Relaxation and Approximation Problems in Spaces of Functions of Bounded Variation

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INTRODUCTION

The variational formulation of many problems in Mathematical Physics, Computer Vision, and Mechanical Engineering takes into account an energy functional depending on a function and a hypersurface, both a priori unknown. Typically these functionals consist of two parts: the first one represents the "volume" energy and is the integral of a potential, depending on the gradient of an unknown scalar or vector function u, with respect to the Lebesgue measure in $\mathbb{R}^n$, n being the number of independent variables; the second one represents the "surface" energy and is the integral with respect to the $(n-1)$-dimensional Hausdorff measure in $\mathbb{R}^n$ of some function computed on a hypersurface, a priori unknown, where the function u is discontinuous. These energies account for several phenomena such as crack growth and crack initiation in the theory of brittle fracture, interface formation between different phases of Cahn-Hilliard fluids, surface tension between small drops of liquid cristals, and are utilized for pattern recognition in Computer Vision to determine surfaces corresponding to sudden changes in the image.

Given a functional modelling the energy of a phenomenon, we look for minimizers which represent the equilibrium states. When a variable of the functional is a hypersurface such a minimum problem is called a "free discontinuity" problem.

This thesis deals with the variational formulation of some of the "static" free discontinuity problems, in the light of resent research on functionals which depend on discontinuous functions.

Let us briefly discuss some models.

Frequently in the literature some static or quasi-static phenomena in damage or fracture mechanics are described by introducing an energy functional of the form

$$(1) \qquad \mathcal{F}_1(u, S) = \int_{\Omega \setminus S} W(Du) \, dx + \int_S \Phi(u^+, u^-) \, d\mathcal{H}^2.$$

Here $\mathcal{H}^2$ denotes the two-dimensional Hausdorff measure, the bounded open subset $\Omega \subset \mathbb{R}^3$ is the reference configuration, the function u represents the displacement, which is differentiable outside the "discontinuity surface" S, and u^+, u^- are the traces of u on the two sides of S. The latter can be interpreted as a crack or a "plasticity" surface. The functions W and Φ represent the bulk and surface energy densities respectively. We emphasize that the surface S does not play the rôle of a parameter: the variable of the problem is the pair (u, S), where u and S are connected by the fact that S is the discontinuity surface of u.

The simplest situation consists in taking $\Phi \equiv C$ a constant; *i.e.*, the surface energy proportional to the surface area of the crack. A model of this kind is found in Griffith's theory of crack propagation [Gri]. In this case the functional $\mathcal{F}_1$ provides a good description of the observed phenomena in the presence of a pre-existing crack, but does not explain the formation of internal quasi-static cracks (see [87], [54], [81], [64], [16]). To avoid supposing a priori the existence of small fractures, we have to consider energy densities which actually depend on the traces of the function u on the surface S. In the spirit of Barenblatt's theory of crack formation we can consider, for example, a function depending on the size of the jump $|u^+ - u^-|$:

(2) $$\Phi(u^+, u^-) = \varphi(|u^+ - u^-|),$$

with $\varphi(t)$ vanishing for $t \to 0$. If this function is approximately linear near 0 it is easy to see that the functional $\mathcal{F}_1$ justifies the formation of small cracks. For a discussion on mathematical models and methods for problems in fracture mechanics we refer to [82], [66].

A model similar to (1) can be introduced for the study of an elastic-plastic plate.

Let us consider a horizontal thin plate whose undeformed shape is a bounded connected open subset $\Omega \subset \mathbb{R}^2$. The plate is submitted to boundary conditions.

In the framework of linear elasticity (see [53], [80]), under appropriate hypotheses, the deformation energy of the plate is given by an integral functional with growth of order two, involving the Hessian matrix of the unknown function, which represents the vertical displacement.

If we take into account also plastic behaviour without hardening (for instance in case of a material subject to Henky's law) the behaviour of the plate can be modelled by introducing an energy functional of the form (see [34])

(3) $$\mathcal{F}_2(u, S) = \int_{\Omega \setminus S} |D^2 u|^2 \, dx + \mathcal{H}^1(S) + \int_{S \cap \Omega} |(Du)^+ - (Du)^-| \, d\mathcal{H}^1.$$

Here $\mathcal{H}^1$ denotes the one-dimensional Hausdorff measure, u represents the vertical displacement, which is continuous on Ω and twice differentiable outside the line S (interpreted as the crease line of the plate), $(Du)^+$ and $(Du)^-$ are the traces of the gradient Du on the two sides of S, and $D^2 u$ denotes the Hessian matrix of u. Different models for the same phenomenon can be found in [51], [52], [90].

This functional corresponds to a linear elastic energy density on the "elastic" set $\Omega \setminus S$, while the energy on the "plasticity" line S takes into account the length of the crease line and the jump of the gradient of the

displacement. The minimum problem associated to the functional in (3) is an example of gradient free discontinuity problem.

The study of functionals depending on a "free discontinuity line" recently arose in pattern recognition in Computer Vision, as a variational approach to the problem of reconstructing the contours of a picture given by a camera.

On a bounded domain $\Omega \subset \mathbb{R}^2$ the image is represented by the grey level function $g \in L^\infty(\Omega)$, which measures the intensity of the light at each point of the screen. D.Mumford and J. Shah [77], [78] suggested the study of the problem

$$(4) \qquad \inf_{(u,S)} \{ \int_{\Omega \backslash S} |Du|^2 \, dx + \mathcal{H}^1(S \cap \Omega) + \int_{\Omega \backslash S} |u - g|^2 \, dx \}$$

where S is a closed subset of $\overline{\Omega}$, $u \in C^1(\Omega \backslash S)$, and $\mathcal{H}^1$ denotes the one-dimensional Hausdorff measure.

Since one expects the function g to be discontinuous along the lines corresponding to sudden changes in the visible surfaces (*e.g.* edges of objects, shadows, different colours), the image segmentation problem consists in finding a pair (u, S) such that S is a set of curves decomposing the image into regions with relatively uniform intensity, while u is a smooth approximation of g on each region. The set S will be interpreted as the union of the lines which give the schematic description of the image. For a general treatment of this subject we refer to [83].

The Mumford-Shah model, though quite simple, is not in some situations a good approximation of the image segmentation problem, in the sense that the qualitative behaviours of the datum g and of the solution u are too different. This happens for instance when the datum g has large gradient in a small region (the Mumford-Shah model presents the so-called "gradient limit" effect, see [24], 4.1.5) or when some crease discontinuities (*i.e.*, lines along which the function is continuous but the first derivative is discontinuous) seem meaningful in the shape of the datum (the solution u never reconstructs them).

In order to overcome the deficiences of such a model, A. Blake and A. Zisserman [24] suggest to modify the functional in (4), including the second order derivatives, instead of the first order ones, and a penalty for unit length of crease discontinuity. Following the ideas of these authors we are led to consider as a new model for the image segmentation problem the functional, defined for every pair of disjoint closed subsets S_0, S_1 of $\overline{\Omega}$ and

for every function $u \in C^0(\Omega \setminus S_0) \cap C^2(\Omega \setminus (S_0 \cup S_1))$ as

$$\mathcal{F}_3(u, S_0, S_1) = \int_{\Omega \setminus (S_0 \cup S_1)} |D^2 u|^2 \, dx + \int_{\Omega \setminus S_0} |u - g|^2 \, dx$$

(5)

$$+ \alpha \, \mathcal{H}^1(S_0) + \beta \, \mathcal{H}^1(S_1),$$

where α and β are positive real numbers. We remark that this model leads to a gradient free discontinuity problem allowing also jumps of the function and this is not the same situation as in (3).

From the point of view of the Calculus of Variations a natural question about the functionals in (1), (3), (4), and (5) is the possibility to apply the so-called Direct Method, which consists in proving lower semicontinuity and coerciveness of the functional with respect to an appropriate topology. This latter requirement guarantees compactness of the minimizing sequences; then, by the lower semicontinuity, each limit point of such a sequence achieves the minimum value.

The problem here is that the known compact topologies on the family of closed sets do not ensure the lower semicontinuity of the terms involving the Hausdorff measure.

For problems of this kind E. De Giorgi and his school have proposed a unified approach based on the use of a new function space, named $SBV(\Omega)$ (see [45], [6]), which allows to transform the minimum problems related to functionals like (1), (3), (4), and (5) into the minimimization of functionals depending only on an unknown function.

Let Ω be an open set in $\mathbb{R}^n$ and $k \geq 1$ be an integer. We recall that a function $u \in L^1(\Omega; \mathbb{R}^k)$ is a function of bounded variation (and we write $u \in BV(\Omega; \mathbb{R}^k)$) if its distributional derivative Du is a finite (matrix-valued) Radon measure on Ω. It turns out that the Lebesgue decomposition of this measure can be written as $Du = \nabla u \, dx + D_s u$, where the density of the absolutely continuous part of Du is denoted by ∇u since it can be interpreted as an approximate differential for u. For a function $u \in BV(\Omega; \mathbb{R}^k)$ it is possible to define a *set of jump points* S_u where u is approximately discontinuous, and it turns out that there exists a countable sequence of C^1 hypersurfaces which covers $\mathcal{H}^{n-1}$-almost all of S_u. Moreover on S_u it is well-defined a "normal" ν_u together with the *traces* u^+, u^- of u on the two sides. In addition $D_s u$ can be decomposed into two mutually singular measures by setting

$$D_s u = (u^+ - u^-) \otimes \nu_u \mathcal{H}^{n-1}_{|S_u} + Cu,$$

where $\mathcal{H}^{n-1}$ denotes the $(n-1)$-dimensional Hausdorff measure, the measure $(u^+ - u^-) \otimes \nu_u \mathcal{H}^{n-1}_{|S_u}$ is the *jump part* and Cu the so-called *Cantor*

part of Du. We recall that the measure Cu is "diffuse" in the sense that $Cu(S) = 0$ if S is a set of Hausdorff dimension $(n-1)$.

The fact that the functionals in (1) and (4) control only the absolutely continuous and the jump part of the derivative, motivates the introduction, due to E. De Giorgi and L. Ambrosio [45], of the subspace $SBV(\Omega; \mathbb{R}^k)$ of the *special* functions of bounded variation that are characterized by the property that $Cu \equiv 0$.

On the space $SBV(\Omega; \mathbb{R}^k)$ it is natural to consider functionals of the form

$$(6) \qquad \int_\Omega f(x, u(x), \nabla u(x))\, dx + \int_{S_u \cap \Omega} g(x, u^+, u^-, \nu_u)\, d\mathcal{H}^{n-1}.$$

For the functionals in (6) a rather complete theory has been developed by L. Ambrosio and A. Braides [9], [10] in the framework of partitions of sets of finite perimeter (in this case the term involving f disappears).

In the general case L. Ambrosio proved in [7] the lower semicontinuity when f is convex (in the last variable) and has a superlinear growth at infinity, and g is *BV-elliptic* and has a superlinear growth for $|u^+ - u^-| \to 0$ (for example if $g \geq c > 0$). These conditions ensure compactness separately for the bulk and jump part of the derivative, so that the two integrals in (6) can be dealt with separately. In [8] this result is extended to the case of f being quasiconvex in the sense of C. B. Morrey (see [75], [74]) in the last variable. It is well known that this hypothesis is the natural assumption in the case of vector-valued u (see [75], [41], [1], [57]) .

How may we treat the minimum problem for the functional in (4)?

The general method proposed by E. De Giorgi consists in the following steps: first we give a weak formulation on the space $SBV(\Omega)$ to the minimum problem; then we look for minimizers trying to apply the Direct Method; finally we study the regularity properties of the minimum points in order to recover a minimizer of the initial problem.

In our case the weak formulation of the minimum problem in (4) consists in the minimization over the space $SBV(\Omega)$ of the functional

$$(7) \qquad \mathcal{F}(u) = \int_\Omega |\nabla u|^2\, dx + \int_\Omega |u - g|^2\, dx + \mathcal{H}^1(S_u).$$

Using the lower semicontinuity theorem of L. Ambrosio, mentioned above, it is easy to prove that the functional $\mathcal{F}$ achieves its minimum on $SBV(\Omega)$. Moreover, in [46] E. De Giorgi, M. Carriero, and A. Leaci have proved the existence of a minimizer for problem (4) by showing that if $u \in SBV(\Omega)$ is a minimum point for $\mathcal{F}$, then $(\tilde{u}, \overline{S_{\tilde{u}}})$, where $\tilde{u}$ denotes the approximately continuous representative of u, is a minimizer of (4) and $\mathcal{H}^1(\overline{S_{\tilde{u}}} \setminus S_{\tilde{u}}) = 0$. Another proof of the same result is due to G. Dal Maso, J. M. Morel, and

S. Solimini [43], with a method which is limited to the two-dimensional case.

Let us try to apply the same general method to the functionals in (1). We consider the weak formulation on $SBV(\Omega)$ by setting

$$(8) \qquad \mathcal{G}_1(u) = \int_\Omega W(\nabla u)\, dx + \int_{S_u \cap \Omega} |u^+(x) - u^-(x)|\, d\mathcal{H}^2$$

(here for simplicity we take $\varphi(t) = t$ in (2)); nevertheless these functionals present the problem of not being lower semicontinuous on $SBV(\Omega)$.

In order to deal with the situation, where the functional is not lower semicontinuous, the Relaxation Methods have been introduced. Let $\mathcal{F}$: $X \to \mathbb{R} \cup \{+\infty\}$ be a functional on a topological space (X, τ). The Relaxation Methods consist in defining a new functional $\overline{\mathcal{F}}$ as the τ-lower semicontinuous envelope of $\mathcal{F}$; i.e., $\overline{\mathcal{F}}$ is the greatest τ-lower semicontinuous functional less than or equal to $\mathcal{F}$. If the functional $\mathcal{F}$ is coercive, then also the relaxed functional satisfies the coerciveness condition; hence it admits a minimum point. Moreover the minimum value of $\overline{\mathcal{F}}$ equals the infimum of $\mathcal{F}$ and each limit point of a minimizing sequence for $\mathcal{F}$ is a minimizer for $\overline{\mathcal{F}}$ (see [32] for a general treatment on this subject).

For the functionals in (8) we can explicitly determine the lower semicontinuous envelope $\overline{\mathcal{G}}_1$ in the L^1-topology, under some convexity hypotheses on W (see Chapter 2, Theorem 2.1). More precisely for every $u \in BV(\Omega)$ we obtain that

$$(9) \qquad \overline{\mathcal{G}}_1(u) = \int_\Omega \overline{W}(\nabla u)\, dx + \int_{S_u \cap \Omega} |D_s u|,$$

where $|D_s u|$ denotes the total variation of the measure $D_s u$ and $\overline{W}$, the bulk energy of $\overline{\mathcal{G}}_1$, is a function, explicitly computed from W, which grows at most linearly at infinity, whatever the form of W. This result can be extended to the vector-valued case under the same hypotheses on W.

We remark that the relaxed functional $\overline{\mathcal{G}}_1$ is finite on the whole $BV(\Omega)$; this means that in general the minimizers may have a diffuse "fractured" zone. Therefore we can not expect to have strong regularity properties of the discontinuity set, e.g. that it is a surface. This difficulty is classically solved by considering only problems where the existence of solutions whose "fracture" remains confined on a surface is supposed a priori.

In order to select between all possible minimizers for $\overline{\mathcal{G}}_1$ those belonging to $SBV(\Omega)$, we can approximate $\overline{\mathcal{G}}_1$ by means of a sequence of functionals obtained by perturbing the functional $\mathcal{G}_1$ with an additional term. These functionals are defined on $BV(\Omega)$ as

$$(10) \qquad \mathcal{G}_{1,\varepsilon}(u) = \begin{cases} \mathcal{G}_1(u) + \varepsilon \int_{S_u \cap \Omega} \Phi(u^+, u^-)\, d\mathcal{H}^2 & u \in SBV(\Omega) \\ +\infty & \text{elsewhere,} \end{cases}$$

where Φ is a nonnegative Lipschitz continuous function in $\mathbb{R}^2$ with $\Phi(v,v) \geq c > 0$ for all v.

The convergence we consider here is a variational convergence, called Γ-convergence, introduced by E. De Giorgi and T. Franzoni in 1975 [48]. Under the additional hypothesis that the approximating sequence is equi-coercive the Γ-convergence ensures convergence of the minimum values and of the minimizers. Therefore to approximate via Γ-convergence represents a choice, among all the possible minima of the limit functional, of those that in particular can be reached following the minimizers of the approximating functionals. We refer to the recent book by G. Dal Maso [42] for a comprehensive introduction to the subject.

In the one-dimensional case we can study the effect of this perturbation on the minimizers of $\overline{\mathcal{G}}_1$ by examining some minimum problems with generalized Dirichlet boundary data, and by characterizing the minimizers for $\overline{\mathcal{G}}_1$ which can be reached following sequences of minimizers for the same problems for $\mathcal{G}_{1,\varepsilon}$. The choice criterion is determined by Φ. Indeed these minimizers are in $SBV(\Omega)$, they have a finite number of jumps, and they minimize (under the same boundary conditions) a functional of the form

$$\sum_{x \in S_u} \overline{\Phi}(u(x+), u(x-))$$

among all SBV-minimizers for $\overline{\mathcal{G}}_1$. The function $\overline{\Phi}$ can be easily computed from the function Φ and is independent of the boundary conditions (see Chapter 2, Sections 3 to 5). Let us remark that we are able to describe the behaviour of such minimizers, but we cannot localize the jumps of a minimum point.

Let us observe that the relaxation result (9) is proved under some convexity hypotheses on the bulk energy. In the vector-valued case a more general relaxation theorem can be obtained for the class of functionals defined on $SBV(\Omega; \mathbb{R}^k)$ by integrals of the form

$$(11) \qquad \int_\Omega f(\nabla u(x))\, dx + \int_{S_u \cap \Omega} g((u^+ - u^-) \otimes \nu_u)\, d\mathcal{H}^{n-1},$$

using the more natural notion of quasiconvexity.

It is proved in [11] the L^1-lower semicontinuity of the integral defined on $BV(\Omega; \mathbb{R}^k)$ by

$$\int_\Omega f(\nabla u(x))\, dx + \int_\Omega f^\infty\left(\frac{D_s u}{|D_s u|}\right) |D_s u|$$

under the assumption of f being quasiconvex and with linear growth (f^∞ is the recession function of f and $\dfrac{D_s u}{|D_s u|}$ denotes the Radon-Nikodym derivative of the measure $D_s u$ with respect to its total variation $|D_s u|$). This

result has been recently generalized in [58], allowing the dependence of f also on x and u.

In the general case, when it is not possible to obtain the effective surface energy density by simply considering the volume energy density, the relaxed functional takes into account, both in its volume and in its surface part, the combined effect of f and g, and it can be written on the whole $BV(\Omega; \mathbb{R}^k)$ as (see Chapter 4)

$$(12) \qquad \int_\Omega \varphi(\nabla u(x))\, dx + \int_\Omega \varphi^\infty \left(\frac{D_s u}{|D_s u|} \right) |D_s u|.$$

Here the function φ is a quasiconvex function with linear growth (whatever the growth conditions satisfied by f may be) characterized as the supremum of all quasiconvex functions less than or equal to f whose recession function is less than or equal to g on rank one matrices.

Let us consider now the minimum problem associated to the functional in (3).

Since this minimum problem is a gradient free discontinuity problem, for the weak formulation we need to introduce another space (different from $SBV(\Omega)$) of functions of bounded variation which allows creasing without fracture. To this purpose let us consider the space $BV^2(\Omega)$ (respectively $SBV^2(\Omega)$) of the functions $u \in W^{1,1}(\Omega)$ with first derivative in $BV(\Omega; \mathbb{R}^n)$ (respectively $SBV(\Omega; \mathbb{R}^n)$). The second derivative $D^2 u$ in the sense of distributions of a function $u \in BV^2(\Omega)$ is a measure admitting the Lebesgue decomposition $D^2 u = \nabla(Du)\, dx + (D^2 u)_s$.

On $SBV^2(\Omega)$ the weak formulation for (3) is obtained by setting

$$\mathcal{G}_2(u) = \int_\Omega |\nabla(Du)|^2\, dx + \mathcal{H}^1(S_{Du}) + \int_{S_{Du} \cap \Omega} |(Du)^+ - (Du)^-|\, d\mathcal{H}^1.$$

By applying the compactness and lower semicontinuity theorems [5], [7] by L. Ambrosio we can prove that the functional $\mathcal{G}_2$ achieves its minimum on $SBV^2(\Omega)$. The study of the regularity properties of the minimizers of $\mathcal{G}_2$ is the object of [34].

In the one-dimensional case the functional in (3) models the energy of a horizontal thin rod, whose undeformed shape is a bounded open interval $I = (a, b)$ of $\mathbb{R}$, submitted to boundary conditions. The same phenomenon can be modelled by introducing the integral energy functional

$$(13) \qquad \mathcal{F}_4(u, S) = \int_{I \backslash S} |u''|^2\, dx + \rho \int_S |(u')^+ - (u')^-|\, d\#,$$

where u', u'' are the first and second derivatives of u, ρ is a positive constant and $\#$ is the counting measure on $\mathbb{R}$. The minimum problem for (13) can

be studied using the same arguments and techniques applied to treat the functionals in (1).

Indeed the weak formulation on $SBV^2(I)$ is obtained considering the functional

$$\mathcal{G}_4(u) = \int_I |\ddot{u}|^2 \, dx + \int_{S_{u'} \cap I} |(u')^+ - (u')^-| \, d\#,$$

where $\ddot{u}$ denotes the density of the absolutely continuous part of u''. This functional is not lower semicontinuous on $SBV^2(I)$ and we can prove (see Chapter 3, Theorem 2.2) that the lower semicontinuous envelope of $\mathcal{G}_4$ in the L^1-topology is given by the functional defined for every $u \in BV^2(\Omega)$ as

$$\overline{\mathcal{G}_4}(u) = \int_I \varphi(\ddot{u}) \, dx + \rho |u_s''|(I),$$

where $|u_s''|$ denotes the total variation of the measure u_s''. Here φ is the convex and everywhere finite scalar function with linear growth at infinity given by

$$\varphi(z) = \begin{cases} z^2 & \text{if } |z| \leq \frac{\rho}{2} \\ \rho(|z| - \frac{\rho}{4}) & \text{if } |z| > \frac{\rho}{2}. \end{cases}$$

Since the relaxed functional $\overline{\mathcal{G}_4}$ is finite on the whole space $BV^2(I)$, the minimizers could not have in general a discontinuity set of the gradient consisting in a finite number of points. We can approximate, via Γ-convergence, the functional $\overline{\mathcal{G}_4}$ by means of a sequence of functionals obtained by perturbing the functional $\mathcal{G}_4$ exactly in the same way we have perturbed $\mathcal{G}_1$ in (10).

We can study the effect of this perturbation on the minimizers of $\overline{\mathcal{G}_4}$ by examining some minimum problems with generalized Dirichlet boundary data and by characterizing the minimizers which can be reached following sequences of minimizers for the same problems for the approximating functionals. We obtain that these minimizers belong to $SBV^2(\Omega)$, they have a finite number of crease points, and they minimize (under the same boundary conditions) a functional of the form

$$\sum_{x \in S_u \cup \{a,b\}} \overline{\Phi}(u'(x+), u'(x-)),$$

among all SBV^2-minimizers for $\overline{\mathcal{G}_4}$ (see Chapter 3, Sections 3 and 4). In particular, in the case $\Phi \equiv 1$ we choose the minimizer with the minimum number of creases and we are able to localize exactly the crease point.

The notion of Γ-convergence has interesting applications also to the study of problem (4).

The numerical treatment of the minimization problem (4) seems quite difficult, because of the lack of convexity and regularity of the functional at hand, mainly due to the term $\mathcal{H}^1(S)$ (see [22], [35], [65], [67], [84], [85], [86]). However, L. Ambrosio and V.M. Tortorelli [12], [13] have shown that the functional $\mathcal{F}$, defined in (7), is the limit, in the sense of Γ-convergence with respect to the L^2-topology, of an equi-coercive sequence of elliptic functionals.

The basic idea is to introduce a new variable s in the approximating functional $\mathcal{F}_\varepsilon$, which controls the unknown set S_u. In view of the variational properties of Γ-convergence, the minimization of $\mathcal{F}$ is then reduced to the minimization of $\mathcal{F}_\varepsilon$, for small ε.

This approximation can be used to attack minimum problem (4) from a numerical viewpoint. Indeed we can show (see Chapter 5) that, if we discretize $\mathcal{F}_\varepsilon$ by means of piecewise linear finite elements, then the discrete functionals Γ-converge to $\mathcal{F}$ and the discrete minimizers converge to a solution of the original problem (4).

Finally, let us consider the minimization problem associated to the functional in (5).

In the one-dimensional case the problem is completely solved in [39], where it is proved that the minimization problem for the functional (5) admits a solution, provided conditions

$$0 < \beta \le \alpha \le 2\beta$$

are satisfied. The proof relies on a semicontinuity theorem, on a compactness theorem, and on regularity arguments. The weak formulation is obtained by setting

$$\mathcal{G}(u) = \int_I |\ddot{u}|^2 \, dx + \alpha \ \#((S_u) \cap I) + \beta \ \#((S_{\dot{u}} \setminus S_u) \cap I) + \int_I |u - g|^2 \, dx,$$

where I is a bounded open interval of $\mathbb{R}$ and u varies over the space $\mathcal{H}^2(I)$ of piecewise H^2 functions. Here $\dot{u}, \ddot{u}$ are the pointwise values of the first and second derivatives of u, and $(S_{\dot{u}} \setminus S_u)$ is the set of crease points of u; *i.e.*, the set of the jump points of $\dot{u}$ which are not jump points of u.

In the one-dimensional case we can also approximate the functional $\mathcal{G}$, in the sense of Γ-convergence with respect to the L^1-topology, by a sequence of elliptic functionals which do not depend on jumps or creases (see Chapter 6). The basic idea is to introduce two new variables which control the unkown sets S_u and $S_{\dot{u}} \setminus S_u$ respectively.

In dimension larger than one, it is not known whether it is possible to give a weak formulation of the minimization problem in an appropriate generalized sense, such as the one proposed for the study of problem (4). This is a difficult open problem and seems to require some new results

about the characterization of functions having gradient in $SBV(\Omega, \mathbb{R}^n)$ by means of their one-dimensional sections (see [20]).

The content of this thesis, which is published in the papers [18], [19], [27], [28], [29], is the result of a research activity carried on by the Author during her graduate studies at the International School for Advanced Studies in Trieste, under the guidance of Prof. Gianni Dal Maso and in collaboration with Dr. Giovanni Bellettini and Prof. Andrea Braides.

CHAPTER 1:

NOTATIONS AND PRELIMINARIES

In this chapter we fix the notation and we recall some definitions and known results concerning the spaces of functions of bounded variation, the notions of quasiconvexity and rank one convexity, the relaxed functional, and Γ-convergence.

1. Notations

The natural numbers n, k will be fixed. We denote by $\{e_i\}$ the canonical basis of $\mathbb{R}^k$, and with $\langle \cdot, \cdot \rangle$ the scalar product in $\mathbb{R}^n$; $|\cdot|$ will be the usual euclidean norm. We shall denote by $M^{k \times n}$ the space of $k \times n$ matrices (k rows, n columns), and by $M_1^{k \times n}$ the subset of $M^{k \times n}$ of all matrices with rank less than or equal to one. We shall identify $M^{k \times n}$ with $\mathbb{R}^{kn}$. If $a \in \mathbb{R}^k$ and $b \in \mathbb{R}^n$ the *tensor product* $a \otimes b \in M_1^{k \times n}$ is the matrix whose entries are $a_i b_j$ with $i = 1, \dots, k$ and $j = 1, \dots, n$. Conversely if a matrix ξ has rank one, there are two vectors $a \in \mathbb{R}^k, b \in \mathbb{R}^n$ such that $\xi = a \otimes b$. If $A \subset \mathbb{R}^k$ and $b \in \mathbb{R}^n$ we will set $A \otimes b = \{a \otimes b : a \in A\} \subset M_1^{k \times n}$; remark that $|a \otimes b| = |a| \, |b|$ (the norms are taken in the proper spaces).

Let Ω be a bounded open subset of $\mathbb{R}^n$; we shall denote with $\mathcal{A}(\Omega)$ (resp. $\mathcal{B}(\Omega)$) the family of the open (resp. Borel) subsets of Ω. We shall use standard notations for the Sobolev and Lebesgue spaces $W^{m,p}(\Omega; \mathbb{R}^k)$ and $L^p(\Omega; \mathbb{R}^k)$. When $k = 1$ we shall drop the target space $\mathbb{R}^k$ in the notation, thus writing simply $W^{m,p}(\Omega)$, $L^p(\Omega)$, and the like.

If u is a scalar function defined on Ω, we shall sometimes use the shorter notation $\{u < t\}$ for $\{x \in \Omega : u(x) < t\}$ (and similar) when no confusion is possible.

The Lebesgue measure and the Hausdorff $(n - 1)$-dimensional measure in $\mathbb{R}^n$ will be denoted by $\mathcal{L}_n$ and $\mathcal{H}^{n-1}$ respectively. We shall use also the notation $|E|$ for $\mathcal{L}_n(E)$, the Lebesgue measure of a measurable set $E \subset \mathbb{R}^n$, and $\#$ for $\mathcal{H}^0$, the counting measure.

Let X be a set, and $E \subset X$; we define the *characteristic function* of E as

$$1_E(z) = \begin{cases} 1 & \text{if } z \in E \\ 0 & \text{if } z \in X \setminus E, \end{cases}$$

and the *indicator function* of E as

$$\chi_E(z) = \begin{cases} 0 & \text{if } z \in E \\ +\infty & \text{if } z \in X \setminus E. \end{cases}$$

If $N \geq 1$ is an integer and $f : \mathbb{R}^N \to [0, +\infty]$ is a convex function, we define $f^\infty : \mathbb{R}^N \to [0, +\infty]$, the *recession function* of f, by setting

$$(1.1) \qquad f^\infty(z) = \lim_{t \to +\infty} \frac{f(tz)}{t}.$$

It is immediate to see that the limit in (1.1) exists for all z; we remark that f^∞ is a Borel function, which is convex and positively homogeneous of degree one.

If $N \geq 1$ is an integer and $f : \mathbb{R}^N \to [0, +\infty]$ is a Borel function, we shall denote by f^{**} the greatest convex and lower semicontinuous function less than or equal to f.

Given a vector-valued Radon measure μ on Ω, we adopt the notation $|\mu|$ for its total variation (see [56], 2.2.5) and we indicate by $\dfrac{\mu}{|\mu|}$ the *Radon-Nikodym derivative* of μ with respect to its total variation. The integral on Ω of a function ψ with respect to the measure $|\mu|$ will be denoted simply by $\int_\Omega \psi |\mu|$.

The symbols $[t]$ and t^+ will denote the *integral part* and the *positive part* of the number $t \in \mathbb{R}$.

2. Functions of Bounded Variation

Let $n, k \geq 1$ be natural numbers, and Ω be a bounded open subset of $\mathbb{R}^n$. We say that $u \in L^1(\Omega; \mathbb{R}^k)$ is a *function of bounded variation* (and we write $u \in BV(\Omega; \mathbb{R}^k)$) if for any $i \in \{1, \dots, k\}$ and $j \in \{1, \dots, n\}$ there is a measure μ_i^j with finite total variation in Ω such that

$$(2.1) \qquad \int_\Omega u^{(i)} \frac{\partial g}{\partial x_j}\, dx = -\int_\Omega g\, d\mu_i^j \qquad \forall g \in \mathcal{C}_c^1(\Omega),$$

where $\mathcal{C}_c^1(\Omega)$ denotes the space of $\mathcal{C}^1$ functions with compact support in Ω.

We denote by Du the $M^{k \times n}$-valued measure whose components are the μ_i^j, and by $|Du|$ its total variation.

$BV(\Omega)$ is a Banach space, if endowed with the BV-norm

$$\|u\|_{BV} = \|u\|_1 + |Du|(\Omega).$$

We denote by S_u the complement of the Lebesgue set of u, that will be sometimes referred to as the set of *jump points* of the function u; ; *i.e.*, $x \notin S_u$ if and only if

$$\lim_{\rho \to 0^+} \rho^{-n} \int_{B_\rho(x)} |u - z| \, dx = 0$$

for some $z \in \mathbb{R}^k$. If such a z exists, it is unique, and we denote it by $\tilde{u}(x)$, the *approximate limit* of u at x. For any function $u \in L^1(\Omega; \mathbb{R}^k)$ the set S_u is negligible and $\tilde{u}$ is a Borel function equal to u almost everywhere. Moreover if $u \in BV(\Omega; \mathbb{R}^k)$, there is a countable sequence of C^1 hypersurfaces Γ_i which covers $\mathcal{H}^{n-1}$-almost all of S_u, *i.e.*,

$$\mathcal{H}^{n-1} \Big(S_u \setminus \bigcup_{i=1}^{\infty} \Gamma_i \Big) = 0.$$

Furthermore, for $\mathcal{H}^{n-1}$-almost every $x \in S_u$ it is possible to find a, $b \in \mathbb{R}^k$ and $\nu \in S^{n-1}$ such that

$$(2.2) \quad \lim_{\rho \to 0^+} \rho^{-n} \int_{B_\rho^\nu(x)} |u - a| \, dx = 0, \qquad \lim_{\rho \to 0^+} \rho^{-n} \int_{B_\rho^{-\nu}(x)} |u - b| \, dx = 0,$$

where $B_\rho^\nu(x) = \{y \in B_\rho(x) : \langle y - x, \nu \rangle > 0\}$. The triplet (a, b, ν) is uniquely determined up to a change of sign of ν and an interchange of a and b, and it will be denoted by $(u^+(x), u^-(x), \nu_u(x))$. If $k = 1$, the triplet (a, b, ν) can be uniquely determined by requiring that $a > b$ (; *i.e.*, the normal ν points towards the larger value of u).

In general, for a function $u \in BV(\Omega; \mathbb{R}^k)$, we have the Lebesgue decomposition

$$(2.3) \qquad Du = D_a u + D_s u = \nabla u \cdot \mathcal{L}_n + D_s u,$$

where we denote by ∇u the density of the absolutely continuous part of Du with respect to the Lebesgue measure; the notation is motivated by the fact that ∇u can be interpreted as an *approximate differential*. The *singular part* of Du with respect to the Lebesgue measure can be further decomposed into to mutually singular measures as

$$(2.4) \qquad D_s u = (u^+ - u^-) \otimes \nu_u \cdot \mathcal{H}^{n-1}{}_{|S_u} + Cu,$$

where $(u^+ - u^-) \otimes \nu_u \cdot \mathcal{H}^{n-1}{}_{|S_u}$ is the *Hausdorff part* and Cu the *Cantor part* of Du. We recall that the measure Cu is "diffuse"; ; *i.e.*, $Cu(S) = 0$ if S is a set of Hausdorff dimension $(n - 1)$.

We will say that a set E is of *finite perimeter in* Ω if $1_E \in BV(\Omega; \mathbb{R})$. We will set $\partial^* E \cap \Omega = S_{1_E} \cap \Omega$ the *reduced boundary of* E in Ω. Remark that $|D 1_E|(\Omega) = \mathcal{H}^{n-1}(\partial^* E \cap \Omega)$ for every E of finite perimeter in Ω. It is easy to check that this notion of perimeter coincides with the elementary one in the smooth case, in particular when E is a polyhedron. A result of E. De Giorgi [47] shows that if E is a set of finite perimeter in Ω, then there exists a sequence of polyhedra (P_h) such that $|((P_h \setminus E) \cup (E \setminus P_h)) \cap \Omega| \to 0$, and

$$(2.5) \qquad \mathcal{H}^{n-1}(\partial^* E \cap \Omega) = \lim_h \mathcal{H}^{n-1}(\partial P_h \cap \Omega).$$

This result demonstrates that the measure theoretic notion of perimeter is a sensible extension of the elementary definition.

We recall that if $u \in BV(\Omega; \mathbb{R})$, then for a.e. $t \in \mathbb{R}$ the set $\{u > t\}$ is of finite perimeter in Ω, and we have the so-called *coarea formula*:

$$(2.6) \qquad |Du|(\Omega) = \int_{-\infty}^{+\infty} \mathcal{H}^{n-1}(\partial^* \{u > t\} \cap \Omega)\, dt.$$

We recall also the *Fleming & Rishel coarea formula*. Let u be a Lipschitz function; then for every $v \in BV(\Omega)$ we have that

$$(2.7) \qquad \int_\Omega v|\nabla u|\, dx = \int_{-\infty}^{+\infty} \int_{\partial^*\{u>t\}\cap\Omega} \tilde{v}\, d\mathcal{H}^{n-1}\, dt$$

(∇u is the a.e. gradient of the function u). Analogous formulas hold with $\{u < t\}$ instead of $\{u > t\}$.

We say that u is a *special* function of bounded variation, and we write $u \in SBV(\Omega; \mathbb{R}^k)$, if $u \in BV(\Omega; \mathbb{R}^k)$ and $Cu \equiv 0$. The space $SBV(\Omega; \mathbb{R}^k)$ was introduced by E. De Giorgi and L. Ambrosio [45].

For the general exposition of the theory of functions of bounded variation we refer to [56], [61], [68], [91] and [92]. For an introduction to the properties of the space SBV we refer to [45], [5], [7].

3. Relaxation

We recall the notion of *relaxed functional*. Let $F : X \to \mathbb{R} \cup \{+\infty\}$ be a functional on a metric space (X, τ). The relaxed functional $\overline{F}$ of F, or *relaxation* of F, (in the τ-topology) is the greatest τ-lower semicontinuous functional less than or equal to F; ; *i.e.*, the greatest functional such that $\overline{F} \leq F$ and $\overline{F}(u) \leq \liminf_h \overline{F}(u_h)$ for every sequence $(u_h)_h$ converging to u in the τ-topology. We point out here only that the relaxed functional $\overline{F}$ allows to describe the behaviour of minimizing sequences for F; indeed minimizing sequences for problems involving F converge, up to a subsequence, to solutions for the corresponding problems for $\overline{F}$. For a general treatment of this subject we refer to the books by G. Buttazzo [32], and by G. Dal Maso [42].

4. Γ-convergence

Let us recall some basic definitions and results about Γ-convergence (we refer to [48], [42] for a bibliography on the subject). Let (X, τ) be a topological space, and let $F_h : X \to [0, +\infty]$ be a sequence of functionals on X.

If $\mathcal{N}(x)$ denotes the set of all open neighbourhoods of x in X, let us define the Γ-*lower limit* and the Γ-*upper limit* at a point $x \in X$ respectively by

$$(\Gamma\text{-}\liminf_{h \to \infty} F_h)(x) = \sup_{U \in \mathcal{N}(x)} \liminf_{h \to \infty} \inf_{y \in U} F_h(y)$$

and

$$(\Gamma\text{-}\limsup_{h \to \infty} F_h)(x) = \sup_{U \in \mathcal{N}(x)} \limsup_{h \to \infty} \inf_{y \in U} F_h(y).$$

The Γ-upper and Γ-lower limits are τ-lower semicontinuous functions.

If we have

$$(\Gamma\text{-}\liminf_{h \to \infty} F_h)(x) = (\Gamma\text{-}\limsup_{h \to \infty} F_h)(x),$$

then we say that the sequence $(F_h)_h$ Γ-converges at x and that the value $F(x)$ of the $(\Gamma\text{-}\liminf_{h \to \infty} F_h)(x)$ is the Γ-*limit* of the sequence $(F_h)_h$ at x. If this Γ-limit exists for all $x \in X$ we will say that F_h Γ-converges to F in X, and we will write

$$F = \Gamma\text{-}\lim_{h \to \infty} F_h.$$

If (X, τ) is a metric space, then we have $F(x) = (\Gamma\text{-}\lim_{h \to \infty} F_h)(x)$ iff the following conditions are satisfied:

a) for every sequence $(x_h)_h$ such that $x_h \to x$ we have

$$F(x) \leq \liminf_{h \to \infty} F_h(x_h);$$

b) there exists a sequence $(\tilde{x}_h)_h$ such that $\tilde{x}_h \to x$ and

$$(4.1) \qquad\qquad F(x) = \lim_{h \to \infty} F_h(\tilde{x}_h).$$

If the Γ-limit exists, it is unique; moreover, if (X, τ) is a separable metric space, every sequence $(F_h)_h$ admits a Γ-converging subsequence.

In the same way as above we define the Γ-limits as $\varepsilon \to 0$ for a family of functionals $(F_\varepsilon)_{\varepsilon > 0}$. We have then that $F = \Gamma\text{-}\lim_{\varepsilon \to 0+} F_\varepsilon$ iff for every sequence (ε_h) of positive numbers converging to 0 we have $F = \Gamma\text{-}\lim_{h \to +\infty} F_{\varepsilon_h}$.

The property which motivates the introduction of Γ-convergence in Calculus of Variations is the following: assume that F_h Γ-converges to F on a metric space (X, τ), and

$$\inf_X F_h = \inf_K F_h \qquad \forall h \in \mathbf{N}$$

for a suitable compact set $K \subset X$. Then

$$(4.2) \qquad \lim_{h \to +\infty} \inf_X F_h = \min\{F(x) : x \in X\}$$

and every sequence $(x_h)_h$ in K such that

$$(4.3) \qquad \lim_{h \to +\infty} F_h(x_h) = \lim_{h \to +\infty} \inf_X F_h$$

admits a subsequence converging to a minimizer of F.

If $F_h = F$ for every $h \in \mathbf{N}$, then the Γ-limit exists and it coincides with $\overline{F}$, the *relaxation* of F.

It is easy to check that we have

$$\Gamma\text{-}\liminf_{h \to +\infty} F_h(x) = \Gamma\text{-}\liminf_{h \to +\infty} \overline{F}_h(x),$$

and the analogous identity holds for the Γ-upper limit.

5. Quasiconvexity and rank one convexity

We recall the notion of *quasiconvex function* (cf. *e.g.* C. B. Morrey [75], [74], B. Dacorogna [40], [41]). We say that a continuous function $\varphi : M^{k \times n} \to [0, +\infty[$ is quasiconvex if for every $\xi \in M^{k \times n}$, A bounded subset of $\mathbb{R}^n$, and $u \in C_c^1(A; \mathbb{R}^k)$ we have the inequality

$$|A|\varphi(\xi) \leq \int_A \varphi(\xi + \nabla u(x)) \, dx.$$

This property is a well-known necessary and sufficient condition for the lower semicontinuity of multiple integrals in Sobolev spaces (cf. Acerbi & Fusco [1], Dacorogna [40]).

Every quasiconvex function $\varphi : M^{k \times n} \to [0, +\infty[$ is *rank one convex*; *i.e.*, it verifies

$$\varphi(\lambda \xi + (1 - \lambda)\zeta) \leq \lambda \varphi(\xi) + (1 - \lambda)\varphi(\zeta)$$

for every $\xi, \zeta \in M^{k \times n}$ such that $rank(\xi - \zeta) \leq 1$, and every $\lambda \in [0, 1]$ (cf. Dacorogna [40], [41]). A recent result by V. Šverák shows that the converse is not true (see [89]).

CHAPTER 2:

A SINGULAR PERTURBATION APPROACH
TO VARIATIONAL PROBLEMS
IN FRACTURE MECHANICS

In this chapter we consider functionals of the form

$$I(u,S) = \int_{\Omega \setminus S} W(Du)dx + \int_S \phi(u_+, u_-)d\mathcal{H}^{n-1},$$

with $\phi(u,v) \sim |u-v|$ for small values of $|u-v|$, which are related to the variational formulation of static or quasi-static phenomena in damage and fracture mechanics. Here Ω is the reference configuration, the function u represents the displacement, which is differentiable outside the "discontinuity surface" S, and u_+, u_- are the traces of u on the two sides of S. The latter can be interpreted as a crack or a plasticity surface. The functions W and ϕ represent the bulk and surface energy densities respectively. These functionals in general are not lower semicontinuous in their natural topology. Hence we may have minimizing sequences with unbounded discontinuity surfaces, and in the limit we could obtain in general a diffuse zone of "non-differentiability". In order to ensure that we obtain solutions whose "fracture" remains confined only on a surface at most, we propose a singular perturbation approach. We approximate the functional I by a sequence of functionals of the form $I_\varepsilon(u,S) = \int_{\Omega \setminus S} W(Du)dx + \int_S \phi_\varepsilon(u_+, u_-)d\mathcal{H}^{n-1}$. We show that in the model case of $\phi(u,v) = |u-v|$, if $\phi_\varepsilon(u,v) \sim |u-v|+\varepsilon\phi_1(u,v)$ the limits of the minimizers of I_ε not only minimize the corresponding problems for I, but they also minimize a "first order" problem involving only an appropriate "surface energy density".

The results of this chapter are contained in [27].

Introduction

This chapter presents some results related to the variational formulation of static or quasi-static phenomena in damage and fracture mechanics. We shall deal with problems which can be described by introducing an integral energy functional of the form

$$(0.1) \qquad I(u,S) = \int_{\Omega \setminus S} W(Du)dx + \int_S \phi(u_+, u_-)d\mathcal{H}^{n-1}.$$

Here $\mathcal{H}^{n-1}$ denotes the $(n-1)$-dimensional Hausdorff measure, Ω is the reference configuration, the function u represents the displacement, which

is differentiable outside the "discontinuity surface" S, and u_+, u_- are the traces of u on the two sides of S. The latter can be interpreted as a crack or a plasticity surface. The functions W and ϕ represent the bulk and surface energy densities respectively.

Models requiring description (0.1) are frequent in the literature. Set aside a discussion on the form of the function W, we can find different requirements on the surface energy density. The simplest situation consists in taking $\phi \equiv C$ a constant; *i.e.*, surface energy is proportional to surface area. A model of this kind is found in Griffith's theory of crack propagation [63]. It is well-known that in this case the functional I, while a good approximation in the presence of a pre-existing crack, does not explain for example the formation of internal quasi-static cracks (see [87], [54], [81], [64], [16]). In order to avoid the postulation of pre-existing small fractures, we have to consider then energy densities which depend actually on the traces of the function u on the surface S, even though this sounds a bit awkward from the viewpoint of the model. In the spirit of Barenblatt's theory of crack formation we can consider for example a function depending on the size of the "jump" $|u_+ - u_-|$:

$$(0.2) \qquad \phi(u_+, u_-) = \varphi(|u_+ - u_-|),$$

with $\varphi(t)$ vanishing for $t \to 0$. If this function φ is approximately linear near 0 it is easy to see that the functional I justifies the formation of small cracks. For a discussion on mathematical models and methods for problems in fracture mechanics we refer to [82] (see also [66]).

We discuss the problem from the viewpoint of the so-called direct method of the Calculus of Variations; *i.e.*, first of all we give to the problem a sufficiently weak formulation, in order to have our functionals defined on a proper space of weakly differentiable functions; then, we look for minimizers trying to exploit lower semicontinuity and coercivity properties. Functionals of the form

$$(0.3) \qquad I_1(u) = \int_{\Omega \setminus S} W(Du)dx + \int_S |u_+ - u_-| d\mathcal{H}^{n-1},$$

(here we take simply $\varphi(t) = t$ in (0.2)) are well defined on the space of *special* functions of bounded variation (note that we consider the surface $S = S_u$ as determined as the jump set of u and we pose $I_1(u) = I(u, S_u)$), but they present the problem of *not* being lower semicontinuous in their natural topology. Hence we may detect minimizing sequences with unbounded discontinuity surfaces, and in the limit we could obtain a diffuse zone of non-differentiability. In particular we have in general non-existence of the solution for minimum problems.

In order to describe the behavior of minimizing sequences we can substitute the functional I_1 with its lower semicontinuous envelope $\overline{I}_1$. Minimizing sequences for problems involving I_1 converge then to solutions for the corresponding problems for $\overline{I}_1$, which in general may have a diffuse "fractured" zone. This difficulty is classically solved by considering only problems where the existence of solutions whose "fracture" remains confined on a surface is supposed *a priori*.

In order to avoid this postulate, we propose a singular perturbation approach. The idea consists in approximating the functional $\overline{I}_1$ with a sequence of functionals I_ε in such a way that the limits of minimizers of I_ε are special minimizers for $\overline{I}_1$ with "fracture" confined on a surface, at most. We propose to consider functionals of the type

$$(0.4) \qquad I_\varepsilon(u) = \int_{\Omega \setminus S} W(Du)dx + \int_S \phi_\varepsilon(u_+, u_-)d\mathcal{H}^{n-1},$$

which, under mild hypotheses on (ϕ_ε), converge to $\overline{I}_1$ as $\varepsilon \to 0$ in a variational sense assuring the convergence of the minima and of the minimizers. Moreover, since the functional in (0.4) may not be lower semicontinuous, we give conditions on ϕ_ε for the lower semicontinuity of these functionals for every $\varepsilon > 0$, and obtain existence theorems.

The main part of this chapter (Sections 3 to 5) is devoted to the description of the effect of this perturbation in the one dimensional case, for which an exact formalization can be obtained.

For every $\varepsilon > 0$ let us consider a function $\varphi_\varepsilon : \mathbb{R} \to \mathbb{R}$ expanding as

$$\varphi_\varepsilon(t) = |t| + \varepsilon\psi(t) + o(\varepsilon)|t|,$$

where ψ is a Lipschitz function such that $\psi(0) = 1$, and the associated functional

$$I_\varepsilon(u) = \int_{\Omega \setminus S} W(Du)dx + \int_S \varphi_\varepsilon(|u_+ - u_-|)d\mathcal{H}^0.$$

We test the effect of the variational convergence of I_ε to $\overline{I}_1$ by studying the asymptotic behavior of the minimum problems with prescribed Dirichlet boundary data. We prove that such minimizers for I_ε converge, up to a subsequence, to a function which not only minimizes the corresponding problem for $\overline{I}_1$, but also minimizes a "first order" problem involving only the jump part.

We can give a mechanical interpretation to this approximating approach. The value $\varphi_\varepsilon(0)$ represents the energy necessary to create a fracture of unit length; in addition it is possible to see that to obtain propagation of a quasi-static fracture we have to postulate a pre-existing fracture with

length proportional to $\varphi_\varepsilon(0)$. Since in our case $\varphi_\varepsilon(0) = \varepsilon$, the passage to the limit for the functionals I_ε as $\varepsilon \to 0$ can be interpreted as the requirement of pre-existing infinitesimal fractures; *i.e.*, as a postulate of the existence of microfractures.

The plan of the chapter is as follows. In the first two sections we consider the weak formulation on $SBV(\Omega)$ of the functional I_1 introduced in (0.3), by posing

$$(0.5) \qquad I_1(u) = \int_\Omega W(D_a u)dx + \int_{S_u \cap \Omega} |u_+ - u_-|d\mathcal{H}^{n-1}.$$

We explicitly determine the lower semicontinuous envelope $\overline{I}_1$ of the functional in (0.5), under some convexity hypotheses on W. More precisely for every $u \in BV(\Omega)$ we obtain that

$$\overline{I}_1(u) = \int_\Omega \overline{W}(D_a u)dx + \int_{S_u \cap \Omega} |D_s u|,$$

where $|D_s u|$ denotes the total variation of the measure $D_s u$ and $\overline{W}$, the bulk energy of $\overline{I}_1$, is a function, explicitly calculated from W, which grows at most linearly at infinity, whatever the form of W.

Note that the relaxed functional $\overline{I}_1$ is finite on the whole $BV(\Omega)$; this means that in general we must incur in somehow "plastic" behaviors of the minimizers. In order to select between all possible minimizers for $\overline{I}_1$ those belonging to $SBV(\Omega)$, we propose to approximate $\overline{I}_1$ by means of a sequence of functionals defined as

$$(0.6) \quad I_\varepsilon(u) = \begin{cases} \displaystyle\int_\Omega W(D_a u)dx + \int_{S_u \cap \Omega} \phi_\varepsilon(u_+, u_-, \nu)d\mathcal{H}^{n-1} \\ \qquad\qquad\qquad\qquad\qquad \text{if } u \in SBV(\Omega) \\ \\ +\infty \qquad\qquad\qquad\qquad\quad \text{elsewhere in } BV(\Omega), \end{cases}$$

with $\phi_\varepsilon : \mathbb{R} \times \mathbb{R} \times S^{n-1} \to [0, +\infty[$. We show that for a large class of ϕ_ε the functionals in (0.6) Γ-converge to $\overline{I}_1$.

