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GEOMETRIC PROBLEMS INVOLVING CURVATURES IN THE CALCULUS OF VARIATIONS

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GEOMETRIC PROBLEMS INVOLVING CURVATURES IN THE CALCULUS OF VARIATIONS.

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1. Introduction. Many geometrical problems in the Calculus of Variations, arising in different fields as geometric measure theory, differential geometry, minimal surfaces, phase transitions, image segmentation, crystal growth, capillarity theory, involve the curvatures of a surface.

Let us start from some very classical examples. The first one, in the plane, is the problem of minimizing the functional

$$\int_0^{l(\gamma)} [1 + |\ddot{\gamma}(s)|^2]\, ds, \tag{1.1}$$

where γ varies over all curves of class C^2 satisfying prescribed boundary conditions, s is the corresponding arc lenght parameter, and $l(\gamma)$ is the lenght of the curve γ. The minimizers, discovered by Euler in 1744, are called *elastica* because of their application to the theory of flexible inextensible rods.

The second example, arising from differential geometry and physics, is the so-called mean curvature flow: to study the motion of a smooth interface $\Sigma(t)$ evolving in time by the law

$$V = \kappa,$$

where V is the normal velocity of the interface $\Sigma(t)$ and κ the sum of its principal curvatures.

The last classical example, the so-called prescribed curvature problem, is tied with the theory of minimal surfaces and capillarity theory, and reads as follows. Given a smooth bounded open set $\Omega \subset \mathbf{R}^n$, a function $\kappa \in L^\infty(\Omega)$, and $\theta \in \mathbf{R}$, consider the minimum problem

$$\min_{A \subseteq \Omega} \tilde{\mathcal{F}}(A), \qquad \text{where} \qquad \tilde{\mathcal{F}}(A) := P_\Omega(A) + \cos(\theta)\mathcal{H}^{n-1}(\partial A \cap \partial\Omega) - \int_A \kappa\, dx. \tag{1.2}$$

Here $P_\Omega(A)$ denotes the perimeter of A in Ω and $\mathcal{H}^{n-1}$ the $(n-1)$-dimensional Hausdorff measure in $\mathbf{R}^n$. If the solution to this problem is a set $A \subseteq \Omega$ with smooth boundary, then the boundary of A has mean curvature κ and contact angle θ at the intersection of ∂A with $\partial\Omega$.

In the last years many efforts have been made to study problems of this type, not only for their own theoretical interest, but also because of their connections with applicative fields.

The study of functionals involving lenght and curvature arised recently in the problem of image segmentation in Computer Vision. Here the main problem is to reconstruct the contours of a picture given by a camera. Given a bounded domain $\Omega \subseteq \mathbf{R}^2$ the image is represented by the grey level function $g \in L^\infty(\Omega)$, which measures the intensity of the light at each point of the screen. Since one expects the function g to be discontinuous along

the lines corresponding to sudden changes in the visible surfaces (e.g. edges of objects, shadows, different colours), the image segmentation problem consists in finding a pair (u, K) such that K is a set of curves decomposing the image into regions with relatively uniform intensity, while u is a smooth approximation of g on each region. The set K will be interpreted as the union of the lines which give the schematic description of the image. Recently Mumford and Nitzberg added an integral term depending on the curvature of K to the energy, in order to reconstruct the hidden boundaries, that is those parts of the image which are occluded by other regions closer to the observer. In this way they assigned an energy to the corners and the cusp's points of K. A simplified version of the problem, when u reduces to the characteristic function of a set E, is then to solve

$$\inf_E \{ \int_{\partial E} [1 + \psi(\kappa)] d\mathcal{H}^1(z) + \int_E g(z)\, dz \}, \tag{1.3}$$

where $E \subseteq \mathbf{R}^2$ is a bounded open set of class $\mathcal{C}^2$, $\kappa(z)$ is the curvature of ∂E at the point z, and ψ is a suitable function of the curvature.

Let us briefly describe the content of the thesis; more detailed informations can be found at the beginning of each Section.

The first chapter is devoted to the study of problem (1.3), mainly from the point of view of relaxation. In Section 1.1 we consider a simplified model, where $\psi(\kappa)$ has superlinear growth, and reduces to $\psi(\kappa) = |\kappa|^p$, for $p > 1$. The study of the minimum problem (1.3) has led us to regard the energy as a function of E rather than of ∂E. In fact, the compactness properties of a minimizing sequence of (1.3) ensure a good convergence for the sets, but the corresponding weak convergence of the boundaries seems not to be appropriate for variational purposes. The main difficulties, in this discussion, are due essentially to the lack of good continuity properties of the map $E \to \partial E$.

In the second part of the first chapter, starting from the suggestions of Mumford and Nitzberg, we study the the functional

$$\mathcal{P}_\psi(E_1, \dots, E_n) = \sum_{i=1}^{n} \int_{\partial E_i} [1 + \psi(\alpha_i, \kappa_i)]\ d\mathcal{H}^1 + \sum_{i=1}^{n+1} \int_{E_i'} (u_i - g)^2\ dz,$$

where $E_1, \dots, E_n \subseteq \mathbf{R}^2$ are bounded open sets of class $\mathcal{C}^2$, $\alpha_i(z)$ is the angle that the tangent unit vector of ∂E_i at $z \in \partial E_i$ forms with the positive direction of the x-axis, $\kappa_i(z)$ is the curvature of ∂E_i at z, ψ is possibly with linear growth with respect to κ_i, and

$$E_1' = E_1,\ E_i' = E_i \setminus \bigcup_{j=1}^{i-1} E_j,\ i = 2, \dots, n, \quad E_{n+1}' = \mathbf{R}^2 \setminus \bigcup_{j=1}^{n} E_j,$$

$$u_i = \fint_{E_i'} g\, dz, \qquad i = 1, \dots, n, \quad u_{n+1} = 0.$$

Each partition $E_1, \ldots, E_n$ is endowed with the following ordering: $E_i > E_j$ (i.e., E_i is closer than E_j to the observer) if $i < j$. This ordering represents relative depth (or occlusion); in this context, if $i = 1, \ldots, n$, the sets E_i' are the "visible" part of E_i, and E_{n+1}' stands for the background.

The second chapter deals with the motion of a surface by its mean curvature. To be precise, it is known that the singularly perturbed reaction-diffusion equation, with the double well potential $\Psi(s) = (1 - s^2)^2$ and forcing term g

$$\partial_t u_\varepsilon - \Delta u_\varepsilon + \tfrac{1}{2\varepsilon^2}\Psi'(u_\varepsilon) = \tfrac{c_0}{2} g \quad \text{in } \mathbf{R}^n \times (0,T), \qquad c_0 = \textstyle\int_{-1}^{1} \sqrt{\Psi(s)}\, ds,$$

provides an approximation for an interface $\Sigma(t)$ evolving by the law

$$V = \kappa + g,$$

where V is the normal velocity of the interface $\Sigma(t)$ and κ the sum of its principal curvatures. Such an equation was introduced by Allen and Cahn in order to describe the motion of antiphase boundaries in crystalline solids, thus showing its relevance in phase transitions. It was independently suggested by De Giorgi as a variational approach to the mean curvature flow. Our goal is to prove a quasi-optimal error estimate, valid before the onset of singularities, for the Hausdorff distance between the surface $\Sigma(t)$ and the approximate interface $\Sigma_\varepsilon(t) = \{\mathbf{x} \in \Omega : u_\varepsilon(\mathbf{x}, t) = 0\}$.

The third chapter of the thesis is devoted to the study of the prescribed curvature problem and some of its generalizations, mainly from the point of view of numerical analysis and discrete approximations by finite element method.

In Section 3.1 we present a numerical approximation to the functional $\tilde{\mathcal{F}}$ described in (1.2). Such a functional is first regularized following an idea of De Giorgi and developed by L. Modica and S. Mortola. It is next discretized by means of piecewise linear finite elements with numerical quadrature, thus allowing the actual implementation on a computer. We demonstrate the convergence of the discrete minimizers to a solution of the continuous problem.

In Section 3.2 we address the numerical minimization of the functional

$$\mathcal{H}^1(S_{\mathbf{v}}) + \int_{\partial\Omega} \mathbf{m} \cdot \mathrm{tr}(\mathbf{v})\, d\mathcal{H}^1(x) - \int_\Omega \kappa \cdot \mathbf{v}\, dx, \quad \forall\, \mathbf{v} \in BV(\Omega; \mathcal{E}), \tag{1.4}$$

where $\mathcal{E}$ is the canonical basis of $\mathbf{R}^d$ $(d \geq 2)$, $BV(\Omega; \mathcal{E})$ is the space of the functions pf bounded variation in Ω with values in $\mathcal{E}$, $S_{\mathbf{v}}$ and $\mathrm{tr}(\mathbf{v})$ are the *jump* set and the trace on $\partial\Omega$ of the vector valued BV function $\mathbf{v} = (v_1, \ldots, v_d)$. In addition, $\mathbf{m} = (m_1, \ldots, m_d) \in BV(\partial\Omega; \mathbf{R}^d)$ is a piecewise constant vector valued BV function, such that $|m_i - m_j| \leq 1$ almost everywhere on $\partial\Omega$, for all $1 \leq i, j \leq d$, and $\kappa = (k_1, \ldots, k_d) \in C^{0,1}(\bar{\Omega}; \mathbf{R}^d)$ is a vector valued Lipschitz continuous function. The minimum problem for $\mathcal{G}$ is related both to the

minimal partition of a set, to the segmentation of images, and to the search for sets with prescribed curvature.

In Section 3.3 we describe different numerical approaches to the prescribed curvature problem, namely a front-tracking algorithm.

Finally, in Section 3.4 we prove that $\tilde{\mathcal{F}}$ can be regularized with a sequence of functionals defined by

$$\int_\Omega \sqrt{\varepsilon^2 + |Dv|^2} + \int_{\partial\Omega} \mu v \, d\mathcal{H}^{n-1} - \int_\Omega \kappa v \, dx,$$

which, in turn, are discretized by continuous linear finite elements. Here the strict convexity in the gradient is the crucial point. The main result is the Γ-convergence of the discrete functionals to $\mathcal{F}$.

CHAPTER 1, SECTION 1.1

SEMICONTINUITY AND RELAXATION PROPERTIES OF A CURVATURE DEPENDING FUNCTIONAL IN 2D*

1. Introduction.

The aim of this paper is to study the functional

$$\mathcal{F}(E) = \int_{\partial E} [1 + |\kappa(z)|^p] \, d\mathcal{H}^1(z), \tag{1.1}$$

where $E \subseteq \mathbf{R}^2$ is a bounded open set of class C^2, $p > 1$ is a real number, $\kappa(z) = \kappa_{\partial E}(z)$ is the curvature of ∂E at the point z, and $\mathcal{H}^1$ denotes the one dimensional Hausdorff measure in $\mathbf{R}^2$.

We are interested in the study of the minimum problem

$$\inf_E \{ \int_{\partial E} [1 + |\kappa(z)|^p] \, d\mathcal{H}^1(z) + \int_E g(z) \, dz \}, \tag{1.2}$$

where $g \in L^\infty(\mathbf{R}^2)$ is a given function. This can be considered as a simplified version of a variational problem proposed in [15] as a criterion for segmentation of images in computer vision. Moreover, it was recently conjectured by E. De Giorgi [4] that problem (1.2) is connected with the asymptotic behaviour of some singular perturbations of minimum problems arising in the theory of phase transitions.

Assume, for simplicity, that g is non-negative for $|z|$ large enough. If we apply the direct method of the calculus of variations to problem (1.2), we are led to consider a minimizing sequence $\{E_h\}_h$, whose elements, according to our hypothesis on g, are contained in a suitable ball independent of h and satisfy

$$\sup_h \mathcal{H}^1(\partial E_h) < +\infty.$$

By a well known compactness theorem there exists a subsequence $\{E_{h_k}\}_k$ which converges in $L^1(\mathbf{R}^2)$ to some bounded set E of finite perimeter. We shall prove that the functional $\mathcal{F}$ is lower semicontinuous with respect to the convergence in $L^1(\mathbf{R}^2)$. This allows us to show

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that, if the limit set E is of class C^2, then E is a minimizer of (1.2). Since, in general, it is hard to prove directly that the limit of a minimizing sequence is of class C^2, we want to extend the functional $\mathcal{F}$ to the set $\mathcal{M}$ of all Lebesgue measurable subsets of $\mathbf{R}^2$, in such a way that the extended functional $\overline{\mathcal{F}}$ is still lower semicontinuous. As usual in the relaxation method (see [2]), we define $\overline{\mathcal{F}} : \mathcal{M} \to [0, +\infty]$ as the lower semicontinuous envelope of $\mathcal{F}$ with respect to the $L^1(\mathbf{R}^2)$-topology, i.e.,

$$\overline{\mathcal{F}}(E) = \inf\{\liminf_{h\to+\infty} \mathcal{F}(E_h) : E_h \to E \text{ in } L^1(\mathbf{R}^2)\}.$$

The main purpose of this paper is to study the functional $\overline{\mathcal{F}}$ and to determine the family of sets E for which $\overline{\mathcal{F}}(E) < +\infty$. The study of the minimum problem (1.2) led us to consider the functional

$$\int_{\partial E} [1 + |\kappa(z)|^p] \, d\mathcal{H}^1(z) \tag{1.3}$$

as a function of E rather than of ∂E. Indeed, the compactness properties of a minimizing sequence of (1.2) ensure a good convergence for the sets E_h, but the corresponding weak convergence of the boundaries ∂E_h does not seem to be appropriate for variational purposes. The main difficulties that we encountered were due essentially to the lack of good continuity properties of the map $E \mapsto \partial E$.

Lower semicontinuity results and existence of minimizers under suitable boundary conditions are much easier to reach for the functional

$$\mathcal{F}(\gamma) = \int_0^{l(\gamma)} [1 + |\tilde{\gamma}(s)|^p] \, ds, \tag{1.4}$$

related to (1.3), where γ varies now over all curves of class C^2 satisfying prescribed boundary conditions and s is the corresponding arc lenght parameter. In the case $p = 2$, this problem is classical and the minimizers, discovered by Euler [8] in 1744, are called *elastica* because of their application to the theory of flexible inextensible rods. For a complete treatment of the elastica we refer to [14,10]. Unfortunately, these results cannot be applied directly to the study of (1.2).

In this paper we have considered the problem in the plane. The extension of our results to the n-dimensional case is a difficult open problem and seems to require the methods of geometric measure theory [9,18]. We want to stress that all our proofs are obtained by using only elementary tools. We cannot exclude that some of these results could also be obtained in a more direct (but less elementary) way by using varifold theory [13,6,7].

We describe now in detail the content of the paper.

In Section 2 we fix the notation and introduce the problem.

Section 3 is devoted to the study of the lower semicontinuity of $\mathcal{F}$. Precisely, we prove (Theorem 3.2) that *given a sequence $\{E_h\}_h$ of bounded open sets of class C^2 converging in $L^1(\mathbf{R}^2)$ to a bounded open set E of class C^2, then*

$$\int_{\partial E} [1 + |\kappa(z)|^p]\ d\mathcal{H}^1(z) \leq \liminf_{h \to +\infty} \int_{\partial E_h} [1 + |\kappa_h(z)|^p]\ d\mathcal{H}^1(z),$$

where κ and κ_h are the curvatures of ∂E and of ∂E_h, respectively.

The definition (1.1) of $\mathcal{F}$ makes sense if E is a set whose boundary can be parametrized, locally, by arcs of regular curves of class $H^{2,p}$. The previuos semicontinuity theorem holds also in this case (Corollary 3.2).

We emphasize that the semicontinuity of $\mathcal{F}$ and the definition of $\overline{\mathcal{F}}$ are considered with respect to the $L^1(\mathbf{R}^2)$-topology. This means that the sequence $\{E_h\}_h$ approximates E if and only if $|E_h \triangle E| \to 0$ as $h \to +\infty$, where $|\cdot|$ denotes the Lebesgue measure and $\triangle$ is the symmetric difference of sets. No further conditions are required on ∂E_h and ∂E.

Simple examples show that there exist sets $E \in \mathcal{M}$ with $\overline{\mathcal{F}}(E) < +\infty$, whose boundary is not smooth. In particular, let us consider the set E of Figure 1.1 and suppose that ∂E is parametrized by arcs of curves of class C^∞, except for the cusp points.

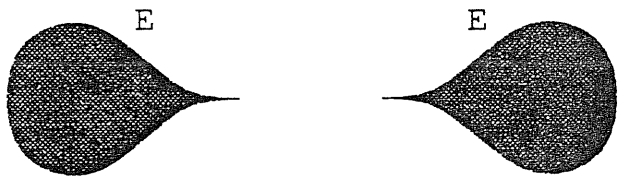

Fig. 1.1: A set E with $\overline{\mathcal{F}}(E) < +\infty$, and whose boundary is not smooth.

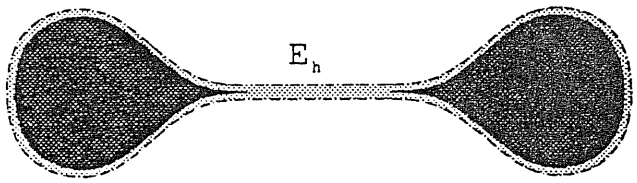

Fig. 1.2: This approximating sequence $\{E_h\}_h$ shows that $\overline{\mathcal{F}}(E) < +\infty$.

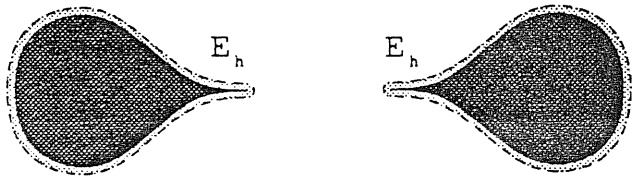

Fig. 1.3: This approximating sequence $\{E_h\}_h$ is such that $\sup_h \int_{\partial E_h} |\kappa_h(z)|^p\ d\mathcal{H}^1(z) = +\infty$.

Figure 1.2 shows that $\overline{\mathcal{F}}(E) < +\infty$. Indeed, the sequence $\{E_h\}_h$ described in Figure 1.2 converges to the set E in the $L^1(\mathbf{R}^2)$ norm and $\sup_h \mathcal{F}(E_h) < +\infty$. On the other hand, the approximating sequence $\{E_h\}_h$ of Figure 1.3 gives raise to an infinite energy. Here the cusp points are smoothed by circular arcs, and it is easy to prove that, if $p > 1$, then $\int_{\partial E_h} |\kappa_h(z)|^p \, d\mathcal{H}^1(z) \to +\infty$ as $h \to +\infty$. Note that Figure 1.2 shows that, in the limit, the sequence $\{\partial E_h\}_h$ creates a hidden arc (with multiplicity two), given by the segment joining the two cusp points.

The main issue is obviously to characterize those subsets E of $\mathbf{R}^2$ such that $\overline{\mathcal{F}}(E) < +\infty$. We find some necessary conditions and a number of different sufficient conditions, but the complete characterization of this class of sets still remains an open problem.

In Section 4 we present some regularity properties of the sets E such that $\overline{\mathcal{F}}(E) < +\infty$. To be precise, we prove the following result (Theorem 4.1, Proposition 4.3): *Let $E \subseteq \mathbf{R}^2$ be a measurable set such that $\overline{\mathcal{F}}(E) < +\infty$. Then, up to a modification of E on a set of measure zero, we have that E is bounded, open, and $\mathcal{H}^1(\partial E) < +\infty$. Moreover, there exists a system of curves $\Gamma = \{\gamma^1, \ldots, \gamma^m\}$ of class $H^{2,p}$ such that ∂E is contained in the union (Γ) of the traces of the curves γ^i and $E = \mathrm{int}(A_\Gamma \cup (\Gamma))$, where A_Γ is the set of all points of $\mathbf{R}^2 \setminus (\Gamma)$ of index 1 with respect to Γ. Finally, ∂E has a continuous unoriented tangent and can have at most a countable set of cusp points.*

At the end of Section 4 we classify the points of ∂E according to the local behaviour of the normal line with respect to ∂E.

In Section 5 we show examples of sets E such that $\overline{\mathcal{F}}(E) < +\infty$, despite of their boundaries being very irregular. Precisely, in Example 1 a set E is described whose irregular boundary points have positive one dimensional Hausdorff measure. In Example 2 a set having an infinite number of cusp points is shown.

In Section 6 we deal with the following problem: which conditions must be satisfied by the boundary of a set E in order to have $\overline{\mathcal{F}}(E) < +\infty$? To answer this question, we first study those systems of curves that can be obtained as limits, in the $H^{2,p}$ norm, of a sequence of boundaries of smooth sets. To this aim we introduce the definition of system of curves satisfying the finiteness property (Definition 6.1), and the compatibility condition (Definition 6.6). Then the following result holds (Theorem 6.2): *let Γ be a system of curves of class $H^{2,p}$ without crossings and satisfying the finiteness property, and define E as the set of all points of $\mathbf{R}^2 \setminus (\Gamma)$ of odd index with respect to Γ. Then $\partial E \subseteq (\Gamma)$. Moreover $\overline{\mathcal{F}}(E) < +\infty$, i.e., there exists a sequence $\{E_h\}_h$ of bounded open sets of class C^∞ such that $E_h \to E$ in $L^1(\mathbf{R}^2)$ as $h \to +\infty$ and $\sup_h \mathcal{F}(E_h) < +\infty$. In addition, up to a suitable surgery operation on the parameter space of Γ, we have that $\partial E_h \to \Gamma$ in $H^{2,p}$ as $h \to +\infty$.*

Then, quite surprisingly, using only elementary properties of graphs, we prove one of the main results of the paper (Theorem 6.5): *Suppose that ∂E is smooth except for a finite*

number n of cusp points. Then

$$\overline{\mathcal{F}}(E) < +\infty \iff n \text{ is even.}$$

In Section 7 we localize the definition of $\mathcal{F}$ to all open subsets of $\mathbf{R}^2$, i.e., we consider the functional

$$\mathcal{F}(E,\Omega) = \int_{\Omega\cap\partial E} [1 + |\kappa(z)|^p]\, d\mathcal{H}^1(z),$$

where $\Omega \subseteq \mathbf{R}^2$ is an open set and E is a bounded open subset of $\mathbf{R}^2$ such that $\Omega \cap \partial E$ is of class C^2. We prove (Theorem 7.1) that $\mathcal{F}(\cdot,\Omega)$ is $L^1(\Omega)$-lower semicontinous.

Let $\overline{\mathcal{F}}(\cdot,\Omega) : \mathcal{M} \to [0,+\infty]$ denote the lower semicontinuos envelope of $\mathcal{F}(\cdot,\Omega)$ with respect to the $L^1(\Omega)$-topology. One of the main results of this section is that, as conjectured by E. De Giorgi in a slightly different context [4, Conjecture 5], there are sets E such that $\overline{\mathcal{F}}(E,\cdot)$, if considered as a set function, is not a measure. Precisely, we construct an example of a set E whose boundary is smooth except for two cusp points and such that

$$\overline{\mathcal{F}}(E,\Omega_1) + \overline{\mathcal{F}}(E,\Omega_2) < \overline{\mathcal{F}}(E,\mathbf{R}^2) < +\infty,$$

where $\Omega_1, \Omega_2 \subseteq \mathbf{R}^2$ are two open sets, with $\Omega_1 \cup \Omega_2 = \mathbf{R}^2$.

This shows that $\overline{\mathcal{F}}(E,\Omega)$ cannot be represented by an integral of the form (1.3). Finally, the value of $\overline{\mathcal{F}}(E,\mathbf{R}^2)$ is computed explicitly in this example (Theorem 7.2).

Acknowledgements

We are indebted to Prof. E. De Giorgi for having suggested us the study of this problem.

2. Notations and preliminaries.

A plane curve $\gamma : [0,1] \to \mathbf{R}^2$ of class C^1 is said to be regular if $\frac{d\gamma(t)}{dt} \neq 0$ for every $t \in [0,1]$. Each closed regular curve $\gamma : [0,1] \to \mathbf{R}^2$ will be identified, in the usual way, with a map $\gamma : \mathbf{S}^1 \to \mathbf{R}^2$, where $\mathbf{S}^1$ denotes the oriented unit circle. By $(\gamma) = \gamma([0,1]) = \{\gamma(t) : t \in [0,1]\}$ we denote the trace of γ and by $l(\gamma)$ its length; s denotes the arc length parameter, and $\dot\gamma$, $\ddot\gamma$ the first and the second derivative of γ with respect to s. Let us fix a real number $p > 1$ and let $p' > 1$ be its conjugate exponent, i.e., $\frac{1}{p} + \frac{1}{p'} = 1$. If the second derivative $\ddot\gamma$ in the sense of distributions belongs to L^p, then the curvature $\kappa(\gamma)$ of γ is given by $|\ddot\gamma|$, and

$$\|\kappa(\gamma)\|_{L^p}^p = \int_0^{l(\gamma)} |\kappa(\gamma)|^p\, ds < +\infty;$$

in this case we say that γ is a curve of class $H^{2,p}$, and we write $\gamma \in H^{2,p}$. Moreover, we put

$$\mathcal{F}(\gamma) = l(\gamma) + \|\kappa(\gamma)\|_{L^p}^p.$$

If $z \in \mathbf{R}^2 \setminus (\gamma)$, $I(\gamma, z)$ is the index of z with respect to γ [3].

$\mathcal{H}^1$ denotes the 1-dimensional Hausdorff measure in $\mathbf{R}^2$ [9]; for any $z_0 \in \mathbf{R}^2, \varrho > 0$, $B_\varrho(z_0) = \{z \in \mathbf{R}^2 : |z - z_0| < \varrho\}$ is the ball centered at z_0 with radius ϱ. We remark that if f is a positive Borel function defined on (γ), then (see for instance [9, Th. 3.2.6])

$$\int_{\gamma(B)} f(z)\, d\mathcal{H}^1(z) \leq \int_B f(\gamma(s))\, ds \tag{2.1}$$

for any Borel set $B \subseteq [0, l(\gamma)]$.

Given a measurable set $E \subseteq \mathbf{R}^2$, χ_E denotes its characteristic function, that is $\chi_E(z) = 1$ if $z \in E$, $\chi_E(z) = 0$ if $z \notin E$; $|E|$ is the Lebesgue measure of E. For any subset C of $\mathbf{R}^2$, we denote by $\mathrm{int}(C)$ the interior of C, by $\overline{C}$ the closure of C, and by ∂C the boundary of C. Let $E \subseteq \mathbf{R}^2$; we say that E is of class $H^{2,p}$ (resp. C^2) if E is open, and, near each point $z \in \partial E$, the set E is the subgraph of a function of class $H^{2,p}$ (resp. C^2) with respect to a suitable orthogonal coordinate system. Note that, if ∂E can be parametrized, locally, by arcs of regular curves of class $H^{2,p}$ (resp. C^2), and E lies locally on one side of its boundary, then E is of class $H^{2,p}$ (resp. C^2).

If E is a bounded subset of $\mathbf{R}^2$ of class $H^{2,p}$, then ∂E can be viewed, locally, as the graph of a function of class $H^{2,p}$. This allows us to define, locally, the notion of curvature of ∂E at $\mathcal{H}^1$ almost every point of ∂E, by using the classical formulas involving second derivatives. It is easy to see that the function $\kappa(z)$ does not depend on the choice of the coordinate system used to describe ∂E as a graph, and belongs to $L^p(\partial E, \mathcal{H}^1)$.

If $\mathcal{M}$ denotes the class of all Lebesgue measurable subets of $\mathbf{R}^2$, we define the map $\mathcal{F} : \mathcal{M} \to [0, +\infty]$ by

$$\mathcal{F}(E) = \begin{cases} \displaystyle\int_{\partial E} [1 + |\kappa(z)|^p]\, d\mathcal{H}^1(z) & \text{if } E \text{ is a bounded open set of class } C^2, \\ +\infty & \text{elsewhere on } \mathcal{M}, \end{cases} \tag{2.2}$$

where $\kappa(z)$ is the curvature of ∂E at the point z.

Let $g \in L^\infty(\mathbf{R}^2)$ be a function such that $\{z \in \mathbf{R}^2 : g(z) < 0\} \subseteq B_R(0)$, for a suitable $R > 0$. Let us consider the minimum problem

$$\inf_E \{ \int_{\partial E} [1 + |\kappa(z)|^p]\, d\mathcal{H}^1(z) + \int_E g(z)\, dz \}, \tag{2.3}$$

where E varies over all bounded open subsets of $\mathbf{R}^2$ with boundary of class C^2. It is clear that (2.3) is equivalent to the problem

$$\inf_{E \in \mathcal{M}} \mathcal{G}(E), \qquad \text{where} \quad \mathcal{G}(E) = \mathcal{F}(E) + \int_E g(z)\, dz, \tag{2.4}$$

in the sense that (2.3) and (2.4) have the same minimum values and the same minimizers. If the region where g is less than a prescribed negative number is large enough, then it is immediate to verify that the minimum value of problem (2.3) is negative, hence every solution of (2.3) must be non-empty.

We shall identify $\mathcal{M}$ with a closed subset of $L^1(\mathbf{R}^2)$ by means of the map $E \mapsto \chi_E$. The $L^1(\mathbf{R}^2)$-topology on $\mathcal{M}$ is, therefore, the topology on $\mathcal{M}$ induced by the distance $d(E_1, E_2) = |E_1 \triangle E_2|$, where $E_1, E_2 \in \mathcal{M}$ and $\triangle$ is the symmetric difference of sets.

Let us prove that any minimizing sequence $\{E_h\}_h$ of $\mathcal{G}$ is relatively compact in $L^1(\mathbf{R}^2)$. Let $\{E_h\}_h$ be such a minimizing sequence; clearly, we can assume that $\sup_h \mathcal{G}(E_h) < +\infty$, hence E_h is a bounded open set of class C^2 for any h. Then, since

$$\int_{\partial E_h} [1 + |\kappa_h(z)|^p] \, d\mathcal{H}^1(z) \leq \sup_h \mathcal{G}(E_h) + \|g\|_\infty |B_R(0)|,$$

it follows that

$$H = \sup_h \mathcal{H}^1(\partial E_h) < +\infty.$$

Now we will show that there exist $h_0 \in \mathbf{N}$ such that $E_h \subseteq B_{R+H}(0)$ for any $h \geq h_0$. If this condition is not satisfied, then there exists a subsequence of $\{E_h\}_h$, still denoted by $\{E_h\}_h$, such that $E_h \setminus B_{R+H}(0) \neq \emptyset$ for every $h \in \mathbf{N}$. As the total lenght of ∂E_h is bounded by H, each set E has a connected component C_h which does not meet $B_R(0)$. Denoting by E_h' the set $E_h' = E_h \setminus C_h$, we get $\mathcal{G}(E_h') < \mathcal{G}(E_h)$ (recall that $g \geq 0$ on $\mathbf{R}^2 \setminus B_R(0)$). Since also $\{E_h'\}_h$ is a minimizing sequence, it follows that necessarily $\mathcal{G}(C_h) = \mathcal{G}(E_h) - \mathcal{G}(E_h') \to 0$ as $h \to +\infty$. On the other hand, one can show (see (3.5)) that C_h gives a positive contribution to the energy $\mathcal{F}$ independent of h, that is $\mathcal{F}(C_h) \not\to 0$, which gives a contradiction.

Hence there exists $h_0 \in \mathbf{N}$ such that

$$E_h \subseteq B_{R+H}(0) \quad \text{for any } h \geq h_0.$$

We deduce that

$$\sup_h [\mathcal{H}^1(\partial E_h) + |E_h|] < +\infty, \qquad E_h \subseteq B_{R+H}(0) \quad \text{for any } h.$$

Using the Rellich compactness theorem in BV (see [12, Th. 1.19]), it follows that there exist a bounded set E of finite perimeter and a subsequence $\{E_{h_k}\}_k$ such that $E_{h_k} \to E$ in $L^1(\mathbf{R}^2)$ as $k \to +\infty$, and this shows that any minimizing sequence of $\mathcal{G}$ is relatively compact in $L^1(\mathbf{R}^2)$.

We denote by $\overline{\mathcal{F}}$ the lower semicontinuous envelope of $\mathcal{F}$ with respect to the topology of $L^1(\mathbf{R}^2)$. It is known that, for every $E \in \mathcal{M}$, we have

$$\overline{\mathcal{F}}(E) = \inf\{\liminf_{h \to +\infty} \mathcal{F}(E_h) : E_h \to E \text{ in } L^1(\mathbf{R}^2) \text{ as } h \to +\infty\}.$$

For the main properties of the relaxed functional we refer to [2]. In particular, one can prove that

$$\inf_{E\in\mathcal{M}}\{\mathcal{F}(E)+\int_E g(z)\ dz\}=\min_{E\in\mathcal{M}}\{\overline{\mathcal{F}}(E)+\int_E g(z)\ dz\}.$$

In addition, every minimizing sequence of $\mathcal{F}(E)+\int_E g(z)\ dz$ has a subsequence converging to a minimum point of $\overline{\mathcal{F}}(E)+\int_E g(z)\ dz$ and every minimum point of $\overline{\mathcal{F}}(E)+\int_E g(z)\ dz$ is the limit of a minimizing sequence of $\mathcal{F}(E)+\int_E g(z)\ dz$.

The main purpose of this paper is to study the properties of the functional $\overline{\mathcal{F}}$. From the definition, it follows immediately that $\overline{\mathcal{F}}(E)<+\infty$ if and only if there exists a sequence $\{E_h\}_h$ of bounded open sets of class C^2 such that $E_h\to E$ in $L^1(\mathbb{R}^2)$ as $h\to+\infty$ and

$$\sup_h \mathcal{H}^1(\partial E_h)<+\infty,\quad \sup_h\int_{\partial E_h}|\kappa_h(z)|^p\ d\mathcal{H}^1(z)<+\infty, \tag{2.5}$$

where κ_h denotes the curvature of ∂E_h.

DEFINITION 2.1. *By a system of curves we mean a finite family $\Gamma=\{\gamma^1,\dots,\gamma^m\}$ of closed regular curves of class C^1 such that $|\frac{d\gamma^i}{dt}|$ is constant on $[0,1]$ for any $i=1,\dots,m$. Denoting by S the disjoint union of m unit circles $\mathrm{S}^1_1,\dots,\mathrm{S}^1_m$, we shall identify Γ with the map $\Gamma:S\to\mathbb{R}^2$ defined by $\Gamma_{|\mathrm{S}^1_i}=\gamma^i$, for $i=1,\dots,m$. The trace (Γ) of Γ is the union of the traces of the curves γ^i, i.e., $(\Gamma)=\bigcup_{i=1}^m(\gamma^i)$.*

DEFINITION 2.2. *If $\Gamma=\{\gamma^1,\dots,\gamma^m\}$ is a system of curves of class $H^{2,p}$, we define*

$$l(\Gamma)=\sum_{i=1}^m l(\gamma^i),\qquad \|\kappa(\Gamma)\|^p_{L^p}=\sum_{i=1}^m\|\kappa(\gamma^i)\|^p_{L^p}=\sum_{i=1}^m\int_0^{l(\gamma^i)}|\tilde\gamma^i(s)|^p\ ds,$$

and

$$\mathcal{F}(\Gamma)=\sum_{i=1}^m\mathcal{F}(\gamma)=\sum_{i=1}^m l(\gamma^i)+\|\kappa(\gamma^i)\|^p_{L^p}.$$

If $z\in\mathbb{R}^2\setminus(\Gamma)$ we define the index of z with respect to Γ as $I(\Gamma,z)=\sum_{i=1}^m I(\gamma^i,z)$.

As $|\frac{d\gamma^i}{dt}|$ is constant on $[0,1]$, we have $s(t)=tl(\gamma^i)$, hence

$$\int_0^{l(\gamma^i)}|\tilde\gamma^i(s)|^p\ ds=l(\gamma^i)^{1-2p}\int_0^1|\frac{d^2\gamma^i}{dt^2}|^p\ dt. \tag{2.6}$$

3. Semicontinuity of $\mathcal{F}$.

DEFINITION 3.1. *By a disjoint system of curves we mean a system of curves* $\Gamma = \{\gamma^1, \dots, \gamma^m\}$ *such that* $(\gamma^i) \cap (\gamma^j) = \emptyset$ *for any* $i, j = 1, \dots, m$, $i \neq j$.

Let $\{E_h\}_h$ be a sequence of sets satisfying (2.5). Note that, for any h, there exists a suitable parametrization Γ_h of ∂E_h such that the sequence $\{\Gamma_h\}_h$ satisfies

$$\sup_h l(\Gamma_h) < +\infty, \qquad \sup_h \|\kappa(\Gamma_h)\|_{L^p}^p < +\infty. \tag{3.1}$$

LEMMA 3.1. *Let* $\Gamma = \{\gamma^1, \dots, \gamma^m\}$ *be a system of curves of class* $H^{2,p}$. *Then*

$$m \leq l(\Gamma)\|\kappa(\Gamma)\|_{L^p}^{p'}(2\pi)^{-p'}. \tag{3.2}$$

PROOF: Let γ be a simple closed regular curve of class $H^{2,p}$; let us prove that

$$l(\gamma) \geq (2\pi)^{p'}\|\kappa(\gamma)\|_{L^p}^{-p'}. \tag{3.3}$$

If $\gamma \in C^2$, then [5, Th. 5.7.3]

$$\int_0^{l(\gamma)} |\tilde{\gamma}(s)|\, ds \geq 2\pi. \tag{3.4}$$

By a standard approximation argument, inequality (3.4) holds for any curve γ of class $H^{2,p}$. Then (3.3) follows from the Hölder inequality, since, if $\gamma \in H^{2,p}$,

$$2\pi \leq \int_0^{l(\gamma)} |\tilde{\gamma}(s)|\, ds \leq l(\gamma)^{\frac{1}{p'}} \Big(\int_0^{l(\gamma)} |\tilde{\gamma}(s)|^p\, ds\Big)^{\frac{1}{p}} = l(\gamma)^{\frac{1}{p'}}\|\kappa(\gamma)\|_{L^p}.$$

Hence $l(\gamma^i) \geq (2\pi)^{p'}\|\kappa(\gamma^i)\|_{L^p}^{-p'}$ for any $i = 1, \dots, m$. Then, recalling Definition 2.2, it follows that

$$l(\Gamma) = \sum_{i=1}^m l(\gamma^i) \geq \sum_{i=1}^m (2\pi)^{p'}\|\kappa(\gamma^i)\|_{L^p}^{-p'} \geq (2\pi)^{p'}\|\kappa(\Gamma)\|_{L^p}^{-p'} m. \quad \square$$

The following generalization of inequality (3.3), as well as Corollary 3.1, will be useful in Section 7.

