t) - u^{I,h}(t)\|_{w^h(t)}\\ &+ \int_s^t \sum_{(x,\tau)\in\mathcal{L}(a^h)} \Big(2\,m + TV(b^h)\Big) \left|a^h(x-,\tau) - \lambda^h(x,\tau)\right| |u^{II,h}(x-,\tau) - u^{I,h}(x-,\tau)|\, d\tau\\ &+ \int_s^t \sum_{(x,\tau)\in\mathcal{J}^I(a^h)} |b^h(x+,\tau) - b^h(x-,\tau)| \qquad (4.53)\\ &\qquad\qquad \big(a^h(x-,\tau) - \lambda^h(x,\tau)\big)\, \big(u^{II,h}(x-,\tau) - u^{I,h}(x-,\tau)\big)\, d\tau\\ &+ \int_s^t \sum_{(x,\tau)\in\mathcal{J}^{II}(a^h)} |b^h(x+,\tau) - b^h(x-,\tau)| \qquad (4.54)\\ &\qquad\qquad \big(a^h(x-,\tau) - \lambda^h(x,\tau)\big)\, \big(u^{I,h}(x-,\tau) - u^{II,h}(x-,\tau)\big)\, d\tau\\ &= \|u^{II,h}(s) - u^{I,h}(s)\|_{w^h(s)} \qquad (4.55) \end{aligned}$$

$$+\int_s^t \sum_{(x,\tau)\in\mathcal{R}(a^h)} \Big(2\,m + TV(b^h)\Big)\,\big|a^h(x-,\tau) - \lambda^h(x,\tau)\big|\,|u^{II,h}(x-,\tau) - u^{I,h}(x-,\tau)|\,d\tau, \tag{4.56}$$

or, with obvious notations,

$$\|u^{II,h}(t) - u^{I,h}(t)\|_{w^h(t)} + \Omega_1^h + \Omega_2^h = \|u^{II,h}(s) - u^{I,h}(s)\|_{w^h(s)} + \Omega_3^h. \tag{4.57}$$

As the maximum strength of rarefaction fronts in $u^{I,h}$ and $u^{II,h}$ vanishes with h (see (4.34)) and rarefaction shocks in a^h arise only from these rarefaction fronts (see (4.6)), we have

$$\Omega_3^h \to 0 \qquad \text{as } h \to 0. \tag{4.58}$$

On the other hand, we can always choose the (initial) approximations at time s in such a way that

$$\bar{w}(s) = w(s) \tag{4.59}$$

and

$$\lim_{h\to 0} \|u^{II,h}(s) - u^{I,h}(s)\|_{w^h(s)} = \|u^{II}(s) - u^I(s)\|_{w(s)}. \tag{4.60}$$

It remains to prove that the limit of the left-hand side of (4.55) is exactly (4.52). This will be established in the following three steps.

Step 2 : We will rely on the local uniform convergence of the front tracking approximations (see Bressan and LeFloch [21]). For all but countably many times τ we have the following properties for u^I (as well as for u^{II}):

(1) For each point of jump z of u^I there exists a sequence $z^h \to z$ such that for each $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\begin{aligned} &|u^{I,h}(x) - u^I(z+)| + |u^I(x) - u^I(z+)| < \varepsilon \quad \text{for all } x - z^h \in (0,\delta),\\ &|u^{I,h}(x) - u^I(z-)| + |u^I(x) - u^I(z-)| < \varepsilon \quad \text{for all } x - z^h \in (-\delta,0) \end{aligned} \tag{4.61}$$

and (clearly)

$$\begin{aligned} &|u^I(x) - u^I(z+)| < \varepsilon \quad \text{for all } x - z \in (0,\delta),\\ &|u^I(x) - u^I(z-)| < \varepsilon \quad \text{for all } x - z \in (-\delta,0). \end{aligned} \tag{4.62}$$

(2) For each point of continuity z of u^I and for each $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|u^{I,h}(x) - u^I(z)| + |u^I(x) - u^I(z)| < \varepsilon \quad \text{for all } x - z \in (-\delta, \delta). \tag{4.63}$$

We also recall from [21] that, for all but countably many times t, the atomic parts of the measures $\bar V^I$ and $\bar V^{II}$ coincide with the one of V^I and V^{II}, that is for each $y \in I\!R$

$$\bar V^I(y+,t) - \bar V^I(y-,t) = V^I(y+,t) - V^I(y-,t), \tag{4.64}$$
$$\bar V^{II}(y+,t) - \bar V^{II}(y-,t) = V^{II}(y+,t) - V^{II}(y-,t). \tag{4.65}$$

Following LeFloch and Liu [53] who established the weak stability of nonconservative products under local uniform convergence, we want to show that

$$\begin{aligned}\Omega_2^h(\tau) :=& \int_{I\!R} \big(a\big(u^{I,h}(y,\tau), u^{II,h}(y,\tau)\big) - f'(u^{I,h}(y,\tau))\big)\,(u^{II,h}(y,\tau) - u^{I,h}(y,\tau))\,dV^{I,h}(y)\\ &+ \int_{I\!R} \big(a\big(u^{I,h}(y,\tau), u^{II,h}(y,\tau)\big) - f'(u^{II,h}(y,\tau))\big)\,(u^{I,h}(y,\tau) - u^{II,h}(y,\tau))\,dV^{II,h}(y)\\ &\longrightarrow \int_{I\!R} \big(a\big(u^{I}(y,\tau), u^{II}(y,\tau)\big) - f'(u^{I}(y,\tau))\big)\,(u^{II}(y,\tau) - u^{I}(y,\tau))\,d\bar V^{I}(y)\\ &+ \int_{I\!R} \big(a\big(u^{I}(y,\tau), u^{II}(y,\tau)\big) - f'(u^{II}(y,\tau))\big)\,(u^{I}(y,\tau) - u^{II}(y,\tau))\,d\bar V^{II}(y).\end{aligned} \tag{4.66}$$

By Lebesgue dominated convergence theorem and since a uniform bound in τ and h is available, it will follow from (4.65) that

$$\begin{aligned}\Omega_2^h = \int_s^t \Omega_2^h(\tau)\,d\tau \longrightarrow& \int_s^t \int_{I\!R} \big(a\big(u^I, u^{II}\big) - f'(u^I)\big)\,(u^{II} - u^I)\,d\bar V^I(y,\tau)\,d\tau\\ &+ \int_s^t \int_{I\!R} \big(a\big(u^I, u^{II}\big) - f'(u^{II})\big)\,(u^{I} - u^{II})\,d\bar V^{II}(y,\tau)\,d\tau.\end{aligned} \tag{4.67}$$

Given $\varepsilon > 0$, select finitely many (large) jumps in u^I or u^{II}, located at $y_1, y_2, \ldots y_n$, so that

$$\sum_{\substack{x \neq y_j\\ j=1,2,\ldots,n}} \big|u^I(x+) - u^I(x-)\big| + \big|u^{II}(x+) - u^{II}(x-)\big| < \varepsilon. \tag{4.68}$$

To each y_j we associate the corresponding discontinuity point y_j^h in $u^{I,h}$ or $u^{II,h}$. To simplify the presentation we will focus on the case where $y_j < y_j^h < y_{j+1} < y_{j+1}^h$ for all j. The other cases can be treated similarly. In view of the local convergence property (4.61)-(4.63) and by extracting a covering of the interval $[y_0, y_n]$, we have also

$$|u^{I,h}(x) - u^I(x)| + |u^{II,h}(x) - u^{II}(x)| \le 2\,\varepsilon, \quad x \in \big(y_j^h, y_{j+1}\big) \subseteq \big(y_j, y_{j+1}\big). \tag{4.69}$$

In view of (4.66) we can construct functions u_ε^I and u_ε^{II} that are continuous everywhere except possibly at the points y_j and such that the following conditions hold with u replaced by either u^I or u^{II}:

$$\begin{aligned} &TV\left(u_\varepsilon; I\!R \setminus \{y_1, \ldots, y_n\}\right) \le C\, TV\left(u; I\!R \setminus \{y_1, \ldots, y_n\}\right), \\ &\|u - u_\varepsilon\|_\infty \le C\,\varepsilon, \quad TV\left(u - u_\varepsilon; I\!R \setminus \{y_1, \ldots, y_n\}\right) \le C\,\varepsilon, \end{aligned} \tag{4.70}$$

where C is independent of ε.

Consider the decompositions

$$\int_{I\!R} \left(a\left(u^{I,h}, u^{II,h}\right) - f'(u^{I,h})\right)\left(u^{II,h} - u^{I,h}\right) dV^{I,h} = \sum_{j=0}^{n} \int_{(y_j^h, y_{j+1}^h)} \cdots + \sum_{j=1}^{n} \int_{\{y_j^h\}} \cdots$$

and

$$\int_{I\!R} \left(a\left(u^{I}, u^{II}\right) - f'(u^{I})\right)\left(u^{II} - u^{I}\right) d\bar{V}^{I} = \sum_{j=0}^{n} \int_{(y_j, y_{j+1})} \cdots + \sum_{j=1}^{n} \int_{\{y_j\}} \cdots.$$