Sections 3 through 5 are devoted to the study, in the 1-dimensional case, of the effect of this perturbation on the minimizers of $\overline{I}_1$. First, in Section 3, we examine the Dirichlet boundary value problems for the functional $\overline{I}_1$, giving a description of the minimizers and showing that we may indeed obtain solutions with diffuse singular part of the derivative; *i.e.* with $Cu \not\equiv 0$. In Section 4 we give necessary and sufficient conditions for the lower semicontinuity of the functionals in (0.6) when inf $\phi_\varepsilon > 0$ and

$\phi_\varepsilon(u,v) \geq |u-v|$, and we obtain some relaxation results. Finally, in Section 5 we deal with the case when we have for ϕ_ε an expansion of the form

$$\phi_\varepsilon(u,v) = |u-v| + \varepsilon\phi(u,v) + o(\varepsilon)|u-v|,$$

with ϕ Lipschitz, and $\phi(u,u) \geq c > 0$ for all u. We explicitly give a characterization of the minimizers for Dirichlet boundary value problems for $\overline{I}_1$ which can be reached following sequences of minimizers for the same problems for I_ε. We show that these minimizers are indeed in $SBV(\Omega)$, they have a finite number of jumps, and they minimize (under the same boundary conditions) a functional of the form

$$\sum_{\{u(x+)\neq u(x-)\}} \overline{\phi}(u(x+), u(x-))$$

among all SBV-minimizers of $\overline{I}_1$. The function $\overline{\phi}$ can be easily computed from the function ϕ, and is independent of the boundary conditions.

1. Preliminaries

For the notation we refer to Chapter 1, Section 1.

Let Ω be a bounded open subset of $\mathbb{R}^n$. We denote by $\mathcal{M}(\Omega)$ the set of the scalar Radon measures on Ω with bounded total variation, and by $\mathcal{M}_+(\Omega)$ the space of the positive Radon measures on Ω with bounded total variation.

The usual *weak* topology* on $\mathcal{M}(\Omega)$ is defined as the weakest topology on $\mathcal{M}(\Omega)$ for which the maps $\mu \mapsto \int_\Omega \psi\, d\mu$ are continuous for every $\psi \in \mathcal{C}_o(\Omega)$ (where $\mathcal{C}_o(\Omega)$ denotes the space of continuous functions vanishing on the boundary of Ω).

With this notation a function $u \in L^1(\Omega)$ is a function of bounded variation (see Chapter 1, Section 2) if for any $i \in \{1,\dots,n\}$ there is a measure $\mu_i \in \mathcal{M}(\Omega)$ such that

$$\int_\Omega u \frac{\partial\varphi}{\partial x_i} dx = -\int_\Omega \varphi\, d\mu_i \qquad \forall\varphi \in \mathcal{C}_c^1(\Omega).$$

We have just observed that $BV(\Omega)$ is a Banach space, if endowed with the norm

$$\|u\|_{BV} = \|u\|_1 + |Du|(\Omega).$$

The product topology of the strong topology of $L^1(\Omega)$ for u, and of the weak* topology of measures for Du will be called the weak* topology of BV, and will be denoted by $BV\text{-}w^*$. Recall that for every sequence $(u_h)_h$

in $BV(\Omega)$ with $\|u_h\|_{BV} \leq c$ there exist a subsequence $(u_{h_k})_k$ and a function $u \in BV(\Omega)$ such that $u_{h_k} \to u$ in $L^1(\Omega)$ and $Du_{h_k} \rightharpoonup Du$ in the weak* topology of measures. We shall denote this convergence by $u_{h_k} \rightharpoonup u$ in $BV\text{-}w^*$.

We refer to Chapter 1, Sections 3,4 for the notions and techniques related to the relaxation and Γ-convergence theories, on which most of the proofs in the chapter are based.

In this chapter we shall consider relaxations in the $BV\text{-}w^*$ topology .

The letter c will denote throughout the chapter a strictly positive constant, whose value may vary from line to line, independent of the parameters of the problems each time considered.

In this chapter, we shall consider functionals F defined on $BV(\Omega)$ for which the estimate

$$(1.1) \qquad\qquad F(u) \geq |Du|(\Omega) - c$$

holds. Note that, for functionals verifying (1.1), it is equivalent to consider sequences converging with respect to the $L^1(\Omega)$-topology and with respect to the $BV\text{-}w^*$ topology. Hence throughout the chapter we will feel free to choose the most suited to the context among the two topologies.

2. Some Relaxation and Γ-convergence Results

In this section we shall state and prove a relaxation and a Γ-convergence result concerning some functionals defined on BV. We show that these functionals can be "reached" starting from functionals defined in SBV.

Let $W : \mathbb{R}^n \to [0, +\infty]$ be a lower semicontinuous convex function such that $W(0) = 0$, and the set

$$K = \{z \in \mathbb{R}^n : W(z) \leq |z| \}$$

is bounded. Then we define the function $\overline{W} : \mathbb{R}^n \to [0, +\infty[$ by setting

$$\overline{W}(z) = (W(z) \wedge |z|)^{**}.$$

Let us remark that, once $W \not\equiv +\infty$, it is not restrictive to suppose $W(0) = 0$. In fact, otherwise we could consider the function $W_1(z) = W(z + z_0) - W(z_0)$, where $W(z_0) = \min W$. Remark also that we have $|z| = \overline{W}^\infty(z)$: it suffices to notice that we have $|z| - R \leq \overline{W}(z) \leq |z|$, where $R > 0$ is such that $B_R \supset K$.

Let us consider a bounded open subset Ω of $\mathbb{R}^n$, and let us define the functional $F : BV(\Omega) \to [0, +\infty]$ by setting

$$(2.1)$$

$$F(u) = \begin{cases} \displaystyle\int_\Omega W(D_a u)dx + \int_{S_u \cap \Omega} |u_+ - u_-| d\mathcal{H}^{n-1} & \text{if } u \in SBV(\Omega) \\ +\infty & \text{elsewhere on } BV(\Omega), \end{cases}$$

and the functional H,

$$(2.2) \qquad H(u) = \int_\Omega \overline{W}(D_a u)dx + \int_\Omega |D_s u|,$$

defined on the whole $BV(\Omega)$.

Theorem 2.1. *For every $u \in BV(\Omega)$ we have $\overline{F}(u) = H(u)$; i.e., the functional H is the relaxation of the functional F with respect to the $L^1(\Omega)$-topology.*

Remark. Theorem 2.1 can be extended to the vector-valued case under the same hypotheses on W. A more general relaxation theorem can be obtained when we suppose W quasiconvex (see Chapter 4).

In order to prove Theorem 2.1 we shall need the following two results about relaxation in BV and $\mathrm{W}^{1,1}$.

Theorem 2.2. *(Goffman & Serrin [62]) Let $V : \mathbb{R}^n \to [0, +\infty[$ be a convex function such that $|z| - c \le V(z) \le c(1 + |z|)$; then the relaxation of the functional*

$$E(u) = \begin{cases} \int_\Omega V(D_a u)dx & \text{if } u \in \mathrm{W}^{1,1}(\Omega) \\ +\infty & \text{if } u \in BV(\Omega) \setminus \mathrm{W}^{1,1}(\Omega) \end{cases}$$

with respect to the L^1-topology is given by

$$\overline{E}(u) = \int_\Omega V(D_a u)dx + \int_\Omega V^\infty\left(\frac{D_s u}{|D_s u|}\right)|D_s u|,$$

for all $u \in BV(\Omega)$, where $\frac{D_s u}{|D_s u|}$ is the Radon-Nikodym derivative of the measure $D_s u$ with respect to its total variation.

Theorem 2.3. *(Buttazzo & Dal Maso [33]) Let $V : \mathbb{R}^n \to [0, +\infty[$ be a Borel function such that $|z| - c \leq V(z) \leq c(1 + |z|)$; then the relaxation of the functional*

$$E(u) = \begin{cases} \int_\Omega V(D_a u)dx & \text{if } u \in \mathcal{C}^1(\Omega) \\ +\infty & \text{if } u \in W^{1,1}(\Omega) \setminus \mathcal{C}^1(\Omega) \end{cases}$$

with respect to the L^1-topology is given by

$$\overline{E}(u) = \int_\Omega V^{**}(D_a u)dx$$

for all $u \in W^{1,1}(\Omega)$.

Proof of Theorem 2.1. By Theorem 2.2 the functional H is lower semi-continuous on $BV(\Omega)$ with respect to the $L^1(\Omega)$-topology. Then the inequality $H \leq \overline{F}$ follows directly from the definition of the relaxed functional, observing that $H \leq F$. The difficulty lies in the proof of the opposite inequality. Let us consider a function $u \in \mathcal{C}^1(\Omega) \cap BV(\Omega)$. For the rest of the proof we return to the 'classical' notation Du for $D_a u$. Let us consider the set

$$\Omega' = \{x \in \Omega : Du(x) \in K\} = \{x \in \Omega : W(Du(x)) \leq |Du(x)|\}.$$

Let us recall that if $x \in \Omega'$, then $|Du(x)| \leq R$. For every $\varepsilon > 0$ we take an open set of finite perimeter $\Omega_\varepsilon \subset \Omega'$ such that $|\Omega' \setminus \Omega_\varepsilon| \leq \varepsilon$ (it suffices to consider for example the set $\Omega_\varepsilon = \{x \in \Omega' : \text{dist}(x, \partial\Omega') > \eta\}$ for $\eta = \eta(\varepsilon) > 0$ small enough). We construct a sequence (u_h) piecewise constant on $A_\varepsilon = \Omega \setminus \overline{\Omega}_\varepsilon$ as follows. For every $k \in \mathbb{Z}$ we find

$$s_k^h \in \left] \frac{k}{h}, \frac{k+1}{h} \right[$$

such that

$$\frac{1}{h} \mathcal{H}^{n-1}(\partial^* \{u > s_k^h\} \cap A_\varepsilon) \leq \int_{\{\frac{k}{h} < u < \frac{k+1}{h}\} \cap A_\varepsilon} |Du| dx$$

(here we use the coarea formula (2.6), Chapter 1). We define then

$$
u_h = \begin{cases} \frac{k}{h} & \text{on } \{s^h_{k-1} < u < s^h_k\} \cap A_\varepsilon, \ k \in \mathbb{Z} \\[2mm] u & \text{on } \overline{\Omega}_\varepsilon. \end{cases}
$$

We can estimate then

$$
F(u_h) = \int_\Omega W(D_a u_h)dx + \int_{S_{u_h} \cap \Omega} |u_{h+} - u_{h-}|d\mathcal{H}^{n-1}
$$

$$
= \int_{\Omega_\varepsilon} W(Du)dx + \int_{S_{u_h} \cap A_\varepsilon} \frac{1}{h}d\mathcal{H}^{n-1} + \int_{(\partial^* \Omega_\varepsilon) \cap \Omega} |u_{h+} - u_{h-}|d\mathcal{H}^{n-1}
$$

$$
\leq \int_{\Omega_\varepsilon} W(Du)dx + \sum_{k \in \mathbb{Z}} \frac{1}{h}\mathcal{H}^{n-1}(\partial^*\{u > s^h_k\} \cap A_\varepsilon) + \frac{2}{h}\mathcal{H}^{n-1}((\partial^*\Omega_\varepsilon) \cap \Omega)
$$

$$
\leq \int_{\Omega_\varepsilon} W(Du)dx + \sum_{k \in \mathbb{Z}} \int_{\{\frac{k}{h} < u < \frac{k+1}{h}\} \cap A_\varepsilon} |Du|dx + \frac{2}{h}\mathcal{H}^{n-1}((\partial^*\Omega_\varepsilon) \cap \Omega)
$$

$$
\leq \int_{\Omega_\varepsilon} W(Du)dx + \int_{A_\varepsilon} |Du|dx + \frac{2}{h}\mathcal{H}^{n-1}((\partial^*\Omega_\varepsilon) \cap \Omega)
$$

$$
\leq \int_{\Omega'} W(Du)dx + \int_{\Omega\setminus\Omega'} |Du|dx + \varepsilon R + \frac{2}{h}\mathcal{H}^{n-1}((\partial^*\Omega_\varepsilon) \cap \Omega)
$$

We have made use of the fact that $|Du| \leq R$ on Ω'. Since $u_h \to u$ in $L^\infty(\Omega)$, we obtain

$$
\overline{F}(u) \leq \liminf_h F(u_h) \leq \int_{\Omega'} W(Du)dx + \int_{\Omega\setminus\Omega'} |Du|dx + \varepsilon R
$$

for any $\varepsilon > 0$. By the arbitrariness of ε we conclude that for every $u \in C^1(\Omega) \cap BV(\Omega)$

$$(2.3) \qquad \overline{F}(u) \leq \int_\Omega \tilde{W}(Du)dx,$$

where the function $\tilde{W}$ is defined by setting

$$
\tilde{W}(z) = W(z) \wedge |z| = \begin{cases} W(z) & \text{if } z \in K \\[2mm] |z| & \text{otherwise.} \end{cases}
$$

Since $\overline{W} = (\tilde{W})^{**}$, by Theorem 2.3 we have that the relaxation of the functional defined by

$$G(u) = \begin{cases} \displaystyle\int_\Omega \tilde{W}(Du)dx & \text{if } u \in C^1(\Omega), \\[2ex] +\infty & \text{elsewhere on } \mathrm{W}^{1,1}(\Omega), \end{cases}$$

with respect to the L^1-topology, is given by H on $\mathrm{W}^{1,1}(\Omega)$; hence, using (2.3), we obtain

$$\overline{F}(u) \leq H(u) \qquad u \in \mathrm{W}^{1,1}(\Omega).$$

Finally by Theorem 2.2 this inequality is valid on the whole $BV(\Omega)$. $\qquad\square$

The functional H, defined in (2.2), can be also considered as Γ-limit of a suitable sequence of functionals, as shown in the following proposition.

Proposition 2.4. *For every $\varepsilon > 0$ let us consider a Borel function $\phi_\varepsilon : \mathbb{R} \times \mathbb{R} \times S^{n-1} \to \mathbb{R}$ such that*

$$(1 - c\varepsilon)|t - s| \leq \phi_\varepsilon(s,t,\nu) \leq (1 + c\varepsilon)|t - s| + \varepsilon$$

for all $(s,t,\nu) \in \mathbb{R} \times \mathbb{R} \times S^{n-1}$, and let us define

$$F_\varepsilon(u) = \begin{cases} \displaystyle\int_\Omega W(D_a u)dx + \int_{S_u \cap \Omega} \phi_\varepsilon(u_+, u_-, \nu)d\mathcal{H}^{n-1} \\[1ex] \qquad\qquad\qquad \text{if } u \in SBV(\Omega) \text{ and } \mathcal{H}^{n-1}(S_u) < +\infty, \\[2ex] +\infty \qquad\qquad\qquad \text{elsewhere on } BV(\Omega). \end{cases}$$

Then, for every $u \in BV(\Omega)$, we have

$$\Gamma\text{-}\lim_{\varepsilon \to 0+} F_\varepsilon(u) = H(u),$$

where the Γ-limit is considered with respect to the $L^1(\Omega)$-topology.

Remark 2.5. As a particular case, we can apply Proposition 2.4 to $\phi_\varepsilon(s,t) = |t - s|$ for all ε, deducing that we can take minimizing sequences (u_h) for $\overline{F}$ with $\mathcal{H}^{n-1}(S_{u_h}) < +\infty$ for every h.

Proof of Proposition 2.4. From the inequality $F_\varepsilon(u) \geq (1 - c\varepsilon)F(u)$, it follows that

$$\Gamma\text{-}\liminf_{\varepsilon \to 0+} F_\varepsilon \geq H.$$

We have to prove then that

$$\Gamma\text{-}\limsup_{\varepsilon\to 0+} F_\varepsilon \le H.$$

Let us take $u \in \mathcal{C}^1(\Omega) \cap BV(\Omega)$, and let us fix a sequence (ε_h) converging to 0. We can consider the sequence (u_h) constructed in the proof of Theorem 2.1; we have

$$S_{u_h} \cap \Omega \subset \left(\bigcup_{k\in\mathbb{Z}} (\partial^*\{u > s_k^h\} \cap A_\eta)\right) \cup ((\partial^*\Omega_\eta) \cap \Omega),$$

where $\eta > 0$ is a fixed positive number sufficiently small. Hence

$$\mathcal{H}^{n-1}(S_{u_h} \cap \Omega) \le \sum_{k\in\mathbb{Z}} \mathcal{H}^{n-1}(\partial^*\{u > s_k^h\} \cap A_\eta) + \mathcal{H}^{n-1}((\partial^*\Omega_\eta) \cap \Omega)$$

$$\le h \sum_{k\in\mathbb{Z}} \int_{\{\frac{k}{h} < u < \frac{k+1}{h}\} \cap A_\eta} |Du|\,dx + c \le hc < +\infty.$$

We can then define a sequence $(w_h) \subset SBV(\Omega)$ by setting $w_0 = u_0 = 0$, and

$$w_{h+1} = \begin{cases} u_{k+1} & \text{if } w_h = u_k \text{ and } k^2\varepsilon_h \le 1, \\[2mm] w_h & \text{otherwise.} \end{cases}$$

We have then

$$\mathcal{H}^{n-1}(S_{w_h} \cap \Omega) = \mathcal{H}^{n-1}(S_{u_k} \cap \Omega) \le ck \le c(\varepsilon_h)^{-\frac{1}{2}},$$

$w_h \to u$ in $\mathrm{L}^1(\Omega)$, and $\int_{S_{w_h} \cap \Omega} |(w_h)_+ - (w_h)_-|\,d\mathcal{H}^{n-1} < c$. Hence

$$\Gamma\text{-}\limsup_{h\to+\infty} F_{\varepsilon_h}(u) \le \limsup_h F_{\varepsilon_h}(w_h)$$

$$\le \limsup_h \left(\int_\Omega W(D_a w_h)\,dx + (1 + c\varepsilon_h)\int_{S_{w_h} \cap \Omega} |(w_h)_+ - (w_h)_-|\,d\mathcal{H}^{n-1}\right.$$

$$\left. + \varepsilon_h \mathcal{H}^{n-1}(S_{w_h} \cap \Omega)\right)$$

$$\le \limsup_h \left(\int_{\Omega_\eta} W(D_a u)\,dx + \int_{S_{w_h} \cap \Omega} |(w_h)_+ - (w_h)_-|\,d\mathcal{H}^{n-1} + c\varepsilon_h + c(\varepsilon_h)^{\frac{1}{2}}\right)$$

$$\le \int_\Omega \tilde{W}(D_a u)\,dx + \eta R.$$

Since this inequality holds for every $\eta > 0$, we obtain that

$$\Gamma\text{-}\limsup_{h \to +\infty} F_{\varepsilon_h}(u) \leq \int_\Omega \tilde{W}(D_a u)dx$$

for every $u \in \mathcal{C}^1(\Omega) \cap BV(\Omega)$. As in the proof of Theorem 2.1, using Theorems 2.3 and 2.2, we conclude that on $BV(\Omega)$

$$\Gamma\text{-}\limsup_{h \to \infty} F_{\varepsilon_h} \leq H.$$

By the arbitrariness of the sequence (ε_h), this inequality concludes the proof. $\square$

Examples 2.6. Let us fix $\rho > 0$ and $z_0 \in \mathbb{R}^n$, and let us consider

$$W(z) = \chi_{B_\rho(z_0)}(z) = \begin{cases} 0 & \text{if } |z - z_0| < \rho \\ +\infty & \text{otherwise,} \end{cases}$$

then we have

$$\overline{W}(z) = \text{dist}(z, B_\rho(z_0)).$$

If $W(z) = |z|^2$, then it is easy to check that

$$\overline{W}(z) = \begin{cases} z^2 & \text{if } |z| \leq \frac{1}{2} \\ |z| - \frac{1}{4} & \text{otherwise.} \end{cases}$$

3. Minimum Problems in Dimension One

In this section and the following ones we use some approximation techniques in dimension 1. Many results presented below could be seen either as particular cases of the theory of SBV-functions in arbitrary dimension (see [7], [45]) or a specialization of the theorems about functionals defined on measures by G. Bouchitté & G. Buttazzo [26] (see also [25]). We prefer to include the proofs since in the one-dimensional case these can be obtained in a simpler and straightforward way.

We shall deal with functionals defined on the space $BV(I)$, where I is a bounded open interval of $\mathbb{R}$; without loss of generality we will take

$I =]0, 1[$. The space $SBV(I)$ reduces to the set of the functions u in $BV(I)$ such that their measure first derivative, denoted by $\dot{u}$, is of the form

$$\dot{u} = \dot{u}_a dt + \sum_{k=1}^{\infty} a_k \delta_{t_k},$$

where $t_k \in I$, $a_k \in \mathbb{R}$, $\sum_{k=1}^{\infty} |a_k| < \infty$, and δ_t is the Dirac measure at t. Note that, if we define

$$u(t+), \quad u(t-)$$

the right-hand and left-hand traces respectively of the function u at t, which exist for all $t \in I$, then $a_k = u(t_k+) - u(t_k-)$. Moreover we can express integration on S_u as a summation; *e.g.*,

$$\int_{S_u \cap I} |u_+(t) - u_-(t)| d\#(t) = \int_{S_u \cap I} |u(t+) - u(t-)| d\#(t)$$

$$= \sum_{t \in S_u \cap I} |u(t+) - u(t-)|.$$

Note that (a quotient space of) the space $BV(I)$ with the BV-w^* topology can be identified with the space $\mathcal{M}(I)$ of all Radon measures on I with bounded total variation equipped with the weak* topology of measures (for example see Lemma 1.2 in [30]). Let us remark moreover that, given $t_0 \in [0, 1[$ and $u_0 \in \mathbb{R}$, there is a 1-1 correspondence between $\mathcal{M}(I)$ and the subspace $\{u \in BV(I) : u(t_0+) = u_0\}$, given by $\mu \mapsto u_\mu$, where

$$u_\mu(t) = \begin{cases} u_0 + \mu(]t_0, t]) & \text{if } t \geq t_0, \\[2mm] u_0 - \mu(]t, t_0]) & \text{otherwise.} \end{cases}$$

In the sequel we will then feel free to define sometimes functions in $BV(I)$ by simply describing their measure derivative and the value $u(t_0+)$ at some point $t_0 \in [0, 1[$ (or equivalently $u(t_0-)$ at some point $t_0 \in]0, 1]$).

Let $g : \mathbb{R} \to [0, +\infty[$ be a convex function such that $g(0) = 0$, the set

$$(3.1) \qquad J = \left\{ t \in \mathbb{R} : g(t) = (g(t) \wedge |t|)^{**} \right\}$$

is bounded, and g is strictly convex on J. Note that $J = [a, b]$ is a closed interval containing 0, and that

$$(3.2) \qquad (g(t) \wedge |t|)^{**} = f(t) = \begin{cases} g(t) & \text{if } a \leq t \leq b \\[2mm] g(b) - b + t & \text{if } t > b \\[2mm] g(a) + a - t & \text{if } t < a \end{cases}$$

We shall consider in this section the functional F defined by

$$F(u) = \begin{cases} \displaystyle\int_I g(\dot{u}_a)dt + \int_{S_u \cap I} |u_+ - u_-|d\# & \text{if } u \in SBV(I) \\ +\infty & \text{elsewhere on BV(I),} \end{cases}$$

and its relaxation (given by Theorem 2.1)

$$\overline{F}(u) = H(u) = \int_I f(\dot{u}_a)dt + \int_I |\dot{u}_s| \qquad u \in BV(I),$$

where the function f is given by (3.2).

We can obtain minimizing sequences for $\overline{F}$ with a finite number of jumps, as shown in the following proposition.

Proposition 3.1. *Let us define on $BV(I)$ the functional F_0 by setting*

$$F_0(u) = \begin{cases} \displaystyle\int_I g(\dot{u}_a)dt + \int_{S_u \cap I} |u_+ - u_-|d\# & \\ & \text{if } u \in SBV(I) \text{ and } \#(S_u) < +\infty, \\ +\infty & \text{otherwise.} \end{cases}$$

Then we still have $H = \overline{F}_0$; i.e., we can take minimizing sequences for $\overline{F}$ with a finite number of jumps.

Proof. Let $(u_h) \subset SBV(I)$ such that $u_h \rightharpoonup u$ in $BV\text{-}w^*$, and $H(u) = \lim\limits_h F(u_h)$. We can write the measure $\dot{u}_h = (\dot{u}_h)_a dt + \sum_{k=1}^\infty a_k^h \delta_{t_k^h}$, with $S_h = \sum_{k=1}^\infty |a_k^h| < +\infty$. Fix $h \in \mathbb{N}$, and let $N_h \in \mathbb{N}$ be such that $|\sum_{k=1}^{N_h} |a_k^h| - S_h| < \frac{1}{h}$. Then there exists a function $v_h \in SBV(I)$ such that

$$\dot{v}_h = (\dot{u}_h)_a + \sum_{k=1}^{N_h} a_k^h \delta_{t_k^h}, \text{ and } \|u_h - v_h\|_\infty < \frac{1}{h}.$$

Of course, we have $v_h \rightharpoonup u$ in $BV\text{-}w^*$, and

$$\lim_h F(v_h) = \lim_h F(u_h) = H(u).$$

$\square$

We can describe the behavior of the functional H by examining some minimum problems with generalized Dirichlet boundary data. Let us fix $\alpha \in \mathbb{R}$, and consider the boundary conditions $u(0) = 0$ and $u(1) = \alpha$. It is well-known that these conditions are not well-posed for problems in $BV(I)$ (see [15]). We have instead to *relax* these conditions by penalizing jumps at $t = 0, 1$, considering the minimum problems

$$(3.3) \qquad m_\alpha = \min\{H(u) + |u(0+)| + |\alpha - u(1-)| : u \in BV(I)\},$$

where the values $u(0+)$, $u(1-)$ represent the inner traces of u in 0 and 1 respectively. We can easily describe the minimum points for problem (3.3), as follows.

Proposition 3.2. *Let us consider the minimum problem in (3.3) (recall that $J = [a, b]$ is given by (3.1)). Then we have*

  i) if $\alpha \in J$ then the unique minimum point is $u(t) = \alpha t$;

  ii) if $\alpha > b$ then the minimum points for (3.3) are all $u \in BV(I)$ such that $u(0+) \geq 0$, $u(1-) \leq \alpha$, $\dot{u} \in M_+(I)$, $\dot{u}_a \geq b$ a.e.;

  iii) if $\alpha < a$ then the minimum points for (3.3) are all $u \in BV(I)$ such that $u(0+) \leq 0$, $u(1-) \geq \alpha$, $-\dot{u} \in M_+(I)$, $\dot{u}_a \leq a$ a.e.

Proof. Let us consider the case $\alpha > 0$ (the opposite case being analogous). We want to show that if u is a minimum point for (3.3) then $\dot{u} \in M_+(I)$, $u(0+) \geq 0$, and $u(1-) \leq \alpha$. This can be restated as the requirement that $\dot{u}^* \in \mathcal{M}_+(\mathbb{R})$, where the function $u^* \in BV_{\mathrm{loc}}(\mathbb{R})$ is obtained by extending u to 0 in $]-\infty, 0]$ and to α in $]1, +\infty[$.

Let $u \in BV(I)$, and let us consider the function $v \in BV_{\mathrm{loc}}(\mathbb{R})$ defined by

$$v(t) = \big((\dot{u}^*)_+([0, t])\big) \wedge \alpha,$$

where $(\dot{u}^*)_+ \in M_+(I)$ is the positive part of the measure $\dot{u}^*$. It is easy to see that $u = v$ iff $\dot{u}^*$ belong to $\mathcal{M}_+(\mathbb{R})$. If $u \neq v$ then we can have either $v < \alpha$ a.e., or not. In the first case we have $\dot{v}_a \leq \dot{u}_a$ a.e., $|\dot{v}_s|(I) \leq |\dot{u}_s|(I)$, and $v(1-) > u(1-)$. This implies that u is not a minimizer. In the second case it is easy to see that $|\dot{v}| \leq |\dot{u}|$ (as measures), the inequality being strict on a non-negligible set. Again this gives that u is not a minimizer.

We can then suppose that $\dot{u}^* \in \mathcal{M}_+(\mathbb{R})$; i.e., $\dot{u} \in \mathcal{M}_+(I)$, $u(0+) \geq 0$, and $u(1-) \leq \alpha$. Let us consider now the set $E = \{\dot{u}_a < b\}$, and $v \in BV(I)$ such that $v(0+) = 0$ and $\dot{v} = w\,dt + \dot{u}_s$, where

$$w(t) = \begin{cases} \dfrac{1}{1 - |E|} \displaystyle\int_{I \setminus E} \dot{u}_a\,ds & \text{if } t \notin E \\[4mm] \dfrac{1}{|E|} \displaystyle\int_E \dot{u}_a\,ds & \text{if } t \in E. \end{cases}$$

We have $\dot{v}^* \in \mathcal{M}_+(\mathbb{R})$, $|u(0+)| + |\alpha - u(1-)| = |v(0+)| + |\alpha - v(1-)|$, and

$$H(u) - H(v) = \int_E f(\dot{u}_a)dt - |E|f\left(\frac{1}{|E|}\int_E \dot{u}_a dt\right.$$

$$= \int_E h(\dot{u}_a)dt - |E|h\left(\frac{1}{|E|}\int_E \dot{u}_a dt\right).$$

By the strict convexity of g we must have then $\dot{u}_a = s < b$ constant a.e. on E. The value $H(u)$ can be computed in terms of s and $|E|$, as

$$(3.4) \quad H(u) + |u(0+)| + |\alpha - u(1-)| = |E|(g(s) - g(b) + b - s) + g(b) - b + \alpha.$$

Recall that we have $g(s) - s > g(b) - b$ for $s \neq b$. If $\alpha > b$ we must have then $|E| = 0$, and this concludes the proof of ii). In the same way we obtain i) when $\alpha = b$.

If $\alpha < b$, let us notice that we must have

$$s|E| + b(1 - |E|) \leq \int_I \dot{u}_a dt \leq \alpha$$

(since on $I \setminus E$ we have $\dot{u}_a \geq b$). Fixed $|E|$ the minimizing choice is $s = b - \frac{1}{|E|}(b - \alpha)$, for which

$$(3.5) \quad H(u) + |u(0+)| + |\alpha - u(1-)| = g(b) - |E|\left(g(b) - h\left(b - \frac{1}{|E|}(b - \alpha)\right)\right);$$

it is easy to check that we must have $|E| = 1$, and $s = \alpha$. $\qquad\square$

Remark 3.3. By Proposition 3.2 we obtain, for all $\alpha \in \mathbb{R}$:
 i) $m_\alpha = f(\alpha)$ (it follows from the descriptions of the minimizers, and (3.4), (3.5));
 ii) we have

$$m_\alpha = \min\{F(u) + |u(0+)| + |\alpha - u(1-)| : u \in SBV(I)\}.$$

This is trivial if $\alpha \in J$, and follows considering for example the function $u(t) = bt$ if $\alpha > b$, and the function $u(t) = at$ if $\alpha < a$.

The following proposition shows that we can consider the functional $H = \overline{F}$, as the limit of a sequence of functionals defined in $SBV(I)$.

Proposition 3.4. *For every $\varepsilon > 0$, let $\phi_\varepsilon : \mathbb{R} \times \mathbb{R} \to [0, +\infty[$ be a Borel function, satisfying*

$$(1 - c\varepsilon)|t - s| \leq \phi_\varepsilon(s, t) \leq (1 + c\varepsilon)|t - s| + \varepsilon,$$

and let us define on $BV(I)$ the functional F_ε by setting

$$F_\varepsilon(u) = \begin{cases} \displaystyle\int_I g(\dot{u}_a)dt + \int_{S_u \cap I} \phi_\varepsilon(u(t+), u(t-))d\#(t) \\ \qquad\qquad\qquad \text{if } u \in SBV(I), \text{ and } \#(S_u) < +\infty, \\ \\ +\infty \qquad\qquad\qquad\qquad \text{otherwise.} \end{cases}$$

Then we have $H = \Gamma\text{-}\lim_{\varepsilon \to 0+} F_\varepsilon$.

Proof. The same proof of Proposition 2.4 holds true. Notice that if we define

$$\tilde{\phi}_\varepsilon(u, v, +1) = \phi_\varepsilon(u, v),$$
$$\tilde{\phi}_\varepsilon(u, v, -1) = \phi_\varepsilon(v, u),$$

we obtain functionals of the general form considered in Proposition 2.4. □

Remark 3.5 The proof of Proposition 3.2 can be easily extended to the case when we do not assume that the function g is strictly convex on J. For instance, if g is linear on a subinterval $[c, d] \subset J$, and $\alpha \in [c, d]$, all the functions $u \in W^{1,1}(I)$ such that $u(0) = 0$, $u(1) = \alpha$, and $\dot{u}(t) \in [c, d]$ for a.e. $t \in I$ are minimum points for (3.3).

4. Semicontinuity and Relaxation in SBV

Proposition 3.4 exhibits a singular perturbation of the functional H, with functionals that may in many ways behave "better" than their limit. Let us recall that by the Γ-convergence, and the equi-coerciveness of the functionals on BV, the approximation of H by the functionals F_ε represents a choice among all the possible minima for the problem in (3.3), of those that in particular can be reached following minimizing sequences for the corresponding problems for F_ε. It is therefore of some interest to briefly examine the structure (and existence theorems) of these functionals [†].

[†] Since we shall focus our attention on the behavior of the jump-part energy, we will limit our analysis, for the sake of simplicity, to the "bulk energy" g verifying the growth condition $g(z) \geq |z|^2 - c$. From Section 2 it will be clear that all the results of Sections 4 and 5 continue to be true if we substitute $|z|^2$ with any convex function h growing more than linearly at infinity, with minor modifications in the proofs.

We shall need some simple results about lower semicontinuity properties of functionals defined in $SBV(\mathrm{I})$. We say that a function $\phi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is *subadditive* if we have

$$(4.1) \qquad \phi(a,b) \leq \phi(a,c) + \phi(c,b) \text{ for all } a,b,c \in \mathbb{R}.$$

Proposition 4.1. *(Lower semicontinuity in $SBV(I)$) Let (u_h) be a sequence in $SBV(I)$ such that $\|u_h\|_{BV} \leq c$, $\|(\dot{u}_h)_a\|_2 \leq c$ and $\#(S_{u_h}) \leq c$; then (possibly passing to a subsequence) there exists $u \in SBV(I)$ such that $u_h \rightharpoonup u$ in $BV\text{-}w^*$, and*

$$(\dot{u}_h)_a \rightharpoonup \dot{u}_a \text{ in } L^2(I), \quad \#(S_u) \leq \liminf_{h \to \infty} \#(S_{u_h}).$$

Moreover, if ϕ is subadditive and l.s.c., then

$$\sum_{t \in S_u} \phi(u(t+), u(t-)) \leq \liminf_{h \to +\infty} \sum_{t \in S_{u_h}} \phi(u_h(t+), u_h(t-)).$$

Proof. Since $\|u_h\|_{BV} \leq c$ we can suppose (possibly passing to a subsequence) that $u_h \rightharpoonup u$ in $BV\text{-}w^*$ for some $u \in BV(I)$.

On the other hand, since $\#(S_{u_h}) \leq c$, we can suppose that

$$\#(S_{u_h}) = \liminf_{k \to \infty} \#(S_{u_k}) = N \in \mathbb{N}$$

independent of $h \in \mathbb{N}$; *i.e.*, $S_{u_h} = \{t_j^h : j = 1 \ldots, N\}$, with $0 < t_1^h < t_2^h < \ldots < t_N^h < 1$, and, without loss of generality, that $t_j^h \to t_j \in [0,1]$ for every $j = 1 \ldots, N$.

Let us define $v_h \in \mathrm{H}^1(I)$ by setting $v_h(0+) = u_h(0+)$ and $\dot{v}_h = (\dot{u}_h)_a dt$ (*i.e.*, no jump part for $\dot{v}_h$). The sequence (v_h) is then bounded in $\mathrm{H}^1(I)$. Passing possibly to a further subsequence, we can suppose then that $v_h \rightharpoonup v$ weakly in $\mathrm{H}^1(I)$; in particular $(\dot{u}_h)_a = (\dot{v}_h)_a \rightharpoonup (\dot{v})_a$ weakly in $L^2(I)$.

The measure $\dot{u} - (\dot{v})_a dt$ must be supported by $S = \{t_j : j = 1, \ldots, N\} \cap I$, since the sequence (u_h) converges weakly in $\mathrm{H}^1_{\mathrm{loc}}$ outside any neighborhood of S. This shows that the Cantor part of $\dot{u}$ must be 0, and $\dot{u}_a = \dot{v}_a$ a.e. Moreover, we obtain that $S_u \subset S$, hence $\#(S_u) \leq N$; *i.e.*, $\#(S_u) \leq \liminf_{h \to \infty} \#(S_{u_h})$. The function u belong then to $SBV(I)$.

Since $\dot{u}_h \rightharpoonup \dot{u}$, and $(\dot{u}_h)_a dt \rightharpoonup \dot{u}_a dt$ in the weak* topology of measures, we must have $(\dot{u}_h)_s \rightharpoonup \dot{u}_s$ in the weak* topology of measures. Let us recall that we can write

$$(\dot{u}_h)_s = \sum_{j=1}^{N} \left(u_h(t_h^j+) - u_h(t_h^j-) \right) \delta_{t_h^j}.$$

If for some $j \in \{1, \ldots, N\}$ we have $t_h^j \to 0$, then the sequence $(u_h(t_h^j+) - u_h(t_h^j-))\delta_{t_h^j}$ does not give any contribution to the limit measure $\dot{u}$. The same is true if $t_h^j \to 1$. If instead $\bar{t} \in S_u \subset S$, then there exist sequences $(t_h^l), (t_h^{l+1}), \ldots (t_h^m)$ converging to $\bar{t}$, hence

$$u(\bar{t}+) - u(\bar{t}-) = \lim_h \sum_{j=l}^{m} (u_h(t_h^j+) - u_h(t_h^j-)).$$

We can also suppose that $u_h(t_h^j+) \to a_j^+$, $u_h(t_h^j-) \to a_j^-$ for all $j = l, \ldots, m$. Remark then that $a_m^+ = u(\bar{t}+)$, $a_l^- = u(\bar{t}-)$, and $a_k^+ = a_{k+1}^-$ for $k = l, \ldots, m-1$, so that

$$u(\bar{t}+) - u(\bar{t}-) = \sum_{j=l}^{m} (a_j^+ - a_j^-).$$

By the subadditivity of ϕ, we have

$$\phi(u(\bar{t}+), u(\bar{t}-)) \le \sum_{j=l}^{m} \phi(a_j^+, a_j^-).$$

Finally, by the lower semicontinuity of ϕ, we obtain

$$\phi(a_j^+, a_j^-) \le \liminf_h \phi(u_h(t_h^j+), u_h(t_h^j-)),$$

and hence

$$\sum_{t \in S_u} \phi(u(t+), u(t-)) \le \liminf_{h \to +\infty} \sum_{t \in S_{u_h}} \phi(u_h(t+), u_h(t-)),$$

the desired inequality. □

For more lower semicontinuity results see Theorem 3.3 by L. Ambrosio in [7]. We refer also to [10] for a discussion on necessary and sufficient conditions for the lower semicontinuity for functionals in dimension higher than one.

Remark 4.2. We can consider also integrands depending on the jump, and on one of the two traces of the function; *i.e.*, of the form

$$(4.2) \qquad\qquad \theta(u(t+) - u(t-), u(t-))$$

under assumptions of lower semicontinuity, and

$$(4.3) \qquad \theta(s + r, u) \leq \theta(s, u) + \theta(r, u + s),$$

for all $r, s, u \in \mathbb{R}$. In the same way we can deal with integrands of the type $\theta(u(t+) - u(t-), u(t+))$. It is easy to check that if we set

$$(4.4) \qquad \phi(x, y) = \theta(x - y, y),$$

then ϕ satisfies (4.1) iff θ satisfies (4.3). We will sometimes prefer the notation (4.2) to the equivalent form (4.4) since it highlights the dependence of the function θ on the jump.

We turn our attention now to the problem of the relaxation of functionals defined in $SBV(I)$.

The *subadditive envelope* $sub\,\phi$ of a function $\phi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is the greatest subadditive function less than or equal to ϕ. It is easy to check that $sub\,\phi \in \mathbb{R} \cup \{-\infty\}$ is given by the formula

$$sub\,\phi(x, y) = \inf\{\sum_{k=1}^{m} \phi(x_k, x_{k-1}) : x_0 = y, x_m = x, m = 1, 2, \ldots\}.$$

Let us recall also that, given a function $\psi : \mathbb{R}^n \to \mathbb{R}$, the *lower semicontinuous envelope* of ψ, that we shall denote by $sc^-(\psi)$, is defined as the greatest lower semicontinuous function less than or equal to ψ. We have

$$sc^-(\psi)(x) = \min\{\liminf_k \psi(x_k) : x_k \to x\}$$
$$= \min\{\lim_k \psi(x_k) : x_k \to x \text{ and } \exists \lim_k \psi(x_k)\}.$$

Given a function $\phi : \mathbb{R} \times \mathbb{R} \to [0, +\infty[$, we shall define the function $\overline{sub}\,\phi : \mathbb{R} \times \mathbb{R} \to [0, +\infty[$ by setting

$$(4.5) \qquad \overline{sub}\,\phi(x, y) = sub\,(sc^-(\phi))(x, y)$$

Remark that, since the values $\phi(x, x)$ are never taken into account in the functionals, we could set by definition $\phi(x, x) = 0$ for all $x \in \mathbb{R}$. This position is frequent in the literature (see [7], [10], [26]), and would affect neither the lower semicontinuity, nor the subadditivity properties of ϕ. Anyway, we *will not* make use of this convention, since we find useful to deal whenever possible with convex or continuous functions.

Proposition 4.3. *If ϕ verifies*

$$\inf \phi > 0, \qquad \phi(x,y) \geq |x - y| \quad \textit{for all } x, y \in \mathbb{R},$$

then we have

i) $\inf(\overline{sub}\,\phi) > 0$ *and* $\overline{sub}\,\phi(x,y) \geq |x - y|$ *for all $x,y \in \mathbb{R}$;*

ii) $\overline{sub}\,\phi$ *is the greatest lower semicontinuous and subadditive function less than or equal to ϕ.*

Proof. i) follows directly from the definition, as the fact that $\overline{sub}\,\phi$ is subadditive, and that $\inf(\mathrm{sc}^-(\phi)) > 0$ and $\mathrm{sc}^-(\phi(x,y)) \geq |x - y|$. Let us prove that $\overline{sub}\,\phi$ is lower semicontinuous. Fixed $x, y \in \mathbb{R}$, and two sequences $x_h \to x$, $y_h \to y$, such that there exists the limit $\lim_h \overline{sub}\,\phi(x_h, y_h)$, we have to prove that

$$\overline{sub}\,\phi(x,y) \leq \lim_h \overline{sub}\,\phi(x_h, y_h).$$

By definition for every $h \in \mathbb{N}$ there exist $x_0^h, \ldots, x_{m_h}^h$ such that

$$x_0^h = y_h \to y, \qquad x_{m_h}^h = x_h \to x,$$

and

$$\sum_{k=1}^{m_h} \mathrm{sc}^-(\phi)(x_k^h, x_{k-1}^h) \leq \overline{sub}\,\phi(x_h, y_h) + \frac{1}{h}.$$

By the condition $\inf(\mathrm{sc}^-(\phi)) > 0$ we have that the sequence (m_h) is bounded. Hence we can suppose $m_h = m$, independent of h. The condition $\mathrm{sc}^-(\phi)(x,y) \geq |x - y|$ implies that all sequences $(x_0^h), \ldots, (x_m^h)$ are bounded. Again, we can suppose then that $x_0^h \to x_0, x_1^h \to x_1, \ldots, x_m^h \to x_m$ for some $x_0, x_1, \ldots, x_m \in \mathbb{R}$. Of course, $x_0 = y$, and $x_m = x$. By semicontinuity we obtain then

$$\mathrm{sc}^-(\phi)(x_k, x_{k-1}) \leq \liminf_h \mathrm{sc}^-(\phi)(x_k^h, x_{k-1}^h),$$

and hence we get

$$\sum_{k=1}^{m} \mathrm{sc}^-(\phi)(x_k, x_{k-1}) \leq \sum_{k=1}^{m} \liminf_h \mathrm{sc}^-(\phi)(x_k^h, x_{k-1}^h)$$

$$\leq \liminf_h \sum_{k=1}^{m} \mathrm{sc}^-(\phi)(x_k^h, x_{k-1}^h) \leq \lim_h \overline{sub}\,\phi(x_h, y_h).$$

By definition this shows that

$$\overline{sub}\,\phi(x,y) \leq \lim_h \overline{sub}\,\phi(x_h, y_h).$$

It is easy to see from the definition that if ϕ^1 is l.s.c. and subadditive, and $\phi^1 \leq \phi$ then $\phi^1 \leq \overline{sub}\,\phi$, and hence ii) is proved. $\qquad \square$

Proposition 4.4.   *(Relaxation in SBV(I))* Let $\phi : \mathbb{R} \times \mathbb{R} \to [0, +\infty[$ *be a Borel function verifying*

$$\inf \phi > 0, \qquad \phi(x,y) \geq |x - y|.$$

Then the relaxation of the functional

$$G(u) = \begin{cases} \displaystyle\int_I g(\dot{u}_a)dt + \int_{S_u \cap I} \phi(u(t+), u(t-))d\#(t) \\ \qquad\qquad\qquad\qquad \text{if } u \in SBV(I), \text{ and } \#(S_u) < +\infty, \\ \\ +\infty \qquad\qquad\qquad\qquad \text{otherwise,} \end{cases}$$

with respect to the topology of $L^1(I)$, is given by

$$\overline{G}(u) = \begin{cases} \displaystyle\int_I g(\dot{u}_a)dt + \int_{S_u \cap I} \overline{sub}\,\phi(u(t+), u(t-))d\#(t) \\ \qquad\qquad\qquad\qquad \text{if } u \in SBV(I), \text{ and } \#(S_u) < +\infty, \\ \\ +\infty \qquad\qquad\qquad\qquad \text{otherwise,} \end{cases}$$

where $\overline{sub}\,\phi$ is given by (4.5).

Proof.   First of all we prove that $\overline{G}$ is L^1-lower semicontinuous. Consider a sequence $u_h \to u$ in $L^1(I)$ such that $\lim_h \overline{G}(u_h) < +\infty$. Since $\overline{sub}\,\phi(x,y) \geq |x - y|$, we have $\|u_h\|_{BV} \leq c$, and hence $u_h \to u$ in $BV\text{-}w^*$. Let us remark that $\inf \overline{sub}\,\phi > 0$, and then we have

$$\#(S_{u_h} \cap I) \leq \overline{G}(u_h)/\inf \overline{sub}\,\phi \leq c < +\infty.$$

The hypotheses of Proposition 4.1 are then satisfied by the sequence (u_h). Since $\|(\dot{u}_h)_a\|_2 \leq c$ we can suppose $(\dot{u}_h)_a \rightharpoonup \dot{u}_a$ weakly in L^2, and then, using the convexity of g

$$\int_I g(\dot{u}_a)dt \leq \liminf_h \int_I g((\dot{u}_h)_a)dt.$$

Using proposition 4.1 we obtain thus $\overline{G}(u) \leq \lim_h \overline{G}(u_h)$. This shows that the functional $\overline{G}$ is less than or equal to the relaxation of G.

We have now to show that for every $u \in SBV(I)$ we can build up a recovery sequence (u_h) such that $u_h \to u$ in $L^1(I)$, and

$$\overline{G}(u) = \lim_{h \to \infty} G(u_h).$$

We can limit ourselves to the case $\overline{G}(u) < +\infty$. We can study the case of a single jump without losing in generality. We can suppose $u \in SBV(I)$, and

$$\dot{u}_s = (u(t_0+) - u(t_0-))\delta_{t_0}$$

for some $t_0 \in I$. By the definition of $\overline{sub}\,\phi$, for every $h \in \mathbb{N}$ there exist real numbers $a_0^h, a_1^h, \ldots, a_{N_h}^h$, and $b_0^h, b_1^h \ldots, b_{N_h}^h$ such that we have

$$|b_0^h - u(t_0-)| \leq \frac{1}{h^2}, \qquad |a_{N_h}^h - u(t_0+)| \leq \frac{1}{h^2}, \qquad |b_j^h - a_{j-1}^h| \leq \frac{1}{h^2}$$

for every $j = 1, \ldots, N_h$, and

$$\overline{sub}\,\phi(u(t_0+), u(t_0-)) + \frac{1}{h} \geq \sum_{j=0}^{N_h} \phi(a_j^h, b_j^h).$$

Let us remark that the hypotheses on ϕ imply that both N_h and $\sum_{j=0}^{N_h} |a_j^h - b_j^h|$ be finite. We can suppose that $N_h = N$ independent of $h \in \mathbb{N}$. Let us fix $M \in \mathbb{N}$ such that $]t_0 - \frac{1}{M}, t_0 + \frac{1}{M}[\subset I$. For every $h \geq MN$ we define $u_h \in SBV(I)$ by setting

$$u_h(0+) = u(0+), \quad \text{and} \quad \dot{u}_h = w_h dt + \sum_{j=0}^{N}(a_j^h - b_j^h)\delta_{\left(t_0 - \frac{N-j}{h}\right)},$$

where

$$w_h(t) = \begin{cases} \dot{u}_a(t) & \text{if } t < t_0 - \frac{1}{M} \\[2mm] \dot{u}_a(t) + \frac{Mh}{h-MN}(b_0^h - u(t_0-)) & \text{if } t_0 - \frac{1}{M} \leq t < t_0 - \frac{N}{h} \\[2mm] (b_j^h - a_{j-1}^h)h & \text{if } t_0 - \frac{N-j+1}{h} \leq t < t_0 - \frac{N-j}{h} \\[2mm] \dot{u}_a(t) + M(u(t_0+) - a_N^h) & \text{if } t_0 \leq t < t_0 + \frac{1}{M} \\[2mm] \dot{u}_a(t) & \text{if } t \geq t_0 + \frac{1}{M}. \end{cases}$$

Remark that $u_h(t) = u(t)$ for $|t_0 - t| > \frac{1}{M}$ for all $h \in \mathbb{N}$. We have then $u_h \rightharpoonup u$ in $BV\text{-}w^*$, and $\overline{G}(u) = \lim_h G(u_h)$. It is clear that in the same way we can treat the general case of more than one jump. $\qquad\square$

Remarks 4.5.   (On subadditive functions and subadditive envelopes) We consider now the special case when

$$\phi(s, t) = \varphi(s - t).$$

In this case it is immediate to check that ϕ is subadditive iff we have

(4.6) $\varphi(x + y) \leq \varphi(x) + \varphi(y)$

for all $x, y, \in \mathbb{R}$. If φ verifies (4.6) we will again say then that it is subadditive. Similarly, we will denote by $sub\,\varphi$ the *subadditive envelope* of φ; i.e.; the greatest function less then or equal to φ verifying (4.6):

$$sub\,\varphi(x) = \inf\Big\{\sum_{k=1}^{m}\varphi(x_k) : \sum_k x_k = x, m = 1, 2, \ldots\Big\},$$

and in the same way

$$\overline{sub}\,\varphi(x) = sub\,(sc^-(\varphi))(x).$$

Notice that $\overline{sub}\,\phi(s, t) = \overline{sub}\,\varphi(s - t)$.

4.5.1. If φ is L-Lipschitz, then so is also $sub\,\varphi$. In fact, let $s, t \in \mathbb{R}$; for every $\eta > 0$ there exist $t_1, \ldots, t_m$ such that $\sum t_j = t$, and $\sum \varphi(t_j) \leq sub\,\varphi(t) + \eta$. Let us define then $s_j = t_j + \frac{s-t}{m}$, so that we have $\sum s_j = s$, and

$$sub\,\varphi(s) \leq \sum \varphi(s_j) \leq \sum \varphi(t_j) + L|t - s| \leq sub\,\varphi(t) + L|t - s| + \eta.$$

This shows that $sub\,\varphi(s) \leq sub\,\varphi(t) + L|t - s|$. In the same way we obtain $sub\,\varphi(t) \leq sub\,\varphi(s) + L|t - s|$.