LEMMA 3.2. *Let* $\gamma : [0,1] \to \mathbf{R}^2$ *be a regular curve of class* $H^{2,p}$ *and let* $\theta : [0,1] \to \mathbf{R}$ *be a continuous function such that* $\theta(t)$ *is the oriented angle between the* x*-axis and the tangent vector of* γ *at* t. *Then*

$$l(\gamma) \geq |\theta_1 - \theta_0|^{p'}\|\kappa(\gamma)\|_{L^p}^{-p'}, \quad \text{where } \theta_0 = \theta(0) \text{ and } \theta_1 = \theta(1).$$

PROOF: Let us write with obvious notation $\dot\gamma(s) = (\cos\theta(s), \sin\theta(s))$. Then, using the Hölder inequality, it follows that

$$\|\kappa(\gamma)\|_{L^p}^p = \int_0^{l(\gamma)} |\dot\theta(s)|^p \, ds \geq l(\gamma)^{-\frac{p}{p'}} \Big(\int_0^{l(\gamma)} |\dot\theta(s)| \, ds\Big)^p \geq$$

$$l(\gamma)^{-\frac{p}{p'}} \Big|\int_0^{l(\gamma)} \dot\theta(s) \, ds\Big|^p = l(\gamma)^{-\frac{p}{p'}} \Big(\int_0^1 \frac{d\theta}{dt}(t) \, dt\Big)^p = |\theta_1 - \theta_0|^p l(\gamma)^{-\frac{p}{p'}}. \quad \square$$

COROLLARY 3.1. *Let $\gamma, \theta_0, \theta_1$ be as in Lemma 3.2. Then*

$$l(\gamma) + \|\kappa(\gamma)\|_{L^p}^p \geq |\theta_1 - \theta_0| \Big[\Big(\frac{p}{p'}\Big)^{\frac{1}{p}} + \Big(\frac{p'}{p}\Big)^{\frac{1}{p'}}\Big]. \tag{3.5}$$

PROOF: Lemma 3.2 implies that

$$l(\gamma) + \|\kappa(\gamma)\|_{L^p}^p \geq l(\gamma) + l(\gamma)^{-\frac{p}{p'}} |\theta_1 - \theta_0|^p.$$

Hence, since the minimum point of the function $\lambda + \lambda^{-\frac{p}{p'}} |\theta_1 - \theta_0|^p$ is reached at $\lambda = |\theta_1 - \theta_0| \big(\frac{p}{p'}\big)^{\frac{1}{p}}$, we obtain

$$l(\gamma) + \|\kappa(\gamma)\|_{L^p}^p \geq |\theta_1 - \theta_0| \Big(\frac{p}{p'}\Big)^{\frac{1}{p}} + |\theta_1 - \theta_0|^p |\theta_1 - \theta_0|^{-\frac{p}{p'}} \Big(\frac{p}{p'}\Big)^{-\frac{1}{p'}} = |\theta_1 - \theta_0| \Big[\Big(\frac{p}{p'}\Big)^{\frac{1}{p}} + \Big(\frac{p'}{p}\Big)^{\frac{1}{p'}}\Big]. \quad \square$$

DEFINITION 3.2. *Let $E \subseteq \mathbf{R}^2$ be a bounded open set of class C^1. We say that a disjoint system of curves Γ is an oriented parametrization of ∂E if each curve of the system is simple, $(\Gamma) = \partial E$, and, in addition,*

$$E = \{z \in \mathbf{R}^2 : I(\Gamma, z) = 1\}, \quad \mathbf{R}^2 \setminus \overline{E} = \{z \in \mathbf{R}^2 : I(\Gamma, z) = 0\}.$$

PROPOSITION 3.1. *Each bounded subset E of $\mathbf{R}^2$ of class $H^{2,p}$ (resp. C^2) admits an oriented parametrization of class $H^{2,p}$ (resp. C^2).*

PROOF: Since E is of class $H^{2,p}$ each connected component of ∂E can be parametrized by a regular closed curve of class $H^{2,p}$. As ∂E is compact and locally connected, it has a finite number of connected components. The statement about the index follows from Jordan's Theorem. $\square$

DEFINITION 3.3. *We say that a sequence $\{\Gamma_h\}_h$ of systems of curves of class $H^{2,p}$ converges weakly in $H^{2,p}$ to a system of curves $\Gamma = \{\gamma^1, \dots, \gamma^m\}$ of class $H^{2,p}$ if the number of curves of each system Γ_h equals the number of curves of Γ for h large enough, i.e., $\Gamma_h = \{\gamma_h^1, \dots, \gamma_h^m\}$, and, in addition, $\gamma_h^i \rightharpoonup \gamma^i$ weakly in $H^{2,p}$ as $h \to +\infty$ for any $i = 1, \dots, m$.*

Note that, if $\{\Gamma_h\}_h$ converges to $\Gamma = \{\gamma^1, \dots, \gamma^m\}$ weakly in $H^{2,p}$, then

$$\gamma_h^i \to \gamma^i \text{ in } C^1 \text{ as } h \to +\infty, \tag{3.6}$$

for any $i = 1, \dots, m$. In particular, $l(\gamma_h^i) \to l(\gamma^i)$ as $h \to +\infty$.

THEOREM 3.1. *Let $\{\Gamma_h\}_h$ be a sequence of systems of curves satisfying (3.1) such that the traces (Γ_h) are contained in a bounded subset of $\mathbf{R}^2$ independent of h. Then $\{\Gamma_h\}_h$ has a subsequence which converges weakly in $H^{2,p}$ to a system of curves Γ.*

PROOF: By (3.2), using (3.1) it follows that the number m_h of curves of the system Γ_h is bounded uniformly with respect to h. Hence, for a subsequence (still denoted by $\{\Gamma_h\}_h$), there exists an integer m such that $\Gamma_h = \{\gamma_h^1, \dots, \gamma_h^m\}$ for any h. Fix $i \in \{1, \dots, m\}$; using (3.3) and (3.1) we get that there exist two positive constants c_1, c_2 such that

$$c_1 \le l(\gamma_h^i) \le c_2 \quad \text{for any } h.$$

Then, using (2.6) we obtain that there exists a positive constant c_3 such that $\int_0^1 |\frac{d^2\gamma_h^i}{dt^2}|^p \, dt \le l(\gamma_h^i)^{2p-1} \|\kappa(\gamma_h^i)\|_{L^p}^p \le c_3 \, c_2^{2p-1}$, whence, as (Γ_h) are bounded uniformly with respect to h by the hypothesis, the family $\{\gamma_h^i\}_h$ is equibounded in $H^{2,p}$. Then, for a subsequence, there exist m curves $\gamma^1, \dots, \gamma^m$ of class $H^{2,p}$ such that $\gamma_h^i \rightharpoonup \gamma^i$ weakly in $H^{2,p}$ as $h \to +\infty$, for any $i = 1, \dots, m$. This shows that $\{\Gamma_h\}_h$ converges to the system of curves $\Gamma = \{\gamma^1, \dots, \gamma^m\}$ weakly in $H^{2,p}$. □

DEFINITION 3.4. *We say that Γ is a limit system of curves of class $H^{2,p}$ if Γ is the weak $H^{2,p}$ limit of a sequence $\{\Gamma_h\}_h$ of oriented parametrizations of bounded open sets of class $H^{2,p}$.*

The following remark is an easy consequence of (3.6).

REMARK 3.1. *If Γ is a limit system of curves of class $H^{2,p}$, then $I(\Gamma, z) \in \{0, 1\}$ for any $z \in \mathbf{R}^2 \setminus (\Gamma)$.*

In order to prove the semicontinuity Theorem 3.2, we need the following Lemma.

LEMMA 3.3. *Let $E \subseteq \mathbf{R}^2$ be a measurable set, let $\{E_h\}_h$ be a sequence of bounded open sets of class $H^{2,p}$ satisfying (2.5), and suppose that $E_h \to E$ in $L^1(\mathbf{R}^2)$ as $h \to +\infty$. Let us define*

$$E^\star = \{z \in \mathbf{R}^2 : \exists\, r > 0 \; |B_r(z) \setminus E| = 0\},$$

$$F^\star = \{z \in \mathbf{R}^2 : \exists\, r > 0 \; |B_r(z) \cap E| = 0\}.$$

Then $E^\star$ and $F^\star$ are open, $E^\star$ is bounded, $|E \triangle E^\star| = 0$, $E^\star = \mathrm{int}(\mathbf{R}^2 \setminus F^\star)$, $F^\star = \mathrm{int}(\mathbf{R}^2 \setminus E^\star)$, and

$$\partial E^\star = \partial F^\star = \{z \in \mathbf{R}^2 : 0 < |B_r(z) \cap E| < |B_r(z)| \ \forall r > 0\}. \tag{3.7}$$

Moreover there exists a limit system Γ of curves of class $H^{2,p}$ with the following properties:

(i) $E^\star = \mathrm{int}\big(A_\Gamma \cup (\Gamma)\big) \supseteq A_\Gamma$, where $A_\Gamma = \{z \in \mathbf{R}^2 \setminus (\Gamma) : I(\Gamma, z) = 1\}$;

(ii) $F^\star = \mathrm{int}\big(B_\Gamma \cup (\Gamma)\big) \supseteq B_\Gamma$, where $B_\Gamma = \{z \in \mathbf{R}^2 \setminus (\Gamma) : I(\Gamma, z) = 0\}$;

(iii) $\partial E^* = \partial F^* = \partial A_\Gamma \cap \partial B_\Gamma \subseteq (\Gamma)$.

PROOF: It follows immediately from the definition that E^* and F^* are open. For any h, by (3.1) and by Lemma 3.1 ∂E_h admits an oriented parametrization Δ_h of class $H^{2,p}$ and the number m_h of curves of the system Δ_h is bounded uniformly with respect to h. Hence, for a subsequence (still denoted by $\{\Delta_h\}_h$), there exists an integer m such that $\Delta_h = \{\gamma_h^1, \ldots, \gamma_h^m\}$ for any h. In order to apply compactness arguments, we shall transform the sequence $\{\Delta_h\}_h$, which is not necessarily bounded, (see Figure 3.1), into a sequence $\{\Gamma_h\}_h$ of oriented parametrizations whose traces are contained in a bounded set independent of h.

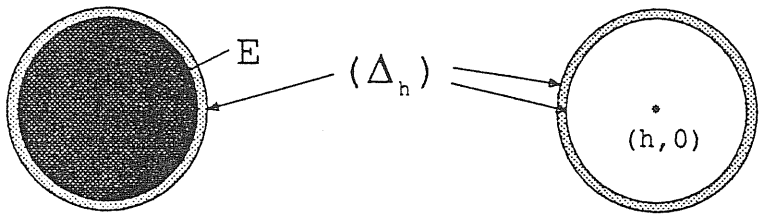

Fig. 3.1: The sequence $\{\Delta_h\}_h$ is not bounded: two connected components of (Δ_h) approaches infinity as $h \to +\infty$ with lenght and curvature uniformly bounded with respect to h.

Let us consider first the sequence of curves $\{\gamma_h^1\}_h$. If the sequence $a_h^1 = \sup_{t\in[0,1]} |\gamma_h^1(t)|$ converges to $+\infty$ as $h \to +\infty$, then we eliminate γ_h^1, i.e., we replace Δ_h by the system $\Delta_h^1 = \{\gamma_h^2, \ldots, \gamma_h^m\}$. Note that, as $l(\gamma_h^1)$ is uniformly bounded with respect to h, the behaviour of $\{a_h^1\}_h$ gives that, for any $r > 0$, there exists $h_r \in \mathbf{N}$ such that $I(\gamma_h^1, z) = 0$ (hence $I(\Delta_h^1, z) = I(\Delta_h, z)$) for any $h \geq h_r$ and for any $z \in B_r(0) \setminus (\Delta_h)$. If $\{a_h^1\}_h$ does not tend to $+\infty$, there exists a subsequence, still denoted by $\{\gamma_h^1\}_h$, such that the traces (γ_h^1) are bounded uniformly with respect to h. In this case we define $\Delta_h^1 = \Delta_h$. Starting from $\{\Delta_h^1\}_h$, we repeat the same procedure for $\{\gamma_h^2\}_h$, obtaining a new sequence of systems of curves $\{\Delta_h^2\}_h$. After m steps, we end up with a sequence of systems $\{\Delta_h^m\}_h$, which we shall denote by $\{\Gamma_h\}_h$. By construction, for every h, Γ_h is a disjoint system of curves of class $H^{2,p}$, and for every $r > 0$ there exists $h_r \in \mathbf{N}$ such that

$$I(\Gamma_h, z) = I(\Delta_h, z) \tag{3.8}$$

for any $h \geq h_r$ and for any $z \in B_r(0) \setminus (\Delta_h)$. It is clear also by construction that the traces (Γ_h) are bounded uniformly with respect to h, i.e., there exists $R > 0$ such that

$$(\Gamma_h) \subseteq B_R(0) \qquad \text{for any } h. \tag{3.9}$$

From (3.8) and (3.9) it follows easily that, for h large enough, for any $z \in \mathbf{R}^2 \setminus (\Gamma_h)$ we have $I(\Gamma_h, z) \in \{0, 1\}$. Let us define $A_h = \{z \in \mathbf{R}^2 \setminus (\Gamma_h) : I(\Gamma_h, z) = 1\}$. Since Γ_h is a

disjoint system of class $H^{2,p}$, A_h is a bounded open set and $\partial A_h = (\Gamma_h)$. Using Theorem 3.1, there exists a subsequence $\{\Gamma_h\}_h$ which converges weakly in $H^{2,p}$ to a limit system Γ of curves of class $H^{2,p}$. By (3.9) we have $(\Gamma) \subseteq \overline{B_R(0)}$. By Remark 3.1, $I(\Gamma, z) \in \{0, 1\}$ for any $z \in \mathbf{R}^2 \setminus (\Gamma)$. Let A_Γ be the open set defined in (i). It is clear that $A_\Gamma \subseteq B_R(0)$. Since $\chi_{A_h}(z) = I(\Gamma_h, z)$ for any $z \in \mathbf{R}^2 \setminus (\Gamma_h)$, and $\chi_{A_\Gamma}(z) = I(\Gamma, z)$ for any $z \in \mathbf{R}^2 \setminus (\Gamma)$, by the continuity property of the index and by the dominated convergence theorem we have that $A_h \to A_\Gamma$ in $L^1(\mathbf{R}^2)$ as $h \to +\infty$. Let us prove that

$$|E \triangle A_\Gamma| = 0. \tag{3.10}$$

By (3.8) and (3.9), for any $r \geq R$, we have that $A_h = E_h \cap B_r(0)$ for h large enough; passing to the limit as $h \to +\infty$, we obtain that $|A_\Gamma \triangle (E \cap B_r(0))| = 0$. Since r is arbitrary, we get (3.10).

Let us prove that $|E \triangle E^*| = 0$. By (3.10), it is enough to prove that

$$|A_\Gamma \triangle E^*| = 0. \tag{3.11}$$

Since A_Γ is open and $|E \triangle A_\Gamma| = 0$, for any $z \in A_\Gamma$ there exists $r > 0$ such that $B_r(z) \subseteq A_\Gamma$, so that $|B_r(z) \setminus E| = |B_r(z) \setminus A_\Gamma| = 0$, hence $z \in E^*$. Therefore

$$A_\Gamma \subseteq E^*. \tag{3.12}$$

To prove (3.11), being $|(\Gamma)| = 0$, it is sufficient to show that

$$E^* \subseteq A_\Gamma \cup (\Gamma). \tag{3.13}$$

If $z \notin A_\Gamma \cup (\Gamma)$, then $I(\Gamma, z) = 0$. Since the index is locally constant, there exists $r > 0$ such that $I(\Gamma, w) = 0$ for any $w \in B_r(z)$. As $|B_r(z) \cap E| = |B_r(z) \cap A_\Gamma| = 0$, we have $z \notin E^*$. This proves (3.13), so we can conclude that $|E \triangle E^*| = 0$. Moreover (3.13) shows that E^* is bounded.

Let us prove (i). The inclusion $\mathrm{int}(A_\Gamma \cup (\Gamma)) \supseteq A_\Gamma$ follows from the fact that A_Γ is open. The inclusion $E^* \subseteq \mathrm{int}\big(A_\Gamma \cup (\Gamma)\big)$ follows from (3.13) and from the fact that E^* is open. To prove the opposite inclusion, let $z \in \mathrm{int}\big(A_\Gamma \cup (\Gamma)\big)$. Then there exists $r > 0$ such that $B_r(z) \subseteq A_\Gamma \cup (\Gamma)$. Hence $|B_r(z) \setminus E| = |B_r(z) \setminus A_\Gamma| = 0$ by (3.10), and, therefore, $z \in E^*$. This shows that $E^* = \mathrm{int}(A_\Gamma \cup (\Gamma))$.

Let us prove (ii). The inclusion $\mathrm{int}(B_\Gamma \cup (\Gamma)) \supseteq B_\Gamma$ follows from the fact that B_Γ is open. Since E^* and F^* are disjoint open sets, we have $F^* \subseteq \mathrm{int}(\mathbf{R}^2 \setminus E^*)$. As $B_\Gamma \cup (\Gamma) = \mathbf{R}^2 \setminus A_\Gamma$, by (3.12) we have $\mathbf{R}^2 \setminus E^* \subseteq B_\Gamma \cup (\Gamma)$, hence

$$F^* \subseteq \mathrm{int}(\mathbf{R}^2 \setminus E^*) \subseteq \mathrm{int}(B_\Gamma \cup (\Gamma)). \tag{3.14}$$

To prove the opposite inclusion, let $z \in \mathrm{int}(B_\Gamma \cup (\Gamma))$. Then there exists $r > 0$ such that $B_r(z) \subseteq B_\Gamma \cup (\Gamma)$, hence $|B_r(z) \cap E| = |B_r(z) \cap A_\Gamma| = 0$ (see (3.10)), and, therefore, $z \in F^*$. This shows that $\mathrm{int}(B_\Gamma \cup (\Gamma)) \subseteq F^*$, which, together with (3.14), gives

$$F^* = \mathrm{int}(\mathbf{R}^2 \setminus E^*) = \mathrm{int}(B_\Gamma \cup (\Gamma)), \tag{3.15}$$

and concludes the proof of (ii).

Since E^* and F^* are disjoint open sets, we have $E^* \subseteq \mathrm{int}(\mathbf{R}^2 \setminus F^*)$. As $A_\Gamma \cup (\Gamma) = \mathbf{R}^2 \setminus B_\Gamma$, by (ii) we have $\mathbf{R}^2 \setminus F^* \subseteq \mathbf{R}^2 \setminus B_\Gamma = A_\Gamma \cup (\Gamma)$, hence $E^* \subseteq \mathrm{int}(\mathbf{R}^2 \setminus F^*) \subseteq \mathrm{int}(A_\Gamma \cup (\Gamma))$. By (i) we conclude that

$$E^* = \mathrm{int}(\mathbf{R}^2 \setminus F^*), \tag{3.16}$$

which, together with (3.15), gives

$$\partial E^* = \mathbf{R}^2 \setminus (E^* \cup \mathrm{int}(\mathbf{R}^2 \setminus E^*)) = \mathbf{R}^2 \setminus (E^* \cup F^*) =$$

$$\mathbf{R}^2 \setminus (\mathrm{int}(\mathbf{R}^2 \setminus F^*) \cup F^*) = \partial F^*$$

and proves (3.7).

We will now prove (iii). Since A_Γ and B_Γ are disjoint open sets, we have $\partial A_\Gamma \cup \partial B_\Gamma \subseteq \mathbf{R}^2 \setminus (A_\Gamma \cup B_\Gamma) = (\Gamma)$. Let us prove that $\partial E^* \subseteq \overline{A}_\Gamma$. For every $z \in \partial E^*$ and for every neighbourhood U of z we select $w \in U \cap E^* = U \cap \mathrm{int}(A_\Gamma \cup (\Gamma))$. Then there exists $r > 0$ such that $B_r(w) \subseteq U$ and $B_r(w) \subseteq A_\Gamma \cup (\Gamma)$. As (Γ) has Lebesgue measure zero, we have $B_r(w) \cap A_\Gamma \neq \emptyset$, hence $U \cap A_\Gamma \neq \emptyset$. This implies that $z \in \overline{A}_\Gamma$.

Let us prove that $\partial E^* \subseteq \overline{B}_\Gamma$. For every $z \in \partial E^*$ and for every neighbourhoof U of z we select $w \in U \setminus E^* = U \setminus \mathrm{int}(A_\Gamma \cup (\Gamma))$. Then there exists $r > 0$ such that $B_r(w) \subseteq U$ and $B_r(w) \cap B_\Gamma = B_r(w) \setminus (A_\Gamma \cup (\Gamma)) \neq \emptyset$. This shows that $U \cap B_\Gamma \neq \emptyset$, hence $z \in \overline{B}_\Gamma$.

Therefore $\partial E^* \subseteq \overline{A}_\Gamma \cap \overline{B}_\Gamma$. Since A_Γ and B_Γ are disjoint open sets, we have $\overline{A}_\Gamma \cap \overline{B}_\Gamma = \partial A_\Gamma \cap \partial B_\Gamma$, hence $\partial E^* \subseteq \partial A_\Gamma \cap \partial B_\Gamma$.

Let us prove that $\partial A_\Gamma \cap \partial B_\Gamma \subseteq \partial E^*$. By (i) we have $\partial A_\Gamma \subseteq \overline{A}_\Gamma \subseteq \overline{E^*}$, by (ii) we have $\partial B_\Gamma \subseteq \overline{B}_\Gamma \subseteq \overline{F^*}$, and by (3.16) we have $E^* = \mathbf{R}^2 \setminus \overline{F^*}$, hence $\overline{F^*} = \mathbf{R}^2 \setminus E^*$. It follows that

$$\partial A_\Gamma \cap \partial B_\Gamma \subseteq \overline{F^*} \cap \overline{E^*} = \overline{E^*} \setminus E^* = \partial E^*$$

hence $\partial E^* = \partial A_\Gamma \cap \partial B_\Gamma$. From (3.7) we obtain $\partial E^* = \partial F^*$, which concludes the proof of (iii). □

LEMMA 3.4. *Let $E \subseteq \mathbf{R}^2$ be a bounded open set of class $H^{2,p}$, and let Γ be a system of curves of class $H^{2,p}$ such that $\partial E \subseteq (\Gamma)$. Then*

$$\int_{\partial E} |\kappa(z)|^p \, d\mathcal{H}^1(z) \leq \|\kappa(\Gamma)\|^p_{L^p}. \tag{3.17}$$

Hence

$$\int_{\partial E} [1 + |\kappa(z)|^p] \, d\mathcal{H}^1(z) \leq \mathcal{F}(\Gamma)$$

(see Definition 2.2).

PROOF: Let $\Gamma = \{\gamma^1, \ldots, \gamma^m\}$. For any measurable set $T \subseteq \partial E$ and any $i = 1, \ldots, m$ let

$$V_i = \{s \in [0, l(\gamma^i)] : \gamma^i(s) \in T\},$$

$$T_i = \{z \in \mathbf{R}^2 : z = \gamma^i(s), s \in V_i\} = (\gamma^i) \cap T.$$

Since $\partial E \subseteq (\Gamma)$, we have $T = \bigcup_{i=1}^m T_i$. To prove the thesis it will be enough to show that

$$\int_{T_i} |\kappa(z)|^p \, d\mathcal{H}^1(z) \leq \int_{V_i} |\tilde{\gamma}^i(s)|^p \, ds \quad \text{for any } i = 1, \ldots, m, \tag{3.18}$$

where T varies over a suitable finite measurable partition $\mathcal{P}$ of ∂E. In fact, from (3.18) it follows that

$$\int_T |\kappa(z)|^p \, d\mathcal{H}^1(z) \leq \sum_{i=1}^m \int_{T_i} |\kappa(z)|^p \, d\mathcal{H}^1(z) \leq \sum_{i=1}^m \int_{V_i} |\tilde{\gamma}^i(s)|^p \, ds,$$

hence

$$\int_{\partial E} |\kappa(z)|^p \, d\mathcal{H}^1(z) = \sum_{T \in \mathcal{P}} \int_{T \cap \partial E} |\kappa(z)|^p \, d\mathcal{H}^1(z) \leq \sum_{i=1}^m \int_0^{l(\gamma^i)} |\tilde{\gamma}^i(s)|^p \, ds = \|\kappa(\Gamma)\|_{L^p}^p,$$

that proves the assertion. We choose the finite partition $\mathcal{P}$ of ∂E as follows. Any element of $\mathcal{P}$ must be contained in a rectangle in which ∂E is a cartesian graph. This means that for any $T \in \mathcal{P}$ we can choose a suitable orthogonal coordinate system, two bounded open intervals I, J, and a function $f : I \to J$ of class $H^{2,p}$ such that

$$(I \times J) \cap E = \{(x, y) \in I \times J : y < f(x)\},$$

$$(I \times J) \cap \partial E = \{(x, f(x)) : x \in I\},$$

$$T \subseteq (I \times J) \cap \partial E.$$

Let us fix $i \in \{1, \ldots, m\}$ and let $\gamma^i(s) = (\gamma_1^i(s), \gamma_2^i(s))$. For any $s \in V_i$ we have $\gamma_2^i(s) = f(\gamma_1^i(s))$. Since the functions γ_2^i and $f(\gamma_1^i)$ are of class $H^{2,p}$ on an open set containing V_i and coincide on V_i, we have [11]

$$\dot{\gamma}_2^i(s) = \frac{d}{ds} f(\gamma_1^i(s)) \ \text{ on } V_i, \qquad \tilde{\gamma}_2^i(s) = \frac{d^2}{ds^2} f(\gamma_1^i(s)) \ \text{ for a.e. } s \in V_i.$$

Then, since $|\dot\gamma^i(s)| = 1$, we obtain

$$\dot\gamma_1^i(s) = \frac{\pm 1}{\sqrt{1+(f'(\gamma_1^i(s)))^2}}, \qquad \dot\gamma_2^i(s) = \frac{\pm f'(\gamma_1^i(s))}{\sqrt{1+(f'(\gamma_1^i(s)))^2}} \quad \text{for any } s \in V_i,$$

and

$$\ddot\gamma_1^i(s) = \frac{-f'(\gamma_1^i(s))f''(\gamma_1^i(s))}{(1+(f'(\gamma_1^i(s)))^2)^2}, \qquad \ddot\gamma_2^i(s) = \frac{f''(\gamma_1^i(s))}{(1+(f'(\gamma_1^i(s)))^2)^2} \quad \text{for a.e. } s \in V_i.$$

It follows that

$$|\ddot\gamma^i(s)| = \frac{|f''(\gamma_1^i(s))|}{(1+(f'(\gamma_1^i(s)))^2)^{\frac{3}{2}}} = |\kappa(\gamma^i(s))| \quad \text{for a.e. } s \in V_i.$$

Hence, using (2.1),

$$\int_{V_i} |\ddot\gamma^i(s)|^p \, ds = \int_{V_i} |\kappa(\gamma^i(s))|^p \, ds \geq \int_{T_i} |\kappa(z)|^p \, d\mathcal{H}^1(z),$$

that is (3.17). This concludes the proof. □

THEOREM 3.2. *The functional $\mathcal{F}(\cdot)$ is $L^1(\mathbf{R}^2)$-lower semicontinuous on the class of all bounded open subsets of $\mathbf{R}^2$ of class C^2, i.e., given a sequence $\{E_h\}_h$ of bounded open sets of class C^2 converging in $L^1(\mathbf{R}^2)$ to a bounded open set E of class C^2, then*

$$\int_{\partial E} (1+|\kappa(z)|^p) \, d\mathcal{H}^1(z) \leq \liminf_{h\to+\infty} \int_{\partial E_h} (1+|\kappa_h(z)|^p) \, d\mathcal{H}^1(z), \tag{3.19}$$

where κ and κ_h are the curvatures of ∂E and of ∂E_h, respectively.

PROOF: Let $\{E_h\}_h$ be a sequence of bounded open sets of class C^2 such that $E_h \to E$ in $L^1(\mathbf{R}^2)$ as $h \to +\infty$. We can suppose that the right hand side in (3.19) is finite, otherwise the result is trivial. Let $\{E_{hk}\}_k$ be a subsequence of $\{E_h\}_h$ with the property that

$$\lim_{k\to+\infty} \mathcal{F}(E_{hk}) = \liminf_{h\to+\infty} \mathcal{F}(E_h) < +\infty.$$

For simplicity, this subsequence (and any further subsequence) will be denoted by $\{E_k\}_k$. For any k, ∂E_k has a finite number of connected components. Let Δ_k be an oriented parametrization of ∂E_k. Since $\{E_k\}_k$ satisfies (2.5), the sequence $\{\Delta_k\}_k$ satisfies (3.1). As in the proof on Theorem 3.1, we can suppose that $\Delta_k = \{\gamma_k^1, \dots, \gamma_k^m\}$ with m independent of k. Let $\{\Gamma_k\}_k = \{\gamma_k^{i_1}, \dots, \gamma_k^{i_n}\}$ be the equibounded sequence and let Γ be the limit system of curves of class $H^{2,p}$ constructed in the proof of Lemma 3.3. Then, by construction, $n \leq m$, $\Gamma = \{\gamma^{i_1}, \dots, \gamma^{i_n}\}$, and for any $j = 1, \dots, n$ we have that $\gamma_k^{i_j} \rightharpoonup \gamma^{i_j}$ weakly in

$H^{2,p}$ as $k \to +\infty$. Using (iii) of Lemma 3.3 and the weak lower semicontinuity of the L^p norm, we have

$$\liminf_{h\to+\infty} \mathcal{F}(E_h) = \lim_{k\to+\infty} \mathcal{F}(E_k) \geq \lim_{k\to+\infty} \sum_{j=1}^{n} l(\gamma_k^{i_j}) + \liminf_{k\to+\infty} \sum_{j=1}^{n} l(\gamma_k^{i_j})^{1-2p} \int_0^1 |\frac{d^2\gamma_k^{i_j}}{dt^2}|^p \, dt$$

$$\geq \sum_{j=1}^{n} l(\gamma^{i_j}) + \sum_{j=1}^{n} l(\gamma^{i_j})^{1-2p} \liminf_{k\to+\infty} \int_0^1 |\frac{d^2\gamma_k^{i_j}}{dt^2}|^p dt$$

$$\geq l(\Gamma) + \sum_{j=1}^{n} l(\gamma^{i_j})^{1-2p} \int_0^1 |\frac{d^2\gamma^{i_j}}{dt^2}|^p dt = l(\Gamma) + \sum_{j=1}^{n} \int_0^{l(\gamma^{i_j})} |\tilde{\gamma}^{i_j}(s)|^p \, ds$$

$$= l(\Gamma) + \|\kappa(\Gamma)\|_{L^p}^p.$$

Since $\|\kappa(\Gamma)\|_{L^p}^p \geq \int_{\partial E} |\kappa(z)|^p \, d\mathcal{H}^1(z)$ by Lemma 3.4 and $l(\Gamma) \geq \mathcal{H}^1(\partial E)$ by (2.1), we conclude that

$$\liminf_{h\to+\infty} \mathcal{F}(E_h) \geq \int_{\partial E} [1 + |\kappa(z)|^p] \, d\mathcal{H}^1(z) = \mathcal{F}(E),$$

which is the assertion of the theorem. □

The following result generalizes Theorem 3.2.

COROLLARY 3.2. *Let $E \subseteq \mathbf{R}^2$ be a bounded open set of class $H^{2,p}$. Then*

$$\overline{\mathcal{F}}(E) = \int_{\partial E} [1 + |\kappa(z)|^p] \, d\mathcal{H}^1(z). \tag{3.20}$$

In particular, $\overline{\mathcal{F}}(E) < +\infty$.

PROOF: Theorem 3.2 holds with the same proof if E is of class $H^{2,p}$, hence, passing to the infimum with respect to the approximating sequence $\{E_h\}_h$ in (3.19), we infer that

$$\int_{\partial E} [1 + |\kappa(z)|^p] \, d\mathcal{H}^1(z) \leq \overline{\mathcal{F}}(E).$$

Let us prove the opposite inequality. Proposition 3.1 implies that there exists an oriented parametrization $\Gamma = \{\gamma^1, \ldots, \gamma^m\}$ of ∂E of class $H^{2,p}$. Hence in particular $\partial E = \bigcup_{i=1}^m (\gamma^i)$, where $\gamma^i : [0,1] \to \mathbf{R}^2$ are closed simple regular disjoint curves of class $H^{2,p}$. For any $i = 1, \ldots, m$, let us consider a sequence $\{\gamma_h^i\}_h$ of curves of class $C^\infty([0,1])$ such that $\gamma_h^i \to \gamma^i$ strongly in $H^{2,p}$ as $h \to +\infty$. It follows that, for h large enough, the approximating system $\Gamma_h = \{\gamma_h^1, \ldots, \gamma_h^m\}$ is a disjoint system of curves. Moreover, the curves γ_h^i are simple for h large enough. If not, there exist a subsequence, still denoted by $\{\gamma_h^i\}_h$, and two sequences $\{s_h\}_h$, $\{t_h\}_h$ of points, with $0 \leq s_h < t_h < 1$, such that $\gamma_h^i(s_h) = \gamma_h^i(t_h)$. By compactness, we may also assume that $s_h \to s$ and $t_h \to t$. Then $\gamma^i(s) = \gamma^i(t)$. If $t < 1$, this implies

$s = t$. By the mean value theorem, there exist points ξ_h and η_h between s_h and t_h such that the first component of the derivative of γ_h^i vanishes on ξ_h and the second component vanishes on η_h. As $\xi_h \to s = t$ and $\eta_h \to s = t$, we conclude that the derivative of γ vanishes at s, which contradicts our assumption on γ (see Definition 2.1). If $t = 1$, then either $s = 1$ or $s = 0$. The former case leads to the same contradiction obtained before. The latter case can be treated with obvious modifications. Hence the curves γ_h^i are simple for h large enough.

By construction, the traces (Γ_h) are equibounded uniformly with respect to h. For any h, let us define $E_h = \{z \in \mathbf{R}^2 \setminus (\Gamma_h) : I(\Gamma_h, z) = 1\}$. Then $\partial E_h = \bigcup_{i=1}^m (\gamma_h^i)$, $\{E_h\}_h$ converges to $\{z \in \mathbf{R}^2 \setminus (\Gamma) : I(\Gamma, z) = 1\} = E$ in $L^1(\mathbf{R}^2)$ as $h \to +\infty$, and $\int_{\partial E_h} [1 + |\kappa_h(z)|^p] \, d\mathcal{H}^1(z) = \sum_{i=1}^m \int [1 + |\tilde{\gamma}_h^i(s)|^p] \, ds$. It follows that

$$\overline{\mathcal{F}}(E) \leq \lim_{h \to +\infty} \sum_{i=1}^m \int_0^{l(\gamma_h^i)} [1 + |\tilde{\gamma}_h^i(s)|^p] \, ds = \int_{\partial E} [1 + |\kappa(z)|^p] \, d\mathcal{H}^1(z),$$

by construction. This concludes the proof of (3.20). □

4. Some properties of the sets E with $\overline{\mathcal{F}}(E) < +\infty$.

We recall the definition of tangent cone for an arbitrary subset of $\mathbf{R}^2$.

DEFINITION 4.1. *Whenever $C \subseteq \mathbf{R}^2$, $z \in \overline{C}$, we define the (unoriented) tangent cone of C at z, denoted by $T_C(z)$, as the set of all $v \in \mathbf{R}^2$ such that for every $\epsilon > 0$ there exist $w \in C$ and $r \in \mathbf{R}$ with $|w - z| < \epsilon$ and $|r(w - z) - v| < \epsilon$. Such vectors v will be called tangent vectors of C at z.*

By the definition, it follows that $T_C(z)$ is a closed subset of $\mathbf{R}^2$ and that, if $v \neq 0$ is an element of $T_C(z)$, then the straight line through the origin containing v is contained in $T_C(z)$.

If $T_C(z)$ is reduced to a straight line, we can write $T_C(z) = \{c\tau(z) : c \in \mathbf{R}\}$, where $\tau(z)$ is a tangent unit vector of C at z, determined up to the sign. In this case we define the normal line $N_C(z)$ of C at z as the straight line through the origin orthogonal to $T_C(z)$. We will make the following convention: once $\tau(z)$ has been chosen, $\nu(z)$ is taken in such a way that $\{\tau(z), \nu(z)\}$ is oriented as the canonical basis of $\mathbf{R}^2$. Of course, we have $N_C(z) = \{c\nu(z) : c \in \mathbf{R}\}$, where $\nu(z)$ is a unit vector orthogonal to $\tau(z)$, determined up to the sign.

REMARK 4.1. *Let C be a subset of $\mathbf{R}^2$, and let $z \in C$ be a point such that the tangent cone $T_C(z)$ is a straight line. It follows immediately from the definition and the uniqueness of the tangent line that there exists $r > 0$ such that $B_r(z) \cap N_C(z) \cap C = \{z\}$.*

Note that if Γ is a system of curves, $z \in (\Gamma)$ and $\Gamma^{-1}(z) = \{t_1, \ldots, t_k\}$, then

$$T_{(\Gamma)}(z) = \bigcup_{i=1}^k \{c \frac{d\Gamma}{dt}(t_i) : c \in \mathbf{R}\}. \tag{4.1}$$

DEFINITION 4.2. *We say that a system of curves Γ is without crossings if $\frac{d\gamma^i(t_1)}{dt}$ and $\frac{d\gamma^j(t_2)}{dt}$ are parallel, whenever $\gamma^i(t_1) = \gamma^j(t_2)$ for some $i \neq j$ and $t_1, t_2 \in [0,1]$.*

Let Γ be a system of curves without crossings, and let $z \in (\Gamma)$; note that by (4.1) the tangent cone of (Γ) at z is a straight line. Observe also that if Γ is a limit system of curves of class $H^{2,p}$ then Γ is without crossings. This follows easily from the fact that Γ is the limit in C^1 of a sequence of disjoint systems of simple curves (see (3.6)).

We want now to list some regularity properties of those sets E such that $\overline{\mathcal{F}}(E) < +\infty$. We shall identify the real projective space $\mathbf{P}^1$ with the set of all one dimensional linear subspaces of $\mathbf{R}^2$.