Here $y_0^h = y_0 = -\infty$ and $y_{n+1}^h = y_{n+1} = +\infty$. Thus in (4.66) we have to estimate

$$\begin{aligned} \Omega_2^h(\tau) &= \int_{I\!R} \left(a\left(u^{I,h}, u^{II,h}\right) - f'(u^{I,h})\right)\left(u^{II,h} - u^{I,h}\right) dV^{I,h} \\ &\quad - \int_{I\!R} \left(a\left(u^{I}, u^{II}\right) - f'(u^{I})\right)\left(u^{II} - u^{I}\right) d\bar{V}^{I} \\ &= T_1^h + T_2^h \end{aligned} \tag{4.71}$$

with

$$\begin{aligned} T_1^h := &\sum_{j=1}^{n} \int_{\{y_j^h\}} \left(a\left(u^{I,h}, u^{II,h}\right) - f'(u^{I,h})\right)\left(u^{II,h} - u^{I,h}\right) dV^{I,h} \\ &- \sum_{j=1}^{n} \int_{\{y_j\}} \left(a\left(u^{I}, u^{II}\right) - f'(u^{I})\right)\left(u^{II} - u^{I}\right) d\bar{V}^{I} \end{aligned}$$

and

$$\begin{aligned} T_2^h := &\sum_{j=0}^{n} \int_{(y_j^h, y_{j+1}^h)} \left(a\left(u^{I}, u^{II}\right) - f'(u^{I,h})\right)\left(u^{II,h} - u^{I,h}\right) dV^{I,h} \\ &- \sum_{j=0}^{n} \int_{(y_j, y_{j+1})} \left(a\left(u^{I}, u^{II}\right) - f'(u^{I})\right)\left(u^{II} - u^{I}\right) d\bar{V}^{I}. \end{aligned}$$

First, relying on the convergence property (4.65) we have immediately

$$
\begin{aligned}
T_1^h = \sum_{j=1}^{n} &\big(a\big(u^{I,h}(y_j^h-), u^{II}(y_j^h-)\big) - \lambda^{I,h}(y_j^h-)\big)\,\big(u^{II,h}(y_j^h-) - u^{I,h}(y_j^h-)\big) \\
&\qquad\qquad |u^{I,h}(y_j^h+) - u^{I,h}(y_j^h-)| \\
&-\big(a\big(u^I(y_j-), u^{II}(y_j-)\big) - \lambda^I(y_j-)\big)\,\big(u^{II}(y_j-) - u^I(y_j-)\big)\,|u^I(y_j+) - u^I(y_j-)|,
\end{aligned}
$$

so that

$$
\big|T_1^h\big| \le C \sum_{j=1}^{n} \sum_{\pm} |u^{I,h}(y_j^h\pm) - u^I(y_j\pm)| + |u^{II,h}(y_j^h\pm) - u^{II}(y_j\pm)|.
$$

Thus, in view of the local convergence at jump points (4.61), for h small enough we obtain

$$
\big|T_1^h\big| \le C\,\varepsilon. \tag{4.72}
$$

Relying on the simplifying assumption $y_j < y_j^h < y_{j+1} < y_{j+1}^h$ for all j, we can decompose T_2^h as follows:

$$
\begin{aligned}
T_2^h \;=\; &\sum_{j=0}^{n} \int_{(y_j^h, y_{j+1})} \big(a\big(u^{I,h}, u^{II,h}\big) - f'(u^{I,h})\big)\,\big(u^{II,h} - u^{I,h}\big)\,dV^{I,h} \\
&\qquad\qquad - \big(a\big(u^I, u^{II}\big) - f'(u^I)\big)\,\big(u^{II} - u^I\big)\,d\bar{V}^I \\
&- \sum_{j=0}^{n} \int_{(y_j, y_j^h]} \big(a\big(u^I, u^{II}\big) - f'(u^I)\big)\,\big(u^{II} - u^I\big)\,d\bar{V}^I \\
&+ \sum_{j=0}^{n} \int_{[y_{j+1}, y_{j+1}^h)} \big(a\big(u^{I,h}, u^{II,h}\big) - f'(u^{I,h})\big)\,\big(u^{II,h} - u^{I,h}\big)\,dV^{I,h} \\
=:\; & T_{2,1}^h + T_{2,2}^h + T_{2,3}^h.
\end{aligned} \tag{4.73}
$$

We first consider $T_{2,2}^h$:

$$
\begin{aligned}
T_{2,2}^h = &-\sum_{j=0}^{n} \int_{(y_j, y_j^h]} \big(a\big(u^I(y_j+), u^{II}(y_j+)\big) - f'(u^I(y_j+))\big)\,\big(u^{II}(y_j+) - u^I(y_j+)\big)\,d\bar{V}^I(y) \\
&+ \sum_{j=0}^{n} \int_{(y_j, y_j^h]} \Big\{ \big(a\big(u^I(y), u^{II}(y)\big) - f'(u^I(y))\big)\,\big(u^{II}(y) - u^I(y)\big) \\
&\qquad - \big(a\big(u^I(y_j+), u^{II}(y_j+)\big) - f'(u^I(y_j+))\big)\,\big(u^{II}(y_j+) - u^I(y_j+)\big) \Big\}\,d\bar{V}^I(y).
\end{aligned}
$$

Therefore, with (4.47), we obtain

$$
\big|T_{2,2}^h\big| \le C \sum_j \big|\bar{V}^I(y_j+) - \bar{V}^I(y_j^h+)\big|
$$

$$+C\,V^I(+\infty)\Big(\sup_{y\in(y_j,y_j^h]}|u^I(y)-u^I(y_j+)|+\sup_{x\in(y_j,y_j^h]}|u^{II}(y)-u^{II}(y_j+)|\Big).$$

Since $y_j^h\to y_j$, we have $\left|\bar V^I(y_j+)-\bar V^I(y_j^h+)\right|\to 0$, therefore for h sufficiently small

$$\left|T_{2,2}^h\right|\le C\,\varepsilon. \tag{4.74}$$

A similar argument for $T_{2,3}^h$ shows that

$$\left|T_{2,3}^h\right|\le C\,\varepsilon. \tag{4.75}$$

Next consider the decomposition

$$\begin{aligned}
&\big(a(u^{I,h},u^{II,h})-f'(u^{I,h})\big)\,(u^{II,h}-u^{I,h})\,dV^{I,h}-\big(a(u^I,u^{II})-f'(u^I)\big)\,(u^{II}-u^I)\,d\bar V^I\\
&=\big(a\big(u^{I,h},u^{II,h}\big)-f'(u^{I,h})\big)\,(u^{II,h}-u^{I,h})\,dV^{I,h}-\big(a(u^I,u^{II})-f'(u^I)\big)\,(u^{II}-u^I)\,dV^{I,h}\\
&\quad+\big(a\big(u^I,u^{II}\big)-f'(u^I)\big)\,(u^{II}-u^I)\,dV^{I,h}-\big(a\big(u_\varepsilon^I,u_\varepsilon^{II}\big)-f'(u_\varepsilon^I)\big)\,(u_\varepsilon^{II}-u_\varepsilon^I)\,dV^{I,h}\\
&\quad+\big(a\big(u_\varepsilon^I,u_\varepsilon^{II}\big)-f'(u_\varepsilon^I)\big)\,(u_\varepsilon^{II}-u_\varepsilon^I)\,dV^{I,h}-\big(a\big(u_\varepsilon^I,u_\varepsilon^{II}\big)-f'(u_\varepsilon^I)\big)\,(u_\varepsilon^{II}-u_\varepsilon^I)\,d\bar V^I\\
&\quad+\big(a\big(u_\varepsilon^I,u_\varepsilon^{II}\big)-f'(u_\varepsilon^I)\big)\,(u_\varepsilon^{II}-u_\varepsilon^I)\,d\bar V^I-\big(a\big(u^I,u^{II}\big)-f'(u^I)\big)\,(u^{II}-u^I)\,d\bar V^I,
\end{aligned}$$

which, with obvious notation, yields a decomposition for $T_{2,1}^h$

$$T_{2,1}^h=M_1^h+M_2^h+M_3^h+M_4^h. \tag{4.76}$$

Using (4.47) and the local convergence property (4.68), we obtain

$$\begin{aligned}
|M_1^h|&\le C\sum_{j=0}^n\int_{(y_j^h,y_{j+1})}\left|dV^{I,h}\right|\Big(\sup_{(y_j^h,y_{j+1})}|u^{I,h}-u^I|+\sup_{(y_j^h,y_{j+1})}|u^{II,h}-u^{II}|\Big)\\
&\le C\,\varepsilon.
\end{aligned}\tag{4.77}$$

Similarly using (4.47) and (4.70) we obtain

$$\begin{aligned}
|M_2^h|&\le C\sum_{j=0}^n\int_{(y_j^h,y_{j+1})}\left|dV^{I,h}\right|\Big(\sup_{(y_j^h,y_{j+1})}|u^I-u_\varepsilon^I|+\sup_{(y_j^h,y_{j+1})}|u^{II}-u_\varepsilon^{II}|\Big)\\
&\le C\,\varepsilon.
\end{aligned}\tag{4.78}$$

Dealing with M_4^h is similar:

$$\begin{aligned}
|M_4^h|&\le C\sum_{j=0}^n\int_{(y_j^h,y_{j+1})}\left|d\bar V^I\right|\Big(\sup_{(y_j^h,y_{j+1})}|u^I-u_\varepsilon^I|+\sup_{(y_j^h,y_{j+1})}|u^{II}-u_\varepsilon^{II}|\Big)\\
&\le C\,\varepsilon.
\end{aligned}\tag{4.79}$$

Finally to treat M_3^h we observe that, since u_ε^I and u_ε^{II} are continuous functions on each interval (y_j^h, y_{j+1}) and since $dV^{I,h}$ is sequence of bounded measures converging weakly-star toward $d\bar{V}^I$, we have for all h sufficiently small

$$|M_3^h| \leq \varepsilon. \tag{4.80}$$

Combining (4.76)-(4.80) we get

$$\left|T_{2,1}^h\right| \leq C\,\varepsilon. \tag{4.81}$$

Combining (4.73)-(4.75) and (4.81) we obtain

$$\left|T_2^h\right| \leq C\,\varepsilon$$

and thus with (4.71)-(4.72)

$$\left|\Omega_2^h(\tau)\right| \leq C\,\varepsilon \quad \text{for all } h \text{ sufficiently small.}$$

Since ε is arbitrary, this completes the proof of (4.66).