4.5.2. If φ is convex then $sub\,\varphi$ can be computed more easily:

$$sub\,\varphi(x) = \inf\{k\varphi(\frac{x}{k}) : k = 1, 2, \ldots\}.$$

This follows immediately by the convexity inequality $k\varphi(\frac{y}{k}) \leq \sum_{j=1}^{k} \varphi(y_j)$, whenever $y = \sum_j y_j$.

4.5.3. If φ is subadditive and locally bounded, than it grows less than linearly at infinity. In fact, for every $y \in \mathbb{R}$ we have

$$\varphi(y) \leq (1 + [|y|])\, \varphi\Big(\frac{y}{1 + [|y|]}\Big),$$

so that $\varphi(y) \leq \sup\{\varphi(t) : |t| \leq 1\}(1 + |t|)$ ([t] is the greatest integer less than or equal to t).

4.5.4. It may happen that $\varphi(t) > 0$ for every $t \in \mathbb{R}$ but there is no $C > 0$ such that $\varphi(t) \geq C(|t| - 1)$ (*e.g.* $\varphi(t) = \log(1 + e^t)$). If we have $\varphi \geq 0$, and $\lim\limits_{t \to +\infty} \varphi(t) = \lim\limits_{t \to -\infty} \varphi(t) = 0$, we must have $\varphi \equiv 0$; in fact $0 \leq \varphi(t) \leq \lim\limits_{x \to +\infty} (\varphi(t - x) + \varphi(x)) = 0$ for all $t \in \mathbb{R}$.

4.5.5. We say that $\psi :]0, +\infty[\to \mathbb{R}$ is *positively subadditive* if we have

$$\psi(a + b) \leq \psi(a) + \psi(b) \text{ for all } a, b \in]0, +\infty[.$$

Notice that if ψ is decreasing and non-negative, then it is positively subadditive. We remark also that if $\psi \geq 0$ is positively subadditive, then the map defined by $\varphi(t) = \psi(|t|)$ for $t \neq 0$, and $\varphi(0) = t_0$ (with the condition $0 \leq t_0 \leq \liminf\limits_{t \to 0+} \psi(t)$) may not be subadditive on the whole $\mathbb{R}$ (but it is if in addition ψ is increasing). Moreover, if $\varphi : \mathbb{R} \to \mathbb{R}$ is such that both $t \mapsto \varphi(t)$ and $t \mapsto \varphi(-t)$ are positively subadditive, then we have

$$sub\,\varphi(t) = \varphi(t) \wedge \inf\{\varphi(x) + \varphi(t - x) : x \in \mathbb{R}\}.$$

Examples 4.6. 4.6.1 If we take $\varphi(t) = 1 + t^2$, it is immediate to see that φ is not subadditive (for example by Remark 4.5.3 above). Since φ is convex we have, by Remark 4.5.2,

$$sub\,\varphi(t) = \min\{k + \frac{t^2}{k} : k = 1, 2, \ldots\}.$$

Note that φ is not Lipschitz continuous, while $sub\,\varphi$ is Lipschitz continuous but not C^1. Let us remark also that $sub\,\varphi$ is asymptotic to $2|t|$ as $t \to \pm\infty$.

4.6.2. If $\varphi(t) = (2|t| - 1) \vee 1$, then by 4.5.2 $sub\,\varphi$ is even and continuous, and in $[0, +\infty[$ we have

$$sub\,\varphi(t) = \begin{cases} 1 & \text{if } t \leq 1 \\ k + 2(t - k) & \text{if } k \leq t \leq k + \frac{1}{2}, \ k = 1, 2, \ldots \\ k & \text{if } k - \frac{1}{2} \leq t \leq k, \ k = 2, 3, \ldots \end{cases}$$

In this case we have

$$|t| \leq sub\,\varphi(t) \leq |t| + \frac{1}{2}$$

for $|t| \geq \frac{1}{2}$. We have $sub\,\varphi(t) = |t|$ for $t = \pm 1, \pm 2, \ldots$, $sub\,\varphi(t) = |t| + \frac{1}{2}$ for $t = \pm\frac{1}{2}, \pm\frac{3}{2}, \ldots$, and hence $sub\,\varphi$ is not asymptotic to a linear function as $t \to \pm\infty$.

4.6.3. If $\varphi(t) = |t - 1|$, then by 4.5.2 we have $sub\,\varphi(t) = \min\{|t - k| : k = 1, 2, \ldots\}$; *i.e.*,

$$sub\,\varphi(t) = \begin{cases} 1 - t & \text{if } t \leq 1 \\ \text{dist}(t, \mathbb{N}) & \text{if } t \geq 1. \end{cases}$$

Notice that in this case the limit $\lim\limits_{t \to +\infty} sub\,\varphi(t)$ does not exist.

4.6.4. If $\varphi(t) = ||t| - 1|$, then we have $sub\,\varphi(t) = \text{dist}(t, \mathbb{Z})$. In fact, it is easy to check that $\overline{\varphi}(t) = \text{dist}(t, \mathbb{Z})$ is subadditive, and hence $\overline{\varphi}(t) \leq sub\,\varphi(t)$. Moreover $sub\,\varphi(t) = 0$ on $\mathbb{Z}$ (since $sub\,\varphi(0) \leq \varphi(1) + \varphi(-1) = 0$, and $sub\,\varphi(\pm k) \leq k\varphi(\pm 1) = 0$, for $k = 1, 2, \ldots$), and hence by 4.5.1 we have also $sub\,\varphi(t) \leq \overline{\varphi}(t)$.

4.6.5. A continuous subadditive function need not be uniformly continuous: take for example

$$\varphi(t) = 3 + \sin(t^2).$$

The next proposition deals with the relaxation of *confined problems*; *i.e.*, with constraints of the type

$$\|u\|_\infty \leq M.$$

Proposition 4.7. *(Relaxation of confined problems in SBV(I)) Let $M > 0$, and let $\phi : [-M, M]^2 \to [0, +\infty[$ be a Borel function with $\inf \phi > 0$. Then the relaxation of the functional*

$$G(u) = \begin{cases} \displaystyle\int_I g(\dot{u}_a)dt + \int_{S_u \cap I} \phi(u(t+), u(t-))\,d\#(t) \\ \qquad \text{if } u \in SBV(I),\ \#(S_u) < +\infty \text{ and } \|u\|_\infty \leq M, \\ +\infty \qquad \text{otherwise} \end{cases}$$

with respect to the topology of $L^1(I)$ is given by

$$\overline{G}(u) = \begin{cases} \displaystyle\int_I g(\dot{u}_a)dt + \int_{S_u \cap I} \overline{\phi}(u(t+), u(t-))d\#(t) \\ \qquad \text{if } u \in SBV(I),\ \#(S_u) < +\infty, \text{ and } \|u\|_\infty \leq M, \\ +\infty \qquad \text{otherwise,} \end{cases}$$

where we have defined $\overline{\phi}$ by setting

$$\overline{\phi}(x, y) = \inf \sum_{k=1}^m sc^- \phi(x_k, x_{k-1})$$

where the infimum is taken over all finite sequences $x_1, \ldots, x_m$ with $x_0 = y$, $x_m = x$, $|x_k| \leq M$ for all $k = 1, \ldots, m$.

Proof. The proof is completely analogous to the proof of Proposition 4.4; it suffices to remark that if we take the sequence (u_h) constructed in Proposition 4.4, and define $v_h = (-M) \vee u_h \wedge M$, then this sequence verifies $\overline{G}(u) = \lim_h G(v_h)$. □

Remark 4.8. We can consider equivalently the jump-part energy density θ given by

$$\theta(s, u) = \phi(u + s, u).$$

We can then describe the relaxed functional by means of the function

$$\overline{\theta}(s, u) = \overline{\phi}(u + s, u) = \inf \sum_{k=1}^{m} \mathrm{sc}^{-}\theta(x_k - x_{k-1}, x_{k-1}),$$

for $|u| \leq M, |u+s| \leq M$, where the infimum is taken over all finite sequences $x_1, \ldots, x_m$ with $x_0 = u$, $x_m = u + s$, $|x_k| \leq M$ for all $k = 1, \ldots, m$.

Remark 4.9. Notice that even when we have $\theta(s, u) = \varphi(s)$, the function $\overline{\theta}$ depends in general on both jump and trace. We give a simple example: consider $M = 1$, and the function

$$\varphi(s) = \begin{cases} 3 - |s| & \text{if } |s| \leq 2 \\ 1 & \text{if } |s| \geq 2. \end{cases}$$

Notice that φ is not subadditive (for example $\varphi(0) = 3 > \varphi(2) + \varphi(-2) = 2$). It is easy to see that

$$\overline{\theta}(s, u) = \varphi(s) \wedge \inf \left\{ \varphi(x - u) + \varphi(u + s - x) : |x| \leq 1 \right\},$$

for every $u \in [-1, 1]$, $s \in [-1 - u, 1 - u]$ (see Remark 4.5.5). An easy computation yields

$$\overline{\theta}(s, u) = \min\{3 - |s|, 4 - 2u - s, 4 + 2u + s\},$$

and

$$\overline{\phi}(v, u) = \min\{3 - |v - u|, 4 - u - v, 4 + u + v\}.$$

In particular we have

$$\overline{\theta}(0, u) = \overline{\phi}(u, u) = \begin{cases} 3 & \text{if } |u| \leq \frac{1}{2} \\ 4 - 2|u| & \text{if } \frac{1}{2} \leq |u| \leq 1. \end{cases}$$

5. Development by Γ-convergence

We turn our attention now to the description of the effect of the singular perturbation introduced in Proposition 3.4. We shall focus our attention on the minima of (relaxed) Dirichlet boundary value problems. We fix $\alpha \in \mathbb{R}$, and we define (as already done for the functional H in (3.3)) the functionals

$$
F_\varepsilon^\alpha(u) = \begin{cases} \displaystyle\int_I g(\dot{u}_a)dt + \int_{S_{u^*}} \phi_\varepsilon(u^*(t+), u^*(t-))d\#(t) \\ \qquad\qquad\qquad \text{if } u \in SBV(I), \text{ and } \#(S_u) < +\infty, \\ \\ +\infty \qquad\qquad\qquad \text{otherwise}, \end{cases}
$$

where (ϕ_ε) are as in Proposition 3.4. We recall that the function $u^* \in BV_{\mathrm{loc}}(\mathbb{R})$ is obtained by extending u to 0 in $]-\infty, 0]$ and to α in $]1, +\infty[$. For instance if $u \in SBV(I)$, $\#(S_u) < +\infty$, $u(0+) \neq 0$ and $u(1-) \neq \alpha$ we get

$$
F_\varepsilon^\alpha(u) = \int_I g(\dot{u}_a)dt + \int_{S_u \cap I} \phi_\varepsilon(u(t+), u(t-))d\#(t)
$$

$$
+ \phi_\varepsilon(u(0+), 0) + \phi_\varepsilon(\alpha, u(1-)).
$$

It is easy to see, as in Proposition 3.4, that the Γ-limit of these functionals is simply given by

$$
H^\alpha(u) = H(u) + |u(0+)| + |\alpha - u(1-)| \qquad \text{for every } u \in BV(I).
$$

In Proposition 3.2 we have described the minimum points for the functional H^α, showing that the minimum value is given by $f(\alpha)$. In order to describe the effect of the Γ-convergence of F_ε^α to H^α, we shall study the behavior as $\varepsilon \to 0$, of the functionals

$$
(5.1) \qquad\qquad \frac{1}{\varepsilon}\big(F_\varepsilon^\alpha(u) - f(\alpha)\big).
$$

The Γ-limit of these functionals —let us call it H_1^α— (if it exists) represents some sort of first order *development by* Γ-*convergence* of the functional H^α. This concept was introduced by G.Anzellotti and S.Baldo [14]; we refer to their a for more examples and a complete introduction to the subject. It is immediate to see that $H_1^\alpha(u) = +\infty$ whenever $H^\alpha(u) \neq f(\alpha)$, and hence it is finite only on minimizers for H^α; moreover the limits of minimizers for F_ε^α are exactly the (minimizers for H^α which are also) minimizers for H_1^α. Hence this Γ-limit describes precisely the effect of the introduction of F_ε^α.

In view of the lower semicontinuity and relaxation results presented in Section 4, we are going to make some additional hypotheses on the function

ϕ_ε in order to ensure that the functionals F_ε^α are really effective: we would like the minimizers of the approximating functionals to belong to $SBV(I)$, and have some compactness properties. We shall suppose then that ϕ_ε verifies an estimate of the type

$$(5.2) \qquad \phi_\varepsilon(x,y) \geq (1 - L\varepsilon)|x - y| + \varepsilon.$$

This inequality allows us to give a lower bound for the functional H_1^α, and in particular to restrict its effective domain. It is sufficient to treat the case when in (5.2) the equality holds, as in the following proposition.

Proposition 5.1. *Let us suppose that* $\phi_\varepsilon(x,y) = \varphi_\varepsilon(x - y)$, *where*

$$\varphi_\varepsilon(t) = (1 - L\varepsilon)|t| + \varepsilon,$$

for some positive constant L independent of ε. Then a necessary and sufficient condition for $H_1^\alpha(u)$ to be finite is that
(5.3)
$$H^\alpha(u) = f(\alpha), \quad u \in SBV(I), \quad \#(S_u) < +\infty \quad and \quad \dot{u}_a(t) \in J \ for \ a.e. \ t.$$

Proof. Let us remark that the case $\alpha \in J$ is trivial since we have a unique minimizer, on which H_1^α is 0. Then we shall suppose throughout the proof that $\alpha \notin J$; for instance let us consider the case $\alpha > b$ (the case $\alpha < a$ being analogous).

The sufficiency of (5.3) is trivial: we can take $u_\varepsilon = u$ for every $\varepsilon > 0$, obtaining

$$H_1^\alpha(u) \leq \liminf_{\varepsilon \to 0} \frac{1}{\varepsilon}\big(F_\varepsilon^\alpha(u) - f(\alpha)\big) \leq \sum_{S_{u^*}}(1 - L|u^*(t+) - u^*(t-)|).$$

To show that condition (5.3) is necessary let us consider, for $k = 1, 2 \ldots$, the minimum problems

$$m_{\varepsilon k}^\alpha = \min\{F_\varepsilon^\alpha(u) \ : \ u \in SBV(I), \ \#(S_{u^*}) = k\}.$$

In order to calculate this minimum value, as in the proof of Proposition 3.2, we can limit our analysis to the case $\dot{u}_a = C$, a constant, and $\dot{u}_s \in \mathcal{M}^+(I)$. Since $u \in SBV(I)$ the latter condition means that all the jumps of u are upwards. By the convexity of φ_ε (when $\varepsilon \leq \frac{1}{L}$) we can suppose that all the k jumps are equal; hence we have

$$m_{\varepsilon k}^\alpha = \min\Big\{g(C) + k\varphi_\varepsilon\big(\frac{\alpha - C}{k}\big) \ : \ 0 \leq C \leq \alpha\Big\}.$$

The computation of this minimum problem is trivial, and we have:

$$(5.4) \qquad m^\alpha_{\varepsilon k} = g(C^\alpha_{\varepsilon k}) - C^\alpha_{\varepsilon k} + \alpha + \varepsilon(k - L(\alpha - C^\alpha_{\varepsilon k})),$$

where the minimizer $C^\alpha_{\varepsilon k}$ is uniquely determined, because of the strict convexity of g, by the requirement that $(1 - L\varepsilon)$ belongs to the subdifferential of g at $C^\alpha_{\varepsilon k}$. Note that b is the unique positive minimizer of the function $g(t) - t$ and that $C^\alpha_{\varepsilon k}$ converges to b as $\varepsilon \to 0$.

Let us consider now a function $u \in BV(I)$ such that $H^\alpha_1(u) < +\infty$; clearly $u \in SBV(I)$. Moreover, as remarked above, u must be a minimizer for H^α. If $\varepsilon_h \to 0$, $u_h \to u$ in $L^1(I)$ and

$$\lim_h \frac{1}{\varepsilon_h}\big(F^\alpha_{\varepsilon_h}(u_h) - f(\alpha)\big) < +\infty,$$

then we must have

$$F^\alpha_{\varepsilon_h}(u_h) \le \varepsilon_h c + f(\alpha) = g(b) - b + \alpha + \varepsilon_h c.$$

By (5.4) we have then, setting $k_h = \#(S_{u^*_h})$,

$$g(C^\alpha_{\varepsilon_h k_h}) - C^\alpha_{\varepsilon_h k_h} + \alpha + \varepsilon_h(\#(S_{u^*_h}) - L(\alpha - C^\alpha_{\varepsilon_h k_h}) = m^\alpha_{\varepsilon_h k_h}$$

$$\le F^\alpha_{\varepsilon_h}(u_h) \le g(b) - b + \alpha + \varepsilon_h c.$$

Hence we obtain

$$\#(S_{u^*_h}) \le \frac{1}{\varepsilon_h}[g(b) - b - (g(C^\alpha_{\varepsilon_h k_h}) - C^\alpha_{\varepsilon_h k_h})] + c + L(\alpha - C^\alpha_{\varepsilon_h k_h}),$$

and, using the estimate

$$g(b) - b - (g(C^\alpha_{\varepsilon_h k_h}) - C^\alpha_{\varepsilon_h k_h}) \le L\varepsilon_h(b - C^\alpha_{\varepsilon_h k_h}),$$

we conclude

$$\#(S_{u^*_h}) \le c + L(\alpha - C^\alpha_{\varepsilon_h k_h}) \le c.$$

Now, since the sequence (u_h) verifies the hypotheses of Proposition 4.1, we have that $\#(S_u) < +\infty$. Moreover, using again Proposition 4.1, we get that (F is the functional defined in (2.1))

$$F(u) + |u(0+)| + |\alpha - u(1-)|$$

$$\le \liminf_h [\int_I g((\dot u_h)_a)dt + \int_{S_{u_h} \cap I} |(u_h)_+ - (u_h)_-|d\# + |u_h(0+)| + |\alpha - u_h(1-)|]$$

$$\le \lim_h F^\alpha_{\varepsilon_h}(u_h) = f(\alpha) = H(u) + |u(0+)| + |\alpha - u(1-)|.$$

Since $F(u) \ge H(u)$, we must have $F(u) = H(u)$, and hence

$$\int_I g(\dot u_a)dt = \int_I f(\dot u_a)dt.$$

This condition implies that $\dot u_a \in J$ a.e.; *i.e.*, $\dot u_a = b$ a.e. $\square$

Remark 5.2. If, as a particular case of Proposition 5.1, we take the "bulk energy" $g(t) = t^2$, the minimum values in (5.4) are given by

$$m_{\varepsilon k}^\alpha = \alpha - \frac{1}{4} + \varepsilon(k - L(\alpha - \frac{1}{2})) - \frac{1}{4}\varepsilon^2 L^2.$$

We can consider as well the functions

$$\varphi_\varepsilon(t) = |t| + \varepsilon(1 - L|t|)^+.$$

The conclusions of Proposition 5.1 are still valid. Notice that in this case the minimum values in (5.4) are given by

$$m_{\varepsilon k}^\alpha = \begin{cases} \alpha - \frac{1}{4} + \varepsilon(k - L(\alpha - \frac{1}{2})) - \frac{1}{4}\varepsilon^2 L^2 & \text{if } k \geq L(\alpha - \frac{1}{2}) + \varepsilon\frac{L^2}{2} \\[2mm] (\alpha - \frac{k}{L})^2 + \frac{k}{L} & \text{if } L(\alpha - \frac{1}{2}) < k < L(\alpha - \frac{1}{2}) + \varepsilon\frac{L^2}{2} \\[2mm] \alpha - \frac{1}{4} & \text{if } 1 \leq k \leq L(\alpha - \frac{1}{2}). \end{cases}$$

Remark that for ε sufficiently small the second condition on k is empty.

We compute now the Γ-limit of the functionals in (5.1), under the hypothesis that φ_ε admit an asymptotic development as $\varepsilon \to 0$.

Theorem 5.3. *Let us suppose that $\phi_\varepsilon(u, v) = \varphi_\varepsilon(u - v)$, and that there exists a Lipschitz function φ such that $\varphi(0) > 0$ and we have*

$$\varphi_\varepsilon(t) = |t| + \varepsilon\varphi(t) + r(\varepsilon, t)$$

with $|r(\varepsilon, t)| \leq o(\varepsilon)|t|$. Then for every $u \in BV(I)$ there exists the limit

$$H_1^\alpha(u) = \Gamma\text{-}\lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\big(F_\varepsilon^\alpha(u) - f(\alpha)\big).$$

When $\alpha \in J$ then this limit is trivially equal to 0 if $u(t) = \alpha t$, and $+\infty$ otherwise.

Let us consider the case $\alpha \notin J$. If we define $\overline{\varphi}$ as the greatest function less than or equal to φ, such that both functions $\overline{\varphi}(t)$ and $\overline{\varphi}(-t)$ are positively subadditive, then we have

(5.5) $$H_1^\alpha(u) = \int_{S_{u^*}} \overline{\varphi}(u^*(t+) - u^*(t-))d\#(t)$$

when $u \in SBV(I)$, $\#(S_u) < +\infty$, $\alpha > b$, $\dot{u}_a = b$ *a.e. on* I, *and* $\dot{u}_s \in \mathcal{M}^+(I)$
(*respectively* $\alpha < a$, $\dot{u}_a = a$ *a.e. on* I, *and* $-\dot{u}_s \in \mathcal{M}^+(I)$),

$$(5.6) \qquad\qquad H_1^\alpha(u) = +\infty$$

elsewhere in $BV(I)$.

Remark 5.4 The function $\overline{\varphi}$ can be described by setting $\overline{\varphi}(0) = \varphi(0)$, and

$$\overline{\varphi}(t) = \inf\left\{\sum_{k=1}^m \varphi(t_k) : t_k > 0, \sum_{k=1}^m t_k = t, \ m = 1, 2, \dots\right\} \qquad \text{if } t > 0,$$

$$\overline{\varphi}(t) = \inf\left\{\sum_{k=1}^m \varphi(t_k) : t_k < 0, \sum_{k=1}^m t_k = t, \ m = 1, 2, \dots\right\} \qquad \text{if } t < 0.$$

We can suppose, and we will, that $\varphi(0) = 1$. Let $\eta > 0$ and $M > 0$ be fixed; then, if ε is sufficiently small, we have (L the Lipschitz constant of φ):

i) φ_ε is increasing on $]0, M[$ (and decreasing on $]-M, 0[$). In fact we have for $t, s > 0$

$$\varphi_\varepsilon(t+s) - \varphi_\varepsilon(t) = s + \varepsilon\big(\varphi(t+s) - \varphi(t)\big) + o(\varepsilon)M \geq s + o(\varepsilon)M - \varepsilon L s > 0;$$

ii) for all $t \in \mathbb{R}$

$$\varphi_\varepsilon(t) \geq (1 - (L+\eta)\varepsilon)|t| + \varepsilon;$$

iii) for all $t \in \mathbb{R}$

$$\varphi_\varepsilon(t) \leq (1 + (L+\eta)\varepsilon)|t| + \varepsilon.$$

Proof of Theorem 5.3 By Remark 5.4 (ii), we can apply Proposition 5.1 and obtain (5.6) (note that by Proposition 3.2 if $u \in SBV(I)$ is a minimum point for H^α and $\dot{u}_a \in J$ a.e., then actually $\dot{u}_a = b$ a.e. if $\alpha > b$, and $\dot{u}_a = a$ a.e. if $\alpha < a$).

Let us consider the case $\alpha > b$ (the case $\alpha < a$ being analogous). We have to compute our Γ-limit for all $u \in SBV(I)$ such that $\dot{u}_a = b$ a.e., $u(0+) \geq 0$, $u(1-) \leq \alpha$, $\#(S_u) < +\infty$, and $u(t+) > u(t-)$ for every $t \in S_u$.

Let us take a sequence $\varepsilon_h \to 0$, and $u_h \in SBV(I)$ with $u_h \to u$ in $L^1(I)$ such that

$$(5.7) \qquad\qquad \lim_h \frac{1}{\varepsilon_h}\big(F_{\varepsilon_h}^\alpha(u_h) - f(\alpha)\big) < +\infty.$$

Then, it is easy to see that we have $\|(\dot{u}_h)_a\|_2 \leq c$, $\#(S_{u_h}) \leq c$, $\|u_h\|_{BV} \leq c$.

In the rest of the proof we shall indicate with v^* the extension to $BV_{\text{loc}}(\mathbb{R})$ of a function $v \in BV(I)$, obtained by setting $v(t) = 0$ for $t \leq 0$, and $v(t) = \alpha$ for $t \geq 1$. We will also denote by $S_{v^*}^-$ the set $\{t \in [0,1] : v^*(t+) < v^*(t-)\}$; *i.e.*, the points where we have a downwards jump for v^*.

Let $\xi_h = \int_I (\dot{u}_h)_a \, dt$. Remembering that

$$\alpha = \xi_h + \sum_{t \in S_{u_h^*} \setminus S_{u_h^*}^-} |u_h^*(t+) - u_h^*(t-)| - \sum_{t \in S_{u_h^*}^-} |u_h^*(t+) - u_h^*(t-)|,$$

we get, using Jensen's inequality and the development of φ_ε, that

$$F_{\varepsilon_h}^\alpha(u_h) - f(\alpha)$$

$$\geq g(\xi_h) - (g(b) - b + \alpha) + \sum_{t \in S_{u_h^*}} |u_h^*(t+) - u_h^*(t-)|$$

$$+ \varepsilon_h \sum_{t \in S_{u_h^*}} \varphi(u_h^*(t+) - u_h^*(t-)) - o(\varepsilon_h)\|u_h\|_{BV}$$

$$\geq g(\xi_h) - (g(b) - b + \xi_h) + 2 \sum_{t \in S_{u_h^*}^-} |u_h^*(t+) - u_h^*(t-)|$$

$$+ \varepsilon_h \sum_{t \in S_{u_h^*}} \varphi(u_h^*(t+) - u_h^*(t-)) - o(\varepsilon_h).$$

Hence, recalling that $g(t) - (g(b) - b + t) \geq 0$ for every t, by (5.7) we must have

$$(5.8) \qquad \sum_{t \in S_{u_h^*}^-} |u_h^*(t+) - u_h^*(t-)| \leq c\varepsilon_h.$$

Now let $\bar{t} \in S_{u^*}$. We can suppose that $\bar{t}$ is the limit of exactly N sequences $(t_h^1), \ldots, (t_h^N)$ with $t_h^j \in S_{u_h^*}$ (see the proof of Proposition 4.1). If some of these points, say $t_h^1, \ldots, t_h^{k_h}$ belong to $S_{u_h^*}^-$, then we can define $v_h \in SBV(I)$ as follows. We choose one of the points of the remaining sequences, say t_h^N, such that

$$u_h^*(t_h^N +) - u_h^*(t_h^N -) \geq \frac{u^*(\bar{t}+) - u^*(\bar{t}-)}{N}$$

and we set

$$\dot{v}_h^* - \dot{u}_h^* = \sum_{j=1}^{k_h}(u_h^*(t_h^j+) - u_h^*(t_h^j-))\delta_{t_h^N} - \sum_{j=1}^{k_h}(u_h^*(t_h^j+) - u_h^*(t_h^j-))\delta_{t_h^j},$$

and $v_h(t) = 0$ for $t \leq 0$. Note that we still have $v_h \to u$ in $L^1(I)$, and that for h sufficiently large

$$u_h^*(t_h^N+) - u_h^*(t_h^N-) + \sum_{j=1}^{k_h}(u_h^*(x_h^j+) - u_h^*(x_h^j-)) > 0$$

by (5.8). By i) of Remark 5.4 we get that the sequence (v_h) verifies, for h sufficiently large, the inequality

$$\sum_{j=k_h+1}^{N} \varphi_\varepsilon(v_h^*(t_h^j+) - v_h^*(t_h^j+))$$

$$= \sum_{j=k_h+1}^{N-1} \varphi_\varepsilon(v_h^*(t_h^j+) - v_h^*(t_h^j+))$$

$$+ \varphi_\varepsilon\left(u_h^*(t_h^N+) - u_h^*(t_h^N-) + \sum_{j=1}^{k_h}(u_h^*(t_h^j+) - u_h^*(t_h^j-))\right)$$

$$\leq \sum_{j=k_h+1}^{N-1} \varphi_\varepsilon(u_h^*(t_h^j+) - u_h^*(t_h^j+)) + \varphi_\varepsilon(u_h^*(t_h^N+) - u_h^*(t_h^N-)).$$

Hence we obtain, for h sufficiently large,

$$\sum_{j=1}^{N}\varphi_\varepsilon(v_h^*(t_h^j+) - v_h^*(t_h^j+)) \leq \sum_{j=1}^{N}\varphi_\varepsilon(u_h^*(t_h^j+) - u_h^*(t_h^j+)).$$

Since $\#(S_{u^*}) < +\infty$, we can repeat this procedure for every $t \in S_{u^*}$, obtaining a sequence (w_h) such that $\dot{w}_h^* \in \mathcal{M}^+(I)$, $w_h \to u$ in $L^1(I)$, and

$$\lim_h \frac{1}{\varepsilon_h}\big(F_{\varepsilon_h}^\alpha(u_h) - f(\alpha)\big) \geq \lim_h \frac{1}{\varepsilon_h}\big(F_{\varepsilon_h}^\alpha(w_h) - f(\alpha)\big).$$

We can then suppose $S^-_{u^*_h} = \varnothing$, and

$$F^\alpha_{\varepsilon_h}(u_h) - f(\alpha) \geq g(\xi_h) - (g(b) - b + \xi_h) + \varepsilon_h \sum_{t \in S_{u^*_h}} \varphi(u^*_h(t+) - u^*_h(t-)) - o(\varepsilon_h).$$

Recalling that φ is continuous and $\overline{\varphi}$ is positively subadditive, it is easy to see that

$$\liminf_h \sum_{t \in S_{u^*_h}} \varphi(u^*_h(t+) - u^*_h(t-)) \geq \sum_{t \in S_{u^*}} \overline{\varphi}(u^*(t+) - u^*(t-)),$$

and hence it is proven that for the functional H^α_1 defined in (5.5) we have

$$H^\alpha_1(u) \leq \lim_h \frac{1}{\varepsilon_h}\big(F^\alpha_{\varepsilon_h}(u_h) - f(\alpha)\big).$$

Let us construct now a recovery sequence (u_h) such that

$$H^\alpha_1(u) \geq \limsup_h \frac{1}{\varepsilon_h}\big(F^\alpha_{\varepsilon_h}(u_h) - f(\alpha)\big).$$

As in the proof of Proposition 4.4 we can consider the case of a single jump; *i.e.*, we can suppose that

$$\dot{u}^*_s = (\alpha - b)\delta_{t_0} \quad \text{for some } t_0 \in [0,1].$$

We will suppose $t_0 \neq 1$ (if $t_0 = 1$ the same proof is valid with obvious changes). By Remark 5.4 for every $h \in \mathbb{N}$ there exist $t^h_1 \ldots, t^h_{m_h} \in]0, +\infty[$ such that

$$\alpha - b = \sum_{k=1}^{m_h} t^h_k, \quad \text{and} \sum_{k=1}^{m_h} \varphi(t^h_k) \leq \overline{\varphi}(\alpha - b) + \frac{1}{h}.$$

Let us consider $M > 0$ such that $t_0 + \frac{1}{M} < 1$. Then for $h \geq M$ we define the sequence $u_h \in SBV(I)$ by setting $u_h(t) = 0$ for $t \leq 0$,

$$(\dot{u}_h)_a = b,$$

and

$$(\dot{u}^*_h)_s = \sum_{k=1}^{m_h} t^h_k \delta_{t_0 + \frac{k-1}{h m_h}}.$$

We have $u_h \rightharpoonup u$ in BV-w^*, and

$$F^\alpha_{\varepsilon_h}(u_h) = g(b) + \sum_{k=1}^{m_h} \varphi_{\varepsilon_h}(t^h_k) \leq g(b) - b + \alpha + \varepsilon_h \sum_{k=1}^{m_h} \varphi(t^h_k) + (\alpha - b)o(\varepsilon_h)$$

$$\leq g(b) - b + \alpha + \varepsilon_h \overline{\varphi}(\alpha - b) + \frac{\varepsilon_h}{h} + (\alpha - b)o(\varepsilon_h).$$

This inequality implies that

$$\overline{\varphi}(\alpha - b) \geq \limsup_h \frac{1}{\varepsilon_h}\big(F^\alpha_{\varepsilon_h}(u_h) - f(\alpha)\big),$$

and hence the thesis (recall that $H^\alpha_1(u) = \overline{\varphi}(\alpha - b)$). □

Remark 5.5. Theorem 5.3 is still valid if we consider functions of the form

$$\phi_\varepsilon(u,v) = |u-v| + \varepsilon\phi(u,v) + r(\varepsilon,u,v),$$

with ϕ Lipschitz, $\phi(u,u) \geq c$, and $|r(\varepsilon,u,v)| \leq o(\varepsilon)|u-v|$. In this case we have

$$H_1^\alpha(u) = \int_{S_{u^*}} \overline{\phi}(u^*(t+), u^*(t-))d\#(t)$$

where the function $\overline{\phi}$ is given by

$$\overline{\phi}(x,y) = \inf\{\sum_{k=1}^m \phi(x_k, x_{k-1}) : x_0 = y, x_m = x, x_0 < x_1 < \ldots < x_m,$$

$$m = 1, 2, \ldots\}.$$

for $x > y$, and by an analogous formula for $x < y$. The proof is the same except in that we have to follow the construction of Proposition 4.4 in order to build a recovery sequence for the Γ-limsup.

Examples 5.6.   5.6.1 Let $\psi :]0,+\infty[\to [0,+\infty[$ be a Lipschitz positively subadditive function, with $\lim_{t\to 0+} \psi(t) > 0$, and let $\varphi_\varepsilon(t) = |t| + \varepsilon\psi(|t|)$ $(t \neq 0)$. Then we have

$$H_1^\alpha(u) = \int_{S_{u^*}} \psi(|u^*(t+) - u^*(t-)|)d\#(t).$$

This remark applies to the functions $\varphi_\varepsilon(t) = |t| + \varepsilon(1 - L|t|)^+$ considered in Remark 5.2, and in particular to the case $\varphi_\varepsilon = |t| + \varepsilon$ ($\psi \equiv 1$), where $H_1^\alpha(u) = \#(S_{u^*})$.

5.6.2 We can consider the functions

$$\varphi_\varepsilon(t) = |t| + \varepsilon(1 - |t|)^2.$$

These functions do not verify the hypotheses of Theorem 5.3 ($t \mapsto (1-|t|)^2$ is not Lipschitz continuous). We can use then Proposition 4.4 and notice that the Γ-limit will not be modified if we consider, in place of φ_ε, the functions $sub(\varphi_\varepsilon)$. By Remark 4.5.2 we have (for small ε)

$$sub(\varphi_\varepsilon)(t) = |t| + \varepsilon\min\{\frac{1}{k}(|t| - k)^2 : k = 1, 2, \ldots\},$$

and hence

$$H_1^\alpha(u) = \int_{S_{u^*}} \overline{\varphi}(|u^*(t+) - u^*(t-)|)d\#(t),$$

where

$$\overline{\varphi}(t) = \min\{\frac{1}{k}(t-k)^2 : k = 1, 2, \ldots\}.$$

5.6.3 If we take

$$\varphi_\varepsilon(t) = \frac{1}{\varepsilon}t^2 + \frac{\varepsilon}{4},$$

then, as in the previous example, we have to consider

$$sub\,\varphi_\varepsilon(t) = |t| + \varepsilon \min\left\{\frac{1}{k}\left(\frac{k}{2} - \frac{|t|}{\varepsilon}\right)^2 : k = 1, 2, \ldots\right\}.$$

In this case it is easy to see that, since we have $sub\,\varphi_\varepsilon(t) = |t|$ for $t = k\frac{\varepsilon}{2}$, the hypotheses of Theorem 5.3 and Proposition 5.1 are not verified, and the limit functional H_1^α is zero on all minimum points for (3.3).

Theorem 5.7. *Let us fix $\alpha \in \mathbb{R}$. Let (u_ε) be a sequence in $SBV(I)$ such that*

$$(5.9) \qquad F_\varepsilon^\alpha(u_\varepsilon) \leq \inf\{F_\varepsilon^\alpha(v) : v \in BV(I)\} + o(\varepsilon).$$

Then, for every sequence (ε_h) of positive numbers converging to 0, there exist a subsequence, that we still denote by (ε_h), and a function $u \in SBV(I)$ with $\#(S_u) < +\infty$, such that $u_{\varepsilon_h} \rightharpoonup u$ in $BV\text{-}w^*$, $(\dot{u}_{\varepsilon_h})_a$ converges to $\dot{u}_a$ weakly in $L^2(I)$, and

$$H^\alpha(u) = \min\{H^\alpha(v) : v \in BV(I)\} = m_\alpha,$$

$$H_1^\alpha(u) = \min\{H_1^\alpha(v) : v \in BV(I)\} = m_\alpha^1.$$

Moreover,

$$(5.10) \qquad F_{\varepsilon_h}^\alpha(u_{\varepsilon_h}) = m_\alpha + \varepsilon_h m_\alpha^1 + o(\varepsilon_h).$$

Proof. Let us fix a sequence (ε_h) of positive numbers converging to 0. By the equicoerciveness of the sequence (F_ε^α), we can find a further subsequence (u_{ε_h}) such that $u_{\varepsilon_h} \rightharpoonup u$ for some $u \in BV(I)$. By (4.2) and (4.3) of Chapter 1, from (5.9) it follows that u is a minimum point for H^α. We can suppose moreover that there exists the limit (possibly infinite)

$$\lim_{h \to +\infty} \frac{1}{\varepsilon_h}(F_{\varepsilon_h}^\alpha(u_{\varepsilon_h}) - m_\alpha).$$

Let us show now that u minimizes also H_1^α; *i.e.*, for every $v \in BV(I)$ we have $H_1^\alpha(v) \geq H_1^\alpha(u)$. Proposition 5.1 shows that it suffices to prove this inequality for all $v \in SBV(I)$ with $\#(S_v) < +\infty$ and $H^\alpha(v) = m_\alpha$. By Theorem 5.3 and (4.1) of Chapter 1, there exists a sequence $v_h \to v$ in $L^1(I)$ such that

$$H_1^\alpha(v) = \lim_h \frac{1}{\varepsilon_h}(F_{\varepsilon_h}^\alpha(v_h) - m_\alpha).$$

Hence by (5.9) we obtain

$$H_1^\alpha(u) \leq \lim_{h \to +\infty} \frac{1}{\varepsilon_h}(F_{\varepsilon_h}^\alpha(u_{\varepsilon_h}) - m_\alpha)$$

(5.11)

$$\leq \lim_{h \to +\infty} \left(\frac{1}{\varepsilon_h}(F_{\varepsilon_h}^\alpha(v_h) - m_\alpha) + \frac{o(\varepsilon_h)}{\varepsilon_h} \right) = H_1^\alpha(v).$$

This inequality shows that $H_1^\alpha(u) = m_\alpha^1$. If we choose $v = u$ in (5.11) we obtain that

$$H_1^\alpha(u) = \lim_{h \to +\infty} \frac{1}{\varepsilon_h}(F_{\varepsilon_h}^\alpha(u_{\varepsilon_h}) - m_\alpha);$$

i.e.,

$$m_\alpha^1 = \frac{1}{\varepsilon_h}(F_{\varepsilon_h}^\alpha(u_{\varepsilon_h}) - m_\alpha) + o(1).$$

Hence we obtain (5.10).   □

CHAPTER 3:

ON THE BENDING OF A ROD:
A SINGULAR PERTURBATION APPROACH

A simple one-dimensional model for an elasto-plastic bar parametrized on a bounded open interval I is given by an energy of the form

$$\mathcal{G}(u) = \int_I \varphi(u''(x)) \, dx,$$

where φ is a convex function with linear growth at infinity. Dirichlet boundary value problems for the functional $\mathcal{G}$ in general admit no solutions. Hence we may have minimizing sequences with unbounded number of discontinuities of the first derivative and in the limit we could obtain a diffuse zone of discontinuity for the first derivative.

In order to assure that we obtain solutions whose first derivative's discontinuity set is a finite number of points, we combine a relaxation argument and a singular perturbation method similar to the one described in Chapter 2. This process leads to approximate the functional $\mathcal{G}$ by suitable simpler integrals. We prove that the limits of the minimizers of the approximating functionals are piecewise C^2 functions minimizing $\mathcal{G}$, which are characterized as minimizers also of another functional involving only an appropriate energy density, which can be explicitly computed, on the discontinuity set of the first derivative. We are able not only to describe the qualitative behaviour of such limit functions, but we can also localize exactly their first derivative's discontinuity points.

The results of this chapter are contained in [29].

Introduction

The variational formulation for problems modelling elasto-plastic bars or plates involves functionals depending on the second derivative of the displacement, with linear-growth integrands. In general, minimum problems for such integrals do not possess a classical solution; a "relaxed" solution must be searched for in the framework of functions with Radon measure second derivatives, or with bounded Hessian ([51], [52], [90], [34]).

A simple one-dimensional model for an elasto-plastic bar parametrized on the interval $(0,1)$ is given by the energy

$$\mathcal{G}(u) = \int_0^1 \varphi(u''(x)) \, dx, \qquad \varphi(z) = \begin{cases} z^2 & \text{if } |z| \le 2 \\ 4(|z|-1) & \text{if } |z| > 2. \end{cases}$$

It is easy to check that for some boundary conditions on u and u' the minimum problem for the functional $\mathcal{G}$

$$\min\Big\{\mathcal{G}(u) \ : \ u(0) = u_0, \ u(1) = u_1, \ u'(0) = \xi_0, \ u'(1) = \xi_1\Big\}$$

admits no solution over the space $\mathcal{C}^2(0,1)$ of twice differentiable functions on $(0,1)$. "Weak" solutions can be obtained by relaxing both the functional and the boundary value conditions in the space $BV^2(0,1)$ of the functions $u \in W^{1,1}(0,1)$ whose second derivative is a Radon measure with finite total variation on $(0,1)$. On this space the relaxed energy takes the form

$$\mathcal{H}(u) = \int_0^1 \varphi(\ddot{u}(x)) \, dx + 4|u_s''|(0,1),$$

where $u'' = \ddot{u}\,dx + u_s''$ is the Lebesgue decomposition of u'' in its absolutely continuous and singular parts, and $|u_s''|$ denotes the total variation of the measure u_s''. The direct methods of the calculus of variations apply to such a functional, but in general the solution to a minimum problem involving $\mathcal{H}$ will not be unique and we cannot expect discontinuities of the derivative only on a finite number of points (the minimizers may have a diffuse singular part).

In this chapter we propose a singular perturbation criterion for a choice among minimizers for $\mathcal{H}$. First, we shall show that problems involving $\mathcal{G}$ and $\mathcal{H}$ are in a sense equivalent to minimum problems for a functional of the form

$$\mathcal{F}(u) = \int_0^1 |\ddot{u}(x)|^2 \, dx + 4 \sum_{u'(x+) \neq u'(x-)} |u'(x+) - u'(x-)|,$$

defined on functions $u \in W^{1,1}(0,1)$ which are piecewise C^2; then we shall perturb the functional $\mathcal{F}$ with an additional term by setting

$$\mathcal{F}_\varepsilon(u) = \mathcal{F}(u) + 4 \sum_{u'(x+) \neq u'(x-)} \varepsilon\,\psi(u'(x+), u'(x-)).$$

We prove that there exist minimizers for $\mathcal{F}_\varepsilon$ under very mild conditions on ψ, and that they converge to particular minimum points of $\mathcal{H}$, which are characterized as minimizers of a second functional, of the form

$$\sum_{u'(x+) \neq u'(x-)} \overline{\psi}(u'(x+), u'(x-)),$$

among piecewise C^2 minimizers of $\mathcal{H}$. The function $\overline{\psi}$ can be explicitly computed and does not depend on the boundary values of the problem. The choice criterion is determined by ψ. In the special case when $\psi \equiv 1$ we still have $\overline{\psi} \equiv 1$, and the singular perturbation approach gives exactly the unique piecewise C^2 minimizer of $\mathcal{H}$ with minimum number of discontinuity of the first derivative.

1. Preliminaries

For the notation we refer to Chapter 1, Section 1.

Let $I = (a, b)$ be a bounded open interval of $\mathbb{R}$; in this chapter we shall use the notation dx for the Lebesgue measure, and $\#$ for the counting measure. We denote by $\mathcal{M}(I)$ the set of the scalar Radon measures on I with bounded total variation.

The usual *weak* topology* on $\mathcal{M}(I)$ is defined as the weakest topology on $\mathcal{M}(I)$ for which the maps $\mu \mapsto \int_I \psi \, d\mu$ are continuous for every $\psi \in C(I)$ such that $\psi(a) = \psi(b) = 0$.

With this notation a function $u \in L^1(I)$ is a function of bounded variation (see Chapter 1, Section 2) if there is a measure $\mu \in \mathcal{M}(I)$ such that

$$\int_I u\varphi' dx = -\int_I \varphi \, d\mu \qquad \forall \varphi \in C^1_c(I).$$

We have just observed that $BV(I)$ is a Banach space, if endowed with the BV norm

$$\|u\|_{BV} = \|u\|_1 + |Du|(I).$$

The product topology of the strong topology of $L^1(I)$ for u, and of the weak* topology of measures for u' will be called the weak* topology of BV, and will be denoted by $BV\text{-}w^*$. Recall that for every sequence $(u_h)_h$ in $BV(I)$ with $\|u_h\|_{BV} \leq c$ there exist a subsequence $(u_{h_k})_k$ and a function $u \in BV(I)$ such that $u_{h_k} \to u$ in $L^1(I)$, and $u'_{h_k} \rightharpoonup u'$ in the weak* topology of measures. We shall denote this convergence by $u_{h_k} \rightharpoonup u$ in $BV\text{-}w^*$.

Let $u \in BV(I)$; in the one-dimensional case, if we denote by S_u the complement of the Lebesgue set of u, we can observe that S_u is at most a sequence of points. Furthermore the function u admits right-hand and left-hand traces $u(x+), u(x-)$ at every $x \in I$ in an approximate sense, which means that

$$\lim_{\rho \to 0^+} \frac{1}{\rho} \int_{x-\rho}^{x} |u(t) - u(x-)| dt = 0, \qquad \lim_{\rho \to 0^+} \frac{1}{\rho} \int_{x}^{x+\rho} |u(t) - u(x+)| dt = 0.$$

Since for every $x \in S_u$ we have $u(x-) \neq u(x+)$, it is clear why the set S_u is sometimes referred to as the set of *jump points* of the function u.

We say that a function $u \in L^1(I)$ belongs to $BV^2(I)$ if its second derivative u'' in the sense of distribution is a Radon measure with finite total variation. Theorem 1.8 of [70] and [69] section 6.1.7 imply that $u \in BV^2(I)$ if and only if $u \in W^{1,1}(I)$ and $u' \in BV(I)$; the measure u'' admits the Lebesgue decomposition $u'' = \ddot{u} \, dx + u''_s$.

It is easy to see that $BV^2(I)$ is a Banach space endowed with the norm

$$\|u\|_{BV^2} = \int_I |u| dx + \int_I |u'| dx + |u''|(I).$$

Notice that if $u \in BV^2(I)$, then $\tilde{u}$ is continuous on the closure of I, hence we can take a continuous representative of u and speak about the value $u(x)$ at every $x \in [a, b]$. Moreover the first derivative u' has approximate right-hand and left-hand traces $u'(x+), u'(x-)$ at every $x \in [a, b]$ (only the right-hand one at a and the left-hand one at b) and we have $u'(x+) \neq u'(x-)$ at most for a sequence of points called *crease points* of u, whose set is denoted by $S_{u'}$.

We consider in $BV^2(I)$ the weak* topology $BV^2\text{-}w^*$ defined as the product topology of the strong topology in $W^{1,1}(I)$ for u and of the weak* topology of measures for u''. Recall that every sequence in $BV^2(I)$ with $\|u_h\|_{BV^2} \leq c$ admits a subsequence converging in $BV^2\text{-}w^*$ to a function $u \in BV^2(I)$.

We refer to Chapter 1 (Sections 3 and 4) for the notions and techniques related to the relaxation and Γ-convergence theories, on which most of the proofs in the chapter are based. In this chapter we shall consider relaxations in the $BV^2 - \omega^*$ topology.

The letter c will denote throughout the chapter a strictly positive constant, whose value may vary from line to line, independent from the parameters of the problems each time considered.

Remark 1.1. In this chapter, we shall consider functionals F defined on $BV^2(I)$ for which the estimate

(1.1) $$F(u) \geq |u''|(I) - c$$

holds. Note that, for functionals verifying (1.1), it is equivalent to consider sequences converging with respect to the $L^1(I)$-topology and with respect to the $BV^2\text{-}w^*$ topology (this follows from the fact that in dimension one we can easily prove on the space $BV^2(\Omega)$ an interpolation inequality for the first derivative). Hence throughout the chapter we will feel free to choose the most suited to the context between the two topologies.

2. Some relaxation, semicontinuity, and Γ-convergence results

Let $I = (a, b)$ be a bounded open interval of $\mathbb{R}$; let us consider the functional $\mathcal{G} : BV^2(I) \to [0, +\infty]$ defined as

(2.1) $$\mathcal{G}(u) = \begin{cases} \int_I \varphi(u''(x)) \, dx & \text{if } u \in C^2(I) \\ +\infty & \text{otherwise on } BV^2(I), \end{cases}$$

with φ given by

(2.2) $$\varphi(z) = \begin{cases} z^2 & \text{if } |z| \leq 2\rho \\ 4\rho(|z| - \rho) & \text{if } |z| > 2\rho, \end{cases}$$

where ρ is a positive constant. As pointed out in the introduction the functional $\mathcal{G}$ models the energy of an elasto-plastic bar parametrized on the interval I.

We consider the functional $\mathcal{G}$ on the space $BV^2(I)$ since it is not coercive on the Sobolev space $W^{2,1}(I)$ with respect to any appropriate topology.

In the space $BV^2(I)$ the functional $\mathcal{G}$ is not lower semicontinuous. To describe the behaviour of minimizing sequences for $\mathcal{G}$ we can substitute the functional $\mathcal{G}$ with its relaxation $\overline{\mathcal{G}}$. Let us consider on the whole $BV^2(I)$ the following functional

$$(2.3) \qquad \mathcal{H}(u) = \int_I \varphi(\ddot{u}(x))\,dx + 4\,\rho\,|u_s''|(I).$$

As an application of the semicontinuity result of [62], or as a particular case of Theorem 4.4 of [4], we obtain the following theorem.

Theorem 2.1. *For every* $u \in BV^2(I)$ *we have* $\overline{\mathcal{G}}(u) = \mathcal{H}(u)$; *; i.e., the functional* $\mathcal{H}$ *is the relaxation of the functional* $\mathcal{G}$ *with respect to the* $L^1(I)$-*topology.*

The aim of this chapter is to single out among the $BV^2(I)$-solutions of some minimum problems with generalized Dirichlet boundary data related to the functional $\mathcal{H}$, exactly those "phisically sound"; in this case we would have a solution with one or more creases rather than with a "diffuse" singular second derivative. This can be done by combining a relaxation argument and a singular perturbation method similar to the one described in Chapter 2.

First we shall observe that in place of the functional $\mathcal{G}$ we can consider an appropriate functional (see (2.4) below) that can be perturbed in a suitable way. To this purpose, let us denote by $SBV^2(I)$ the space of the functions $u \in BV^2(I)$ such that $u' \in SBV(I)$. We define a new functional $\mathcal{F}$ on $BV^2(I)$ by setting

(2.4)

$$\mathcal{F}(u) = \begin{cases} \displaystyle\int_I |\ddot{u}(x)|^2\,dx + 4\rho \sum_{x \in S_{u'} \cap I} |u'(x+) - u'(x-)| \\ \qquad\qquad \text{if } u \in SBV^2(I) \text{ and } \#(S_{u'} \cap I) < +\infty \\ +\infty \qquad\quad \text{otherwise.} \end{cases}$$

Theorem 2.2. *The functional* $\mathcal{F}$ *is not lower semicontinuous on* $BV^2(I)$ *with respect to the* $L^1(I)$-*topology and its relaxation is given by* $\mathcal{H}$.