DEFINITION 4.3. *Let C be a subset of $\mathbf{R}^2$. We say that C has a continuous unoriented tangent if at each point $z \in C$ the tangent cone $T_C(z)$ of C at z is a straight line and the map $T_C : z \to T_C(z)$ from C into $\mathbf{P}^1$ is continuous.*

PROPOSITION 4.1. *Let Γ be a system of curves of class C^1 without crossings and let C be a closed subset of (Γ) with no isolated points. Then C has a continuous unoriented tangent.*

PROOF: It is clear that the tangent cone $T_C(z)$ of C is a straight line at each point $z \in C$. We have to prove that the map T_C is continuous. Let $z \in C$, $\Gamma^{-1}(z) = \{t_1, \dots, t_k\}$. As $\frac{d\Gamma}{dt}(t_1), \dots, \frac{d\Gamma}{dt}(t_k)$ are proportional, we have $T_C(z) = \pi(\frac{d\Gamma}{dt}(t_1)) = \cdots = \pi(\frac{d\Gamma}{dt}(t_k))$, where π denotes the canonical projection of $\mathbf{R}^2 \setminus \{0\}$ into $\mathbf{P}^1$, i.e., π associates with each vector $v \in \mathbf{R}^2 \setminus \{0\}$ the straight line through the origin containing π. Let $U \subseteq \mathbf{P}^1$ be an open neighbourhood of $T_C(z)$. The map $t \to T_C(\Gamma(t))$ is continuous because $T_C(\Gamma(t)) = \pi(\frac{d\Gamma}{dt}(t))$. Hence for any $\varepsilon > 0$, there exist positive numbers $\delta_1, \dots, \delta_k$ such that $T_C(\Gamma(t)) \in U$ if $|t - t_i| < \delta_i$, for some $i = 1, \dots, k$, so that, if $V = \bigcup_{i=1}^k \Gamma(t_i - \delta_i, t_i + \delta_i)$, then $T_C(V \cap C) \subseteq U$. Let us take δ_i so small that the intervals $]t_i - \delta_i, t_i + \delta_i[$ are pairwise disjoint; since Γ is a system of curves parametrized with constant velocity, the implicit function theorem implies that V is a neighbourhood of z in (Γ) for the induced topology from $\mathbf{R}^2$. This concludes the proof. □

THEOREM 4.1. *Let $E \subseteq \mathbf{R}^2$ be a measurable set such that $\overline{\mathcal{F}}(E) < +\infty$, and let $E^\star = \{z \in \mathbf{R}^2 : \exists r > 0 \ |B_r(z) \setminus E| = 0\}$. Then $E^\star$ satisfies the following properties:*

(i) $E^\star$ is bounded, open, and $|E \triangle E^\star| = 0$;

(ii) $\mathcal{H}^1(\partial E^\star) < +\infty$;

(iii) there exists a limit system of curves Γ of class $H^{2,p}$ such that $(\Gamma) \supseteq \partial E^\star$ and $E^\star = \mathrm{int}(A_\Gamma \cup (\Gamma))$, where $A_\Gamma = \{z \in \mathbf{R}^2 \setminus (\Gamma) : I(\Gamma, z) = 1\}$;

(iv) $\partial E^\star$ has a continuous unoriented tangent;

(v) $\overline{\mathcal{F}}(E) \geq \inf\{\mathcal{F}(\Gamma) : \Gamma \in \mathcal{A}(E)\}$, where $\mathcal{A}(E)$ is the collection of all limit systems of curves Γ of class $H^{2,p}$ satisfying (iii).

PROOF: Assertions (i), (ii), and (iii) follow from (2.5) and Lemma 3.3. Assertion (iv) follows from Proposition 4.1, and from Lemma 3.3 (iii) (note that (3.7) guarantees that ∂E^* has no isolated poins). Let us prove (v). Since $\overline{\mathcal{F}}(E) < +\infty$, there exists a sequence $\{E_h\}_h$ of bounded open sets of class C^2 satisfying (2.5) and such that $E_h \to E$ in $L^1(\mathbf{R}^2)$ as $h \to +\infty$. Using the same notation as in the proof of Theorem 3.2, for a subsequence $\{E_k\}_k$ we have, from the weak lower semicontinuity of the L^p norm,

$$\liminf_{h\to+\infty} \mathcal{F}(E_h) = \lim_{k\to+\infty} \mathcal{F}(E_k) = l(\Gamma) + \liminf_{k\to+\infty} \int_{\partial E_k} |\kappa_k(z)|^p \, d\mathcal{H}^1(z)$$

$$\geq l(\Gamma) + \| \kappa(\Gamma) \|^p_{L^p} \geq \inf\{\mathcal{F}(\Gamma) : \Gamma \in \mathcal{A}(E)\}.$$

Taking the infimum with respect to the approximating sequence $\{E_h\}_h$, we get (v). □

DEFINITION 4.4. *Let C be a subset of $\mathbf{R}^2$, let $z \in \partial C$, and suppose that the tangent cone of ∂C at z is a straight line. Let $\nu(z)$ be a normal unit vector of ∂C at z, and, for any $\varrho > 0$, let*

$$N^+_\varrho(z) = \{z + s\nu(z) : 0 < s < \varrho\}, \quad N^-_\varrho(z) = \{z - s\nu(z) : 0 < s < \varrho\}.$$

Then we define

$$\partial_{00}C = \{z \in \partial C : \exists r > 0 \; N^-_r(z) \cup N^+_r(z) \subseteq \mathbf{R}^2 \setminus \overline{C}\},$$

$$\partial_{11}C = \{z \in \partial C : \exists r > 0 \; N^-_r(z) \cup N^+_r(z) \subseteq \mathrm{int}(C)\},$$

$$\partial_{01}C = \{z \in \partial C : \exists r > 0 \; N^-_r(z) \subseteq \mathbf{R}^2 \setminus \overline{C}, \; N^+_r(z) \subseteq \mathrm{int}(C) \text{ or conversely }\}.$$

REMARK 4.2. *Let C be a subset of $\mathbf{R}^2$ such that for any $z \in \partial C$ the tangent cone of ∂C at z is a straight line. Then $\partial C = \partial_{00}C \cup \partial_{11}C \cup \partial_{01}C$. The inclusion $\partial_{00}C \cup \partial_{11}C \cup \partial_{01}C \subseteq \partial C$ is obvious. The opposite inclusion follows from Remark 4.1. In particular, if $E \subseteq \mathbf{R}^2$ is measurable and $\overline{\mathcal{F}}(E) < +\infty$, then $\partial E^* = \partial_{00}E^* \cup \partial_{11}E^* \cup \partial_{01}E^*$.*

PROPOSITION 4.2. *Let $E \subseteq \mathbf{R}^2$ be a measurable set such that $\overline{\mathcal{F}}(E) < +\infty$, and let Γ and A_Γ be as Theorem 4.1 (iii). Then the sets $\partial_{00}E^*$, $\partial_{11}E^*$, and $\partial_{01}E^*$ remain unchanged if we replace in their definition $\mathrm{int}(E^*)$ by A_Γ and $\mathbf{R}^2 \setminus E^*$ by $B_\Gamma = \{z \in \mathbf{R}^2 \setminus (\Gamma) : I(\Gamma, z) = 0\}$.*

PROOF: Let $z \in \partial E$, let $z \in \partial_{11}E$, and let $r > 0$ be such that $N^-_r(z) \cup N^+_r(z) \subseteq \mathrm{int}(E^*) = E^*$. We will show that $N^-_r(z) \cup N^+_r(z) \subseteq A_\Gamma$. As the tangent cone of (Γ) is a straight line, by Remark 4.1 there exists $r > 0$ such that $B_r(z) \cap N_{(\Gamma)}(z) \cap (\Gamma) = \{z\}$. Since $E^* = \mathrm{int}\big(A_\Gamma \cup (\Gamma)\big)$, the assertion follows. The other cases are analogous. □

DEFINITION 4.5. *Let Γ be a system of curves without crossings, let $z \in (\Gamma)$, let $\tau(z)$ be a tangent unit vector of (Γ) at z and let $\nu(z)$ be the corresponding oriented normal unit vector. If $t \in S = S_1^1 \cup \cdots \cup S_m^1$ satisfies $\Gamma(t) = z$ and $\frac{d\Gamma}{dt}(t) \cdot \tau(z) > 0$ (resp. $\frac{d\Gamma}{dt}(t) \cdot \tau(z) < 0$), then we say that Γ points to the right (resp. to the left) with respect to $\nu(z)$ at t. If there are k points $t_1, \ldots, t_k \in S$ such that $\Gamma(t_i) = z$ and Γ points to the right (resp. to the left) with respect to $\nu(z)$ at each point t_i, then we say that Γ points k times to the right (resp. to the left) with respect to $\nu(z)$ at z.*

A particular case of the next Lemma (the case in which $k = 0$, $d = 1$) can be found in [1, Lemma 9.2.5]. The general case can be proved using the same methods.

LEMMA 4.1. *Let Γ be a system of curves without crossings, let $z \in (\Gamma)$, and let $z_1, z_2 \neq z$ be points on $N_{(\Gamma)}(z)$, one on each side of z, and close enough to z. Suppose that Γ points k-times to the right, and d-times to the left with respect to $\nu(z)$ at z. Then $|I(\Gamma, z_1) - I(\Gamma, z_2)| = |k - d|$.*

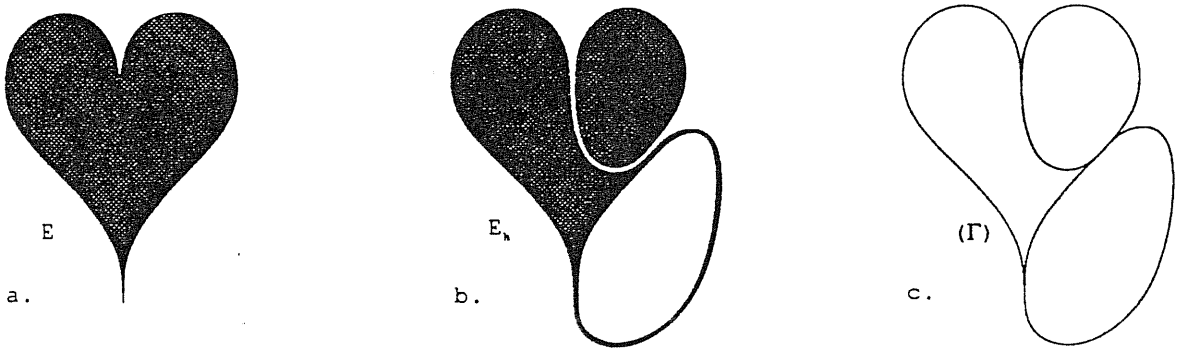

Fig. 4.1: a. A set E with $\overline{\mathcal{F}}(E) < +\infty$.
b. An example of approximating sequence $\{E_h\}_h$ satisfying (2.5).
c. The trace of the limit system Γ of curves.

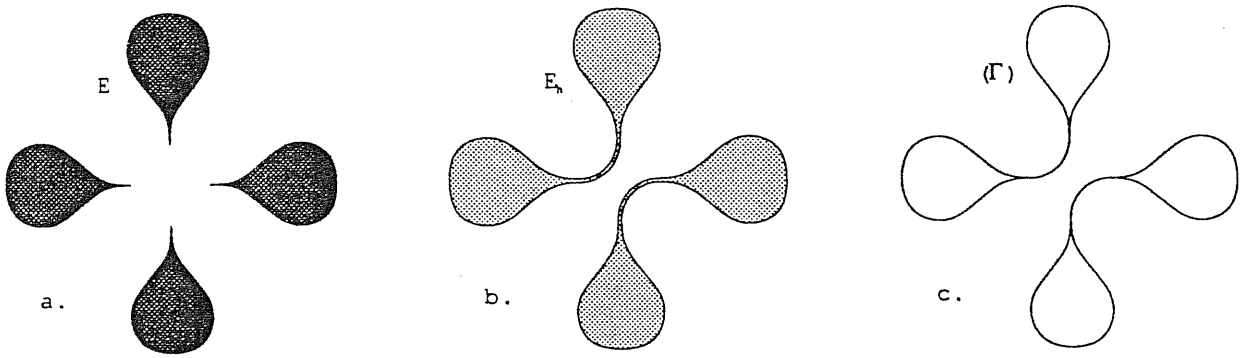

Fig. 4.2; a. A set E with $\overline{\mathcal{F}}(E) < +\infty$.
b. An example of approximating sequence $\{E_h\}_h$ satisfying (2.5).
c. The trace of the limit system Γ of curves.

Let $E \subseteq \mathbf{R}^2$ be a measurable set such that $\overline{\mathcal{F}}(E) < +\infty$, let $\Gamma = \{\gamma^1, \ldots, \gamma^m\}$ be one of the limit systems of curves satisfying condition (iii) of Theorem 4.1, and let $z \in \partial_{01} E^*$.

Using Proposition 4.1, we deduce that, if $\Gamma^{-1}(z)$ has k elements, then k is odd. More precisely, we get that $k = 2n+1$, and Γ points n-times to the right with respect to $\nu(z)$ at z, and $(n+1)$-times to the left (or viceversa). Conversely, if $z \in \partial_{00}E^*$ or $z \in \partial_{11}E^*$, then $k = 2n$, $n \geq 1$, and Γ points n-times to the right with respect to $\nu(z)$ at z, and n-times to the left.

Let $z \in (\Gamma) \setminus \partial E^*$; then from Theorem 4.1 (iii) there exists $r > 0$ such that either $I(\Gamma, w) = 0$ for any $w \in N_{(\Gamma)}(z) \cap \big(B_r(z) \setminus (\Gamma)\big)$, or $I(\Gamma, w) = 1$ for any $w \in N_{(\Gamma)}(z) \cap \big(B_r(z) \setminus (\Gamma)\big)$. This shows that (Γ) has even multiplicity at any point $z \in (\Gamma) \setminus \partial E$.

In figures 4.1 and 4.2 we show some examples of sets E with $\overline{\mathcal{F}}(E) < +\infty$, their approximation in the $L^1(\mathbf{R}^2)$ norm by a sequence $\{E_h\}_h$ satisfying (2.5), and the limit systems of curves Γ of Theorem 4.1 (iii).

DEFINITION 4.6. *Let $E \subseteq \mathbf{R}^2$ be a measurable set such that $\overline{\mathcal{F}}(E) < +\infty$, let $q \in \partial E^*$, and let $\tau(q)$ be a tangent unit vector of ∂E^* at q. We say that q is a cusp point of ∂E^* if there exists $r > 0$ such that*

$$\text{either} \quad B_r^+(q) \cap \partial E^* = \emptyset, \quad \text{or} \quad B_r^-(q) \cap \partial E^* = \emptyset,$$

where

$$B_r^+(q) = \{z \in B_r(q) : (z-q) \cdot \tau(q) > 0\}, \; B_r^-(q) = \{z \in B_r(q) : (z-q) \cdot \tau(q) < 0\}.$$

PROPOSITION 4.3. *Let $E \subseteq \mathbf{R}^2$ be a measurable set such that $\overline{\mathcal{F}}(E) < +\infty$. Then the set of the cusp points of ∂E^* is at most countable.*

PROOF: Let $\Gamma = \{\gamma^1, \ldots, \gamma^m\}$ be one of the limit systems of curves satisfying condition (iii) of Theorem 4.1, and let C be the set of the cusp points of ∂E^* belonging to (γ^1). For any $q \in C$ let $t_q \in [0,1]$ be such that $\gamma^1(t_q) = q$. Assume for simplicity that $t_q \in]0,1[$. Since $\frac{d\gamma^i(t_q)}{dt} \neq 0$, by the definition of cusp point there exists $\epsilon_q > 0$ such that either $\gamma^1(I_q^-) \cap \partial E^* = \emptyset$, or $\gamma^1(I_q^+) \cap \partial E^* = \emptyset$, where $I_q^- =]t_q - \epsilon_q, t_q[$, $I_q^+ =]t_q, t_q + \epsilon_q[$. Let $C^- = \{q \in C : \gamma^1(I_q^-) \cap \partial E^* = \emptyset\}$, $C^+ = \{q \in C : \gamma^1(I_q^+) \cap \partial E^* = \emptyset\}$. Note that in particular $\gamma^1(I_q^-) \cap C^- = \emptyset$ for any $q \in C^-$, and $\gamma^1(I_q^+) \cap C^+ = \emptyset$ for any $q \in C^+$. Hence, for any $q_1, q_2 \in C^+$, $q_1 \neq q_2$, $I_{q_2}^+ \cap I_{q_1}^+ = \emptyset$, and a similar property holds for C^-. This implies that C^+ and C^- are at most countable. □

We will see at the end of the next section that there exist measurable sets $E \subseteq \mathbf{R}^2$ such that $\overline{\mathcal{F}}(E) < +\infty$ whose boundary has an infinite number of cusp points.

DEFINITION 4.7. *Let $E \subseteq \mathbf{R}^2$ be a measurable set such that $\overline{\mathcal{F}}(E) < +\infty$, let $q \in \partial E$, and let $T(q)$ be the tangent line of ∂E at q. We say that q is a branch point of ∂E if there exist $r > 0$ and a tangent unit vector $\tau(q)$ of ∂E at q such that $B_r^+(q) \cap \partial E$ is a cartesian graph with respect to $T(q)$ and $B_r^-(q) \cap \partial E$ is not a cartesian graph with respect to $T(q)$.*

An example of branch point is shown in Figure 6.2.

The proof of the following proposition is similar to the proof of Proposition 4.3.

PROPOSITION 4.4. *Let $E \subseteq \mathbf{R}^2$ be a measurable set such that $\overline{\mathcal{F}}(E) < +\infty$. Then ∂E^* can have at most a countable number of branch points.*

5. Some critical examples.

In this section we show two pathological examples of measurable sets E such that the boundary of the set E^* defined in Theorem 4.1 is very irregular and, despite of this fact, $\overline{\mathcal{F}}(E) < +\infty$.

EXAMPLE 1. *There exists a measurable set $E \subseteq \mathbf{R}^2$ such that $\overline{\mathcal{F}}(E) < +\infty$ and $\mathcal{H}^1(\partial_{00}E^* \cup \partial_{11}E^*) > 0$.*

PROOF: Let A be a dense, open subset of $[0,1]$ such that $A = \bigcup_{k=1}^{\infty} I_k$, where $I_k =]a_k, b_k[$ are pairwise disjoint, and $\mathcal{H}^1([0,1] \setminus A) > 0$.

For any k, let $x_k = \frac{a_k+b_k}{2}$, and let $\phi : \mathbf{R} \to [0,1]$ be a function of class C^∞, such that $\phi(0) = 1$, $\phi(x) > 0$ if $|x| < 1$, $\phi(x) = 0$ if $|x| \geq 1$, $\phi(x) = \phi(-x)$ for any $x \in [0, +\infty]$.

For any $x \in \mathbf{R}$, let $\phi_k(x) = \phi(\frac{2}{b_k-a_k}(x - x_k))$. It is possible to find a sequence of positive real numbers $\{c_k\}_k$ such that the function

$$\Phi(x) = \sum_{k=1}^{\infty} c_k \phi_k(x)$$

is of class $C^\infty(\mathbf{R})$. Finally, let us define (see Figure 5.1)

$$E = \{(x,y) : x \in A, -\Phi(x) < y < \Phi(x)\}.$$

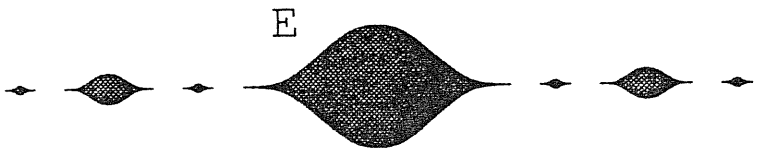

Fig. 5.1: A pathological set E with $\overline{\mathcal{F}}(E) < +\infty$.

By construction, $\partial_{00}E^* = ([0,1] \setminus A) \times \{0\}$, $\partial_{11}E^* = \emptyset$; hence $\mathcal{H}^1(\partial_{00}E^* \cup \partial_{11}E^*) = \mathcal{H}^1([0,1] \setminus A) > 0$. Note that, as $\partial_{01}E^* = (A \times \mathbf{R}) \cap \partial E^*$ and A is dense in $[0,1]$, we have that $\overline{\partial_{01}E^*} = \partial E^*$.

Let us prove that $\overline{\mathcal{F}}(E) < +\infty$. We will show that there exists a sequence $\{E_h\}_h$ of bounded open sets of class C^∞ such that $E_h \to E$ in $L^1(\mathbf{R}^2)$ as $h \to +\infty$, and $\sup_h \mathcal{F}(E_h) < +\infty$. Let γ be any simple closed curve of class C^∞ whose trace (γ) contains the segment $[0,1] \times \{0\}$, and, for any $h \in \mathbf{N}$, let

$$E_h = \{(x,y) : x \in [0,1], -\Phi(x) - \tfrac{1}{h} < y < \Phi(x) + \tfrac{1}{h}\} \bigcup \{(x,y) : \mathrm{dist}((x,y),(\gamma)) < \tfrac{1}{h}\}$$

(see Figure 5.2).

Then E_h has a boundary of class C^∞ for h large enough, and the sequence $\{E_h\}_h$ satisfies the required properties. □

EXAMPLE 2. *There exists a measurable set E such that $\overline{\mathcal{F}}(E) < +\infty$ and ∂E^* has an infinite number of cusp points.*

PROOF: Consider the family of intervals $I_k =]\frac{1}{2k-1}, \frac{1}{2k}[=]a_k, b_k[$, $k \in \mathbf{N}$, $k \geq 1$. Using the same notation of Example 1, we can construct the functions ϕ_k and Φ, and we can define E as follows:

$$E = \{(x, y) : x \in A, -\Phi(x) < y < \Phi(x)\}.$$

Then E verifies the required properties. □

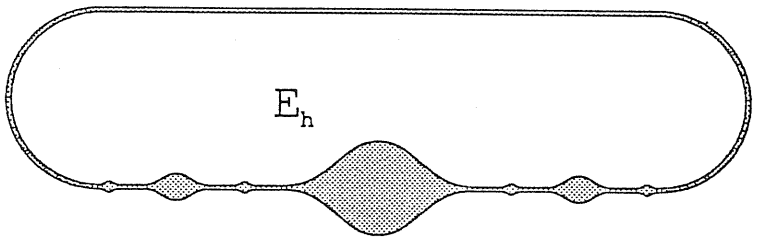

Fig. 5.2: The approximation of E in the $L^1(\mathbf{R}^2)$ norm by a sequence $\{E_h\}_h$ satisfying (2.5).

6. Systems of curves that can be approximated by boundaries of sets.

Let $E \subseteq \mathbf{R}^2$ be a bounded measurable set. The aim of this section is to find which conditions must be satisfied by ∂E in order to have that $\overline{\mathcal{F}}(E) < +\infty$. To do that, we need the following definition.

DEFINITION 6.1. *We say that a system of curves Γ satisfies the finiteness property if there exists a finite set F such that $(\Gamma) \setminus F$ is a one dimensional submanifold of $\mathbf{R}^2$ of class C^1.*

As the curves γ^i of Γ have constant velocity, it is easy to see that $(\Gamma) \setminus F$ has a finite number of connected components. By adding, if necessary, a finite number of points to F, we may assume that each connected component of $(\Gamma) \setminus F$ is diffeomorphic to an open interval.

REMARK 6.1. *Let S be the disjoint union of m unit circles $\mathbf{S}^1_1, \ldots, \mathbf{S}^1_m$, and let $\Gamma : S \to \mathbf{R}^2$ be a system of curves. By the elementary properties of one dimensional submanifolds and of their parametrizations with constant velocity, it follows that Γ satisfies the finiteness property if and only if there exists a finite number of points $t_1, \ldots, t_n$ in S such that each circle $\mathbf{S}^1_i$ contains at least two of these points, and the unique finite partition*

$\mathcal{P}$ *of* $S \setminus \{t_1, \ldots, t_n\}$ *composed of open arcs of circles having endpoints in* $\{t_1, \ldots, t_n\}$ *satisfies the following properties:*

(i) for any $I, H \in \mathcal{P}$ *either* $\Gamma(I) \cap \Gamma(H) = \emptyset$ *or* $\Gamma(I) = \Gamma(H)$;

(ii) Γ *is injective on the closure* $\overline{I}$ *of any element* I *of* $\mathcal{P}$;

(iii) $\Gamma(t_i) \notin \Gamma(H)$ *for every* $H \in \mathcal{P}$ *and for every* $i = 1, \ldots, n$.

Note that if $\Gamma = \{\gamma^1, \ldots, \gamma^m\}$ is a system of curves satisfying the finiteness property, then no curve γ^i is closed.

DEFINITION 6.2. *Let* Γ *be a system of curves satisfying the finiteness property, and let* $t_1, \ldots, t_n$ *and* $\mathcal{P}$ *be as in Remark 6.1. For every* $I \in \mathcal{P}$ *the set* $\Gamma(I)$ *will be called a branch of* Γ, *and for every* $i = 1, \ldots, n$ *the point* $\Gamma(t_i)$ *will be called a node of* Γ. *The set of all branches will be denoted by* $\mathcal{B}(\Gamma)$ *and the set of all nodes will be denoted by* $\mathcal{N}(\Gamma)$.

DEFINITION 6.3. *Let* Γ *be a system of curves satisfying the finiteness property, and let* J *be an open and connected subset of* $S \setminus \{t_1, \ldots, t_n\}$. *The set* $\Gamma(J)$ *will be called an arc of* Γ.

Let A be an arc of Γ, and let $z \in A$. Then the number of elements of the set $\Gamma^{-1}(z)$ depends only on A and does not depend on z. This number will be called the multiplicity of the arc A. In particular, since A is contained in a branch B, the multiplicity of A is the multiplicity of B.

We will consider now the problem of the approximation, in the $H^{2,p}$ norm, of a system of curves Γ by a sequence $\{\Gamma_h\}_h$ whose elements are boundaries of smooth bounded open sets (see also [19, Ch. 8.9.4] and [17] for a similar approach, in a very different context). We begin with the following Lemma.

LEMMA 6.1. *Let* $\Gamma = \{\gamma^1, \ldots, \gamma^m\} : S \to \mathbf{R}^2$ *be a system of curves of class* $H^{2,p}$ *without crossings and satisfying the finiteness property, and let* $\{t_1, \ldots, t_n\} = \Gamma^{-1}(\mathcal{N}(\Gamma))$. *Then there exist* $\varepsilon_0 > 0$ *and a sequence* $\Gamma_\varepsilon : S \to \mathbf{R}^2$, $0 < \varepsilon < \varepsilon_0$ *of systems of curves of class* $H^{2,p}$ *without crossings, satisfying the finiteness property, such that for every* $0 < \varepsilon < \varepsilon_0$

(i) $\Gamma_\varepsilon^{-1}(\mathcal{N}(\Gamma_\varepsilon)) = \{t_1^\varepsilon, \ldots, t_n^\varepsilon\}$, *and* $t_i^\varepsilon \to t_i$ *as* $\varepsilon \to 0$, *for any* $i = 1, \ldots, n$;

(ii) $\Gamma_\varepsilon \to \Gamma$ *strongly in* $H^{2,p}(S, \mathbf{R}^2)$ *as* $\varepsilon \to 0$;

(iii) $\Gamma_\varepsilon(t_i^\varepsilon) = \Gamma(t_i)$ *for any* $i = 1, \ldots, n$;

(iv) $\Gamma_\varepsilon\big|_{S \setminus \{t_1^\varepsilon, \ldots, t_n^\varepsilon\}} : S \setminus \{t_1^\varepsilon, \ldots, t_n^\varepsilon\} \to \Gamma_\varepsilon(S \setminus \{t_1^\varepsilon, \ldots, t_n^\varepsilon\})$ *is a homeomorphism.*

Lemma 6.1 shows that the system Γ can be approximated in the $H^{2,p}$ norm by a sequence of systems of curves $\{\Gamma_\varepsilon\}_\varepsilon$ of class $H^{2,p}$ satisfying the following properties: the systems Γ_ε are defined on the same parameter space S, the nodes of Γ_ε coincide with the nodes of Γ (condition (iii)), all branches of Γ_ε have multiplicity one (condition (iv)), and

$\Gamma_\varepsilon^{-1}(\mathcal{N}(\Gamma_\varepsilon))$ is a small perturbation of $\Gamma^{-1}(\mathcal{N}(\Gamma))$ (condition (i)). In particular, any branch of Γ with multiplicity k is approximated by k different branches of Γ_ε with multiplicity one.

PROOF: We will construct the approximation for a pair (q_1, q_2) of consecutive nodes of Γ, and for one of the branches joining q_1 and q_2. Then it will be sufficient to repeat this procedure for all pairs of consecutive nodes and for all the branches.

Let $q_1, q_2 \in \mathcal{N}(\Gamma)$ be two consecutive nodes, let B be a branch joining q_1 and q_2, and let $\nu : B \to \mathbf{R}^2$ be a continuous unit normal vector field on B. As $\overline{B}$ is homeomorphic to a segment, we can extend ν as a uniformly continuous vector field defined on $\mathbf{R}^2$ such that $|\nu(z)| = 1$ for any $z \in \mathbf{R}^2$. To overcome regularity problems for the approximating curves, we regularize ν as follows. We take a function $\widetilde{\nu} \in C^\infty(\mathbf{R}^2, \mathbf{R}^2)$ such that $|\widetilde{\nu}(z)| \leq 1$ and $\widetilde{\nu}(z) \cdot \nu(z) \geq \frac{1}{2}$ for any $z \in \mathbf{R}^2$. Such a function exists: in fact, defining $\nu_\eta = \nu * \varrho_\eta$, where $\{\varrho_\eta\}_\eta$ is a sequence of mollifiers, we get $\nu_\eta \in C^\infty(\mathbf{R}^2, \mathbf{R}^2)$ and $\nu_\eta \cdot \nu \to 1$ uniformly on $\mathbf{R}^2$ as $\eta \to 0$. Then it is enough to define $\widetilde{\nu} = \nu_\eta$ for η small enough.

Let $d : \mathbf{R}^2 \to [0, +\infty[$ be the function defined by

$$d(z) = \min\{\operatorname{dist}(z, B') : B' \in \mathcal{B}(\Gamma), B' \neq B\}.$$

From the definition it follows that d is continuous, $d(z) > 0$ for any $z \in B$, and $d(z) = 0$ if $z = q_1$ or $z = q_2$. In order to guarantee that the approximating sequence has the same regularity as the original system of curves Γ, and that approximations of different branches do not overlap, we introduce a function $h : \mathbf{R}^2 \to [0, +\infty[$ of class C^∞ having the following properties: $h(q_1) = h(q_2) = 0$, $0 < h(z) < d(z)$ for any $z \in B$, and all the derivatives of h at the points q_1, q_2 vanish. Let k be the multiplicity of B. Then $\Gamma^{-1}(B) = \bigcup_{k=1}^k I_j$, where $I_j \in \mathcal{P}$ for any $j = 1, \ldots, k$. We define the k approximating branches $B_\varepsilon^1, \ldots, B_\varepsilon^k$ of B as follows: for any $j = 1, \ldots, k$ and $\varepsilon > 0$ let

$$B_\varepsilon^j = \Gamma_\varepsilon(I_j), \quad \text{where} \quad \Gamma_\varepsilon(t) = \Gamma(t) + \varepsilon \tfrac{j}{k} h(\Gamma(t)) \widetilde{\nu}(\Gamma(t)), \quad t \in \overline{I_j}.$$

Let us consider the function $g : \overline{I_j} \times \mathbf{R} \to \mathbf{R}^2$ defined by

$$g(t, \varrho) = \Gamma(t) + \varrho \widetilde{\nu}(\Gamma(t)).$$

Since the jacobian of g is strictly positive at the points of the form $(t, 0)$, by the implicit function theorem it follows that g is locally invertible. As $\overline{I_j}$ is compact, there exists $\varepsilon_0 > 0$ such that the function g is injective on $\overline{I_j} \times [-\varepsilon_0, \varepsilon_0]$. Since $\Gamma_\varepsilon(t) = g(t, \varepsilon \frac{j}{k} h(\Gamma(t)))$ for any $t \in I_j$, it follows that, if $\varepsilon < \varepsilon_0$, then Γ_ε is injective on $\overline{I_j}$. This implies that $\Gamma_{\varepsilon | I_j} : I_j \to \Gamma_\varepsilon(I_j)$ is a homeomorphism. Moreover $\Gamma_\varepsilon(I_j) \cap \Gamma_{\varepsilon'}(I_j) = \emptyset$ for any $0 < \varepsilon$, $\varepsilon' < \varepsilon_0$, $\varepsilon \neq \varepsilon'$. Note that, if $\varepsilon < \frac{1}{2}$ and $B' \in \mathcal{B}(\Gamma)$, with $B' \neq B$, then

$$\operatorname{dist}(z, B) < \operatorname{dist}(z, B') \quad \forall z \in B_\varepsilon^j. \tag{6.1}$$

In fact, if $z = \Gamma_\varepsilon(t) \in B_\varepsilon^j$ and $B' \neq B$, we have $d(\Gamma(t)) \leq \operatorname{dist}(\Gamma(t), B') \leq \operatorname{dist}(\Gamma(t), z) + \operatorname{dist}(z, B') \leq \varepsilon h(\Gamma(t)) + \operatorname{dist}(z, B') < \varepsilon d(\Gamma(t)) + \operatorname{dist}(z, B')$. Hence

$$\operatorname{dist}(z, B') > d(\Gamma(t)) - \varepsilon d(\Gamma(t)). \tag{6.2}$$

Moreover, if $\varepsilon < \frac{1}{2}$, then $d(\Gamma(t)) - \varepsilon d(\Gamma(t)) > \varepsilon d(\Gamma(t))$. Since $\varepsilon d(\Gamma(t)) \geq \operatorname{dist}(z, B)$, using (6.2) we get (6.1).

Let us repeat this procedure for all pairs of consecutive nodes of Γ and for all branches. From inequality (6.1) it follows that the approximating branches corresponding to different branches of Γ are disjoint. At the end, we obtain a family $\Gamma_\varepsilon = \{\gamma_\varepsilon^1, \ldots, \gamma_\varepsilon^m\}$ of regular closed curves, which satisfy all conditions of the lemma except the condition that $\left|\frac{d\gamma_\varepsilon^i}{dt}\right|$ is constant. This last requirement can be fulfilled by taking a suitable reparametrization of Γ_ε. □

Let $q \in \mathcal{N}(\Gamma)$; we denote by $\mathcal{B}(q)$ the set of all branches of Γ having endpoint q.

DEFINITION 6.4. *Let $q \in \mathcal{N}(\Gamma)$ and $B_1, B_2 \in \mathcal{B}(q)$. We say that B_1 and B_2 lie on the same side with respect to q if there exist a tangent unit vector $\tau(q)$ of (Γ) at q and a neighbourhood U of q such that $(z-q)\cdot\tau(q) > 0$ for every $z \in (B_1 \cup B_2) \cap U$. If this does not happen, we say that B_1 and B_2 lie on opposite sides with respect to q.*

DEFINITION 6.5. *Let B, B' be two branches of Γ having multiplicity one. We say that B and B' are consecutive if there exist two consecutive arcs of circles $I, H \in \mathcal{P}$ in S such that $B = \Gamma(I)$ and $B' = \Gamma(H)$.*

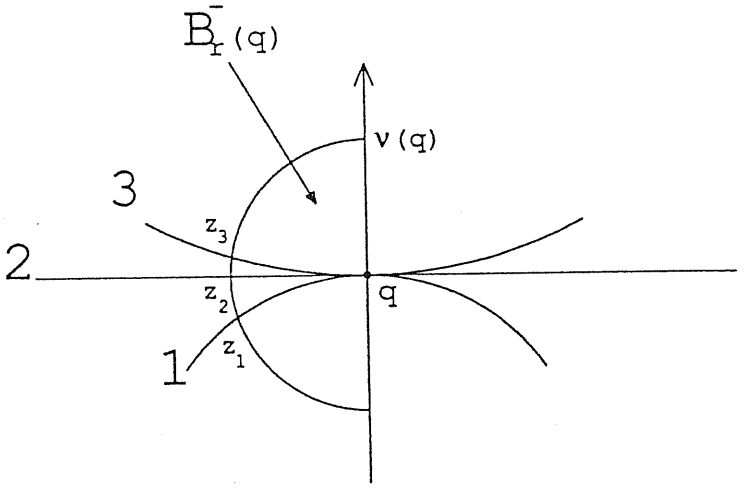

Fig. 6.1: the ordering number of the elements of $\mathcal{B}^-(q)$.

Note that two consecutive branches have a common endpoint and lie on opposite sides with respect to it. Clearly the converse is not true in general.

Suppose that any branch of Γ has multiplicity one; it is easy to prove that, given $B \in \mathcal{B}(q)$, the number of the branches of $\mathcal{B}(q)$ lying on the same side of B with respect to q is the same of the number the branches of $\mathcal{B}(q)$ lying on the opposite side.

We want now to order the branches of $\mathcal{B}(q)$ which lie on the same side of B with respect to q. Let us fix a tangent unit vector $\tau(q)$ of (Γ) at q such that $z \cdot \tau(q) > 0$ for every $z \in B$ near q, and let $\nu(q)$ be a normal unit vector of (Γ) at q (this time we do not require that $\{\tau(q), \nu(q)\}$ has a particular orientation). Let $B_r^+(q) = \{z \in B_r(q) : (z-q) \cdot \tau(q) > 0\}$. Using the implicit function theorem, one can prove that, when r is sufficiently small, the set $B_r^+(q) \cap ((\Gamma) \setminus \{q\})$ is composed of k different arcs $A_1, \ldots, A_k$ having q as an endpoint. For any $i = 1, \ldots, k$ let $z_i = (\partial B_r^+(q) \cap \partial B_r(q)) \cap \overline{A_i}$ (see Figure 6.1). We say that $z_i \prec z_j$ if and only if $z_i \cdot \nu(q) < z_j \cdot \nu(q)$, and that $A_i \prec A_j$ if and only if $z_i \prec z_j$. Since each branch B' of $\mathcal{B}(q)$ lying on the same side of B with respect to q contains one and only one arc A_i, we can associate with B' the ordering number corresponding to A_i. This number will be called the ordering number of B' with respect to q. Of course, the ordering number depends on the orientation of the normal vector $\nu(q)$.

DEFINITION 6.6. *Suppose that any branch of Γ has multiplicity one. We say that Γ satisfies the compatibility condition if consecutive arcs have the same ordering number.*

Note that, while the ordering number of a branch depends on the choice of $\nu(q)$, the compatibility condition is independent of the choice of the unit normal vectors of (Γ) at its nodes.