Step 3 : Consider now the term

$$\Omega_1(\tau) = \sum_{(x,\tau)\in\mathcal{L}(a^h)\cap\mathcal{J}(u^{I,h})} \Big(2\,m+TV(b^h)\Big)\left|a^h(x-,\tau)-\lambda^h(x,\tau)\right| |u^{II,h}(x-,\tau)-u^{I,h}(x-,\tau)|. \tag{4.82}$$

On one hand, observe that

$$TV(b^h(\tau)) = TV(u^{I,h}(\tau)) + TV(u^{II,h}(\tau)) \longrightarrow \bar{V}^I(+\infty,\tau) + \bar{V}^{II}(+\infty,\tau). \tag{4.83}$$

For all but countably many τ the following holds. Extracting a subsequence if necessary we can always assume that for each j either $(y_j^h,\tau) \in \mathcal{L}(a^h)$ for all h, or else $(y_j^h,\tau) \notin \mathcal{L}(a^h)$ for all h. Then consider the following three sets: denote by J_1 the set of indices j such that $(y_j^h,\tau) \in \mathcal{L}(a^h)$ and $(y_j,\tau) \in \mathcal{L}(a)$. Let J_2 the set of indices j such that $(y_j^h,\tau) \notin \mathcal{L}(a^h)$ and $(y_j,\tau) \in \mathcal{L}(a)$. Finally J_3 is the set of indices j such that $(y_j^h,\tau) \in \mathcal{L}(a^h)$ and $(y_j,\tau) \notin \mathcal{L}(a)$.

The local convergence property (4.61)-(4.62) implies

$$\begin{aligned}&\sum_{j\in J_1}\left|a\big(u^{I,h}(y_j^h-), u^{II,h}(y_j^h-)\big) - a\big(u^{I,h}(y_j^h-), u^{I,h}(y_j^h+)\big)\right| |u^{II,h}(y_j^h-) - u^{I,h}(y_j^h-)| \\ &\longrightarrow \sum_{j\in J_1}\left|a\big(u^I(y_j-), u^{II}(y_j-)\big) - a\big(u^I(y_j-), u^I(y_j+)\big)\right| |u^{II}(y_j-) - u^I(y_j-)|.\end{aligned} \tag{4.84}$$

(Indeed, given $\varepsilon > 0$, choose finitely many jump points as in (4.68) and use (4.61)-(4.62) with ε replaced with $\varepsilon\,|u^I(z+) - u^I(z+)|$).

On the other hand for indices in J_2 or J_3 we have

$$\sum_{j\in J_2\cup J_3} \big|a\big(u^{I,h}(y_j^h-),u^{II,h}(y_j^h-)\big)-a\big(u^{I,h}(y_j^h-),u^{I,h}(y_j^h+)\big)\big|\,|u^{II,h}(y_j^h-)-u^{I,h}(y_j^h-)| \longrightarrow 0 \tag{4.85}$$

but

$$\sum_{j\in J_2\cup J_3} \big|a\big(u^I(y_j-),u^{II}(y_j-)\big) - a\big(u^I(y_j-),u^I(y_j+)\big)\big|\,|u^{II}(y_j-) - u^I(y_j-)| = 0. \tag{4.86}$$

Indeed, for each $j \in J_2$, y_j is a Lax shock but y_j^h is not. Extracting a subsequence if necessary, it must be that the Lax inequalities are violated on the left or on the right side of y_j^h for all h. So it must be that, assuming that it is the case on the left side, $a\big(u^I(y_j-),u^{II}(y_j-)\big) - a\big(u^I(y_j-),u^I(y_j+) \geq 0$ while $a\big(u^I(y_j^h-),u^{II}(y_j^h-)\big) - a\big(u^I(y_j^h-),u^I(y_j^h+) \leq 0$ for all h. But the latter converges toward the former by the local uniform convergence, which proves that $a\big(u^I(y_j-),u^{II}(y_j-)\big) - a\big(u^I(y_j-),u^I(y_j+) = 0$.

Combining (4.82)-(4.86) yields

$$\Omega_1^h \to \int_0^t \sum_{(x,\tau)\in\mathcal{L}(a)} \bar q(\tau)\,\big|a(x-,\tau) - \lambda(x,\tau)\big|\,|u^{II}(x,\tau) - u^I(x,\tau)|\,d\tau, \tag{4.87}$$

where $\bar q := 2\,m + \bar V^I(+\infty) + \bar V^{II}(+\infty)$.

Step 4 : Continuity of the weighted norm.

Fix some time t. Recall that the weight $\bar w(t)$ is defined based on the total variation functions $\bar V^{II}$ and $\bar V^I$ and on the function $u^{II}(t) - u^I(t)$. The weight $w^h(t)$ is defined based on the total variation functions $V^{II,h}$ and $V^{I,h}$ and on the function $u^{II,h}(t) - u^{I,h}(t)$. On the other hand, $u^{II,h} - u^{I,h}(t) \to u^{II} - u^I(t)$, $V^{II,h} \to \bar V^{II}$ and $V^{I,h} \to \bar V^I$. Therefore we have

$$w(x,t) = \bar w(x,t) \quad \text{whenever } u^{II}(x,t) - u^I(x,t) \neq 0. \tag{4.88}$$

Combining (4.88) and the $\mathbf{L}^1$ convergence $u^{II,h} - u^{I,h}(t) \to u^{II} - u^I(t)$, we have

$$\|u^{II}(t) - u^I(t)\|_{\bar w(t)} = \lim_{h\to 0} \|u^{II,h}(t) - u^{I,h}(t)\|_{w^h(t)}. \tag{4.89}$$

Step 5 : The left-hand side of (4.45) is bounded above by (4.52).

First of all, the inequality

$$\int_0^t \int_{\mathbb{R}} \big(a(u^I, u^{II}) - f'(u^I)\big)\,(u^{II} - u^I)\, dV^I(y,\tau) d\tau$$
$$\leq \int_0^t \int_{\mathbb{R}} \big(a(u^I, u^{II}) - f'(u^I)\big)\,(u^{II} - u^I)\, d\bar{V}^I(y,\tau) d\tau \tag{4.90}$$

is a direct consequence of (4.49) and the definition of the nonconservative product in (4.43)-(4.44).

On the other hand, by the definition of the weighted norm and because of (4.50)-(4.51), similarly to (4.88) we have the inequality

$$w(x,t) \leq \bar{w}(x,t) \qquad \text{whenever } u^{II}(x,t) - u^I(x,t) \neq 0. \tag{4.91}$$

Hence, by (4.90) and (4.91) the left-hand side of (4.45) is bounded above by (4.52). This completes the proof of Theorem 4.3.5. □

4.4 Generalized characteristics and Maximum Principle

We now return to the setting in Section 4.2 and aim at extending the analysis therein to arbitrary functions of bounded variation. For exact solutions of the hyperbolic equation

$$\psi_t + \big(a\,\psi\big)_x = 0, \tag{4.92}$$

we will establish a maximum principle: Any solution of (4.92) remains non-negative for all times if it is so initially. For a more precise (local) statement, our proof will make use of Dafermos-Filippov theory of generalized characteristics.

Our main assumption throughout this section is the following:

$$\text{There exists a constant } E \text{ such that } a_x \leq \frac{E}{t}. \tag{4.93}$$

This is nothing but a generalization of the well-known Oleinik's entropy inequality. To motivate (4.93), let us recall the following result.

Let f be a strictly convex function and u be an entropy solution (with bounded variation for all times) of the conservation law

$$u_t + f(u)_x = 0, \qquad u(x,t) \in \mathbb{R}. \tag{4.94}$$

Then is is known that there exists a constant $C = C(u)$ such that

$$u_x \leq \frac{C}{t}. \tag{4.95}$$

Lemma 4.4.1 *If u^I and u^{II} are two entropy solutions of the conservation law (4.94), then the averaging speed*

$$a = a(u^I, u^{II}) := \frac{f(u^{II}) - f(u^I)}{u^{II} - u^I}. \tag{4.96}$$

satisfies our assumption (4.93), with $E = \sup f'' \big(C(u^I) + C(u^{II})\big)/2$.

PROOF OF LEMMA 4.4.1. Let us fix some time $t > 0$. On each Borel set consisting of points of continuity of both u^I and u^{II}, the following holds:

$$\begin{aligned}
\partial_x a &= \partial_x \int_0^1 f'\big(\theta\, u^I + (1-\theta)\, u^{II}\big)\, d\theta \\
&= \int_0^1 f''\big(\theta\, u^I + (1-\theta)\, u^{II}\big)\, \big(\theta\, \partial_x u^I + (1-\theta)\, \partial_x u^{II}\big)\, d\theta \\
&\le \int_0^1 \sup f''\, \big(\theta\, \frac{C(u^I)}{t} + (1-\theta)\, \frac{C(u^{II})}{t}\big)\, d\theta \\
&\le \sup f''\, \frac{C(u^I) + C(u^{II})}{2\,t}.
\end{aligned}$$

On the other hand, at a point x where one of u^I or u^{II} is discontinuous, we have with an obvious notation

$$a_+ - a_- = \int_0^1 f'\big(\theta\, u^I_+ + (1-\theta)\, u^{II}_+\big)\, d\theta - \int_0^1 f'\big(\theta\, u^I_- + (1-\theta)\, u^{II}_-\big)\, d\theta \le 0,$$

since f' is an increasing function and (for instance by (4.95)) both u^I and u^{II} satisfy $u^I_+ \le u^I_-$ and $u^{II}_+ \le u^{II}_-$. □

By definition, a generalized characteristic $y = y(t)$ associated with the coefficient a must satisfy for almost every t (in its domain of definition)

$$a_+(y(t), t) \le y'(t) \le a_-(y(t), t). \tag{4.97}$$

According to Filippov's theory of differential equations [36], through each point $(\bar{x}, \bar{t})$ there pass a maximal and a minimal generalized characteristic.