Proof. Since we have just observed that the functional $\mathcal{H}$ is L^1-lower semicontinuous on $BV^2(I)$ it is clear that $\mathcal{H} \leq \overline{\mathcal{F}}$. To prove the opposite inequality let us consider a function $u \in \mathcal{C}^2(I) \cap BV^2(I)$. We follow the proof of Theorem 2.1 in Chapter 2. We can construct a sequence $(u_h)_h$, by approximating u' by piecewise constant functions on the open set

$$A = \{x \in I : x^2 < 4\rho|x|\},$$

in the following way. First, we consider the function $v = u' \in \mathcal{C}^1(I) \cap BV(I)$ and we construct, as in Chapter 2, Th. 2.1, a sequence $(v_h)_h$ converging to v in $L^\infty(I)$. Then we define $u_h(x) = \int_a^x v_h(t)\, dt$; clearly $u_h \in BV^2(I)$ and $u_h \to u$ in $L^\infty(I)$. In addition, with the same calculation as in Chapter 2, Th. 2.1, we obtain that

$$\mathcal{F}(u) \leq \int_A |u''(x)|^2\, dx + 4\rho \int_{I\backslash A} |u''(x)|\, dx = \int_I \tilde{\varphi}(u''(x))\, dx,$$

for every $u \in \mathcal{C}^2 \cap BV^2(I)$, where

$$\tilde{\varphi}(z) = \begin{cases} z^2 & \text{if } |z| \leq 2\rho \\ 4\rho|z| & \text{if } |z| > 2\rho. \end{cases}$$

Finally, by applying successively the relaxation theorems [62] and [33] we can prove that

$$\mathcal{F}(u) \leq \int_I (\tilde{\varphi})^{**}(u''(x))\, dx = \int_I \varphi(u''(x))\, dx,$$

on $W^{1,1}(I)$ (here $(\tilde{\varphi})^{**}$ denotes the greatest convex function less than or equal to $\tilde{\varphi}$) and extend this inequality on the whole $BV(I)$. $\square$

Theorem 2.2 shows that, in order to study the behaviour of minimizing sequences, functionals $\mathcal{G}$ and $\mathcal{F}$ are equivalent.

The functional $\mathcal{H}$, defined in (2.3), can be also considered as the Γ-limit of a suitable sequence of functionals obtained perturbing the functional $\mathcal{F}$ with an additional term. More precisely the following proposition holds.

Proposition 2.3.   *For every $\varepsilon > 0$ let us consider a Borel function $\psi_\varepsilon : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ such that*

$$(2.5) \qquad -\varepsilon|\xi - \eta| \leq \psi_\varepsilon(\xi,\eta) \leq \varepsilon(|\xi - \eta| + 1)$$

for every $(\xi,\eta) \in \mathbb{R}^2$, and let us define

$$(2.6) \quad \mathcal{F}_\varepsilon(u) = \begin{cases} \mathcal{F}(u) + \displaystyle\sum_{x \in S_{u'} \cap I} \psi_\varepsilon(u'(x+), u'(x-)) \\ \qquad\qquad\qquad \text{if } u \in SBV^2(I) \text{ and } \#(S_{u'} \cap I) < +\infty \\ +\infty \qquad\qquad \text{elsewhere on } BV^2(I). \end{cases}$$

Then, for every $u \in BV^2(I)$, we have

$$(\Gamma\text{-}\lim_{\varepsilon \to 0+} \mathcal{F}_\varepsilon)(u) = \mathcal{H}(u),$$

where the Γ-limit is considered with respect to the $L^1(I)$-topology.

Proof. From the inequality $\mathcal{F}_\varepsilon(u) \geq (1 - \varepsilon)\mathcal{F}(u)$, it follows that

$$\Gamma - \liminf_{\varepsilon \to 0} \mathcal{F}_\varepsilon \geq \mathcal{H}.$$

To prove that

$$\Gamma - \limsup_{\varepsilon \to 0} \mathcal{F}_\varepsilon \leq \mathcal{H}$$

it suffices to adapt the proof of Proposition 2.4 in Chapter 2. Let us take $u \in C^2(I) \cap BV^2(I)$, and let us fix a sequence $(\varepsilon_h)_h$ converging to 0. If we consider the sequence $(u_h)_h$ constructed in the proof of Theorem 2.2, we obtain that

$$\#(S_{u'_h} \cap I) \leq hc < +\infty.$$

By defining the sequence $(\omega_h)_h$ exactly as in Chapter 2, Prop. 2.4, with the same calculation we obtain that

$$(\Gamma - \limsup_{h \to +\infty} \mathcal{F}_{\varepsilon_h})(u) \leq \int_I \tilde{\varphi}(u''(x))\, dx$$

for every $u \in C^2(I) \cap BV^2(I)$ ($\tilde{\varphi}$ is defined in the proof of Theorem 2.2). As in the proof of Theorem 2.2, using [62] and [33], we conclude that on $BV^2(I)$

$$\Gamma - \limsup_{h \to +\infty} \mathcal{F}_{\varepsilon_h} \leq \mathcal{H}.$$

By the arbitrariness of the sequence $(\varepsilon_h)_h$, the proof is complete. □

We want to study the lower semicontinuity of the functionals $\mathcal{F}_\varepsilon$. Let us suppose that the functions ψ_ε are of the form

$$(2.7) \qquad \psi_\varepsilon(\xi, \eta) = \varepsilon\, \psi(\xi, \eta),$$

where ψ is a Lipschitz continuous function such that

$$(2.8) \qquad \psi(\xi, \xi) \geq c > 0$$

for every $\xi \in \mathbb{R}$. In addition suppose that for a fixed $\varepsilon_0 > 0$ the function $(\xi, \eta) \mapsto \psi_{\varepsilon_0}(\xi, \eta) + 4|\xi - \eta|$ is subadditive; ; *i.e.*, for every $\xi, \eta \in \mathbb{R}$ we have

$$(2.9) \quad \psi_{\varepsilon_0}(\xi, \eta) + 4|\xi - \eta| \leq \psi_{\varepsilon_0}(\xi, \zeta) + \psi_{\varepsilon_0}(\zeta, \eta) + 4|\xi - \zeta| + 4|\zeta - \eta|$$

for any $\zeta \in \mathbb{R}$. Note that if ψ_{ε_0} verifies (2.9) then every ψ_ε does for $0 < \varepsilon < \varepsilon_0$.

In these hypotheses we are able to prove the following proposition.

Proposition 2.4. *Let $\psi : \mathbb{R} \times \mathbb{R} \to [0, +\infty[$ be a Lipschitz continuous function verifying (2.8), and for every $\varepsilon > 0$ let ψ_ε be defined by (2.7). If for some $\varepsilon_0 > 0$ the function ψ_{ε_0} satisfies (2.9), then for every $0 < \varepsilon \le \varepsilon_0$ the functional $\mathcal{F}_\varepsilon$ is lower semicontinuous in the $L^1(I)$-topology on $BV^2(I)$.*

Proof. We follow the line of the first part of the proof of Proposition 4.4 in Chapter 2. Let us consider a sequence $u_h \to u$ in $L^1(I)$ such that $\lim_{h \to +\infty} \mathcal{F}_\varepsilon(u_h) < +\infty$. Then we have that $|u_h''|(I) \le c$, which implies that $(u_h)_h$ is bounded in $BV^2(I)$. Indeed, using the interpolation inequality we can bound the L^1-norm of the first derivative u_h' by the sum of the L^1-norm of u_h and of the total variation on I of the measure u_h'' up to a positive constant depending only on the length of the interval I. Hence $u_h \to u$ in $BV^2 - \omega^*$. In addition $\#(S_{u_h'} \cap I) \le c < +\infty$ and $\|\ddot{u}_h\|_2 \le c$. Therefore we can apply Prop. 4.1 of Chapter 2 to the sequence $(u_h')_h$ and conclude that $\mathcal{F}_\varepsilon(u) \le \lim_h \mathcal{F}_\varepsilon(u_h)$. $\qquad\square$

3. Minimum Problems

We can describe the behaviour of the functionals $\mathcal{H}$ and $(\mathcal{F}_\varepsilon)_{\varepsilon > 0}$ by examining some minimum problems with generalized Dirichlet boundary data. Without loss of generality we will deal with problems on the interval $I = (0, 1)$ and we will fix $\rho = 1$ (see Remark 3.10 for the general case). Let us fix $(v_0, v_1, \xi_0, \xi_1) \in \mathbb{R}^4$ and consider the boundary conditions $u(0) = v_0, u(1) = v_1, u'(0-) = \xi_0$, and $u'(1+) = \xi_1$. Since boundary conditions on u' are not preserved under passage to the limit in $BV^2 - \omega^*$ the boundary conditions must be "relaxed"; the corresponding minimum problem for $\mathcal{H}$ is given by

$$(3.1) \qquad m(v_0, v_1, \xi_0, \xi_1) = \min \left\{ \int_I \varphi(\ddot{u}(x))\, dx + 4|(u^*)_s''|([0,1]) \right\},$$

where the minimum is taken among all functions $u \in BV^2(0,1)$ such that $u(0) = v_0, u(1) = v_1$, and the function $u^* \in BV^2_{loc}(\mathbb{R})$ is obtained by extending u to $x \mapsto (\xi_0 x + v_0)$ in $]-\infty, 0]$ and to $x \mapsto [\xi_1 x + (v_1 - \xi_1)]$ in $[1, +\infty[$. For the minimum problems related to the functionals $(\mathcal{F}_\varepsilon)_\varepsilon$ we can prove the following theorem.

Theorem 3.1. *In the hypotheses of Proposition 2.4, for every $0 < \varepsilon \le \varepsilon_0$ and every choice of real numbers v_0, v_1, ξ_0, ξ_1 there exists $u \in SBV^2(0,1)$*

with $\#(S_{u'} \cap (0,1)) < +\infty$ *verifying the minimum value*

(3.2)
$$m_\varepsilon(v_0, v_1, \xi_0, \xi_1) = \min \left\{ \int_0^1 |\ddot{u}(x)|^2 \, dx + \right.$$
$$\left. \sum_{x \in S_{(u^*)'}} \left[4 |(u^*)'(x+) - (u^*)'(x-)| + \psi_\varepsilon((u^*)'(x+), (u^*)'(x-)) \right] \right\},$$

where the minimum is taken among all functions $u \in SBV^2(0,1)$ *such that* $\#(S_{u'} \cap (0,1)) < +\infty$, $u(0) = v_0$, *and* $u(1) = v_1$.

Proof. Let us fix $0 < \varepsilon \leq \varepsilon_0$; by Proposition 2.4 the functional we minimize in (3.2) is lower semicontinuous in the L^1-topology. We have to prove that this functional is coercive. To this purpose let us fix a positive number t and let us consider a sequence $(u_h)_h$ in the sublevel set of the functional corresponding to t. As in the proof of Proposition 2.4 we obtain that $|u_h''|(I) \leq c$ and this implies that the sequence $(u_h)_h$ is bounded in $SBV^2(I)$ (here we make use not only of the interpolation inequality, but also of the fact that the boundary conditions guarantees that the sequence $(u_h)_h$ is bounded in $L^1(I)$). Moreover we have that $\#(S_{u'_h} \cap I) \leq c$. Therefore by Prop. 4.1 of Chapter 2, up to a subsequence we have that $u_h \to u$ in $SBV^2 - \omega^*$, where u is a function in $SBV^2(I)$ such that $\#(S_{u'} \cap (0,1)) < +\infty$, $u(0) = v_0$, and $u(1) = v_1$. $\qquad\square$

The aim of this section is to describe the minimizers for problem (3.1) and the behaviour as $\varepsilon \to 0$ of the minimizers of (3.2). First we reduce (3.2) to a sequence of simpler minimum problems which do not depend on ε.

Up to a translation and a rotation (*i.e.*, addition of an affine function) it is not restrictive to suppose that at $x = 0$ we have the conditions $u(0) = (u^*)'(0-) = 0$, so that problem (3.2) is equivalent to the minimum problem

$$m_\varepsilon(0, v_1 - (v_0 + \xi_0), 0, \xi_1 - \xi_0) = \min \left\{ \int_0^1 |\ddot{u}(x)|^2 \, dx + \right.$$
$$\left. \sum_{x \in S_{(u^*)'}} \left[4 |(u^*)'(x+) - (u^*)'(x-)| + \psi_\varepsilon((u^*)'(x+), (u^*)'(x-)) \right] \right\},$$

where the minimum is taken among all functions $u \in SBV^2(0,1)$ such that $\#(S_{u'} \cap (0,1)) < +\infty$, $u(0) = 0, u(1) = v_1 - (v_0 + \xi_0)$ (recall that $(u^*)'(0-) = 0$ and $(u^*)'(1+) = \xi_1 - \xi_0$).

In order to simplify the calculations, let us consider the case $\psi \equiv 1$ (see Section 4 for a discussion on the general case); since now the minimum problem depends only on the boundary data at the point $x = 1$, for every

choice of $(v, \xi) \in \mathbb{R}^2$ we have to study

(3.3)
$$m_\varepsilon(v, \xi) = \min \Big\{ \int_0^1 |\ddot{u}(x)|^2 \, dx + \\ 4 \sum_{x \in S_{(u^*)'}} |(u^*)'(x+) - (u^*)'(x-)| + \varepsilon \#(S_{(u^*)'}) \Big\},$$

where the minimum is taken among all functions $u \in SBV^2(0, 1)$ such that $\#(S_{u'} \cap (0, 1)) < +\infty$, $u(0) = 0, u(1) = v$ (the extension of u gives $(u^*)'(0-) = 0$ and $(u^*)'(1+) = \xi$). We consider then for every $n \in \mathbb{N}$ the problem

(3.4) $$M(n, v, \xi) = \inf \Big\{ \int_0^1 |\ddot{u}(x)|^2 \, dx + 4 \sum_{x \in S_{(u^*)'}} |(u^*)'(x+) - (u^*)'(x-)| \Big\},$$

where the minimum is taken among the same functions as in (3.3) with the additional condition $\#(S_{(u^*)'}) = n$, so that (3.3) becomes

$$m_\varepsilon(v, \xi) = \min_{n \in \mathbb{N}} \{ M(n, v, \xi) + \varepsilon n \}.$$

Let us consider the minimum problem (3.4); the following proposition concerns the case $n = 0$ (no creases).

Proposition 3.2. *We have that the minimum value*

(3.5) $$M(0, v, \xi) = 4(3v^2 - 3v\xi + \xi^2)$$

and it is achieved at the function $u(x) = (-2v + \xi) x^3 + (3v - \xi) x^2$.

Proof. In the case $n = 0$ we have to minimize $\int_0^1 |u''(x)|^2 \, dx$ over the Sobolev space $H^2(0, 1)$ with the boundary conditions $u(0) = u'(0) = 0, u(1) = v$, and $u'(1) = \xi$. It is well known that the minimizer of such a problem is unique and that it is the only function verifying at the same time the Euler Equation

(3.6) $$\int_0^1 u''(x) \, \varphi''(x) \, dx = 0 \qquad \forall \varphi \in C_c^\infty(I)$$

and the boundary conditions. As a consequence of (3.6) u satisfies the equation $u^{IV} \equiv 0$ in the sense of distributions, which implies that u is a polynomial with degree less than or equal to 3. Therefore the minimizer is $u(x) = (-2v + \xi)x^3 + (3v - \xi)x^2$ and the corresponding minimum value is given by

$$M(0, v, \xi) = \int_0^1 |6(-2v + \xi) x + 2(3v - \xi)|^2 \, dx = 4(\xi^2 - 3v\xi + 3v^2). \quad \square$$

Now we turn our attention to the case of a single crease ($n = 1$).

Proposition 3.3. *If $n = 1$ in (3.4), then*

(3.7)
$$M(1, v, \xi) = 4 \inf \left\{ 3w^2 - 3w\eta + \eta^2 + |\xi - \eta| \quad : \right.$$
$$\left. w, \eta \in \mathbb{R}, \eta \neq \xi, 0 \leq \frac{v - w}{\xi - \eta} \leq 1 \right\}.$$

Proof. Let $u \in SBV^2(0, 1)$ be such that $\#(S_{(u^*)'}) = 1$; this means that

$$(u^*)'' = \ddot{u} \, dx + \zeta \delta_{x_0} \qquad \text{with } 0 \leq x_0 \leq 1 \text{ and } \zeta \neq 0.$$

Let us consider the function

$$u_a(x) = u(x) - \zeta(x - x_0)^+;$$

since $u_a'' = \ddot{u} \, dx + \zeta \delta_{x_0} - \zeta \delta_{x_0} = \ddot{u} \, dx$ (here δ_x denotes the Dirac measure at x), we deduce that $u_a \in H^2(0, 1)$. Moreover we have that $u_a(0) = u_a'(0) = 0, u_a(1) = v - \zeta(1 - x_0)$, and $u_a'(1) = \xi - \zeta$.

Since we are interested in the computation of $M(1, v, \xi)$, we can suppose that the function u_a realizes the minimum $M(0, v - \zeta(1 - x_0), \xi - \zeta)$. Therefore, using Proposition 3.2, we deduce that

$$M(1, v, \xi) = 4 \inf \left\{ 3(v - \zeta(1 - x_0))^2 - 3(v - \zeta(1 - x_0))(\xi - \zeta) \right.$$
$$\left. + (\xi - \zeta)^2 + |\zeta| \right\}$$

where the infimum is taken for $\zeta \in \mathbb{R} \setminus \{0\}$. By posing $\eta = \xi - \zeta$ and $w = v - \zeta(1 - x_0)$, we obtain (3.7). Indeed the condition $0 \leq x_0 \leq 1$ becomes $0 \leq \frac{v - w}{\xi - \eta} \leq 1$. □

With simple calculations we can specify the values of $M(1, v, \xi)$.

Proposition 3.4. *Let us suppose that $v \geq 0$ and $\xi \in \mathbb{R}$. Then the following assertions hold.*

(i) If $v > 1$ and $\xi > v + 1$, then

$$M(1, v, \xi) = 4(\xi - 1);$$

the minimum is reached by the function

$$u(x) = x^2 + (\xi - 2) \left(x - \frac{\xi - v - 1}{\xi - 2} \right)^+,$$

whose crease point is $x_0 = \dfrac{\xi - v - 1}{\xi - 2} = 1 - \dfrac{v - 1}{\xi - 2}.$

(ii) *If* $v \geq 0$ *and* $\xi < \min\{0, (3v - 1)/2\}$, *then*

$$M(1, v, \xi) = 3v^2 + 6v - 4\xi - 1;$$

the minimum is reached by the function

$$u(x) = -\frac{1}{2}(v + 1)\,x^3 + \frac{1}{2}(3v + 1)\,x^2.$$

The crease point is at $x_0 = 1$.

(iii) *If* $v > \dfrac{1}{3}$ *and* $0 \leq \xi \leq \min\{v + 1, 3v - 1\}$, *then*

$$M(1, v, \xi) = 3(\xi - v)^2 - 2\xi + 6v - 1;$$

the minimum is reached by the function

$$u(x) = \frac{1}{2}(\xi - v - 1)\,x^3 + x^2 + \frac{1}{2}(3v - \xi - 1)\,x.$$

The crease point is at $x_0 = 0$.

(iv) *If* $0 \leq v \leq 1$ *and* $\max\{\dfrac{3v - 1}{2}, 3v - 1\} \leq \xi \leq \dfrac{3v + 1}{2}$, *then the infimum is not reached and*

$$M(1, v, \xi) = 4(v^2 - 3v\xi + \xi^2) = M(0, v, \xi).$$

(v) *If* $0 \leq v \leq 1$ *and* $\xi > \frac{3v+1}{2}$, *then*

$$M(1, v, \xi) = 3v^2 - 6v + 4\xi - 1;$$

The minimum is reached by the function

$$u(x) = \frac{1}{2}(1 - v)x^3 + \frac{1}{2}(3v - 1)x^2.$$

The crease point is at the point $x_0 = 1$.

Remark 3.5. The case $v \leq 0$ is solved by a symmetry argument. Indeed the minimum problem we consider is symmetric with respect to the origin in the plane of the boundary value system (v, ξ). For instance if $v < -1$ and $\xi < v - 1$, then we have

$$M(1, v, \xi) = 4(-\xi - 1)$$

and if $v \leq 0$ and $\xi > \max\{0, \dfrac{3v + 1}{2}\}$, then

$$M(1, v, \xi) = 3v^2 - 6v + 4\xi - 1.$$

Let us consider now the case $n \geq 2$ (two or more creases). Let $\mathcal{I}$: $SBV^2_{\text{loc}}(\mathbb{R}) \to [0, +\infty]$ be the functional we minimize in problem (3.4); ; *i.e.*,

$$\mathcal{I}(u) = \int_0^1 |\ddot{u}(x)|^2 \, dx + 4 \sum_{x \in S_{u'} \cap [0,1]} |u'(x+) - u'(x-)|,$$

and let us denote by $\mathcal{S}(v, \xi)$ the space of the functions $u \in SBV^2_{\text{loc}}(\mathbb{R})$ such that $u(0) = u'(0-) = 0, u(1) = v$, and $u'(1+) = \xi$.

Proposition 3.6. *Let us fix $n \geq 2$ and $(v, \xi) \in \mathbb{R}^2$. Let us consider a function $u \in SBV^2(0, 1)$ such that $u(0) = 0$, $u(1) = v$, and $\#(S_{(u^*)'}) = n$, where u^* denotes the extension of u associated to (v, ξ) in such a way that $u^* \in \mathcal{S}(v, \xi)$. Then there exists a function $U \in \mathcal{S}(v, \xi)$ such that $\#(S_{U'}) = 2$, $S_{U'} = \{0, 1\}$, and $\mathcal{I}(u^*) - \mathcal{I}(U) \geq 0$.*

Proof. Let us consider a function u satisfying the hypotheses of the proposition. Then we have that

$$(u^*)'' = \ddot{u} \, dx + \sum_{k=1}^n \zeta_k \, \delta_{x_k},$$

with $\zeta_k \neq 0$ and $x_k \in [0, 1]$ for every $k = 1, \dots, n$. Let us define the function $u_a \in SBV^2_{\text{loc}}(\mathbb{R})$ as

$$u_a(x) = u^*(x) - \sum_{k=1}^n \zeta_k (x - x_k)^+.$$

We have that $u_a'' = \ddot{u} \, dx$, hence $u_a \in H^2(0, 1)$. Our aim is to substitute all crease points of u with two crease points at $x = 0, 1$. This can be done by defining a new function $U \in SBV^2_{\text{loc}}(\mathbb{R})$ as

$$U(x) = u_a(x) + \left(\sum_{k=1}^n \zeta_k(1 - x_k)\right)x^+ + \left(\sum_{k=1}^n \zeta_k x_k\right)(x - 1)^+.$$

Indeed we have that $U(0) = U'(0-) = 0, U(1) = u_a(1) + \sum_{k=1}^n (\zeta_k(1 - x_k)) = v$, and $U'(1+) = u_a'(1) + \sum_{k=1}^n (\zeta_k(1 - x_k)) + \sum_{k=1}^n (\zeta_k x_k) = u_a'(1) + \sum_{k=1}^n \zeta_k = \xi$. Moreover

$$U'' = \ddot{u} \, dx + \left(\sum_{k=1}^n \zeta_k(1 - x_k)\right)\delta_0 + \left(\sum_{k=1}^n \zeta_k x_k\right)\delta_1,$$

hence $\#(S_{U'}) = 2$ and $S_{U'} = \{0, 1\}$. Finally we have that

$$(3.8) \quad \mathcal{I}(u^*) - \mathcal{I}(U) = 4\left(\sum_{k=1}^n |\zeta_k| - |\sum_{k=1}^n \zeta_k(1 - x_k)| - |\sum_{k=1}^n \zeta_k x_k|\right) \geq 0.$$

It is easy to see that in all possible cases the last inequality in (3.8) is satisfied and this concludes the proof. □

Proposition 3.6 implies that for every $(v, \xi) \in \mathbb{R}^2$ we have

$$M(n, v, \xi) \geq M(2, v, \xi) \qquad \forall n > 2;$$

furthermore it follows also that in the computation of $M(2, v, \xi)$ we can consider only functions $u \in SBV^2(0, 1)$ such that $u(0) = 0, u(1) = v$, and $S_{(u^*)'} = \{0, 1\}$. For these functions we have that

$$(u^*)'' = \ddot{u} \, dx + \zeta \delta_0 + \eta \delta_1,$$

with $\zeta, \eta \neq 0$. Let us consider the function

$$u_a(x) = u^*(x) - \zeta x^+ - \eta(x - 1)^+;$$

since $u_a'' = \ddot{u} \, dx$, we deduce that $u_a \in H^2(0, 1)$. Moreover we have that $u_a(0) = u_a'(0) = 0, u_a(1) = v - \zeta$, and $u_a'(1) = \xi - \zeta - \eta$. Therefore we can suppose that u_a realizes the minimum $M(0, v - \zeta, \xi - \zeta - \eta)$. Using Proposition 3.2 we deduce that

$$M(2, v, \xi) = 4 \, \inf \Big\{ 3(v - \zeta)^2 - 3(v - \zeta)(\xi - \zeta - \eta) +$$
$$(\xi - \zeta - \eta)^2 + |\zeta| + |\eta| \Big\},$$

where the infimum is taken for $\zeta, \eta \in \mathbb{R} \setminus \{0\}$. Finally, if we change variables by setting $w = v - \zeta$ and $\lambda = \xi - \zeta - \eta$, we obtain that

$$(3.9) \quad M(2, v, \xi) = 4 \, \inf \Big\{ 3 \, w^2 - 3 \, w\lambda + \lambda^2 + |v - w| + |\xi - v - \lambda + w| \Big\},$$

where the infimum is taken for $w \in \mathbb{R} \setminus \{v\}, \lambda \in \mathbb{R} \setminus \{\xi\}$.

The following proposition concerns the values of $M(n, v, \xi)$ for any $n \geq 2$.

Proposition 3.7. *Let us suppose that $v \geq 0$ and $\xi \in \mathbb{R}$. Then the following assertions hold.*

(i) If $v > \dfrac{1}{3}$ and $\xi < v - \dfrac{1}{3}$, then

$$M(2, v, \xi) = 4(2v - \xi - \frac{1}{3}) < \min_{n \neq 2} M(n, v, \xi);$$

the minimum in $M(2, v, \xi)$ is reached by the function

$$u(x) = -\frac{2}{3}x^3 + x^2 + (v - \frac{1}{3})x^+.$$

(ii) If $v > 1$ and $\xi > v + 1$, then

$$M(n, v, \xi) = M(2, v, \xi) = 4(\xi - 1) = M(1, v, \xi);$$

the minimum in $M(2, v, \xi)$ is reached by the function

$$u(x) = x^2 + (v - 1)x^+,$$

while the minimum in $M(n, v, \xi)$ is reached on all functions of the form

$$u(x) = x^2 + \left(\sum_{k=1}^{n} \zeta_k (x - x_k)^+\right)$$

with $\zeta_k > 0, x_k \in [0, 1]$ for every $k = 1, \ldots, k$, $\sum_{k=1}^{n} \zeta_k = \xi - 2$, and $\sum_{k=1}^{n} \zeta_k (1 - x_k) = v - 1$.

(iii) In all other cases, for every $n \geq 2$ we have

$$M(n, v, \xi) = M(1, v, \xi),$$

but the minimum is not reached on functions with more than one crease.

We can summarize our results in the following theorem (see also fig.1).

Theorem 3.8.   *Let $(v, \xi) \in \mathbb{R}^2$; then the following assertions hold.*

(i) If

$$v > \frac{1}{3} \text{ and } \xi < v - \frac{1}{3}$$

(symmetrically if $v < -\frac{1}{3}$ and $\xi > v + \frac{1}{3}$),

then the minimum value of the functional $\mathcal{I}$ over $\mathcal{S}(v, \xi)$ is achieved only on functions with two crease points and the minimum point is unique.

(ii) If

$$-1 \leq v \leq \frac{1}{3} \text{ and } \xi < \frac{3v - 1}{2}$$

(symmetrically if $-\frac{1}{3} \leq v \leq 1$ and $\xi > \frac{3v + 1}{2}$),

or if

$$\frac{1}{3} < v \leq 1 \text{ and } v - \frac{1}{3} \leq \xi < 3v - 1$$

(symmetrically if $-1 \leq v < -\frac{1}{3}$ and $3v + 1 < \xi \leq v + \frac{1}{3}$),

or if

$$v > 1 \text{ and } v - \frac{1}{3} \le \xi \le v + 1$$

$$(\text{symmetrically if } v < -1 \text{ and } v - 1 \le \xi \le v + \frac{1}{3}),$$

then the minimum value of the functional $\mathcal{I}$ over $\mathcal{S}(v, \xi)$ is achieved only on functions with one crease point and the minimum point is unique.

(iii) If

$$-1 \le v \le 1 \text{ and } \max\{3v - 1, \frac{3v - 1}{2}\} \le \xi \le \min\{3v + 1, \frac{3v + 1}{2}\},$$

then the minimum value of the functional $\mathcal{I}$ over $\mathcal{S}(v, \xi)$ is achieved only on C^2 functions and the minimum point is unique.

(iv) If

$$v > 1 \text{ and } \xi > v + 1$$

$$(\text{symmetrically if } v < -1 \text{ and } \xi < v - 1),$$

then the minimum value of the functional $\mathcal{I}$ over $\mathcal{S}(v, \xi)$ is achieved on one or more functions with n crease points, for any $n \in \mathbb{N}$.
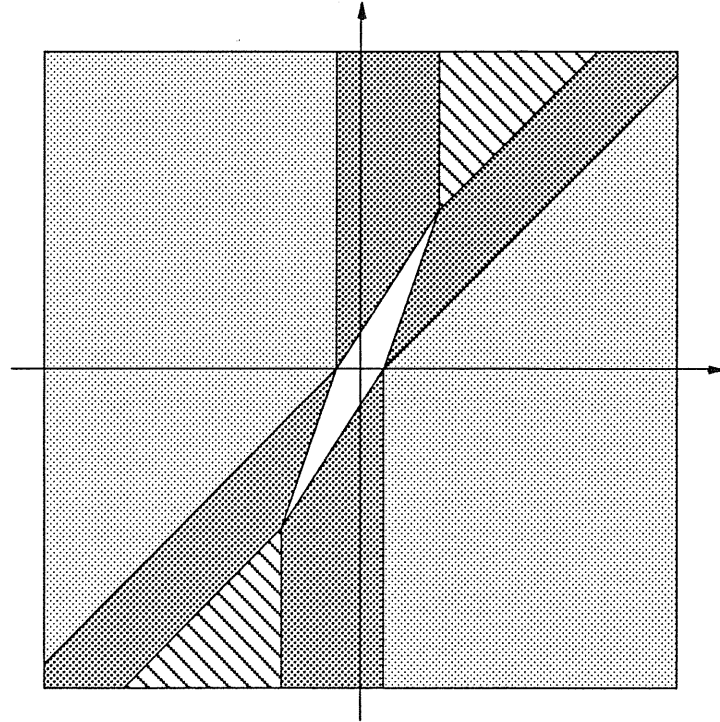

Fig. 1. The minimizers of the functional $\mathcal{I}$ over $\mathcal{S}(v, \xi)$ in the plane of the boundary value system (v, ξ)

The rhomb with vertex $A = (1,2), B = (1/3,0), A' = (-1,-2), B' = (-1/3,0)$ is identified as the set of boundary value system (v, ξ) such that the (unbent) solutions u to $M(0, v, \xi)$ verifies $\|u''\|_\infty \le 2$, that is

$$\mathcal{F}(u) = \int_0^1 \varphi(u''(x)) \, dx = \int_0^1 |u''(x)|^2 \, dx.$$

In particular the point A corresponds to the solution $u(x) = x^2$, with $u''(x) \equiv 2$, and the point B to the solution $u(x) = \frac{2}{3}x^3 + x^2$, with $u''(x) = -4x + 2$.

Remark 3.9. Theorem 3.8 provides also a description of the minimizers for the corresponding boundary value problems for the functional $\mathcal{H}$ (see (3.1)). More precisely we have uniqueness of the solution in $BV^2(0,1)$ in the regions where we have uniqueness of the functional $\mathcal{I}$ on $\mathcal{S}(v, \xi)$, while in the region $\{(v, \xi) \in \mathbb{R}^2 : v > 1, \xi > v + 1\}$ (resp. $\{(v, \xi) \in \mathbb{R}^2 : v < -1, \xi < v - 1\}$) the minimizers are all the functions of the form

$$u(x) = x^2 + \int_0^x \mu([0, t]) \, dt,$$

where μ is any positive (resp. negative) Borel measure on $[0, 1]$ such that

$$\int_0^1 \mu([0, t]) \, dt = v - 1 \quad \text{and} \quad \mu([0, 1]) = \xi - 2.$$

In particular, when μ is of the form $\sum_{k=1}^n \zeta_k \, \delta_k$ with $\zeta_k > 0$ (resp. $\zeta_k < 0$) for $k = 1, \ldots, n$, $x_k \in [0, 1]$, $\sum_{k=1}^n \zeta_k (1 - x_k) = v - 1$, and $\sum_{k=1}^n \zeta_k = \xi - 2$, the minimizers are of the form

$$u(x) = x^2 + \sum_{k=1}^n \zeta_k (x - x_k)^+.$$

Remark 3.10. Let us fix an open bounded interval $I = (0, a)$ and a positive constant ρ. Let us consider the problem

$$(3.9) \qquad \inf \left\{ \int_0^a |\ddot{u}(x)|^2 \, dx + 4\rho \sum_{x \in S_{(u^*)'}} |(u^*)'(x+) - (u^*)'(x-)| \right\}$$

where the infimum is taken among all functions $u \in SBV^2(I)$ such that $\#(S_{u'} \cap I) < +\infty$, $u(0) = 0$, and $u(a) = v$ (the function $u^* \in SBV^2_{loc}(\mathbb{R})$ is obtained by extending u to $x \mapsto 0$ in $]-\infty, 0]$ and to $x \mapsto [\xi(x-a) + v]$ on $[a, +\infty[$).

We can easily describe, with a graph similar to that of fig. 1, the solutions to problem (3.9), where the point A is replaced by $(\rho a^2, 2\rho a)$, the point B by $(\frac{1}{3}\rho a^2, 0)$, and the slopes of the lines passing through these two points are still 1. Let us remark that we can not simply rescale the functional; indeed, if the infimum in (3.9) is a minimum, then it is achieved on the function $U(x) = a^2 u(\frac{x}{a})$, where the function u is the minimizer of the functional $\mathcal{I}$ on $\mathcal{S}(v/a^2, \xi/a)$. The general case of an interval $I = (a, b)$ can be reduced to (3.9) by a simple translation, since the problem depends only on the interval length.

4. The Main Result

Proposition 2.3 exhibits a singular perturbation of the functional $\mathcal{H}$ with a sequence of equi-coercive functionals. Therefore to approximate $\mathcal{H}$ by the functionals $\mathcal{F}_\varepsilon$ via Γ-convergence represents a choice among all the possible minimizers for problem (3.1), of those that in particular can be reached following minimizing sequences for the corresponding problems for $\mathcal{F}_\varepsilon$.

We say that a function $\Phi :]0, +\infty[\to \mathbb{R}$ is *positively subadditive* if we have

$$\Phi(a + b) \leq \Phi(a) + \Phi(b) \qquad \text{for all } a, b \in]0, +\infty[.$$

Given a function $\psi : \mathbb{R} \to \mathbb{R}$, we denote by $\overline{\psi}$ the greatest lower semicontinuous function less than or equal to ψ such that both functions $\overline{\psi}(t)$, $\overline{\psi}(-t)$ are positively subadditive. We refer to Chapter 2, Section 4 for remarks and examples about positively subadditive functions.

Theorem 4.1. *Let $\psi : \mathbb{R} \times \mathbb{R} \to [0, +\infty[$ be a Lipschitz continuous function verifying (2.8), and for every $\varepsilon > 0$ let $\psi_\varepsilon(\xi, \eta) = \varepsilon\psi(\xi, \eta)$. Let us consider the functionals $\mathcal{F}_\varepsilon$ as defined in (2.6), and let us suppose that the functions ψ_ε satisfies the hypotheses of Proposition 2.4. Let us fix an interval $I = (a, b)$ and $(v_1, v_2, \xi_1, \xi_2) \in \mathbb{R}^4$. For every $\varepsilon > 0$ small enough, let u_ε be the solution to the problem*

$$m_\varepsilon = \min \left\{ \int_a^b |\ddot{u}(x)|^2 \, dx + 4\rho \sum_{x \in S_{(u^*)'}} |(u^*)'(x+) - (u^*)'(x-)| \right.$$

$$\left. + \sum_{x \in S_{(u^*)'}} \psi_\varepsilon((u^*)'(x+), (u^*)'(x-)) \right\}$$

where the minimum is taken among all functions $u \in SBV^2(a, b)$ *such that* $\#(S_{u'} \cap (a, b)) < +\infty$, $u(a) = v_1$, *and* $u(b) = v_2$. *Then, up to a subsequence,* $(u_\varepsilon)_\varepsilon$ *converges weakly in* $BV^2(a, b)$ *to a function* $u \in SBV^2(a, b)$, $(\ddot{u}_\varepsilon)_\varepsilon$ *converges to* $\ddot{u}$ *weakly in* $L^2(a, b)$, $(u_\varepsilon)''_s$ *converges to* u''_s *weakly in the sense of measures, and* u *verifies the minimum*

$$m = \inf\left\{\int_a^b |\ddot{u}(x)|^2\, dx + 4\rho \sum_{x \in S_{(u^*)'}} |(u^*)'(x+) - (u^*)'(x-)|\right\}$$

$$= \min\left\{\int_a^b \varphi(\ddot{u}(x))\, dx + 4\rho |(u^*)''_s|([a, b])\right\},$$

where the infimum and the minimum are taken over the same class above. If

$$(4.1) \quad \xi_2 - \xi_1 > v_2 - v_1 - \xi_1(b - a) + \rho(b - a)(2 - (b - a)) > 2\rho(b - a)$$

or symmetrically if

$$\xi_2 - \xi_1 < v_2 - v_1 - \xi_1(b - a) - \rho(b - a)(2 - (b - a)) < -2\rho(b - a),$$

then u *is characterized by minimizing also*

$$\sum_{x \in S_{(u^*)'}} \overline{\psi}((u^*)'(x+), (u^*)'(x-))$$

among all functions $u \in SBV^2(a, b)$ *which verify the minimum value* m. *In all other cases the minimum point verifying* m *belongs to* $SBV^2(a, b)$ *and it is unique.*

Proof. To prove the theorem it suffices to repeat the same arguments of Section 5 of Chapter 2, using the results of Section 3. $\qquad\square$

Remark 4.2. In the region (4.1) the minimizers $u \in SBV^2(a, b)$ of the functional $\mathcal{H}$ take the form

$$u(x) = \rho\, x^2 + (\xi_1 - 2\rho\, a)\, x + \rho\, a^2 - \xi_1\, a + v_1 + \sum_{k=1}^n \zeta_k(x - x_k)^+$$

with

$$\zeta_k > 0, \quad \sum_{k=1}^n \zeta_k = \xi_2 - \xi_1 - 2\rho(b - a),$$

$$\sum_{k=1}^{n} \zeta_k (1 - x_k) = v_2 - v_1 - \xi_1(b - a) - \rho(b - a)^2.$$

If we suppose that $\psi(\zeta, \eta) = \varphi(|\zeta - \eta|)$ the singular perturbation approach selects between these minimizers those minimizing also the functional

$$\sum_{k=1}^{n} \overline{\varphi}(|\zeta_k|).$$

Example 4.3. In the particular case when $\psi \equiv 1$, we obtain $\overline{\psi} \equiv 1$ and we select exactly the unique minimizer of $\mathcal{H}$ with minimum number of crease points; *i.e.*,

$$u(x) = \rho\, x^2 + (\xi_1 - 2\,\rho\, a)\, x + \rho\, a^2 - \xi_1\, a + v_1$$
$$+ \left((\xi_2 - \xi_1 - 2\,\rho(b - a))(x - b) + v_2 - v_1 - \xi_1(b - a) - \rho(b - a)^2 \right)^{+}.$$

Examples 4.4. **4.4.1.** If we take $\psi(\zeta, \eta) = \varphi(|\zeta - \eta|)$, where $\varphi(t) = 1 + t^2$, then we can compute $\overline{\psi}$ obtaining that

$$\overline{\psi}(\zeta, \eta) = \min\{k + \frac{|\zeta - \eta|^2}{k} : k \in \mathbb{N} \setminus \{0\}\}.$$

4.4.2 We can consider also non symmetric functions ψ. For instance let us take $\psi(\zeta, \eta) = \varphi(\zeta - \eta)$, where $\varphi(t) = |t - 1|$, which gives

$$\overline{\psi}(\zeta, \eta) = \begin{cases} 1 - (\zeta - \eta) & \text{if } \zeta - \eta \leq 1 \\ \text{dist}(\zeta - \eta, \mathbb{N}) & \text{if } \zeta - \eta > 1. \end{cases}$$

4.4.3 We can allow the dependence of ψ also on the variable x, by considering functions of the form $\psi(x, \zeta, \eta)$. Such functions can be useful to treat the case of inhomogeneous materials.

CHAPTER 4:

THE INTERACTION BETWEEN
BULK ENERGY AND SURFACE ENERGY
IN MULTIPLE INTEGRALS

In this chapter we study some integral functionals defined on $SBV(\Omega; \mathbb{R}^k)$, the space of vector-valued special functions with bounded variation on the open set $\Omega \subset \mathbb{R}^n$, of the form

$$F(u) = \int_\Omega f(\nabla u(x))dx + \int_{S_u \cap \Omega} g((u^+ - u^-) \otimes \nu_u) \, d\mathcal{H}^{n-1}.$$

On f we suppose only that it is finite at one point, and on g we assume that it is positively 1-homogeneous, and that it is locally bounded on the sets $\mathbb{R}^k \otimes \nu_m$, where $\{\nu_1, \dots, \nu_n\} \subset S^{n-1}$ is a basis of $\mathbb{R}^n$. We prove that the lower semicontinuous envelope of F in the $L^1(\Omega; \mathbb{R}^k)$-topology is finite and with linear growth on the whole $BV(\Omega; \mathbb{R}^k)$, and that it admits the integral representation

$$\overline{F}(u) = \int_\Omega \varphi(\nabla u(x))dx + \int_\Omega \varphi^\infty \left(\frac{D_s u}{|D_s u|} \right) |D_s u|$$

A formula for φ is given, which takes into account the interaction between the bulk energy density f and the surface energy density g.

The results of this chapter are contained in [BC2].

Introduction

Many problems in Mathematical Physics, Computer Vision, and Mechanical Engineering take into account surface energies on some "free boundary" or "free discontinuity" set. These energies account for several phenomena such as crack growth and crack initiation in the theory of brittle fracture, interface formation between different phases of Cahn-Hilliard fluids, surface tension between small drops of liquid crystals, and are utilized for pattern recognition in computer vision to determine surfaces corresponding to sudden changes in the image (*e.g.* the edges of the objects, shadows, different colours).

We are interested in a variational formulation for some of the *static* free discontinuity problems, in the light of recent research on functionals which depend on discontinuous functions. From the point of view of the calculus of variations, a rather complete theory has been developed by L. Ambrosio

& A. Braides [9], [10] in absence of a "volume" counterpart of the surface energy, in the framework of partitions of sets of finite perimeter. When we allow the presence of a bulk energy, it is natural to take into account spaces of functions of bounded variation. We recall that if Ω is an open set in $\mathbb{R}^n$, a function u belongs to $BV(\Omega; \mathbb{R}^k)$ if it is an integrable function, and its distributional derivative Du is a finite (matrix-valued) Radon measure on Ω. It turns out that the Lebesgue decomposition of this measure can be written as $Du = \nabla u \, dx + D_s u$, where the density of the absolutely continuous part of Du is denoted by ∇u since it can be interpreted as an approximate differential for u. For a function $u \in BV(\Omega; \mathbb{R}^k)$ it is possible moreover to define a *set of jump points* S_u where u is approximately discontinuous, and on which it is well-defined a "normal" ν_u together with the *traces* u^+, u^- of u on the two sides. Recently, E. De Giorgi & L. Ambrosio [45] have introduced the subspace $SBV(\Omega; \mathbb{R}^k)$ of *special* functions of bounded variation, that are characterized by the property that the singular part of Du can be written as

$$(0.1) \qquad D_s u = (u^+ - u^-) \otimes \nu_u \, \mathcal{H}^{n-1}{}_{|S_u}$$

($\mathcal{H}^{n-1}{}_{|S_u}$ denotes the restriction to S_u of the $(n-1)$-dimensional Hausdorff measure). Remark that in general $D_s u$ contains also a diffuse "Cantor" part. On the space $SBV(\Omega; \mathbb{R}^k)$ it is natural to consider functionals of the form

$$(0.2) \qquad \int_\Omega f(x, u(x), \nabla u(x)) \, dx + \int_{S_u \cap \Omega} g(x, u^+, u^-, \nu_u) \, d\mathcal{H}^{n-1}.$$

These integrals model many of the problems so far considered in the literature, and provide a good functional setting for problems that had been before considered only under additional un-natural hypotheses, imposed to obtain *a priori* smoothness on S_u.

A natural question for the functionals above is the possibility of application of the so-called Direct Method of the Calculus of Variations, that is summarized in the equation

lower semicontinuity + compactness = existence of minimizers,

and hence the study of necessary and sufficient conditions for their lower semicontinuity in suitable topologies. A general lower semicontinuity result it is not yet available. Partial results are due to L. Ambrosio [7], which ensure the lower semicontinuity when f is *convex* (in the last variable) and has a *superlinear growth* at infinity, and g is *BV-elliptic* and has a *superlinear growth* for $|u^+ - u^-| \to 0$ (for example if $g \geq c > 0$). These conditions guarantee compactness separately for the bulk and jump part of

the derivative, so that the two integrals in (0.2) can be dealt with separately. Nevertheless, in the case of *vector-valued u* (see for instance [75], [40], [1], [57]) it is well-known that the natural assumption on f is the *quasiconvexity* in the sense of C. B. Morrey (see [75], [74]). If $M^{k \times n}$ denotes the space of $k \times n$ matrices, we recall that a continuous function $f : M^{k \times n} \rightarrow [0, +\infty[$ is said to be *quasiconvex* if for every $\xi \in M^{k \times n}$, A bounded subset of $\mathbb{R}^n$, and $v \in \mathcal{C}_c^1(A; \mathbb{R}^k)$ we have the inequality

$$|A|f(\xi) \leq \int_A f(\xi + \nabla v(x)) \, dx.$$

In this framework, an interesting result, due to L. Ambrosio & G. Dal Maso [11], is the L^1-lower semicontinuity of the integral defined on $BV(\Omega; \mathbb{R}^k)$ by

$$(0.3) \qquad \int_\Omega f(\nabla u(x)) \, dx + \int_\Omega f^\infty \left(\frac{D_s u}{|D_s u|} \right) |D_s u|,$$

under the assumption of f being *quasiconvex* and with linear growth (f^∞ is the recession function of f and $\dfrac{D_s u}{|D_s u|}$ denotes the Radon-Nikodym derivative of the measure $D_s u$ with respect to its total variation $|D_s u|$). This result has been recently generalized by I. Fonseca & S. Müller [58], allowing the dependence of f also on x and u. Considering the restriction of the functional (0.3) to $SBV(\Omega; \mathbb{R}^k)$, we have

$$(0.4) \qquad g(u^+, u^-, \nu) = f^\infty((u^+ - u^-) \otimes \nu_u).$$

Condition (0.4) is satisfied in some models, but in general it is not possible to obtain the *effective* surface energy density by simply considering the volume energy density. No lower semicontinuity results on $SBV(\Omega; \mathbb{R}^k)$ are known so far under the assumption of f being *quasiconvex*.

Purpose of this work is to give a *relaxation* and *integral representation* result on the special yet meaningful class of functionals defined on $SBV(\Omega; \mathbb{R}^k)$ by integrals of the form

$$(0.5) \qquad F(u) = \int_\Omega f(\nabla u(x)) \, dx + \int_{S_u \cap \Omega} g((u^+ - u^-) \otimes \nu_u) \, d\mathcal{H}^{n-1}.$$

The lower semicontinuous envelope in the L^1-topology of the functional F, ; *i.e.*, the greatest L^1-lower semicontinuous functional less than or equal to F, is defined by relaxation as

$$\overline{F}(u) = \inf \{ \liminf_h F(u_h) : (u_h) \text{ in } SBV(\Omega; \mathbb{R}^k), u_h \rightarrow u \text{ in } L^1(\Omega, \mathbb{R}^k) \}.$$

Under the only hypotheses (besides the necessary measurability conditions)

$f : M^{k \times n} \to [0, +\infty]$ finite at one point, say at 0,

$g : M^{k \times n} \to [0, +\infty]$ positively 1-homogeneous

and locally bounded in n independent directions of $\mathbb{R}^n$,

we prove that $\overline{F}$ can be represented as an integral on the whole $BV(\Omega; \mathbb{R}^k)$. In this case, the relaxation of F takes into account, both in its volume and in its surface part, of the combined effects of f and g, and it can be written on the whole $BV(\Omega; \mathbb{R}^k)$ as

$$(0.6) \qquad \overline{F}(u) = \int_\Omega \varphi(\nabla u(x))\, dx + \int_\Omega \varphi^\infty \Big(\frac{D_s u}{|D_s u|}\Big) |D_s u|.$$

The function φ is a quasiconvex function with linear growth (whatever the growth conditions satisfied by f may be), and it satisfies the formula

$$(0.7) \qquad \varphi(\xi) = \sup\{\psi(\xi) \ : \ \psi \text{ quasiconvex, } \psi \le f \text{ on } M^{k \times n},$$
$$\psi^\infty(w) \le g(w) \text{ if } rank(w) \le 1\}.$$

The chapter is divided as follows. Section 1 is devoted to the statement of our main result, Theorem 2.1. In the second section we recall some relaxation results in BV and Sobolev spaces. In Section 3 we prove the main theorem in several steps. The first one is to establish that under the very weak hypotheses on f and g the relaxation $\overline{F}$ is of linear growth, and indeed finite, on the whole $BV(\Omega; \mathbb{R}^k)$. Then we prove by a measure theoretic approach, and localization technique, that the study of the relaxed functional at a fixed u can be reduced to the study of a regular Borel measure on Ω. This fact allows us to use some integral representation arguments by G.Buttazzo & G. Dal Maso [33] and to write the restriction of $\overline{F}$ to $W^{1,1}$ as an integral. The final step is to use the lower semicontinuity results by L. Ambrosio & G. Dal Maso [11] in order to obtain upper and lower bounds for $\overline{F}$ on the whole $BV(\Omega; \mathbb{R}^k)$; the use of formula (0.7) shows that these bounds coincide, and gives (0.6). Let us remark that in the scalar case (; *i.e.*, when $k = 1$) this result can be obtained, together with a simpler formula for φ, using a direct construction of the "recovery sequences" for $\overline{F}(u)$ (see Chapter 2, Theorem 2.1); this approach is not possible in the vector-valued case. Finally in Section 4 we specialize formula (0.7) to some particular f and g, and we provide some applications of our result.

1. The Main Relaxation and Integral Representation Result

We refer to Chapter 1 for the notation, the properties of the functions of bounded variation, the notions of quasiconvexity, of rank one convexity, and of relaxed functional. The letter c will denote throughout the chapter a strictly positive constant, whose value may vary from line to line, and which is independent of the parameters of the problems each time considered.

The main result of this chapter is to characterize the relaxation in the L^1-topology of some functionals defined in $SBV(\Omega; \mathbb{R}^k)$.