DEFINITION 6.7. *Let Γ be a system of curves without crossings and satisfying the finiteness property. The finite undirected graph $\mathcal{G}_\Gamma$ whose vertices are the nodes of Γ, and whose edges are the branches of Γ, counted with the corresponding multiplicity, will be called the graph associated to Γ.*

DEFINITION 6.8. *Let $\Gamma_1 : S_1 \to \mathbf{R}^2$ and $\Gamma_2 : S_2 \to \mathbf{R}^2$ be two systems of curves without crossings, satisfying the finiteness property and having all branches with multiplicity one. We say that Γ_1 is equivalent to Γ_2 if $\mathcal{G}_{\Gamma_1} = \mathcal{G}_{\Gamma_2}$.*

We now prove the following crucial result.

THEOREM 6.1. *Let $\Gamma : S \to \mathbf{R}^2$ be a system of curves without crossings, satisfying the finiteness property and having all branches with multiplicity one. Then there exists a system of curves equivalent to Γ and satisfying the compatibility condition.*

PROOF: For any $q \in \mathcal{N}(\Gamma)$ let us fix a unit normal vector $\nu(q)$ of (Γ) at q. To prove the theorem, one has to show that there exists a finite number of cyclic paths of edges of the graph $\mathcal{G}_\Gamma$ with the following properties:

(i) each edge of $\mathcal{G}_\Gamma$ appears once and only once in the family of paths;

(ii) if A and B are two edges of the same path with a common vertex q, then A and B lie on opposite sides with respect to q and have the same ordering number.

We shall construct the cyclic paths using the following algorithm. Let us choose $B_1 \in \mathcal{B}(\Gamma)$ in an arbitrary way, and let q_0 and q_1 be the vertices of B_1. Suppose that $B_1, \ldots, B_i$

and $q_0, \ldots, q_i$ have been defined. If $q_i = q_0$ and B_i lies on the opposite side of B_1 with respect to q_0 and the ordering number of B_i with respect to q_i coincides with the ordering number of B_1 with respect to q_0, then $B_1, \ldots, B_i$ is a cyclic path, and the algorithm stops. Otherwise, we define B_{i+1} as the unique branch of $\mathcal{B}(q_i)$ lying on the opposite side of B_i with respect to q_i and having the same ordering number. Next, we define q_{i+1} as the vertex of B_{i+1} different from q_i. Let us show that $B_i \neq B_j$ for any $j = 1, \ldots, i-1$. We argue by contradiction. Let $i \geq 1$ be the smallest integer such that there exists $j \in \mathbf{N}$, $1 \leq j < i$ such that $B_i = B_j$. Note that $i \geq 2$. There are two possibilities: either $q_{i-1} = q_j$ or $q_{i-1} = q_{j-1}$.

Suppose that $q_{i-1} = q_j$. Let $q = q_{i-1} = q_j$ and $B = B_i = B_j$. By the definition of the algorithm, we have that B_{i-1} and B_{j+1} lie on the opposite side of B with respect to q and have the same ordering number of B, hence they coincide. By the minimality of i, this implies that $i - 1 = j + 1$. The endpoints of B_{j+1} are $q_j = q_{i-1}$ and $q_{j+1} = q_{i-1}$, so that B_{j+1} is a loop, in a contradiction with the hypothesis.

Suppose that $q_{i-1} = q_{j-1}$, and suppose that $j > 1$. Let $q = q_{i-1} = q_{j-1}$ and $B = B_i = B_j$. Then B_{i-1} and B_{j-1} lie on the opposite side of B with respect to q and have the same ordering number of B, hence they coincide, and this contradicts the minimality of i. If $j = 1$ we have $q_{i-1} = q_0$. Let $q = q_{i-1} = q_0$ and $B = B_i = B_1$. Then B_{i-1} and B_1 lie on the opposite side of B_1 with respect to q and have the same ordering number, so that the algorithm would have stopped at the step $i - 1$, a contradiction.

In this way we get a cyclic path satisfying property (ii), and such that no edge of $\mathcal{G}_\Gamma$ appears twice in the path. Then we repeat the algorithm, starting from an edge not contained in the first path. It is clear that after a finite number of implementations of the algorithm all edges are reached, since one can prove, arguing as above, that all the cyclic paths obtained are pairwise disjoint.

For any $i = 1, \ldots, m$ let $r_i = \frac{l(\gamma^i)}{2\pi}$, let S_i be an oriented circle of radius r_i and let $S(r_1, \ldots, r_m)$ be the disjoint union of the circles S_i's. Let us reparametrize Γ on $S(r_1, \ldots, r_m)$ in such a way that $\left|\frac{d\Gamma}{ds}(s)\right| = 1$ for any $s \in S(r_1, \ldots, r_m)$. Let us still denote by Γ this parametrization, by $\mathcal{P}$ the partition of $S(r_1, \ldots, r_m)$ into arcs of circles, and by $I_1, \ldots, I_k$ the arcs of circles of $S(r_1, \ldots, r_m)$ corresponding to $B_1, \ldots, B_k$.

The family of cyclic paths obtained above corresponds to the following surgery operations on the parameter space S of Γ. Let $\Gamma = \{\gamma^1, \ldots, \gamma^m\}$ and $S = \mathbf{S}^1_1 \cup \cdots \cup \mathbf{S}^1_m$. Let $B_1, \ldots, B_k$ represent a cyclic path of edges of $\mathcal{G}_\Gamma$, and let $I_1, \ldots, I_k$ be the corresponding arcs of circles in S (recall that there exists a bijection between $\mathcal{B}(\Gamma)$ and the partition $\mathcal{P}$ of S). For any $j = 1, \ldots, k$, let q_{j-1}, q_j be the vertices of B_j, and let $I_j = (a_{j-1}, b_{j-1})$. Obviously, $q_0, q_1, \ldots, q_k = q_0$ are ordered in the cyclic path. Let us possibly change the orientation of some I_j (and let us still denote it by $I_j = (a_{j-1}, b_{j-1})$) in such a way that a_{j-1} corresponds to q_{j-1} and b_{j-1} to q_j. Next, for any $j = 1, \ldots, k$, let us glue together a_j with b_{j-1}. Note that, as $\left|\frac{d\gamma^i}{ds}\right| = 1$, $\frac{d\Gamma}{ds}(a_j) = \frac{d\Gamma}{ds}(b_{j-1})$, for any j. What we get is

an oriented closed circle. Repeating this procedure for all the cyclic paths, we obtain a new parameter space formed by a disjoint union of circles having different radii. With a reparametrization, we get the system of curves of the statement. □

We stress that the system of curves of Theorem 6.1 has the same vertices, branches, and multiplicities (in this case all equal to one) of Γ altough, in general, it is defined on a different parameter space. In particular $l(\Gamma)$ and $\|\kappa(\Gamma)\|_{L^p}$ remain unchanged when we pass to the new equivalent system of curves. Note also that the orientation of some branches of Γ can be reversed. Let $z \notin (\Gamma)$ and let α be a continuous curve connecting z with ∞ such that all the intersections between (Γ) and (α) are trasversal. Since $I(\Gamma, z)$ (mod 2) can be computed using the parity of the number of the intersections of (Γ) with (α), we also deduce that the set of points of odd index with respect to the new system of curves coincides with the set $\{z \in \mathbf{R}^2 \setminus (\Gamma) : I(\Gamma, z) \equiv 1 \pmod 2\}$.

LEMMA 6.2. *Let $\Gamma : S \to \mathbf{R}^2$ be a system of curves of class $H^{2,p}$ without crossings, satisfying the finiteness property, having all branches with multiplicity one and satisfying the compatibility condition. Then there exist $\eta_0 > 0$ and a sequence $\Gamma^\eta : S \to \mathbf{R}^2$, $0 < \eta < \eta_0$, of disjoint systems of simple curves of class C^∞ such that $\Gamma^\eta \to \Gamma$ strongly in $H^{2,p}$ as $\eta \to 0$.*

PROOF: We will construct the approximation near a fixed node q, and this procedure must be repeated for any $q \in \mathcal{N}(\Gamma)$.

Let $q \in \mathcal{N}(\Gamma)$, $\Gamma^{-1}(q) = \{t_1, \dots, t_k\}$. Without loss of generality, we shall suppose that $q = 0$ and $\nu(q) = \mathrm{e}_2$. Let $B_1^- \dots, B_k^-$ (resp. $B_1^+ \dots, B_k^+$) be the branches of Γ lying on the left (resp. on the right) of q, labelled by their ordering number with respect to 0. Let $R(q) = R = \{(x, y) \in \mathbf{R}^2 : |x| < c_1, |y| < c_2\}$ be a rectangle centered at 0, and let $R^- = \{(x, y) \in R : x < 0\}$, $R^+ = \{(x, y) \in R : x > 0\}$. If R is sufficiently small, using the implicit function theorem one can show that, for any $i = 1, \dots, k$, $R^- \cap B_i^- = A_i^-$, $R^+ \cap B_i^+ = A_i^+$, where $A_1^-, \dots, A_k^-$ and $A_1^+, \dots, A_k^+$ are cartesian graph of a function of class $H^{2,p}$ with respect to the x-axis. Hence, if $i < j$, then A_j lies above A_i. For any $j = 1, \dots, k$, let $J_i^- = \Gamma^{-1}(A_i^-)$, $J_i^+ = \Gamma^{-1}(A_i^+)$, $J_i = J_i^- \cup J_i^+ \cup \{t_i\}$. Note that, since Γ satisfies the compatibility condition, J_i is an open interval. Let $\pi : R \to]-c_1, c_1[$ be the projection of R onto $]-c_1, c_1[$. Let $\phi :]-c_1, c_1[\to [0, 1]$ be a C^∞ function with the following properties: $\phi(0) = 1$, $\phi(x) = 0$ for any $\frac{c_1}{2} \le x < c_1$, $\phi'(x) < 0$ for any $0 < x < \frac{c_1}{2}$, and $\phi(-x) = \phi(x)$ for any $x \in [0, c_1[$. For any $\eta > 0$ let us define

$$\Gamma^\eta(t) = \begin{cases} \Gamma(t) + \eta \mathrm{e}_2 \frac{i}{l} \phi(\pi(\Gamma(t))) & \text{if } t \in J_i \text{ for some } i = 1, \dots, k, \\ \Gamma(t) & \text{elsewhere on } S. \end{cases}$$

Note that, if η is sufficiently small, the functions $\Gamma_{|J_i}$ are injective on J_i and $\Gamma^\eta(J_i) \cap \Gamma^\eta(J_j) = \emptyset$ for any $i \neq j$. Repeating this procedure for any node, and taking the rectangles $R(q)$ pairwise disjoint, after a reparametrization we obtain that $\Gamma^\eta : S \to \mathbf{R}^2$ is a disjoint system of simple curves of class $H^{2,p}$ such that $\Gamma^\eta \to \Gamma$ strongly in $H^{2,p}$ as $\eta \to 0$.

Furthermore, any system Γ^η can be approximated in the $H^{2,p}$ norm by a disjoint system of simple curves of class C^∞ (see the proof of Corollary 3.2). By a diagonal argument, we get the assertion. □

Let $\Gamma : S \to \mathbf{R}^2$ be a system of curves of class $H^{2,p}$ without crossings and satisfying the finiteness property. Let

(6.3)
$$E = \{z \in \mathbf{R}^2 \setminus (\Gamma) : I(\Gamma, z) \equiv 1 \pmod 2\}, \quad F = \{z \in \mathbf{R}^2 \setminus (\Gamma) : I(\Gamma, z) \equiv 0 \pmod 2\},$$

$$E^* = \{z \in \mathbf{R}^2 : \exists r > 0 \ |B_r(z) \setminus E| = 0\}, \quad F^* = \{z \in \mathbf{R}^2 : \exists r > 0 \ |B_r(z) \setminus F| = 0\}. \tag{6.4}$$

Then E, E^*, F, and F^* are open, E^* is bounded,

$$|E \triangle E^*| = 0, \quad E^* = \operatorname{int}(\mathbf{R}^2 \setminus F^*), \quad F^* = \operatorname{int}(\mathbf{R}^2 \setminus E^*), \tag{6.5}$$

$$\partial E^* = \partial F^* = \{z \in \mathbf{R}^2 : 0 < |B_r(z) \cap E| < |B_r(z)| \ \forall r > 0\} = \partial E \cap \partial F \subseteq (\Gamma).$$

The fact that E, E^*, F, and F^* are open is trivial. To prove that $|E \triangle E^*| = 0$ it is enough to observe that $E \subseteq E^* \subseteq E \cup (\Gamma)$ (see (3.12), (3.13)). In particular, E^* is bounded. To prove that $E^* = \operatorname{int}(\mathbf{R}^2 \setminus F^*)$, $F^* = \operatorname{int}(\mathbf{R}^2 \setminus E^*)$ and $\partial E^* = \partial F^* = \{z \in \mathbf{R}^2 : 0 < |B_r(z) \cap E| < |B_r(z)| \ \forall r > 0\}$, it is sufficient to repeat the proof of (3.15), (3.16) and (3.7), respectively. Moreover, the relation $\partial E^* = \partial F^* = \partial E \cap \partial F \subseteq (\Gamma)$ can be proved as in Lemma 3.3 (iii).

THEOREM 6.2. *Let $\Gamma : S \to \mathbf{R}^2$ be a system of curves of class $H^{2,p}$ without crossings and satisfying the finiteness property. Let E, E^*, F, and F^* be defined as in (6.3), (6.4). Then*

$$\overline{\mathcal{F}}(E) = \overline{\mathcal{F}}(E^*) < +\infty, \tag{6.6}$$

hence there exists a sequence $\{E_h\}_h$ of bounded open sets of class C^∞ such that $E_h \to E$ in $L^1(\mathbf{R}^2)$ as $h \to +\infty$ and $\sup_h \mathcal{F}(E_h) < +\infty$. In addition, there exist oriented parametrizations Γ_h of ∂E_h defined on the same parameter space $\widetilde{S}$, such that $\{\Gamma_h\}_h$ converges strongly in $H^{2,p}$ to a system of curves equivalent to Γ, defined on $\widetilde{S}$, and whose trace contains ∂E^*. Finally,

$$\overline{\mathcal{F}}(E) \le \inf\{\mathcal{F}(\Delta) : \Delta \in \mathcal{Q}(E)\}, \tag{6.7}$$

where $\mathcal{Q}(E)$ is the collection of those systems of curves Δ of class $H^{2,p}$ without crossings, satisfying the finiteness property and such that $E^* = \operatorname{int}(A_\Delta \cup (\Delta))$, where $A_\Delta = \{z \in \mathbf{R}^2 \setminus (\Delta) : I(\Delta, z) \equiv 1 \pmod 2\}$.

PROOF: Let $\{t_1, \dots, t_n\} = \Gamma^{-1}(\mathcal{N}(\Gamma))$, let $\Gamma_\varepsilon : S \to \mathbf{R}^2$, $0 < \varepsilon < \varepsilon_0$ be the approximating sequence of systems of curves of Lemma 6.1, and let $\{t_1^\varepsilon, \dots, t_n^\varepsilon\} = \Gamma_\varepsilon^{-1}(\mathcal{N}(\Gamma_\varepsilon))$.

For any $0 < \varepsilon < \varepsilon_0$, there exists, by Theorem 6.1, a system curves equivalent to Γ_ε and satisfying the compatibility condition. Let us prove that these systems of curves are defined on a same parameter space. It is easy to see that the graphs associated to these systems of curves are isomorphic. If we identify isomorphic graphs, then the cyclic paths of edges constructed in the proof of Theorem 6.1 do not depend on ε, as well as the ordering numbers of any node $q \in \mathcal{N}(\Gamma_\varepsilon) = \mathcal{N}(\Gamma)$. It follows that the parameter space obtained at the end of the proof of Theorem 6.1, on which the new systems of curves are defined, does not depend on ε, and we shall denote it by $\widetilde{S}$. Repeating on $S \setminus \{t_1, \ldots, t_n\}$ the same surgery operations made on $S \setminus \{t_1^\varepsilon, \ldots, t_n^\varepsilon\}$ (recall that $t_i^\varepsilon \to t_i$ for any $i = 1, \ldots, n$ as $\varepsilon \to 0$), we get a system of curves defined on $\widetilde{S}$ and equivalent to Γ. Let us still denote this system by the symbol Γ. Then $\Gamma_\varepsilon \to \Gamma$ strongly in $H^{2,p}(\widetilde{S}, \mathbf{R}^2)$ as $\varepsilon \to 0$.

For any $\varepsilon > 0$, let $\{\Gamma_\varepsilon^\eta\}_\eta$ be the sequence of disjoint systems of simple curves of Lemma 6.2 converging to Γ_ε strongly in $H^{2,p}(\widetilde{S}, \mathbf{R}^2)$ as $\eta \to 0$. By a diagonal argument, there exists a sequence $\Gamma_h : \widetilde{S} \to \mathbf{R}^2$ of disjoint systems of simple curves of class C^∞ such that $\Gamma_h \to \Gamma$ strongly in $H^{2,p}(\widetilde{S}, \mathbf{R}^2)$ as $h \to +\infty$. For any h, let us define

$$E_h = \{z \in \mathbf{R}^2 \setminus (\Gamma_h) : I(\Gamma_h, z) \equiv 1 \pmod 2\}.$$

Note that this definition does not depend on the orientation of each circle composing the parameter space $\widetilde{S}$. By the continuity property of the index and by the dominated convergence theorem, it follows that $\{E_h\}_h$ converges to the set $\{z \in \mathbf{R}^2 \setminus (\Gamma) : I(\Gamma, z) \equiv 1 \pmod 2\}$, as $h \to +\infty$. Since this set coincides with the set of all points of $\mathbf{R}^2 \setminus (\Gamma)$ of odd index with respect to the original system of curves, by the definition of E we get that $E_h \to E$ in $L^1(\mathbf{R}^2)$ as $h \to +\infty$. Moreover, by construction, $\sup_h \mathcal{F}(E_h) < +\infty$; it follows that $\overline{\mathcal{F}}(E) < +\infty$. Since $|E \triangle E^*| = 0$ (see 6.5), we have that $\overline{\mathcal{F}}(E) = \overline{\mathcal{F}}(E^*)$, and (6.6) is proven.

In addition, since $\Gamma_h \to \Gamma$ strongly in $H^{2,p}(\widetilde{S}, \mathbf{R}^2)$ as $h \to +\infty$, it follows that

$$l(\Gamma) + \|\kappa(\Gamma)\|_{L^p}^p = \lim_{h \to +\infty} \int_{\partial E_h} [1 + |\kappa_h(z)|^p] \, d\mathcal{H}^1(z) \geq \overline{\mathcal{F}}(E).$$

Using the same arguments, we deduce that, if $\Delta \in \mathcal{Q}(E)$, then $\overline{\mathcal{F}}(E) = \overline{\mathcal{F}}(A_\Delta) \leq l(\Delta) + \|\kappa(\Delta)\|_{L^p}^p$ (recall that $|A_\Delta \triangle E| = 0$). Whence, passing to the infimum with respect to $\Delta \in \mathcal{Q}(E)$, we get (6.7), and this concludes the proof. □

Let $\mathcal{G}$ be a finite graph. We denote by $\mathcal{B}(\mathcal{G})$ the edges of $\mathcal{G}$ and by $\mathcal{N}(\mathcal{G})$ the vertices of $\mathcal{G}$. For any $q \in \mathcal{N}(\mathcal{G})$, we denote by $\mathcal{B}(q)$ the edges of $\mathcal{G}$ having endpoint q. We shall designate by $\varrho_\mathcal{G}(q)$ the number of elements of $\mathcal{B}(q)$, counted with their multiplicity. In this definition, any loop at q will be counted as a double edge.

Suppose that $\mathcal{G}$ (considered as a subset of $\mathbf{R}^2$) has a continuous unoriented tangent, and let $\nu(q)$ be a unit normal vector of $\mathcal{G}$ at q. We denote by $\mathcal{B}^-(q)$ (resp. $\mathcal{B}^+(q)$) the edges of $\mathcal{B}(q)$ lying on the left (resp. on the right) of q with respect to $\nu(q)$. We shall designate

by $\varrho_{\mathcal{G}}^-(q)$ (resp. by $\varrho_{\mathcal{G}}^+(q)$) the number of elements of $\mathcal{B}^-(q)$ (resp. of $\mathcal{B}^+(q)$) counted with their multiplicity.

DEFINITION 6.9. *Let $\mathcal{G}$ be a finite graph. We say that $\mathcal{G}$ is regular if $\varrho_{\mathcal{G}}^-(q) = \varrho_{\mathcal{G}}^+(q)$ for any $q \in \mathcal{N}(\mathcal{G})$.*

Let us observe that if Γ is a system of curves without crossings and satisfying the finiteness property, and $\mathcal{G}_\Gamma$ is the graph associated to Γ, then $\mathcal{G}_\Gamma$ is regular.

DEFINITION 6.10. *We say that a bounded open set $E \subseteq \mathbf{R}^2$ has a piecewise $H^{2,p}$ boundary if $\partial E = \bigcup_{i=1}^k (\alpha^i)$, where $\alpha^1, \dots, \alpha^k : [0,1] \to \mathbf{R}^2$ are regular simple curves of class $H^{2,p}$ such that, for any $i, j \in \{1, \dots, k\}$, $i \neq j$, $\alpha^i(]0,1[) \cap \alpha^j(]0,1[) = \emptyset$, $\alpha^i(0) \neq \alpha^i(1)$, and for any $t \in]0,1[$ there exists $r > 0$ such that $B_r(\alpha^i(t)) \setminus \partial E$ has exactly two connected components, $B_r(\alpha^i(t)) \cap E$ and $B_r(\alpha^i(t)) \setminus \overline{E}$.*

Note that, if $E \subseteq \mathbf{R}^2$ is a bounded open set with a piecewise $H^{2,p}$ boundary, then $\alpha^i(0), \alpha^i(1) \notin \alpha^j(]0,1[)$, for any $i \neq j$. If, in addition, ∂E has a continuous unoriented tangent, then, for any $t_1, t_2 \in \{0,1\}$ such that $\alpha^i(t_1) = \alpha^j(t_2)$, the vectors $\frac{d\alpha^i}{dt}(t_1)$ and $\frac{d\alpha^j}{dt}(t_2)$ are parallel.

Note also that the curves $\alpha^1, \dots, \alpha^k$ can be reordered and suitably parametrized and glued together to form a family of curves $\{\gamma^1, \dots, \gamma^m\}$, with $1 \leq m < k$, satisfying the following properties: for any $i = 1, \dots, m$, γ^i is continuous, closed, and of class $H^{2,p}$ up a finite number of points, and

$$E = \{z \in \mathbf{R}^2 \setminus \bigcup_{i=1}^m (\gamma^i) : \sum_{i=1}^m I(\gamma^i, z) \equiv 1 \pmod 2\}. \tag{6.8}$$

As a notation, for any $i = 1, \dots, k$, the sets $\alpha^i(]0,1[)$ will be called the branches of ∂E and will be denoted by $\mathcal{B}(\partial E)$, and the points $\alpha^i(0), \alpha^i(1)$ will be called the vertices of ∂E and will be denoted by $\mathcal{N}(\partial E)$. For any $q \in \mathcal{N}(\partial E)$, we denote by $\mathcal{B}(q)$ the branches of ∂E having endpoint q. Assume now that ∂E has a continuous unoriented tangent, and Let $\nu(q)$ be a unit normal vector of ∂E at q. We denote by $\mathcal{B}^-(q)$ (resp. $\mathcal{B}^+(q)$) the branches of $\mathcal{B}(q)$ lying on the left (resp. on the right) of q with respect to $\nu(q)$, and by $\varrho_{\partial E}^-(q)$ (resp. $\varrho_{\partial E}^+(q)$) the number of elements of $\mathcal{B}^-(q)$ (resp. of $\mathcal{B}^+(q)$). Note that, if $\overline{\mathcal{F}}(E) < +\infty$, then the cusp and the branch points of ∂E (see Definitions 4.6 and 4.7) are vertices. It is clear that there can be vertices in a neighbourhood of which ∂E is regular (some of them must be artificially introduced because of the condition $\alpha^i(0) \neq \alpha^i(1)$). These can be regarded as "artificial" vertices. The simplest "genuine" vertex is the *simple cusp point*. This is defined as a vertex q such that $\varrho_{\partial E}^+(q) = 2$ and $\varrho_{\partial E}^-(q) = 0$ for a suitable choice of the normal vector $\nu(q)$. Note that the simple cusp points are cusp points according to Definition 4.6.

Observe that $|\varrho^+_{\partial E}(q) - \varrho^-_{\partial E}(q)|$ is even, for any $q \in \mathcal{N}(\partial E)$. In fact, given $q \in \mathcal{N}(\partial E)$, there exists $r > 0$ such that $B_r(q)$ contains both points of E and points of $\mathbf{R}^2 \setminus E$, $\partial B_r(q)$ meets trasversally all branches of ∂E, hence the number of points of the set $\partial E \cap \partial B_r(q)$ is even. If r is sufficiently small, this number coincides with $\varrho^+_{\partial E}(q) + \varrho^-_{\partial E}(q)$, which has the same parity of $|\varrho^+_{\partial E}(q) - \varrho^-_{\partial E}(q)|$.

Intuitively, more complex vertices can be seen as the result of the collapse of a finite number of simple cusp points, with the possibile addition of some artificial vertices. We now introduce the balanced multiplicity $\omega_{\partial E}(q)$ of a vertex q, whose intuitive meaning is the difference between the number of simple cusps having vertex at q and lying on opposite sides of q with respect to $\nu(q)$. The formal definition is

$$\omega_{\partial E}(q) = \frac{|\varrho^+_{\partial E}(q) - \varrho^-_{\partial E}(q)|}{2}.$$

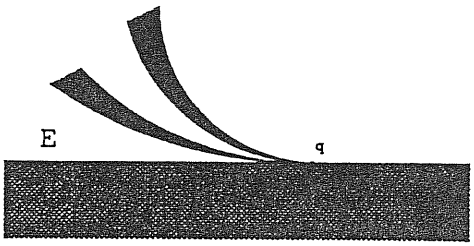

Fig. 6.2: $\omega_{\partial E}(q) = 2$.

Note that if q is a simple cusp point of ∂E, then $\omega_{\partial E}(q) = 1$, and if q is a vertex of ∂E in a neighbourhood of which ∂E is regular, then $\omega_{\partial E}(q) = 0$.

DEFINITION 6.11. *Let $E \subseteq \mathbf{R}^2$ be a bounded open set with a piecewise $H^{2,p}$ boundary ∂E having a continuous unoriented tangent. The undirected graph $\mathcal{G}_{\partial E}$ whose vertices are the vertices of ∂E, and whose edges are the branches of ∂E with multiplicity one, will be called the graph associated to ∂E.*

Note that, in general, $\mathcal{G}_{\partial E}$ is not a regular graph.

THEOREM 6.3. *Let $E \subseteq \mathbf{R}^2$ be a bounded open set with a piecewise $H^{2,p}$ boundary ∂E having a continuous unoriented tangent. Then*

$$\sum_{q \in \mathcal{N}(\partial E)} \omega_{\partial E}(q) \quad \text{is even} \quad \Longrightarrow \quad \overline{\mathcal{F}}(E) < +\infty.$$

PROOF: We shall define a regular graph $\mathcal{G}$ whose set of edges contains $\mathcal{B}(\partial E)$ and such that each edge of $\mathcal{B}(\mathcal{G}) \setminus \mathcal{B}(\partial E)$ has even multiplicity.

For any vertex q of ∂E, let us fix a normal unit vector $\nu(q)$ of ∂E at q in such a way that $\varrho^+_{\partial E}(q) \geq \varrho^-_{\partial E}(q)$. Denote by $\{q_1, \ldots, q_m\}$ the set of all vertices of ∂E with $\omega_{\partial E}(q) \geq 2$. We construct m graphs $\mathcal{G}'_1, \ldots, \mathcal{G}'_m$ inductively. Let $\mathcal{G}'_1 = \mathcal{G}_{\partial E}$, and suppose that the graph $\mathcal{G}'_{i-1}$ has been defined. If $i \leq m$, we define the graph $\mathcal{G}'_i$ as follows. Let $\gamma^i : [0,1] \to \mathbf{R}^2$ be a smooth curve lying on the left of q with respect to $\nu(q)$, such that $\gamma^i(0) = \gamma^i(1) = q_i$, $\frac{d\gamma^i}{dt}(0) = -\frac{d\gamma^i}{dt}(1)$, $(\gamma^i) \cap \partial E = \{q_i\}$, and such that the tangent lines of ∂E and (γ^i) at q_i coincide.

Let us associate with (γ^i) a multiplicity defined by $\omega_{\partial E}(q_i)$, if $\omega_{\partial E}(q_i)$ is even, and by $\omega_{\partial E}(q_i) - 1$, if $\omega_{\partial E}(q_i)$ is odd. To eliminate loops, add a new vertex $z_i \neq q_i$ on (γ^i). Let $\mathcal{G}'_i$ be the graph whose vertices are the vertices of $\mathcal{G}'_{i-1}$ together with z_i, and whose edges are the edges of $\mathcal{G}'_{i-1}$ together with the two arcs on the curve (γ^i) determined by the point z_i, with the even multiplicity previously defined. Clearly, $\varrho^+_{\mathcal{G}'_i}(z_i) = \varrho^-_{\mathcal{G}'_i}(z_i)$.

At the end of this algorithm, we get a graph $\mathcal{G}' = \mathcal{G}'_m$ whose set of edges contains $\mathcal{B}(\partial E)$, such that each edge of $\mathcal{B}(\mathcal{G}') \setminus \mathcal{B}(\partial E)$ has even multiplicity, and such that

$$\varrho^+_{\mathcal{G}'}(q) - \varrho^-_{\mathcal{G}'}(q) = \begin{cases} 0 & \text{if } \omega_{\partial E}(q) \text{ is even,} \\ 2 & \text{if } \omega_{\partial E}(q) \text{ is odd,} \end{cases}$$

for any $q \in \mathcal{N}(\partial E)$.

By the hypothesis, the number of the set of vertices q of ∂E with $\omega_{\partial E}(q)$ odd is even. Let us denote by $\{p_1, \ldots, p_{2k}\}$ these vertices. We shall construct $\mathcal{G}$ inductively using the following algorithm.

Define $\mathcal{G}_1 = \mathcal{G}'$, and suppose that the graph $\mathcal{G}_{i-1}$ has been defined. If $i \leq k$, we define the graph $\mathcal{G}_i$ as follows. Let $\gamma^i : [0,1] \to \mathbf{R}^2$ be a smooth curve such that $\gamma^i(0) = p_{2i-1}$, $\gamma^i(1) = p_{2i}$, the direction of $\frac{d\gamma^i}{dt}(0)$ (resp. of $\frac{d\gamma^i}{dt}(1)$) coincides with the direction of $T_{\partial E}(p_{2i-1})$ (resp. $T_{\partial E}(p_{2i})$), $(\gamma^i) \cap \mathcal{N}(\partial E) = \{p_{2i-1}, p_{2i}\}$, and such that (γ^i) meets the edges of $\mathcal{G}'$ in a finite number of points, say $\{z_1, \ldots, z_{h(i)}\}$, in such a way that the tangent lines of (γ^i) and of the branches of $\mathcal{G}'$ coincide at the intersection points.

Let us associate with (γ^i) the multiplicity two. Let $\mathcal{G}_i$ be the graph whose vertices are the vertices of $\mathcal{G}_{i-1}$ together with $\{z_1, \ldots, z_{h(i)}\}$, and whose edges are the edges of $\mathcal{G}_{i-1}$ together with the arcs on the curve (γ^i) determined by the points $z_1, \ldots, z_{h(i)}$ with the multiplicity two.

At the end of this algorithm, we get a regular graph $\mathcal{G}$ whose set of edges contains $\mathcal{B}(\partial E)$ and such that each edge of $\mathcal{B}(\mathcal{G}) \setminus \mathcal{B}(\partial E)$ has even multiplicity. As $\mathcal{G}$ is regular, there exists a system of curves Γ of class $H^{2,p}$ associated to $\mathcal{G}$ (see Definition 6.7). Since, by contruction, all branches of Γ having even multiplicity do not meet the boundary of E, using (6.8) we deduce that

$$|E \triangle \{z \in \mathbf{R}^2 \setminus (\Gamma) : I(\Gamma, z) \equiv 1 \pmod 2\}| = 0.$$

The thesis now follows from Theorem 6.2. □

Let $E \subseteq \mathbf{R}^2$ be a bounded open set with a piecewise $H^{2,p}$ boundary ∂E having a continuous unoriented tangent. Assume that $\overline{\mathcal{F}}(E) < +\infty$. Let Γ be one of the limit systems of curves of Theorem 4.1 (iii), and suppose that Γ satisfies the finiteness property. Observe that, for any $B \in \mathcal{B}(\Gamma)$, we can assume that either $B \subseteq \partial E$ or $B \cap \partial E = \emptyset$. Moreover, by Theorem 4.1 (iii) and Lemma 4.1, it follows that, if $B \subseteq \partial E$ then B has odd multiplicity, and if $B \cap \partial E = \emptyset$, then B has even multiplicity. Now we can prove the following result.

THEOREM 6.4. *Let $E \subseteq \mathbf{R}^2$ be a bounded open set with a piecewise $H^{2,p}$ boundary ∂E having a continuous unoriented tangent. Let $\mathcal{N}(\partial E) = \{q, \dots, q_n\}$, and suppose that each q_i is a simple cusp point of ∂E. Then*

$$\overline{\mathcal{F}}(E) < +\infty \quad \Longrightarrow \quad n \text{ is even.}$$

PROOF: Let Γ be one of the limit systems of curves satisfying (iii) of Theorem 4.1, and let us suppose first that Γ satisfies the finiteness property. We shall define an auxiliary graph $\mathcal{G}$ such that $\mathcal{N}(\partial E)$ coincides with the set of those vertices q of $\mathcal{G}$ such that $\varrho_{\mathcal{G}}(q)$ is odd. We construct $\mathcal{G}$ in such a way that the vertices of $\mathcal{G}$ are the vertices of $\mathcal{G}_\Gamma$, and whose edges are those edges B of $\mathcal{G}_\Gamma$ having multiplicity $m_{\mathcal{G}_\Gamma}(B) > 1$. Precisely, let $B \in \mathcal{B}(\mathcal{G}_\Gamma)$; the multiplicity $m_{\mathcal{G}}(B)$ of any edge of $\mathcal{G}$ is defined as follows:

$$m_{\mathcal{G}}(B) = \begin{cases} \frac{m_{\mathcal{G}_\Gamma}(B)}{2} & \text{if } m_{\mathcal{G}_\Gamma}(B) \text{ is even,} \\ \frac{m_{\mathcal{G}_\Gamma}(B)-1}{2} & \text{if } m_{\mathcal{G}_\Gamma}(B) \text{ is odd.} \end{cases}$$

Hence, if $m_{\mathcal{G}_\Gamma}(B) = 1$, then B is deleted. Note that $\mathcal{G}$ is not necessarily regular.

Let $q \in \mathcal{N}(\mathcal{G}_\Gamma)$. If $q \notin \partial E$, then all edges of $\mathcal{G}_\Gamma$ meeting at q have even multiplicity. Hence $\varrho^+_{\mathcal{G}_\Gamma}(q)$ and $\varrho^-_{\mathcal{G}_\Gamma}(q)$ are even. As $\mathcal{G}_\Gamma$ is regular, we have $\varrho^+_{\mathcal{G}_\Gamma}(q) = \varrho^-_{\mathcal{G}_\Gamma}(q)$. Therefore $\varrho_{\mathcal{G}}(q)$ is even.

If $q \in \partial E$ and ∂E is regular in a neighbourhood of q, then exactly two of the edges of $\mathcal{G}_\Gamma$ meeting at q have odd multiplicity and lie one on each side of q with respect to $\nu(q)$; let us denote by B_1 and B_2 these edges, and let $2k+1$ and $2k'+1$ be their multiplicities, respectively. All the other edges meeting at q have even multiplicity; let $2i$ (resp. $2j$) be the number (counted with multiplicities) of all these branches lying on the same (resp. on the opposite) side of B_1, B_2 with respect to $\nu(q)$. As $\mathcal{G}_\Gamma$ is regular, we have that $\varrho^-_{\mathcal{G}_\Gamma}(q) = \varrho^+_{\mathcal{G}_\Gamma}(q)$, i.e., $2k+1+2i = 2k'+1+2j$. Hence $\varrho_{\mathcal{G}}(q) = k+k'+i+j = 2(k+i)$ is even.

If $q \in \partial E$ is a simple cusp point of ∂E, then there exist exactly two edges $B_1, B_2 \in \mathcal{B}(\mathcal{G}_\Gamma)$ having odd multiplicity and lying both on one side of q with respect to $\nu(q)$. Let $2k+1$, $2k'+1$ be these multiplicities, and let $2i$ (resp. $2j$) be the number (counted with multiplicities) of all the other branches of $\mathcal{G}_\Gamma$ lying on the same (resp. on the opposite)

side of B_1, B_2 with respect to $\nu(q)$ (see Figure 6.3). As $\mathcal{G}_\Gamma$ is regular, we have that $\varrho^-_{\mathcal{G}_\Gamma}(q) = \varrho^+_{\mathcal{G}_\Gamma}(q)$, i.e., $2j = 2i + 2k + 1 + 2k' + 1$.

Hence $\varrho_{\mathcal{G}}(q) = j + i + k + k' = 2(i + k + k') + 1$ is odd. It follows that the simple cusp points of ∂E are exactly the vertices q of $\mathcal{G}$ such that $\varrho_{\mathcal{G}}(q)$ is odd. Since in any finite graph $\mathcal{G}$ the number of the set of vertices q with $\varrho_{\mathcal{G}}(q)$ odd is even ([16, Th. 1.2.1]), we deduce that n is even.

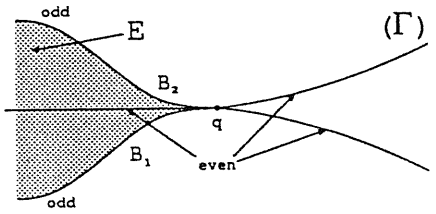

Fig. 6.3: Computation of $\varrho^-_{\mathcal{G}}(q)$ and $\varrho^+_{\mathcal{G}}(q)$ when $q \in \partial E$ is a simple cusp point of ∂E.

We still have to prove the theorem in the most delicate case, i.e., when the system of curves $\Gamma = \{\gamma^1, \ldots, \gamma^m\}$ does not satisfy the finiteness property. We want to modify Γ into a system of curves Λ of class C^1, satisfying the finiteness property, such that $(\Lambda) \subseteq (\Gamma)$, and still

$$|E \triangle \{z \in \mathbf{R}^2 \setminus (\Lambda) : I(\Lambda, z) \equiv 1 \pmod 2\}| = 0. \tag{6.9}$$

Then the argument of the previuos part of the proof can be repeated for Λ (the C^1 regularity is enough for this purpose), so we can conclude as before that n is even.