Definition 4.4.2 *A generalized characteristic is said to be genuine iff for almost every t it satisfies*

$$y'(t) \in \big\{a_-(y(t), t), a_+(y(t), t)\big\}. \tag{4.98}$$

Proposition 4.4.3 *Any minimal backward generalized characteristic is genuine and for almost every t satisfies*

$$y'(t) = a_-(y(t), t). \tag{4.99}$$

Similarly, for a maximal backward generalized characteristic we have $y'(t) = a_+(y(t), t)$.

Proof of Proposition 4.4.3. Here we only rely on the following consequence of (4.93): $a_+ \le a_-$ at each discontinuity point of the function a. Geometrically, this condition prevents the existence of rarefaction-shocks in a. On the other hand, rarefaction centers (also prevented by (4.93) for $t > 0$) could still be allowed for the present purpose.

Consider $(\bar{x}, \bar{t}) \in (-\infty, +\infty) \times (0, \infty)$, and let $y(t) := y(t; \bar{x}, \bar{t})$ be the minimal backward characteristic through $(\bar{x}, \bar{t})$. We prove that it is genuine on its domain $(s, \bar{t}]$. We proceed as in [28] and assume by contradiction that there is a measurable set J, $\bar{J} \subset (s, \bar{t}]$ of positive Lebesgue measure, and $\varepsilon > 0$ such that

$$a_-(y(t), t) - y'(t) > 2\varepsilon, \qquad t \in J. \tag{4.100}$$

For each $t \in J$ there exists $\delta(t) > 0$ with the property

$$a_+(x, t) \ge a_-(y(t), t) - \varepsilon, \qquad x \in (y(t) - \delta(t),\ y(t)). \tag{4.101}$$

Finally, there is a subset $I \subset J$ with $\mu^*(I) > 0$ (here μ^* denotes the outer measure) and $\bar{\delta} > 0$ such that $\delta(t) > \bar{\delta}$ for $t \in I$.

Let τ be a density point of I, with respect to μ^*. Thus there exists $\bar{r}$, $0 < \bar{r} < \bar{t} - \tau$, so that

$$\frac{\mu^*(I \cap [\tau, \tau + r])}{r} > \frac{2|\alpha| + \varepsilon}{2|\alpha| + 2\varepsilon}, \qquad 0 < r \le \bar{r}, \tag{4.102}$$

where

$$\alpha := \inf\big\{a_+(x, t) - a_-(y(t), t) :\ s < t \le \bar{t},\ y(t) - \bar{\delta} \le x < y(t)\big\}.$$

Now take a point $y \in (y(\tau) - \bar{\delta},\ y(\tau))$ with the property $y > y(\tau) - \frac{1}{2}\varepsilon\bar{r}$, and consider a forward characteristic $z(\cdot)$ through (y, τ). We first observe that

$$z(t) < y(t), \qquad t > \tau,$$

since $y(t)$ is the minimal backward characteristic through $(\bar{x}, \bar{t})$.

In addition, we have

$$z(t) > y(t) - \bar{\delta}, \qquad t >\in [\tau,\ \tau + \bar{r}].$$

Indeed, suppose by contradiction that for some $r \in (0, \bar{r}]$, $z(t) > y(t) - \bar{\delta}$ for $t > \in [\tau,\ \tau + r)$, but $z(\tau + r) = y(\tau + r) - \bar{\delta}$. Then

$$
\begin{aligned}
0 &= z(\tau + r) - y(\tau + r) - \bar{\delta} = y + \int_{\tau}^{\tau + r} z'(t)dt \ - \ y(t) \ - \ \int_{\tau}^{\tau + r} y'(t)dt \ + \ \bar{\delta} \\
&> \int_{\tau}^{\tau + r} (z'(t) - y'(t))dt \\
&= \int_{I \cap [\tau, \tau + r]} (z'(t) - a_{-}(y(t), t) + a_{-}(y(t), t) - y'(t))dt \\
&\quad + \int_{[\tau, \tau + r] \setminus I} (z'(t) - a_{-}(y(t), t) + a_{-}(y(t), t) - y'(t))dt \\
&\geq \varepsilon \mu^{*}\big(I \cap [\tau, \tau + r]\big) + \alpha\big(r - \mu^{*}(I \cap [\tau, \tau + r])\big) > 0,
\end{aligned}
$$

by (4.100)-(4.102), which leads to a contradiction. In the same way one obtains

$$
\begin{aligned}
0 &> z(\tau + \bar{r}) - y(\tau + \bar{r}) = y + \int_{\tau}^{\tau + \bar{r}} z'(t)dt \ - \ y(t) \ - \ \int_{\tau}^{\tau + \bar{r}} y'(t)dt \\
&> \varepsilon \mu^{*}\big(I \cap [\tau, \tau + \bar{r}]\big) + \alpha\big(\bar{r} - \mu^{*}(I \cap [\tau, \tau + r])\big) - \frac{1}{2}\varepsilon \bar{r} > 0,
\end{aligned}
$$

which gives another contradiction. For the maximal backward characteristic the proof is similar. □

Proposition 4.4.4 *Forward characteristics leaving from some* $(\bar{x}, \bar{t})$ *are unique when* $\bar{t} > 0$.

PROOF OF PROPOSITION 4.4.4. Suppose there were two forward characteristics $y(\cdot)$ and $z(\cdot)$ through $(\bar{x}, \bar{t})$ with $y(\tau) < z(\tau)$ for some $\tau > \bar{t}$. By (4.93) we have

$$z'(\tau) - y'(\tau) \leq a_{-}(z(t), t) - a_{+}(y(t), t) \leq C_{\bar{t}}\big(z(\tau) - y(\tau)\big). \tag{4.103}$$

Integrating (4.103) from $\bar{t}$ to τ one gets $z(\tau) - y(\tau) = 0$, which gives a contradiction. □

Theorem 4.4.5 *Let* $\psi = \psi(x, t)$ *be a solution of (4.92) such that on some interval* $[\xi_0, \zeta_0]$ *we have*

$$\psi(x, 0) \geq 0, \qquad x \in [\xi_0, \zeta_0]. \tag{4.104}$$

Let $\xi = \xi(t)$ be any forward generalized characteristic leaving from $(\xi_0, 0)$, and $\zeta = \zeta(t)$ be any forward generalized characteristic leaving from $(\zeta_0, 0)$.

Then we have for all $t \geq 0$

$$\psi(x,t) \geq 0, \qquad x \in \big(\xi(t), \zeta(t)\big). \tag{4.105}$$

Note that it may happen that $\xi(t) = \zeta(t)$ for t large enough.

Proof of Theorem 4.4.5. Observe that the two characteristics cannot cross and fix any time $t > 0$ such that $\xi(t) < \zeta(t)$. Fix also any two points such that $\xi(t) < \bar{y} < \bar{z} < \zeta(t)$. Let $y(t)$ and $z(t)$ be the maximal and minimal backward characteristics emanating from $\bar{y}$ and $\bar{z}$, respectively. These characteristics can not leave the region limited by $\xi(t)$ and $\zeta(t)$.

Integrating (4.92) in the domain bounded by the characteristics $y(t)$ and $z(t)$, and using that these characteristics are genuine, so that the flux terms along the vertical boundaries vanish identically, we arrive at

$$\int_{\bar{y}}^{\bar{z}} \psi(x,t)\, dx = \int_{y(0)}^{z(0)} \psi(x,0)\, dx \geq 0. \tag{4.106}$$

The last inequality is due to the fact that $\psi(.,0) \geq 0$ and the inequalities $\xi_0 = \xi(0) \leq y(0) \leq z(0) \leq \zeta(0) = \zeta_0$. Since $\bar{y}$ and $\bar{z}$ are arbitrary, we obtain (4.105). □

4.5 A sharp $\mathbf{L}^1$ estimate for hyperbolic linear equations

Based on the maximum principle established in Section 4.4, we now derive a sharp estimate for the weighted norm introduced in Section 4.2. We restrict attention again to the situation where u^I and u^{II} are two entropy solutions of the conservation law (4.94) and a is the averaging speed given in (4.96). We define a weight by analogy with what was done in Section 4.2 in the special case of piecewise constant solutions.

Given a solution ψ of the equation (4.92), we introduce weighted $\mathbf{L}^1$ norm in the following way. Set

$$V^I(x,t) = TV_{-\infty}^x(u^I(t)), \qquad V^{II}(x,t) = TV_{-\infty}^x(u^{II}(t)) \tag{4.107}$$

and fix some parameter $m \geq 0$. Then consider the weight-function defined, for each

$t \geq 0$ and each point of continuity x for $u^I(t)$ and $u^{II}(t)$, by

$$w(x,t) = \begin{cases} m + V^I(\infty,t) - V^I(x,t) + V^{II}(x,t) \text{ if } \psi(x,t) > 0, \\ \\ m + V^I(x,t) + V^{II}(\infty,t) - V^{II}(x,t) \text{if } \psi(x,t) \leq 0. \end{cases} \tag{4.108}$$

It is immediate to see that

$$m \leq w(x,t) \leq m + TV(u^I(t)) + TV(u^{II}(t)), \qquad x \in I\!R. \tag{4.109}$$

Finally the weighted norm on the solutions ψ of (4.92) is defined by

$$\|\psi(t)\|_{w(t)} := \int_{I\!R} |\psi(x,t)|\, w(x,t)\, dx.$$

Note that the weight depends on the fixed solutions u^I and u^{II}, but also on the solution ψ.