Theorem 1.1. *Let $f : M^{k \times n} \to [0, +\infty]$ be a positive Borel function such that $f(0) \neq +\infty$, and let $g : M_1^{k \times n} \to [0, +\infty]$ be a positively 1-homogeneous Borel function. On the function g we suppose moreover that there exist n linearly independent vectors $\nu_1, \ldots, \nu_n$ in S^{n-1} such that g is locally bounded on $\mathbb{R}^k \otimes \nu_m$ for all $m = 1, \ldots, n$.*

Let Ω be a bounded open subset of $\mathbb{R}^n$, and let us define the functional $\mathcal{F} : BV(\Omega; \mathbb{R}^k) \to [0, +\infty]$ by setting

$$\mathcal{F}(u) = \begin{cases} \displaystyle\int_\Omega f(\nabla u(x))\, dx + \int_{S_u \cap \Omega} g((u^+ - u^-) \otimes \nu_u)\, d\mathcal{H}^{n-1} \\ \qquad\qquad\qquad\qquad \text{if } u \in SBV(\Omega; \mathbb{R}^k) \\ \\ +\infty \qquad\qquad\qquad\quad \text{if } u \in BV(\Omega; \mathbb{R}^k) \setminus SBV(\Omega; \mathbb{R}^k). \end{cases}$$

Then the lower semicontinuous envelope of $\mathcal{F}$ in the $L^1(\Omega; \mathbb{R}^k)$-topology is given by

$$(1.1) \qquad \overline{\mathcal{F}}(u) = \int_\Omega \varphi(\nabla u(x))\, dx + \int_\Omega \varphi^\infty\Big(\frac{D_s u}{|D_s u|}\Big) |D_s u|$$

for every $u \in BV(\Omega; \mathbb{R}^k)$, where the function $\varphi : M^{k \times n} \to [0, +\infty[$ is given by

$$(1.2) \qquad \begin{aligned} \varphi(\xi) = \sup\{\psi(\xi) \ &: \ \psi \text{ quasiconvex}, \ \psi \leq f \text{ on } M^{k \times n}, \\ &\quad \psi^\infty(w) \leq g(w) \text{ if } rank(w) \leq 1\}, \end{aligned}$$

and satisfies $0 \leq \varphi(\xi) \leq c(1 + |\xi|)$ for all $\xi \in M^{k \times n}$.

Remark 1.2. Let us suppose that, in addition, the function f is convex and the function g is rank one convex on $\mathbb{R}^k \otimes \nu$ for every $\nu \in S^{n-1}$, which means that

$$g(\lambda a \otimes \nu + (1 - \lambda)b \otimes \nu) \leq \lambda g(a \otimes \nu) + (1 - \lambda)g(b \otimes \nu),$$

for every $a, b \in \mathbb{R}^k$, $\lambda \in [0, 1]$, and $\nu \in S^{n-1}$. Such a property is satisfied, for instance, if g is (the restriction to $M_1^{k \times n}$ of) a quasiconvex function.

The functional F is convex on $SBV(\Omega; \mathbb{R}^k)$; it is straightforward to check that its relaxation $\overline{F}$ must be convex too, and hence also the integrand φ. Then we have the formula

$$\varphi(\xi) = \sup\{\psi(\xi) \; : \; \psi \text{ convex}, \; \psi \leq f \text{ on } M^{k \times n}, \psi^\infty \leq g \text{ on } M_1^{k \times n}\}.$$

As a corollary to Theorem 1.1, in the scalar case (; *i.e.*, $k = 1$) we get the following result, which generalizes the relaxation Theorem 2.1 of [27] (see Chapter 2, Theorem 2.1), where a simpler formula for φ is obtained using a direct construction of the "recovery sequences" for $\overline{\mathcal{F}}(u)$ (see also [25]).

Corollary 1.3. *Let $k = 1$. Under the hypotheses of Theorem 1.1 the function φ verifies the formula*

$$\varphi(z) = \big(f \wedge (f(0) + g)\big)^{**}(z)$$

for all $z \in \mathbb{R}^n$ (h^{**} denotes the greatest convex and lower semicontinuous function less than or equal to h).

Proof. Let ψ be a convex function such that $\psi \leq f$ and $\psi^\infty \leq g$ on $\mathbb{R}^n$. In particular we have $\psi(0) \leq f(0)$ and it is easy to check then that $\psi \leq f(0) + g$. Recalling Remark 1.2 we conclude that

$$\varphi(z) = \sup\{\psi(z) : \; \psi \text{ convex}, \; \psi \leq f \wedge (f(0) + g)\}$$
$$= \big(f \wedge (f(0) + g)\big)^{**}(z). \qquad \square$$

For additional remarks and examples, see Section 4.

2. Preliminary Results on Relaxation

In order to prove Theorem 1.1 we shall make use of some relaxation results. The first one deals with functionals defined on Sobolev spaces.

Theorem 2.1. *(G. Buttazzo & G. Dal Maso [33] Theorem 1.1 and [32] Theorem 4.3.2) Let $\Omega \subset \mathbb{R}^n$ be an open set, and $F : W^{1,1}(\Omega; \mathbb{R}^k) \times \mathcal{A}(\Omega) \to [0, +\infty[$ be a functional satisfying for every $u, v \in W^{1,1}(\Omega; \mathbb{R}^k)$ and for every $A \in \mathcal{A}(\Omega)$:*

(i) (linear growth condition) $|F(u, A)| \leq c(|A| + \int_A |\nabla u(x)| \, dx)$;

(ii) (locality) $F(u, A) = F(v, A)$ whenever $u = v$ on A;

(iii) (semicontinuity) $F(\cdot, A)$ is $W^{1,1}$-sequentially lower semicontinuous;

(iv) (translation invariance) $F(u+b, A) = F(u, A)$ for every constant vector $b \in \mathbb{R}^k$;

(v) $F(u, \cdot)$ is the restriction to $\mathcal{A}(\Omega)$ of a regular Borel measure.

Then there exists a Carathéodory function $\psi : \Omega \times M^{k \times n} \to [0, +\infty[$, quasiconvex in the second variable for a.e. $x \in \Omega$, such that the integral representation

$$F(u, A) = \int_A \psi(x, \nabla u(x)) \, dx$$

holds for every $u \in W^{1,1}(\Omega; \mathbb{R}^k)$ and for every $A \in \mathcal{A}(\Omega)$.

The second result we recall is a lower semicontinuity and relaxation theorem for quasiconvex integrals on the space of vector-valued *BV*-functions.

Theorem 2.2. *(L. Ambrosio & G. Dal Maso [11] Theorem 4.1) Let $\varphi : M^{k \times n} \to [0, +\infty[$ be a quasiconvex function satisfying*

(2.1) $$0 \le \varphi(\xi) \le c(1 + |\xi|) \qquad \text{for every } \xi \in M^{k \times n},$$

and let us define on $BV(\Omega; \mathbb{R}^k)$ the functional $\mathcal{F}$ by setting

(2.2) $$\mathcal{F}(u) = \int_\Omega \varphi(\nabla u(x)) \, dx + \int_\Omega \varphi^\infty \left(\frac{D_s u}{|D_s u|} \right) |D_s u|.$$

Then $\mathcal{F}$ is $L^1(\Omega; \mathbb{R}^k)$-lower semicontinuous on $BV(\Omega; \mathbb{R}^k)$, and we have

(2.3) $$F = \overline{F + \chi_{W^{1,1}(\Omega; \mathbb{R}^k)}};$$

; i.e., F coincides with the relaxation of its restriction to the Sobolev space $W^{1,1}(\Omega; \mathbb{R}^k)$.

Remark that in order to have a good definition of $\mathcal{F}$ in (2.2) (and of $\overline{\mathcal{F}}$ in (1.1)) we have to extend the notion of $\varphi^\infty(\xi)$ to a quasiconvex φ. This quantity is well-defined by (2.1) if the rank of ξ is less than or equal to one since quasiconvex functions are convex in rank one directions. In general the limit in (2.1) does not exist for all $\xi \in M^{k \times n}$ (cf. Müller [76]). Nevertheless, a recent result by G. Alberti [3] ensures that the matrix $\dfrac{D_s u}{|D_s u|}$ is of rank 1 $|D_s u|$-a.e., and hence every quantity is well-defined in (2.2) and (1.1).

3. Proof of the Main Result

We start by giving an upper bound for the functional $\overline{\mathcal{F}}$. We shall consider the functional $\mathcal{G} \geq \mathcal{F}$ defined by

$$
\mathcal{G}(u) = \begin{cases} \displaystyle\int_\Omega f(\nabla u(x))\,dx + \int_{S_u \cap \Omega} g((u^+ - u^-) \otimes \nu_u)\,d\mathcal{H}^{n-1} \\[2mm] \qquad\qquad \text{if } u \in SBV(\Omega;\mathbb{R}^k) \text{ and } \mathcal{H}^{n-1}(S_u \cap \Omega) < +\infty \\[4mm] +\infty \qquad\qquad \text{elsewhere in } BV(\Omega;\mathbb{R}^k). \end{cases}
$$

Obviously an upper bound for $\overline{\mathcal{G}}$ will do as well.

Proposition 3.1. *We have* $\overline{\mathcal{G}}(u) \leq c(1 + |Du|(\Omega))$ *for all functions* $u \in BV(\Omega;\mathbb{R}^k)$.

Proof. We first prove the proposition under the additional hypothesis

(3.1) g locally bounded on rank one matrices.

We define then the constant $M < +\infty$ by setting

(3.2) $M = \sup\{g(a \otimes \nu) \ : \ a \in S^{k-1}, \ \nu \in S^{n-1}\}.$

We shall deal first with the scalar case ($k = 1$), and then extend the proof to the case of vector-valued u.

Step 1: $k = 1$. In this case g is defined on the whole $\mathbb{R}^n = \mathbb{R} \otimes \mathbb{R}^n$. Let us consider a function $u \in BV(\Omega) \cap C^1(\Omega)$, and let us fix $h \in \mathbb{N}$; by the coarea formula (2.6), Chapter 1 we have

$$
|Du|(\Omega) = \sum_{j \in \mathbb{Z}} \int_{\frac{j}{h}}^{\frac{j+1}{h}} \mathcal{H}^{n-1}(\partial^*\{u > t\} \cap \Omega)\,dt.
$$

Hence, by the mean value theorem, for every $j \in \mathbb{Z}$ we can find $s_j^h \in \ ]\frac{j}{h}, \frac{j+1}{h}[$ such that

$$
\frac{1}{h}\mathcal{H}^{n-1}(\partial^*\{u > s_j^h\} \cap \Omega) \leq \int_{\frac{j}{h}}^{\frac{j+1}{h}} \mathcal{H}^{n-1}(\partial^*\{u > t\} \cap \Omega)\,dt,
$$

so that

(3.3) $\displaystyle\sum_{j \in \mathbb{Z}} \frac{1}{h}\mathcal{H}^{n-1}(\partial^*\{u > s_j^h\} \cap \Omega) \leq |Du|(\Omega).$

Let us construct now the sequence (u_h) in $SBV(\Omega)$ by setting

$$(3.4) \qquad u_h(x) = \frac{j}{h} \quad \text{on } \{s^h_{j-1} < u < s^h_j\}.$$

It is clear that for every $h \in \mathbb{N}$ we have $\nabla u_h(x) = 0$ for a.e. $x \in \Omega$,

$$S_{u_h} \cap \Omega = \bigcup_{j \in \mathbb{Z}} \partial^* \{u > s^h_j\} \cap \Omega,$$

and

$$Du_h = D_s u_h = \sum_{j \in \mathbb{Z}} \frac{1}{h} \nu^j_h \, \mathcal{H}^{n-1}{\lfloor}_{\partial^* \{u > s^h_j\} \cap \Omega},$$

where ν^j_h is defined by

$$D1_{\{u > s^h_j\}} = \nu^j_h \mathcal{H}^{n-1}{\lfloor}_{\partial^* \{u > s^h_j\}}.$$

Hence we obtain

$$\mathcal{F}(u_h) = \int_\Omega f(\nabla u_h(x)) \, dx + \int_{S_{u_h} \cap \Omega} g((u^+_h - u^-_h) \otimes \nu_{u_h}) \, d\mathcal{H}^{n-1}$$

$$(3.5) \qquad \begin{aligned} &= f(0)|\Omega| + \sum_{j \in \mathbb{Z}} \int_{\partial^* \{u > s^h_j\} \cap \Omega} g\left(\frac{1}{h} \nu^j_h\right) d\mathcal{H}^{n-1} \\ &\le f(0)|\Omega| + \sum_{j \in \mathbb{Z}} \frac{1}{h} M \, \mathcal{H}^{n-1}(\partial^* \{u > s^h_j\} \cap \Omega) \\ &\le f(0)|\Omega| + M|Du|(\Omega). \end{aligned}$$

We have made use here of (3.2), (3.3), and of the positive homogeneity of g. Remark also that for every h

$$\mathcal{H}^{n-1}(S_{u_h} \cap \Omega) = \sum_{j \in \mathbb{Z}} \mathcal{H}^{n-1}(\partial^* \{u > s^h_j\} \cap \Omega) \le h|Du|(\Omega) < +\infty,$$

hence $\mathcal{G}(u_h) = \mathcal{F}(u_h)$.

Since $u_h \to u$ in $L^\infty(\Omega)$, by the definition of $\overline{\mathcal{G}}$ we conclude that

$$\overline{\mathcal{G}}(u) \le \liminf_h \mathcal{G}(u_h) \le f(0)|\Omega| + M \, |Du|(\Omega).$$

For a general $u \in BV(\Omega)$ it suffices to recall that there exists a sequence (v_h) in $\mathcal{C}^\infty(\Omega) \cap BV(\Omega)$ (for example obtained by convolution from u; see [61]) such that $v_h \to u$ in $\mathrm{L}^1(\Omega)$ and

$$|Du|(\Omega) = \lim_h |Dv_h|(\Omega) = \lim_h \int_\Omega |\nabla v_h|\, dx.$$

By the lower semicontinuity of $\overline{\mathcal{G}}$ we obtain then

$$\overline{\mathcal{G}}(u) \leq \liminf_h \overline{\mathcal{G}}(v_h) \leq \lim_h (f(0)|\Omega| + M\,|Dv_h|(\Omega))$$
$$= f(0)|\Omega| + M\,|Du|(\Omega).$$

Step 2: $k \geq 2$. In this case we can proceed "componentwise". Let us fix a function $u = (u_{(1)}, \ldots, u_{(k)}) \in \mathcal{C}^1(\Omega; \mathbb{R}^k) \cap BV(\Omega; \mathbb{R}^k)$. Proceeding as in Step 1, for every $h \in \mathbb{N}$, $j \in \mathbb{Z}$, and every $i = 1, \ldots, k$ we can find $s_j^{i,h} \in]\frac{j}{h}, \frac{j+1}{h}[$ such that

$$\frac{1}{h}\mathcal{H}^{n-1}(\partial^* \{u_{(i)} > s_j^{i,h}\} \cap \Omega) \leq \int_{\frac{i}{h}}^{\frac{i+1}{h}} \mathcal{H}^{n-1}(\partial^* \{u_{(i)} > t\} \cap \Omega)\, dt.$$

We can therefore define the function $u_h \in SBV(\Omega; \mathbb{R}^k)$ by setting

$$(3.6) \qquad u_{h(i)}(x) = \frac{j}{h} \quad \text{on } \{s_{j-1}^{i,h} < u_{(i)} < s_j^{i,h}\}.$$

By (3.6) we have that

$$Du_h = D_s u_h = \sum_{i=1}^k \sum_{j\in\mathbb{Z}} \frac{1}{h} e_i \otimes \nu_{i,h}^j \mathcal{H}^{n-1}\big|_{\partial^* \{u_{(i)} > s_j^{i,h}\}\cap\Omega},$$

where $\nu_{i,h}^j(x)$ is defined by

$$D\mathbf{1}_{\{u_{(i)} > s_j^{i,h}\}} = \nu_{i,h}^j \mathcal{H}^{n-1}\big|_{\partial^* \{u_{(i)} > s_j^{i,h}\}}.$$

We can proceed now as in Step 1 and obtain

$$\overline{\mathcal{G}}(u) \leq \liminf_h \mathcal{G}(u_h) \leq f(0)|\Omega| + \sqrt{k}\, M\,|Du|(\Omega).$$

The same inequality holds on the whole $BV(\Omega; \mathbb{R}^k)$ by approximation.

Step 3: the case of g not locally bounded. Under the general hypotheses of Theorem 1.1 the function g is not necessarily locally bounded on $M_1^{k \times n}$, but it is on the subspaces $\mathbb{R}^k \otimes \nu_m$ for $m = 1, \ldots, n$. We have to modify the proof of the previous steps in order to have jump part densities of the form $a \otimes \nu_m$. It suffices to consider for every i, h, and j a polyhedron $P_j^{i,h}$ with

$$\{u_{(i)} > \frac{j+1}{h}\} \subset P_j^{i,h} \subset \{u_{(i)} > \frac{j}{h}\},$$

and such that

$$\mathcal{H}^{n-1}(\partial P_j^{i,h} \cap \Omega) \leq \mathcal{H}^{n-1}(\partial^* \{u_{(i)} > s_j^{i,h}\} \cap \Omega) + \frac{1}{h} 2^{-|j|}.$$

It is clear that each of these polyhedra can be approximated by polyhedra each of whose faces is orthogonal to one of the vectors $\nu_1, \ldots, \nu_n$, increasing the surface area by at most a constant factor depending on this basis. We can suppose then that each of the faces of $P_j^{i,h}$ is orthogonal to some ν_m, and that

$$\mathcal{H}^{n-1}(\partial P_j^{i,h} \cap \Omega) \leq c\big(\mathcal{H}^{n-1}(\partial^* \{u_{(i)} > s_j^{i,h}\} \cap \Omega) + \frac{1}{h} 2^{-|j|}\big).$$

We can then define $u_h \in SBV(\Omega; \mathbb{R}^k)$ by setting

$$u_{h(i)}(x) = \frac{j}{h} \quad \text{on } P_{j-1}^{i,h} \setminus P_j^{i,h},$$

and conclude the proof as in Step 2, taking now

(3.7) $$M = \sup \{ g(a \otimes \nu_m) : a \in S^{k-1}, \quad m = 1, \ldots, n\}.$$

$\square$

Remark. If we take some extra care in Step 2 of the previous proposition, we can obtain approximating sequences which jump only in the coordinate directions of the target space $\mathbb{R}^k$ (; *i.e.*, their jump part densities have the form $e_i \otimes b$). It suffices to choose the $s_j^{i,h}$ so that

$$\mathcal{H}^{n-1}((\partial^* \{u_{(i)} > s_j^{i,h}\} \cap \partial^* \{u_{(l)} > s_m^{l,h}\}) \cap \Omega) = 0$$

for every $m, j \in \mathbb{Z}$ and for every $i, l \in \{1, \ldots, k\}$ with $i \neq l$.

Taking into account the construction of Step 3 of the previous proposition we can obtain jump part densities of the form

$$\frac{1}{h} e_i \otimes \nu_m.$$

Hence the conclusion of Proposition 3.1 still holds true under the only hypothesis of g to be finite on the set

$$\{e_i \otimes \nu_m : i = 1, \ldots, k, \ m = 1, \ldots, n\}.$$

We localize the functional $\mathcal{G}$ by defining for every open subset A of Ω

$$(3.8) \qquad \mathcal{G}(u, A) = \int_A f(\nabla u(x))\, dx + \int_{S_u \cap A} g((u^+ - u^-) \otimes \nu_u)\, d\mathcal{H}^{n-1}$$

if $u \in SBV(\Omega; \mathbb{R}^k)$ and $\mathcal{H}^{n-1}(S_u \cap \Omega) < +\infty$, and by setting

$$\mathcal{G}(u, A) = +\infty \qquad \text{elsewhere on } BV(\Omega; \mathbb{R}^k).$$

In the same way we define

$$\overline{\mathcal{G}}(u, A) = \inf\{\liminf_h \mathcal{G}(u_h, A) \ : \ u_h \to u \text{ in } \mathrm{L}^1(A), u_h \in BV(\Omega; \mathbb{R}^k)\}.$$

Localizing the proof of Proposition 3.1 we get the linear growth condition

$$(3.9) \qquad\qquad |\overline{\mathcal{G}}(u, A)| \leq c(|A| + |Du|(A))$$

for every $u \in BV(\Omega; \mathbb{R}^k)$ and every $A \in \mathcal{A}(\Omega)$.

Proposition 3.2.   *For every $u \in BV(\Omega; \mathbb{R}^k)$ the set function $\overline{\mathcal{G}}(u, \cdot)$ is (the restriction to the family of the open subsets of Ω of) a regular Borel measure on Ω.*

Proof. *Step 1: $\overline{\mathcal{G}}(u, \cdot)$ is regular;* ; *i.e.,* for every open set $A \subset \Omega$, we have

$$(3.10) \qquad \overline{\mathcal{G}}(u, A) = \sup\{\overline{\mathcal{G}}(u, A') \ : \ A' \text{ open}, A' \subset\subset A\}.$$

We shall first consider the case of g locally bounded; ; *i.e.,* there exists a constant M, defined as in (3.2), such that

$$g(a \otimes \nu) \leq M\,|a|.$$

Let us remark that $\overline{\mathcal{G}}(u, \cdot)$ is an increasing set function; ; *i.e.,* $\overline{\mathcal{G}}(u, A') \leq \overline{\mathcal{G}}(u, A)$ if $A' \subset A$, hence the inequality "$\geq$" in (3.10) is trivial. Let us prove now the opposite inequality. Fixed K a compact subset of A, let us define $\delta = \frac{1}{2}\mathrm{dist}\,(\partial A, K)$, $\mathrm{d}_K(x) = \mathrm{dist}\,(x, K)$,

$$B(t) = \{x \in A \ : \ \mathrm{d}_K(x) < t\} \qquad t \in]0, \delta[,$$

and $B = B(\delta) = \{x \in A \ : \ \mathrm{d}_K(x) < \delta\}$.

Let us choose two sequences of functions $(u_h), (v_h)$ in $SBV(\Omega; \mathbb{R}^k)$ such that $u_h \to u$ in $\mathrm{L}^1(B)$, $v_h \to u$ in $\mathrm{L}^1(A \setminus K)$, and

$$\overline{\mathcal{G}}(u, B) = \lim_h \mathcal{G}(u_h, B)$$

$$\overline{\mathcal{G}}(u, A \setminus K) = \lim_h \mathcal{G}(v_h, A \setminus K).$$

Since we have $\mathcal{H}^{n-1}(S_{u_h} \cap \Omega) + \mathcal{H}^{n-1}(S_{v_h} \cap \Omega) < +\infty$, then for every h the set of $t \in]0, \delta[$ that do not verify

$$(3.11) \qquad \mathcal{H}^{n-1}(S_{u_h} \cap \partial^* B(t)) + \mathcal{H}^{n-1}(S_{v_h} \cap \partial^* B(t)) = 0$$

is at most countable (in fact, the set of t for which this quantity is larger than $\frac{1}{h}$ is finite). In the same way we have that

$$(3.12) \qquad |Du_h|(\partial^* B(t)) + |Dv_h|(\partial^* B(t)) = 0$$

except for at most a countable set of $t \in]0, \delta[$.

We can apply Fleming & Rishel coarea formula (2.7), Chapter 1 to the integral

$$\int_{B \setminus K} |u_h - v_h|\, dx = \int_{B \setminus K} |u_h - v_h| |\nabla d_K|\, dx$$

$$= \int_0^\delta \int_{\partial^* B(t)} |\tilde{u}_h(x) - \tilde{v}_h(x)|\, d\mathcal{H}^{n-1}(x)\, dt$$

(recall that the a.e. gradient ∇d_K of the Lipschitz function d_K has unit length a.e.). By the mean value theorem for every h we can choose $t_h \in]0, \delta[$ such that (3.11) and (3.12) hold, $B(t_h)$ is a set of finite perimeter, and

$$(3.13) \qquad \int_{\partial^* B(t_h)} |\tilde{u}_h - \tilde{v}_h|\, d\mathcal{H}^{n-1} \le \frac{1}{\delta} \int_{B \setminus K} |u_h - v_h|\, dx.$$

We can define the sequence (w_h) in $\mathrm{L}^1(A)$ by setting

$$w_h = \begin{cases} u_h & \text{in } B(t_h) \\ v_h & \text{in } A \setminus B(t_h). \end{cases}$$

Note that for every h we have $w_h \in SBV(\Omega; \mathbb{R}^k)$ and

$$\nabla w_h = \nabla u_h\, 1_{B(t_h)} + \nabla v_h 1_{A \setminus B(t_h)};$$

moreover the Hausdorff part of the measure Dw_h is given by

$$(u_h^+ - u_h^-) \otimes \nu_{u_h} \cdot \mathcal{H}^{n-1}{}_{|S_{u_h} \cap B(t_h)} + (v_h^+ - v_h^-) \otimes \nu_{v_h} \cdot \mathcal{H}^{n-1}{}_{|S_{v_h} \cap (A \setminus B(t_h))}$$

$$+ (\tilde{u}_h - \tilde{v}_h) \otimes \nu_{\partial^* B(t_h)} \cdot \mathcal{H}^{n-1}{}_{|\partial^* B(t_h)},$$

where $\nu_{\partial^* B(t_h)}$ denotes the normal to $\partial^* B(t_h)$ pointing inwards $B(t_h)$.

We obtain then

$$\mathcal{G}(w_h, A) \leq \mathcal{G}(u_h, B) + \mathcal{G}(v_h, A \setminus K) + M \int_{\partial^* B(t_h)} |w_h{}^+ - w_h{}^-| \, d\mathcal{H}^{n-1}$$

$$= \mathcal{G}(u_h, B) + \mathcal{G}(v_h, A \setminus K) + M \int_{\partial^* B(t_h)} |\tilde{u}_h - \tilde{v}_h| \, d\mathcal{H}^{n-1}$$

$$\leq \mathcal{G}(u_h, B) + \mathcal{G}(v_h, A \setminus K) + M \frac{1}{\delta} \int_{B \setminus K} |u_h - v_h| \, dx. \qquad (3.14)$$

Since $w_h \to u$ in $\mathrm{L}^1(A)$, and $(u_h - v_h) \to 0$ in $\mathrm{L}^1(B \setminus K)$, we have, by taking the limit as $h \to +\infty$,

$$\overline{\mathcal{G}}(u, A) \leq \liminf_h \mathcal{G}(w_h, A) \leq \overline{\mathcal{G}}(u, B) + \overline{\mathcal{G}}(u, A \setminus K).$$

By (3.9) we get then

$$\overline{\mathcal{G}}(u, A) \leq \overline{\mathcal{G}}(u, B) + c(|A \setminus K| + |Du|(A \setminus K)).$$

Since the last term in this inequality can be taken arbitrarily small and $B \subset\subset A$, we obtain the desired inequality in (3.10).

We can remove now the hypothesis of local boundedness of g, assuming the only hypotheses of Theorem 1.1. Remark that the only place where we made use of the local boundedness of g is inequality (3.14). We choose now a closed polyhedron P with faces orthogonal to the directions $\nu_1, \ldots, \nu_n$ such that $K \subset P \subset\subset A$. Let us consider, in place of the usual distance, the new distance $\mathrm{dist}^\nu(x, y) = \sup_m |\langle x - y, \nu_m \rangle|$, and $\mathrm{d}_P(x) = \min\{\mathrm{dist}^\nu(x, y) : y \in P\}$. With this new definition of the distance we can proceed as above with P instead of K, remarking that the sets $B(t)$ are all polyhedra with faces orthogonal to the directions $\nu_1, \ldots, \nu_n$. It is clear that we take into account only of the values of g on the sets $\mathbb{R}^k \otimes \nu_m$, $m = 1, \ldots, n$, and therefore we obtain (3.14) with M defined as in (3.7).

Step 2: $\overline{\mathcal{G}}(u, \cdot)$ is a subadditive set function; ; i.e., we have

$$\overline{\mathcal{G}}(u, A_1 \cup A_2) \leq \overline{\mathcal{G}}(u, A_1) + \overline{\mathcal{G}}(u, A_2)$$

for every pair of open subsets A_1, A_2 of Ω.

By the regularity of $\overline{\mathcal{G}}(u, \cdot)$ (Step 1), it is sufficient to prove that

$$\overline{\mathcal{G}}(u, A) \leq \overline{\mathcal{G}}(u, A_1) + \overline{\mathcal{G}}(u, A_2)$$

for every open set $A \subset\subset A_1 \cup A_2$. This inequality can be proved by arguing as in Step 1, choosing $K = A \setminus A_2$, and

$$B = \{x \in A \ : \ \mathrm{dist}(x, K) < \frac{1}{2}\mathrm{dist}(K, A \setminus A_1)\}.$$

Moreover it is clear that $\overline{\mathcal{G}}(u, \cdot)$ is additive on disjoint sets; ; *i.e.,*

$$\overline{\mathcal{G}}(u, A_1 \cup A_2) = \overline{\mathcal{G}}(u, A_1) + \overline{\mathcal{G}}(u, A_2)$$

if $A_1 \cap A_2 = \emptyset$.

Step 3: $\overline{\mathcal{G}}(u, \cdot)$ is the restriction to the open subsets of Ω of a regular Borel measure. It suffices to remark that the set function $\overline{\mathcal{G}}(u, \cdot)$ verifies:

(a) $\overline{\mathcal{G}}(u, \cdot)$ is a positive and increasing set function defined on $\mathcal{A}(\Omega)$;

(b) $\overline{\mathcal{G}}(u, \cdot)$ is regular (Step 1);

(c) $\overline{\mathcal{G}}(u, \cdot)$ is subadditive, and it is additive on disjoint sets (Step 2),
and apply Theorem 5.6 by E. De Giorgi & G. Letta [50]. $\qquad\square$

Proposition 3.3. *There exists a quasiconvex function $\psi : M^{k \times n} \to [0, +\infty[$ such that we have*

$$(3.15) \qquad \overline{\mathcal{G}}(u, A) = \int_A \psi(\nabla u(x)) \, dx$$

for every $A \in \mathcal{A}(\Omega)$ and $u \in W^{1,1}(\Omega; \mathbb{R}^k)$. The function ψ verifies

$$(3.16) \qquad 0 \le \psi(\xi) \le c(1 + |\xi|)$$

for every $\xi \in M^{k \times n}$.

Proof. We want to apply Theorem 2.1, which ensures the representation

$$(3.17) \qquad \overline{\mathcal{G}}(u, A) = \int_A \psi(x, \nabla u(x)) \, dx$$

for a suitable quasiconvex Carathéodory function $\psi : \Omega \times M^{k \times n} \to [0, +\infty[$. Let us check the hypotheses of Theorem 2.1. We have already proved conditions (i) and (v) (see (3.9) and Proposition 3.2). Property (ii) follows from the definition of $\overline{\mathcal{G}}$, while (iii) is verified since $\overline{\mathcal{G}}(\cdot, A)$ is L^1-lower semicontinuous. Finally, it is easy to check that $\mathcal{G}$ verifies $\mathcal{G}(u + z, A) = \mathcal{G}(u, A)$ for every $u \in BV(\Omega; \mathbb{R}^k)$, $A \in \mathcal{A}(\Omega)$, and for every constant vector $z \in \mathbb{R}^k$, and so does $\overline{\mathcal{G}}$. Hence we obtain the representation formula (3.17) for every $u \in W^{1,1}(\Omega; \mathbb{R}^k)$.

In order to prove that ψ does not depend on x, we observe that, by the definition of $\mathcal{G}$ and $\overline{\mathcal{G}}$, if we compute $\overline{\mathcal{G}}$ on the linear function $u_\xi(x) = \xi x$, we obtain

$$\int_{B_1} \psi(x, \xi) \, dx = \int_{B_2} \psi(x, \xi) \, dx$$

on any pair of congruent balls $B_1, B_2 \subset \Omega$. This equality implies that $\psi(x, \xi) = \psi(y, \xi)$ at every pair of Lebesgue points of the function $\psi(\cdot, \xi)$. If we choose a dense sequence (ξ_h) in $M^{k \times n}$, using the continuity of $\psi(x, \cdot)$ we get the existence of a set $N \subset \Omega$ with $|N| = 0$, and such that

$$\psi(x, \xi) = \psi(y, \xi)$$

for every $\xi \in M^{k \times n}$ and for every $x, y \in \Omega \setminus N$. Hence it is not restrictive to suppose

$$\psi(x, \xi) = \psi(\xi),$$

obtaining (3.15). The inequalities in (3.16) follow from the integral representation (3.15), using estimate (3.9) and the positivity of $\overline{\mathcal{G}}$. $\square$

Proposition 3.4. *The function ψ in Proposition 3.3 verifies*

(3.18) $\psi(\xi) \leq f(\xi)$ *for every* $\xi \in M^{k \times n}$,

(3.19) $\psi^\infty(a \otimes \nu) \leq g(a \otimes \nu)$ *for every* $a \in \mathbb{R}^k, \nu \in S^{n-1}$.

Proof. Inequality (3.18) follows for example from

$$|\Omega| \, \psi(\xi) = \overline{\mathcal{G}}(\xi x, \Omega) \leq \mathcal{G}(\xi x, \Omega) = |\Omega| \, f(\xi).$$

As for (3.19), let us fix $a \in \mathbb{R}^k$ and $\nu \in S^{n-1}$. For every $t > 0$ we can consider the linear function

$$u_t(x) = ta\langle x, \nu \rangle,$$

so that we have $Du_t = ta \otimes \nu$. We can approximate u_t in $L^1(\Omega; \mathbb{R}^k)$ with a sequence (u_t^h) in $SBV(\Omega; \mathbb{R}^k)$ defined by

$$u_t^h(x) = \frac{1}{h} ta \, [h\langle x, \nu \rangle],$$

which has jumps of size $\frac{1}{h}$ in the direction a, along hyperplanes orthogonal to ν at a regular distance $\frac{1}{h}$. Let now Q_ν be any open cube contained in Ω with an edge parallel to ν. It is easy to see then that we have

$$\mathcal{G}(u_t^h, Q_\nu) = f(0)|Q_\nu| + g(\frac{1}{h} t a \otimes \nu) \, \mathcal{H}^{n-1}(S_{u_t^h} \cap Q_\nu)$$

$$\leq f(0)|Q_\nu| + t \, g(a \otimes \nu)|Q_\nu|$$

(note that we use here only the positive homogeneity of g). Hence we obtain

$$\psi(t \, a \otimes \nu) = |Q_\nu|^{-1} \overline{\mathcal{G}}(u_t, Q_\nu) \leq |Q_\nu|^{-1} \liminf_h \mathcal{G}(u_t^h, Q_\nu) \leq f(0) + t \, g(a \otimes \nu).$$

Dividing by t, and letting $t \to +\infty$ we obtain

$$\psi^\infty(a \otimes \nu) = \lim_{t \to +\infty} \frac{\psi(t \, a \otimes \nu)}{t} \leq \lim_{t \to +\infty} \frac{f(0) + t \, g(a \otimes \nu)}{t} = g(a \otimes \nu),$$

that is, the inequality in (3.19). $\square$

Proof of Theorem 1.1. We can now prove Theorem 1.1. By the definition of $\overline{\mathcal{F}}$, and by (2.3), for every $u \in BV(\Omega; \mathbb{R}^k)$ we have

$$\overline{\mathcal{F}}(u) \leq \overline{\mathcal{G}}(u) \leq \overline{(\overline{\mathcal{G}} + \chi_{W^{1,1}(\Omega;\mathbb{R}^k)})}(u)$$

$$= \int_\Omega \psi(\nabla u(x))\, dx + \int_\Omega \psi^\infty\left(\frac{D_s u}{|D_s u|}\right) |D_s u|.$$

Let now φ be the function defined by formula (1.2). By Proposition 3.4 we have that $\psi \leq \varphi$, hence

$$(3.20) \qquad \overline{\mathcal{F}}(u) \leq \int_\Omega \varphi(\nabla u(x))\, dx + \int_\Omega \varphi^\infty\left(\frac{D_s u}{|D_s u|}\right) |D_s u|.$$

For the opposite inequality we have to prove that φ satisfies the hypotheses of Theorem 2.2. Inequalities $0 \leq \varphi \leq f$ follow immediately from the definition of φ. Let us prove that φ verifies the linear growth condition

$$(3.21) \qquad 0 \leq \varphi(\xi) \leq c(1 + |\xi|),$$

for every $\xi \in M^{k \times n}$. It suffices to show that $\phi(\xi) \leq c(1 + |\xi|)$ on $M^{k \times n}$ for every quasiconvex ϕ such that $\phi \leq f$ on $M^{k \times n}$ and $\phi^\infty \leq g$ on $M_1^{k \times n}$. Let us fix such a ϕ. By the rank one convexity of ϕ it follows that for every $\xi \in M^{k \times n}$, $a \in \mathbb{R}^k$, and $m = 1, \ldots, n$ the function $t \mapsto \phi(\xi + t\, a \otimes \nu_m)$ is convex on $\mathbb{R}$ and Lipschitz with constant $\max\{g(a \otimes \nu_m), g(-a \otimes \nu_m)\}$. Since every $\xi \in M^{k \times n}$ can be uniquely decomposed as $\xi = \sum_{m=1}^n a_m \otimes \nu_m$, for suitable vectors $a_m \in \mathbb{R}^k$, by the Lipschitz condition on ϕ we get

$$\phi(\xi) \leq \phi(0) + \sum_{m=1}^n g(a_m \otimes \nu_m) \leq f(0) + cM|\xi|,$$

where M is defined as in (3.7). Hence φ verifies (3.21), and it is quasiconvex. Finally it is easy to see that $\varphi^\infty \leq g$ on $M_1^{k \times n}$. Therefore we have

$$(3.22) \qquad \int_\Omega \varphi(\nabla u(x))\, dx + \int_\Omega \varphi^\infty\left(\frac{D_s u}{|D_s u|}\right) |D_s u| \leq \mathcal{F}(u).$$

Moreover, by Theorem 2.2 the left-hand side of (3.22) gives a L^1-lower semicontinuous functional on $BV(\Omega; \mathbb{R}^k)$, hence we obtain, by definition of relaxation,

$$\int_\Omega \varphi(\nabla u(x))\, dx + \int_\Omega \varphi^\infty\left(\frac{D_s u}{|D_s u|}\right) |D_s u| \leq \overline{\mathcal{F}}(u).$$

This inequality, together with (3.20), concludes the proof of Theorem 1.1.
$\square$

4. Additional Remarks

In this section we provide some examples and remarks for some classes of special f and g.

Remark 4.1. *(Positively 1-homogeneous functionals)* If the bulk energy density f is positively 1-homogeneous, then it is easy to see that φ is positively 1-homogeneous. In fact it is immediate to check that ψ is a quasiconvex function such that $\psi \leq f$ on $M^{k \times n}$ and $\psi^\infty \leq g$ on $M_1^{k \times n}$ if and only if for every fixed $\lambda > 0$ such is the function $\phi(\xi) = \frac{1}{\lambda}\psi(\lambda\xi)$.

Therefore we obtain the following formula for φ:

$$\varphi(\xi) = \sup\{\psi(\xi) : \psi \text{ quasiconvex and positively 1-homogeneous,}$$
$$\psi \leq f \text{ on } M^{k \times n}, \psi^\infty \leq g \text{ on } M_1^{k \times n}\},$$

hence (recall that $\psi^\infty = \psi$ for ψ positively 1-homogeneous)

$$\varphi(\xi) = \sup\{\psi(\xi) : \psi \text{ quasiconvex and positively 1-homogeneous,}$$
$$\psi \leq f \wedge g \text{ on } M^{k \times n}\}$$

(the function g is extended to $+\infty$ on $M^{k \times n} \setminus M_1^{k \times n}$).

Note that the fact that φ is positively 1-homogeneous and quasiconvex does not imply that φ is convex (cf. Müller [76]).

Remark 4.2. *(Partitions)* Let us consider the case

$$f(\xi) = \begin{cases} 0 & \text{if } \xi = 0 \\ +\infty & \text{elsewhere.} \end{cases}$$

Then the functional $\mathcal{F}$ is finite only on functions in the space $SBV(\Omega; \mathbb{R}^k)$ with $\nabla u = 0$ a.e. These functions can be identified with "partitions of Ω in sets of finite perimeter" (see Ambrosio & Braides [9], [10], Congedo & Tamanini [38]). Every such function can be expressed as

$$(4.1) \qquad\qquad u = \sum_{j \in \mathbb{N}} c_j 1_{E_j},$$

where $c_j \in \mathbb{R}^k$, and (E_j) is a partition of Ω in sets of finite perimeter. The functional can be rewritten then in the form

$$\mathcal{F}(u) = \sum_{i,j \in \mathbb{N}} \frac{1}{2} \int_{(\partial^* E_i \cap \partial^* E_j) \cap \Omega} g\Big((c_j - c_i) \otimes \nu_j\Big) \, d\mathcal{H}^{n-1},$$

where ν_j is the interior normal to E_j, and $\partial^* E_j$ denotes the reduced boundary of E_j.

Since f is positively 1-homogeneous, by Remark 4.1 the relaxed functional $\overline{\mathcal{F}}$ is given by

$$\overline{\mathcal{F}}(u) = \int_\Omega \varphi(Du) = \int_\Omega \varphi(\nabla u)\,dx + \int_\Omega \varphi\left(\frac{D_s u}{|D_s u|}\right)|D_s u|,$$

with

$$\varphi(\xi) = \sup\{\psi(\xi) : \psi \text{ quasiconvex and positively 1-homogeneous},$$

$$\psi \leq g \text{ on } M_1^{k\times n}\}$$

(note that φ is positively 1-homogeneous, hence $\varphi = \varphi^\infty$).

Remark that we obtain as a by-product of Theorem 1.1 (but also directly from [11]; see also I. Fonseca [57]) that if g is quasiconvex then the corresponding functional defined on partitions is lower semicontinuous with respect to the L^1-convergence. Hence the integrand $\tilde{g}(u, v, \nu) = g((v - u) \otimes \nu)$ is *BV-elliptic* (see Ambrosio & Braides [10]).

Remark 4.3. *(Partitions in Polyhedral Sets)* As a particular case of functionals defined on partitions, we can consider a function g finite and locally bounded only for n linearly independent directions $\nu_1, \ldots, \nu_n$ in $\mathbb{R}^n$; ; *i.e.*,

$$g(a\otimes\nu) < +\infty \implies \nu = \nu_m \text{ for some } m \in \{1, \ldots, n\},$$

$$\sup\{\, g(a \otimes \nu_m) : a \in S^{k-1}, m = 1, \ldots, n \,\} < +\infty.$$

The domain of the functional $\mathcal{F}$ is then the set $\mathcal{P}_\nu$ of all partitions of Ω of the form (4.1) into polyhedra whose faces are orthogonal to the directions $\nu_1, \ldots, \nu_n$.

If for example $g(\xi) \geq c|\xi|$ on $M^{k\times n}$ (this hypothesis ensures the existence of a minimum in (4.2)), we can apply Remark 4.2 and prove the equivalence between segmentation problems of the type

$$\inf\left\{ \sum_{i,j\in\mathbb{N}} \int_{\partial(E_i \cap \partial E_j)\cap\Omega} g((c_j - c_i) \otimes \nu_j)\, d\mathcal{H}^{n-1} \right.$$

$$\left. + \sum_{j\in\mathbb{N}} \int_{E_j} |c_j - \alpha(x)|\, dx : u = \sum_{j\in\mathbb{N}} c_j 1_{E_j} \in \mathcal{P}_\nu \right\},$$

where α is a given L^1 function, and the corresponding minimum problems in $BV(\Omega; \mathbb{R}^k)$

$$(4.2) \qquad \min\left\{ 2\int_\Omega \varphi(Du) + \int_\Omega |u(x) - \alpha(x)|\, dx : u \in BV(\Omega; \mathbb{R}^k)\right\}.$$

As an example, if $\{e_i\}_{i=1}^n$ is the canonical basis in $\mathbb{R}^n$ and

$$g(\xi) = \begin{cases} |a| & \text{if } \xi = a \otimes e_m, \, m = 1, \dots, n \\ +\infty & \text{otherwise,} \end{cases}$$

the function φ can be computed explicitly. Indeed, by Remarks 4.1 and 4.2 we have

$$\varphi(\xi) = g^{**}(\xi) = \sum_{j=1}^n \sqrt{\sum_{i=1}^k \xi_{ij}^2}.$$

A similar representation for φ can be obtained for an arbitrary choice of the basis $\nu_1, \dots, \nu_n$ in $\mathbb{R}^n$ instead of the canonical one.

Remark 4.4. *(The two-well problem)* Let A and $B \in M^{k \times n}$ be two matrices such that $rank(A - B) \geq 2$. If we take

$$f(\xi) = \begin{cases} 0 & \text{if } \xi = A \text{ or } \xi = B, \\ +\infty & \text{otherwise on } M^{k \times n}, \end{cases}$$

and

$$g(a \otimes \nu) = |a|,$$

then the function φ gived by formula (1.2) is quasiconvex but not convex.

To see this, let us denote by $Q\psi$ the *quasiconvexification* of (*i.e.*, the greatest quasiconvex function less than or equal to) the function

$$\psi(\xi) = \min\{|\xi - A|, |\xi - B|\}.$$

It is already proved (see [88]) that $Q\psi$ vanishes only at A and B. Moreover it is clear that $Q\psi \leq f$ on $M^{k \times n}$ and that $(Q\psi)^\infty \leq g$ on $M_1^{k \times n}$. It follows that $\varphi \geq Q\psi$; since $\varphi(A) = \varphi(B) = 0$, we conclude that $\varphi(\xi) = 0$ if and only if $\xi = A$ or $\xi = B$, hence φ is not convex.

CHAPTER 5:

DISCRETE APPROXIMATION
OF A FREE DISCONTINUITY PROBLEM

In this chapter we approximate by discrete Γ-convergence a functional proposed by Mumford-Shah for a variational approach to image segmentation. Such a functional is first relaxed with a sequence of nonconvex functionals, which in turn, are discretized by piecewise linear finite elements. Under a suitable relation between the relaxation parameter ε and the meshsize h, the convergence of the discrete functionals and the compactness of any sequence of discrete minimizers are proved. The proof relies on the techniques of Γ-convergence and on the properties of the Lagrange interpolation and Clement operators.

The results of this chapter are contained in [19].

Introduction

A fundamental problem in Computer Vision is to reconstruct the contours of a picture given by a camera [24], [83]. Given a bounded domain $\Omega \subseteq \mathbb{R}^2$ the image is represented by the grey level function $g \in L^\infty(\Omega)$, which measures the intensity of the light at each point of the screen. Since one expects the function g to be discontinuous along the lines corresponding to sudden changes in the visible surfaces (e.g. edges of objects, shadows, different colours), the image segmentation problem consists in finding a pair (u, K) such that K is a set of curves decomposing the image into regions with relatively uniform intensity, while u is a smooth approximation of g on each region. The set K will be interpreted as the union of the lines which give the schematic description of the image.

Many problems in image segmentation can be solved by minimizing a functional depending on u and K, as pointed out by S. and D. Geman [59].

D. Mumford and J. Shah [77], [78] developed this variational approach by suggesting the study of the problem

$$(0.1) \qquad \inf_{(u,K)} \{ \int_{\Omega \setminus K} |\nabla u|^2 \, dx + \mathcal{H}^1(K \cap \Omega) + \int_{\Omega \setminus K} |u - g|^2 \, dx \},$$

where K is a closed subset of $\overline{\Omega}$, $u \in \mathcal{C}^1(\Omega \setminus K)$, and $\mathcal{H}^1$ denotes the one-dimensional Hausdorff measure (see [56]).

Such problem has been studied by several authors (see, among others, [5], [43], [44], [72]), and falls within the general n-dimensional setting of

free discontinuity problems proposed by E. De Giorgi in the last years [44], [45], [46]. In this context one minimizes the functional defined by

$$(0.2) \qquad \mathcal{F}(u) = \int_{\Omega} |\nabla u|^2 \, dx + \mathcal{H}^{n-1}(S_u) + \int_{\Omega} |u - g|^2 \, dx,$$

on the class $SBV(\Omega)$ of the special functions u of bounded variation in Ω [6], [45]; here S_u is the jump set of u and ∇u is the gradient of u in an approximate sense. By relying on a general compactness and semicontinuity theorem due to L. Ambrosio [5], one can show that the functional $\mathcal{F}$ in (0.2) achieves its minimum on $SBV(\Omega)$. Moreover, by studying the regularity properties of the minimizers, E. De Giorgi, M. Carriero, and A. Leaci [46] proved that problem (0.1) is essentially equivalent to minimize $\mathcal{F}$.

The numerical treatment of (0.2) seems quite difficult, because of the lack of convexity and regularity of the functional at hand, mainly due to the term $\mathcal{H}^{n-1}(S_u)$ (see [22], [35], [65], [67], [84], [85], [86]). However, L. Ambrosio and V.M. Tortorelli [12], [13] have shown that $\mathcal{F}$ is the limit, as $\varepsilon \to 0$, of a sequence $\{\mathcal{F}_\varepsilon\}_\varepsilon$ (see (2.1)) of regular elliptic functionals in the sense of Γ-convergence (ε is the relaxation parameter). The basic idea is to introduce a new variable s in the approximating functional $\mathcal{F}_\varepsilon$, which controls the unknown set S_u (see also [55]). In view of the variational properties of Γ-convergence, the minimization of $\mathcal{F}$ is then reduced to the minimization of $\mathcal{F}_\varepsilon$, for small ε.

In this chapter we show that if we discretize $\mathcal{F}_\varepsilon$ by means of piecewise linear finite elements, then the discrete functionals Γ-converge to $\mathcal{F}$ and the discrete minimizers converge to a solution of the original problem (0.1). More precisely, let $\mathcal{F}_{\varepsilon,h}$ denote the discretization of $\mathcal{F}_\varepsilon$ (see (2.3)), where h denotes the meshsize, and let $\mathcal{R}$ be the class of all piecewise $\mathcal{C}^2$ submanifolds of $\mathbb{R}^n$ of dimension $n - 1$. Using the notion of discrete Γ-convergence introduced in [21], we prove the following result:

Theorem 0.1. *Let* $h = o(\varepsilon)$; *then the sequence* $\{\mathcal{F}_{\varepsilon,h}\}_\varepsilon$ *Γ-converges in* $L^2(\Omega)$ *to* $\mathcal{F}$ *as* $\varepsilon \to 0$ *on the class of all functions* $u \in SBV(\Omega) \cap L^\infty(\Omega)$ *such that* $S_u \in \mathcal{R}$. *Moreover, under the additional hypothesis that there exists at least a minimum point of the functional* $\mathcal{F}$ *whose jump set belongs to* $\mathcal{R}$, *we have that any family* $\{u_{\varepsilon,h}\}_\varepsilon$ *of absolute minimizers of* $\{\mathcal{F}_{\varepsilon,h}\}_\varepsilon$ *is relatively compact in* $L^2(\Omega)$, *and each of its limit points minimizes* $\mathcal{F}$.

From the numerical point of view, the fact that the convergence takes place on the class of all functions whose jump set belongs to $\mathcal{R}$ is not a restriction, since any $u \in SBV(\Omega) \cap L^\infty(\Omega)$ such that S_u is polygonal has this property.

Let us briefly describe the content of this chapter.

In Sections 1 and 2 we give some notation, and we introduce the continuous and discretized functionals.

In Section 3 we prove a lemma on the extension of $SBV(\Omega)$ functions, which is useful in the constructive part of the proof of Theorem 0.1.

In Section 4 we prove that the functional $\mathcal{F}$ is less than or equal to the Γ-lower limit of the sequence $\{\mathcal{F}_{\varepsilon,h}\}_{\varepsilon}$ on all functions $u \in SBV(\Omega) \cap L^{\infty}(\Omega)$.

In Section 5 we prove that the functional $\mathcal{F}$ is greater than or equal to the Γ-upper limit of the sequence $\{\mathcal{F}_{\varepsilon,h}\}_{\varepsilon}$ on the class of all functions $u \in SBV(\Omega) \cap L^{\infty}(\Omega)$ such that $S_u \in \mathcal{R}$.

Finally, in Section 6, we prove that if there exists at least a minimizer of the functional $\mathcal{F}$ whose jump set belongs to $\mathcal{R}$, then the minimum values of $\mathcal{F}_{\varepsilon,h}$ converge to the minimum value of $\mathcal{F}$ as $\varepsilon \to 0$. Moreover any family $\{u_{\varepsilon,h}\}_{\varepsilon}$ of absolute minimizers of $\{\mathcal{F}_{\varepsilon,h}\}_{\varepsilon}$ is relatively compact in $L^2(\Omega)$, and each of its limit points minimizes $\mathcal{F}$. Thus we achieve the conclusion of the proof of Theorem 0.1.