Condition (6.9) will be achieved by requiring that each branch of Λ with even multiplicity does not meet ∂E and each branch with odd multiplicity is included in ∂E (recall the method for computing $I(\Lambda, z)$ mentioned before Lemma 6.2).

Let C be the set of those points $q \in (\Gamma)$ such that, for any neighbourhood U_q of q, $(\Gamma) \cap U_q$ cannot be written as a cartesian graph with respect to the tangent line $T_{(\Gamma)}(q)$ of (Γ) at q, and let K be the set of all accumulation points of C. Note that K is a compact set.

For any $q \in K$, let $R(q)$ be a small rectangle centered at q, having two sides parallel to $T_{(\Gamma)}(q)$ and such that each curve γ^i meeting $R(q)$ is a cartesian graph in $R(q)$ with respect to $T_{(\Gamma)}(q)$. Such a rectangle exists, since $\gamma^1, \ldots, \gamma^m$ are regular curves of class $H^{2,p}$ parametrized with constant velocity. Furthermore, we can choose $R(q)$ in such a way that, if γ^i is a curve of Γ passing through $R(q)$, then γ^i meets the point q and does not meet the two sides of $R(q)$ which are parallel to $T_{(\Gamma)}(q)$. Finally, we may also assume that, if q is not a cusp point of ∂E, then $R(q)$ does not contain any cusp point of ∂E.

As K is a compact set, there exist $q_1, \ldots, q_N$ points of K such that $U = \bigcup_{i=1}^N R(q_i)$ contains K. In order to construct the modified system of curves Λ, we argue by induction.

Let $\Lambda_0 = \Gamma$, let $i \leq N$, and suppose that Λ_{i-1} has been defined. Then Λ_i is obtained by modifying Λ_{i-1} only on $R(q_i)$, i.e., $(\Lambda_{i-1}) \setminus R(q_i) = (\Lambda_i) \setminus R(q_i)$, in such a way that $\partial E \subseteq (\Lambda_i) \subseteq (\Lambda_{i-1})$, and that each branch of $(\Lambda_i) \cap R(q_i)$ with even multiplicity does not meet ∂E, while each branch of $(\Lambda_i) \cap R(q_i)$ with odd multiplicity is contained in ∂E. Moreover, by the construction of Λ_i, there exist a finite set F_i such that $(\Lambda_i) \cap R(q_i) \setminus F_i$ is a one dimensional C^1 submanifold. We define $\Lambda = \Lambda_N$. The previous remark implies that there exists a finite set F such that $(\Lambda) \cap U \setminus F$ is a one dimensional C^1 submanifold. By the definition of K, the set $C \setminus U$ has no accumulation points, so it is finite. It follows that Λ satisfies the finiteness property. As in the points of $(\Lambda) \setminus U$ we already have the correct multiplicity by Lemma 4.1, it is clear that $\Lambda = \Lambda_N$ satisfies the required properties.

Let us suppose first that $q_i \in K$ is such that $q_i \in (\Gamma) \setminus \overline{E}$. To simplify the notation, we assume that $q_i = 0$, that $T_{(\Lambda_i)}(0)$ coincides with the x-axis, and that $\nu(0) = e_2$. Let $R(0) =]-a, a[\times]-b, b[$; we can assume that $R(0) \cap \overline{E} = \emptyset$. We shall work on the set $(\Lambda_{i-1}) \cap ([0, a[\times]-b, b[)$, since the modification of Λ_{i-1} on the set $(\Lambda_{i-1}) \cap]-a, 0] \times]-b, b[$ is analogous. Because of the assumptions on $R(0)$ and the inclusion $(\Lambda_{i-1}) \subseteq (\Gamma)$, the set $(\Lambda_{i-1}) \cap \{(x, y) \in R(0) : x = a\}$ consists of a finite number of points $\{z_1, \ldots, z_h\}$, labelled by their y-coordinate. Let $\lambda^1, \ldots, \lambda^k$ be the curves of the system Λ_{i-1} which meet $R(0)$ and let $f_1, \ldots, f_k \in H^{2,p}(]0, a[)$ be the functions whose graphs are the traces of $\lambda^1, \ldots, \lambda^k$, i.e., $(\lambda^j) \cap R(0) = \mathrm{graph}(f_j)$. Note that $\{z_1, \ldots, z_h\} = \{f_1(a), \ldots, f_k(a)\}$. As (Λ_{i-1}) has a continuous unoriented tangent, it follows that $f_j'(x) = f_r'(x)$ whenever $f_j(x) = f_r(x)$.

For any $j \in \{1, \ldots, h\}$ let d_j be the number of elements of the set $\Lambda_{i-1}^{-1}(z_j)$. By the inductive property of Λ_{i-1} we have that d_j is even, for any $j \in \{1, \ldots, h\}$.

For any $j \in \{1, \ldots, h\}$ let S_j be the set of all continuous functions $g : [0, a] \to]-b, b[$ such that

$$\mathrm{graph}(g) \subseteq \bigcup_{r=1}^{h} \mathrm{graph}(f_r) \qquad \text{and} \qquad g(b) = z_j. \tag{6.10}$$

It is easy to see that S_j is non-empty and that each function $g \in S_j$ is of class C^1 and satisfies

$$\sup |g'| \leq \max_{1 \leq r \leq h} |f_r'| < +\infty \tag{6.11}$$

(recall that $f_j'(x) = f_r'(x)$ whenever $f_j(x) = f_r(x)$). Let $g_j : [0, a] \to [-b, b]$ be the function defined by

$$g_j(x) = \sup_{g \in S_j} g(x).$$

As all functions in S_j are continuous, by Lindelöf's Theorem there exists a sequence $\{g_j^k\}_k$ such that $g_j(x) = \sup_k g_j^k(x)$. As S_j is a lattice, we may assume that $g_j^k \leq g_j^{k+1}$ for every k. This implies that $g_j^k \to g_j$ pointwise as $k \to +\infty$, hence g_j satisfies (6.10). By (6.11)

and by Ascoli-Arzelà Theorem it follows that g_j is continuous. Hence g_j is the greatest element of S_j, and, consequently, $g_j \in C^1([0,a])$.

By the maximality of g_j it follows that

$$g_j(c) = g_l(c) \qquad \Longrightarrow \qquad g_j(x) = g_l(x) \quad \forall x \in [0,c]. \tag{6.12}$$

We are now in a position to replace the branches of (Λ_{i-1}) contained in $[0,a] \times [-b,b]$ by the curves given by the graphs of the functions g_j, $j = 1,\dots,h$, each curve counted with the multiplicity d_j equal to the multiplicity of z_j for the system of curves (Λ_{i-1}). The system obtained after the same operation on $[-a,a] \times [-b,b]$ will be Λ_i.

It is easy to see that (6.12) implies that there exists a finite set F_i such that $(\Lambda_i) \cap R(0) \setminus F_i$ is a one dimensional C^1 submanifold.

This concludes the modification when $q_i \in (\Gamma) \setminus \overline{E}$. The previous arguments can be repeated exactly when $q_i \in \mathrm{int}(E)$, obviously taking $R(q_i)$ small enough such that $R(q_i) \subseteq \mathrm{int}(E)$.

We still have to consider the case in which $q_i \in \partial E$. We shall work in the most difficult case, i.e., when $q_i \in K$ is a simple cusp point of ∂E, the case in which ∂E is regular near q_i being easier.

We shall adopt the notation of the previous case, with the following modifications. We shall suppose that E lies on the right of 0, and that $R \cap \partial E = \mathrm{graph}(\phi_1) \cup \mathrm{graph}(\phi_2)$, where $\phi_1, \phi_2 :]0,a[\to [-b,b]$ are functions of class $H^{2,p}$, with $\phi_1 \le \phi_2$. As ∂E is contained in (Λ_{i-1}), we have that $\phi_1(b), \phi_2(b) \in \{z_1,\dots,z_h\}$, hence there exist $j_1 < j_2$ such that $\phi_1(b) = z_{j_1}$ and $\phi_2(b) = z_{j_2}$. Note that, in general, $\mathrm{graph}(\phi_1)$ is not a subset of the trace of a unique curve γ^i, and the same holds for $\mathrm{graph}(\phi_2)$. Observe also that there can be a curve γ^i which passes alternatively above $\mathrm{graph}(\phi_2)$ and below $\mathrm{graph}(\phi_1)$ an infinite number of times.

We will divide the set $\{1,\dots,h\}$ into three disjoint subsets, taking into account the part of (Λ_{i-1}) lying below the cusp, the part lying above the cusp, and the part lying inside the cusp. Let $S_1 = \{1,\dots,j_1\}$, $S_2 = \{j_2,\dots,h\}$, $S = \{j_1+1,\dots,j_2-1\}$.

For any $j \in \{1,\dots,h\}$ let d_j be the number of elements of the set $\Lambda_{i-1}^{-1}(z_j)$. By the inductive property of Λ_{i-1} we have that d_{j_1} and d_{j_2} are odd, while d_j is even, for any $j \neq j_1, j_2$. For any $j \in S_1$ let S_j be the set of all continuous functions $g : [0,a] \to]-b,b[$ such that

$$\mathrm{graph}(g) \subseteq \bigcup_{r=1}^{h} \mathrm{graph}(f_r \wedge \phi_1) \qquad \text{and} \qquad g(b) = z_j.$$

Then, as before, S_j is non-empty, if $g \in S_j$, then $g \in C^1$,

$$\sup |g'| \le \max\{\sup |\phi_1'|, \max_{1 \le r \le h} |f_r'|\} < +\infty$$

(recall that $f_j'(x) = f_r'(x)$ whenever $f_j(x) = f_r(x)$), and $f_j'(x) = \phi_1'(x)$ whenever $f_j(x) = \phi_1(x)$).

Let, as before, $g_j : [0,a] \to [-b,b]$ be the function defined by

$$g_j(x) = \sup_{g \in S_j} g(x).$$

For what concerns the definition of g_j for $j \in S_2$, we define S_j as the set of all continuous functions $g : [0,a] \to]-b,b[$ such that

$$\text{graph}(g) \subseteq \bigcup_{r=1}^{h} \text{graph}(f_r \vee \phi_2) \qquad \text{and} \qquad g(b) = z_j,$$

and

$$g_j(x) = \inf_{g \in S_j} g(x).$$

Finally, inside the cusp we need a further attention, to treat the case in which a curve γ^i meets $\text{graph}(\phi_1)$ an infinite number of times. Hence, for any $j \in S$, let S_j be the set of all continuous functions $g : [0,a] \to]-b,b[$ such that

$$\text{graph}(g) \subseteq \bigcup_{r=1}^{h} \text{graph}((f_r \wedge \phi_1) \vee \phi_2) \qquad \text{and} \qquad g(b) = z_j.$$

Define $\tilde{g}_j(x) = \sup_{g \in S_j} g_j(x)$, and let

$$\overline{x} = \sup\{x \in [0,a] : \tilde{g}_j(x) = \phi_1(x)\}.$$

Define

$$g_j(x) = \begin{cases} \tilde{g}_j(x) & \text{if } \overline{x} < x \le a, \\ \phi_1(x) & \text{if } 0 \le x \le \overline{x}. \end{cases}$$

The system of curves Λ_i is the system obtained from Λ_{i-1} by replacing the branches of (Λ_{i-1}) contained in $[0,a] \times [-b,b]$ with the curves given by the graphs of the functions g_j, for $j \in S_1 \cup S_2 \cup S$, each curve counted with the multiplicity d_j equal to the multiplicity of z_j for the system of curves (Λ_{i-1}). We conclude, as before, that there exists a finite set F_i such that $(\Lambda_i) \cap R(0) \setminus F_i$ is a one dimensional C^1 submanifold. □

As a particular case of Theorems 6.3 and 6.4, we get one of the main results of the paper, namely,

THEOREM 6.5. *Let $E \subseteq \mathbb{R}^2$ be a bounded open set with a piecewise $H^{2,p}$ boundary ∂E having a continuous unoriented tangent. Let $\mathcal{N}(\partial E) = \{q_1, \dots, q_n\}$, and suppose that each q_i is a simple cusp point of ∂E. Then*

$$\overline{\mathcal{F}}(E) < +\infty \iff n \text{ is even.}$$

7. Localization.

Some results of Sections 3, 4 can be localized. Let $\Omega \subseteq \mathbf{R}^2$ be an open set, and let $\mathcal{M}$ be the class of all measurable subsets of $\mathbf{R}^2$. The $L^1(\Omega)$-topology on $\mathcal{M}$ is the topology induced by the pseudo-distance $d(E_1, E_2) = |\Omega \cap (E_1 \triangle E_2)|$. We say that $E \in \mathcal{M}$ is of class $C^2(\Omega)$, and we write $E \in C^2(\Omega)$, if E is bounded, open and $\Omega \cap \partial E$ is of class C^2, i.e., near a point $z \in \Omega \cap \partial E$ the set $\Omega \cap E$ is the subgraph of a function of class C^2 with respect to a suitable orthogonal coordinate system.

We define the map $\mathcal{F}(\cdot, \Omega) : \mathcal{M} \to [0, +\infty]$ by

$$\mathcal{F}(E, \Omega) = \begin{cases} \displaystyle\int_{\Omega \cap \partial E} (1 + |\kappa(z)|^p)\, d\mathcal{H}^1(z) & \text{if } E \in C^2(\Omega), \\ +\infty & \text{elsewhere on } \mathcal{M}. \end{cases}$$

Note that

$$\mathcal{F}(E, \Omega) = \sup_{n \in \mathbf{N}} \mathcal{F}(E, \Omega_n) \tag{7.1}$$

for every sequence $\{\Omega_n\}_n$ of open sets invading Ω. Moreover $\mathcal{F}(E, \cdot)$ is increasing if considered as a set function, i.e., if Ω_1, Ω_2 are open,

$$\Omega_1 \subseteq \Omega_2 \quad \Rightarrow \mathcal{F}(E, \Omega_1) \leq \mathcal{F}(E, \Omega_2). \tag{7.2}$$

By $\overline{\mathcal{F}}(\cdot, \Omega)$ we denote the lower semicontinuous envelope of $\mathcal{F}(\cdot, \Omega)$ with respect to the topology of $L^1(\Omega)$. It is known that, for every $E \in \mathcal{M}$, we have

$$\overline{\mathcal{F}}(E, \Omega) = \inf\{\liminf_{h \to +\infty} \mathcal{F}(E_h, \Omega) : E_h \to E \text{ in } L^1(\Omega) \text{ as } h \to +\infty\}.$$

Note that $\overline{\mathcal{F}}(E, \Omega) < +\infty$ if and only if there exists a sequence $\{E_h\}_h$ of bounded open sets of class $C^2(\Omega)$ such that $E_h \to E$ in $L^1(\Omega)$ as $h \to +\infty$, and

$$\sup_h \mathcal{H}^1(\Omega \cap \partial E_h) < +\infty, \quad \sup_h \int_{\Omega \cap \partial E_h} |\kappa_h(z)|^p\, d\mathcal{H}^1(z) < +\infty, \tag{7.3}$$

where κ_h denotes the curvature of $\Omega \cap \partial E_h$.

THEOREM 7.1. *Let E be a bounded open set of class $C^2(\Omega)$. Then*

$$\mathcal{F}(E, \Omega) \leq \liminf_{h \to +\infty} \mathcal{F}(E_h, \Omega) \tag{7.4}$$

for any sequence $\{E_h\}_h$ of bounded open sets of class $C^2(\Omega)$ such that $E_h \to E$ in $L^1(\Omega)$ as $h \to +\infty$.

PROOF: Let $\{E_h\}_h$ be a sequence of bounded open sets of class $C^2(\Omega)$ such that $E_h \to E$ in $L^1(\Omega)$ as $h \to +\infty$. We can suppose that the right hand side of (7.4) is finite,

otherwise the result is trivial. Given a sequence $\{\Omega_n\}_n$ of relatively compact open sets invading Ω, it will be sufficient to prove that

$$\mathcal{F}(E,\Omega_n) \le \liminf_{h\to+\infty} \mathcal{F}(E_h,\Omega_n), \tag{7.5}$$

for every $n \in \mathbf{N}$. In fact, (7.4) follows from (7.5), since, by (7.1) and (7.2),

$$\mathcal{F}(E,\Omega) = \sup_{n\in\mathbf{N}} \mathcal{F}(E,\Omega_n) \le \sup_{n\in\mathbf{N}} \liminf_{h\to+\infty} \mathcal{F}(E_h,\Omega_n) \le \liminf_{h\to+\infty} \mathcal{F}(E_h,\Omega).$$

Let us fix $n \in \mathbf{N}$. Let $\{E_{hk}\}_k$ be a subsequence of $\{E_h\}_h$ with the property that

$$\lim_{k\to+\infty} \mathcal{F}(E_{hk},\Omega_n) = \liminf_{h\to+\infty} \mathcal{F}(E_h,\Omega_n) < +\infty.$$

For simplicity, this subsequence (and any further subsequence) will be denoted by $\{E_k\}_k$. Since $\{E_k\}_k$ satisfies (7.3), it follows that, for any k,

$$\Omega_n \cap \partial E_k \subseteq \bigcup_{i=1}^{m_k} (\gamma_k^i) \cup \bigcup_{j=1}^{r_k} (\beta_k^j),$$

where

$\{\gamma_k^1,\dots,\gamma_k^{m_k}\}$ is a disjoint system of curves of class C^2 such that $\overline{\Omega}_n \cap (\gamma_k^i) \neq \emptyset$ (see Lemma 3.1);

$\{\beta_k^1,\dots,\beta_k^{r_k}\}$ is a finite family of simple regular curves of class C^2 such that $|\frac{d\beta_k^j}{dt}|$ is constant, $\beta_k^j(0),\beta_k^j(1) \in \partial\Omega$ and $(\beta_k^j) \cap \partial\Omega_n \neq \emptyset$ for any $j = 1,\dots,r_k$;

all the sets $(\gamma_k^1),\dots,(\gamma_k^{m_k})$, $(\beta_k^1),\dots,(\beta_k^{r_k})$ are pairwise disjoint;

$$\sum_{i=1}^{m_k} l(\gamma_k^i) + \sum_{j=1}^{r_k} l(\beta_k^j) + \sum_{i=1}^{m_k} \|\kappa(\gamma_k^i)\|_{L^p}^p + \sum_{j=1}^{r_k} \|\kappa(\beta_k^j)\|_{L^p}^p < +\infty. \tag{7.6}$$

By Lemma 3.1 we have that $\{m_k\}_k$ is uniformly bounded with respect to k. Since $l(\beta_k^j) \ge 2\mathrm{dist}(\partial\Omega_n,\partial\Omega) > 0$ for any $j = 1,\dots,r_k$, using (7.6) we get that $\{r_k\}_k$ is also uniformly bounded with respect to k. Passing to a suitable subsequence, we can assume that m_k and r_k are independent of k. These numbers will be denoted by m and r, respectively. Since all curves of the family $\{\gamma_k^1,\dots,\gamma_k^m,\beta_k^1,\dots,\beta_k^r\}$ meet $\overline{\Omega}_n$, by (7.6) their traces are contained in a bounded subset of $\mathbf{R}^2$ independent of k. Repeating the arguments of Theorem 3.1, by compactness there exists a family $\Gamma = \{\gamma^1,\dots,\gamma^m,\beta^1,\dots,\beta^r\}$ of regular curves of class $H^{2,p}$ such that $\gamma_k^i \rightharpoonup \gamma^i$ and $\beta_k^j \rightharpoonup \beta^j$ weakly in $H^{2,p}$ as $k \to +\infty$, for any $i = 1,\dots,m$ and any $j = 1,\dots,r$. Then, since by hypothesis $E \in C^2(\Omega)$, it follows that

$\Omega_n \cap (\Gamma) \supseteq \Omega_n \cap \partial E$. In fact, suppose by contradiction that there exists $z \in \Omega_n \cap (\partial E \setminus (\Gamma))$. Let $r > 0$ be such that $B_r(z) \subseteq \Omega_n$, $B_r(z) \cap (\Gamma) = \emptyset$, and

$$|B_{\frac{r}{2}}(z) \cap E| > 0, \quad |B_{\frac{r}{2}}(z) \setminus E| > 0. \tag{7.7}$$

As $\gamma_k^i \to \gamma^i$ and $\beta_k^j \to \beta^j$ in C^1 as $k \to +\infty$, there exists $k_0 \in \mathbf{N}$ such that $B_{\frac{r}{2}}(z) \cap \partial E_k = \emptyset$ for any $k \geq k_0$. This implies that either $B_{\frac{r}{2}}(z) \subseteq E_k$, or $B_{\frac{r}{2}}(z) \cap E_k = \emptyset$, which, together with (7.7), contradicts the fact that $E_k \to E$ in $L^1(\Omega)$ as $k \to +\infty$. Repeating the last part of the proof of Theorem 3.2 relatively to Ω_n we obtain (7.5). □

COROLLARY 7.1. *Let $E \subseteq \mathbf{R}^2$ be a bounded open set of class $H^{2,p}$ which is relatively compact in Ω. Then*

$$\overline{\mathcal{F}}(E, \Omega) = \int_{\partial E} [1 + |\kappa(z)|^p] \, d\mathcal{H}^1(z).$$

In particular, $\overline{\mathcal{F}}(E, \Omega) < +\infty$.

PROOF: Theorem 7.1 holds with the same proof if E is of class $H^{2,p}$, hence, passing to the infimum with respect to the approximating sequence $\{E_h\}_h$ in (7.4) we infer that $\int_{\partial E} [1 + |\kappa(z)|^p] \, d\mathcal{H}^1(z) \leq \overline{\mathcal{F}}(E, \Omega)$. The opposite inequality can be proved as in Corollary 3.2, using the hypothesis that E is relatively compact in Ω. □

The last part of this section is devoted to find an example of a set E in which we can calculate $\overline{\mathcal{F}}(E) = \mathcal{F}(E, \mathbf{R}^2)$. Moreover, as we shall prove, for this set E one has that $\overline{\mathcal{F}}(E, \cdot)$ is not subadditive (see [4, Conjecture 5]). This shows that $\overline{\mathcal{F}}(E, \cdot)$ cannot be a measure and in particular that $\overline{\mathcal{F}}(E, \Omega)$ can not be represented as in integral of the form (1.3) for a suitable choice of the function $\kappa(z)$.

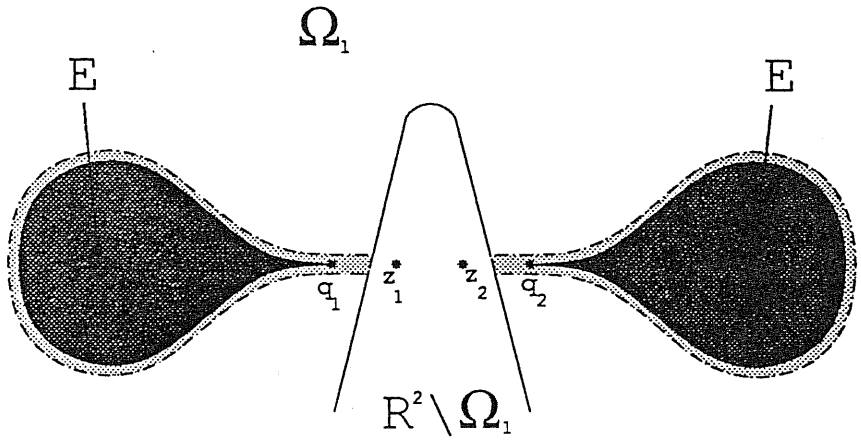

Fig. 7.1: Each rectilinear segment of ∂E_h near the points q_1 and q_2 gives a contribution to the functional $\mathcal{F}(E_h)$ equal to its lenght, which is less than $\frac{1}{3}$

Let $E \subseteq \mathbf{R}^2$ be the set of Figure 7.1, where $q_1 = (0,0)$ and $q_2 = (1,0)$, and let C_1, C_2 be the two connected components of ∂E. Note that the unoriented tangent lines in q_1 and q_2 coincide with the x-axis. As in Theorem 4.1 (v), we denote by $\mathcal{A}(E)$ the set of all limit systems Γ of curves of class $H^{2,p}$ such that $(\Gamma) \supseteq \partial E$ and $E = \mathrm{int}(A_\Gamma \cup (\Gamma))$, where $A_\Gamma = \{z \in \mathbf{R}^2 \setminus (\Gamma) : I(\Gamma, z) = 1\}$.

LEMMA 7.1. *Let E be the set of Figure 7.1, and let $\Gamma = \{\gamma^1, \dots, \gamma^m\} \in \mathcal{A}(E)$. Then either there exist $i, j \in \{1, \dots, m\}$, $i \neq j$ and $t_1, t_2 \in [0,1]$ such that $\gamma^i(t_1) = \gamma^j(t_2) = q_1$, or there exist $i \in \{1, \dots, m\}$ and $t_1, t_2 \in [0,1]$ with $0 \leq t_1 < t_2 < 1$ such that $\gamma^i(t_1) = \gamma^i(t_2) = q_1$. The same holds for the point q_2.*

PROOF: Let $\{z_h^{(1)}\}_h$, $\{z_h^{(2)}\}_h$ be two sequences of points such that, for any h, $z_h^{(1)} = (x_h, y_h^{(1)})$, $z_h^{(2)} = (x_h, y_h^{(2)})$, $z_h^{(1)} \in C_1 \cap \{y \leq 0\}$, $z_h^{(2)} \in C_1 \cap \{y \geq 0\}$, and $z_h^{(1)} \to q_1$, $z_h^{(2)} \to q_1$ as $h \to +\infty$. Since $(\Gamma) \supseteq \partial E$, passing to suitable subsequences, there exist $i, j \in \{1, \dots, m\}$ and two sequences $\{s_h\}_h$, $\{t_h\}_h$ of points, with $s_h, t_h \in [0,1]$, such that $z_h^{(1)} = (x_h, y_h^{(1)}) = \gamma^i(s_h)$, $z_h^{(2)} = (x_h, y_h^{(2)}) = \gamma^j(t_h)$ for any h. By compactness, $s_h \to t_1$, $t_h \to t_2$ as $h \to +\infty$, so that $\gamma^i(t_1) = \gamma^j(t_2) = q_1$. If $i \neq j$ the proof is complete. Let us suppose that $i = j$, and assume by contradiction that $t_1 = t_2$. By the mean value theorem, for any h there exists a point ξ_h between s_h and t_h such that the first component of the derivative of γ^i vanishes at ξ_h. As $\xi_h \to t_1 = t_2$, we conclude that the first component of the derivative of γ^i at t_1 vanishes. This is a contradiction, since any tangent unit vector of ∂E at q_1 is parallel to the x-axis. □

An important property of the set E is that C_1 lies on the left with respect to the y-axis, and that C_2 lies on the right with respect to the normal line of ∂E at q_2, which is the line $\{x = 1\}$. Let us denote by $\partial_r E$ the set of the regular points of ∂E, i.e., $\partial_r E = \partial E \setminus \{q_1, q_2\}$. Then the following result holds.

THEOREM 7.2. *Let E be the set of Figure 7.1. Then*

$$\overline{\mathcal{F}}(E) = \mathcal{H}^1(\partial E) + \int_{\partial_r E} |\kappa(z)|^p \, d\mathcal{H}^1(z) + 2.$$

PROOF: We shall show that

$$\overline{\mathcal{F}}(E) = \inf\{\mathcal{F}(\Gamma) : \Gamma \in \mathcal{A}(E)\} = \inf\{\mathcal{F}(\Gamma) : \Gamma \in \mathcal{Q}(E)\}, \tag{7.8}$$

where $\mathcal{Q}(E)$ is the collection of those systems of curves Γ of class $H^{2,p}$ without crossings, satisfying the finiteness property and such that $\{z \in \mathbf{R}^2 : \exists r > 0 \ |B_r(z) \setminus E| = 0\} = \mathrm{int}(A_\Gamma \cup (\Gamma))$, with $A_\Gamma = \{z \in \mathbf{R}^2 \setminus (\Gamma) : I(\Gamma, z) \equiv 1 \pmod 2\}$ (see Theorem 6.2).

By Theorem 4.1 (v) and (6.7) we have

$$\inf\{\mathcal{F}(\Gamma) : \Gamma \in \mathcal{Q}(E)\} \geq \overline{\mathcal{F}}(E) \geq \inf\{\mathcal{F}(\Gamma) : \Gamma \in \mathcal{A}(E)\}.$$

Since

$$\inf\{\mathcal{F}(\Gamma) : \Gamma \in \mathcal{Q}(E)\} \leq \mathcal{H}^1(\partial E) + \int_{\partial_r E} |\kappa(z)|^p \, d\mathcal{H}^1(z) + 2,$$

to prove (7.8) it will be enough to show that, for any $\Gamma \in \mathcal{A}(E)$,

$$\mathcal{F}(\Gamma) \geq \mathcal{H}^1(\partial E) + \int_{\partial_r E} |\kappa(z)|^p \, d\mathcal{H}^1(z) + 2. \tag{7.9}$$

Let $\Gamma = \{\gamma^1, \ldots, \gamma^m\} \in \mathcal{A}(E)$. Let us use Lemma 7.1. Suppose that there exist $i, j \in \{1, \ldots, m\}$, $i \neq j$, and $t_1, t_2 \in [0,1]$ such that $\gamma^i(t_1) = \gamma^j(t_2) = q_1$. Up to a reparametrization of the curve γ^i, we can suppose that $t_1 = 0$, that the tangent unit vector of γ^i at $t = 0$ is $(1,0)$, and that γ^i is parametrized by the arc lenght s. Let $[0, l(\gamma^i)] = A \cup B$, where $A = \{s \in [0, l(\gamma^i)] : \gamma^i(s) \in \partial E\}$, and $B = \{s \in [0, l(\gamma^i)] : \gamma^i(s) \notin \partial E\}$. Clearly $B \neq \emptyset$. Let us prove that

$$\mathcal{H}^1(B) + \int_B |\dot{\gamma}^i(s)|^p \, ds \geq 1. \tag{7.10}$$

There are two possibilities: either $(\gamma^i) \cap C_2 \neq \emptyset$, or $(\gamma^i) \cap C_2 = \emptyset$. If $(\gamma^i) \cap C_2 \neq \emptyset$ one obviously gets $\mathcal{H}^1(B) + \int_B |\dot{\gamma}^i(s)|^p \, ds \geq |q_2 - q_1| = 1$, that is (7.10).

Suppose now that $(\gamma^i) \cap C_2 = \emptyset$, and denote by γ_1^i the x-component of the curve γ^i. By construction, we have that $\dot{\gamma}_1^i(0) = 1$. Since γ^i is closed, we can consider the smallest parameter $\sigma \in]0, l(\gamma^i)[$ such that $\gamma_1^i(\sigma) > 0$ and $\dot{\gamma}_1^i(\sigma) = 0$. It is easy to prove that $\gamma_1^i(s) > 0$ for any $s \in]0, \sigma]$. Since C_1 lies on the left with respect to the y-axis, we get $C_1 \cap \gamma^i(]0, \sigma]) = \emptyset$. Moreover, by assumption, we have also that $C_2 \cap \gamma^i([0, \sigma]) = \emptyset$. Hence, $[0, \sigma] \subseteq B$, and using (3.5) we get

$$\mathcal{H}^1(B) + \int_B |\dot{\gamma}^i(s)|^p \, ds \geq \sigma + \int_0^\sigma |\dot{\gamma}^i(s)|^p \, ds \geq \frac{\pi}{2}[(\frac{p}{p'})^{\frac{1}{p}} + (\frac{p'}{p})^{\frac{1}{p'}}] > 1,$$

that gives (7.10).

Using the same arguments also for the curve γ^j, (7.9) follows from (7.10) applied to both γ^i and γ^j.

Let us suppose now that there exist $i \in \{1, \ldots, m\}$ and $t_1, t_2 \in [0,1]$, $0 \leq t_1 < t_2 < 1$ such that $\gamma^i(t_1) = \gamma^j(t_2) = q_1$. If there exist $k \in \{1, \ldots, m\}$, $k \neq i$, and $t \in [0,1]$ such that $\gamma^k(t) = q_2$, then (7.9) follows from the previous arguments. The last possibility (see Lemma 7.1) is whenever there exist $t_3, t_4 \in [0,1]$, $0 \leq t_3 < t_4 < 1$, $t_3 \neq t_1$, $t_4 \neq t_2$ such that $\gamma^i(t_3) = \gamma^i(t_4) = q_2$, and this case can be treated with obvious modifications. □

THEOREM 7.3. *Let E be the set of Figure 7.1. Then there exist two open sets $\Omega_1, \Omega_2 \subseteq \mathbf{R}^2$, with $\Omega_1 \cup \Omega_2 = \mathbf{R}^2$, such that*

$$\overline{\mathcal{F}}(E, \Omega_1) + \overline{\mathcal{F}}(E, \Omega_2) < \overline{\mathcal{F}}(E, \mathbf{R}^2) < +\infty.$$

PROOF:

As we saw in Section 1, $\overline{\mathcal{F}}(E) = \overline{\mathcal{F}}(E, \mathbf{R}^2) < +\infty$.

Let $z_1 = (0, \frac{1}{3})$ and $z_2 = (0, \frac{2}{3})$. Let us fix two open subsets Ω_1, Ω_2 of $\mathbf{R}^2$ such that $z_1, z_2 \notin \Omega_1$, $\Omega_2 \cap E = \emptyset$, and $\Omega_1 \cup \Omega_2 = \mathbf{R}^2$. Then

$$\overline{\mathcal{F}}(E, \Omega_2) = 0, \tag{7.11}$$

and from Theorem 7.2 we have

$$\overline{\mathcal{F}}(E, \mathbf{R}^2) = \mathcal{H}^1(\partial E) + \int_{\partial_r E} |\kappa(z)|^p \, d\mathcal{H}^1(z) + 2. \tag{7.12}$$

Moreover one easily constructs a sequence $\{E_h\}_h$ of bounded open subsets of $\mathbf{R}^2$ of class $C^2(\Omega_1)$ with $E_h \to E$ in $L^1(\Omega_1)$ as $h \to +\infty$, such that (see Figure 7.1)

$$\mathcal{H}^1(\partial E) + \int_{\partial_r E} |\kappa(z)|^p \, d\mathcal{H}^1(z) + \frac{4}{3} = \lim_{h \to +\infty} \mathcal{F}(E_h, \Omega_1) \geq \overline{\mathcal{F}}(E, \Omega_1). \tag{7.13}$$

The conclusion of the theorem follows now from (7.11), (7.12), (7.13). □

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CHAPTER 1, SECTION 1.2

VARIATIONAL PROPERTIES OF AN IMAGE SEGMENTATION FUNCTIONAL DEPENDING ON CONTOURS CURVATURE*

1. Introduction. A great attention has been recently devoted to the reconstruction of the contours of a picture given by a camera. One of the approaches considered to solve this problem if of variational type (see, among others, [16,6,24,23,21,13,19,25,26,27,12,1, 4,10]), where the solution is viewed as a minimum point of suitable energy functionals. More recently, in a paper by Mumford and Nitzberg [22] an integral term depending on the curvature of the contours has been added in the energy, in order to reconstruct the hidden boundaries, that is those parts of the image which are occluded by other regions closer to the observer. In this paper, starting from the suggestions of [22], we study the behaviour of a functional involving lenght and curvature of boundaries of (possibly non smooth) sets, from the viewpoint of relaxation theory. More precisely, let $\psi : \mathbf{R} \times \mathbf{R} \to [0, +\infty]$ be a Borel function having the following properties:

(i) $\psi(\eta, \cdot)$ is convex on $\mathbf{R}$ for any $\eta \in \mathbf{R}$;

(ii) ψ is lower semicontinuous on $\mathbf{R} \times \mathbf{R}$;

(iii) $\psi(\cdot, 0)$ is locally bounded;

(iv) there exist two constants $a > 0$ and $b \geq 0$ such that

$$\psi(\eta, \xi) \geq a|\xi| - b \geq 0 \qquad \forall (\eta, \xi) \in \mathbf{R} \times \mathbf{R}; \tag{1.1}$$

(v) $\psi(\eta, \xi) = \psi(\eta + \pi, \xi)$ for any $(\eta, \xi) \in \mathbf{R} \times \mathbf{R}$;

(vi) $\psi(\eta, \xi) = \psi(\eta, -\xi)$ for any $(\eta, \xi) \in \mathbf{R} \times \mathbf{R}$.

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Let $n \in \mathbf{N}$, $n \geq 1$, and let $g \in L^\infty(\mathbf{R}^2)$ be a function with compact support. The aim of this paper is to study the functional $\mathcal{P}_\psi$ defined by

$$\mathcal{P}_\psi(E_1,\dots,E_n) = \sum_{i=1}^{n} \int_{\partial E_i} [1+\psi(\alpha_i,\kappa_i)]\ d\mathcal{H}^1 + \sum_{i=1}^{n+1} \int_{E_i'} (u_i - g)^2\ dz,$$

where $E_1,\dots,E_n \subseteq \mathbf{R}^2$ are bounded open sets of class $\mathcal{C}^2$, $\alpha_i(z)$ is the angle that the tangent unit vector of ∂E_i at $z \in \partial E_i$ forms with the positive direction of the x-axis, $\kappa_i(z)$ is the curvature of ∂E_i at z, $\mathcal{H}^1$ denotes the one-dimensional Hausdorff measure in $\mathbf{R}^2$, and

$$\begin{aligned} &E_1' = E_1,\ E_i' = E_i \setminus \bigcup_{j=1}^{i-1} E_j,\ i = 2,\dots,n, \quad E_{n+1}' = \mathbf{R}^2 \setminus \bigcup_{j=1}^{n} E_j, \\ &u_i = \fint_{E_i'} g\ dz, \qquad i = 1,\dots,n, \quad u_{n+1} = 0 \end{aligned} \tag{1.2}$$

(see Section 7). Note that conditions (v) and (vi) ensure that the functional $\mathcal{P}_\psi$ is independent of the orientation of the boundaries ∂E_i.

In [22] the function ψ has the specific form

$$\psi(\eta,\xi) = \psi(|\xi|) = \begin{cases} c_1\xi^2 & \text{if } |\xi| < T, \\ c_2|\xi| - c_3 & \text{if } |\xi| \geq T, \end{cases} \tag{1.3}$$

with $T > 0$, $c_1 > 0$, $c_1T^2 = c_2T - c_3$, $c_2 = 2c_1T$. Each partition $E_1,\dots,E_n$ is endowed with the following ordering: $E_i > E_j$ (i.e., E_i is closer than E_j to the observer) if $i < j$. This ordering represents relative depth (or occlusion); in this context, if $i = 1,\dots,n$, the sets E_i' defined in (1.2) are the "visible" part of E_i, and E_{n+1}' stands for the background. The partition minimizing the energy is called an optimal overlapping segmentation of the image g.