Our sharp estimate will involve the nonconservative product

$$\mu_\psi^I(t) = \big(a - f'(u^I(t))\big)\, \psi(t)\, dV^I(t)$$

defined for all almost every $t \geq 0$ by

(1) If B is a Borel set included in the set of continuity points of $u^I(t)$ then

$$\mu_\psi^I(t)(B) = \int_B \big(a(t) - f'(u^I(t))\big)\, \psi(t)\, dV^I(t), \tag{4.110}$$

where the integral is defined in a classical sense;

(2) If x is a point of jump of $u^I(t)$, then

$$\mu_\psi^I(t)(\{x\}) = \big(a(x-,t) - \lambda^I(x,t)\big)\, \psi(x-,t)\, \big|u^I(x+,t) - u^I(x-,t)\big|. \tag{4.111}$$

Here $\lambda^I(x,t)$ is a the shock speed of the discontinuity in u^I located at (x,t). The measure $\mu_\psi^{II}(t)$ is defined similarly. Regarding the expression (4.111), it is worth noting that if (x,t) is a point of approximate jump of u^I and ψ, then the jump relation for the equation (4.92) reads

$$\big(a(x-,t) - \lambda^I(x,t)\big)\, \psi(x-,t) = \big(a(x+,t) - \lambda^I(x,t)\big)\, \psi(x+,t). \tag{4.112}$$

In the same way we define

$$\mu_\psi^{II}(t) = \big(f'(u^{II}(t)) - a\big)\, \psi(t)\, dV^{II}(t).$$

We now prove:

Theorem 4.5.1 *Let u^I and u^{II} be two entropy solutions of (4.1) such that $u^{II} - u^I$ admits finitely many changes of sign. Let ψ be any solution of bounded variation of the hyperbolic equation (4.92) satisfying the constrain*

$$\psi\left(u^{II} - u^I\right) \geq 0. \tag{4.113}$$

Then for all $0 \leq s \leq t$

$$\begin{aligned} \|\psi(t)\|_{w(t)} &+ \int_s^t \sum_{(x,\tau)\in\mathcal{L}(a)} \Big(2\,m + TV(a)\Big) \big|a_-(x,\tau) - \lambda(x,\tau)\big|\,|\psi_-(x,\tau)|\,d\tau \\ &+ \int_s^t \int_{I\!R} \big(a(\tau) - f'(u^I(\tau))\big)\,\psi(\tau)\,dV^I(\tau)d\tau \\ &+ \int_s^t \int_{I\!R} \big(a(\tau) - f'(u^{II}(\tau))\big)\,\psi(\tau)\,dV^{II}(\tau)d\tau \\ &\leq \|\psi(s)\|_{w(s)}. \end{aligned} \tag{4.114}$$

The assumption (4.113) is clearly satisfied with the choice $\psi = u^{II} - u^I$. Therefore our previous result in Theorem 4.3.5 (derived via a completely different proof) can be regarded as a corollary of Theorem 4.5.1.

It is interesting to observe that, when $u^{II} = u^I$, the weight (4.108) becomes constant, and therefore (4.114) reduces to the $\mathbf{L}^1$ estimate.

$$\begin{aligned} \|\psi(t)\|_{\mathbf{L}^1} &+ \int_s^t \sum_{(x,\tau)\in\mathcal{L}(a)} \Big(2\,m + TV(a)\Big) \big|a_-(x,\tau) - \lambda(x,\tau)\big|\,|\psi_-(x,\tau)|\,d\tau \\ &\leq \|\psi(s)\|_{\mathbf{L}^1}. \end{aligned}$$

Also, note that under the assumption (4.113) $\mu_\psi^I(t)$ and $\mu_\psi^{II}(t)$ are positive except at points $(x,t) \in \mathcal{L}(a) \cup \mathcal{R}(a)$. However, these negative terms are offset in (4.114) by the positve terms under the first integral.

Proof of Theorem 4.5.1. Fix any positive time t. By assumption we have finitely many points $-\infty = y_0 < y_1 < \ldots < y_n < y_{n+1} = +\infty$ such that, on each interval (y_i, y_{i+1}), we have $\psi(t) \geq 0$ when i is odd and $\psi(t) \geq 0$ when i is even. For every $i = 1, \cdots, n$, consider the (unique by Proposition 4.4.4) forward characteristic $y_i(\cdot)$ associated with the coefficient a and issuing from the initial point (y_i, t).

We will focus attention on some interval (y_i, y_{i+1}) with i odd, say, and with $-\infty < y_i < y_{i+1} < +\infty$. Except when specified differently, all of the characteristics to be considered from now on are associated with the solution u^{II}. For definiteness we will first study the case that the forward characteristic $\chi_0(\cdot)$ (associated with u^{II} and) issuing from the point (y_i, t) is located on the right-side of the curve y_i, that is,

$$y_i(\tau) \leq \chi_0(\tau), \qquad t \leq \tau \leq t + \delta$$

for some $\delta > 0$ sufficiently small.

Fix some (sufficiently small) $\varepsilon > 0$ and denote by $y_i < z_1 < \ldots < z_N < y_{i+1}$ the points where u^I has a jump larger or equal to ε, that is,

$$u_-^{II}(z_I, t) - u_+^{II}(z_I, t) \geq \varepsilon, \qquad I = 1, \ldots, N. \tag{4.115}$$

For each $I = 1, \ldots, N$, consider also the forward characteristic $\chi_I(\cdot)$ issuing from the point (z_I, t). For definiteness, we will also assume that the forward characteristic $\chi_{N+1}(\cdot)$ issuing from (y_{i+1}, t) satisfies

$$\chi_{N+1}(\tau) \leq y_{i+1}(\tau), \qquad t \leq \tau \leq t + \delta$$

for some $\delta > 0$ sufficiently small.

Next, let us select a time $s > t$ with $s - t$ so small that the following properties hold:

(a) No intersection among the characteristics $y_i, \chi_0, \chi_1, \ldots, \chi_N, \chi_{N+1}, y_{i+1}$ occurs in the time interval $[t, s]$.

(b) For $I = 1, \ldots, N$, let $\zeta_I(\cdot)$ and $\xi_I(\cdot)$ be the minimal and the maximal backward characteristics emanating from the point $(\chi_I(s), s)$. Then the total variation of $u^{II}(\cdot, t)$ over the intervals $(\zeta_I(t), z_I)$ and $(z_I, \xi_I(t))$ should not exceed $\frac{\varepsilon}{N}$.

(c) Let $\zeta_0(\cdot)$ be the minimal backward characteristic emanating from $(y_i(s), s)$ and $\xi_0(\cdot)$ be the maximal backward characteristic emanating from $(\chi_0(s), s)$. Then the total variation of $u^{II}(\cdot, t)$ over the intervals $(y_i, \xi_0(t))$ and $(\zeta_0(t), y_i)$ should not exceed ε.

(d) Let $\zeta_{N+1}(\cdot)$ be the minimal backward characteristic emanating from the point $(\chi_{N+1}(s), s)$ and $\xi_{N+1}(\cdot)$ be the maximal backward characteristic emanating from $(y_{i+1}(s), s)$. Then the total variation of $u^{II}(\cdot, t)$ over the intervals $(\zeta_{N+1}(t), y_{i+1})$ and $(y_{i+1}, \xi_{N+1}(t))$ should not exceed ε.

For $I = 0, \ldots, N$, and some integer k to be fixed later, consider a mesh of the form

$$\chi_I(s) = x_I^0 < x_I^1 < \ldots < x_I^k < x_I^{k+1} = \chi_{I+1}(s). \tag{4.116}$$

For $I = 0, \ldots, N$ and $j = 1, \ldots, k$, consider also the maximal backward characteristic $\xi_I^j(\cdot)$ emanating from the point (x_I^j, s) and identify its intercept $z_I^j = \xi_I^j(t)$ by the horizontal line at time t. Finally set also

$$z_0^0 = y_i, \quad z_N^{k+1} = y_{i+1}, \quad z_{I-1}^{k+1} = z_I^0 = z_I, \qquad I = 1, \ldots, N.$$

To start the proof, we integrate the equation (4.92) satisfied by the function ψ, successively in each domain limited by the characteristics introduced above. Applying Green's theorem, we arrive at the following five formulas:

(i) Integrating (4.92) on the region

$$\big\{(x,\tau)\,/\,t < \tau < s, \quad y_i(\tau) < x < \chi_0(\tau)\big\}$$

and multiplying by $V^{II}(y_i, t)$ one gets

$$\int_{y_i(s)}^{\chi_0(s)} \psi(x,s)\, V^{II}(y_i,t)\, dx + \int_t^s (y_i' - a_+)\, \psi_+(y_i(\tau),\tau)\, V^{II}(y_i,t)\, d\tau$$
$$+ \int_t^s (a_- - \lambda_0)\, \psi_-(\chi_0(\tau),\tau)\, V^{II}(y_i,t)\, d\tau = 0. \tag{4.117}$$

(ii) Integrating (4.92) on each of the regions

$$\big\{(x,\tau)\,/\,t < \tau < s, \quad \xi_I^j(\tau) < x < \xi_I^{j+1}(\tau)\big\}$$

for $I = 0, \ldots, N$ and $j = 1, \ldots, k$, and then multiplying by $V^{II}(z_I^j+, t)$, one gets

$$\int_{x_I^j}^{x_I^{j+1}} \psi(x,s)\, V^{II}(z_I^j+,t)\, dx - \int_{z_I^j}^{z_I^{j+1}} \psi(x,t)\, V^{II}(z_I^j+,t)\, dx$$
$$+ \int_t^s (\lambda_I^j - a_+)\psi_+(\xi_I^j(\tau),\tau)\, V^{II}(z_I^j+,t)\, d\tau$$
$$+ \int_t^s (a_- - \lambda_I^{j+1})\, \psi_-(\xi_I^{j+1}(\tau),\tau)\, V^{II}(z_I^j+,t)\, d\tau = 0. \tag{4.118}$$