1. Notations

In $\mathbb{R}^n$ we denote by $|\cdot|$ the usual euclidean norm; for any $x \in \mathbb{R}^n$ and any $\varrho > 0$ we indicate by $B_{\varrho}(x) = \{z \in \mathbb{R}^n : |z - x| < \varrho\}$ the ball centered at x with radius ϱ. Given two sets $A, B \subseteq \mathbb{R}^n$, we denote by $\overline{A}$ the closure of A and by $A \subset\subset B$ we mean that $\overline{A}$ is a compact set contained in B. For any $t > 0$ we denote by $(A)_t$ the tubular neighbourhood of A defined as $\{z \in \mathbb{R}^n : \text{dist}(z, A) < t\}$. If A, B are open sets we indicate by $\text{Lip}(\overline{A}, \overline{B})$ the space of the Lipschitz continuous functions $f : \overline{A} \to \overline{B}$ and by $\text{Lip}(f)$ the Lipschitz constant of $f \in \text{Lip}(\overline{A}, \overline{B})$. If a, b are two vectors in $\mathbb{R}^n$ we recall that the tensor product $a \otimes b$ is the $(n \times n)$-matrix whose entries are $a_i b_j$ with $i, j \in \{1, \ldots, n\}$; we remark that $|a \otimes b| = |a||b|$, where we consider the space of $(n \times n)$-matrices endowed with the usual euclidean norm.

Let $S \subseteq \mathbb{R}^n$; we say that $S \in \mathcal{R}$ if $S = f(C)$, where $f : \mathbb{R}^n \to \mathbb{R}^n$ is a diffeomorphism of class $\mathcal{C}^2$, and C is a finite union of $(n-1)$-dimensional simplices (not necessarily closed) in $\mathbb{R}^n$ [79] such that the intersection of any two of these simplices is a face (not necessarily closed) of each of them.

Throughout the chapter, the letter C will stand for a strictly positive constant, whose value may vary from line to line and which is independent of the parameters involved.

Let $\{c_{\varepsilon}\}_{\varepsilon}$ be a sequence of real numbers depending on the continuous parameter $\varepsilon \to 0$; when we write $c_{\varepsilon} = o(1)$ we mean that $\lim_{\varepsilon \to 0} c_{\varepsilon} = 0$, while by $c_{\varepsilon} = O(1)$ we mean that $|c_{\varepsilon}| \leq C$ for every $\varepsilon > 0$.

If S is a compact subset of $\mathbb{R}^n$ we denote by $\Pi_S(x) = \{z \in S : |z - x| = \text{dist}(x, S)\}$ for any $x \in \mathbb{R}^n$.

Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set. If $u \in W^{2,1}(\Omega)$ we indicate by $D^2 u$ the Hessian matrix of u.

We refer to Chapter 1 for the properties of the functions of bounded variation (see Section 2) and for the definition of Γ-convergence of a sequence of functionals (see Section 4).

2. Position of the Problem

Let $g \in L^\infty(\Omega)$; the map $\mathcal{F} : L^\infty(\Omega) \to [0, +\infty]$ is defined by

$$\mathcal{F}(u) = \begin{cases} \displaystyle\int_\Omega |\nabla u|^2 dx + \mathcal{H}^{n-1}(S_u) + \int_\Omega |u-g|^2 dx & \text{if } u \in SBV(\Omega) \cap L^\infty(\Omega) \\ +\infty & \text{elsewhere.} \end{cases}$$

We add a formal extra variable s to $\mathcal{F}$ by setting

$$\mathcal{F}(u,s) = \begin{cases} \mathcal{F}(u) & \text{if } s \equiv 1 \\ +\infty & \text{elsewhere on } L^\infty(\Omega) \times L^\infty(\Omega;[0,1]). \end{cases}$$

Let $\{\kappa_\varepsilon\}_\varepsilon$ be a sequence of positive numbers converging to zero as $\varepsilon \to 0$ such that

$$\lim_{\varepsilon \to 0} \frac{\kappa_\varepsilon}{\varepsilon} = 0.$$

For any $\varepsilon > 0$ the relaxed functional $\mathcal{F}_\varepsilon : L^\infty(\Omega) \times L^\infty(\Omega;[0,1]) \to [0,+\infty]$ reads as follows:

$$(2.1) \quad \begin{aligned} \mathcal{F}_\varepsilon(u,s) &= \int_\Omega (s^2 + \kappa_\varepsilon)\,|\nabla u|^2\,dx + \int_\Omega \left[\varepsilon|\nabla s|^2 + \frac{1}{4\varepsilon}(1-s)^2\right]\,dx \\ &\quad + \int_\Omega |u-g|^2\,dx \end{aligned}$$

if $(u,s) \in H^1(\Omega) \times H^1(\Omega;[0,1])$, and $\mathcal{F}_\varepsilon = +\infty$ elsewhere. The following result is proved in [12], [13].

Theorem 2.1. *We have*

$$\Gamma - \lim_{\varepsilon \to 0} \mathcal{F}_\varepsilon = \mathcal{F}$$

with respect to the $L^2(\Omega) \times L^2(\Omega;[0,1])$-topology. Precisely, given $(u,s) \in L^\infty(\Omega) \times L^\infty(\Omega;[0,1])$, the following two properties hold:
(i) for any sequence $\{(u_\varepsilon, s_\varepsilon)\}_\varepsilon$ in $H^1(\Omega) \times H^1(\Omega;[0,1])$ converging to (u,s) in $L^2(\Omega) \times L^2(\Omega;[0,1])$ we have

$$\mathcal{F}(u,s) \leq \liminf_{\varepsilon \to 0} \mathcal{F}_\varepsilon(u_\varepsilon, s_\varepsilon);$$

(ii) there exists a sequence $\{(u_\varepsilon, s_\varepsilon)\}_\varepsilon$ in $H^1(\Omega) \times H^1(\Omega; [0,1])$ converging to (u, s) in $L^2(\Omega) \times L^2(\Omega; [0,1])$ such that

$$\mathcal{F}(u, s) \geq \limsup_{\varepsilon \to 0} \mathcal{F}_\varepsilon(u_\varepsilon, s_\varepsilon).$$

Moreover, if $(u_\varepsilon, s_\varepsilon)$ is an absolute minimizer of $\mathcal{F}_\varepsilon$ for any $\varepsilon > 0$, then the sequence $\{(u_\varepsilon, s_\varepsilon)\}_\varepsilon$ is relatively compact in $L^2(\Omega) \times L^2(\Omega; [0,1])$ and each of its limit points minimizes $\mathcal{F}$.

For technical reasons, we shall modify the functional $\mathcal{F}_\varepsilon$ as follows. Let

$$\mathcal{M}_\varepsilon(s) = \int_\Omega \left[\varepsilon |\nabla s|^2 + \frac{1}{4\varepsilon} \omega(s) \right] dx \qquad \forall s \in H^1(\Omega; [0,1]),$$

where $\omega(t) = 1 - t^2$ for any $t \in [0,1]$, and set

$$c_0 = \int_0^1 \sqrt{\omega(t)} \, dt.$$

We define the new functional $\mathcal{F}_\varepsilon : L^\infty(\Omega) \times L^\infty(\Omega; [0,1]) \to [0, +\infty]$ as

$$\mathcal{F}_\varepsilon(u, s) = \int_\Omega (s + \kappa_\varepsilon) |\nabla u|^2 \, dx + \frac{1}{2c_0} \mathcal{M}_\varepsilon(s) + \int_\Omega |u - g|^2 \, dx$$

if $(u, s) \in H^1(\Omega) \times H^1(\Omega; [0,1])$ and $\mathcal{F}_\varepsilon = +\infty$ elsewhere.

Note that the terms $\int_\Omega s^2 |\nabla u|^2 \, dx$ and $\int_\Omega \frac{1}{4\varepsilon}(1 - s)^2 \, dx$ have been replaced by the terms $\int_\Omega s |\nabla u|^2 \, dx$ and $\int_\Omega \frac{1}{4\varepsilon} \omega(s) \, dx$, respectively. It is not difficult to prove that these modifications on the functional $\mathcal{F}_\varepsilon$ do not affect the statement of Theorem 2.1.

We introduce a discretization of $\mathcal{F}_\varepsilon$ by piecewise linear finite elements. For the sake of simplicity we shall assume that Ω is a polyhedron. Let us denote by $\{\mathcal{S}_h\}_{h>0}$ a regular family of partitions of Ω into simplices ([36] p. 132), so that $\overline{\Omega} \equiv \Omega_h := \cup_{S \in \mathcal{S}_h} S$, for all $h > 0$. Let $V_h(\Omega) \subset H^1(\Omega)$ indicate the linear finite element space over $\mathcal{S}_h$, let $V_h(\Omega; [0,1]) = \{v \in V_h(\Omega) : v(x) \in [0,1] \; \forall x \in \Omega\}$, and let $p_h : C^0(\overline{\Omega}) \to V_h$ be the Lagrange interpolation operator. By $r_h : L^2(\Omega) \to V_h(\Omega)$ we indicate the Clement operator (see [37]). It is well known that there exists a positive constant $\eta \geq 1$, depending only on the space-dimension n and on the minimum of the angles of the elements of $\mathcal{S}_h$, such that if $f \equiv 0$ on a ball $B_R(x) \subseteq \Omega$, then $r_h(f) \equiv 0$ on any $S \in \mathcal{S}_h$ such that $S \subseteq B_{R-\eta h}(x)$.

For any $\varepsilon > 0$ let $g_\varepsilon \in C_0^\infty(\Omega)$ approximate the function $g \in L^\infty(\Omega)$ so that (see [31] Theorem IV.30) $g_\varepsilon \to g$ in $L^2(\Omega)$ and

$$(2.2) \qquad \|g_\varepsilon\|_{L^\infty(\Omega)} \leq \|g\|_{L^\infty(\Omega)}, \quad \|\nabla g_\varepsilon\|_{L^\infty(\Omega)} \leq \frac{C}{\varepsilon}.$$

For any $\varepsilon, h > 0$ set

$$\mathcal{M}_{\varepsilon,h}(s) = \int_\Omega \left[\varepsilon |\nabla s|^2 + \frac{1}{4\varepsilon} p_h(\omega(s)) \right] dx \qquad \forall s \in V_h(\Omega; [0,1]).$$

The discretized version $\mathcal{F}_{\varepsilon,h} : L^\infty(\Omega) \times L^\infty(\Omega; [0,1]) \to [0, +\infty]$ is then defined by

$$(2.3) \quad \mathcal{F}_{\varepsilon,h}(u,s) = \int_\Omega (s + \kappa_\varepsilon)|\nabla u|^2 dx + \frac{1}{2c_0} \mathcal{M}_{\varepsilon,h}(s) + \int_\Omega p_h((u - g_\varepsilon)^2) dx$$

if $(u,s) \in V_h(\Omega) \times V_h(\Omega; [0,1])$ and $\mathcal{F}_{\varepsilon,h} = +\infty$ elsewhere. The integrals in (2.3) can be evaluated via the vertex quadrature rule, which is exact for piecewise linear functions. On the other hand, the interpolation operator p_h in (2.3) will introduce extra difficulties in proving the main Γ-convergence Theorem 0.1.

3. An Extension Lemma

Let us prove the following result on the extension of $SBV(\Omega)$ functions.

Lemma 3.1. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set with Lipschitz continuous boundary, and let $S \subseteq \overline{\Omega}$ be a closed set such that $S \cap \partial\Omega \neq \emptyset$ and $\mathcal{H}^{n-1}(S) < +\infty$. Let $u \in L^\infty(\Omega) \cap H^1(\Omega \setminus S)$. Then there exist a bounded open set $\Omega' \supset\supset \Omega$, a closed set $S' \subseteq \overline{\Omega'}$ with $S' \cap \overline{\Omega} = S \cap \overline{\Omega}$ and $\mathcal{H}^{n-1}(S') < +\infty$, a function $U \in L^\infty(\Omega') \cap H^1(\Omega' \setminus S')$, and a real number $N \geq 2$ depending only on Ω, such that $U = u$ on Ω, and*

$$(3.1) \qquad \left\{ x \in \Omega : \mathrm{dist}(x, S') < \frac{\delta}{N} \right\} \subseteq \left\{ x \in \Omega : \mathrm{dist}(x, S) < \frac{\delta}{2} \right\}$$

for every $\delta > 0$ sufficiently small.

Proof. Let $Q =]-1,1[^n$ be the unit cube in $\mathbb{R}^n$, $Q^0 = \{z \in Q : z_n = 0\}$, $Q^- = \{z \in Q : z_n < 0\}$, $Q^+ = \{z \in Q : z_n > 0\}$, and denote by $R : Q^+ \cup Q^0 \to Q^- \cup Q^0$ the map $R(z_1, \ldots, z_{n-1}, z_n) = (z_1, \ldots, z_{n-1}, -z_n)$. As $\partial\Omega$ is compact and Lipschitz continuous, there exist k bounded open subsets $\{U_i\}_{i=1,\ldots,k}$ of $\mathbb{R}^n$ and k functions $\{h_i\}_{i=1,\ldots,k}$ such that $\partial\Omega \subseteq \bigcup_{i=1}^{k} U_i$, and for every i

$$h_i \in \mathrm{Lip}(\overline{Q}, \overline{U_i}), \quad h_i^{-1} \in \mathrm{Lip}(\overline{U_i}, \overline{Q}),$$
$$h_i(Q^+) = U_i \cap \Omega = U_i^+, \quad h_i(Q^0) = U_i \cap \partial\Omega.$$

For every $i \in \{1, \ldots, k\}$ let $S_i^+ = S \cap \overline{U_i^+}$ and $S_i^- = h_i \circ R \circ h_i^{-1}(S_i^+)$; then S_i^+ and S_i^- are (possibly empty) closed sets. Let us define

$$\Omega' = \Omega \cup \left(\bigcup_{i=1}^{k} U_i \right)$$

and

$$S' = S \cup \left(\bigcup_{i=1}^{k} S_i^- \right).$$

Let us observe that Ω' is a bounded open set, $\Omega' \supset\supset \Omega$, and that S' is a closed subset of $\overline{\Omega'}$ with $S' \cap \Omega = S \cap \Omega$ and $\mathcal{H}^{n-1}(S') < +\infty$. We claim that there exists a finite family $B_1, \ldots, B_m$ of balls in $\mathbb{R}^n$ centered at appropriate points of $S \cap \partial\Omega$ such that $S \cap \partial\Omega \subseteq \bigcup_{j=1}^{m} B_j$ and the following property holds: for any $j \in \{1, \ldots, m\}$ there exists $i \in \{1, \ldots, k\}$ with $B_j \subseteq U_i$ and

$$(3.2) \qquad x \in B_j \implies \Pi_S(x), \Pi_{S'\backslash\Omega}(x) \in U_i.$$

Property (3.2) means that for any $j \in \{1, \ldots, m\}$ there exists $i \in \{1, \ldots, k\}$ such that $\Pi_S \equiv \Pi_{S_i^+}$ and $\Pi_{S'\backslash\Omega} \equiv \Pi_{S_i^-}$ on B_j. In order to prove (3.2) let us fix $y \in S \cap \partial\Omega$; then $y \in S_i^+ \cap \partial\Omega$ for some index $i \in \{1, \ldots, k\}$. We shall distinguish two cases. If $y \notin \partial U_i$, let $L = \text{dist}(y, \partial U_i) > 0$. The map

$$g(x) = \max\{\text{dist}(x, S), \text{dist}(x, S' \backslash \Omega)\} \qquad \forall x \in U_i$$

is continuous and $g(y) = 0$. Therefore there exists $\delta > 0$ such that $g(x) < \frac{L}{3}$ whenever $x \in B_\delta(y)$. Define $r = r(i, y) = \min\{\delta, \frac{L}{3}\}$; if $x \in B_r(y)$ then $\Pi_S(x), \Pi_{S'\backslash\Omega}(x) \in B_{\frac{L}{3}}(x) \subseteq B_{\frac{2L}{3}}(y) \subseteq U_i$. If $y \in \partial U_i$, there exists an index $l = l(y) \in \{1, \ldots, k\}$ such that $y \in U_l$, hence we can repeat the previous argument with i replaced by l. The claim then follows from the compactness of $S \cap \partial\Omega$.

Let us prove (3.1). For every $j \in \{1, \ldots, m\}$ set $B_j^+ = B_j \cap \Omega$. We first show that there exists a real number $N \geq 2$ such that for every $j \in \{1, \ldots, m\}$ we have

$$(3.3) \qquad \text{dist}(x, S) \leq \frac{N}{2} \text{dist}(x, S' \backslash \Omega) \qquad \forall x \in B_j^+.$$

Let us fix $j \in \{1, \ldots, m\}$ and $x \in B_j^+$; let $i \in \{1, \ldots, k\}$ be such that $B_j \subseteq U_i$ and property (3.2) holds. Let $p \in \Pi_S(x)$ and $p' \in \Pi_{S'\backslash\Omega}(x)$; by (3.2) we have that $p \in S_i^+$ and $p' \in S_i^-$. If $y = h_i^{-1}(x) \in Q^+$, $q \in \Pi_{h_i^{-1}(S_i^+)}(y)$, and $q' \in \Pi_{R(h_i^{-1}(S_i^+))}(y)$, then we have

$$(3.4) \quad \text{dist}(x, S) = |x - p| \leq |x - h_i(q)| = |h_i(y) - h_i(q)| \leq \text{Lip}(h_i)|y - q|.$$

Let us observe that $|y - q| \leq |y - q'|$. Indeed, using the properties of the map R, if we consider the continuous function $\psi : Q \to Q$ defined by $\psi(z) = \mathrm{dist}(z, h_i^{-1}(S_i^+)) - \mathrm{dist}(z, R(h_i^{-1}(S_i^+)))$ we have that $Q^0 = \{z \in Q : \psi(z) = 0\}$, $Q^- = \{z \in Q : \psi(z) > 0\}$, and $Q^+ = \{z \in Q : \psi(z) < 0\}$. As $y \in Q^+$, we conclude that $\mathrm{dist}(y, h_i^{-1}(S_i^+)) - \mathrm{dist}(y, R(h_i^{-1}(S_i^+))) = |y - q| - |y - q'| < 0$. Now, using the fact that $h_i^{-1}(p') \in R(h_i^{-1}(S_i^+))$, from (3.4) we deduce that

$$\mathrm{dist}(x, S) \leq \mathrm{Lip}(h_i)|y - q'| \leq \mathrm{Lip}(h_i)|h_i^{-1}(x) - h_i^{-1}(p')|$$
$$\leq \mathrm{Lip}(h_i)\mathrm{Lip}(h_i^{-1})|x - p'| = \alpha_i \, \mathrm{dist}(x, S' \setminus \Omega),$$

where $\alpha_i = \mathrm{Lip}(h_i)\mathrm{Lip}(h_i^{-1})$. If we choose a real number $N \geq 2$ such that $\frac{N}{2} \geq \max\{\alpha_i : i = 1, \dots, k\}$, we get (3.3). Observe that there exists a constant $\beta > 0$ such that if $\overline{x}$ does not belong to $\cup_{j=1}^m B_j^+$ then $\mathrm{dist}(\overline{x}, S' \setminus \Omega) \geq \beta$. As a consequence, for $\delta > 0$ sufficiently small, if $\overline{x} \in \{x \in \Omega : \mathrm{dist}(x, S' \setminus \Omega) < \frac{\delta}{N}\}$, then $\overline{x} \in \bigcup_{j=1}^m B_j^+$, so that, using (3.3),

$$\mathrm{dist}(\overline{x}, S) \leq \frac{N}{2}\mathrm{dist}(\overline{x}, S' \setminus \Omega) < \frac{N}{2}\frac{\delta}{N} = \frac{\delta}{2},$$

hence $\overline{x} \in \{x \in \Omega : \mathrm{dist}(x, S) < \frac{\delta}{2}\}$. Finally, observing that $S' \cap \Omega = S \cap \Omega$ and $N \geq 2$, we have (3.1). In addition (3.1) allows to conclude that $S' \cap \overline{\Omega} = S \cap \overline{\Omega}$.

The function U which extends u on Ω' is simply defined by $U(x) = u(x)$ if $x \in \Omega$ and $U(x) = u(h_i(R(h_i^{-1}(x))))$ if $x \in (\Omega' \setminus \Omega) \cap U_i$ for $i \in \{1, \dots, k\}$. By construction we have $U \in L^\infty(\Omega') \cap H^1(\Omega' \setminus S')$ and this concludes the proof. $\qquad\square$

4. The Lower Inequality

In this section we prove that the Γ-lower limit of the sequence $\mathcal{F}_{\varepsilon,h}$ is larger than or equal to the functional $\mathcal{F}$ on the whole $L^\infty(\Omega) \times L^\infty(\Omega; [0,1])$.

Theorem 4.1. *Let $h = o(\kappa_\varepsilon)$. Let $(u, s) \in L^\infty(\Omega) \times L^\infty(\Omega; [0,1])$ and for every $\varepsilon > 0$ let $(u_{\varepsilon,h}, s_{\varepsilon,h}) \in V_h(\Omega) \times V_h(\Omega; [0,1])$ be such that the sequence $\{(u_{\varepsilon,h}, s_{\varepsilon,h})\}_\varepsilon$ converges to (u, s) in $L^2(\Omega) \times L^2(\Omega; [0,1])$. Then we have*

$$(4.1) \qquad \mathcal{F}(u, s) \leq \liminf_{\varepsilon \to 0} \mathcal{F}_{\varepsilon,h}(u_{\varepsilon,h}, s_{\varepsilon,h}).$$

Proof. We can suppose that the right hand-side of (4.1) is finite, otherwise the result is trivial. Passing to a suitable subsequence (still denoted by $\{(u_{\varepsilon,h}, s_{\varepsilon,h})\}_\varepsilon$), we can assume that

$$(4.2) \qquad \lim_{\varepsilon \to 0} \mathcal{F}_{\varepsilon,h}(u_{\varepsilon,h}, s_{\varepsilon,h}) = \liminf_{\varepsilon \to 0} \mathcal{F}_{\varepsilon,h}(u_{\varepsilon,h}, s_{\varepsilon,h}) < +\infty.$$

Therefore, by the definition of $\mathcal{F}_{\varepsilon,h}$, we have

$$(4.3) \qquad \int_\Omega |\nabla u_{\varepsilon,h}|^2 \, dx \le \frac{C}{\kappa_\varepsilon} \qquad \forall \varepsilon > 0,$$

$$(4.4) \qquad \int_\Omega |\nabla s_{\varepsilon,h}|^2 \, dx \le \frac{C}{\varepsilon} \qquad \forall \varepsilon > 0.$$

Let us split $\mathcal{F}_{\varepsilon,h}(u_{\varepsilon,h}, s_{\varepsilon,h})$ as follows:

$$
\begin{aligned}
\mathcal{F}_{\varepsilon,h}(u_{\varepsilon,h}, s_{\varepsilon,h}) &= \mathcal{F}_\varepsilon(u_{\varepsilon,h}, s_{\varepsilon,h}) \\
&\quad + \frac{1}{2c_0} \int_\Omega \frac{1}{4\varepsilon} \left[p_h(\omega(s_{\varepsilon,h})) - \omega(s_{\varepsilon,h}) \right] \, dx \\
&\quad + \int_\Omega \left[p_h((u_{\varepsilon,h} - g_\varepsilon)^2) - (u_{\varepsilon,h} - g)^2 \right] \, dx \\
&=: \mathcal{F}_\varepsilon(u_{\varepsilon,h}, s_{\varepsilon,h}) + \frac{1}{2c_0} \mathrm{I}_{\varepsilon,h} + \mathrm{II}_{\varepsilon,h}.
\end{aligned}
$$

(4.5)

In view of Theorem 2.1(i), to show (4.1) it will be enough to prove that $\lim_{\varepsilon \to 0} \mathrm{I}_{\varepsilon,h} = \lim_{\varepsilon \to 0} \mathrm{II}_{\varepsilon,h} = 0$. Using [36] Theorem 3.1.5, we have

$$
\begin{aligned}
|\mathrm{I}_{\varepsilon,h}| &\le \frac{1}{4\varepsilon} \int_\Omega |p_h(\omega(s_{\varepsilon,h})) - \omega(s_{\varepsilon,h})| \, dx \\
&= \frac{1}{4\varepsilon} \sum_{S \in \mathcal{S}_h} \int_S |p_h(\omega(s_{\varepsilon,h})) - \omega(s_{\varepsilon,h})| \, dx \\
&\le \frac{1}{4\varepsilon} \sum_{S \in \mathcal{S}_h} |S| \, \|p_h(\omega(s_{\varepsilon,h})) - \omega(s_{\varepsilon,h})\|_{L^\infty(S)} \\
&\le \frac{Ch^2}{\varepsilon} \sum_{S \in \mathcal{S}_h} |S| \, \|D^2(\omega(s_{\varepsilon,h}))\|_{L^\infty(S)}.
\end{aligned}
$$

(4.6)

Note that, as $\nabla s_{\varepsilon,h}$ is constant on each $S \in \mathcal{S}_h$, we have $\|D^2(\omega(s_{\varepsilon,h}))\|_{L^\infty(S)} = 2\|\nabla s_{\varepsilon,h} \otimes \nabla s_{\varepsilon,h}\|_{L^\infty(S)} = 2|\nabla s_{\varepsilon,h}|^2$ on S. Therefore, using (4.4), we deduce

$$
\begin{aligned}
|\mathrm{I}_{\varepsilon,h}| &\le \frac{Ch^2}{\varepsilon} \sum_{S \in \mathcal{S}_h} |S| \, |\nabla s_{\varepsilon,h}|^2 \\
&= \frac{Ch^2}{\varepsilon} \sum_{S \in \mathcal{S}_h} \int_S |\nabla s_{\varepsilon,h}|^2 \, dx = \frac{Ch^2}{\varepsilon} \int_\Omega |\nabla s_{\varepsilon,h}|^2 \, dx \le \frac{Ch^2}{\varepsilon^2} = o(1)
\end{aligned}
$$

(4.7)

as, by assumption, $h = o(\varepsilon)$. We deduce that $\lim\limits_{\varepsilon \to 0} I_{\varepsilon,h} = 0$. Hence, by (4.2),
we find that $\dfrac{1}{4\varepsilon} \displaystyle\int_\Omega \omega(s_{\varepsilon,h}) \, dx$ is uniformly bounded with respect to ε; as a
consequence $s \equiv 1$.

Let us prove that $\lim\limits_{\varepsilon \to 0} II_{\varepsilon,h} = 0$. We have

$$|II_{\varepsilon,h}| \leq \int_\Omega \left| p_h((u_{\varepsilon,h} - g_\varepsilon)^2) - (u_{\varepsilon,h} - g_\varepsilon)^2 \right| \, dx$$

$$+ \int_\Omega \left| (u_{\varepsilon,h} - g_\varepsilon)^2 - (u_{\varepsilon,h} - g)^2 \right| \, dx =: II^{(1)}_{\varepsilon,h} + II^{(2)}_{\varepsilon,h}.$$

Let us show that $\lim\limits_{\varepsilon \to 0} II^{(1)}_{\varepsilon,h} = 0$. Using the linearity of the Lagrange interpolation operator we get

$$
\begin{aligned}
II^{(1)}_{\varepsilon,h} &\leq \int_\Omega \left| p_h(u^2_{\varepsilon,h}) - u^2_{\varepsilon,h} \right| \, dx + \int_\Omega \left| p_h(g^2_\varepsilon) - g^2_\varepsilon \right| \, dx \\
&\quad + 2 \int_\Omega \left| p_h(u_{\varepsilon,h} g_\varepsilon) - u_{\varepsilon,h} g_\varepsilon \right| \, dx =: A_{\varepsilon,h} + B_{\varepsilon,h} + C_{\varepsilon,h}.
\end{aligned}
$$
(4.8)

From [36] Theorem 3.1.5, the fact that $u_{\varepsilon,h}$ is piecewise linear, and (4.3) we deduce that

$$
\begin{aligned}
A_{\varepsilon,h} &\leq \sum_{S \in \mathcal{S}_h} |S| \, \|p_h(u^2_{\varepsilon,h}) - u^2_{\varepsilon,h}\|_{L^\infty(S)} \\
&\leq Ch^2 \sum_{S \in \mathcal{S}_h} |S| \, \|D^2(u^2_{\varepsilon,h})\|_{L^\infty(S)} \leq Ch^2 \sum_{S \in \mathcal{S}_h} |S| \, |\nabla u_{\varepsilon,h}|^2 \\
&= Ch^2 \int_\Omega |\nabla u_{\varepsilon,h}|^2 \, dx \leq \frac{Ch^2}{\kappa_\varepsilon} = o(1);
\end{aligned}
$$
(4.9)

using (2.2) we have
(4.10)
$$
\begin{aligned}
B_{\varepsilon,h} &\leq \sum_{S \in \mathcal{S}_h} |S| \, \|p_h(g^2_\varepsilon) - g^2_\varepsilon\|_{L^\infty(S)} \leq Ch \sum_{S \in \mathcal{S}_h} |S| \, \|D(g^2_\varepsilon)\|_{L^\infty(S)} \\
&\leq Ch \sum_{S \in \mathcal{S}_h} |S| \, \|g_\varepsilon \nabla g_\varepsilon\|_{L^\infty(S)} \leq Ch \sum_{S \in \mathcal{S}_h} |S| \, \|g_\varepsilon\|_{L^\infty(S)} \|\nabla g_\varepsilon\|_{L^\infty(S)} \\
&\leq Ch \sum_{S \in \mathcal{S}_h} |S| \, \|\nabla g_\varepsilon\|_{L^\infty(S)} \leq \frac{Ch}{\varepsilon} |\Omega| = o(1).
\end{aligned}
$$

In addition, setting $g_{\varepsilon,h} = p_h(g_\varepsilon)$ and noting that $p_h(u_{\varepsilon,h} g_\varepsilon) = p_h(u_{\varepsilon,h} g_{\varepsilon,h})$, we have

$$C_{\varepsilon,h} \leq 2 \int_\Omega \left| p_h(u_{\varepsilon,h} g_{\varepsilon,h}) - u_{\varepsilon,h} g_{\varepsilon,h} \right| \, dx + 2 \int_\Omega \left| u_{\varepsilon,h} g_{\varepsilon,h} - u_{\varepsilon,h} g_\varepsilon \right| \, dx$$

$$=: 2(C^{(1)}_{\varepsilon,h} + C^{(2)}_{\varepsilon,h}).$$

Then, using the fact that $D^2(u_{\varepsilon,h}g_{\varepsilon,h}) = \nabla u_{\varepsilon,h} \otimes \nabla g_{\varepsilon,h} + \nabla g_{\varepsilon,h} \otimes \nabla u_{\varepsilon,h}$, the estimate $|\nabla g_{\varepsilon,h}| \leq \|\nabla g_\varepsilon\|_{L^\infty(\Omega)} \leq \dfrac{C}{\varepsilon}$, the Hölder inequality, and (4.3), we obtain
(4.11)

$$
\begin{aligned}
C_{\varepsilon,h}^{(1)} &\leq \sum_{S\in\mathcal{S}_h} |S|\, \|p_h(u_{\varepsilon,h}g_{\varepsilon,h}) - u_{\varepsilon,h}g_{\varepsilon,h}\|_{L^\infty(S)} \\
&\leq Ch^2 \sum_{S\in\mathcal{S}_h} |S|\, \|D^2(u_{\varepsilon,h}g_{\varepsilon,h})\|_{L^\infty(S)} \leq Ch^2 \sum_{S\in\mathcal{S}_h} |S|\, |\nabla u_{\varepsilon,h}||\nabla g_{\varepsilon,h}| \\
&\leq \frac{Ch^2}{\varepsilon} \sum_{S\in\mathcal{S}_h} |S|\, |\nabla u_{\varepsilon,h}| = \frac{Ch^2}{\varepsilon} \int_\Omega |\nabla u_{\varepsilon,h}|\, dx \\
&\leq \frac{Ch^2}{\varepsilon}\Big(\int_\Omega |\nabla u_{\varepsilon,h}|^2\, dx\Big)^{\frac{1}{2}} \leq C\frac{h^2}{\varepsilon \kappa_\varepsilon^{\frac{1}{2}}} = o(1).
\end{aligned}
$$

Finally, as the sequence $\{u_{\varepsilon,h}\}_\varepsilon$ is convergent in $L^2(\Omega)$, we deduce

$$
\begin{aligned}
C_{\varepsilon,h}^{(2)} &\leq \sum_{S\in\mathcal{S}_h} \|p_h(g_\varepsilon) - g_\varepsilon\|_{L^\infty(S)} \int_S |u_{\varepsilon,h}|\, dx \\
&\leq Ch \sum_{S\in\mathcal{S}_h} \|\nabla g_\varepsilon\|_{L^\infty(S)} \int_S |u_{\varepsilon,h}|\, dx \\
&\leq \frac{Ch}{\varepsilon} \int_\Omega |u_{\varepsilon,h}|\, dx \leq \frac{Ch}{\varepsilon} = o(1).
\end{aligned}
$$
(4.12)

From (4.8), (4.9), (4.10), (4.11), and (4.12), it follows that $\lim\limits_{\varepsilon\to 0} \mathrm{II}_{\varepsilon,h}^{(1)} = 0$.

Moreover, by the Hölder inequality and (2.2) we have

$$
\begin{aligned}
\mathrm{II}_{\varepsilon,h}^{(2)} &\leq \int_\Omega |g_\varepsilon^2 - g^2|\, dx + 2\int_\Omega |u_{\varepsilon,h}|\, |g_\varepsilon - g|\, dx \leq \int_\Omega |g_\varepsilon^2 - g^2|\, dx \\
&\quad + 2\Big(\int_\Omega |u_{\varepsilon,h}|^2\, dx\Big)^{\frac{1}{2}} \Big(\int_\Omega |g_\varepsilon - g|^2\, dx\Big)^{\frac{1}{2}} = o(1),
\end{aligned}
$$
(4.13)

which proves that $\lim\limits_{\varepsilon\to 0} \mathrm{II}_{\varepsilon,h}^{(2)} = 0$ and concludes the proof. $\qquad\square$

5. The Upper Inequality

Obviously the Γ-upper limit of the sequence $\mathcal{F}_{\varepsilon,h}$ is less than or equal to the functional $\mathcal{F}$ on the set where $\mathcal{F}$ is not finite. Moreover the following theorem states that this inequality holds also for every (u,s) such that $S_u \in \mathcal{R}$ and $s \equiv 1$.

Theorem 5.1. *Let* $h = o(\kappa_\varepsilon)$. *Let* $(u, s) \in L^\infty(\Omega) \times L^\infty(\Omega; [0, 1])$; *assume that* $u \in SBV(\Omega)$ *and that* $S_u \in \mathcal{R}$. *Then for every* $\varepsilon > 0$ *there exists* $(u_{\varepsilon,h}, s_{\varepsilon,h}) \in V_h(\Omega) \times V_h(\Omega; [0, 1])$ *such that the sequence* $\{(u_{\varepsilon,h}, s_{\varepsilon,h})\}_\varepsilon$ *converges to* (u, s) *in* $L^2(\Omega) \times L^2(\Omega; [0, 1])$ *and*

$$(5.1) \qquad \mathcal{F}(u, s) \geq \limsup_{\varepsilon \to 0} \mathcal{F}_{\varepsilon,h}(u_{\varepsilon,h}, s_{\varepsilon,h}).$$

Proof. We can assume that the left-hand side of (5.1) is finite, otherwise the result is trivial. Hence, we shall suppose $s \equiv 1$ and $u \in SBV(\Omega) \cap H^1(\Omega \setminus \overline{S_u})$. As $S_u \in \mathcal{R}$, we remark that $\mathcal{H}^{n-1}(\overline{S_u} \setminus S_u) = 0$.

Let us introduce some notation. We set

$$d(x) = \text{dist}(x, S_u) \qquad \forall x \in \Omega,$$

$$\mathcal{P} = \{x \in S_u : S_u \notin \mathcal{C}^2 \text{ in } x\},$$

$$T_t = \{x \in (S_u)_t \cap \Omega : \text{dist}(\Pi_{\overline{S}_u}(x), \mathcal{P}) < t\} \qquad \forall t > 0.$$

Note that, for any $x \in ((S_u)_t \setminus T_t) \cap \Omega$, the map $\Pi_{\overline{S}_u}$ is single-valued for t sufficiently small. For any $A \subseteq \Omega$ we define

$$A_h = \cup\{S \in \mathcal{S}_h : S \cap A \neq \emptyset\},$$

$$A^h = \cup\{S \in \mathcal{S}_h : S \subseteq A\}.$$

Let $\{b_\varepsilon\}_\varepsilon$ be a sequence of positive real numbers converging to zero and such that

$$(5.2) \qquad \lim_{\varepsilon \to 0} \frac{b_\varepsilon}{\varepsilon} = 0, \qquad \lim_{\varepsilon \to 0} \frac{\kappa_\varepsilon}{b_\varepsilon} = 0.$$

For any $\varepsilon > 0$ let us define on Ω the function

$$s_\varepsilon(x) = \begin{cases} 0 & \text{if } x \in (S_u)_{b_\varepsilon}, \\ \sigma_\varepsilon(d(x)) & \text{if } x \in (S_u)_{b_\varepsilon + \varepsilon\pi} \setminus (S_u)_{b_\varepsilon}, \\ 1 & \text{if } x \in \Omega \setminus (S_u)_{b_\varepsilon + \varepsilon\pi}, \end{cases}$$

where $\sigma_\varepsilon(t) = \sin(\frac{t - b_\varepsilon}{2\varepsilon})$ is the solution of the problem

$$\sigma_\varepsilon'(t) = \frac{1}{2\varepsilon}\sqrt{1 - \sigma_\varepsilon^2(t)} \qquad \sigma_\varepsilon(b_\varepsilon) = 0.$$

Note that $\sigma_\varepsilon(b_\varepsilon + \varepsilon\pi) = 1$, $s_\varepsilon \in H^1(\Omega; [0, 1])$, $s_\varepsilon \to 1$ in $L^2(\Omega; [0, 1])$, and $\text{Lip}(s_\varepsilon) = O(\varepsilon^{-1})$. Reasoning as in [13] Theorem 3.1 one has

$$\lim_{\varepsilon \to 0} \frac{1}{2c_o} \mathcal{M}_\varepsilon(s_\varepsilon) = \mathcal{H}^{n-1}(S_u).$$

For any ε and any h we define on $\Omega \setminus T_{b_\varepsilon + \varepsilon\pi + h}$ the functions

$$
s_\varepsilon^h(x) = \begin{cases} 0 & \text{if } x \in (S_u)_{b_\varepsilon + h}, \\ \sigma_\varepsilon^h(d(x)) & \text{if } x \in [(S_u)_{b_\varepsilon + \varepsilon\pi + h} \setminus (S_u)_{b_\varepsilon + h}] \setminus T_{b_\varepsilon + \varepsilon\pi + h}, \\ 1 & \text{if } x \in \Omega \setminus (S_u)_{b_\varepsilon + \varepsilon\pi + h}, \end{cases}
$$

where $\sigma_\varepsilon^h(t) = \sin(\frac{t - (b_\varepsilon + h)}{2\varepsilon})$ for any $t \in [b_\varepsilon + h, b_\varepsilon + \varepsilon\pi + h]$. Note that $s_\varepsilon^h \in H^1(\Omega \setminus T_{b_\varepsilon + \varepsilon\pi + h}; [0,1])$ and $\text{Lip}(s_\varepsilon^h) = O(\varepsilon^{-1})$. Using Mac-Shane's Extension Theorem for Lipschitz continuous functions [56] Theorem 2.10.43, s_ε^h can be extended on the whole Ω as a Lipschitz continuous function (still denoted by s_ε^h) so that $s_\varepsilon^h \in H^1(\Omega; [0,1])$, and $\text{Lip}(s_\varepsilon^h) = O(\varepsilon^{-1})$. In addition $s_\varepsilon^h \to 1$ in $L^2(\Omega)$ as $\varepsilon \to 0$. Reasoning again as in [13] Theorem 3.1 one has

$$(5.3) \qquad \lim_{\varepsilon \to 0} \frac{1}{2c_0} \mathcal{M}_\varepsilon(s_\varepsilon^h) = \mathcal{H}^{n-1}(S_u).$$

Finally, for any ε and any h define

$$s_{\varepsilon,h} = p_h(s_\varepsilon^h).$$

This means that on Ω we have

$$
s_{\varepsilon,h}(x) = \begin{cases} 0 & \text{if } x \in [(S_u)_{b_\varepsilon + h}]^h, \\ p_h(s_\varepsilon^h(x)) & \text{if } x \in [(S_u)_{b_\varepsilon + \varepsilon\pi + h}]_h \setminus [(S_u)_{b_\varepsilon + h}]^h, \\ 1 & \text{if } x \in [\Omega \setminus (S_u)_{b_\varepsilon + \varepsilon\pi + h}]^h. \end{cases}
$$

Observe that $s_{\varepsilon,h} \to 1$ in $L^2(\Omega)$ as $\varepsilon \to 0$ and that $[(S_u)_{b_\varepsilon + h}]^h \supseteq (S_u)_{b_\varepsilon}$, hence

$$(5.4) \qquad \Omega \setminus [(S_u)_{b_\varepsilon + h}]^h \subseteq \Omega \setminus (S_u)_{b_\varepsilon}.$$

Let us prove that

$$(5.5) \qquad \lim_{\varepsilon \to 0} \frac{1}{2c_0} \mathcal{M}_{\varepsilon,h}(s_{\varepsilon,h}) = \mathcal{H}^{n-1}(S_u).$$

In view of (5.3) it will be enough to show that

$$\lim_{\varepsilon \to 0} |\mathcal{M}_{\varepsilon,h}(s_{\varepsilon,h}) - \mathcal{M}_\varepsilon(s_\varepsilon^h)| = 0.$$

We have

$$|\mathcal{M}_{\varepsilon,h}(s_{\varepsilon,h}) - \mathcal{M}_\varepsilon(s_\varepsilon^h)| \le \varepsilon \int_\Omega |\, |\nabla s_{\varepsilon,h}|^2 - |\nabla s_\varepsilon^h|^2 \,| \, dx$$

$$+ \frac{1}{4\varepsilon} \int_\Omega |(p_h(\omega(s_{\varepsilon,h})) - \omega(s_\varepsilon^h))| \, dx =: \text{I}_{\varepsilon,h} + \text{II}_{\varepsilon,h}.$$

Let us prove that $\lim_{\varepsilon \to 0} I_{\varepsilon,h} = 0$. For simplicity of notation, set

$$(S_u)_{\varepsilon,h} := [(S_u)_{b_\varepsilon + \varepsilon\pi + h}]_h \setminus [(S_u)_{b_\varepsilon + h}]^h,$$

$$(S_u)^1_{\varepsilon,h} := (S_u)_{\varepsilon,h} \setminus [T_{b_\varepsilon + \varepsilon\pi + h}]_h,$$

$$(S_u)^2_{\varepsilon,h} := (S_u)_{\varepsilon,h} \cap [T_{b_\varepsilon + \varepsilon\pi + h}]_h.$$

As $s_{\varepsilon,h} = s^h_\varepsilon$ on the complement of $(S_u)_{\varepsilon,h}$ and $\|\nabla s_{\varepsilon,h}\|_{L^\infty(\Omega)} \leq \|\nabla s^h_\varepsilon\|_{L^\infty(\Omega)}$ $= O(\varepsilon^{-1})$, we have

$$I_{\varepsilon,h} = \varepsilon \int_{(S_u)_{\varepsilon,h}} \left| |\nabla s_{\varepsilon,h}|^2 - |\nabla s^h_\varepsilon|^2 \right| dx$$

$$= \varepsilon \int_{(S_u)_{\varepsilon,h}} |\nabla s_{\varepsilon,h} - \nabla s^h_\varepsilon| \, |\nabla s_{\varepsilon,h} + \nabla s^h_\varepsilon| dx \leq C \int_{(S_u)_{\varepsilon,h}} |\nabla s_{\varepsilon,h} - \nabla s^h_\varepsilon| dx$$

$$= C \int_{(S_u)_{\varepsilon,h}} |\nabla p_h(s^h_\varepsilon) - \nabla s^h_\varepsilon| \, dx = C \Big(\int_{(S_u)^1_{\varepsilon,h}} |\nabla p_h(s^h_\varepsilon) - \nabla s^h_\varepsilon| \, dx$$

$$+ \int_{(S_u)^2_{\varepsilon,h}} |\nabla p_h(s^h_\varepsilon) - \nabla s^h_\varepsilon| \, dx \Big) =: A_{\varepsilon,h} + B_{\varepsilon,h}.$$

Then, using well known properties of the Lagrange interpolation operator, we obtain

$$A_{\varepsilon,h} \leq C \sum_{S \subseteq (S_u)^1_{\varepsilon,h}} |S| \, \|\nabla p_h(s^h_\varepsilon) - \nabla s^h_\varepsilon\|_{L^\infty(S)} \leq Ch \sum_{S \subseteq (S_u)^1_{\varepsilon,h}} |S| \, \|D^2 s^h_\varepsilon\|_{L^\infty(S)}$$

$$\leq C\Big(\frac{h}{\varepsilon^2} + \frac{h}{\varepsilon}\Big) \mathcal{H}^n((S_u)^1_{\varepsilon,h}) = O\Big(\frac{h}{\varepsilon}\Big),$$

because $(D^2 s^h_\varepsilon)(x) = (\sigma^h_\varepsilon)''(d(x))|\nabla d(x)|^2 + (\sigma^h_\varepsilon)'(d(x))(D^2 d(x))$ on $(S_u)^1_{\varepsilon,h}$, $|\nabla d| = 1$ almost everywhere, and $\|D^2 d\|_{L^\infty((S_u)^1_{\varepsilon,h})} \leq C$ (see [60]). In addition

$$B_{\varepsilon,h} \leq C \sum_{S \subseteq (S_u)^2_{\varepsilon,h}} |S| \, \|\nabla p_h(s^h_\varepsilon) - \nabla s^h_\varepsilon\|_{L^\infty(S)} \leq \frac{C}{\varepsilon} \mathcal{H}^n((S_u)^2_{\varepsilon,h}) \leq C\varepsilon,$$

because $\mathcal{H}^n((S_u)^2_{\varepsilon,h}) = O(\varepsilon^2)$. This implies that $\lim_{\varepsilon \to 0} I_{\varepsilon,h} = 0$.

Let us prove that $\lim_{\varepsilon \to 0} II_{\varepsilon,h} = 0$. We have

$$II_{\varepsilon,h} \leq \frac{1}{4\varepsilon} \int_{(S_u)_{\varepsilon,h}} |p_h(\omega(s_{\varepsilon,h})) - \omega(s_{\varepsilon,h})| dx + \frac{1}{4\varepsilon} \int_{(S_u)_{\varepsilon,h}} |\omega(s_{\varepsilon,h}) - \omega(s^h_\varepsilon)| \, dx$$

$$=: C_{\varepsilon,h} + D_{\varepsilon,h}.$$

Therefore, reasoning as in (4.6) and (4.7), using the estimate $\|\nabla s_{\varepsilon,h}\|_{L^\infty(\Omega)} \leq \frac{C}{\varepsilon}$ we have that

$$C_{\varepsilon,h} \leq \frac{Ch^2}{\varepsilon} \sum_{S \subseteq (S_u)_{\varepsilon,h}} |S| \, \|D^2(\omega(s_{\varepsilon,h}))\|_{L^\infty(S)} \leq \frac{Ch^2}{\varepsilon} \int_{(S_u)_{\varepsilon,h}} |\nabla s_{\varepsilon,h}|^2 \, dx$$

$$\leq \frac{Ch^2}{\varepsilon^3} \mathcal{H}^n((S_u)_{\varepsilon,h}) = \frac{Ch^2}{\varepsilon^2} = o(1).$$

This proves that $\lim_{\varepsilon \to 0} C_{\varepsilon,h} = 0$. Moreover

$$D_{\varepsilon,h} \leq \frac{1}{4\varepsilon} \sum_{S \subseteq (S_u)_{\varepsilon,h}} |S| \, \|\omega(s_{\varepsilon,h}) - \omega(s_\varepsilon^h)\|_{L^\infty(S)}$$

$$\leq \frac{1}{4\varepsilon} \sum_{S \subseteq (S_u)_{\varepsilon,h}} |S| \, \mathrm{Lip}(\omega) \|s_{\varepsilon,h} - s_\varepsilon^h\|_{L^\infty(S)}$$

$$\leq \frac{Ch}{\varepsilon} \sum_{S \subseteq (S_u)_{\varepsilon,h}} |S| \, \|\nabla s_\varepsilon^h\|_{L^\infty(S)} \leq \frac{Ch}{\varepsilon^2} \mathcal{H}^n((S_u)_{\varepsilon,h}) \leq \frac{Ch}{\varepsilon} = o(1).$$

Then $\lim_{\varepsilon \to 0} D_{\varepsilon,h} = 0$; as a consequence $\lim_{\varepsilon \to 0} \mathrm{II}_{\varepsilon,h} = 0$.

If η is the constant taking into account the quasi-locality of the Clement operator (see Section 2), we define on Ω the functions

$$u_\varepsilon^h(x) = \begin{cases} 0 & \text{if } x \in (S_u)_{\frac{b_\varepsilon}{2}+\eta h}, \\ (1 - \psi_\varepsilon^h(x))u(x) & \text{if } x \in (S_u)_{b_\varepsilon - \eta h} \setminus (S_u)_{\frac{b_\varepsilon}{2}+\eta h}, \\ u(x) & \text{if } x \in \Omega \setminus (S_u)_{b_\varepsilon - \eta h}, \end{cases}$$

where $\psi_\varepsilon^h \in \mathcal{C}_0^\infty(\mathbb{R}^n)$, $0 \leq \psi_\varepsilon^h \leq 1$, $\psi_\varepsilon^h \equiv 1$ on $(S_u)_{\frac{b_\varepsilon}{2}+\eta h} \cap \Omega$, $\psi_\varepsilon^h \equiv 0$ on $\Omega \setminus (S_u)_{b_\varepsilon - \eta h}$, and $\|\nabla \psi_\varepsilon^h\|_{L^\infty(\Omega)} \leq \frac{C}{b_\varepsilon}$. The sequence $\{u_\varepsilon^h\}_\varepsilon$ converges to u in $L^2(\Omega)$. For any ε and any h define

$$u_{\varepsilon,h} = r_h(u_\varepsilon^h),$$

where r_h denotes the Clement operator (see Section 2).

Note that, by the property of quasi-locality of r_h and the definition of $u_{\varepsilon,h}$, we have

(5.6) $u_{\varepsilon,h} = r_h(u)$ on a set containing $\Omega \setminus (S_u)_{b_\varepsilon}$.