The special case $\psi(\eta,\xi) = |\xi|^2$ and $n = 1$ has been studied in [3]. Here the quadratic growth of ψ at infinity is the main difference with (1.3), and reveals very different effects with respect to the linear case (see also [2]).

To study the functional $\mathcal{P}_\psi$, which, in our approach is considered as a function of the sets $E_1,\dots,E_n$ rather than of their boundaries $\partial E_1,\dots,\partial E_n$, we need a detailed study of the variational properties of the functional

$$F_\psi(E) = \int_{\partial E} [1+\psi(\alpha,\kappa)]\ d\mathcal{H}^1.$$

As usual in the relaxation method (see [7]), if $\mathcal{M}$ denotes the class of all Lebesgue measurable subsets of $\mathbf{R}^2$, we define $\overline{F_\psi} : \mathcal{M} \to [0,+\infty]$ as the lower semicontinuous envelope of F_ψ with respect to the $L^1(\mathbf{R}^2)$-topology, i.e.,

$$\overline{F_\psi}(E) = \inf\{\liminf_{h\to+\infty} F_\psi(E_h) : E_h \to E \text{ in } L^1(\mathbf{R}^2)\}. \tag{1.4}$$

The main purpose of this paper is to study the functional $\overline{F_\psi}$ and to compute $\overline{F_\psi}(E)$ for some special sets E.

Simple examples show that there exist sets $E \in \mathcal{M}$ with $\overline{F_\psi}(E) < +\infty$, whose boundary is not smooth, as polygons and sets with cusp points. In particular, let us consider the set of Figure 1.1, and let ψ be as in (1.3). Then the approximating sequence $\{E_h\}_h$ of Figure 1.2, constructed by smoothing the corner p with circular arcs, shows that

$$\overline{F_\psi}(E) \leq \mathcal{H}^1(\partial E) + \int_{\partial E \setminus \{p\}} \psi(\alpha, \kappa)\, d\mathcal{H}^1 + c_2(\pi - \alpha).$$

Note that the corner p is penalized, in terms of the energy, of a factor which is proportional to $\pi - \alpha$.

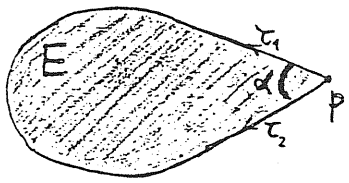

Fig. 1.1: A set E with $\overline{F_\psi}(E) < +\infty$ and whose boundary is not smooth.

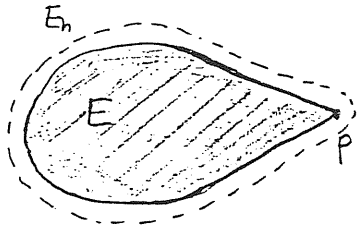

Fig. 1.2: This approximating sequence $\{E_h\}_h$ shows that $\overline{F_\psi}(E) < +\infty$.

This penalization corresponds to the variation made by the unit tangent vector of the curves parametrizing ∂E_h in a neighbourhood of p, in the limit as $h \to +\infty$. This example shows that, if one deals with curves parametrized by arc lenght, it can be useful to identify the tangent vectors as points of the unit circle $\mathbf{S}^1$ and to consider the jumps at the corners in terms of the corresponding angles. We are then led to look at the set ∂E of Figure 1.1 as the trace of a continuous closed curve γ with $\dot\gamma \in BV$ (the space of the functions of bounded variation) and with the constraint $|\dot\gamma| = 1$ (see also [17]). The convergence in this class of curves must be considered (mod 2π), as explained in Section 4, thus showing that some care is needed in the definition of the energies.

We describe now in detail the content of the paper.

In Section 2 we fix some notation and, in particular, we define the concept of system of curves of class $\mathcal{B}$, that is a finite family of closed curves of class $W^{1,1}$ whose derivatives are functions of class BV (Definition 2.2).

In Section 3 we introduce the argument of $\dot\gamma$ (see Lemma 3.1), where γ is a curve of class $\mathcal{B}$ parametrized with constant velocity.

In Section 4 we study some properties of the space $BV([a,b];\mathbf{R}/2\pi)$, which, in turn, are useful for the definition of the energy functional given in Section 5.

Section 6 is devoted to the study of the lower semicontinuity of F_ψ. Precisely, we prove (Theorem 6.1) that *given a sequence $\{E_h\}_h$ of bounded open sets of class C^2 converging in $L^1(\mathbf{R}^2)$ to a bounded open set E of class C^2, then*

$$F_\psi(E) \le \liminf_{h\to+\infty} F_\psi(E_h).$$

A similar result holds if we localize F_ψ on an open subset Ω of $\mathbf{R}^2$ (Corollary 6.1).

In Section 7 we show how to reduce the study of the functional $\mathcal{P}_\psi$ to the study of the simpler functional F_ψ.

In Section 8 we show that, if E is a bounded open set that can be written, locally, as the subgraph of a Lipschitz function having derivative of class BV, then $\overline{F_\psi}(E) < +\infty$.

Finally, in Section 9 we focus our attention to the case (1.3), and we prove the following results (see Theorem 9.1 and Corollary 9.2): *let E be the disjoint union of a finite number of curvilinear polygons $P_1,\dots,P_q$. For any $i=1,\dots,q$ denote by $\alpha_i^1,\dots,\alpha_i^{h(i)}$ the angles of the polygon P_i. Then*

$$\mathcal{H}^1(\partial E) + \int_{\partial_r E} \psi(|\kappa|)\, d\mathcal{H}^1(z) + c_2 \sum_{i=1}^{q}\sum_{j=i}^{h(i)} \min(\alpha_i^j, 2\pi-\alpha_i^j, |\pi-\alpha_i^j|) \le \overline{F_\psi}(E) \le$$

$$\le \mathcal{H}^1(\partial E) + \int_{\partial_r E} \psi(|\kappa|)\, d\mathcal{H}^1(z) + c_2 \sum_{i=1}^{q}\sum_{j=i}^{h(i)} |\pi-\alpha_i^j|,$$

where $\partial_r E$ denotes ∂E without the vertices, and κ denotes the curvature of $\partial_r E$. If in particular $\frac{\pi}{2} \le \alpha_i^j \le \frac{3\pi}{2}$ for any $i=1,\dots,q$ and any $j=1,\dots,h(i)$, then

$$\overline{F_\psi}(E) = \mathcal{H}^1(\partial E) + \int_{\partial_r E} \psi(|\kappa|)\, d\mathcal{H}^1(z) + c_2 \sum_{i=1}^{q}\sum_{j=i}^{h(i)} |\pi-\alpha_i^j|. \tag{1.5}$$

Moreover, we show that (1.5) does not necessarily hold if $\alpha_i^j < \frac{\pi}{2}$ or $\alpha_i^j > \frac{3}{2}\pi$ for some i,j, the result depending on the relative distance between the corners. Indeed (Theorem 9.2) *let $E = E_1 \cup E_2$ be the set of Figure 9.1, and assume that*

$$0 < \alpha < \frac{\pi}{2}. \tag{1.6}$$

Then if the distance D betweeen p_1 and p_2 is sufficiently small one has

$$\overline{F_\psi}(E) = \mathcal{H}^1(\partial E) + \int_{\partial_r E} \psi(|\kappa|)\, d\mathcal{H}^1(z) + 2D + 2c_2\alpha,$$

where $\partial_r E = \partial E \setminus \{p_1, p_2\}$.

In particular, if D is sufficiently small,

$$\overline{F_\psi}(E) < \mathcal{H}^1(\partial E) + \int_{\partial_r E} \psi(|\kappa|)\, d\mathcal{H}^1(z) + 2c_2(\pi - \alpha).$$

As the proof of this result has local character, we deduce that there are non convex sets E having some angles α_i verifying condition (1.6) such that the their energy (i.e., the value of $\overline{F_\psi}(E)$) is strictly less to the energy of the smooth part of the boundary $\partial_r E$ plus the extra term $c_2 \sum_i (\pi - \alpha_i)$. This lower value of the energy can be reached, ideally, by joining properly the angles with a "hidden" line.

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2. Notations and preliminary definitions.

$\mathcal{H}^h$ denotes the h-dimensional Hausdorff measure in $\mathbf{R}^2$ for $h = 0, 1$ [15]; $\mathcal{L}^2$ denotes the Lebesgue measure in $\mathbf{R}^2$. For any $z_0 \in \mathbf{R}^2$ and $\varrho > 0$, $B_\varrho(z_0) = \{z \in \mathbf{R}^2 : |z - z_0| < \varrho\}$ is the ball centered at z_0 with radius ϱ.

$\mathcal{M}$ denotes the class of all Lebesgue measurable subets of $\mathbf{R}^2$. We shall identify $\mathcal{M}$ with a closed subset of $L^1(\mathbf{R}^2)$ by means of the map $E \mapsto \chi_E$, where χ_E denotes the characteristic function of E, i.e., $\chi_E(z) = 1$ if $z \in E$, $\chi_E(z) = 0$ if $z \notin E$. The $L^1(\mathbf{R}^2)$-topology on $\mathcal{M}$ is, therefore, the topology on $\mathcal{M}$ induced by the distance $d(E_1, E_2) = \mathcal{L}^2(E_1 \triangle E_2)$, where $E_1, E_2 \in \mathcal{M}$ and $\triangle$ is the symmetric difference of sets.

For any subset C of $\mathbf{R}^2$, we denote by $\mathrm{int}(C)$ the interior of C, by $\overline{C}$ the closure of C, and by ∂C the topological boundary of C. Let $E \subseteq \mathbf{R}^2$; we say that E is of class $\mathcal{C}^2$ if E is open, and, near each point $z \in \partial E$, the set E is the subgraph of a function of class $\mathcal{C}^2$ with respect to a suitable orthogonal coordinate system. Note that, if ∂E can be parametrized, locally, by arcs of regular curves of class $\mathcal{C}^2$, and E lies locally on one side of its boundary, then E is of class $\mathcal{C}^2$. We denote by $\mathcal{C}^2_b(\mathbf{R}^2)$ the class of all bounded sets of class $\mathcal{C}^2$.

2.1. Systems of curves. A plane curve $\gamma : [0,1] \to \mathbf{R}^2$ of class $\mathcal{C}^1$ is said to be regular if $\frac{d\gamma(t)}{dt} \neq 0$ for every $t \in [0,1]$. Each closed regular curve $\gamma : [0,1] \to \mathbf{R}^2$ will be identified, in the usual way, with a map $\gamma : \mathbf{S}^1 \to \mathbf{R}^2$, where $\mathbf{S}^1$ denotes the oriented unit circle. By $(\gamma) = \gamma([0,1]) = \{\gamma(t) : t \in [0,1]\}$ we denote the trace of γ and by $l(\gamma)$ its length; s denotes the arc length parameter.

If $z \in \mathbf{R}^2 \setminus (\gamma)$, $I(\gamma, z)$ is the index of z with respect to γ (see, for instance, [5], and [8, III.8]).

DEFINITION 2.1. *A system of curves is a finite family* $\Gamma = \{\gamma^1, \dots, \gamma^m\}$ *of closed curves of class* $W^{1,1}$ *such that* $|\frac{d\gamma^i}{dt}|$ *is constant almost everywhere on* $[0,1]$ *for any* $i = 1, \dots, m$. *The trace* (Γ) *of* Γ *is defined as* $\bigcup_{i=1}^m (\gamma^i)$, *and its lenght* $l(\Gamma)$ *as* $\sum_{i=1}^m l(\gamma^i)$. *If* $z \in \mathbf{R}^2 \setminus (\Gamma)$, *we define the index* $I(\Gamma, z)$ *of* z *with respect to* Γ *as* $\sum_{i=1}^m I(\gamma^i, z)$.

We say that a system of curves $\Gamma = \{\gamma^1, \dots, \gamma^m\}$ is disjoint if $(\gamma^i) \cap (\gamma^j) = \emptyset$ for any $i, j = 1, \dots, m$, $i \neq j$.

Let $E \in \mathcal{C}_b^2(\mathbf{R}^2)$; we say that a disjoint system Γ of curves is an oriented parametrization of ∂E if any curve of the system is simple, $(\Gamma) = \partial E$, and

$$E = \{z \in \mathbf{R}^2 : I(\Gamma, z) = 1\}, \quad \mathbf{R}^2 \setminus \overline{E} = \{z \in \mathbf{R}^2 : I(\Gamma, z) = 0\}.$$

It is not difficult to prove that each $E \in \mathcal{C}_b^2(\mathbf{R}^2)$ admits an oriented parametrization of class C^2.

2.2. The space BV. If λ is a scalar or vector-valued Radon measure, its total variation will be denoted by $|\lambda|$. Let I be a real interval, $B \subseteq I$ be a Borel set and $f : B \to \mathbf{R}$ be a Borel function; the integral of f on B with respect to λ will be indicated by $\int_B f\lambda$. By $\lambda \llcorner B$ we mean the measure defined by $\lambda \llcorner B(F) = \lambda(B \cap F)$ for any Borel set F.

If μ is a scalar Radon measure on I, we have a unique decomposition $\lambda = \lambda^a + \lambda^s$, where λ^a is absolutely continuous and λ^s is singular with respect to μ. The density of λ^a with respect to μ, which is a function belonging to L^1_μ, will be indicated by $\frac{d\lambda}{d\mu}$ or by $\frac{\lambda}{\mu}$, and will be called the Radon-Nikodym derivative of λ with respect to μ. Then

$$\frac{d\lambda}{d\mu}(t) = \lim_{\varrho \to 0} \frac{\lambda(B_\varrho(t))}{\mu(B_\varrho(t))} \qquad \text{for } \mu - \text{ a.e. } t \in I, \tag{2.1}$$

and $\lambda(B) = \int_B \frac{d\lambda}{d\mu} d\mu + \lambda^s(B)$ for every Borel set $B \subseteq I$.

The space $BV(I)$ is defined as the space of the functions $f \in L^1_{\text{loc}}(I)$ whose distributional gradient $\dot{f}$ is a Radon measure with bounded total variation in I. We say that $f = (f_1, f_2) \in BV(I; \mathbf{R}^2)$ if $f_i \in BV(I)$ for any $i = 1, 2$. We recall that any function in $BV(I)$ coincides with a function of bounded variation in the classical sense.

If $f \in BV(I)$ we indicate by S_f the jump set of f, and by $f(t-)$, $f(t+)$ the approximate lower and upper limits of f at the point $t \in I$. It is well known that S_f is at most countable. If $f = (f_1, f_2) \in BV(I; \mathbf{R}^2)$, by S_f we mean $S_{f_1} \cup S_{f_2}$. We shall write

$$\dot{f} = \dot{f}^a \, dt + \dot{f}^s.$$

Here $\dot{f}^a$ denotes the density of the absolutely continuous part of $\dot{f}$ with respect to the Lebesgue measure dt and coincides almost everywhere with the pointwise derivative of the function $f(t+)$; the singular part $\dot{f}^s$ can be written on a Borel set $B \subseteq I$ as

$$\dot{f}^s(B) = \int_{B \cap S_f} (f(t+) - f(t-)) d\mathcal{H}^0(t) + (Cf)(B),$$

where Cf is the so-called Cantor part, and is a measure such that $(Cf)(\{t\}) = 0$ for any $t \in B$.

We say that a sequence $\{f_h\}_h \subseteq BV(I;\mathbf{R}^2)$ is weakly convergent in BV to a function $f \in BV(I;\mathbf{R}^2)$ if $f_h \to f$ in L^1 and $\dot{f}_h \rightharpoonup \dot{f}$ weakly as measures as $h \to +\infty$.

We recall the following result (see [28, Sec. 13.2]).

THEOREM 2.1. *Let $f \in BV(I)$, and let $\phi \in \mathcal{C}^1(\mathbf{R})$ be a Lipschitz continuous function. Then the composite function $\phi(f)$ is of class $BV(I)$, and its distributional derivative $\dot{\phi(f)}$ can be written as*

$$\dot{\phi(f)} = \dot{\phi}(f)\dot{f}\llcorner(I \setminus S_f) + \sum_{t \in S_f} [\phi(f(t+)) - \phi(f(t-))]\delta_t, \tag{2.2}$$

where δ_t is the Dirac mass at the point $t \in S_f \subseteq I$.

For the definitions and the main properties of the functions of bounded variation we refer to [28,9,18,15,20].

2.3. Systems of curves of class $\mathcal{B}$.

DEFINITION 2.2. *We say that a system of curves $\Gamma = \{\gamma^1, \dots, \gamma^m\}$ is of class $\mathcal{B}$ if $\gamma^i \in W^{1,1}([0,1])$ and $\frac{d\gamma^i}{dt} \in BV([0,1];\mathbf{R}^2)$ for any $i = 1, \dots, m$.*

DEFINITION 2.3. *We say that a sequence $\{\Gamma_h\}_h$ of systems of curves of class $\mathcal{C}^2$ is weakly convergent to a system of curves $\Gamma = \{\gamma^1, \dots, \gamma^m\}$ if the number of curves of each system Γ_h is the same as the number of curves of Γ for h large enough, i.e., $\Gamma_h = \{\gamma_h^1, \dots, \gamma_h^m\}$ for any h, and, in addition, $\gamma_h^i \to \gamma^i$ uniformly in $\mathcal{C}^0$, $\frac{d\gamma_h^i}{dt} \rightharpoonup \frac{d\gamma^i}{dt}$ weakly in BV as $h \to +\infty$, for any $i = 1, \dots, m$.*

DEFINITION 2.4. *We say that Γ is a limit system of curves if Γ is the weak limit of a sequence of oriented parametrizations of bounded open sets of class $\mathcal{C}^2$.*

It follows from the definitions that, if Γ is a limit system of curves, then $\Gamma \in \mathcal{B}$, and $I(\Gamma, z) \in \{0,1\}$ for any $z \in \mathbf{R}^2 \setminus (\Gamma)$.

Note that if the sequence of systems $\{\Gamma_h\}_h$ of class $\mathcal{C}^2$ is weakly convergent to $\Gamma = \{\gamma^1, \dots, \gamma^m\} \in \mathcal{B}$, then, for any $i = 1, \dots, m$ we have

$$l(\gamma_h^i) = \int_0^1 |\frac{d\gamma_h^i}{dt}|\, dt \to \int_0^1 |\frac{d\gamma^i}{dt}|\, dt = l(\gamma^i) \qquad \text{as } h \to +\infty.$$

Possibly passing to a subsequence, we have that $|\frac{d\gamma_h^i}{dt}| \to |\frac{d\gamma^i}{dt}|$ almost everywhere on $[0,1]$ as $h \to +\infty$. Hence $|\frac{d\gamma^i}{dt}|$ is constant almost everywhere on $[0,1]$, so that each curve γ^i can be parametrized by arc lenght.

2.4. Relaxation. Given a functional $\mathcal{L} : \mathcal{M} \to [0, +\infty]$, we denote by $\overline{\mathcal{L}}$ the lower semicontinuous envelope (or relaxed functional) of $\mathcal{L}$ with respect to the topology of $L^1(\mathbf{R}^2)$ (see (1.4)). For the main properties of the relaxed functionals we refer to [7].

3. The argument functions.

Let us prove the following Lemma.

LEMMA 3.1. *Let $[a,b] \subseteq \mathbf{R}$ be a bounded interval, and let $f \in BV([a,b];\mathbf{R}^2)$ be such that $|f(t)| = R$ for almost every $t \in [a,b]$, for a suitable $R > 0$. Then there exists a function $\Theta : [a,b] \to \mathbf{R}$, that we shall call an argument of f, having the following properties:*

(a) $\Theta \in BV([a,b])$ and $f(t) = (R\cos\Theta(t), R\sin\Theta(t))$ for almost every $t \in [a,b]$;

(b) $S_\Theta = S_f$;

(c) $|\Theta(t+) - \Theta(t-)| \leq \pi$ for every $t \in]a,b[$.

Moreover condition (c) can be replaced by one of the following stronger conditions:

$$-\pi < \Theta(t+) - \Theta(t-) \leq \pi \qquad \forall t \in]a,b[, \tag{3.1}$$

or

$$-\pi \leq \Theta(t+) - \Theta(t-) < \pi \qquad \forall t \in]a,b[. \tag{3.2}$$

When (3.1) or (3.2) hold then Θ is unique up to a global translation of an integer multiple of 2π.

PROOF. Set $O_1 = \partial B_R(0) \cap \{x > 0\}$, $O_2 = \partial B_R(0) \cap \{y > 0\}$, $O_3 = \partial B_R(0) \cap \{x < 0\}$, $O_4 = \partial B_R(0) \cap \{y < 0\}$. If the image of $f = (f_1, f_2)$ is contained in one of the sets O_i, for instance in O_1, then the function Θ is simply defined as $\Theta(t) = \arcsin(\frac{1}{R} f_2(t))$.

Let us consider now the general case. Let us fix $0 < \sigma < \frac{R}{4}$. We claim that there exists a finite set $\{s_1, \ldots, s_m\}$ of points of $]a,b[$ such that

$$|\dot{f}|(J) < 2\sigma < \frac{R}{2}, \tag{3.3}$$

where J is any open interval contained in $[a,b] \setminus \{s_1, \ldots, s_m\}$.

The set $S_f = \{t \in]a,b[: |\dot{f}|(\{t\}) > 0\}$ is at most countable, and we denote it by $\{t_i\}_i$, for $i \in \mathbf{N}, i \geq 1$, and $t_i < t_{i+1}$. Hence there exist a sequence $\{v_i\}_i \subseteq \mathbf{R}^2$ and an $\mathbf{R}^2$-valued Radon measure μ on $[a,b]$ such that

$$\dot{f} = \mu + \sum_{i=1}^{+\infty} v_i \delta_i, \qquad \sum_{i=1}^{+\infty} |v_i| < +\infty, \qquad \text{and} \qquad \mu(\{t\}) = 0 \quad \forall t \in [a,b],$$

where δ_i is the Dirac mass at the point t_i.

Let $k \in \mathbf{N}$, $k \geq 1$, be such that

$$\sum_{i=k}^{+\infty} |v_i| < \sigma. \tag{3.4}$$

Let $g : [a,b] \to \mathbf{R}$ be defined by $g(t) = |\mu|([a,t]) = |\mu|(]a,t[)$. Since $\mu(\{t\}) = 0$ for any $t \in [a,b]$, g is continuous, and hence uniformly continuous on $[a,b]$. As a consequence, there exists a finite number of points $\{\tau_1, \dots, \tau_r\} \subseteq]a,b[$ such that, if I denotes any interval contained in $[a,b] \setminus \{\tau_1, \dots, \tau_r\}$, then

$$|\mu|(I) < \sigma. \tag{3.5}$$

Denote by $\{s_1, \dots, s_m\}$ the ordered set of points of $]a,b[$ given by the union of $\{t_1, \dots, t_k\}$ and $\{\tau_1, \dots, \tau_r\}$, $m \leq k + r$. Set $s_0 = a$, $s_{m+1} = b$. Then, for any $j = 0, \dots, m$ and any $t \in]s_j, s_{j+1}]$, by (3.5) and (3.4) we have

$$|f(t-) - f(s_j+)| \leq |\dot{f}|(]s_j, t[) \leq |\mu|(]s_j, t[) + \sum_{i \geq k : t_i \in]s_j, t[} |v_i| \leq 2\sigma < \frac{R}{2}, \tag{3.6}$$

and this proves claim (3.3).

From (3.6) it follows that

$$\forall j = 0, \dots, m \quad \exists n \in \{1, \dots, 4\} \text{ such that } f(]s_j, s_{j+1}[) \subset\subset O_n.$$

It is then sufficient to define the function Θ separately on each interval $]s_j, s_{j+1}[$ by means of functions Θ_j, $0 \leq j \leq m$, and then to glue together all these functions, possibly adding to the values of Θ_j suitable integer multiples of 2π, in order to fulfill condition (c). Assertion (3.1) (respectively (3.2)) can be easily obtained by properly gluing the functions Θ_j. The uniqueness of Θ (up to global translations of an integer multiple of 2π) follows from the previous contruction. □

Let γ be a curve of class B such that $|\dot{\gamma}(s)| = 1$ for almost every $s \in [0, l(\gamma)] = I$. Let Θ be an argument of $\dot{\gamma}$. Using formula (2.2), (a) of Lemma 3.1, and the uniqueness of the Lebesgue decomposition of a measure, it is not difficult to show that

$$\ddot{\gamma}^a = (-\sin\Theta, \cos\Theta)\dot{\Theta}^a \qquad \text{a.e. in } I \tag{3.7}$$

and, if $B \subseteq I$ is a Borel set, then

$$\begin{aligned} (\ddot{\gamma}^s)(B) = &\int_{B \cap (I \setminus S_\Theta)} (-\sin\Theta, \cos\Theta)\dot{\Theta}^s + \\ &\sum_{s \in B \cap S_\Theta} (\cos\Theta(s+) - \cos\Theta(s-), \sin\Theta(s+) - \sin\Theta(s-))\delta_s. \end{aligned} \tag{3.8}$$

Therefore (3.7) yields

$$|\ddot{\gamma}^a| = |\dot{\Theta}^a| \qquad \text{a.e in } I.$$

Note also that

$$|\ddot{\gamma}^s|(\{s\}) < |\dot{\Theta}^s|(\{s\}) \qquad \forall s \in S_\Theta.$$

Indeed, for any $s \in S_\Theta$, using (3.8) and the equality $2(1 - \cos\phi) = 4\sin^2(\frac{\phi}{2})$, we have

$$\begin{aligned} &(|\ddot{\gamma}^s|(\{s\}))^2 = |(\cos\Theta(s+) - \cos\Theta(s-), \sin\Theta(s+) - \sin\Theta(s-))|^2 = \\ &4\sin^2\left(\frac{\Theta(s+) - \Theta(s-)}{2}\right) \leq (\Theta(s+) - \Theta(s-))^2 = (|\dot{\Theta}^s|(\{s\}))^2. \end{aligned}$$

4. The space $BV([a,b]; \mathbf{R}/2\pi)$.

DEFINITION 4.1. *Let $\theta_1, \theta_2 \in BV([a,b])$. We say that θ_1 and θ_2 are equivalent, and we write $\theta_1 \sim \theta_2$ if for almost every $t \in [a,b]$ there exists $k(t) \in \mathbf{Z}$ such that $\theta_1(t) = \theta_2(t) + 2k(t)\pi$.*

Given $\theta \in BV([a,b])$, we denote by $[\theta]$ the equivalence class of θ, i.e., $[\theta] = \{\underline{\theta} \in BV([a,b]) : \underline{\theta} \sim \theta\}$. We set

$$BV([a,b];\mathbf{R}/2\pi) = \{[\theta] : \theta \in BV([a,b])\}.$$

Let $f \in BV([a,b];\mathbf{R}^2)$ be such that $|f(t)| = R > 0$ for almost every $t \in [a,b]$. Note that, if $\theta_1, \theta_2 : [a,b] \to \mathbf{R}$ are two arguments of f, then $\theta_1 \sim \theta_2$. Even more, the function $\theta_1 - \theta_2$ is constant on each open interval contained in $[a,b] \setminus \{t \in [a,b] : f(t+) = -f(t-)\}$, and its values are integer multiples of 2π.

DEFINITION 4.2. *Let $[\theta] \in BV([a,b];\mathbf{R}/2\pi)$, and let $\{[\theta_h]\}_h \subseteq BV([a,b];\mathbf{R}/2\pi)$ be a sequence. We say that $[\theta_h] \to [\theta]$ in $L^1([a,b];\mathbf{R}/2\pi)$ as $h \to +\infty$ if, for any $\underline{\theta} \in [\theta]$ and any $h \in \mathbf{N}$, there exists $\underline{\theta}_h \in [\theta_h]$ such that $\underline{\theta}_h \to \underline{\theta}$ in $L^1([a,b])$ as $h \to +\infty$.*

Note that if $\theta \in BV([a,b])$, $\{\theta_h\}_h \subseteq BV([a,b])$, and $\theta_h \to \theta$ in $L^1([a,b])$ as $h \to +\infty$, then $[\theta_h] \to [\theta]$ in $L^1([a,b];\mathbf{R}/2\pi)$ as $h \to +\infty$.

THEOREM 4.1. *Let $[\theta] \in BV([a,b];\mathbf{R}/2\pi)$. Then there exists a function $\Theta \in [\theta]$, that we shall call a minimal representative of $[\theta]$, such that*

$$|\dot{\Theta}|([a,b]) = \inf\{|\dot{\underline{\theta}}|([a,b]) : \underline{\theta} \in [\theta]\}. \tag{4.1}$$

PROOF. Let $\underline{\theta} \in [\theta]$; as $\underline{\theta} \in BV([a,b])$, the set $\{t \in]a,b[: |\underline{\theta}(t+) - \underline{\theta}(t-)| > \pi\}$ has a finite number of elements $\{t_1, \dots, t_N\}$, where $a < t_1 < \cdots < t_N < b$. Starting from $\underline{\theta}$, we construct $\Theta \sim \underline{\theta}$, with $S_\Theta \subseteq S_{\underline{\theta}}$, by adding to $\underline{\theta}$ separately on each connected component of $[a,b] \setminus \{t_1, \dots, t_N\}$, a suitable integer multiple of 2π, in such a way that

$$|\Theta(t+) - \Theta(t-)| \leq \pi \qquad \forall t \in]a,b[\tag{4.2}$$

(for instance, set $\Theta = \underline{\theta}$ on $[a,t_1[$, and $\Theta = \underline{\theta} + 2n_1\pi$ on $]t_1,b]$, where $n_1 \in \mathbf{Z}$ is such that $|\Theta(t_1+) - \Theta(t_1-)| \leq \pi$, and continue).

Let us prove (4.1). Assume by contradiction that there exists $\theta_1 \in [\theta]$ such that

$$|\dot{\theta}_1|([a,b]) < |\dot{\Theta}|([a,b]). \tag{4.3}$$

As $\Theta \in BV([a,b])$, there exists a finite set of points $a < \tau_0 < \cdots < \tau_n < b$ such that

$$|\dot{\Theta}|(]\tau_{i-1}, \tau_i[) < \frac{\pi}{2} \qquad \forall i = 1, \dots, n. \tag{4.4}$$

As $\theta_1 \sim \Theta$, for any $i = 1, \dots, n$ there exists $k_i \in \mathbf{Z}$ such that $\theta_1(\tau_i+) - \theta_1(\tau_i-) = \Theta(\tau_i+) - \Theta(\tau_i-) + 2k_i\pi$; since $|\Theta(\tau_i+) - \Theta(\tau_i-)| \leq \pi$ by (4.2), we get $|\theta_1(\tau_i+) - \theta_1(\tau_i-)| \geq |\Theta(\tau_i+) - \Theta(\tau_i-)|$. Therefore, by (4.3) we deduce that there exists $j \in \{1, \dots, n\}$ such that, letting $I =]\tau_{j-1}, \tau_j[$,

$$|\dot{\theta}_1|(I) < |\dot{\Theta}|(I). \tag{4.5}$$

Using (4.4), we have that $\Theta(I)$ is contained, up to a set of zero Lebesgue measure, into a real interval of lenght less than $\frac{\pi}{2}$, and by (4.5) the same property holds for $\theta_1(I)$. Let us fix $\tau \in I$ a continuity point of Θ and θ_1, let $k \in \mathbf{Z}$ be such that $\theta_1(\tau) = \Theta(\tau) + 2k\pi$, and let $\theta_2 = \theta_1 - 2k\pi$. Then $\theta_2 \sim \theta_1 \sim \Theta$, $\dot{\theta}_2 = \dot{\theta}_1$, and $\theta_2(I) \cap \Theta(I)$ is non empty, since it contains $\Theta(\tau)$. Hence $\theta_2(I) \cup \Theta(I)$ is contained, up to a set of zero Lebesgue measure, into a real interval of lenght less than π. As $\theta_2 \sim \Theta$, we necessarily have that $\theta_2(I)$ coincides with $\Theta(I)$ up to a set of zero Lebesgue measure. It follows that $|\dot{\theta}_1|(I) = |\dot{\theta}_2|(I) = |\dot{\Theta}|(I)$, and this contradicts (4.5). □

We shall denote by $\mathcal{M}[\theta]$ the set of the minimal representatives Θ of $[\theta] \in BV([a,b]; \mathbf{R}/2\pi)$ such that $\Theta(a+) \in [0, 2\pi[$.

Let $\theta \in BV([a,b]; \mathbf{R}/2\pi)$, and let $\underline{\theta} \in [\theta]$. Observe that, if $\underline{\theta}(a+) \in [0, 2\pi[$ and $|\underline{\theta}(t+) - \underline{\theta}(t-)| \leq \pi$ for any $t \in [a,b]$, then $\underline{\theta} \in \mathcal{M}[\theta]$. In addition, if $\Theta_1, \Theta_2 \in \mathcal{M}[\theta]$, then

$$|\dot{\Theta}_1^s|(B) = |\dot{\Theta}_2^s|(B) \qquad \text{for any Borel set } B \subseteq [a,b]. \tag{4.6}$$

Note that, given $[\theta] \in BV([a,b]; \mathbf{R}/2\pi)$ and $\{[\theta_h]\}_h \subseteq BV([a,b]; \mathbf{R}/2\pi)$, then $[\theta_h] \to [\theta]$ in $L^1([a,b]; \mathbf{R}/2\pi)$ as $h \to +\infty$ if and only if for any $\Theta \in \mathcal{M}[\theta]$ and any $h \in \mathbf{N}$ there exists $\underline{\theta}_h \in [\theta_h]$ such that $\underline{\theta}_h \to \Theta$ in $L^1([a,b])$ as $h \to +\infty$. Note also that, in general, if $\Theta_h \in \mathcal{M}[\theta_h]$, the sequence $\{\Theta_h\}_h$ does not converge to a minimal representative Θ of $[\theta]$ in $L^1([a,b])$, unless all jumps of Θ are less than or equal to π and $\Theta(a+) \in [0, 2\pi[$.

LEMMA 4.1. *Let $[\theta] \in BV([a,b]; \mathbf{R}/2\pi)$ and let $\theta_1, \theta_2 \in [\theta]$. Then*

$$|\theta_1^a| = |\theta_2^a| \qquad \text{a.e. in } [a,b]. \tag{4.7}$$

PROOF. As θ_1, $\theta_2 \in BV([a,b])$ there exists a finite set of points $a = \tau_0 < \tau_1 < \cdots < \tau_n < \tau_{n+1} = b$ such that $|\dot{\theta}_i|(]\tau_j, \tau_{j+1}[) < \frac{\pi}{4}$ for any $i = 1, 2$ and any $j = 0, \dots, n$. The assertion then follows reasoning as at the end of the proof of Theorem 4.1 (see (4.5) and below). □

We conclude this section with a result that will be useful in the proof of Theorem 6.1.

PROPOSITION 4.1. *Let $\gamma \in \mathcal{B}$, let $\{\gamma_h\}_h \subseteq \mathcal{B}$ be a sequence of curves such that $\dot{\gamma}_h \to \dot{\gamma}$ in $L^1([0,1]; \mathbf{R}^2)$ as $h \to +\infty$. Assume that γ and γ_h are parametrized with constant*

velocity on $[0,1]$, *for any* $h \in \mathbf{N}$. *Let* $\theta, \theta_h \in BV([0,1])$ *be arguments of* $\dot\gamma, \dot\gamma_h$ *respectively, for any* $h \in \mathbf{N}$. *Then* $\theta_h \to \theta$ *in* $L^1([0,1];\mathbf{R}/2\pi)$ *as* $h \to +\infty$.