(iii) Integrating (4.92) on each of the regions

$$\big\{(x,\tau)\,/\,t < \tau < s, \quad \chi_I(\tau) < x < \xi_I^1(\tau)\big\}$$

for $I = 0, \ldots, N$, and multiplying by $V^{II}(z_I+, t)$ one gets

$$\int_{\chi_I(s)}^{x_I^1} \psi(x,s)\, V^{II}(z_I+,t)\, dx - \int_{z_I}^{z_I^1} \psi(x,t)\, V^{II}(z_I+,t)\, dx$$
$$+ \int_t^s (\lambda_I - a_+)\, \psi_+(\chi_I(\tau),\tau)\, V^{II}(z_I+,t)\, d\tau$$
$$+ \int_t^s (a_- - \lambda_I^1)\, \psi_-(\xi_I^1(\tau),\tau)\, V^{II}(z_I+,t)\, d\tau = 0. \tag{4.119}$$

(iv) Integrating (4.92) on the regions

$$\big\{(x,\tau)\,/\,t < \tau < s, \quad \xi_I^k(\tau) < x < \chi_{I+1}(\tau)\big\}$$

for $I = 0, \ldots, N$, and multiplying by $V^{II}(z_I^k+, t)$ one gets

$$\int_{x_I^k}^{\chi_{I+1}(s)} \psi(x,s)\, V^{II}(z_I^k+,t)\, dx - \int_{z_I^k}^{z_{I+1}} \psi(x,t)\, V^{II}(z_I^k+,t)\, dx$$
$$+ \int_t^s (\lambda_I^k - a_+)\, \psi_+(\xi_I^k(\tau),\tau)\, V^{II}(z_I^k+,t)\, d\tau$$
$$+ \int_t^s (a_- - \lambda_{I+1})\, \psi_-(\chi_{I+1}(\tau),\tau)\, V^{II}(z_I^k+,t)\, d\tau = 0. \tag{4.120}$$

(v) Finally integrating (4.92) on the last region

$$\big\{(x,\tau)\,/\,t < \tau < s, \quad \chi_{N+1}(\tau) < x < y_{i+1}(\tau)\big\}$$

and multiplying by $V^{II}(y_{i+1}, t)$ one gets

$$\int_{\chi_{N+1}(s)}^{y_{i+1}(s)} \psi(x,s)\, V^{II}(y_{i+1},t)\, dx + \int_t^s (\lambda_{N+1} - a_+)\, \psi_+(\chi_{N+1}(\tau),\tau)\, V^{II}(y_{i+1},t)\, d\tau$$
$$+ \int_t^s (a_- - y_{i+1}')\, \psi_-(y_{i+1}(\tau),\tau)\, V^{II}(y_{i+1},t)\, d\tau = 0. \tag{4.121}$$

Next, summing all of the formulas (4.117)-(4.121) leads us to the general identity:

$$\int_{y_i(s)}^{\chi_0(s)} \psi(x,s)\, V^{II}(y_i,t)\, dx + \sum_{I=0}^{N} \sum_{j=0}^{k} \int_{x_I^j}^{x_I^{j+1}} \psi(x,s)\, V^{II}(z_I^j+,t)\, dx$$
$$+ \int_{\chi_{N+1}(s)}^{y_{i+1}(s)} \psi(x,s)\, V^{I}(y_{i+1},t)\, dx - \sum_{I=0}^{N} \sum_{j=0}^{k} \int_{z_I^j}^{z_I^{j+1}} \psi(x,t)\, V^{II}(z_I^j+,t)\, dx$$
$$= - \sum_{I=0}^{N} \sum_{j=1}^{k} \int_t^s [V^{II}(z_I^j+,t) - V^{II}(z_I^{j-1}+,t)]\, (\lambda_I^j - a_-)\, \psi_-(\xi_I^j(\tau),\tau)\, d\tau$$

$$
\begin{aligned}
&-\sum_{I=0}^{N}\int_t^s [V^{II}(z_I+,t) - V^{II}(z_{I-1}^k+,t)]\,(\lambda_I^j - a_-)\,\psi_-(\chi_I(\tau),\tau)\,d\tau \\
&-\int_t^s [V^{II}(y_i+,t) - V^{II}(y_i,t)]\,(\lambda_0 - a_-)\,\psi_-(\chi_0(\tau),\tau)\,d\tau \\
&-\int_t^s [V^{II}(y_{i+1}+,t) - V^{II}(z_N^k,t)]\,(\lambda_{N+1} - a_-)\,\psi_-(\chi_{N+1}(\tau),\tau)\,d\tau \\
&-\int_t^s (y_i' - a_+)\,\psi_+(y_i(\tau),\tau)\,V^{II}(y_i,t)\,d\tau \\
&-\int_t^s (a_- - y_{i+1}')\,\psi_-(y_{i+1}(\tau),\tau)\,V^{II}(y_{i+1},t)\,d\tau.
\end{aligned}
\tag{4.122}
$$

To estimate the right-hand side of (4.122), we recall that the solution u^I of a scalar conservation laws satisfies

$$
V^{II}(y_i,t) \geq V^{II}(\chi_0(s),s), \qquad V^{II}(z_I^j+,t) \geq V^{II}(x_I^j+,s),
$$

for $I = 0,\dots,N$ and $j = 0,\dots,k$. Hence, choosing the difference $x_I^{j+1} - x_I^j$ in (4.116) sufficiently small and since the function $V^{II}(\cdot,t)$ is nondecreasing, we conclude that the left-hand side of (4.122) can be bounded from below, as follows:

$$
\text{L.H.S.} \geq \int_{y_i(s)}^{y_{i+1}(s)} \psi(x,s)V^{II}(x,s)dx - (s-t)\varepsilon - \int_{y_i(t)}^{y_{i+1}(t)} \psi(x,t)V^{II}(x,t)dx. \tag{4.123}
$$

Estimating the right-hand side of (4.122) is more involved. First note that each term arising in the left-hand side of (4.122) is non-positive. This follows from our condition (4.113). Indeed, consider a point (x,s) of approximate jump or approximate continuity of u^I, u^{II} and ψ. If all of these functions are continuous, the result is trivial. Call λ the discontinuity speed. Based on the jump relation (4.112), we see that either $\psi_-\,(\lambda - a_-) = \psi_+\,(\lambda - a_+) = 0$, or else all of the terms ψ_-, $\lambda - a_-$, ψ_+, and $\lambda - a_+$ are distinct from zero.

Suppose first that (x,s) is a point in the interior of the region limited by the two curves $y_i(.)$ and $y_{i+1}(.)$. In the latter case, since $\psi \geq 0$ in the region under consideration, we deduce that $\psi_- > 0$ and $\psi_+ > 0$, while the terms $\lambda - a_-$ and $\lambda - a_+$ are either both negative or both positive. Actually, in view of the sign condition (4.113), we have $u_\pm^{II} - u_\pm^I \geq 0$ and, therefore, $\lambda - a_\pm \geq$ as follows from (4.38) (here we are dealing with a jump of u^{II}).

Consider next a point of the boundary y_i, for instance. So we now have $\psi_- < 0$ and $\psi_+ > 0$, while the terms $\lambda - a_-$ and $\lambda - a_+$ opposite sign. Since no rarefaction-shock can arise, the discontinuity must be a Lax shock and so $\lambda - a_- < 0$ and $\lambda - a_+ > 0$. Again the corresponding term in (4.122) has a favorable sign. (Observe that the condition (4.113) was not used in this second case.)

Then, for all $I = 0, \dots, N$ and $j = 1, \dots, k$, let $\theta_I^j(\cdot)$ be the (maximal, for definiteness) backward characteristic associated with u^I and issuing from the point $(\xi_I^j(\tau), \tau)$. Denote also by $\theta(z_I^j; \tau)$ its intercept with the horizontal line at time t. Setting

$$\tilde{a}(x, t; \tau) := \frac{f(u^{II}(x,t)) - f(u^I(\theta(x,\tau),t))}{u^{II}(x,t) - u^I(\theta(x;\tau),t)}$$

and using that the solution u^I remains constant along the characteristic $\theta_I^j(\cdot)$, we obtain

$$(\lambda_I^j - a_-)(\xi_I^j(\tau)) = \lambda_I^j(z_I^j) - \tilde{a}(z_I^j, t; \tau). \tag{4.124}$$

Then consider the (maximum, for definiteness) backward characteristic $y_I^j(\cdot)$ associated with a and issuing from the point $(\xi_I^j(\tau), \tau)$. By integrating ψ along the characteristic $y_I^j(\cdot)$ and using the inequality (4.95), we arrive at a lower bound for ψ

$$\psi(\xi_I^j(\tau), \tau) \geq \psi(y_I^j(t), t) \left(\frac{t}{\tau}\right)^E, \qquad t < \tau < s. \tag{4.125}$$

Upon choosing $x_I^{j+1} - x_I^j$ in (4.116) so small that the oscillation of $V_c^{II}(\cdot)$ over each interval $(z_I^j - z_I^{j+1})$ does not exceed ε and recalling the standard estimates on Stieltjes integrals we deduce from (4.122)-(4.124) that

$$\begin{aligned}
&\sum_{I=0}^{N}\sum_{j=1}^{k}\int_t^s [V^{II}(z_I^j+,t) - V^{II}(z_I^{j-1}+,t)]\ \big((\lambda_I^j - a_-)\,\psi_-(\xi_I^j(\tau),\tau)\big)\,d\tau \\
&\geq \sum_{I=0}^{N}\sum_{j=1}^{k}\int_t^s [V_c^{II}(z_I^j,t) - V_c^{II}(z_I^{j-1},t)]\ \big(\lambda_I^j(z_I^j) - \tilde{a}(z_I^j,t,\tau)\big)\psi(y_I^j(t),t)\ \left(\frac{t}{\tau}\right)^E\ d\tau \\
&\geq \int_t^s \sum_{I=0}^{N}\sum_{j=1}^{k}\left(\int_{z_I^{j-1}}^{z_I^j}\big(\lambda_I^j(x) - \tilde{a}(x,t,\tau)\big)\,\psi(x,t)\,dV_c^{II}(x,t) - c\,\varepsilon\right)\left(\frac{t}{\tau}\right)^E d\tau \\
&= \int_t^s \left(\int_{y_i}^{y_{i+1}}\big(\lambda_I^j(x) - \tilde{a}(x,t,\tau)\big)\psi(x,t)\,dV_c^{II}(x,t) - c(y_{i+1} - y_i)\varepsilon\right)\left(\frac{t}{\tau}\right)^E d\tau.
\end{aligned} \tag{4.126}$$