Observe that $u_{\varepsilon,h} \to u$ in $L^2(\Omega)$ as $\varepsilon \to 0$. Indeed

$$\int_\Omega |u_{\varepsilon,h} - u|^2 \, dx = \int_{\Omega \setminus (S_u)_{b_\varepsilon}} |r_h(u) - u|^2 dx + \int_{(S_u)_{b_\varepsilon} \cap \Omega} |r_h(u_\varepsilon^h) - u|^2 dx$$

$$\leq \int_\Omega |r_h(u) - u|^2 dx + 2 \int_{(S_u)_{b_\varepsilon} \cap \Omega} |r_h(u_\varepsilon^h) - u_\varepsilon^h|^2 dx$$

$$+ 2 \int_{(S_u)_{b_\varepsilon} \cap \Omega} |u_\varepsilon^h - u|^2 dx =: \mathrm{I}_{\varepsilon,h} + \mathrm{II}_{\varepsilon,h} + \mathrm{III}_{\varepsilon,h}.$$

It is obvious that $\lim_{\varepsilon \to 0} \text{III}_{\varepsilon,h} = 0$, and by the properties of the Clement operator we have $\lim_{\varepsilon \to 0} \text{I}_{\varepsilon,h} = 0$. In addition (see [37] Theorem 1)

$$\text{II}_{\varepsilon,h} \leq 2 \int_{[(S_u)_{b_\varepsilon}]_h} |r_h(u_\varepsilon^h) - u_\varepsilon^h|^2 dx \leq C h^2 \int_{[(S_u)_{b_\varepsilon}]_h} |\nabla u_\varepsilon^h|^2 dx$$

$$\leq C h^2 \Big(\int_{[(S_u)_{b_\varepsilon}]_h} |\nabla \psi_\varepsilon^h|^2 |\nabla u|^2 dx + \int_{[(S_u)_{b_\varepsilon}]_h} (1 - \psi_\varepsilon^h)^2 |\nabla u|^2 dx \Big)$$

$$\leq C \Big(\frac{h^2}{b_\varepsilon^2} + h^2 \Big) = o(1).$$

Claim:

$$(5.7) \qquad \lim_{\varepsilon \to 0} \int_{\Omega \backslash (S_u)_{b_\varepsilon}} |\nabla(r_h(u) - u)|^2 \, dx = 0.$$

Proof of the claim. Let us consider the set $\overline{S_u}$. If $\overline{S_u} \cap \partial \Omega \neq \emptyset$, applying Lemma 3.1 to the bounded open set Ω and to the closed set $\overline{S_u}$ there exist a bounded open set $\Omega' \supset\supset \Omega$, a closed set $S' \subseteq \overline{\Omega'}$, a real number $N \geq 2$, and a function $U \in L^\infty(\Omega') \cap H^1(\Omega' \backslash S')$ such that $S' \cap \overline{\Omega} = S \cap \overline{\Omega}$, $U = u$ on Ω and

$$(5.8) \qquad \Omega \backslash (S')_{\frac{2b_\varepsilon}{N}} \supseteq \Omega \backslash (S_u)_{b_\varepsilon}$$

for every ε sufficiently small.

Let $\varepsilon > 0$ small enough; consider the restriction of U on the Lispchitz set $O_\varepsilon = \{x \in \Omega' : \text{dist}(x, \partial\Omega' \cup S') > \frac{b_\varepsilon}{N} - h\}$ and let us extend it on the whole $\mathbb{R}^n$ to a function $U_\varepsilon \in H^1(\mathbb{R}^n)$ with compact support (see [31] Theorem IX.7) such that $\|U_\varepsilon\|_{L^2(\mathbb{R}^n)} \leq C \|U\|_{L^2(\Omega')}$.

If $\overline{S_u} \cap \partial\Omega = \emptyset$, then for every ε sufficiently small the set $\Omega \backslash (S_u)_{b_\varepsilon}$ has a Lipschitz continuous boundary. In this case we consider the function u on $\Omega \backslash (S_u)_{b_\varepsilon}$, and we extend it on the whole $\mathbb{R}^n$ to a function $U_\varepsilon \in H^1(\mathbb{R}^n)$ having the previous properties.

Let $\{\varrho_\varepsilon\}_\varepsilon$ be a sequence of mollifiers defined by $\varrho_\varepsilon(x) = (\frac{b_\varepsilon}{N})^{-n} \varrho(\frac{xN}{b_\varepsilon})$, and set $V_\varepsilon = U_\varepsilon \star \varrho_\varepsilon$. Then $V_\varepsilon \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ for any $\varepsilon > 0$.

Let us verify that

$$(5.9) \qquad \lim_{\varepsilon \to 0} \int_{[\Omega \backslash (S_u)_{b_\varepsilon}]_h} |\nabla(V_\varepsilon - u)|^2 \, dx = 0,$$

$$(5.10) \qquad \Big(\int_{[\Omega \backslash (S_u)_{b_\varepsilon}]_h} |D^2 V_\varepsilon|^2 \Big)^{\frac{1}{2}} = O(b_\varepsilon^{-1}).$$

Observe that, in view of (5.8), if $x \in [\Omega \backslash (S_u)_{b_\varepsilon}]_h$ then $x \in [\Omega \backslash (S')_{\frac{2b_\varepsilon}{N}}]_h$, so that $\text{dist}(x, S') \geq \frac{2b_\varepsilon}{N} - h$, and hence $B_{\frac{b_\varepsilon}{N}}(x)$ is contained in O_ε. Therefore

$V_\varepsilon = U \star \varrho_\varepsilon$ on $[\Omega \setminus (S_u)_{b_\varepsilon}]_h$; moreover $U \in H^1(\Omega \setminus \overline{S_u})$ so that $\nabla(U \star \varrho_\varepsilon) = \nabla U \star \varrho_\varepsilon$ on $[\Omega \setminus (S_u)_{b_\varepsilon}]_h$. We then have

$$\int_{[\Omega \setminus (S_u)_{b_\varepsilon}]_h} |\nabla(V_\varepsilon - u)|^2 \, dx$$

$$= \int_{[\Omega \setminus (S_u)_{b_\varepsilon}]_h} |\nabla(U_\varepsilon \star \varrho_\varepsilon) - \nabla U|^2 \, dx$$

$$= \int_{[\Omega \setminus (S_u)_{b_\varepsilon}]_h} |\nabla U \star \varrho_\varepsilon - \nabla U|^2 \, dx \to 0$$

as $\varepsilon \to 0$. This proves (5.9).

Let $i, j \in \{1, \dots, n\}$; using well known properties of the convolutions, the Hölder inequality, and Fubini-Tonelli's Theorem, we have

$$\int_{[\Omega \setminus (S_u)_{b_\varepsilon}]_h} |D_{ij} V_\varepsilon|^2 dx$$

$$\leq \int_{[\Omega \setminus (S_u)_{b_\varepsilon}]_h} \left(\int_{\mathbb{R}^n} |D_i U_\varepsilon(y)| |D_j \varrho_\varepsilon(x - y)|^{\frac{1}{2}} |D_j \varrho_\varepsilon(x - y)|^{\frac{1}{2}} \, dy \right)^2 dx$$

$$\leq \|D_j \varrho_\varepsilon\|_{L^1(\mathbb{R}^n)} \int_{[\Omega \setminus (S_u)_{b_\varepsilon}]_h} \int_{\mathbb{R}^n} |D_i U_\varepsilon(y)|^2 |D_j \varrho_\varepsilon(x - y)| dy dx$$

$$= O(b_\varepsilon^{-1}) \int_{\mathbb{R}^n} |D_i U_\varepsilon(y)|^2 \int_{[\Omega \setminus (S_u)_{b_\varepsilon}]_h} |D_j \varrho_\varepsilon(x - y)| \, dx dy$$

$$\leq O(b_\varepsilon^{-2}) \int_{([\Omega \setminus (S_u)_{b_\varepsilon}]_h)_{\frac{b_\varepsilon}{N}}} |D_i U_\varepsilon(y)|^2 \, dy$$

$$= O(b_\varepsilon^{-2}) \int_{([\Omega \setminus (S_u)_{b_\varepsilon}]_h)_{\frac{b_\varepsilon}{N}}} |D_i U|^2 dy = O(b_\varepsilon^{-2}),$$

which proves (5.10).

Then, using [37] Theorem 1, (5.10) and (5.9), we deduce

$$\|\nabla(r_h(u) - u)\|_{L^2(\Omega \setminus (S_u)_{b_\varepsilon})} \leq \|\nabla(r_h(V_\varepsilon) - V_\varepsilon)\|_{L^2([\Omega \setminus (S_u)_{b_\varepsilon}]_h)}$$

$$+ \|\nabla r_h(u - V_\varepsilon) - \nabla(u - V_\varepsilon)\|_{L^2([\Omega \setminus (S_u)_{b_\varepsilon}]_h)} \leq Ch \|D^2 V_\varepsilon\|_{L^2([\Omega \setminus (S_u)_{b_\varepsilon}]_h)}$$

$$+ C \|\nabla(V_\varepsilon - u)\|_{L^2([\Omega \setminus (S_u)_{b_\varepsilon}]_h)} = O\left(\frac{h}{b_\varepsilon}\right) + o(1).$$

As $\lim_{\varepsilon \to 0} \dfrac{h}{b_\varepsilon} = 0$, the claim is proved. $\qquad \square$

In order to prove (5.1) we shall show that

$$(5.11) \qquad \limsup_{\varepsilon \to 0} \int_\Omega (s_{\varepsilon,h} + \kappa_\varepsilon)|\nabla u_{\varepsilon,h}|^2 \, dx \leq \int_\Omega |\nabla u|^2 \, dx,$$

$$(5.12) \qquad \lim_{\varepsilon \to 0} \int_\Omega p_h((u_{\varepsilon,h} - g_\varepsilon)^2) \, dx = \int_\Omega |u - g|^2 \, dx.$$

Let us prove (5.11). Using the definition of $s_{\varepsilon,h}$, (5.4), and (5.6), we have

$$\int_\Omega s_{\varepsilon,h}|\nabla u_{\varepsilon,h}|^2 \, dx = \int_{\Omega \setminus [(S_u)_{b_\varepsilon}+h]^h} s_{\varepsilon,h}|\nabla u_{\varepsilon,h}|^2 \, dx$$

$$\leq \int_{\Omega \setminus [(S_u)_{b_\varepsilon}+h]^h} |\nabla u_{\varepsilon,h}|^2 \, dx \leq \int_{\Omega \setminus (S_u)_{b_\varepsilon}} |\nabla u_{\varepsilon,h}|^2 \, dx$$

$$= \int_{\Omega \setminus (S_u)_{b_\varepsilon}} |\nabla r_h(u)|^2 \, dx.$$

Therefore, by (5.7) we deduce

$$\limsup_{\varepsilon \to 0} \int_\Omega s_{\varepsilon,h}|\nabla u_{\varepsilon,h}|^2 \, dx \leq \limsup_{\varepsilon \to 0} \int_{\Omega \setminus (S_u)_{b_\varepsilon}} |\nabla r_h(u)|^2 \, dx = \int_\Omega |\nabla u|^2 \, dx.$$

To conclude the proof of (5.11) we must show that

$$\lim_{\varepsilon \to 0} \kappa_\varepsilon \int_\Omega |\nabla u_{\varepsilon,h}|^2 \, dx = 0.$$

Using the properties of the Clement operator (see [37] Theorem 1), the definition of u_ε^h, and the fact that $\|\nabla \psi_\varepsilon^h\|_{L^\infty(\Omega)} = O(b_\varepsilon^{-1})$, we have

$$\kappa_\varepsilon \int_\Omega |\nabla u_{\varepsilon,h}|^2 \, dx = \kappa_\varepsilon \int_\Omega |\nabla r_h(u_\varepsilon^h)|^2 \, dx \leq C\kappa_\varepsilon \int_\Omega |\nabla u_\varepsilon^h|^2 \, dx$$

$$(5.13) \qquad \begin{aligned} = C\kappa_\varepsilon \Big(&\int_{\Omega \setminus (S_u)_{b_\varepsilon - \eta h}} |\nabla u|^2 \, dx \\ &+ \int_{((S_u)_{b_\varepsilon - \eta h} \setminus (S_u)_{\frac{b_\varepsilon}{2} + \eta h}) \cap \Omega} |\nabla (1 - \psi_\varepsilon^h)u|^2 \, dx \Big) \end{aligned}$$

$$\leq C(\kappa_\varepsilon + \frac{\kappa_\varepsilon}{b_\varepsilon}) \to 0$$

as $\varepsilon \to 0$ (recall (5.2)). This concludes the proof of (5.11).

Let us prove (5.12). We have

$$\Big| \int_\Omega p_h((u_{\varepsilon,h} - g_\varepsilon)^2) - (u - g)^2 \, dx \Big| \leq \int_\Omega |p_h((u_{\varepsilon,h} - g_\varepsilon)^2) - (u_{\varepsilon,h} - g_\varepsilon)^2| \, dx$$

$$+ \int_\Omega |(u_{\varepsilon,h} - g_\varepsilon)^2 - (u - g)^2| \, dx =: \mathrm{I}_{\varepsilon,h} + \mathrm{II}_{\varepsilon,h}.$$

Obviously, as $u_{\varepsilon,h} \to u$ and $g_\varepsilon \to g$ in $L^2(\Omega)$ as $\varepsilon \to 0$, we have $\lim_{\varepsilon \to 0} II_{\varepsilon,h} = 0$. Moreover, using the same notation as in (4.8) and below, we write

$$I_{\varepsilon,h} \le A_{\varepsilon,h} + B_{\varepsilon,h} + 2C^{(1)}_{\varepsilon,h} + 2C^{(2)}_{\varepsilon,h}.$$

As in (4.9), (4.10), (4.11), and (4.12), recalling that $\int_\Omega |\nabla u_{\varepsilon,h}|^2 \, dx = O(b_\varepsilon^{-1})$ (see (5.13)), we find $A_{\varepsilon,h} = O(\frac{h^2}{b_\varepsilon})$, $B_{\varepsilon,h} = O(\frac{h}{\varepsilon})$, $C^{(1)}_{\varepsilon,h} = O(\frac{h^2}{\varepsilon b_\varepsilon^{\frac{1}{2}}})$, and $C^{(2)}_{\varepsilon,h} = O(\frac{h}{\varepsilon})$. This concludes the proof of (5.12). Therefore, (5.1) is a consequence of (5.5), (5.11), and (5.12).   □

Observe that to show the Γ-convergence of $\{\mathcal{F}_{\varepsilon,h}\}_\varepsilon$ to $\mathcal{F}$ on the whole $L^\infty(\Omega) \cap SBV(\Omega)$ one should have to prove a result of this type, which, to our knowledge, is not known:

let $u \in SBV(\Omega) \cap L^\infty(\Omega)$ be such that $\mathcal{F}(u, 1) < +\infty$. Then there exists a sequence $\{(u_\varepsilon, s_\varepsilon)\}_\varepsilon$ in $(L^\infty(\Omega) \cap SBV(\Omega)) \times L^\infty(\Omega; [0, 1])$ converging to u in $L^2(\Omega) \times L^2(\Omega; [0, 1])$ such that for any ε the set S_{u_ε} belongs to $\mathcal{R}$ and $\lim_{\varepsilon \to 0} \mathcal{F}(u_\varepsilon, s_\varepsilon) = \mathcal{F}(u, 1)$.

Indeed, assume that the previous result holds. By the L^1-lower semicontinuity of the Γ-upper limit of the sequence $\{\mathcal{F}_{\varepsilon,h}\}_\varepsilon$, denoted by $\mathcal{F}''$, and by the inequality $\mathcal{F}(u_\varepsilon, s_\varepsilon) \ge \mathcal{F}''(u_\varepsilon, s_\varepsilon)$ proved in Theorem 5.1, it follows that

$$\mathcal{F}''(u, 1) \le \liminf_{\varepsilon \to 0} \mathcal{F}''(u_\varepsilon, s_\varepsilon) \le \liminf_{\varepsilon \to 0} \mathcal{F}(u_\varepsilon, s_\varepsilon) = \mathcal{F}(u, 1).$$

6. Convergence of Minimum Values and of Minimizers

The first result of this section is the following compactness theorem.

Theorem 6.1. *Let* $h = o(\kappa_\varepsilon)$. *For any* $\varepsilon > 0$ *let* $(u_{\varepsilon,h}, s_{\varepsilon,h}) \in V_h(\Omega) \times V_h(\Omega; [0, 1])$ *be a minimum point of* $\mathcal{F}_{\varepsilon,h}$. *Then there exist a subsequence* $\{(u_{\varepsilon,h}, s_{\varepsilon,h})\}_\varepsilon$ *and a function* $u \in SBV(\Omega) \cap L^\infty(\Omega)$ *such that*

(6.1) $(u_{\varepsilon,h}, s_{\varepsilon,h}) \to (u, 1)$ *in* $L^2(\Omega) \times L^2(\Omega; [0, 1])$ *as* $\varepsilon \to 0$.

Proof. As $(u_{\varepsilon,h}, s_{\varepsilon,h})$ is a minimizer of $\mathcal{F}_{\varepsilon,h}$, using (2.2) we have

$$\mathcal{F}_{\varepsilon,h}(u_{\varepsilon,h}, s_{\varepsilon,h}) \le \mathcal{F}_{\varepsilon,h}(0, 1) = \int_\Omega p_h(g_\varepsilon^2) \, dx \le C \|p_h(g_\varepsilon^2)\|_{L^\infty(\Omega)}$$

$$\le C \|g_\varepsilon^2\|_{L^\infty(\Omega)} \le C.$$

Then from $\int_\Omega \frac{1}{4\varepsilon}(1 - s_{\varepsilon,h}^2)\, dx \le C$ it follows that $s_{\varepsilon,h} \to 1$ in $L^2(\Omega; [0,1])$ as $\varepsilon \to 0$. Moreover we have the estimates

$$(6.2) \qquad \int_\Omega |\nabla u_{\varepsilon,h}|^2\, dx \le \frac{C}{\kappa_\varepsilon}, \qquad \int_\Omega |\nabla s_{\varepsilon,h}|^2\, dx \le \frac{C}{\varepsilon}.$$

Since $(u_{\varepsilon,h}, s_{\varepsilon,h})$ is a minimizer of $\mathcal{F}_{\varepsilon,h}$, by a well known truncation argument we see that $\|u_{\varepsilon,h}\|_{L^\infty(\Omega)} \le \|g_\varepsilon\|_{L^\infty(\Omega)} \le C$ for any $\varepsilon > 0$.

Using the same notation of (4.5), we have

$$\mathcal{F}_{\varepsilon,h}(u_{\varepsilon,h}, s_{\varepsilon,h}) = \mathcal{F}_\varepsilon(u_{\varepsilon,h}, s_{\varepsilon,h}) + \frac{1}{2c_0} \mathrm{I}_{\varepsilon,h} + \mathrm{II}_{\varepsilon,h}.$$

By (6.2), reasoning as in the proof of Theorem 4.1 (see (4.7), (4.8), (4.9), (4.10), (4.11), (4.12), and (4.13)) we have that $\lim_{\varepsilon \to 0} \mathrm{I}_{\varepsilon,h} = \lim_{\varepsilon \to 0} \mathrm{II}_{\varepsilon,h} = 0$ (we remark that the equi-boundedness in $L^2(\Omega)$ of the sequence $\{u_{\varepsilon,h}\}_\varepsilon$ follows from the fact that $\|u_{\varepsilon,h}\|_{L^\infty(\Omega)} \le C$). Therefore

$$(6.3) \qquad \mathcal{F}_\varepsilon(u_{\varepsilon,h}, s_{\varepsilon,h}) \le C \qquad \forall \varepsilon > 0.$$

As for any $t \in [0,1]$ we have $t^2 \le t$ and $(1-t)^2 \le 1 - t^2$, we deduce that (6.3) holds also if the functionals $\mathcal{F}_\varepsilon$ are as in (2.1). Finally, assertion (6.1) follows from Theorem 2.1 (see [12], [13]). □

Theorems 4.1 and 5.1 do not assert the Γ-convergence of the sequence $\{\mathcal{F}_{\varepsilon,h}\}_\varepsilon$ to $\mathcal{F}$ on the whole $SBV(\Omega) \cap L^\infty(\Omega)$. Nevertheless, we can prove the following theorem concerning the convergence of the minimum values and of the minimizers.

Theorem 6.2. *Let $h = o(\kappa_\varepsilon)$. Let us assume that there exists at least a minimizer $\bar{u}$ of the functional $\mathcal{F}$ such that $S_{\bar{u}} \in \mathcal{R}$. Then the minimum values of $\mathcal{F}_{\varepsilon,h}$ converge to the minimum value of $\mathcal{F}$ as $\varepsilon \to 0$. Moreover any family $\{(u_{\varepsilon,h}, s_{\varepsilon,h})\}_\varepsilon$ of absolute minimizers of $\{\mathcal{F}_{\varepsilon,h}\}_\varepsilon$ is relatively compact in $L^2(\Omega) \times L^2(\Omega; [0,1])$, and each of its limit points minimizes $\mathcal{F}$.*

Proof. Let us prove first that

$$(6.4) \qquad \lim_{\varepsilon \to 0}(\min \mathcal{F}_{\varepsilon,h}) = \min \mathcal{F}.$$

Observe that

$$(6.5) \qquad \liminf_{\varepsilon \to 0}(\min \mathcal{F}_{\varepsilon,h}) \ge \min \mathcal{F}.$$

Indeed, let $(u_{\varepsilon,h}, s_{\varepsilon,h}) \in V_h(\Omega) \times V_h(\Omega; [0,1])$ be a minimizer of $\mathcal{F}_{\varepsilon,h}$; up to subsequences, using Theorem 6.1 we can suppose that there exists a function $u \in SBV(\Omega) \cap L^\infty(\Omega)$ such that (6.1) holds and that the liminf in the left hand-side of (6.5) is a limit.

Therefore, by Theorem 4.1 we get

$$\liminf_{\varepsilon \to 0}(\min \mathcal{F}_{\varepsilon,h}) = \lim_{\varepsilon \to 0} \mathcal{F}_{\varepsilon,h}(u_{\varepsilon,h}, s_{\varepsilon,h}) \geq \mathcal{F}(u,1) \geq \min \mathcal{F}.$$

By hypothesis there exists a minimizer $\overline{u}$ of $\mathcal{F}$ whose jump set $S_{\overline{u}} \in \mathcal{R}$. Then for every $\varepsilon > 0$ there exists $(v_{\varepsilon,h}, \sigma_{\varepsilon,h}) \in V_h(\Omega) \times V_h(\Omega; [0,1])$ such that the sequence $\{(v_{\varepsilon,h}, \sigma_{\varepsilon,h})\}_\varepsilon$ converges to $(u,1)$ in $L^2(\Omega) \times L^2(\Omega; [0,1])$ and (5.1) holds. Using (6.5) we then obtain that

$$\min \mathcal{F} = \mathcal{F}(\overline{u},1) \geq \limsup_{\varepsilon \to 0} \mathcal{F}_{\varepsilon,h}(v_{\varepsilon,h}, \sigma_{\varepsilon,h})$$
$$\geq \limsup_{\varepsilon \to 0}(\min \mathcal{F}_{\varepsilon,h}) \geq \liminf_{\varepsilon \to 0}(\min \mathcal{F}_{\varepsilon,h}) \geq \min \mathcal{F},$$

which proves (6.4).

Finally the fact that any family $\{(u_{\varepsilon,h}, s_{\varepsilon,h})\}_\varepsilon$ of absolute minimizers of $\{\mathcal{F}_{\varepsilon,h}\}_\varepsilon$ is relatively compact in $L^2(\Omega) \times L^2(\Omega; [0,1])$ follows immediately from Theorem 6.1. If (u,s) is a limit point of such a sequence (observe that we cannot conclude that $S_u \in \mathcal{R}$), using Theorem 4.1 and (6.4) we have that

$$\mathcal{F}(u,s) \leq \liminf_{\varepsilon \to 0} \mathcal{F}_{\varepsilon,h}(u_{\varepsilon,h}, s_{\varepsilon,h}) = \lim_{\varepsilon \to 0}(\min \mathcal{F}_{\varepsilon,h}) = \min \mathcal{F},$$

which implies that $s \equiv 1$ and $(u,1)$ is a minimum point of $\mathcal{F}$. $\qquad\square$

CHAPTER 6:

APPROXIMATION OF A FUNCTIONAL
DEPENDING ON JUMPS AND CORNERS

We consider the functional $\mathcal{G}$, related to segmentation problems, defined by

$$G(u) = \int_a^b |\ddot{u}|^2 \, dt + \alpha \#(S_u) + \beta \#(S_{\dot{u}} \setminus S_u) + \int_a^b |u - g|^2 \, dt,$$

where $g \in L^2(a, b)$, u is a piecewise H^2 function, $\dot{u}, \ddot{u}$ are the pointwise values of the first and second derivatives of u, and $\#(S_u)$ (resp. $\#(S_{\dot{u}} \setminus S_u)$) denotes the number of the jump points of u (resp. corner points of u, i.e., jump points of $\dot{u}$ which are not jump points of u).

We prove that the functional $\mathcal{G}$ can be approximated, via De Giorgi's Γ-convergence, by an equicoercive sequence of elliptic functionals which do not depend on jumps or corners.

The results of this chapter are contained in [18].

Introduction

In this chapter we will show how to approximate, in a variational sense, a functional recently proposed as a model of a segmentation problem in dimension one.

Precisely, let $g \in L^2(a, b)$ be a given function, and let α, β be two real numbers, with

(0.1) $$0 < \beta \leq \alpha \leq 2\beta.$$

Let us consider the functional

$$G(u) = \int_a^b |\ddot{u}|^2 \, dt + \alpha \#(S_u) + \beta \#(S_{\dot{u}} \setminus S_u) + \int_a^b |u - g|^2 \, dt,$$

where u varies over the space $\mathcal{H}^2(a, b)$ of piecewise H^2 functions. Here $\dot{u}, \ddot{u}$ are the pointwise values of the first and second derivatives of u, and $\#(S_u)$ (resp. $\#(S_{\dot{u}} \setminus S_u)$) denotes the number of the jump points of u (resp. corner points of u, i.e., jump points of $\dot{u}$ which are not jump points of u).

We prove that the functional G can be approximated, via De Giorgi's Γ-convergence [48], [49], [42], by an equi-coercive sequence of elliptic functionals which do not depend on jumps or corners.

The minimum problem

$$(0.2) \qquad \inf\{G(u) : u \in \mathcal{H}^2(a, b)\}$$

has been suggested in [24] and [84] as a variational approach to the segmentation problem corresponding to the datum g. For a mathematical treatment of problem (0.2) we refer to [39].

A segmentation problem consists in subdividing the interval $]a, b[$ into appropriate subintervals, and in approximating, on each subinterval, the function g by a smooth function. Such a problem arises when one has to approximate a discontinuous datum g, eliminating the less relevant details but preserving some properties of its behaviour; indeed, in this situation, one can obtain a better approximation of g by means of piecewise smooth functions rather than by globally smooth functions.

In the expression of the functional G the first term requires u to vary smoothly on each connected component of $]a, b[\backslash(S_u \cup S_{\dot{u}})$, while the last term forces u to be close to g. The other terms are introduced to avoid a subdivision of $]a, b[$ into too many parts.

In the one-dimensional case a segmentation problem arises in the perception of speech, which requires segmenting time (the domain of the speech signal) into intervals during which a single phoneme is being pronounced. In dimension two, in the setting of Computer Vision, the function g, defined on a plane domain Ω, represents the grey level of an image given by a camera, and the image segmentation problem consists in reconstructing g by a function u which is smooth on appropriate regions with relatively uniform light intensity.

A variational approach to a segmentation problem consists in minimizing a suitable energy functional, as pointed out by S. Geman and D. Geman [59]. In the study of the image segmentation problem D. Mumford and J. Shah [78], [77] developed this variational idea by suggesting the study of the two-dimensional problem

$$(0.3) \qquad \inf_{(u,K)} \{ \int_{\Omega\backslash K} |\nabla u|^2 \, dx + \mathcal{H}^1(K) + \int_{\Omega\backslash K} |u - g|^2 \, dx \},$$

where K is a closed subset of $\overline{\Omega}$, $u \in \mathcal{C}^1(\overline{\Omega} \backslash K)$, and $\mathcal{H}^1$ denotes the one dimensional Hausdorff measure in $\mathbb{R}^2$ [56]. The idea is to find a pair (u, K) such that K is a set of curves decomposing Ω into appropriate regions, and u is a smooth approximation of g on each region. The set K will be interpreted as the union of the lines giving the schematic description of the image.

Problem (0.3) has been studied by many authors; we mention [44], [46], [43], [72], [73] for some references about this subject. In particular, we point out the n-dimensional setting proposed by E. De Giorgi [45] and developed

in [46], [5], [6], which is based on a weak formulation of (0.3) by means of a problem of the form

$$(0.4) \quad \inf_u F(u), \quad \text{where} \quad F(u) = \int_\Omega |\nabla u|^2 \, dx + \mathcal{H}^{n-1}(S_u) + \int_\Omega |u - g|^2 \, dx.$$

Here u belongs to the space $SBV(\Omega)$ of the special bounded variation functions [6], S_u is the jump set of u in an approximate sense, and $\mathcal{H}^{n-1}$ denotes the $(n-1)$-dimensional Hausdorff measure in $\mathbb{R}^n$ [56].

L. Ambrosio and V.M. Tortorelli [12], [13] approximated F by a sequence of elliptic functionals via Γ-convergence. The approximating sequence $\{F_\varepsilon\}_\varepsilon$ proposed in [13] reads as follows:

$$(0.5) \quad F_\varepsilon(u, s) = \int_\Omega (s^2 + \lambda_\varepsilon)|\nabla u|^2 \, dx + \mathcal{M}_\varepsilon(s) + \int_\Omega |u - g|^2 \, dx,$$

where ε is the relaxation parameter, λ_ε is a sequence of positive numbers vanishing faster than ε, $u \in \mathcal{C}^1(\Omega)$, $s \in \mathcal{C}^1(\Omega, [0, 1])$, and

$$\mathcal{M}_\varepsilon(s) = \int_\Omega [\varepsilon|\nabla s|^2 + \frac{(s - 1)^2}{4\varepsilon}] \, dx.$$

The main difficulties in the approximation are due to the term $\mathcal{H}^{n-1}(S_u)$, representing the measure of an unknown hypersurface. The idea is to introduce a new variable s, which controls the jump set S_u in the following sense: if $\{(u_\varepsilon, s_\varepsilon)\}_\varepsilon$ is a minimizing sequence such that $\lim_{\varepsilon \to 0} F_\varepsilon(u_\varepsilon, s_\varepsilon) < +\infty$, then, up to subsequences, $\{s_\varepsilon\}_\varepsilon$ converges to $s \equiv 1$ a.e. on Ω, $\{u_\varepsilon\}_\varepsilon$ converges to some function u strongly in $L^1(I)$, and the level sets $\{s_\varepsilon < \eta_\varepsilon\}$ approximate S_u, for a suitable sequence $\{\eta_\varepsilon\}_\varepsilon$ of positive numbers converging to zero as $\varepsilon \to 0$.

We point out that the choice of $\mathcal{M}_\varepsilon$ relies on a pioneering work by L. Modica and S. Mortola [71], who suggested how to approximate functionals of area type by means of elliptic functionals. The sequence $\{\mathcal{M}_\varepsilon\}_\varepsilon$ is indeed the correct approximation of the term $\mathcal{H}^{n-1}(S_u)$, appearing in the limit F.

By the properties of Γ-convergence, a sequence of minimizers of the functionals F_ε converges, possibly passing to subsequences, to a minimum point of F, as $\varepsilon \to 0$. Therefore the approximation (0.5) can be used to attack problem (0.4) from a numerical viewpoint [67], [84], [19] (see also [17], [22] for the applications of Γ-convergence to numerical analysis).

We are now in a position to explain our choice of the approximating sequence $\{G_\varepsilon\}_\varepsilon$ converging to G.

It has been proved in [39] that the minimization problem (0.2) admits a solution. Precisely, the functional G is coercive and $L^1(a, b)$-lower semicontinuous on $\mathcal{H}^2(a, b)$, provided inequalities (0.1) are satisfied. This fact

suggests to consider the $L^1(a,b)$-convergence as a good topology for an approximation theorem of Γ-convergence.

Following the approach of [12], [13], our first attempt was to consider functionals of the form

$$\int_a^b (s^2 + \lambda_\varepsilon)|u''|^2 \, dt + \mathcal{M}_\varepsilon(s) + \int_a^b |u - g|^2 \, dt.$$

It is not difficult to see that, with this definition, the variable s controls the set $S_u \cup S_{\dot u}$ in the same sense explained above, and $\{\mathcal{M}_\varepsilon\}_\varepsilon$ should give rise, in the limit, to a term of the form $\#(S_u \cup S_{\dot u})$. This is obviously unsatisfactory and does not permit to treat the case $\alpha \neq \beta$. The main idea is then to distinguish S_u from $S_{\dot u}$, by introducing another control variable σ for the set S_u. The approximating sequence $\{G_\varepsilon\}_\varepsilon$ becomes then

$$G_\varepsilon(u,s,\sigma) = \int_a^b (s^2 + \lambda_\varepsilon)|u''|^2 dt + \beta\mathcal{M}_\varepsilon(s) + (\alpha - \beta)\mathcal{M}_\varepsilon(\sigma)$$

$$+ \mu_\varepsilon \int_a^b \sigma^2 |u'|^2 dt + \int_a^b |u - g|^2 dt,$$

where $u \in H^2(a,b)$, $s, \sigma \in H^1(a,b)$, $0 \leq s \leq 1$, $0 \leq \sigma \leq 1$, and $\{\lambda_\varepsilon\}_\varepsilon$, $\{\mu_\varepsilon\}_\varepsilon$ are suitable sequences of positive numbers converging to zero, as $\varepsilon \to 0$ (see (2.4)).

The term $\mu_\varepsilon \int_a^b \sigma^2 |u'|^2 \, dt$ requires some comments (see the proof of Lemma 3.2 (i)). Suppose that $\{(u_\varepsilon, s_\varepsilon, \sigma_\varepsilon)\}_\varepsilon$ is a minimizing sequence such that $\lim_{\varepsilon \to 0} G_\varepsilon(u_\varepsilon, s_\varepsilon, \sigma_\varepsilon) < +\infty$. A term of the form $\int_a^b \sigma^2 |u'|^2 \, dt$ would guarantee that the level sets $\{\sigma_\varepsilon \leq \eta_\varepsilon\}$ approximate S_u, for a suitable sequence $\{\eta_\varepsilon\}_\varepsilon$ of positive numbers converging to zero, but, as no integrals involving $\dot u$ appear in G, this term should vanish in the limit. This is the reason why we multiply $\int_a^b \sigma^2 |u'|^2 \, dt$ by the factor μ_ε which converges to zero. However, with this choice, the level sets of σ_ε might not approximate the set S_u. The problem is solved by proving that, if $\{\sigma_\varepsilon\}_\varepsilon$ does not tend to vanish near the points of S_u as $\varepsilon \to 0$, then the functions s_ε must approach zero at least twice (see (3.20)) in a neighbourhood of S_u, under a suitable choice of the rate of convergence of $\{\mu_\varepsilon\}_\varepsilon$ (see (2.4)). At this stage, the inequalities (0.1) between the coefficients become crucial. Observe that this remark shows that the two control functions s and σ are not completely independent.

Then our main result reads as follows:

Theorem 0.1.   *The sequence $\{G_\varepsilon\}_\varepsilon$ Γ-converges to G in L^1 as $\varepsilon \to 0$. Moreover, the functionals G_ε are equi-coercive with respect to the L^1-topology.*

In view of well known properties of Γ-convergence, from Theorem 0.1 it follows that *any family of absolute minimizers of G_ε is relatively compact in L^1 and each of its limit points minimizes G.*

We have considered the problem in dimension one. The extension of our results to the n-dimensional case is a difficult open problem, and seems to require some new results about the characterization of functions having gradient in $SBV(\Omega; \mathbb{R}^n)$ by means of their one-dimensional sections (see [20]).

Let us briefly describe the contents of the chapter.

In Section 1 we give some notation and we recall an interpolation inequality for the intermediate derivatives.

In Section 2 we introduce the problem.

In Section 3 we prove that the functional G is less than or equal to the Γ-lower limit of the sequence $\{G_\varepsilon\}_\varepsilon$.

In Section 4 we prove that the functional G is greater than or equal to the Γ-upper limit of the sequence $\{G_\varepsilon\}_\varepsilon$.

Finally, in Section 5, we prove that the sequence $\{G_\varepsilon\}_\varepsilon$ is equi-coercive and this concludes the proof of Theorem 0.1.

1. Notations and Preliminaries

Let $I =]a, b[$ be a bounded open interval of $\mathbb{R}$. Let $u \in L^2(I)$; by u' and u'' we mean the first and second derivative of u in the sense of distributions. By $H^1(I)$ (resp. $H^2(I)$) we denote the Sobolev space of the functions $u \in L^2(I)$ such that u' is (resp. both u' and u'' are) representable by a square-integrable function. We denote by $H^1(I, [0, 1])$ the convex set $\{s \in H^1(I) : s(t) \in [0, 1] \; \forall t \in I\}$. We point out an interpolation inequality which will be useful in the sequel (see [2] Lemma 4.10).

Proposition 2.1.   *Let $u \in L^2(I)$ be such that $u'' \in L^2(I)$. Then $u \in H^2(I)$ and*

$$\int_I |u'|^2 \, dt \leq R \left(\int_I |u|^2 \, dt + \int_I |u''|^2 \, dt \right),$$

where $R = 2 \cdot 9^2 \max\{(b - a)^2, (b - a)^{-2}\}$.

We designate by $\mathcal{H}^2(I)$ the space of the functions $u \in L^2(I)$ such that there exists a finite partition $x_0 = a < x_1 < \ldots < x_{n+1} = b$ of I with the property that $u|_{]x_i, x_{i+1}[}$ is of class $H^2(x_i, x_{i+1})$, for any $i = 0, \ldots, n$.

Let $u \in \mathcal{H}^2(I)$; as u is of class $\mathcal{C}^1$ on each subinterval $]x_i, x_{i+1}[$, the pointwise derivative $\dot{u}$ is defined at all points of I except possibly at $x_1, \ldots, x_n$. Note that $\dot{u} = u'$ almost everywhere on $]x_i, x_{i+1}[$, and that u'

is a measure on I including some concentrated masses at the points x_i. Note also that $\dot{u}$ is the absolutely continuous part of u' with respect to the Lebesgue measure.

Since $\dot{u}$ is absolutely continuous on $]x_i, x_{i+1}[$, there exist the left and right limits $\dot{u}^-(x_i)$ and $\dot{u}^+(x_i)$ of $\dot{u}$ at each point x_i of the partition, and they are finite. Moreover, the pointwise derivative $\ddot{u}$ of $\dot{u}$ is defined a.e. on I. The function $\ddot{u}$ coincides with u'' a.e. on each $]x_i, x_{i+1}[$, hence $\ddot{u} \in L^2(I)$.

Let $u \in \mathcal{H}^2(I)$; we denote by S_u the jump set of u, i.e.,

$$S_u = \{x \in I : u^-(x) \neq u^+(x)\},$$

where $u^-(x)$ and $u^+(x)$ are the left and right limits of u at the point x. We denote by $S_{\dot{u}}$ the jump set of $\dot{u}$, i.e.,

$$S_{\dot{u}} = \{x \in I : \dot{u}^-(x) \neq \dot{u}^+(x)\}.$$

2. Position of the Problem

Let $g \in L^2(I)$ be a given function, and let α, β be two real numbers, with

$$(2.1) \qquad\qquad 0 < \beta \leq \alpha \leq 2\beta.$$

Let $G : \mathcal{H}^2(I) \to [0, +\infty[$ be the functional defined by

$$G(u, I) = \int_I |\ddot{u}|^2 \, dt + \alpha \#(S_u) + \beta \#(S_{\dot{u}} \setminus S_u) + \int_I |u - g|^2 \, dt,$$

where $\#$ is the counting measure on $\mathbb{R}$. Observe that $(S_{\dot{u}} \setminus S_u)$ is the set of the corner points of u, i.e., the jump points of $\dot{u}$ which are not jump points of u.

The minimum problem

$$(2.2) \qquad\qquad \inf\{G(u, I) : u \in \mathcal{H}^2(I)\}$$

admits a solution, provided conditions (2.1) are satisfied (see [39]). The direct method of the Calculus of Variations for problem (2.2) is applied with respect to the topology of $L^1(I)$ on $\mathcal{H}^2(I)$. In addition $G(\cdot, I)$ is not coercive with respect to the topology of $L^2(I)$.

Observe that, for any $u \in \mathcal{H}^2(I)$, the functional G can be rewritten as

$$G(u, I) = \int_I |\ddot{u}|^2 \, dt + (\alpha - \beta)\#(S_u) + \beta \#(S_u \cup S_{\dot{u}}) + \int_I |u - g|^2 \, dt.$$

As explained in the introduction, our aim is to approximate G, in the sense of Γ-convergence, by a sequence of elliptic functionals. To do this, we need some preparations.

We denote by $X(I)$ the convex subset of $(L^1(I))^3$ defined by $H^2(I) \times H^1(I, [0, 1]) \times H^1(I, [0, 1])$.

Let $\mathcal{G} : (L^1(I))^3 \to [0, +\infty]$ be the map defined by

$$\mathcal{G}(u, s, \sigma, I) = \begin{cases} G(u, I) & \text{if } u \in \mathcal{H}^2(I), \ s \equiv 1, \ \sigma \equiv 1, \\ +\infty & \text{elsewhere on } (L^1(I))^3. \end{cases}$$

For any $\varepsilon > 0$ and any function $s \in H^1(I, [0, 1])$, let us define

$$\mathcal{M}_\varepsilon(s, I) = \int_I [\varepsilon(s')^2 + \frac{(s-1)^2}{4\varepsilon}] \, dt.$$

Note that, using Young's inequality, it follows that

$$(2.3) \qquad \mathcal{M}_\varepsilon(s, I) \geq \int_I (1 - s)|s'| \, dt.$$

Let $\{\lambda_\varepsilon\}_\varepsilon$, $\{\mu_\varepsilon\}_\varepsilon$ be two sequences of positive numbers converging to zero such that

$$(2.4) \qquad \lim_{\varepsilon \to 0} \frac{\lambda_\varepsilon}{\varepsilon^3} = 0, \quad \lim_{\varepsilon \to 0} \frac{\varepsilon}{\mu_\varepsilon} = 0.$$

We are now in a position to introduce the approximating sequence $\{\mathcal{G}_\varepsilon\}_\varepsilon$. For any $\varepsilon > 0$, let

$$\mathcal{G}_\varepsilon(u, s, \sigma, I) = \int_I (s^2 + \lambda_\varepsilon)|u''|^2 dt + \beta \mathcal{M}_\varepsilon(s, I) + (\alpha - \beta)\mathcal{M}_\varepsilon(\sigma, I)$$

$$+ \mu_\varepsilon \int_I \sigma^2 |u'|^2 dt + \int_I |u - g|^2 dt$$

if $(u, s, \sigma) \in X(I)$, and set $\mathcal{G}_\varepsilon(u, s, \sigma, I) = +\infty$ if $(u, s, \sigma) \in (L^1(I))^3 \setminus X(I)$.

We denote by $\mathcal{G}_-$ and $\mathcal{G}_+$ respectively the Γ-lower limit and the Γ-upper limit of the sequence $\{\mathcal{G}_\varepsilon\}_\varepsilon$ with respect to the topology of $(L^1)^3$. For the main properties of the Γ-convergence we refer to Chapter 1, Section 4.

Observe that $\mathcal{G}_-(u, s, \sigma, \cdot)$, if considered as a set function, is increasing, i.e.,

$$(2.5) \qquad I_1 \subseteq I_2 \Longrightarrow \mathcal{G}_-(u, s, \sigma, I_1) \leq \mathcal{G}_-(u, s, \sigma, I_2),$$

and superadditive (see [42] Prop. 16.12), i.e.,

$$(2.6) \quad I_1 \cap I_2 = \emptyset \Longrightarrow \mathcal{G}_-(u, s, \sigma, I_1 \cup I_2) \geq \mathcal{G}_-(u, s, \sigma, I_1) + \mathcal{G}_-(u, s, \sigma, I_2).$$

3. Proof of the Lower Inequality

We begin with a Lemma whose original proof can be found in [12]. For the sake of completeness we give here a simpler proof due to G. Dal Maso.

Lemma 3.1. *Let $\{\phi_h\}_h$ be a sequence of functions of class $H^1(I)$ such that $\phi_h \to 0$ a.e. on I as $h \to +\infty$, and suppose that*

$$(3.1) \qquad \sup_h \int_I |\phi_h'| \, dt < +\infty.$$

Then, there exists a subsequence, still denoted by $\{\phi_h\}_h$, with the following property: for any $\delta > 0$, we can find a finite set $F \subseteq I$ such that, if K is a compact set contained in $I \setminus F$, then $K \subseteq \{t \in I : |\phi_h(t)| \le \delta\}$ for h large enough.

Proof. For any $h \in \mathbb{N}$, let ν_h be the finite positive Radon measure on I defined by $\nu_h(B) = \int_B |\phi_h'| \, dt$ for every Borel subset B of I. By (3.1), it follows that there exist a finite positive Radon measure ν on I and a subsequence (still denoted by $\{\nu_h\}_h$) such that $\nu_h \rightharpoonup \nu$ weakly in the sense of measures as $h \to +\infty$. As for every $r > 0$ the set $\{t \in I : \nu(\{t\}) \ge r\}$ is finite, the set $\{t \in I : \nu(\{t\}) > 0\} = \bigcup_{n \in \mathbb{N}} \{t \in I : \nu(\{t\}) \ge \frac{1}{n}\}$ is at most countable, and we denote it by $\{t_i\}_{i \in \mathbb{N}}$. Hence, there exist a sequence $\{c_i\}_i$ of real numbers and a Radon measure μ on I such that

$$(3.2) \qquad \nu = \mu + \sum_{i=1}^{+\infty} c_i \delta_{t_i}, \quad \sum_{i=1}^{+\infty} c_i < +\infty, \quad \text{and} \quad \mu(\{t\}) = 0 \;\; \forall t \in I,$$

where δ_{t_i} denotes the Dirac measure at the point t_i. Let us fix $\delta > 0$. We define the set F as

$$F = \{t \in I : \nu(\{t\}) \ge \frac{\delta}{2}\};$$

as just noticed, we have that F is a finite set.

To prove the assertion we shall argue by contradiction. Let us suppose that there exist a compact set $K \subseteq I \setminus F$ and a sequence $\{t_h\}_h$ of points of K such that $|\phi_h(t_h)| > \delta$ for any $h \in \mathbb{N}$. By the compactness of K, the sequence $\{t_h\}_h$ has a subsequence which converges to a point $\bar{t} \in K$. Let us still denote by $\{t_h\}_h$ this subsequence. Since $\bar{t} \in I \setminus F$, it follows that $\nu(\{\bar{t}\}) < \frac{\delta}{2}$. Using (3.2) and the hypothesis that $\phi_h \to 0$ a.e. on I as $h \to +\infty$, we can find, in each neighbourhood of $\bar{t}$, points $t_1, t_2 \in I$ such that

$$t_1 < \bar{t} < t_2, \qquad \nu(\{t_1\}) = \nu(\{t_2\}) = 0, \qquad \lim_{h \to +\infty} \phi_h(t_1) = 0.$$

In addition, since $\nu(\{\bar{t}\}) = \nu(\bigcap_{n\in\mathbb{N}}]\bar{t}-\frac{1}{n},\bar{t}+\frac{1}{n}[) = \lim_{n\to+\infty}\nu(]\bar{t}-\frac{1}{n},\bar{t}+\frac{1}{n}[)$, we can fix $t_1, t_2 \in I$ with the further property that

$$\nu([t_1, t_2]) < \nu(\{\bar{t}\}) + \frac{\delta}{2} < \delta.$$

Since $\nu(\{t_1\}) = \nu(\{t_2\}) = 0$ and $\nu_h \rightharpoonup \nu$ as $h \to +\infty$, it follows that (see [23] Th.2.1)

(3.3) $$\lim_{h\to+\infty} \nu_h([t_1, t_2]) = \nu([t_1, t_2]) < \delta.$$

Let $h_0 \in \mathbb{N}$ be such that $t_h \in]t_1, t_2[$ for any $h \geq h_0$. By the definition of total variation, we have that, for any $h \geq h_0$,
(3.4)

$$\nu_h([t_1, t_2]) = \int_{t_1}^{t_2} |\phi'_h|\, dt$$

$$\geq |\phi_h(t_1) - \phi_h(t_h)| + |\phi_h(t_h) - \phi_h(t_2)| \geq |\phi_h(t_1) - \phi_h(t_h)|.$$

Since, by contradiction, $|\phi_h(t_h)| > \delta$ for any $h \in \mathbb{N}$, and $\lim_{h\to+\infty}\phi_h(t_1) = 0$, from (3.4) it follows that

$$\lim_{h\to+\infty} \nu_h([t_1, t_2]) \geq \liminf_{h\to+\infty} |\phi_h(t_1) - \phi_h(t_h)| \geq \delta,$$

which contradicts (3.3), and concludes the proof of the Lemma.   □
   The main result of this section is the following theorem.

Theorem 3.1.   *For any triple of functions $(u, s, \sigma) \in (L^1(I))^3$ we have*

(3.5) $$\mathcal{G}(u, s, \sigma, I) \leq \mathcal{G}_-(u, s, \sigma, I).$$

Moreover, if $\mathcal{G}_-(u, s, \sigma, I) < +\infty$, then $u \in \mathcal{H}^2(I)$, $s \equiv 1$, and $\sigma \equiv 1$.

Proof.   Let $(u, s, \sigma) \in (L^1(I))^3$; we can assume that $\mathcal{G}_-(u, s, \sigma, I) < +\infty$, otherwise (3.5) is trivial. By the definition of $\mathcal{G}_-$ there exist a sequence $\{\varepsilon_h\}_h$ of positive numbers converging to zero as $h \to +\infty$ and a sequence $\{(u_h, s_h, \sigma_h)\}_h$ of elements of $(L^1(I))^3$, such that $(u_h, s_h, \sigma_h) \to (u, s, \sigma)$ in $(L^1(I))^3$ and

$$\lim_{h\to+\infty} \mathcal{G}_{\varepsilon_h}(u_h, s_h, \sigma_h, I) = \mathcal{G}_-(u, s, \sigma, I),$$

$$(s_h, \sigma_h) \to (s, \sigma) \quad \text{a.e. on } I \text{ as } h \to +\infty.$$

If s (or σ) is not identically 1, then, by the Fatou's Lemma,

$$\mathcal{G}_-(u,s,\sigma,I) \geq \liminf_{h\to+\infty} \frac{1}{4\varepsilon_h} \int_I (s_h - 1)^2 \, dt + \liminf_{h\to+\infty} \frac{1}{4\varepsilon_h} \int_I (\sigma_h - 1)^2 \, dt \geq$$

$$\left(\int_I (s-1)^2 \, dt + \int_I (\sigma-1)^2 \, dt\right) \lim_{h\to+\infty} \frac{1}{4\varepsilon_h} = +\infty,$$

which contradicts the assumption that $\mathcal{G}_-(u,s,\sigma,I)$ is finite. Hence, we can assume that $s \equiv 1$ and $\sigma \equiv 1$, otherwise both $\mathcal{G}(u,s,\sigma,I)$ and $\mathcal{G}_-(u,s,\sigma,I)$ are equal to $+\infty$.

The idea of the proof of Theorem 3.1 is to test inequality (3.5) separately near the regular points of u, near the jump points of u, and near the jump points of $\dot{u}$. More precisely, the proof of the inequality $\mathcal{G} \leq \mathcal{G}_-$ is based on the following lemma.

Lemma 3.2. *Let $u \in \mathcal{H}^2(I)$, and let $\omega = (u,1,1)$. Let $x \in I$ and $l \in \mathbb{R}^+$ be such that $]x-l, x+l[\subseteq I$ and $]x-l, x+l[\cap(S_u \cup S_{\dot{u}}) \subseteq \{x\}$. The following assertions hold:*
(i) if $x \in S_u$ then

$$\mathcal{G}_-(\omega,]x-\varrho, x+\varrho[) \geq \alpha \qquad \text{for any } \varrho \in]0, l[;$$

(ii) if $x \in S_{\dot{u}} \setminus S_u$ then

$$\mathcal{G}_-(\omega,]x-\varrho, x+\varrho[) \geq \beta \qquad \text{for any } \varrho \in]0, l[;$$

(iii) if $x \notin S_u \cup S_{\dot{u}}$ then

$$\mathcal{G}_-(\omega,]x-\varrho, x+\varrho[) \geq \int_{x-\varrho}^{x+\varrho} |u''|^2 \, dt + \int_{x-\varrho}^{x+\varrho} |u-g|^2 \, dt \qquad \text{for any } \varrho \in]0, l[.$$

Proof of Lemma 3.2. Let us prove (i). As $x \in S_u$, we have that $u \notin H^1(x-\varrho, x+\varrho)$ for any $\varrho \in]0, l[$. Let us fix $\varrho \in]0, l[$; we can suppose that $\mathcal{G}_-(\omega,]x-\varrho, x+\varrho[) < +\infty$, otherwise the result is trivial. Recalling the definition of $\mathcal{G}_-$ we have to prove that $\liminf_{\varepsilon\to 0} \mathcal{G}_\varepsilon(\omega_\varepsilon,]x-\varrho, x+\varrho[) \geq \alpha$ for every sequence $\{\omega_\varepsilon = (u_\varepsilon, s_\varepsilon, \sigma_\varepsilon)\}_\varepsilon$ of elements of $(L^1(x-\varrho, x+\varrho))^3$ converging to ω in $(L^1(x-\varrho, x+\varrho))^3$ such that $\liminf_{\varepsilon\to 0} \mathcal{G}_\varepsilon(\omega_\varepsilon,]x-\varrho, x+\varrho[) < +\infty$. Let $\{\omega_\varepsilon\}_\varepsilon$ be such a sequence. Let $\{\varepsilon_h\}_h$ be a sequence of positive numbers converging to zero as $h \to +\infty$ such that

$$(3.6) \quad \lim_{h\to+\infty} \mathcal{G}_{\varepsilon_h}(\omega_h,]x-\varrho, x+\varrho[) = \liminf_{\varepsilon\to 0} \mathcal{G}_\varepsilon(\omega_\varepsilon,]x-\varrho, x+\varrho[) = H < +\infty,$$

and

$$(s_h, \sigma_h) \to (1,1) \qquad \text{a.e. on }]x - \varrho, x + \varrho[,$$

where $\omega_h = \omega_{\varepsilon_h} \in X(x - \varrho, x + \varrho)$, for any h.