PROOF. Write $\dot\gamma(t) = (R\cos\theta(t), R\sin\theta(t))$, and $\dot\gamma_h(t) = (R_h\cos\theta_h(t), R_h\sin\theta_h(t))$ for almost every $t \in [0,1]$ and any $h \in \mathbf{N}$ (see Lemma 3.1). We have $\lim_{h\to+\infty} R_h = R$. Using the notation of Lemma 3.1, there exists a finite set $\{s_1,\dots,s_m\}$ of points of $]0,1[$ such that, if J is any open interval contained in $[a,b]\setminus\{s_1,\dots,s_m\}$, then

$$\dot\gamma(J) \subset\subset A \subset\subset O_n, \tag{4.8}$$

for a suitable arc $A \subseteq \partial B_R(0)$ and a suitable $n \in \{1,\dots,4\}$. Let $\underline\theta \in [\theta]$; we have to prove that for any $h \in \mathbf{N}$ there exists $\underline\theta_h \in [\theta_h]$ such that $\underline\theta_h \to \underline\theta$ in $L^1([0,1])$ as $h \to +\infty$. It will be sufficient to construct the sequence $\{\underline\theta_h\}_h$ on each interval J, and to prove that $\underline\theta_h \to \underline\theta$ in $L^1(J)$. Let us fix such an interval J; without loss of generality, we can assume that $\dot\gamma(J) \subset\subset A \subset\subset O_1$. Let $B \subseteq \partial B_R(0)$ be an arc such that $A \subset\subset B \subset\subset O_1$ and, for any $h \in \mathbf{N}$, let $A_h \subset\subset B_h \subset\subset O_1^h = \partial B_{R_h}(0) \cap \{x > 0\}$ be the corresponding arcs on $\partial B_{R_h}(0)$ omothetic to A, B, and O_1, respectively. Let $\tau \in I$ be a continuity point of $\underline\theta$. For any $h \in \mathbf{N}$, define $\underline\theta_h : J \to \mathbf{R}$, $\underline\theta_h \in [\theta_h]$, as follows:

$$\underline\theta_h(t) = \begin{cases} \theta_h(t) & \text{for a.e. } t \in J \text{ such that } \dot\gamma_h(t) \in B_h, \\ \alpha_h(t) & \text{for a.e. } t \in J \text{ such that } \dot\gamma_h(t) \notin B_h, \end{cases}$$

where $\alpha_h \in BV([0,1])$ is an arbitrary function belonging to $[\theta_h]$ such that

$$|\alpha_h(t) - \theta(\tau)| \le \pi \qquad \text{for a.e. } t \in J. \tag{4.9}$$

To prove that $\underline\theta_h \to \underline\theta$ in $L^1(J)$ as $h \to +\infty$, we show that there exists a constant $M > 0$ such that

$$|\underline\theta_h(t) - \underline\theta(t)| \le M|\dot\gamma_h(t) - \dot\gamma(t)| \qquad \text{for a.e. } t \in J,\ \forall h \in \mathbf{N}. \tag{4.10}$$

Let $h \in \mathbf{N}$ and $t \in J$; if $\dot\gamma_h(t) \in B_h$, then (4.10) follows from the lipschitzianity of arcsin (see the definition of the argument in the proof of Lemma 3.1), since B_h is relatively compact in O_1^h, and $R_h \to R$ as $h \to +\infty$. If $\dot\gamma_h(t) \notin B_h$, then, as $\dot\gamma(J) \subset\subset A$ and $R_h \to R$ as $h \to +\infty$, there exists a constant $c > 0$ depending only on $B \setminus A$ such that $|\dot\gamma_h(t) - \dot\gamma(t)| \ge c$. From (4.8) we have that $|\underline\theta(\tau) - \underline\theta(t)| < \pi$; hence, using (4.9), we obtain

$$|\underline\theta_h(t) - \underline\theta(t)| \le |\alpha_h(t) - \underline\theta(\tau)| + |\underline\theta(\tau) - \underline\theta(t)| \le \pi + \pi \le \frac{2\pi}{c}|\dot\gamma_h(t) - \dot\gamma(t)|,$$

and this concludes the proof of (4.10). □

5. The energy functionals.

Let $\psi : \mathbf{R} \times \mathbf{R} \to [0, +\infty]$ be a Borel function satisfying properties (i)-(vi) listed in Section 1. Denote by ψ_∞ the recession function of ψ with respect to ξ, i.e.,

$$\psi_\infty(\eta, \xi) = \lim_{t \to 0+} t\psi(\eta, \frac{\xi}{t}) \qquad \forall (\eta, \xi) \in \mathbf{R} \times \mathbf{R}.$$

Let $\theta \in BV([a,b]; \mathbf{R}/2\pi)$, $\underline{\theta} \in [\theta]$ and $\Theta \in \mathcal{M}[\theta]$; for any open interval $I \subseteq [a,b]$ define

$$K_\psi(\theta, I) = K^a_\psi(\theta, I) + K^s_\psi(\theta, I),$$

where

$$K^a_\psi(\theta, I) = \int_I \psi(\underline{\theta}, \dot{\underline{\theta}}^a)\, dt,$$

and

$$K^s_\psi(\theta, I) = \int_I \int_{\Theta(s-)}^{\Theta(s+)} \psi_\infty(\tau, \frac{d\dot{\Theta}^s}{d|\dot{\Theta}^s|})\, d\tau\, d|\dot{\Theta}^s|.$$

Observe that by condition (v) and by (4.7) the term $K^a_\psi(\theta, I)$ does not depend on the choice of $\underline{\theta} \in [\theta]$, and by (4.6) the term $K^s_\psi(\theta, I)$ does not depend on the choice of $\Theta \in \mathcal{M}[\theta]$. In addition

$$K^s_\psi(\theta, I) \le \int_I \int_{\underline{\theta}(s-)}^{\underline{\theta}(s+)} \psi_\infty(\tau, \frac{d\dot{\underline{\theta}}^s}{d|\dot{\underline{\theta}}^s|})\, d\tau\, d|\dot{\underline{\theta}}^s| \qquad \forall\, \underline{\theta} \in [\theta]. \tag{5.1}$$

Note that, using condition (vi),

$$K^s_\psi(\theta, I) = \int_{I \setminus S_\Theta} \psi_\infty(\Theta(s+), 1)\, d|\dot{\Theta}^s| + \sum_{s \in I \cap S_\Theta} \int_{\Theta(s-)}^{\Theta(s+)} \psi_\infty(\tau, 1)\, d\tau, \tag{5.2}$$

and, if $\theta \in C^2([a,b])$, then

$$K_\psi(\theta, I) = K^a_\psi(\theta, I) = \int_I \psi(\theta, \dot{\theta})\, dt.$$

5.1. Semicontinuity and relaxation of $K_\psi(\theta, [a,b])$.

THEOREM 5.1. *Let $\psi : \mathbf{R} \times \mathbf{R} \to [0, +\infty]$ be a Borel function satisfying properties (i)-(vi), and let $I \subseteq [a,b]$ be an open interval. Let $[\theta] \in BV(I; \mathbf{R}/2\pi)$, and let $\{[\theta_h]\}_h \subseteq BV(I; \mathbf{R}/2\pi)$ be such that $[\theta_h] \to [\theta]$ in $L^1(I; \mathbf{R}/2\pi)$ as $h \to +\infty$. Then*

$$K_\psi(\theta, I) \le \liminf_{h \to +\infty} K_\psi(\theta_h, I). \tag{5.3}$$

PROOF. We can suppose that the right hand side of (5.3) is finite, otherwise the result is trivial. Let $\{[\theta_k]\}_k$ be a subsequence of $\{[\theta_h]\}_h$ such that

$$\lim_{k\to+\infty} K_\psi(\theta_k, I) = \liminf_{h\to+\infty} K_\psi(\theta_h, I) < +\infty.$$

Let $\Theta \in \mathcal{M}[\theta]$ and $\underline{\theta}_k \in [\theta_k]$ be such that $\underline{\theta}_k \to \Theta$ in $L^1(I)$ as $k \to +\infty$. By (1.1) we have

$$\sup_k |\dot{\Theta}_k|(I) \le \sup_k |\dot{\theta}_k|(I) < +\infty.$$

Hence [18] there exist a subsequence (still denoted by $\{\Theta_k\}_k$) and a function $\alpha \in BV(I)$ such that $\Theta_k \to \alpha$ in $L^1(I)$ as $k \to +\infty$. It follows that $[\Theta_k] \to [\alpha]$ in $L^1(I; \mathbf{R}/2\pi)$ as $k \to +\infty$, so that $\alpha \in [\theta]$. Note that α does not necessarily belong to $\mathcal{M}[\theta]$. Using (5.1) and [11, Theorem 3.1] we then have

$$K_\psi(\theta, I) = \int_I \psi(\alpha, \dot{\alpha}^a)\, dt + \int_I \fint_{\Theta(s-)}^{\Theta(s+)} \psi_\infty(\tau, \frac{d\dot{\Theta}^s}{d|\dot{\Theta}^s|})\, d\tau\, d|\dot{\Theta}^s| \le$$

$$\int_I \psi(\alpha, \dot{\alpha}^a)\, dt + \int_I \fint_{\alpha(s-)}^{\alpha(s+)} \psi_\infty(\tau, \frac{d\dot{\alpha}^s}{d|\dot{\alpha}^s|})\, d\tau\, d|\dot{\alpha}^s| \le$$

$$\liminf_{k\to+\infty} \left(\int_I \psi(\Theta_k, \dot{\Theta}_k^a)\, dt + \int_I \fint_{\Theta_k(s-)}^{\Theta_k(s+)} \psi_\infty(\tau, \frac{d\dot{\Theta}_k^s}{d|\dot{\Theta}_k^s|})\, d\tau\, d|\dot{\Theta}_k^s| \right) =$$

$$\lim_{k\to+\infty} K_\psi(\theta_k, I) = \liminf_{h\to+\infty} K_\psi(\theta_h, I).$$

□

THEOREM 5.2. *Let $\psi : \mathbf{R} \times \mathbf{R} \to [0, +\infty]$ be a Borel function satisfying properties (i),(ii), (v), (vi), such that for any η the function $\xi \to \psi(\eta, \xi)$ is positively homogeneous of degree one on $\mathbf{R}$, and such that $0 \le \psi(\eta, \xi) \le M(\eta)|\xi|$ for all $(\eta, \xi) \in \mathbf{R} \times \mathbf{R}$, for a suitable continuous function $M : \mathbf{R} \to \mathbf{R}^+$. Let $I \subseteq [a, b]$ be an open interval. Then for any $[\theta] \in BV(I; \mathbf{R}/2\pi)$ we have*

$$K_\psi(\theta, I) = \inf\{\liminf_{h\to+\infty} K_\psi(\theta_h, I) : \{[\theta_h]\}_h \subseteq BV(I; \mathbf{R}/2\pi), [\theta_h] \to [\theta] \text{ in } L^1(I; \mathbf{R}/2\pi)\}.$$

PROOF. Let $[\theta] \in BV(I; \mathbf{R}/2\pi)$; by Theorem 5.1 we deduce

$$K_\psi(\theta, I) \le \inf\{\liminf_{h\to+\infty} K_\psi(\theta_h, I) : \{[\theta_h]\}_h \subseteq BV(I; \mathbf{R}/2\pi), [\theta_h] \to [\theta] \text{ in } L^1(I; \mathbf{R}/2\pi)\}.$$

Let us prove the opposite inequality. Let $G : BV(I) \to [0, +\infty]$ be the map defined by

$$G(\theta, I) = \begin{cases} \displaystyle\int_I \psi(\theta, \dot{\theta})\, dt & \text{if } \theta \in W^{1,1}(I), \\ +\infty & \text{elsewhere on } BV(I). \end{cases}$$

In view of the assumptions on ψ, by [11, Theorem 3.5] we have, for any $\theta \in BV(I)$,

$$K_\psi(\theta, I) = \overline{G}(\Theta, I) =$$
$$\inf\{\liminf_{h\to+\infty} G(\theta_h, I) : \{\theta_h\}_h \subseteq W^{1,1}(I), \theta_h \to \Theta \text{ in } L^1(I)\} =$$
$$\inf\{\liminf_{h\to+\infty} K_\psi(\theta_h, I) : \{\theta_h\}_h \subseteq W^{1,1}(I), \theta_h \to \Theta \text{ in } L^1(I)\} \geq$$
$$\inf\{\liminf_{h\to+\infty} K_\psi(\theta_h, I) : \{\theta_h\}_h \subseteq BV(I), \theta_h \to \Theta \text{ in } L^1(I)\} \geq$$
$$\inf\{\liminf_{h\to+\infty} K_\psi(\theta_h, I) : \{\theta_h\}_h \subseteq BV(I), [\theta_h] \to [\theta] \text{ in } L^1(I; \mathbb{R}/2\pi)\},$$

and this concludes the proof. □

Let $\Gamma = \{\gamma^1, \dots, \gamma^m\}$ be a system of curves of class $\mathcal{B}$ parametrized by arc lenght, and, for any $i = 1, \dots, m$ let Θ_i be an argument of $\dot\gamma^i$ on $[0, l(\gamma^i)]$. We define

$$K_\psi(\Gamma) = K^a_\psi(\Gamma) + K^s_\psi(\Gamma), \qquad F_\psi(\Gamma) := l(\Gamma) + K_\psi(\Gamma),$$

where

$$K^a_\psi(\Gamma) = \sum_{i=1}^m K^a_\psi(\Theta_i, [0, l(\gamma^i)]), \qquad K^s_\psi(\Gamma) = \sum_{i=1}^m K^s_\psi(\Theta_i, [0, l(\gamma^i)]).$$

Fix $i \in \{1, \dots, m\}$, and let for simplicity $\gamma^i = \gamma$, $\Theta_i = \Theta$, $\lambda = l(\gamma^i) > 0$. Let $T^\lambda : [0,1] \to [0, \lambda]$ be the map defined by $T^\lambda(t) = \lambda t$ for any $t \in [0,1]$, and denote by g the reparametrization of γ on $[0,1]$, i.e., $g(t) = \gamma(T^\lambda(t))$. Let ω be an argument of $\dot g$ on $[0,1]$. It is not difficult to prove that $\dot\omega(B) = \dot\Theta(T^\lambda(B))$ for any Borel set $B \subseteq [0,1]$. From the uniqueness of the Lebesgue decomposition of a measure, it follows that $\dot\omega^s(B) = \dot\Theta^s(T^\lambda(B))$. Hence, by (2.1), for almost every $t \in [0,1]$, we have

$$\dot\omega^a(t) = \lim_{\varrho\to0+} \frac{\dot\omega(]t-\varrho, t+\varrho[)}{2\varrho} = \lambda \lim_{\varrho\to0+} \frac{\dot\Theta(]\lambda t - \lambda\varrho, \lambda t + \lambda\varrho[)}{2\lambda\varrho} = \lambda\dot\Theta^a(T^\lambda(t)),$$

and, for $|\dot\omega^s|$-almost every $t \in [0,1]$, we have

$$\frac{d\dot\omega^s}{d|\dot\omega^s|}(t) = \lim_{\varrho\to0+} \frac{\dot\omega^s(]t-\varrho, t+\varrho[)}{|\dot\omega^s|(]t-\varrho, t+\varrho[)} = \lim_{\varrho\to0+} \frac{\dot\Theta^s(]\lambda t - \lambda\varrho, \lambda t + \lambda\varrho[)}{|\dot\Theta^s|(]\lambda t - \lambda\varrho, \lambda t + \lambda\varrho[)} = \frac{d\dot\Theta^s}{d|\dot\Theta^s|}(T^\lambda(t)).$$

We then have

$$K^a_\psi(\gamma) = \int_0^{l(\gamma)} \psi(\Theta, \dot\Theta^a)\, ds = l(\gamma) \int_0^1 \psi(\omega, l(\gamma)^{-1}\dot\omega^a)\, dt, \tag{5.4}$$

and

$$K^s_\psi(\gamma) = \int_0^{l(\gamma)} \fint_{\Theta(s-)}^{\Theta(s+)} \psi_\infty(\tau, \frac{d\dot\Theta^s}{d|\dot\Theta^s|})\, d\tau d|\dot\Theta^s| = \int_0^1 \fint_{\omega(t-)}^{\omega(t+)} \psi_\infty(\tau, \frac{d\dot\omega^s}{d|\dot\omega^s|})\, d\tau d|\dot\omega^s|. \tag{5.5}$$

6. Definition and semicontinuity of $F_\psi(E)$.

Let $E \in \mathcal{C}_b^2(\mathbf{R}^2)$, and let $\Gamma = \{\gamma^1, \dots, \gamma^m\}$ be an oriented parametrization of ∂E. The normal vector $n(z)$ at $z \in \partial E$ will point towards the inside of E. For any $z \in \partial E$ let $\alpha(z)$ be the angle that the tangent unit vector at $z \in \partial E$ forms with the positive direction of the x-axis, and let $\kappa(z)$ be the curvature of ∂E. If ψ is a function satisfying properties (i)-(vi), we define the map $F_\psi : \mathcal{M} \to [0, +\infty]$ as

$$F_\psi(E) = \begin{cases} \mathcal{H}^1(\partial E) + \displaystyle\int_{\partial E} \psi(\alpha, \kappa)\, d\mathcal{H}^1(z) & \text{if } E \in \mathcal{C}_b^2(\mathbf{R}^2), \\ \\ +\infty & \text{elsewhere on } \mathcal{M}. \end{cases} \tag{6.1}$$

We set $F_\psi(\emptyset) = 0$. Note that if $E \in \mathcal{C}_b^2(\mathbf{R}^2)$ and $\Gamma = \{\gamma^1, \dots, \gamma^m\}$ is an oriented parametrization of ∂E of class $\mathcal{C}^2$ such that each γ^i is parametrized by arc lenght, i.e., $\gamma^i(s) = (\cos \Theta_i(s), \sin \Theta_i(s))$ for any $s \in [0, l(\gamma^i)]$, then

$$\int_{\partial E} \psi(\alpha, \kappa) d\mathcal{H}^1(z) = \sum_{i=1}^m \int_0^{l(\gamma^i)} \psi(\Theta_i, \dot{\Theta}_i)\, ds = K_\psi(\Gamma),$$

$$F_\psi(E) = l(\Gamma) + K_\psi(\Gamma) = F_\psi(\Gamma).$$

Note that if $p > 1$ and $\psi(\eta, \xi) = |\xi|^p$ for any $(\eta, \xi) \in \mathbf{R} \times \mathbf{R}$, then

$$F_\psi(E) = \int_{\partial E} [1 + |\kappa|^p]\, d\mathcal{H}^1(z),$$

which reduces to the case studied in [3].

LEMMA 6.1. *Let $\Gamma = \{\gamma^1, \dots, \gamma^m\}$ be a system of curves of class $\mathcal{C}^2$. Then*

$$m \le \frac{\max(1, b)}{2\pi a} F_\psi(\Gamma), \tag{6.2}$$

where a and b are defined in (1.1).

PROOF. For any $i \in \{1, \dots, m\}$, if Θ_i is an argument of $\dot{\gamma}^i$, we have ([14, Th. 5.7.3])

$$2\pi \le \int_0^{l(\gamma^i)} |\dot{\Theta}_i(s)|\, ds.$$

Using (1.1) it follows

$$2\pi a \le \int_0^{l(\gamma^i)} \psi(\Theta_i, \dot{\Theta}_i)\, d\mathcal{L}^1(s) + l(\gamma^i) b \le \max(1, b) F_\psi(\gamma^i).$$

Summing over $i = 1, \dots, m$ we get

$$2\pi a m \le \max(1, b) F_\psi(\Gamma),$$

which is the assertion. □

PROPOSITION 6.1. *Let $\psi : \mathbf{R} \times \mathbf{R} \to [0, +\infty]$ be a Borel function satisfying conditions (i)-(vi). Let $\{\Gamma_h\}_h$ be a sequence of systems of curves of class $\mathcal{C}^2$ such that*

$$\sup_h l(\Gamma_h) < +\infty, \qquad \sup_h K_\psi(\Gamma_h) < +\infty, \tag{6.3}$$

and such that

(*) the traces (Γ_h) are contained in a bounded subset of $\mathbf{R}^2$ independent of h;

(**) there exists a positive constant c such that each curve γ_h^i of the system Γ_h satisfies $l(\gamma_h^i) \geq c$, for any h.

Then $\{\Gamma_h\}_h$ has a subsequence weakly convergent to a system of curves Γ of class $\mathcal{B}$.

PROOF. From (6.2), using (6.3) it follows that the number m_h of the curves of the system Γ_h is uniformly bounded with respect to h. Hence, for a subsequence $\{\Gamma_h\}_h$, there exists an integer m such that $\Gamma_h = \{\gamma_h^1, \dots, \gamma_h^m\}$ for any h. Let us fix $i \in \{1, \dots, m\}$; using condition (**) and (6.3) we have $c \leq l(\gamma_h^i) \leq C$ for any h, where C is an absolute positive constant. Hence the curve γ_h^i can be parametrized with constant velocity on $[0,1]$, and from (1.1) and (6.3), we get $\sup_h \int_0^1 |\frac{d^2\gamma_h^i}{d^2t}|\, dt < +\infty$. Therefore, using (*), the family $\{\gamma_h^i\}_h$ is uniformly bounded in $H^{2,1}$ with respect to h. It follows that ([18, Th. 1.19]), for a subsequence, there exist m curves $\gamma^1, \dots, \gamma^m$ of class $\mathcal{B}$ such that $\gamma_h^i \to \gamma^i$ uniformly in $\mathcal{C}^0$, $\frac{d\gamma_h^i}{dt} \rightharpoonup \frac{d\gamma^i}{dt}$ weakly in BV for $h \to +\infty$, for any $i = 1, \dots, m$. The system $\Gamma = \{\gamma^1, \dots, \gamma^m\}$ verifies the assertion. □

The following result gives some properties of those sets E such that $\overline{F_\psi}(E) < +\infty$.

LEMMA 6.2. *Let $E \subseteq \mathbf{R}^2$ be a measurable set, let $\{E_h\}_h \subseteq \mathcal{C}_b^2(\mathbf{R}^2)$ be a sequence satisfying*

$$\sup_h \mathcal{H}^1(\partial E_h) < +\infty, \qquad \sup_h \int_{\partial E_h} \psi(\alpha_h, \kappa_h)\, d\mathcal{H}^1(z), \tag{6.4}$$

and suppose that $E_h \to E$ in $L^1(\mathbf{R}^2)$ as $h \to +\infty$. Let us define

$$E^\star = \{z \in \mathbf{R}^2 : \exists r > 0\ \mathcal{L}^2(B_r(z) \setminus E) = 0\}, \quad F^\star = \{z \in \mathbf{R}^2 : \exists r > 0\ \mathcal{L}^2(B_r(z) \cap E) = 0\}.$$

Then $E^\star$ and $F^\star$ are open, $E^\star$ is bounded, $\mathcal{L}^2(E \triangle E^\star) = 0$, $E^\star = \mathrm{int}(\mathbf{R}^2 \setminus F^\star)$, $F^\star = \mathrm{int}(\mathbf{R}^2 \setminus E^\star)$, and

$$\partial E^\star = \partial F^\star = \{z \in \mathbf{R}^2 : 0 < \mathcal{L}^2(B_r(z) \cap E) < \mathcal{L}^2(B_r(z))\ \forall r > 0\}.$$

Moreover there exists a limit system of curves Γ of class $\mathcal{B}$ with the following properties:

$E^\star = \mathrm{int}\big(A_\Gamma \cup (\Gamma)\big) \supseteq A_\Gamma$, where $A_\Gamma = \{z \in \mathbf{R}^2 \setminus (\Gamma) : I(\Gamma, z) = 1\}$;

$F^\star = \mathrm{int}\big(B_\Gamma \cup (\Gamma)\big) \supseteq B_\Gamma$, where $B_\Gamma = \{z \in \mathbf{R}^2 \setminus (\Gamma) : I(\Gamma, z) = 0\}$;

$$\partial E^* = \partial F^* = \partial A_\Gamma \cap \partial B_\Gamma \subseteq (\Gamma).$$

PROOF. We shall prove the existence of the limit system of curves $\Gamma \in \mathcal{B}$ such that $\mathcal{L}^2(E \triangle A_\Gamma) = 0$, since all the remaining assertions can be proven as in [3, Lemma 3.3].

For any h, from (6.3) and from Lemma 6.1 we have that ∂E_h has an oriented parametrization Δ_h of class C^2, and the number m_h of curves of the system Δ_h is uniformly bounded with respect to h. Hence, for a subsequence (still denoted by $\{\Delta_h\}_h$), there exists an integer m such that $\Delta_h = \{\gamma_h^1, \dots, \gamma_h^m\}$ for any h.

For any h, we want to replace the system Δ_h with a suitable disjoint system of curves Γ_h satisfying all hypotheses of Proposition 6.1.

Following [3, Lemma 3.3], we can replace Δ_h with a disjoint system of curves $\Lambda_h = \{\gamma_h^{i_1}, \dots, \gamma_h^{i_k}\}$ of class C^2, with $1 \le i_1 < \cdots < i_k \le m$, having the following properties:

(6.5) there exists $R > 0$ such that $(\Lambda_h) \subseteq B_R(0)$ for any h;

for any $M > 0$ there exists $h_M \in \mathbf{N}$ such that

$$I(\Lambda_h, z) = I(\Delta_h, z) \tag{6.6}$$

for any $h \ge h_M$ and any $z \in B_M(0) \setminus (\Delta_h)$. Note that from (6.5) and (6.6) it follows that, if h is large enough, $I(\Lambda_h, z) \in \{0, 1\}$ for any $z \in \mathbf{R}^2 \setminus (\Lambda_h)$.

For simplicity, write $\Lambda_h = \{\gamma_h^1, \dots, \gamma_h^k\}$. We need a further substitution, in order to fulfill hypothesis (**) of Proposition 6.1.

Let us consider the sequence $\{\gamma_h^1\}_h$. If $\lim_{h \to +\infty} l(\gamma_h^1) = 0$, replace Λ_h by the system $\Lambda_h^1 = \{\gamma_h^2, \dots, \gamma_h^k\}$. Observe that (6.5) yields $(\gamma_h^1) \subseteq \overline{B_R(0)}$ for any h, so that, possibly passing to a subsequence $\{\gamma_h^1\}_h$, we can find $r_1 > 0$ e $z_1 \in B_R(0)$ such that, for any $0 < r \le r_1$, there exists $h_r \in \mathbf{N}$ with $(\gamma_h^1) \subseteq B_r(z_1)$ for any $h \ge h_r$. Thus $I(\gamma_h^1, z) = 0$ for any $h \ge h_r$ and any $z \in \mathbf{R}^2 \setminus B_r(z_1)$.

If $\{l(\gamma_h^1)\}_h$ does not converge to zero as $h \to +\infty$, there exist a subsequence $\{\gamma_h^1\}_h$ and a constant $c_1 > 0$ such that $l(\gamma_h^1) \ge c_1$ for any h. In this case, let us define $\Lambda_h^1 = \Lambda_h$. Starting from $\{\Lambda_h^1\}_h$, we repeat the same procedure for $\{\gamma_h^2\}_h$, and we obtain a new sequence of systems of curves $\{\Lambda_h^2\}_h$. After k steps, we end up with a sequence of systems of curves $\{\Lambda_h^k\}_h$, which we shall denote by $\{\Gamma_h\}_h$.

By construction, for any h, Γ_h is a disjoint system of curves of class C^2, with $\Gamma_h = \{\gamma_h^{i_1}, \dots, \gamma_h^{i_n}\}$, $1 \le i_1 < \cdots < i_n \le k$, and $(\Gamma_h) \subseteq B_R(0)$. Let us suppose that $l = k - n$ eliminations have been made to replace the system Λ_h with the system Γ_h, and let $r_1, \dots, r_l$ be the corresponding numbers (respectively $z_1, \dots, z_l$ be the corresponding points) defined above. Then, for any $0 < r \le \min_{j=1,\dots,l} r_j$, there exists $h_r \in \mathbf{N}$ sufficiently large such that

$$I(\Gamma_h, z) = I(\Lambda_h, z) \in \{0, 1\} \tag{6.7}$$

for any $h \geq h_r$ and any $z \in \left(\mathbf{R}^2 \setminus \bigcup_{j=1}^{l} B_r(z_j)\right) \setminus (\Lambda_h)$. Using (6.6), for any $M > 0$ and any $0 < r \leq \min_{j=1,\dots,l} r_j$, there exists $h_{r,M} \in \mathbf{N}$ such that

$$I(\Gamma_h, z) = I(\Delta_h, z) \tag{6.8}$$

for any $h \geq h_{r,M}$ and any $z \in \left(B_M(0) \setminus \bigcup_{j=1}^{l} B_r(z_j)\right) \setminus (\Delta_h)$. Define $A_h = \{z \in \mathbf{R}^2 \setminus (\Gamma_h) : I(\Gamma_h, z) = 1\}$. Then $A_h \in \mathcal{C}_b^2(\mathbf{R}^2)$, and $\partial A_h = (\Gamma_h)$. Since $\sup_h F_\psi(\Gamma_h) \leq \sup_h F_\psi(E_h) < +\infty$ by (6.4), by Proposition 6.1 there exists a subsequence $\{\Gamma_h\}_h$ weakly convergent to a limit system of curves $\Gamma \in \mathcal{B}$; moreover $(\Gamma) \subseteq \overline{B_R(0)}$ and, using (6.7), we have that, if $z \in \mathbf{R}^2 \setminus (\Gamma)$, then $I(\Gamma, z) \in \{0, 1\}$. Let $A_\Gamma = \{z \in \mathbf{R}^2 \setminus (\Gamma) : I(\Gamma, z) = 1\}$. It is clear that $A_\Gamma \subseteq B_R(0)$. Since $\chi_{A_h}(z) = I(\Gamma_h, z)$ for any $z \in \mathbf{R}^2 \setminus (\Gamma_h)$, and $\chi_{A_\Gamma}(z) = I(\Gamma, z)$ for any $z \in \mathbf{R}^2 \setminus (\Gamma)$, by the continuity properties of the index and by the Dominated Convergence Theorem, we have $A_h \to A_\Gamma$ in $L^1(\mathbf{R}^2)$ as $h \to +\infty$.

Let us prove that $\mathcal{L}^2(E \triangle A_\Gamma) = 0$. By (6.8), for any $M > 0$ and any $0 < r \leq \min_{j=1,\dots,l} r_j$, we have $(A_h \cap B_M(0)) \setminus \bigcup_{j=1}^{l} B_r(z_j) = (E_h \cap B_M(0)) \setminus \bigcup_{j=1}^{l} B_r(z_j)$ for $h \geq h_{r,M}$ large enough. Passing to the limit as $r \to 0$, we get $\mathcal{L}^2\big((E \triangle A_\Gamma) \cap B_M(0))\big) = 0$. As M is arbitrary, we deduce $\mathcal{L}^2(E \triangle A_\Gamma) = 0$. □

The next comparison result is crucial for the proof of the semicontinuity Theorem 6.1.

LEMMA 6.3. *Let $\psi : \mathbf{R} \times \mathbf{R} \to [0, +\infty]$ be a Borel function satisfying conditions (i)-(vi). Let $E \in \mathcal{C}_b^2(\mathbf{R}^2)$, let $\Gamma \in \mathcal{B}$ be a system of curves such that $(\Gamma) \supseteq \partial E$. Then*

$$\int_{\partial E} \psi(\alpha, \kappa)\, d\mathcal{H}^1(z) \leq K_\psi^a(\Gamma). \tag{6.9}$$

PROOF. Let $\Gamma = \{\gamma^1, \dots, \gamma^m\}$, and, for any $i = 1, \dots, m$ let Θ_i be an argument of $\dot{\gamma}^i$. For any measurable set $T \subseteq \partial E$ and any $i = 1, \dots, m$ let

$$V_i = \{s \in [0, l(\gamma^i)] : \gamma^i(s) \in T\}, \qquad T_i = \gamma^i(V_i).$$

Since $\partial E \subseteq (\Gamma)$, we have $T = \bigcup_{i=1}^{m} T_i$. To prove the thesis it will be enough to show that

$$\int_{T_i} \psi(\alpha, \kappa)\, d\mathcal{H}^1(z) \leq K_\psi^a(\Theta_i, V_i) \quad \text{for any } i = 1, \dots, m, \tag{6.10}$$

where T varies over a suitable finite measurable partition $\mathcal{P}$ of ∂E. In fact, from (6.10) it follows that

$$\int_T \psi(\alpha, \kappa)\, d\mathcal{H}^1(z) \leq \sum_{i=1}^{m} \int_{T_i} \psi(\alpha, \kappa)\, d\mathcal{H}^1(z) \leq \sum_{i=1}^{m} K_\psi^a(\Theta_i, V_i),$$

hence

$$\int_{\partial E} \psi(\alpha,\kappa)\, d\mathcal{H}^1(z) = \sum_{T\in\mathcal{P}} \int_{T\cap\partial E} \psi(\alpha,\kappa)\, d\mathcal{H}^1(z) \le \sum_{i=1}^{m} K^a_\psi(\Theta_i,[0,l(\gamma^i)]) = K^a_\psi(\Gamma),$$

that proves the assertion. We choose the finite partition $\mathcal{P}$ of ∂E as follows. Any element of $\mathcal{P}$ must be contained in a rectangle in which ∂E is a cartesian graph. This means that for any $T \in \mathcal{P}$ we can choose a suitable orthogonal coordinate system, two bounded open intervals I, J, and a function $f : I \to J$ of class C^2 such that

$$(I\times J)\cap E = \{(x,y)\in I\times J : y < f(x)\},$$

$$(I\times J)\cap \partial E = \{(x,f(x)) : x\in I\}, \qquad T\subseteq (I\times J)\cap\partial E.$$

Let us fix $i \in \{1,\dots,m\}$ and let $\gamma^i(s) = (\gamma^i_1(s),\gamma^i_2(s))$. For any $s \in V_i$ we have $\gamma^i_2(s) = f(\gamma^i_1(s))$. Hence $\dot\gamma^i_2(s) = \frac{d}{ds} f(\gamma^i_1(s))$ for almost every $s \in V_i$, and, since $|\dot\gamma(s)| = 1$, we have

$$\cos\Theta_i(s) = \dot\gamma^i_1(s) = \frac{\pm 1}{\sqrt{1+(f'(\gamma^i_1(s)))^2}}, \qquad \sin\Theta_i(s) = \dot\gamma^i_2(s) = \frac{\pm f'(\gamma^i_1(s))}{\sqrt{1+(f'(\gamma^i_1(s)))^2}},$$

for almost every $s \in V_i$.

Set

$$V_i^{\pm} = \{s\in V_i : \dot\gamma^i_1(s) = \frac{\pm 1}{\sqrt{1+(f'(\gamma^i_1(s)))^2}}\}, \qquad T_i^{\pm} = \gamma^i(V_i^{\pm}).$$

Note that, from $\dot\gamma^i_2(s) = \frac{d}{ds} f(\gamma^i_1(s))$, we have

$$\dot\gamma^i_2(s) = \frac{\pm f'(\gamma^i_1(s))}{\sqrt{1+(f'(\gamma^i_1(s)))^2}} \qquad \text{for a.e. } s\in V_i^{\pm}. \tag{6.11}$$

As $\dot\gamma^i_1 \in BV$, and $\frac{\pm 1}{\sqrt{1+(f'(\gamma^i_1))^2}} \in W^{1,1}$ and these two functions coincide almost everywhere on $V_i^{\pm}$, it follows (see [15, Th. 4.5.9]) that their approximate derivatives (i.e., the absolutely continuous part of these derivatives with respect to the Lebesgue measure) must coincide almost everywhere on $V_i^{\pm}$. We then have

$$\ddot\gamma^{i\,a}_1(s) = \frac{\mp f''(\gamma^i_1(s))f'(\gamma^i_1(s))\dot\gamma^i_1(s)}{(1+(f'(\gamma^i_1(s)))^2)^{\frac{3}{2}}} = \frac{\mp f''(\gamma^i_1(s))\dot\gamma^i_2(s)}{(1+(f'(\gamma^i_1(s)))^2)^{\frac{3}{2}}} \qquad \text{for a.e. } s\in V_i^{\pm}.$$

Similarly, by (6.11),

$$\ddot\gamma^{i\,a}_2(s) = \frac{\pm f''(\gamma^i_1(s))\dot\gamma^i_1(s)}{(1+(f'(\gamma^i_1(s)))^2)^{\frac{3}{2}}} \qquad \text{for a.e. } s\in V_i^{\pm}.$$

It follows that

$$\ddot{\gamma}^{ia}(s) = (-\sin\Theta_i(s), \cos\Theta_i(s))\frac{\pm f''(\gamma_1^i(s))}{(1+(f'(\gamma_1^i(s)))^2)^{\frac{3}{2}}} \qquad \text{for a.e. } s \in V_i^{\pm}.$$

On the other hand, from (3.7) we have

$$\ddot{\gamma}^{ia}(s) = (-\sin\Theta_i(s), \cos\Theta_i(s))\dot{\Theta}_i(s) \qquad \text{for a.e. } s \in V_i^{\pm},$$

so we conclude

$$\dot{\Theta}_i^a(s) = \frac{\pm f''(\gamma_1^i(s))}{(1+(f'(\gamma_1^i(s)))^2)^{\frac{3}{2}}} = \pm\kappa(\gamma^i(s)) \quad \text{for a.e. } s \in V_i^{\pm}. \tag{6.12}$$

Since

$$\Theta_i(s) = \alpha(\gamma^i(s)) \pmod{2\pi} \text{ for a.e. } s \in V_i^{\pm}, \tag{6.13}$$

by (6.13), (6.12) and properties (v) and (vi) (see Section 1) we find

$$K_\psi^a(\Theta_i, V_i) = \int_{V_i} \psi(\Theta_i, \dot{\Theta}_i^a)\, ds = \int_{V_i} \psi(\alpha(\gamma^i), \kappa(\gamma^i))\, ds. \tag{6.14}$$

But [15, Th. 3.2.6] ensures

$$\int_{V_i} \psi(\alpha(\gamma^i), \kappa(\gamma^i))\, ds \geq \int_{T_i} \psi(\alpha, \kappa)\, d\mathcal{H}^1(z).$$

This, together with (6.14), gives (6.10) and concludes the proof of the Lemma. □

THEOREM 6.1. *Let $\psi : \mathbf{R}\times\mathbf{R} \to [0,+\infty]$ be a Borel function satisfying properties (i)-(vi). Let $E \in \mathcal{C}_b^2(\mathbf{R}^2)$, and let $\{E_h\}_h \subseteq \mathcal{C}_b^2(\mathbf{R}^2)$ be a sequence converging to E in $L^1(\mathbf{R}^2)$. Then*

$$F_\psi(E) \leq \liminf_{h\to+\infty} F_\psi(E_h). \tag{6.15}$$

PROOF. We can suppose that the right hand side of (6.15) is finite, otherwise the result is trivial. Let $\{E_{hk}\}_k$ be a subsequence of $\{E_h\}_h$ with the property that

$$\lim_{k\to+\infty} F_\psi(E_{hk}) = \liminf_{h\to+\infty} F_\psi(E_h) < +\infty.$$

For simplicity, this subsequence (and any further subsequence) will be denoted by $\{E_k\}_k$. Using Lemma 6.1, we have that, for any k, ∂E_k has a finite number of connected components. Let Δ_k be an oriented parametrization of ∂E_k. Since $\{E_k\}_k$ satisfies (6.4), the

sequence $\{\Delta_k\}_k$ satisfies (6.3). Reasoning as in the proof of Proposition 6.1, we can suppose that $\Delta_k = \{\gamma_k^1, \dots, \gamma_k^m\}$ with m independent of k. Let $\{\Gamma_k\}_k = \{\gamma_k^{i_1}, \dots, \gamma_k^{i_n}\}$ be the sequence and let $\Gamma \in \mathcal{B}$ be the limit system of curves constructed in the proof of Lemma 6.2. Then $n \leq m$, $\Gamma = \{\gamma^{i_1}, \dots, \gamma^{i_n}\}$, and, for any $j = 1, \dots, n$ we have ${\gamma_k}^{i_j} \to \gamma^{i_j}$ uniformly in C^0 and $\frac{d{\gamma_k}^{i_j}}{dt} \rightharpoonup \frac{d\gamma^{i_j}}{dt}$ weakly in BV as $k \to +\infty$. Denote by $\Theta_k^{i_j}$ (respectively Θ_{i_j}) an argument of $\dot{\gamma}_k^{i_j}$ (respectively of $\dot{\gamma}^{i_j}$). From Proposition 4.1 we have that $\Theta_k^{i_j} \to \Theta_{i_j}$ in L^1 as $k \to +\infty$. We have

$$\liminf_{h\to+\infty} F_\psi(E_h) = \lim_{k\to+\infty} F_\psi(E_k) \geq \lim_{k\to+\infty} \sum_{j=1}^{n} l(\gamma_k^{i_j}) +$$

$$\liminf_{k\to+\infty} \sum_{j=1}^{n} \int_0^{l(\gamma_k^{i_j})} \psi(\Theta_k^{i_j}, \dot{\Theta}_k^{i_j})\, ds = \sum_{j=1}^{n} l(\gamma^{i_j}) +$$

$$\liminf_{k\to+\infty} \sum_{j=1}^{n} l(\gamma_k^{i_j}) \int_0^1 \psi(\Theta_k^{i_j}, l(\gamma_k^{i_j})^{-1}\dot{\Theta}_k^{i_j})\, dt \geq$$

$$l(\Gamma) + \sum_{j=1}^{n} l(\gamma^{i_j}) \liminf_{k\to+\infty} \int_0^1 \psi(\Theta_k^{i_j}, l(\gamma_k^{i_j})^{-1}\dot{\Theta}_k^{i_j})\, dt.$$

Using (5.3), (5.4), and (5.5), we then get

$$\liminf_{h\to+\infty} F_\psi(E_h) \geq l(\Gamma) + \sum_{j=1}^{n} K_\psi(\Theta_{i_j}, [0, l(\gamma^{i_j})]) = l(\Gamma) + K_\psi(\Gamma) = F_\psi(\Gamma).$$

Recalling that Lemma 6.2 ensures that $(\Gamma) \subseteq \partial E$, from Lemma 6.3 we obtain $F_\psi(\Gamma) \geq F_\psi(E)$, and this concludes the proof. □

THEOREM 6.2. *Let $E \subseteq \mathbf{R}^2$ be a measurable set such that $\overline{F_\psi}(E) < +\infty$. Then $E^\star$ satisfies the following properties:*

$E^\star$ is bounded, open, and $\mathcal{L}^2(E \triangle E^\star) = 0$;

$\mathcal{H}^1(\partial E^\star) < +\infty$;

there exists a limit system of curves Γ of class $\mathcal{B}$ such that $(\Gamma) \supseteq \partial E^\star$ and $E^\star = \mathrm{int}(A_\Gamma \cup (\Gamma))$, where $A_\Gamma = \{z \in \mathbf{R}^2 \setminus (\Gamma) : I(\Gamma, z) = 1\}$;

$\overline{F_\psi}(E) \geq \inf\{F_\psi(\Gamma) : \Gamma \in \mathcal{A}(E)\}$, where $\mathcal{A}(E)$ is the collection of all limit systems of curves Γ of class $\mathcal{B}$ satisfying (iii).