We now combine (4.117)-(4.122) and (4.126), divide the resulting inequality by $s - t$, and let $s \searrow t$, $\varepsilon \to 0$, obtaining the following inequality:

$$\tfrac{d^+}{dt}\int_{y_i(t)}^{y_{i+1}(t)} \psi(x,t)V^{II}(x,t)dx$$

$$
\begin{aligned}
\leq & -\int_{y_i}^{y_{i+1}} \big(\lambda_I^j(x) - a(x,t)\big)\, \psi(x,t)\, dV_c^{II}(x,t) \\
& - \sum_{(x,t)\in\mathcal{J}(u^{II})} \big(u_-^{II}(x,t) - u_+^{II}(x,t)\big)\, (\lambda^I - a_-)(x,t)\, \psi_-(x,t) \\
& -\big(u_-^{II}(y_i,t) - u_+^{II}(y_i,t)\big)\, (\lambda^I - a_-)(y_i,t)\, \psi_-(y_i,t) \\
& -\big(u_-^{II}(y_{i+1},t) - u_+^{II}(y_{i+1},t)\big)\, (\lambda^I - a_-)(y_{i+1},t)\, \psi_-(y_{i+1},t) \\
& -(y_i' - a_+)\, \psi_+(y_i,t)\, V^{II}(y_i,t) \\
& -(a_- - y_{i+1}')\, \psi_-(y_{i+1},t)\, V^{II}(y_{i+1},t).
\end{aligned}
\tag{4.127}
$$

The third and fourth terms in the right-hand side of (4.127) are due to the fact that χ_0 and χ_{N+1} lie inside the region limited by y_i and y_{i+1}.

We can next focus on the intervals (y_i, y_{i+1}) with i even. Based on a completely symmetric argument and using now the weight $m + V^{II}(\infty,t) - V^{II}(\cdot,t)$ instead of $V^{II}(\cdot,t)$, we obtain

$$
\begin{aligned}
& \tfrac{d^+}{dt} \int_{y_i(t)}^{y_{i+1}(t)} \big(-\psi(x,t)\big) \Big(m + V^{II}(\infty,t) - V^{II}(x,t)\Big) dx \\
\leq & \int_{y_i}^{y_{i+1}} \big(\lambda_I^j(x) - a(x,t)\big) \big(-\psi(x,t)\big) dV_c^{II}(x,t) \\
& + \sum_{(x,t)\in\mathcal{J}(u^{II})} \big(u_-^{II}(x,t) - u_+^{II}(x,t)\big)\, (\lambda^I - a_-)(-\psi_-)(x,t) \\
& +\big(u_-^{II}(y_i,t) - u_+^{II}(y_i,t)\big)\, (\lambda^I - a_-)(-\psi_-)(y_i,t) \\
& +\big(u_-^{II}(y_{i+1},t) - u_+^{II}(y_{i+1},t)\big)\, (\lambda^I - a_-)(-\psi_-)(y_{i+1},t) \\
& -(y_i' - a_+)(-\psi_+)(y_i,t) \Big(m + V^{II}(\infty,t) - V^{II}(y_i,t)\Big) \\
& -(a_- - y_{i+1}')(-\psi_-)(y_{i+1},t) \Big(m + V^{II}(\infty,t) - V^{II}(y_{i+1},t)\Big).
\end{aligned}
\tag{4.128}
$$

By summation over $i = 1, \ldots, n$ in (4.127) for i odd and in (4.128) for i even respectively, we obtain

$$
\begin{aligned}
& \tfrac{d^+}{dt} \int_{-\infty}^{+\infty} \big[\psi(x,t)\big]^+ V^{II}(x,t) + \big[-\psi(x,t)\big]^+ \Big(m + V^{II}(\infty,t) - V^{II}(x,t)\Big) dx \\
\leq & -\sum_{(x,t)\in\mathcal{L}(a)\cap\mathcal{J}(u^{II})} \Big(m + V^{II}(\infty,t)\Big)\, \big|\lambda(x,t) - a_-(x,t)\big|\, |\psi_-(x,t)| \\
& - \sum_{(x,t)\in\mathcal{J}(u^{II})} \big(u_-^{II}(x,t) - u_+^{II}(x,t)\big) \big(\lambda^I(x,t) - a_-(x,t)\big)\, \psi_-(x,t) \\
& - \int_{I\!R} \big(f'(u^{II}(y,t)) - a(y,t)\big)\, \psi(y,t)\, dV_c^{II}(y,t),
\end{aligned}
\tag{4.129}
$$

where the superscript + denotes the positive part of the functions ψ and $-\psi$ respectively.

Consider now the case where

$$\chi_0(\tau) \le y_i(\tau), \qquad t \le \tau \le t+\delta,$$

and

$$y_{i+1}(\tau) \le \chi_{N+1}(\tau), \qquad t \le \tau \le t+\delta.$$

Assume that there exists a time $\bar{\tau} > t$ such that

$$\chi_0(\bar{\tau}) < y_i(\bar{\tau}), \qquad\qquad y_{i+1}(\bar{\tau}) < \chi_{N+1}(\bar{\tau})$$

(otherwise the curves of the two pairs will coincide, and we can reduce to the previous case). Let now $\xi_0(\cdot)$ be the maximal backward characteristic emanating from $(y_i(\bar{\tau}), \bar{\tau})$, and $\zeta_{N+1}(\cdot)$ be the minimal backward characteristic emanating from the point $(y_{i+1}(\bar{\tau}), \bar{\tau})$. Since characteristics cannot cross, we have that

$$y_i(t) < \xi_0(t), \ \zeta_{N+1}(t) < y_{i+1}(t).$$

Then, by finite propagation speed, there exists a time $s > t$ such that

$$y_i(\tau) < \xi_0(\tau), \ \zeta_{N+1}(\tau) < y_{i+1}(\tau), \qquad t \le \tau < s,$$

$$y_i(s) = \xi_0(s), \ \zeta_{N+1}(s) = y_{i+1}(s).$$

Instead of properties (c), (d), we will require that s satisfies the following:

(c') Let $\zeta_0(\cdot)$ be the minimal backward characteristic emanating from $(\chi_0(s), s)$. Then the total variation of $u^{II}(\cdot, t)$ over the intervals $(y_i, \xi_0(t))$ and $(\zeta_0(t), y_i)$ should not exceed ε.

(d') Let $\xi_{N+1}(\cdot)$ be the maximal backward characteristic emanating from $(\chi_{N+1}(s), s)$. Then the total variation of $u^{II}(\cdot, t)$ over the intervals $(\zeta_{N+1}(t), y_{i+1})$ and $(y_{i+1}, \xi_{N+1}(t))$ should not exceed ε.

From then on we can proceed as before. Finally we write the inequality in (4.129) exchanging the roles of u^I and u^{II}, and combining it with (4.128) we arrive exactly at the desired inequality (4.114) and the proof of Theorem 4.5.1 is completed. $\square$

Bibliography

[1] D. Amadori, Initial-boundary value problems for nonlinear systems of conservation laws, *NoDEA* **4** (1997), 1-42.

[2] D. Amadori and R. M. Colombo, Continuous dependence for 2×2 conservation laws with boundary, *J. Differential Equations* **138** (1997), no. 2, 229-266.

[3] D. Amadori and R. M. Colombo, Characterization of viscosity solutions for conservation laws with boundary, *Rend. Sem. Mat. Univ. Padova* **99** (1998), 219-245.

[4] F. Ancona and P. Goatin, $\mathbf{L}^1$ stability for Temple class systems with $\mathbf{L}^\infty$ initial and boundary data, preprint.

[5] F. Ancona and A. Marson, On the attainable set for scalar non-linear conservation laws with boundary control, *SIAM Journal on Control and Optimization* **36** (1998), no. 1, 290-312.

[6] F. Ancona and A. Marson, Scalar non-linear conservation laws with integrable boundary data, *Nonlinear Anal.* **35** (1999), 687-710.

[7] F. Ancona and A. Marson, Well-posedness for general 2×2 systems of conservation laws, submitted.

[8] P. Baiti and A. Bressan, The semigroup generated by a Temple class system with large data, *Differ. Integ. Equat.* **10** (1997), 401-418.

[9] P. Baiti and H. K. Jenssen, Well-posedness for a class of 2×2 conservation laws with $\mathbf{L}^\infty$ data, *J. Differential Equations* **140** (1997), 161-185.

[10] A. Bressan, Global solutions of systems of conservation laws by wave-front tracking, *J. Math. Anal. Appl.* **170** (1992), 414-432.

[11] A. BRESSAN, The unique limit of the Glimm scheme, *Arch. Rational Mech. Anal.* **130** (1995), 205-230.

[12] A. BRESSAN, Analysis of solutions to hyperbolic systems by the front tracking method, in *Analysis of Systems of Conservation Laws (Aachen 1997)* , H. Freistüler Editor, Chapman Hall/CRC Monogr. Surv. Pure Appl. Math. 99, Boca Raton, FL (1999), 1-48.

[13] A. BRESSAN, On the Cauchy problem for nonlinear hyperbolic systems, Proceedings CANum'97, C. Carasso Ed., http://www.emath.fr/Maths/Proc/proc.html

[14] A. BRESSAN, Hyperbolic Systems of Conservation Laws - The one-dimensional Cauchy problem, Oxford Univ. Press, 2000.

[15] A. BRESSAN AND R. M. COLOMBO, The semigroup generated by 2×2 conservation laws, *Arch. Rational Mech. Anal.* **133** (1995), 1-75.