Then (see [12], [13])

$$(3.7) \qquad \lim_{h \to +\infty} \inf_{t \in]x - \varrho, x + \varrho[} s_h(t) = 0.$$

Indeed, if by contradiction $\limsup_{h \to +\infty} \inf_{t \in]x - \varrho, x + \varrho[} s_h(t) > 0$, then there exist a positive constant c and a subsequence (still denoted by $\{s_h\}_h$) such that $\inf_{t \in]x - \varrho, x + \varrho[} s_h(t) \geq c$ for any h. Therefore we get

$$(3.8) \qquad \int_{x - \varrho}^{x + \varrho} |u_h''|^2 \, dt \leq \frac{1}{c^2} \int_{x - \varrho}^{x + \varrho} s_h^2 |u_h''|^2 \, dt \leq \frac{H}{c^2}.$$

Moreover, as $g \in L^2(I)$ and $\int_I |u_h - g|^2 \, dt \leq H$, it follows that

$$(3.9) \qquad \sup_h \int_I |u_h|^2 \, dt \leq 2 \sup_h (\int_I |u_h - g|^2 \, dt + \int_I |g|^2 \, dt) < +\infty,$$

which implies, using (3.8) and Proposition 2.1, that the sequence $\{u_h\}_h$ is bounded in $H^2(x - \varrho, x + \varrho)$. Whence $u \in H^2(x - \varrho, x + \varrho)$, and this contradicts the assumption of (i).

Observe that for any $r \in]0, \varrho[$ we have $u \notin H^1(x - r, x + r)$, hence, as in (3.7),

$$(3.10) \qquad \lim_{h \to +\infty} \inf_{t \in]x - r, x + r[} s_h(t) = 0 \qquad \forall r \in]0, \varrho[.$$

It follows that, for any $h \in \mathbb{N}$, there exists a point x_h of $]x - \frac{\varrho}{2}, x + \frac{\varrho}{2}[$ such that

$$(3.11) \qquad \lim_{h \to +\infty} s_h(x_h) = 0.$$

In addition, as $s_h \to 1$ a.e. on $]x - \varrho, x + \varrho[$, for any $h \in \mathbb{N}$ there exist points y_h, z_h of $]x - \varrho, x + \varrho[$ such that $y_h < x_h < z_h$, and

$$(3.12) \qquad \lim_{h \to +\infty} s_h(y_h) = \lim_{h \to +\infty} s_h(z_h) = 1.$$

Repeating the arguments of [13], we have

$$\int_{x - \varrho}^{x + \varrho} (1 - s_h)|s_h'| \, dt \geq \int_{y_h}^{x_h} (1 - s_h)|s_h'| \, dt + \int_{x_h}^{z_h} (1 - s_h)|s_h'| \, dt \geq$$

$$\int_{s_h(x_h)}^{s_h(y_h)} (1-t)\, dt + \int_{s_h(x_h)}^{s_h(z_h)} (1-t)\, dt = s_h(y_h) - s_h(x_h) + s_h(z_h) -$$

$$s_h(x_h) - \frac{s_h(y_h)^2 - s_h(x_h)^2}{2} - \frac{s_h(z_h)^2 - s_h(x_h)^2}{2}.$$

Then, passing to the limit as $h \to +\infty$ and using (3.11) and (3.12) we get

$$(3.13) \qquad \liminf_{h \to +\infty} \int_{x-\varrho}^{x+\varrho} (1 - s_h)|s_h'|\, dt \geq 1.$$

Now, we must consider the sequence $\{\sigma_h\}_h$. Here, we cannot exclude the case in which $\limsup_{h \to +\infty} \inf_{t \in]x-\varrho, x+\varrho[} \sigma_h(t) > 0$, since the functions σ_h do not tend, in general, to vanish in a neighbourhood of the points of S_u as $h \to +\infty$. We shall distinguish two cases. If $\lim_{h \to +\infty} \inf_{t \in]x-r, x+r[} \sigma_h(t) = 0$ for any $r \in]0, \varrho]$, reasoning as before, we conclude that

$$(3.14) \qquad \liminf_{h \to +\infty} \int_{x-\varrho}^{x+\varrho} (1 - \sigma_h)|\sigma_h'|\, dt \geq 1.$$

Using (2.3), from (3.13) and (3.14) it follows that

$$\lim_{h \to +\infty} \mathcal{G}_{\varepsilon_h}(\omega_h,]x - \varrho, x + \varrho[) \geq \beta \liminf_{h \to +\infty} \int_{x-\varrho}^{x+\varrho} (1 - s_h)|s_h'|\, dt +$$

$$(\alpha - \beta) \liminf_{h \to +\infty} \int_{x-\varrho}^{x+\varrho} (1 - \sigma_h)|\sigma_h'|\, dt \geq \beta + (\alpha - \beta) = \alpha.$$

Hence, recalling the definition of $\mathcal{G}_-$, we get (i).

We still have to consider the most delicate case, i.e., when

$$\exists \varrho_0 \in]0, \varrho] \text{ such that } \limsup_{h \to +\infty} \inf_{t \in]x-\varrho_0, x+\varrho_0[} \sigma_h(t) > 0.$$

Up to a subsequence, we can suppose that there exists a constant $d > 0$ such that

$$(3.15) \qquad \sigma_h(t) \geq d \quad \forall t \in]x - \varrho_0, x + \varrho_0[, \quad \forall h \in \mathbb{N}.$$

We shall prove that

$$(3.16) \qquad \limsup_{h \to +\infty} \mathcal{G}_{\varepsilon_h}(\omega_h,]x - \varrho_0, x + \varrho_0[) \geq 2\beta,$$

which, in view of (2.1), (3.6), and the fact that $\mathcal{G}_-$ is increasing as a set function, will conclude the proof of (i).

To simplify the notation, let us denote ϱ_0 by the symbol ϱ. To prove (3.16) it is enough to show that

$$(3.17) \qquad \limsup_{h \to +\infty} \mathcal{G}_{\varepsilon_h}(\omega_h,]x - \varrho, x + \varrho[) \geq \beta(2 - 4\theta) \qquad \text{for any } \theta \in]0, \frac{1}{2}[.$$

Let us fix $\theta \in]0, \frac{1}{2}[$; suppose that there exist a subsequence of $\{s_h\}_h$, still denoted by $\{s_h\}_h$, and a real number $0 < r < \varrho$ such that, for any $h \in \mathbb{N}$, we can find x_h, q_h, p_h, y_h, z_h points of $]x - \varrho, x + \varrho[$ with

$$(3.18) \qquad \lim_{h \to +\infty} s_h(p_h) = \lim_{h \to +\infty} s_h(y_h) = \lim_{h \to +\infty} s_h(z_h) = 1,$$

$$(3.19) \qquad x - \varrho < p_h < x_h < y_h < q_h < z_h < x + \varrho,$$

$$(3.20) \qquad x_h, q_h \in \{t \in [x - r, x + r] : s_h(t) \leq \theta\}.$$

Then, for every $h \in \mathbb{N}$, using (2.3) and (3.19), we have

$$\mathcal{G}_{\varepsilon_h}(\omega_h,]x - \varrho, x + \varrho[)$$

$$\geq \beta \int_{x-\varrho}^{x+\varrho} (\varepsilon_h |s'_h|^2 + \frac{(s_h - 1)^2}{4\varepsilon_h}) \, dt \geq \beta \int_{x-\varrho}^{x+\varrho} (1 - s_h)|s'_h| \, dt$$

$$\geq \beta [\int_{p_h}^{x_h} (1 - s_h)|s'_h| \, dt + \int_{x_h}^{y_h} (1 - s_h)|s'_h| \, dt$$

$$+ \int_{y_h}^{q_h} (1 - s_h)|s'_h| \, dt + \int_{q_h}^{z_h} (1 - s_h)|s'_h| dt]$$

$$\geq \beta [\int_{s_h(x_h)}^{s_h(p_h)} (1 - t) dt + \int_{s_h(x_h)}^{s_h(y_h)} (1 - t) dt$$

$$+ \int_{s_h(q_h)}^{s_h(y_h)} (1 - t) dt + \int_{s_h(q_h)}^{s_h(z_h)} (1 - t) dt].$$

Passing to the limit as $h \to +\infty$ and using (3.18) and (3.20), we get

$$\lim_{h \to +\infty} \mathcal{G}_{\varepsilon_h}(\omega_h,]x - \varrho, x + \varrho[) \geq \beta [4(1 - \theta) - 2] = \beta(2 - 4\theta),$$

that gives (3.17) and concludes the proof of (i).

Hence the problem reduces to find a subsequence $\{s_h\}_h$, a real number $0 < r < \varrho$, and, for any $h \in \mathbb{N}$, points x_h, q_h, p_h, y_h, z_h of $]x - \varrho, x + \varrho[$ such that conditions (3.18), (3.19), and (3.20) hold.

To simplify the notation, let

$$[u] = |u^+(x) - u^-(x)|, \qquad M = \max(|\dot{u}^-(x)|, |\dot{u}^+(x)|).$$

Let us show that for any $\eta > 0$ there exist a subsequence of $\{u_h\}_h$, still denoted by $\{u_h\}_h$, and a real number $0 < r < \frac{\eta}{C}$, where C is the positive constant $C = \frac{\sqrt{H}}{\theta} + 2\eta + M$, such that the following conditions are satisfied:

$$(3.21) \qquad |u(x - r) - u^-(x)| < \eta, \qquad |u(x + r) - u^+(x)| < \eta,$$

$$(3.22) \qquad |\dot{u}(x - r) - \dot{u}^-(x)| < \eta, \qquad |\dot{u}(x + r) - \dot{u}^+(x)| < \eta,$$

$$(3.23) \quad \sup_h |u_h(x - r) - u(x - r)| < \eta, \qquad \sup_h |u_h(x + r) - u(x + r)| < \eta,$$

$$(3.24) \quad \sup_h |u_h'(x - r) - \dot{u}(x - r)| < \eta, \qquad \sup_h |u_h'(x + r) - \dot{u}(x + r)| < \eta.$$

Inequalities (3.21) and (3.22) are immediate; in fact, since by hypothesis $(S_u \cup S_{\dot{u}}) \cap]x - \varrho, x + \varrho[= \{x\}$, u and $\dot{u}$ are absolutely continuous on the intervals $[x - \varrho, x[$ and $]x, x + \varrho]$.

Let us prove (3.23) and (3.24). Define $\phi_h = (1 - s_h)^2$; as $s_h \in H^1(I, [0, 1])$ and $s_h \to 1$ a.e. on I as $h \to +\infty$, the function ϕ_h is of class $H^1(I)$ for any $h \in \mathbb{N}$, and $\phi_h \to 0$ a.e. on I as $h \to +\infty$. In addition, using (2.3),

$$(3.25) \quad \sup_h \int_I |\phi_h'| \, dt = \sup_h \int_I 2(1 - s_h)|s_h'| \, dt \le 2 \sup_h \mathcal{M}_{\varepsilon_h}(s_h, I) \le 2H.$$

Applying Lemma 3.1 there exists a subsequence, still denoted by $\{s_h\}_h$, with the following property: for any $\delta > 0$ we can find a finite set $F \subseteq I$ such that, if K is a compact set contained in $I \setminus F$, then $K \subseteq \{t \in I : |\phi_h(t)| \le \delta\}$ for any h sufficiently large.

Let $\delta > 0$ be fixed and let A be an open set relatively compact in $I \setminus F$. Then, for every h sufficiently large we have that

$$s_h(t) \ge 1 - \sqrt{\delta} \qquad \forall \, t \in A,$$

hence

$$(3.26) \qquad \int_A |u_h''|^2 \, dt \le \frac{H}{(1 - \sqrt{\delta})^2}.$$

This, together with (3.9) and Proposition 2.1, gives that the sequence $\{u_h\}_h$ is bounded in $H^2(A)$. Possibly passing to a subsequence, we deduce that $u_h \to u$ in $C^1(A)$ as $h \to +\infty$.

As F is a finite set, and since (3.26) holds for any open set A relatively compact in $I \setminus F$, it follows that we can fix $r < \frac{\eta}{C}$ in such a way that (3.23) and (3.24) are satisfied, for a suitable subsequence $\{u_h\}_h$.

Let $\eta = \frac{[u]}{8}$; let $\{(u_h, s_h, \sigma_h)\}_h$ and r be such that conditions (3.21)-(3.24) are satisfied for this value of η. Note that, from (3.10), it follows that there exists a further subsequence of $\{s_h\}_h$, still denoted by $\{s_h\}_h$, such that the set $\{t \in]x - r, x + r[: s_h(t) < \theta\}$ is non empty, for any $h \in \mathbb{N}$.

Since $s_h \to 1$ a.e. on I as $h \to +\infty$, we can find points p_h, z_h such that $p_h \in]x - \varrho, x - r[$, $z_h \in]x + r, x + \varrho[$, and $\lim_{h \to +\infty} s_h(p_h) = \lim_{h \to +\infty} s_h(z_h) = 1$.

Define

$$x_h = \inf\{t \in]x - r, x + r[: s_h(t) < \theta\}, \quad q_h = \sup\{t \in]x - r, x + r[: s_h(t) < \theta\}.$$

Then, for any $h \in \mathbb{N}$, we get $x - \varrho < p_h < x_h < q_h < z_h < x + \varrho$, and x_h, q_h satisfy (3.20). It remains to construct the sequence $\{y_h\}_h$ satisfying (3.18) and (3.19). We begin by proving that the following further condition on $\{x_h\}_h$, $\{q_h\}_h$ holds (recall that d is the constant defined in (3.15)):

$$(3.27) \qquad q_h - x_h \ge E\mu_{\varepsilon_h} \quad \text{for any } h \in \mathbb{N}, \qquad \text{where } E = \frac{d^2 [u]^2}{16H}.$$

The estimate (3.27) of the distance between x_h and q_h, together with (2.4) and the inequality $\int_I (s_h - 1)^2 \le 4H\varepsilon_h$, will be used to construct the sequence $\{y_h\}_h$.

In order to prove (3.27) we shall estimate from below the quantity $|u_h(q_h) - u_h(x_h)|$. Let $h \in \mathbb{N}$ be fixed.

If $x_h = x - r$ and $q_h = x + r$, then, using (3.21) and (3.23), we get immediately that

$$(3.28) \quad |u_h(q_h) - u_h(x_h)| = |u_h(x + r) - u_h(x - r)| \ge [u] - 4\eta = \frac{[u]}{2} > 0.$$

If $x - r < x_h$ or if $q_h < x + r$, we need some more calculations to estimate the differences $|u_h(x_h) - u_h(x - r)|$ or $|u_h(x + r) - u_h(q_h)|$.

Let

$$m_h = x_h - (x - r), \qquad n_h = (x + r) - q_h.$$

If $m_h > 0$ (resp. $n_h > 0$), since $s_h \geq \theta$ on the set $]x - r, x_h[\cup]q_h, x + r[$ (recall the definition of x_h and q_h), we have

$$(3.29) \qquad \int_{x-r}^{x_h} |u_h''|^2 \, dt \leq \frac{H}{\theta^2}, \qquad (\text{resp.} \quad \int_{q_h}^{x+r} |u_h''|^2 \, dt \leq \frac{H}{\theta^2}).$$

If $m_h > 0$, using the first inequality in (3.29) we have

$$(3.30) \qquad |u_h'(t) - u_h'(x - r)|^2 \leq m_h \int_{x-r}^{x_h} |u_h''|^2 dt \leq \frac{m_h H}{\theta^2},$$

for any $t \in [x - r, x_h]$.

Analogously, if $n_h > 0$, using the second inequality in (3.29), we have

$$(3.31) \qquad |u_h'(x + r) - u_h'(t)|^2 \leq \frac{n_h H}{\theta^2},$$

for any $t \in [q_h, x + r]$.

Then, using the triangle inequality for the L^2 norm, we get

$$|u_h(x_h) - u_h(x - r)| = |\int_{x-r}^{x_h} u_h' \, dt| \leq m_h^{\frac{1}{2}} (\int_{x-r}^{x_h} |u_h'|^2 \, dt)^{\frac{1}{2}} \leq$$

$$m_h^{\frac{1}{2}} (\int_{x-r}^{x_h} |u_h'(t) - u_h'(x-r)|^2 \, dt)^{\frac{1}{2}} + m_h^{\frac{1}{2}} (\int_{x-r}^{x_h} |u_h'(x-r) - \dot{u}(x-r)|^2 \, dt)^{\frac{1}{2}} +$$

$$m_h^{\frac{1}{2}} (\int_{x-r}^{x_h} |\dot{u}(x - r)|^2 \, dt)^{\frac{1}{2}}.$$

Hence, from (3.30), (3.24), and (3.22) we obtain

$$(3.32) \qquad |u_h(x_h) - u_h(x - r)| \leq m_h(\frac{\sqrt{H}}{\theta} + 2\eta + |\dot{u}^-(x)|) \leq m_h C,$$

recalling that $C = \frac{\sqrt{H}}{\theta} + 2\eta + M$. Here we use also the fact that $m_h < 2r \leq 1$.

Analogously, using (3.31), (3.24), and (3.22) we get

$$(3.33) \qquad |u_h(x + r) - u_h(q_h)| \leq n_h(\frac{\sqrt{H}}{\theta} + 2\eta + |\dot{u}^+(x)|) \leq n_h C.$$

From the inequality

$$|u_h(x + r) - u_h(x - r)|$$
$$\leq |u_h(x + r) - u_h(q_h)| + |u_h(q_h) - u_h(x_h)| + |u_h(x_h) - u_h(x - r)|,$$

using (3.28), (3.32) and (3.33), it follows that

(3.34) $$|u_h(q_h) - u_h(x_h)| \geq \frac{[u]}{2} - (m_h + n_h)C.$$

Recalling that $\eta = \frac{[u]}{8}$ and $r < \frac{\eta}{C}$, in any case we get

$$(m_h + n_h)C \leq 2rC \leq 2\eta = \frac{[u]}{4}.$$

Hence, from (3.34), it follows that

(3.35) $$|u_h(q_h) - u_h(x_h)| \geq \frac{[u]}{4} \qquad \text{for every } h \in \mathbb{N}.$$

Then, using (3.15) and (3.35), we deduce that

$$H \geq \mu_{\varepsilon_h} \int_{x_h}^{q_h} \sigma_h^2 |u_h'|^2 \, dt \geq d^2 \mu_{\varepsilon_h} \int_{x_h}^{q_h} |u_h'|^2 \, dt \geq \frac{d^2 \mu_{\varepsilon_h}}{q_h - x_h} |\int_{x_h}^{q_h} u_h' \, dt|^2 =$$

$$\frac{d^2 \mu_{\varepsilon_h}}{q_h - x_h} |u_h(q_h) - u_h(x_h)|^2 \geq \frac{d^2 [u]^2}{16} \frac{\mu_{\varepsilon_h}}{q_h - x_h}.$$

Hence

$$q_h - x_h \geq \frac{d^2 [u]^2}{16H} \mu_{\varepsilon_h} = E\mu_{\varepsilon_h} \qquad \text{for any } h \in \mathbb{N},$$

and (3.27) is proven.

We are now in a position to find the sequence $\{y_h\}_h$ satisfying (3.18) and (3.19). For any $h \in \mathbb{N}$, let $y_h \in [x_h + \frac{q_h - x_h}{4}, q_h - \frac{q_h - x_h}{4}]$ be such that

$$s_h(y_h) = \max\{s_h(t) : t \in [x_h + \frac{q_h - x_h}{4}, q_h - \frac{q_h - x_h}{4}]\}.$$

Then $\{y_h\}_h$ satisfies (3.19). Let us prove that $\lim_{h \to +\infty} s_h(y_h) = 1$. We argue by contradiction, and suppose that $\liminf_{h \to +\infty} s_h(y_h) < 1$. Then there exist a subsequence (still denoted by $\{s_h(y_h)\}_h$) and a constant $\gamma < 1$ such that $s_h(y_h) \leq \gamma$ for any $h \in \mathbb{N}$. Then, by the definition of y_h, we have that $s_h(t) \leq \gamma$ for any $t \in [x_h + \frac{q_h - x_h}{4}, q_h - \frac{q_h - x_h}{4}]$ and any $h \in \mathbb{N}$. We deduce that

$$H \geq \liminf_{h \to +\infty} \int_{x_h + \frac{q_h - x_h}{4}}^{q_h - \frac{q_h - x_h}{4}} \frac{(s_h - 1)^2}{4\varepsilon_h} \, dt \geq \frac{(\gamma - 1)^2}{8} \liminf_{h \to +\infty} \frac{(q_h - x_h)}{\varepsilon_h}.$$

Finally, from (3.27) we obtain

$$H \geq \frac{(\gamma - 1)^2}{8} E \liminf_{h \to +\infty} \frac{\mu_{\varepsilon_h}}{\varepsilon_h},$$

which contradicts the assumption on $\{\mu_{\varepsilon_h}\}_h$ (see (2.4)). Hence we have proved that there exist a subsequence $\{s_h\}_h$ and a real number $0 < r < \varrho$ such that, for any $h \in \mathbb{N}$, we can find x_h, q_h, p_h, y_h, z_h points of $]x - \varrho, x + \varrho[$ satisfying (3.18), (3.19), and (3.20), and this concludes the proof of (i).

Let us prove (ii). As $x \in S_{\dot{u}} \setminus S_u$, we have that $u \in H^1(x - \varrho, x + \varrho) \setminus H^2(x - \varrho, x + \varrho)$ for any $\varrho \in]0, l[$. Then, repeating the arguments of the beginning of the proof of (i), it follows that (3.7) and (3.13) hold. Finally (ii) follows from (2.3).

Let us prove (iii). We closely follow [12], [13]. By assumption, $u \in H^2(x - \varrho, x + \varrho)$ for any $\varrho \in]0, l[$. Let $\varrho \in]0, l[$. We can suppose that $\mathcal{G}_-(\omega,]x - \varrho, x + \varrho[) < +\infty$, otherwise the result is trivial. We have to prove that

$$\liminf_{\varepsilon \to 0} \mathcal{G}_\varepsilon(\omega_\varepsilon,]x - \varrho, x + \varrho[) \geq \int_{x-\varrho}^{x+\varrho} |u''|^2 \, dt + \int_{x-\varrho}^{x+\varrho} |u - g|^2 \, dt,$$

for every sequence $\{\omega_\varepsilon = (u_\varepsilon, s_\varepsilon, \sigma_\varepsilon)\}_\varepsilon$ of elements of $(L^1(x - \varrho, x + \varrho))^3$ converging to ω in $(L^1(x - \varrho, x + \varrho))^3$ as $\varepsilon \to 0$, such that $\liminf_{\varepsilon \to 0} \mathcal{G}_\varepsilon(\omega_\varepsilon,]x - \varrho, x + \varrho[) < +\infty$. Let $\{\omega_\varepsilon\}_\varepsilon$ be such a sequence. Let $\{\varepsilon_h\}_h$ be a sequence of positive numbers converging to zero as $h \to +\infty$, such that

$$\lim_{h \to +\infty} \mathcal{G}_{\varepsilon_h}(\omega_h,]x - \varrho, x + \varrho[) = \liminf_{\varepsilon \to 0} \mathcal{G}_\varepsilon(\omega_\varepsilon,]x - \varrho, x + \varrho[) < +\infty,$$

there exist the limits

$$\lim_{h \to +\infty} \int_{x-\varrho}^{x+\varrho} s_h^2 |u_h''|^2 \, dt, \quad \lim_{h \to +\infty} \int_{x-\varrho}^{x+\varrho} |u_h - g|^2 \, dt,$$

and

$$(u_h, s_h, \sigma_h) \to \omega \quad \text{a.e. on }]x - \varrho, x + \varrho[,$$

where $\omega_h = \omega_{\varepsilon_h} \in X(x - \varrho, x + \varrho)$, for any $h \in \mathbb{N}$. Since

$$\lim_{h \to +\infty} \mathcal{G}_{\varepsilon_h}(\omega_h,]x - \varrho, x + \varrho[) \geq \lim_{h \to +\infty} \int_{x-\varrho}^{x+\varrho} s_h^2 |u_h''|^2 \, dt + \lim_{h \to +\infty} \int_{x-\varrho}^{x+\varrho} |u_h - g|^2 \, dt,$$

and, by the Fatou's Lemma $\int_{x-\varrho}^{x+\varrho} |u - g|^2 \, dt \leq \lim_{h \to +\infty} \int_{x-\varrho}^{x+\varrho} |u_h - g|^2 \, dt$, to prove the assertion it will be enough to show that

$$(3.36) \qquad \lim_{h \to +\infty} \int_{x-\varrho}^{x+\varrho} s_h^2 |u_h''|^2 \, dt \geq \int_{x-\varrho}^{x+\varrho} |u''|^2 \, dt.$$

Let $\phi_h = (1 - s_h)^2$; using the same arguments as in the proof of (i) (see (3.25)), there exists a subsequence, still denoted by $\{s_h\}_h$, with the following property: for any fixed $\delta > 0$ we can find a finite set $F \subseteq I$ such that if A is an open set relatively compact in $I \setminus F$, then

$$(3.37) \qquad \int_{x-\varrho}^{x+\varrho} s_h^2 |u_h''|^2 \, dt \geq \int_A s_h^2 |u_h''|^2 \, dt \geq (1 - \sqrt{\delta})^2 \int_A |u_h''|^2 \, dt,$$

for any h sufficiently large. Let A be an open set relatively compact in $]x - \varrho, x + \varrho[\setminus F$. As $\lim_{h \to +\infty} \int_{x-\varrho}^{x+\varrho} s_h^2 |u_h''|^2 dt < +\infty$, from (3.37), (3.9), and Proposition 2.1, it follows that the sequence $\{u_h\}_h$ is bounded in $H^2(A)$, hence it converges to u weakly in $H^2(A)$. Using the weak lower semicontinuity of the L^2 norm, from (3.37) we deduce that
(3.38)

$$\lim_{h \to +\infty} \int_{x-\varrho}^{x+\varrho} s_h^2 |u_h''|^2 \, dt \geq \liminf_{h \to +\infty} \int_A s_h^2 |u_h''|^2 \, dt \geq \quad (1 - \sqrt{\delta})^2 \int_A |u''|^2 \, dt.$$

Then (3.36) follows from (3.38), taking the limit first as $A \nearrow]x - \varrho, x + \varrho[\setminus F$, and then as $\delta \searrow 0$. This concludes the proof of the Lemma. $\square$

Conclusion of the Proof of Theorem 3.1. Let $u \in L^1(I)$ such that $\mathcal{G}_-(\omega, I) < +\infty$, where $\omega = (u, 1, 1)$. Firstly, we shall prove that $u \in \mathcal{H}^2(I)$. Let $F = \{x \in I : u \notin H^2(x - \varrho, x + \varrho) \text{ for any } \varrho > 0\} = F_1 \cup F_2$, where $F_1 = \{x \in I : u \notin H^1(x - \varrho, x + \varrho) \text{ for any } \varrho > 0\}$ and $F_2 = \{x \in I : u \in H^1(x - \varrho, x + \varrho) \setminus H^2(x - \varrho, x + \varrho) \text{ for any } \varrho > 0 \text{ sufficiently small}\}$. Let us prove that F is finite. In fact, however we choose p elements in F_1 and q elements in F_2, using the fact that $\mathcal{G}_-$ is increasing and superadditive as a set function (see (2.5) and (2.6)), and conditions (i) and (ii) of Lemma 3.2, we obtain that

$$\alpha p + \beta q \leq \mathcal{G}_-(\omega, I) < +\infty.$$

By definition, for any $x \in I \setminus F$ there exists $\varrho > 0$ such that $u \in H^2(x - \varrho, x + \varrho)$. It follows that $u \in H^2_{\text{loc}}(I \setminus F)$. Let us show that $u \in H^2(I \setminus F)$. As $I \setminus F$ is a finite union of disjoint intervals, it will be enough to prove that u is of class H^2 on each interval. Let $]x_1, x_2[$ be one of these intervals; by assumption $u \in H^2(x_1 + \delta, x_2 - \delta)$ for any $0 < \delta < \frac{x_2 - x_1}{2}$. Therefore, using condition (iii) of Lemma 3.2 and the fact that $\mathcal{G}_-$ is increasing, we have that

$$+\infty > \mathcal{G}_-(\omega, I) \geq \mathcal{G}_-(\omega,]x_1 + \delta, x_2 - \delta[) \geq \int_{x_1+\delta}^{x_2-\delta} |u''|^2 \, dt + \int_{x_1+\delta}^{x_2-\delta} |u - g|^2 \, dt,$$

for any $0 < \delta < \frac{x_2 - x_1}{2}$. This implies that $\int_{x_1}^{x_2} |u''|^2 \, dt + \int_{x_1}^{x_2} |u|^2 \, dt < +\infty$. Using Proposition 2.1, we get $u \in H^2(x_1, x_2)$. Therefore $u \in H^2(I \setminus F)$, and, in particular, $u \in \mathcal{H}^2(I)$.

Let $\{t_1, \ldots, t_n\} = S_u$, $\{t_{n+1}, \ldots, t_{n+m}\} = S_{\dot{u}} \setminus S_u$, and, for any $i = 1, \ldots, n + m$, let $I_i =]t_i - a_i, t_i + a_i[$ be pairwise disjoint open intervals contained in I. Let $J_1, \ldots, J_{n+m+1}$ be the open intervals composing $I \setminus \bigcup_{i=1}^{n+m} \overline{I}_i$. Then, using (2.5), (2.6), and Lemma 3.2, we get

$$\mathcal{G}_-(\omega, I) \geq \sum_{i=1}^{n+m} \mathcal{G}_-(\omega, I_i) + \sum_{i=1}^{n+m+1} \mathcal{G}_-(\omega, J_i) \geq$$

$$\alpha \#(S_u) + \beta \#(S_{\dot{u}} \setminus S_u) + \sum_{i=1}^{n+m+1} \int_{J_i} (|u''|^2 + |u - g|^2) \, dt.$$

Then, letting $a_i \searrow 0$ for any $i = 1, \ldots, n + m$, we deduce that

$$\mathcal{G}_-(\omega, I) \geq \alpha \#(S_u) + \beta \#(S_{\dot{u}} \setminus S_u) + \int_I |\ddot{u}|^2 \, dt + \int_I |u - g|^2 \, dt,$$

that gives (3.5), and concludes the proof of the Theorem. $\square$

4. Proof of the Upper Inequality

Let $I =]a, b[\subseteq \mathbb{R}$ be a bounded open interval. In this section we shall prove that $\mathcal{G}(u, s, \sigma, I) \geq \mathcal{G}_+(u, s, \sigma, I)$ for every $(u, s, \sigma) \in (L^1(I))^3$. To do this, we shall assume that $u \in \mathcal{H}^2(I)$, $s \equiv 1$, and $\sigma \equiv 1$, since $\mathcal{G}(u, s, \sigma, I)$ is finite only on this class of functions.

Theorem 4.1. *For any $u \in \mathcal{H}^2(I)$, there exists a sequence $\{(u_\varepsilon, s_\varepsilon, \sigma_\varepsilon)\}_\varepsilon$ of elements of $X(I)$ converging to $\omega = (u, 1, 1)$ in $(L^1(I))^3$ as $\varepsilon \to 0$ such that*

(4.1) $$\limsup_{\varepsilon \to 0} \mathcal{G}_\varepsilon(u_\varepsilon, s_\varepsilon, \sigma_\varepsilon, I) \leq \mathcal{G}(\omega, I).$$

Proof. We closely follow the ideas of [12], [13]. For any $A \subseteq \mathbb{R}$ and any $r \in \mathbb{R}^+$, we define

$$(A)_r = \{x \in \mathbb{R} : \text{dist}(x, A) < r\}.$$

Let $u \in \mathcal{H}^2(I)$, let $S_u = \{t_1, \ldots, t_n\}$, and let $S_{\dot{u}} \setminus S_u = \{t_{n+1}, \ldots, t_{n+m}\}$. We choose three sequences of positive numbers converging to zero $\{\eta_\varepsilon\}_\varepsilon$, $\{a_\varepsilon\}_\varepsilon$, and $\{b_\varepsilon\}_\varepsilon$, as follows: $\eta_\varepsilon = o(\varepsilon^{\frac{1}{2}})$, $a_\varepsilon = -2\varepsilon \log \eta_\varepsilon$, and b_ε intermediate between ε and $(\lambda_\varepsilon)^{\frac{1}{3}}$ (recall that, from (2.4), $\lim_{\varepsilon \to 0} \frac{\lambda_\varepsilon}{\varepsilon^3} = 0$).

For any ε small enough, let us define

$$s_\varepsilon(t) = \begin{cases} 0 & \text{if } t \in (S_u \cup S_{\dot{u}})_{b_\varepsilon}, \\ 1 - e^{\frac{t-t_i+b_\varepsilon}{2\varepsilon}} & \text{if } t \in]t_i - (a_\varepsilon + b_\varepsilon), t_i - b_\varepsilon], \ i = 1,\ldots, n+m, \\ 1 - e^{\frac{b_\varepsilon+t_i-t}{2\varepsilon}} & \text{if } t \in [t_i + b_\varepsilon, t_i + a_\varepsilon + b_\varepsilon)[, \ i = 1,\ldots, n+m, \\ 1 - \eta_\varepsilon & \text{if } t \in I \setminus (S_u \cup S_{\dot{u}})_{a_\varepsilon+b_\varepsilon}, \end{cases}$$

$$\sigma_\varepsilon(t) = \begin{cases} 0 & \text{if } t \in (S_u)_{b_\varepsilon}, \\ 1 - e^{\frac{t-t_i+b_\varepsilon}{2\varepsilon}} & \text{if } t \in]t_i - (a_\varepsilon + b_\varepsilon), t_i - b_\varepsilon], \ i = 1,\ldots, n, \\ 1 - e^{\frac{b_\varepsilon+t_i-t}{2\varepsilon}} & \text{if } t \in [t_i + b_\varepsilon, t_i + a_\varepsilon + b_\varepsilon)[, \ i = 1,\ldots, n, \\ 1 - \eta_\varepsilon & \text{if } t \in I \setminus (S_u)_{a_\varepsilon+b_\varepsilon}. \end{cases}$$

The function s_ε in the set $(S_u \cup S_{\dot{u}})_{a_\varepsilon+b_\varepsilon} \setminus (S_u \cup S_{\dot{u}})_{b_\varepsilon}$ is the solution of a Cauchy problem (see [13] Th. 3.1). For instance, on the interval $[t_i + b_\varepsilon, t_i + a_\varepsilon + b_\varepsilon]$, the function s_ε solves the problem

$$y'(t) = \frac{1 - y(t)}{2\varepsilon}, \quad y(t_i + b_\varepsilon) = 0, \quad y(t_i + b_\varepsilon + a_\varepsilon) = 1 - \eta_\varepsilon,$$

for every $i = 1,\ldots, n+m$. The same holds for σ_ε on the set $(S_u)_{a_\varepsilon+b_\varepsilon} \setminus (S_u)_{b_\varepsilon}$.

It is easy to see that $s_\varepsilon, \sigma_\varepsilon \in H^1(I, [0,1])$ for any ε small enough, and $s_\varepsilon \to 1$, $\sigma_\varepsilon \to 1$ in $L^2(I)$ as $\varepsilon \to 0$.

Let us define

$$u_\varepsilon(t) = \begin{cases} f_\varepsilon(t) & \text{if } t \in (S_u)_{b_\varepsilon}, \\ g_\varepsilon(t) & \text{if } t \in (S_{\dot{u}} \setminus S_u)_{b_\varepsilon}, \\ u(t) & \text{if } t \in I \setminus (S_u \cup S_{\dot{u}})_{b_\varepsilon}. \end{cases}$$

Here, for any $t_i \in S_u$ (resp. $t_i \in S_{\dot{u}}$), the function f_ε (resp. g_ε) is the unique cubic joining the point $u(t_i - b_\varepsilon)$ with the point $u(t_i + b_\varepsilon)$, and having the same derivative as u at the points $(t_i \pm b_\varepsilon)$. To be precise, if $t_i \in S_u$, denoting by $A = u(t_i + b_\varepsilon) + u(t_i - b_\varepsilon)$, $B = u(t_i + b_\varepsilon) - u(t_i - b_\varepsilon)$, $C = \dot{u}(t_i + b_\varepsilon) + \dot{u}(t_i - b_\varepsilon)$, $D = \dot{u}(t_i + b_\varepsilon) - \dot{u}(t_i - b_\varepsilon)$, we have

$$f_\varepsilon(t) = \left(\frac{1}{4b_\varepsilon^2}C - \frac{1}{4b_\varepsilon^3}B\right)(t-t_i)^3 + \frac{D}{4b_\varepsilon}(t-t_i)^2 - \left(\frac{C}{4} - \frac{3}{4b_\varepsilon}B\right)(t-t_i) + \frac{A}{2} - \frac{b_\varepsilon}{4}D,$$

for any $t \in (S_u)_{b_\varepsilon}$. If $t_i \in S_{\dot{u}} \setminus S_u$, then g_ε, defined on $(S_{\dot{u}} \setminus S_u)_{b_\varepsilon}$, has exactly the same expression of f_ε.

Clearly $u_\varepsilon \in H^2(I)$. In addition we can prove that

$$(4.2) \qquad |f_\varepsilon'| = O(b_\varepsilon^{-1}), \quad |f_\varepsilon''| = O(b_\varepsilon^{-2}) \qquad \text{on } (S_u)_{b_\varepsilon},$$

$$(4.3) \qquad |g_\varepsilon'| = O(1), \quad |g_\varepsilon''| = O(b_\varepsilon^{-1}) \qquad \text{on } (S_{\dot{u}} \setminus S_u)_{b_\varepsilon}.$$

In fact, since $|t - t_i| < b_\varepsilon$ on $(S_u)_{b_\varepsilon}$, one can verify that $|f'_\varepsilon(t)| \leq |C| + \frac{1}{2}|D| + \frac{3|B|}{2b_\varepsilon} = O(b_\varepsilon^{-1})$, for any $t \in (S_u)_{b_\varepsilon}$. Moreover, if $t \in (S_{\dot{u}})_{b_\varepsilon}$, there exists a constant $c > 0$ such that $|B| \leq cb_\varepsilon$, hence $|g'_\varepsilon| = O(1)$ on $(S_{\dot{u}} \setminus S_u)_{b_\varepsilon}$. This proves the first equalities in (4.2) and in (4.3). The relations concerning f''_ε and g''_ε can be proved analogously. In addition, since f_ε and g_ε are uniformly bounded on $(S_u)_{b_\varepsilon}$ and $(S_{\dot{u}} \setminus S_u)_{b_\varepsilon}$ respectively, we get easily

$$\lim_{\varepsilon \to 0} [\int_{(S_u)_{b_\varepsilon}} |f_\varepsilon(t) - u(t)|^2 \, dt + \int_{(S_{\dot{u}} \setminus S_u)_{b_\varepsilon}} |g_\varepsilon(t) - u(t)|^2 \, dt] = 0.$$

This implies that $u_\varepsilon \to u$ in $L^2(I)$ as $\varepsilon \to 0$, hence $\lim_{\varepsilon \to 0} \int_I |u_\varepsilon - g|^2 \, dt = \int_I |u - g|^2 \, dt$.

Then to prove (4.1), it is enough to show that

$$(4.4) \qquad \limsup_{\varepsilon \to 0} \mathcal{M}_\varepsilon(s_\varepsilon, I) \leq \#(S_u \cup S_{\dot{u}}),$$

$$(4.5) \qquad \limsup_{\varepsilon \to 0} \mathcal{M}_\varepsilon(\sigma_\varepsilon, I) \leq \#(S_u),$$

$$(4.6) \qquad \lim_{\varepsilon \to 0} \mu_\varepsilon \int_I \sigma_\varepsilon^2 |u'_\varepsilon|^2 \, dt = 0,$$

and

$$(4.7) \qquad \lim_{\varepsilon \to 0} \int_I (s_\varepsilon^2 + \lambda_\varepsilon)|u''_\varepsilon|^2 \, dt = \int_I |\ddot{u}|^2 \, dt.$$

Let us prove (4.4). Recall that $n = \#(S_u)$ and $n + m = \#(S_u \cup S_{\dot{u}})$. For any ε sufficiently small, by the definition of s_ε it follows that

$$\mathcal{M}_\varepsilon(s_\varepsilon, I) \leq \frac{(n+m)(b-a)\eta_\varepsilon^2}{4\varepsilon} + \frac{b_\varepsilon(n+m)}{2\varepsilon}$$

$$+ \sum_{i=1}^{n+m} 2 \int_{t_i + b_\varepsilon}^{t_i + a_\varepsilon + b_\varepsilon} [\varepsilon(s'_\varepsilon)^2 + \frac{(s_\varepsilon - 1)^2}{4\varepsilon}] \, dt$$

$$= \frac{(n+m)(b-a)\eta_\varepsilon^2}{4\varepsilon} + \frac{b_\varepsilon(n+m)}{2\varepsilon}$$

$$+ \sum_{i=1}^{n+m} \frac{1}{\varepsilon} \int_{t_i + b_\varepsilon}^{t_i + a_\varepsilon + b_\varepsilon} e^{\frac{b_\varepsilon + t_i - t}{\varepsilon}} \, dt.$$

As $\eta_\varepsilon = o(\varepsilon^{\frac{1}{2}})$, $b_\varepsilon = o(\varepsilon)$ and $\frac{1}{\varepsilon} \int_{b_\varepsilon + t_i}^{a_\varepsilon + b_\varepsilon + t_i} e^{\frac{b_\varepsilon + t_i - t}{\varepsilon}} \, dt = 1 - e^{\frac{-a_\varepsilon}{\varepsilon}} = 1 - \eta_\varepsilon^2$, we get

$$\limsup_{\varepsilon \to 0} \mathcal{M}_\varepsilon(s_\varepsilon, I) \leq n + m,$$

that is (4.4).

Using similar arguments, one can prove (4.5).

Let us show that

$$(4.8) \qquad \lim_{\varepsilon \to 0} \int_I \sigma_\varepsilon^2 |u'_\varepsilon|^2 \, dt = \int_I |\dot{u}|^2 \, dt.$$

We have, using obvious notation,

$$\int_I \sigma_\varepsilon^2 |u'_\varepsilon|^2 \, dt = \int_{I \backslash (S_u)_{b_\varepsilon}} \sigma_\varepsilon^2 |u'_\varepsilon|^2 \, dt$$

$$= \int_{I \backslash (S_u \cup S_{\dot{u}})_{b_\varepsilon}} \sigma_\varepsilon^2 |u'|^2 \, dt + \int_{(S_{\dot{u}} \backslash S_u)_{b_\varepsilon}} \sigma_\varepsilon^2 |g'_\varepsilon|^2 \, dt$$

$$= I_\varepsilon + II_\varepsilon.$$

Since $\sigma_\varepsilon \to 1$ in $L^2(I)$ as $\varepsilon \to 0$, and $0 \leq \sigma_\varepsilon \leq 1$, it follows that

$$(4.9) \qquad \lim_{\varepsilon \to 0} I_\varepsilon = \int_{I \backslash (S_u \cup S_{\dot{u}})} |u'|^2 \, dt = \int_I |\dot{u}|^2 \, dt.$$

Moreover, using (4.3), we get $II_\varepsilon = O(b_\varepsilon)$, and this, together with (4.9), concludes the proof of (4.8). Then (4.6) follows immediately from (4.8).

Finally, let us show (4.7). By the definitions of s_ε and u_ε, we have

$$\int_I (s_\varepsilon^2 + \lambda_\varepsilon) |u''_\varepsilon|^2 \, dt = \int_{I \backslash (S_u \cup S_{\dot{u}})_{b_\varepsilon}} (s_\varepsilon^2 + \lambda_\varepsilon) |u''|^2 \, dt + \lambda_\varepsilon \int_{(S_u \cup S_{\dot{u}})_{b_\varepsilon}} |u''_\varepsilon|^2 \, dt =$$

$$III_\varepsilon + IV_\varepsilon.$$

Since $s_\varepsilon \to 1$ in $L^2(I)$ as $\varepsilon \to 0$, and $0 \leq s_\varepsilon \leq 1$, we get

$$(4.10) \qquad \lim_{\varepsilon \to 0} III_\varepsilon = \int_{I \backslash (S_u \cup S_{\dot{u}})} |u''|^2 \, dt = \int_I |\ddot{u}|^2 \, dt.$$

In addition,

$$IV_\varepsilon = \lambda_\varepsilon \int_{(S_u)_{b_\varepsilon}} |f''_\varepsilon|^2 \, dt + \lambda_\varepsilon \int_{(S_{\dot{u}} \backslash S_u)_{b_\varepsilon}} |g''_\varepsilon|^2 \, dt = IV_\varepsilon^{(1)} + IV_\varepsilon^{(2)}.$$

Using (4.2) and (4.3), we deduce that

$$(4.11) \qquad IV_\varepsilon^{(1)} = O(\frac{\lambda_\varepsilon}{b_\varepsilon^3}), \quad IV_\varepsilon^{(2)} = O(\frac{\lambda_\varepsilon}{b_\varepsilon}).$$

As $\lambda_\varepsilon = o(b_\varepsilon^3)$, passing to the limit as $\varepsilon \to 0$ in (4.11), we obtain that $\lim_{\varepsilon \to 0} IV_\varepsilon = 0$. This, together with (4.10), gives (4.7), and concludes the proof of the theorem. $\qquad \square$

5. Equi-coerciveness

Let $I \subseteq \mathbb{R}$ be a bounded open interval. In this section we prove the following result.

Theorem 5.1.   *Let $c \in]0, +\infty[$, and let $\{(u_\varepsilon, s_\varepsilon, \sigma_\varepsilon)\}_\varepsilon$ be a sequence of elements of $X(I)$ such that*

$$(5.1) \qquad \mathcal{G}_\varepsilon(u_\varepsilon, s_\varepsilon, \sigma_\varepsilon, I) \leq c \qquad \text{for any } \varepsilon > 0.$$

Then there exist a subsequence $\{(u_{\varepsilon_h}, s_{\varepsilon_h}, \sigma_{\varepsilon_h})\}_h$ and a function $u \in \mathcal{H}^2(I)$ such that $(s_{\varepsilon_h}, \sigma_{\varepsilon_h}) \to (1, 1)$ in $(L^2(I))^2$, and $u_{\varepsilon_h} \to u$ in $L^p(I)$ as $h \to +\infty$, for any $p \in [1, 2[$.

Proof. By the definition of the functionals $\mathcal{G}_\varepsilon$, from (5.1) it follows immediately that $(s_\varepsilon, \sigma_\varepsilon) \to (1, 1)$ in $(L^2(I))^2$ as $\varepsilon \to 0$.

Reasoning as in the proof of Lemma 3.2, for any $h \in \mathbb{N}$ let $\phi_h = (1 - s_{\varepsilon_h})^2$, where $\{\varepsilon_h\}_h$ is a suitable sequence of positive numbers converging to zero such that $s_{\varepsilon_h} \to 1$ a.e. on I as $h \to +\infty$. Then ϕ_h is of class $H^1(I)$ for any h, $\phi_h \to 0$ a.e. on I as $h \to +\infty$, and $\sup_h \int_I |\phi_h'| \, dt < +\infty$ (see (3.25)).

Then, by Lemma 3.1, there exists a subsequence of $\{s_{\varepsilon_h}\}_h$, still denoted by $\{s_{\varepsilon_h}\}_h$, with the following property: for any $\delta > 0$ we can find a finite set $F \subseteq I$ such that, if A is a relatively compact open set contained in $I \setminus F$, then for every $t \in A$ we have $s_{\varepsilon_h}(t) \geq 1 - \sqrt{\delta}$ for any h. Therefore

$$\int_A |u_{\varepsilon_h}''|^2 \, dt \leq \frac{1}{(1 - \sqrt{\delta})^2} \int_A s_{\varepsilon_h}^2 |u_{\varepsilon_h}''|^2 \, dt \leq \frac{c}{(1 - \sqrt{\delta})^2},$$

which implies that the sequence $\{u_{\varepsilon_h}\}_h$ is bounded in $H^2(A)$ (see (3.9)).

Let us consider an increasing sequence $\{A_k\}_k$ of open sets relatively compact in $I \setminus F$ with $I \setminus F = \cup_k A_k$. For every k, we can find a subsequence, still denoted by $\{u_{\varepsilon_h}\}_h$, and a function $u \in H^2(A_k)$, such that $u_{\varepsilon_h} \to u$ uniformly on A_k as $h \to +\infty$.

Using a diagonal argument, we construct a subsequence $\{u_{\varepsilon_h}\}_h$ and a function $u \in H^2_{\text{loc}}(I \setminus F)$ such that $u_{\varepsilon_h} \to u$ pointwise on $I \setminus F$ as $h \to +\infty$.

Since $\{u_{\varepsilon_h}\}_h$ is uniformly bounded in $L^2(I)$ (see (3.9)), and it converges to u pointwise on $I \setminus F$, from the Fatou's Lemma we get $u \in L^2(I)$. In addition, for every subset J of I we get

$$\int_J |u_{\varepsilon_h}|^p \, dt \leq |J|^{\frac{1}{p} - \frac{1}{2}} \int_J |u_{\varepsilon_h}|^2 \, dt,$$

which implies that the sequence $\{|u_{\varepsilon_h}|^p\}_h$ is equi-integrable on I for every $p \in [1, 2[$. Therefore, by the dominated convergence theorem we conclude that the sequence $\{u_{\varepsilon_h}\}_h$ converges to u in $L^p(I)$, for every $p \in [1, 2[$.

Finally, since the sequence $\{(u_{\varepsilon_h}, s_{\varepsilon_h}, \sigma_{\varepsilon_h})\}_h$ converges to $\omega = (u, 1, 1)$ in $(L^1(I))^3$ as $h \to +\infty$, from the hypothesis (5.1) and the definition of Γ-lower limit, it follows that $\mathcal{G}_-(\omega, I) < +\infty$. Using Theorem 3.1, we conclude that $u \in \mathcal{H}^2(I)$ and the proof is complete. $\square$

As the sequence $\{\mathcal{G}_\varepsilon\}_\varepsilon$ Γ-converges to the functional $\mathcal{G}$, using the equi-coerciveness proved in Theorem 5.1 we get the following theorem (see [49] Cor.2.4, [42] Cor.7.20).

Theorem 5.2.   *Let I be a bounded open interval $\subseteq \mathbb{R}$. For any $\varepsilon > 0$ let us consider a minimizer $\omega_\varepsilon = (u_\varepsilon, s_\varepsilon, \sigma_\varepsilon)$ of the functional $\mathcal{G}_\varepsilon(\cdot, I)$.*

Then, there exist a sequence $\{\varepsilon_h\}_h$ of positive numbers converging to zero and a minimum point u of the functional G such that $\{u_{\varepsilon_h}\}_h$ converges to u strongly in $L^1(I)$.

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