[16] A. BRESSAN AND R. M. COLOMBO, Decay of positive waves in nonlinear systems of conservation laws, *Ann. Scuola Norm. Sup. Pisa* **26** (1998), 133-160.

[17] A. BRESSAN, G. CRASTA, AND B. PICCOLI, Well posedness of the Cauchy problem for $n \times n$ systems of conservation laws, *Memoir. Amer. Math. Soc.*, **694** (2000).

[18] A. BRESSAN AND P. GOATIN, Oleinik type estimates and uniqueness for $n \times n$ conservation laws, *J. Differential Equations* **156** (1999), 26-49.

[19] A. BRESSAN AND P. GOATIN, Stability of $\mathbf{L}^\infty$ solutions of Temple class systems, *Differ. Integ. Equat.*, to appear.

[20] A. BRESSAN AND P.G. LEFLOCH, Uniqueness of weak solutions to hyperbolic systems of conservation laws, *Arch. Rational Mech. Anal.* **140** (1997), 301-317.

[21] A. BRESSAN AND P.G. LEFLOCH, Structural stability and regularity of entropy solutions to systems of conservation laws, *Indiana Univ. Math. J.* **48** (1999), 43-84.

[22] A. BRESSAN AND M. LEWICKA, A uniqueness condition for hyperbolic systems of conservation laws, Discr. Cont. Dynam. Syst. **6**, n. 3 (2000), 673-682.

[23] A. BRESSAN, T. P. LIU AND T. YANG, $\mathbf{L}^1$ stability estimates for $n\times n$ conservation laws, *Arch. Rational Mech. Anal.* **149** (1999), 1-22.

[24] A. Bressan and W. Shen, Uniqueness for discontinuous O.D.E. and conservation laws, *Nonlinear Analysis* **34** (1998), 637-652.

[25] G. Crasta and P.G. LeFloch, Existence theory for a class of strictly hyperbolic systems, in preparation.

[26] G. Crasta and P.G.LeFloch, in preparation.

[27] C.M. Dafermos, Polygonal approximations of solutions of the initial value problem for a conservation law, *J. Math. Anal. Appl.* **38** (1972), 33-41.

[28] C.M. Dafermos, Generalized characteristics in hyperbolic conservation laws: a study of the structure and the asymptotic behavior of solutions, in "Nonlinear Analysis and Mechanics: Heriot-Watt symposium", ed. R.J. Knops, Pitman, London, Vol. 1 (1977), 1-58.

[29] C.M. Dafermos, Hyperbolic Conservation Laws in Continuum Physics, Grundlehren Math. Wissen., Vol. 325, Springer Verlag, 2000.

[30] C.M. Dafermos and X. Geng, Generalized characteristics, uniqueness and regularity of solutions in a hyperbolic system of conservation laws, *Ann. Inst. Henri Poincaré - Nonlinear Analysis* **8** (1991), 231-269.

[31] G. Dal Maso, P.G. LeFloch and F. Murat, Definition and weak stability of nonconservative products, *J. Math. Pures Appl.* **74** (1995), 483-548.

[32] R. J. DiPerna, Global existence of solutions to nonlinear hyperbolic systems of conservation laws, *J. Differential Equations* **20** (1976), 187-212.

[33] R. J. DiPerna, Uniqueness of solutions to hyperbolic conservation laws, *Indiana Univ. Math. J.* **28** (1979), 137-188.

[34] F. Dubois and P.G. LeFloch, Boundary conditions for non-linear hyperbolic systems of conservation laws, *J. Differential Equations* **71** (1988), 93-122.

[35] L. C. Evans and R. F. Gariepy, Measure Theory and Fine Properties of Functions, C.R.C. Press, 1992.

[36] A.F. Filippov, Differential equations with discontinuous right hand-side, *Math USSR-Sb.* **51** (1960), 99–128. English transl. in *A.M.S. Transl., Ser. 2*, **42**, 199-231.

[37] J. Glimm, Solutions in the large for nonlinear hyperbolic systems of equations, *Comm. Pure Appl. Math.* **18** (1965), 697-715.

[38] P. Goatin and P.G. LeFloch, Sharp $\mathbf{L}^1$ stability estimates of for hyperbolic conservation laws, *J. Amer. Math. Soc.*, submitted.

[39] P. Goatin and P.G. LeFloch, The sharp $\mathbf{L}^1$ continuous dependence of **BV** solutions for systems of conservation laws, submitted.

[40] A. Heibig, Existence and uniqueness of solutions for some hyperbolic systems of conservation laws, *Arch. Rational Mech. Anal.* **126** (1994), 79-101.

[41] J.-X. Hu and P.G. LeFloch, $\mathbf{L}^1$ continuous dependence for systems of conservation laws, *Arch. Rational Mech. Anal.* **151** (2000), 45-93.

[42] K.T. Joseph and P.G. LeFloch, Boundary layers in weak solutions of hyperbolic conservation laws, *Arch. Rational Mech. Anal.* **147** (1999), 47-88.

[43] B.L.. Keyfitz, Solutions with shocks, *Comm. Pure Appl. Math.* **24** (1971), 125-132.

[44] A. N. Kolmogorov and S. V. Fomin, Introductory Real Analysis, Dover, New York, 1975.

[45] H.O. Kreiss, Initial-boundary value problems for hyperbolic systems, *Comm. Pure Appl. Math.* **23** (1970), 277-298.

[46] S. Kruzhkov, First-order quasilinear equations with several space variables, *Mat. Sb.* **123** (1970), 228–255. English transl. in *Math. USSR Sb.* **10** (1970), 217–273.

[47] P.D. Lax, Hyperbolic systems of conservation laws II, *Comm. Pure Appl. Math.* **10** (1957), 537-566.

[48] P.D. Lax, Shock wave and entropy, in "Contributions to Nonlinear Functional Analysis", ed. E. Zarantonello, Acad. Press, New York, 1971, pp. 603-634.

[49] P.G. LeFloch, Explicit formula for scalar non-linear conservation laws with boundary condition, *Math. Methods Appl. Sci.* **10** (1988), 265-287.

[50] P.G. LeFloch, An existence and uniqueness result for two nonstrictly hyperbolic systems, IMA Volumes in Math. and its Appl. 27,"Nonlinear evolution equations that change type", ed. B.L. Keyfitz and M. Shearer, Springer Verlag (1990), pp. 126-138.

[51] P.G. LeFloch, An introduction to nonclassical shocks of systems of conservation laws, Proc. International School on "Theory and Numerics for Conservation Laws", Freiburg, Germany, 20-24 Oct. 97, D. Kröner, M. Ohlberger and C. Rohde ed., Lecture Notes in Computational Science and Engineering, Springer Verlag, 1999.

[52] P.G. LeFloch, Hyperbolic Systems of Conservation Laws: Classical and Nonclassical Shocks, and Kinetic Relations, Lecture notes, in preparation.

[53] P.G. LeFloch and T.P. Liu, Existence theory for nonlinear hyperbolic systems in nonconservative form, *Forum Math.* **5** (1993), 261-280.

[54] P.G. LeFloch and Z.P. Xin, Uniqueness via the adjoint problems for systems of conservation laws, *Comm. Pure Appl. Math.* **46** (1993), 1499-1533.

[55] V. Liapidevskii The continuous dependence on the initial conditions of the generalized solutions of the gas-dynamic system of equations, *USSR Comput. Math. and Math. Phys.* **14** (1974), 158-167.

[56] T. P. Liu, Uniqueness of weak solutions of the Cauchy problem for general 2×2 conservation laws, *J. Differential Equations* **20** (1976), 369–388.

[57] T. P. Liu, Admissible solutions of hyperbolic conservation laws, *Amer. Math. Soc. Memoir* **240** (1981).

[58] T.P. Liu and T. Yang, A new entropy functional for scalar conservation laws, *Comm. Pure Appl. Math.* **52** (1999), 1427-1442.

[59] T.-P. Liu and T. Yang, $\mathbf{L}_1$ stability of conservation laws with coinciding Hugoniot and characteristic curves, *Indiana Univ. Math. J.*, **48**, n.1 (1999), 237-247.

[60] T.-P. Liu and T. Yang, $\mathbf{L}_1$ stability of weak solutions for 2×2 systems of hyperbolic conservation laws, *J. Amer. Math. Soc.* **12** (1999), 729-774.

[61] T.-P. Liu and T. Yang, Well-posedness theory for hyperbolic conservation laws, *Comm. Pure Appl. Math.* **52** (1999), 1553-1580.

[62] O. OLEINIK, Discontinuous solutions of nonlinear differential equations, *Usp. Mat. Nauk.* **12** (1957), 3-73; English transl. in *Amer. Math. Soc. Transl. Ser.2,* **26**, 95-172.

[63] O. OLEINIK, On the uniqueness of the generalized solution of the Cauchy problem for a nonlinear system of equations occurring in mechanics, *Usp. Mat. Nauk (N.S.)* **12** (1957), 169–176 (in Russian).

[64] N. H. RISEBRO, A front-tracking alternative to the random choice method, *Proc. Amer. Math. Soc.* **117**, no. 4 (1993), 1125-1139.

[65] M. SABLÉ-TOUGERON, Méthode de Glimm et probléme mixte, *Ann. Inst. Henri Poincaré* **10**, no. 4, (1993), 423-443.

[66] D. SERRE, Systemes de Lois de Conservation, Diderot Editeur, 1996.

[67] J. SMOLLER, Shock Waves and Reaction-Diffusion Equations, Springer-Verlag, New York, 1983.

[68] B. TEMPLE, Systems of conservation laws with invariant submanifolds, *Trans. Amer. Math. Soc.* **280** (1983), 781-795.

[69] A. I. VOLPERT, The space **BV** and quasilinear equations, *Math. USSR Sb.* **2** (1967), 257-267.