PROOF. The first three assertions follow from Lemma 6.2. Let us prove the last assertion. Since $\overline{F_\psi}(E) < +\infty$, there exists a sequence $\{E_h\}_h \subseteq C_b^2(\mathbf{R}^2)$ satisfying (6.4) and converging to E in $L^1(\mathbf{R}^2)$. Using the same notation as in the proof of Theorem 6.1, for a subsequence $\{E_k\}_k$ we have

$$\liminf_{h\to+\infty} F_\psi(E_h) = \lim_{k\to+\infty} F_\psi(E_k) \geq l(\Gamma) + \liminf_{k\to+\infty} K_\psi(\Gamma_k) \geq l(\Gamma) + K_\psi(\Gamma) = F_\psi(\Gamma).$$

Taking the infimum first with respect to the system $\Gamma \in \mathcal{A}(E)$ and then with respect to the approximating sequence $\{E_h\}_h$, the last assertion follows. □

6.1. Localization. Let $\Omega \subseteq \mathbf{R}^2$ be an open set. We say that a measurable set $E \subseteq \mathbf{R}^2$ is of class $\mathcal{C}^2(\Omega)$ if E is bounded, open, and $\Omega \cap \partial E$ is of class $\mathcal{C}^2$. We define

$$F_\psi(E,\Omega) = \int_{\Omega\cap\partial E} (1+\psi(\alpha,\kappa))\, d\mathcal{H}^1(z).$$

COROLLARY 6.1. *Let E be a set of class $\mathcal{C}^2(\Omega)$, and let $\{E_h\}_h$ be a sequence of sets of class $\mathcal{C}^2(\Omega)$ converging to E in $L^1(\Omega)$. Then*

$$F_\psi(E,\Omega) \le \liminf_{h\to+\infty} F_\psi(E_h,\Omega). \tag{6.16}$$

PROOF. We can assume that the right hand side of (6.16) is finite, otherwise the result is trivial. It will be enough to show that

$$F_\psi(E,A) \le \liminf_{h\to+\infty} F_\psi(E_h,A) \tag{6.17}$$

for any open set A which is relatively compact in Ω. Let $\{E_{hk}\}_k$ be subsequence of $\{E_h\}_h$ such that

$$\lim_{k\to+\infty} F_\psi(E_{hk},A) = \liminf_{h\to+\infty} F_\psi(E_h,A) < +\infty.$$

For simplicity, this subsequence (and any further subsequence) will be denoted by $\{E_k\}_k$. Since

$$\sup_h \mathcal{H}^1(\Omega\cap\partial E_h) < +\infty, \quad \sup_h \int_{\Omega\cap\partial E_h} \psi(\alpha_h,\kappa_h)\, d\mathcal{H}^1(z) < +\infty,$$

it follows that, for any k,

$$A\cap\partial E_k \subseteq \bigcup_{i=1}^{m_k}(\gamma_k^i) \cup \bigcup_{j=1}^{r_k}(\beta_k^j),$$

where

$\{\gamma_k^1,\dots,\gamma_k^{m_k}\}$ is a disjoint system of curves of class $\mathcal{C}^2$ such that $\overline{A}\cap(\gamma_k^i)\neq\emptyset$ for any $i=1,\dots,m_k$;

$\{\beta_k^1,\dots,\beta_k^{r_k}\}$ is a finite family of simple regular curves of class $\mathcal{C}^2$ such that $|\frac{d\beta_k^j}{dt}|$ is constant, $\beta_k^j(0),\beta_k^j(1)\in\partial\Omega$ and $(\beta_k^j)\cap\partial A\neq\emptyset$ for any $j=1,\dots,r_k$;

the sets $(\gamma_k^1),\dots,(\gamma_k^{m_k})$, $(\beta_k^1),\dots,(\beta_k^{r_k})$ are mutually disjoint;

$$F_\psi(\gamma_k^i) + \sum_{j=1}^{r_k} l(\beta_k^j) + \sum_{j=1}^{r_k}\int_0^{l(\beta_k^j)} \psi(\dot\beta_k^j,\ddot\beta_k^j)\, ds < +\infty. \tag{6.18}$$

Using Lemma 6.1 it follows that $\{m_k\}_k$ is uniformly bounded with respect to k. As $l(\beta_k^j) \geq 2\text{dist}(\partial A, \partial\Omega) > 0$ for any $j = 1, \dots, r_k$, using (6.18) we have that $\{r_k\}_k$ is also uniformly bounded with respect to k. Possibly passing to a suitable subsequence, we can suppose that m_k and r_k are independent of k. Denote these numbers by m and r, respectively. As all curves of the family $\{\gamma_k^1, \dots, \gamma_k^m, \beta_k^1, \dots, \beta_k^r\}$ meet $\overline{A}$, from (6.18) it follows that their traces are contained in a bounded subset of $\mathbf{R}^2$ independent of k. Let us consider only the subfamily $\{\gamma_k^{i_1}, \dots, \gamma_k^{i_n}\}$ of $\{\gamma_k^1, \dots, \gamma_k^m\}$ of those curves which do not have infinitesimal lenght; repeating the arguments of Proposition 6.1, by compactness there exists a family $\Gamma = \{\gamma^{i_1}, \dots, \gamma^{i_n}, \beta^1, \dots, \beta^r\}$ of curves of class $\mathcal{B}$ such that $\gamma_k^{i_j} \to \gamma^{i_j}$ and $\beta_k^l \to \beta^l$ uniformly in $\mathcal{C}^0$, $\frac{d\gamma_k{}^{i_j}}{dt} \rightharpoonup \frac{d\gamma^{i_j}}{dt}$ and $\frac{d\beta_k^l}{dt} \rightharpoonup \frac{d\beta^l}{dt}$ weakly in BV as $k \to +\infty$, for any $j = 1, \dots, n$ and any $l = 1, \dots, r$. Therefore, as $E \in \mathcal{C}^2(\Omega)$, it follows that $A \cap (\Gamma) \supseteq A \cap \partial E$ (see [3, Th. 7.1]). Repeating the last part of the proof of Theorem 6.1 relatively to A, we get (6.17). □

7. Energy of an ordered partition.

In this section, following the approach of [22], we study the lower semicontinuous envelope of the functional $\mathcal{P}_\psi$ defined in Section 1.

Let $n \in \mathbf{N}$, $n \geq 1$, let $g \in L^\infty(\mathbf{R}^2)$ be a function with compact support, and let us define the map $\mathcal{P}_\psi : (L^1(\mathbf{R}^2))^n \to [0, +\infty]$ by

$$\mathcal{P}_\psi(E_1, \dots, E_n) = \begin{cases} \displaystyle\sum_{i=1}^{n} F_\psi(E_i) + \sum_{i=1}^{n+1} \int_{E_i'} (u_i - g)^2 \, dz & \text{if } E_1, \dots, E_n \in (\mathcal{C}_b^2(\mathbf{R}^2))^n, \\ +\infty & \text{elsewhere on } \mathcal{M}^n, \end{cases}$$

where, for any $i = 1, \dots, n+1$, the sets E_i' and the numbers u_i are defined in (1.2).

THEOREM 7.1. *If $\overline{\mathcal{P}_\psi}$ denotes the lower semicontinuous envelope of $\mathcal{P}_\psi$ with respect to the product topology induced on $\mathcal{M}^n$ by $(L^1(\mathbf{R}^2))^n$, we have*

$$\overline{\mathcal{P}_\psi}(E_1, \dots, E_n) = \sum_{i=1}^{n} \overline{F_\psi}(E_i) + \sum_{i=1}^{n} \int_{E_i'} (u_i - g)^2 \, dz.$$

PROOF. By the definition of the product topology and the continuity of the term $\int_{E_i'} (u_i - g)^2 \, dz$ with respect to the $L^1(\mathbf{R}^2)$-topology, one easily obtains

$$\overline{\mathcal{F}_\psi}(E_1, \dots, E_n) \geq \sum_{i=1}^{n} \overline{F_\psi}(E_i) + \sum_{i=1}^{n+1} \int_{E_i'} (u_i - g)^2 \, dz.$$

Let us prove the opposite inequality. Let $(E_1, \dots, E_n) \in \mathcal{M}^n$; we can assume that $\overline{F_\psi}(E_i) < +\infty$ for any $i = 1, \dots, n$, otherwise the result is trivial. For any $i = 1, \dots, n$ let $\{E_i^h\}_h \subseteq$

$\mathcal{C}^2_b(\mathbf{R}^2)$ be a sequence converging to E_i in $L^1(\mathbf{R}^2)$ and such that $\lim_{h\to+\infty} F_\psi(E_i^h) = \overline{F_\psi}(E_i) < \infty$. Then, as $(E_1^h,\dots,E_n^h) \to (E_1,\dots,E_n)$ in $(L^1(\mathbf{R}^2))^n$ as $h \to +\infty$, we have

$$\sum_{i=1}^n \overline{F_\psi}(E_i) + \int_{E_i'} (u_i - g)^2 \, dz = \lim_{h\to+\infty} \left[\sum_{i=1}^n F_\psi(E_i^h) + \sum_{i=1}^{n+1} \int_{E_i^{h\prime}} (u_i - g)^2 \, dz \right]$$
$$\geq \overline{\mathcal{P}_\psi}(E_1,\dots,E_n).$$

This concludes the proof. □

This result shows that, to compute $\overline{\mathcal{P}_\psi}$, one needs to compute $\overline{F_\psi}$. This will be the goal of the next sections.

8. An upper bound for $\overline{F_\psi}$.

LEMMA 8.1. *Let γ be a regular curve of class $\mathcal{C}^2$ parametrized by arc lenght, and set $\dot\gamma(s) = (\cos\Theta(s), \sin\Theta(s))$ for any $s \in [0, l(\gamma)]$. Let $s_1, s_2 \in]0, l(\gamma)[$, $s_1 < s_2$, be such that $\gamma(s_1) = \gamma(s_2)$. Then there exist $\sigma_1, \sigma_2 \in [s_1, s_2]$ such that $|\Theta(\sigma_1) - \Theta(\sigma_2)| \geq \pi$. In particular, $\int_{s_1}^{s_2} |\dot\Theta| ds \geq \pi$.*

PROOF. Since $\gamma(s) = \int_0^s (\cos\Theta(\tau), \sin\Theta(\tau)) d\tau + \gamma(0)$ for any $s \in [0, l(\gamma)]$, and $\gamma(s_1) = \gamma(s_2)$, we have

$$\int_{s_1}^{s_2} \cos\Theta(\tau) d\tau = \int_{s_1}^{s_2} \sin\Theta(\tau) d\tau = 0.$$

Therefore $\int_{s_1}^{s_2} \sin(\Theta(\tau) - \alpha) d\tau = 0$ for any $\alpha \in [0, 2\pi]$. It follows that, for any $\alpha \in [0, 2\pi]$, there exists $\sigma_\alpha \in [s_1, s_2]$ such that $\sin(\Theta(\sigma_\alpha) - \alpha) = 0$, hence $\Theta(\sigma_\alpha) = \alpha + k\pi$, for some $k \in \mathbf{Z}$. Since this equality holds for any $\alpha \in [0, 2\pi]$, we have that, given $p \in \mathbf{S}^1$, then either $p \in \Theta([s_1, s_2])$, or $p + \pi \in \Theta([s_1, s_2])$. This easily implies that there exist $\sigma_1, \sigma_2 \in [s_1, s_2]$ such that $|\Theta(\sigma_1) - \Theta(\sigma_2)| \geq \pi$. □

LEMMA 8.2. *Let $\gamma \in \mathcal{B}$ and let Θ be an argument of $\dot\gamma$ on $[0, l(\gamma)]$. Let $s_1, s_2 \in]0, l(\gamma)[$, $s_1 < s_2$, be such that $\gamma(s_1) = \gamma(s_2)$. Then $|\dot\Theta|([s_1, s_2]) \geq \pi$.*

PROOF. Let $k \in \mathbf{Z}$ be such that $|\Theta(0+) - \Theta(l(\gamma)-) - 2k\pi| \leq \pi$. We need to extend Θ on $\mathbf{R}$; one possible extension is the following: $\Theta(s + l(\gamma)) = \Theta(s) + 2k\pi$. Let $\{\varrho_h\}_h$ be a sequence of symmetric mollifiers, and define $\alpha^h = \Theta \star \varrho_h$. Then $\alpha^h \rightharpoonup \Theta$ weakly in $BV_{\mathrm{loc}}(\mathbf{R})$ as $h \to +\infty$, $\lim_{h\to+\infty} \alpha_h(s) = \dfrac{\Theta(s+) + \Theta(s-)}{2}$ for any $s \in \mathbf{R}$, and $\alpha^h(s + l(\gamma)) = \alpha^h(s) + 2k\pi$. Set

$$f_h(s) = \int_0^s (\cos\alpha^h, \sin\alpha^h) \, d\tau + \gamma(0) \qquad \forall s \in \mathbf{R}.$$

Then $f_h : [0, l(\gamma)] \to \mathbf{R}^2$ is parametrized by arc lenght, $\lim_{h\to+\infty} f_h = \gamma$ uniformly on $[0, l(\gamma)]$, $\lim_{h\to+\infty} \dot f_h = \dot\gamma$ in $L^1([0, l(\gamma))]$, $\lim_{h\to+\infty} \dot f_h(s) = (\cos(\dfrac{\Theta(s+) + \Theta(s-)}{2}), \sin(\dfrac{\Theta(s+) + \Theta(s-)}{2}))$

for any $s \in [0, l(\gamma)]$, and f_h is, in general, neither closed nor simple. For any h define

$$d_h = \int_{s_1}^{s_2} (\cos \alpha^h, \sin \alpha^h)\, d\tau, \qquad g_h(t) = f_h(t) - \frac{t}{s_2 - s_1} d_h \qquad \forall t \in \mathbf{R}.$$

By definition it follows that $\lim_{h\to+\infty} |d_h| = 0$, $\lim_{h\to+\infty} g_h = \gamma$ uniformly on $[0, l(\gamma)]$, $\lim_{h\to+\infty} \dot{g}_h = \dot{\gamma}$ in $L^1([0, l(\gamma))]$, $\lim_{h\to+\infty} \dot{g}_h(s) = (\cos(\frac{\Theta(s+) + \Theta(s-)}{2}), \sin(\frac{\Theta(s+) + \Theta(s-)}{2}))$ for any $s \in [0, l(\gamma)]$, and $g_h(s_1) = g_h(s_2)$, so that $g_{h|[s_1,s_2]}$ is a regular curve of class $\mathcal{C}^2$. Let us reparametrize $g_{h|[s_1,s_2]}$ by arc lenght, that is, write

$$\gamma_h(s) = g_h(t_h(s)),$$

where $t_h(s)$ is the inverse of the map $s_h = s_h(t) = \int_{s_1}^{t} |\dot{g}_h|\, d\tau$ for any $t \in [s_1, s_2]$. Set $\dot{\gamma}_h(s) = (\cos \phi_h(s), \sin \phi_h(s))$ for any $s \in [0, l(\gamma_h)]$. Let $\{\sigma_1^h\}_h$, $\{\sigma_2^h\}_h$ be two sequences of points of $[0, l(\gamma_h)]$ given by Lemma 8.1, and, up to subsequences, let $\sigma_1 = \lim_{h\to+\infty} \sigma_1^h$, $\sigma_2 = \lim_{h\to+\infty} \sigma_2^h$. We have $\sigma_1, \sigma_2 \in [s_1, s_2]$. By Lemma 8.1 we have

$$\pi \le \lim_{h\to+\infty} |\phi_h(\sigma_1^h) - \phi_h(\sigma_2^h)| = \frac{1}{2} |\Theta(\sigma_1+) + \Theta(\sigma_1-) - \Theta(\sigma_2+) - \Theta(\sigma_2-)| \le |\dot{\Theta}|([s_1, s_2]). \quad \square$$

In the sequel of this section we shall suppose that $\psi(\eta, \cdot)$ is Lipschitz continuous, with a Lipschitz constant C independent of $\eta \in \mathbf{R}$; moreover $E \subseteq \mathbf{R}^2$ will be a bounded open set that can be written, locally, as the subgraph of a Lipschitz function having derivative of class BV, with respect to a suitable orthogonal coordinate system. As ∂E is locally Lipschitz continuous, ∂E can have corners but can not have cusp points. One can prove that there exists a disjoint system of Lipschitz curves $\Gamma = \{\gamma^1, \dots, \gamma^m\}$ such that $\dot{\gamma}^i \in BV$ for any $i = 1, \dots, m$, $(\Gamma) = \partial E$, and

$$E = \{z \in \mathbf{R}^2 \setminus (\Gamma) : I(\Gamma, z) \equiv 1 \pmod 2\}.$$

For any $i = 1, \dots, m$ let Θ_i be an argument of $\dot{\gamma}_i$. Note that, as ∂E is locally Lipschitz, we have that $|\Theta_i(s+) - \Theta_i(s-)| < \pi$ for any $s \in]0, l(\gamma^i)[$ and any $i = 1, \dots, m$. Moreover we can suppose that for any $i = 1, \dots, m$ there exists $k_i \in \mathbf{Z}$ such that $|\Theta_i(0+) - \Theta_i(l(\gamma^i)-) - 2k_i\pi| < \pi$. The aim of this section is to prove

$$\overline{F_\psi}(E) \le \sum_{i=1}^{m} F_\psi(\Theta_i, [0, l(\gamma^i)]), \tag{8.1}$$

which in particular gives $\overline{F_\psi}(E) < +\infty$.

As $\Theta_i \in BV([0, l(\gamma^i)])$, we can find $\delta > 0$ such that

$$|\Theta_i(s+) - \Theta_i(s-)| \le \pi - \delta \qquad \forall s \in [0, l(\gamma^i)],\ \forall i = 1, \dots, m. \tag{8.2}$$

We shall use the same notation and some arguments of Lemma 8.2, with γ replaced by γ^i, and Θ replaced by Θ_i. Let us extend Θ_i by periodicity $\pmod{2\pi}$ on $\mathbf{R}$. Inequality (8.2) holds then for any $s \in \mathbf{R}$.

Fix $i \in \{1, \dots, m\}$; define $\alpha_i^h = \Theta_i \star \varrho_h$. Then $\alpha_i^h \rightharpoonup \Theta_i$ weakly in $BV_{\rm loc}(\mathbf{R})$ as $h \to +\infty$, and $\lim_{h\to+\infty} |\dot\alpha_i^h|(A) = |\dot\Theta_i|(A)$ for any bounded open set $A \subseteq \mathbf{R}$ such that $|\dot\Theta_i|(\partial A) = 0$. Following [11, Section 3] one can check that $\{\alpha_i^h\}_h$ is a minimizing sequence for $K_\psi(\Theta_i, [0, l(\gamma^i)])$, i.e.,

$$\lim_{h\to+\infty} K_\psi(\alpha_i^h, [0, l(\gamma^i)]) = K_\psi(\Theta_i, [0, l(\gamma^i)]) \qquad \forall i = 1, \dots, m.$$

In particular

$$\sup_i \sup_h K_\psi(\alpha_i^h, [0, l(\gamma^i)]) < +\infty. \tag{8.3}$$

Set

$$f_h^i(s) = \int_0^s (\cos\alpha_i^h, \sin\alpha_i^h)\, d\tau + \gamma^i(0) \qquad \forall s \in \mathbf{R},$$

$$d_h^i = \int_0^{l(\gamma^i)} (\cos\alpha_i^h, \sin\alpha_i^h)\, d\tau, \qquad g_h^i(t) = f_h^i(t) - \frac{t}{l(\gamma^i)} d_h^i \qquad \forall t \in \mathbf{R}.$$

By definition it follows that $\lim_{h\to+\infty} |d_h^i| = 0$, that g_h^i is closed, i.e., $g_h^i(0) = g_h^i(l(\gamma^i))$, and that $\lim_{h\to+\infty} g_h^i = \gamma^i$ and $\lim_{h\to+\infty} |\dot\gamma_h^i| = 1$ uniformly on $[0, l(\gamma^i)]$. Let us reparametrize g_h^i by arc lengh, that is, write

$$\gamma_h^i(s) = g_h^i(t_h(s)),$$

where $t_h(s)$ is the inverse of the map $s_h = s_h(t) = \int_0^t |\dot g_h^i|\, d\tau$. Set also

$$\dot\gamma_h^i(s) = (\cos\phi_i^h(s), \sin\phi_i^h(s)) \qquad \forall s \in [0, l(\gamma_h^i)].$$

Using the fact that $\dot f_h^i \cdot \ddot f_h^i = 0$, we have

$$\dot t_h(s) = \frac{1}{|\dot g_h^i(t_h(s))|}, \qquad \ddot t_h(s) = -\frac{\dot g_h^i(t_h(s)) \cdot \ddot g_h^i(t_h(s))}{|\dot g_h^i(t_h(s)|^4} = \frac{\ddot f_h^i \cdot d_h^i}{l(\gamma^i)|\dot g_h^i(t_h(s))|^4}.$$

Hence

$$\ddot\gamma_h^i(s) = (\dot t_h(s))^2 \ddot f_h^i(t_h(s)) + \ddot t_h(s)\dot g_h^i(t_h(s)) = \left((\dot t_h(s))^2 \mathrm{Id} + \frac{\dot g_h^i \otimes d_h^i}{l(\gamma^i)|\dot g_h^i(t_h(s))|^4}\right) \ddot f_h^i(t_h(s)),$$

where, if $a, b \in \mathbf{R}^2$, the symbol $a \otimes b$ denotes the matrix whose entries are $a_i b_j$, for $i, j = 1, 2$. We then have

$$
\begin{aligned}
&|\,|\dot\phi_i^h(s)| - |\dot\alpha_i^h(t_h(s))|\,| = |\,|\ddot\gamma_h^i(s)| - |\ddot f_h^i(t_h(s))|| \\
&= |\ddot f_h^i(t_h(s))|\Big|\mathrm{Id} - |(\dot t_h(s))^2 \mathrm{Id} + \frac{\dot g_h^i \otimes d_h^i}{l(\gamma^i)|\dot g_h^i(t_h(s))|^4}\Big|\,|.
\end{aligned} \tag{8.4}
$$

Let us prove that $\{\phi_i^h\}_h$ is a minimizing sequence for $K_\psi(\Theta_i, [0, l(\gamma^i)])$. Making the change of variable $\sigma = t_h(s)$, and recalling that $l(\gamma_h^i) = t_h^{-1}(l(\gamma^i))$, we have

$$
\begin{aligned}
&\lim_{h\to+\infty} \Big|\int_0^{l(\gamma_h^i)} \psi(\phi_i^h, \dot\phi_i^h) ds - \int_0^{l(\gamma^i)} \psi(\alpha_i^h, \dot\alpha_i^h) d\sigma\Big| \\
&= \Big|\int_0^{l(\gamma_h^i)} \psi(\phi_i^h(s), \dot\phi_i^h(s)) - \psi(\alpha_i^h(t_h(s)), \dot\alpha_i^h(t_h(s))\dot t_h(s) ds\Big| \\
&\le \int_0^{l(\gamma_h^i)} |\psi(\phi_i^h(s), \dot\phi_i^h(s)) - \psi(\alpha_i^h(t_h(s), \dot\alpha_i^h(t_h(s))| ds \\
&+ \int_0^{l(\gamma_h^i)} |1 - \dot t_h(s)||\psi(\alpha_i^h(t_h(s), \dot\alpha_i^h(t_h(s))| ds \\
&=: \mathrm{I}_h + \mathrm{II}_h.
\end{aligned}
$$

Since $\lim_{h\to+\infty} \dot t_h = 1$ uniformly, using (8.4) we get that $\lim_{h\to+\infty} \mathrm{II}_h = 0$. Moreover, by the Lipschitz assumption on $\psi(\eta, \cdot)$, by condition (vi), and by (8.5) we get

$$
\begin{aligned}
\mathrm{II}_h &\le C \int_0^{l(\gamma_h^i)} |\,|\dot\phi_i^h(s)| - |\dot\alpha_i^h(t_h(s))|\,| ds \\
&\le \int_0^{l(\gamma_h^i)} |\ddot f_h^i(t_h(s))|\,\Big|\mathrm{Id} - |(\dot t_h(s))^2 \mathrm{Id} + \frac{\dot g_h^i \otimes d_h^i}{l(\gamma^i)|\dot g_h^i(t_h(s))|^4}\Big| ds.
\end{aligned}
$$

Since $\lim_{h\to+\infty} \dot t_h = 1$ uniformly and $\lim_{h\to+\infty} |d_h^i| = 0$, using (8.4), we deduce that $\lim_{h\to+\infty} \mathrm{II}_h = 0$.

It follows that

$$
\lim_{h\to+\infty} \Big|\int_0^{l(\gamma_h^i)} \psi(\phi_i^h, \dot\phi_i^h) ds - \int_0^{l(\gamma^i)} \psi(\alpha_i^h, \dot\alpha_i^h) d\sigma\Big| = 0,
$$

and hence $\{\phi_i^h\}_h$ is also a minimizing sequence for $K_\psi(\Theta_i, [0, l(\gamma^i)])$, i.e.,

$$
\lim_{h\to+\infty} K_\psi(\phi_i^h, [0, l(\gamma_h^i)]) = K_\psi(\Theta_i, [0, l(\gamma^i)]) \qquad \forall i = 1, \dots, m.
$$

In addition we have also $\lim_{h\to+\infty} |\dot\phi_i^h|(A) = |\dot\Theta_i|(A)$ for any bounded open set $A \subseteq \mathbf{R}$ such that $|\dot\Theta_i|(\partial A) = 0$.

Let us prove that the curves γ_h^i are simple for h large enough. Assume by contradiction that there exist two convergent sequences $\{s_h\}_h, \{\sigma_h\}_h$ of points of $]0, l(\gamma_h^i)]$ such that $\gamma_h^i(s_h) = \gamma_h^i(\sigma_h)$ for any h. Since γ^i is simple by assumption, it follows that either $\lim_{h\to+\infty} s_h = \lim_{h\to+\infty} \sigma_h = s_0$, or $\lim_{h\to+\infty} s_h = 0$ and $\lim_{h\to+\infty} \sigma_h = l(\gamma)$. Let us consider the first case, the second case being similar. Possibly passing to a subsequence, we can assume that there exist $\phi_1 = \lim_{h\to+\infty} \phi_i^h(s_h), \phi_2 = \lim_{h\to+\infty} \phi_i^h(\sigma_h)$. By Lemma 8.1 we have $|\phi_i^h(s_h) - \phi_i^h(\sigma_h)| \geq \pi$ for any h, so that $|\phi_1 - \phi_2| \geq \pi$. Hence $|\Theta_i(s_0-) - \Theta_i(s_0+)| \geq \pi$, a contradiction with (8.2).

We conclude that $\{\gamma_h^i\}_h$ is a sequence of closed simple curves of class $\mathcal{C}^\infty$; moreover, since $\gamma_h^i \to \gamma^i$ uniformly for any $i = 1, \dots, m$ we have that $(\gamma_h^i) \cap (\gamma_h^j) = \emptyset$ for any $i \neq j$, provided h is sufficiently large. Consider the disjoint system of curves $\Gamma_h = \{\gamma_h^1, \dots, \gamma_h^m\}$, and define

$$E_h = \{z \in \mathbf{R}^2 \setminus (\Gamma_h) : I(\Gamma_h, z) \equiv 1 \pmod 2\}.$$

By the continuity property of the index and by the Dominated Convergence Theorem (see also [8, III.8.8 (v)]), it follows that $E_h \to \{z \in \mathbf{R}^2 \setminus (\Gamma) : I(\Gamma, z) \equiv 1 \pmod 2\}$ in $L^1(\mathbf{R}^2) = E$ as $h \to +\infty$.

In addition $F_\psi(E_h) = F_\psi(\Gamma_h)$, so that

$$\overline{F_\psi}(E) \leq \lim_{h\to+\infty} F_\psi(E_h) = \lim_{h\to+\infty} F_\psi(\Gamma_h) = l(\Gamma) + K_\psi(\Gamma),$$

which proves (8.1).

9. Computation of $\overline{F_\psi}$ in some model examples.

Let $T > 0$; in what follows we shall consider the convex function $\psi : \mathbf{R} \times \mathbf{R} \to [0, +\infty[$ defined by

$$\psi(\eta, \xi) = \begin{cases} c_1 \xi^2 & \text{if } |\xi| < T, \\ c_2 |\xi| - c_3 & \text{if } |\xi| \geq T, \end{cases} \tag{9.1}$$

where $c_1 > 0$, $c_1 T^2 = c_2 T - c_3$, $c_2 \geq 2c_1 T$.

This function has been recently suggested in [22] to study some problems in image segmentation (see Section 1). Obviously $\psi_\infty(\eta, \xi) = c_2|\xi|$, and ψ is of class $\mathcal{C}^1$ if and only if $c_2 = 2c_1 T$. As ψ depend only on $|\xi|$, we shall write $\psi = \psi(|\xi|)$.

Let $\Gamma = \{\gamma^1, \dots, \gamma^m\} \in \mathcal{B}$, and for any $i = 1, \dots, m$ let Θ_i be an argument of $\dot\gamma^i$; for any $p \in (\Gamma)$ we set

$$F_\psi(\Gamma, \{p\}) = c_2 \sum_{i=1}^m \sum_{s \in \gamma^{i-1}(p)} |\Theta_i(s+) - \Theta_i(s-)|,$$

and

$$T_{(\Gamma)}(p) = \bigcup_{i=1}^m \bigcup_{s \in \gamma^{i-1}(p)} \dot\gamma^i(s+) \cup (-\dot\gamma^i(s-)).$$

THEOREM 9.1. *Let ψ be defined as in (9.1), and let E be the set of Figure 1.1, with $0 \le \alpha < 2\pi$. Then*

$$\mathcal{H}^1(\partial E) + \int_{\partial_r E} \psi(|\kappa|)\, d\mathcal{H}^1(z) + c_2 \min(\alpha, 2\pi - \alpha, |\pi - \alpha|) \le \overline{F_\psi}(E) \le$$
$$\mathcal{H}^1(\partial E) + \int_{\partial_r E} \psi(|\kappa|)\, d\mathcal{H}^1(z) + c_2|\pi - \alpha|,$$

where $\partial_r E = \partial E \setminus \{p\}$.

PROOF. It is not difficult (see Figure 1.2) to find a sequence $\{E_h\}_h \subseteq \mathcal{C}_b^2(\mathbf{R}^2)$ converging to E in $L^1(\mathbf{R}^2)$ and such that

$$\lim_{h\to+\infty} F_\psi(E_h) = \mathcal{H}^1(\partial E) + \int_{\partial_r E} \psi(|\kappa|)\, d\mathcal{H}^1(z) + c_2|\pi - \alpha|.$$

Consequently

$$\overline{F_\psi}(E) \le \mathcal{H}^1(\partial E) + \int_{\partial_r E} \psi(|\kappa|)\, d\mathcal{H}^1(z) + c_2|\pi - \alpha|.$$

It remains to show

$$\overline{F_\psi}(E) \ge \mathcal{H}^1(\partial E) + \int_{\partial_r E} \psi(|\kappa|)\, d\mathcal{H}^1(z) + c_2 \min(\alpha, 2\pi - \alpha, |\pi - \alpha|). \tag{9.2}$$

Let $\{E_h\}_h \subseteq \mathcal{C}_b^2(\mathbf{R}^2)$ be a sequence converging to E in $L^1(\mathbf{R}^2)$ such that $\overline{F_\psi}(E) = \lim_{h\to+\infty} F_\psi(E_h) < +\infty$. For any h let Γ_h be an oriented parametrization of ∂E_h of class $\mathcal{C}^2$, let $\Gamma = \{\gamma^1, \dots, \gamma^m\}$ be the limit system of curves of class $\mathcal{B}$ given by Lemma 6.2. For any $i = 1, \dots, m$ let Θ_i be an argument of $\dot\gamma^i$. We recall (see Lemma 6.2) that

$$E = \{z \in \mathbf{R}^2 \setminus (\Gamma) : I(\Gamma, z) = 1\}. \tag{9.3}$$

Theorem 5.1 gives

$$\overline{F_\psi}(E) = \lim_{h\to+\infty} F_\psi(E_h) = \lim_{h\to+\infty} F_\psi(\Gamma_h) \ge F_\psi(\Gamma) = l(\Gamma) + K_\psi^a(\Gamma) + K_\psi^s(\Gamma). \tag{9.4}$$

Using the inclusion $\partial E \subseteq (\Gamma)$ proven in Lemma 6.2 and reasoning as in Lemma 6.3 we deduce

$$l(\Gamma) + K_\psi^a(\Gamma) \ge \mathcal{H}^1(\partial E) + \int_{\partial_r E} \psi(|\kappa|)\, d\mathcal{H}^1(z). \tag{9.5}$$

Moreover, the definition of K_ψ^s yields

$$K_\psi^s(\Gamma) \ge F_\psi(\Gamma, \{p\}). \tag{9.6}$$

From (9.4), (9.5) and (9.6) it follows

$$\overline{F_\psi}(E) \geq \mathcal{H}^1(\partial E) + \int_{\partial_r E} \psi(|\kappa|)\, d\mathcal{H}^1(z) + F_\psi(\Gamma, \{p\}). \tag{9.7}$$

In view of (9.7), to conclude the proof of (9.2) it remains to show

$$F_\psi(\Gamma, \{p\}) \geq c_2 \min(\alpha, 2\pi - \alpha, |\pi - \alpha|). \tag{9.8}$$

We shall restrict ourselves to the case $m = 1$, i.e., when the system Γ reduces to one closed curve γ of class $\mathcal{B}$ (the case $m > 1$ is similar). We denote by Θ an argument of $\dot\gamma$.

Observe that there exists a positive constant ω such that

$$1 + \psi(|\xi|) \geq \omega|\xi| \qquad \forall \xi \in \mathbf{R}, \tag{9.9}$$

hence, if $s_0, s_1 \in [0, l(\gamma)]$, $s_0 < s_1$, we have

$$F_\psi(\gamma_{]]s_0,s_1[}) \geq \omega \int_{s_0}^{s_1} |\dot\Theta| \geq \omega |\Theta(s_1-) - \Theta(s_0+)|.$$

Using the previous inequality and Lemma 8.2, one can show that the set $\gamma^{-1}(p)$ is finite.

If $j \in \{\pm 1\}$, as a notation we set

$$\dot\gamma(s \text{ sign}(j)) = \begin{cases} \dot\gamma(s+) & \text{if } j = 1, \\ \dot\gamma(s-) & \text{if } j = -1. \end{cases}$$

For any $v \in \mathbf{S}^1$, let

$$A_v = \{(s,j) \,:\, s \in \gamma^{-1}(p),\ j \in \{\pm 1\},\ v = j\dot\gamma(s \text{ sign}(j))\}.$$

In the sequel we shall consider only those unit vectors v such that $A_v \neq \emptyset$ (which exist by the inclusion $\partial E \subseteq (\gamma)$).

Then A_v is a finite set, it is composed by distinct elements, and $v_1 \neq v_2 \Rightarrow A_{v_1} \cap A_{v_2} = \emptyset$. If $s_0 \in \gamma^{-1}(p)$ and γ is of class $\mathcal{C}^1$ in a neighbourhood of s_0, then, if $v = \dot\gamma(s_0)$, and if $\dot\gamma(s+), \dot\gamma(s-) \neq v$ for any $s \in \gamma^{-1}(p)$, $s \neq s_0$, we have $A_v = (s_0, +)$.

If γ stops at p and goes back along the same direction $\mathbf{R}v$, i.e., $v = \dot\gamma(s+) = -\dot\gamma(s-)$ for some $s \in \gamma^{-1}(p)$, the corresponding elements of A_v are $(s,+1)$ and $(s,-1)$, which are obviously distinct. Geometrically, given $v \in \mathbf{S}^1$, each element of A_v is in correspondence with a small arc of (γ) containing p lying on the same part of v and having v as the tangent unit vector at p, considered with its orientation. Define $N(v)$ as the number of elements of A_v.

Let τ_1, τ_2 be the two tangent unit vectors of ∂E at p chosen as in Figure 1.1. We claim that for any $v \in \mathbf{S}^1$ such that $v \neq \tau_1, \tau_2$, then $N(v)$ is even, while $N(\tau_1)$ and $N(\tau_2)$ are odd, unless $\tau_1 = \tau_2$ (i.e., p is a cusp point of ∂E, and $\alpha = 0$). Let $v \in \mathbf{S}^1$, $v \neq \tau_1, \tau_2$, and assume that v points towards the complement of E (the case in which v points towards E is similar). Suppose by contradiction that $N(v)$ is odd. Since $\gamma^{-1}(p)$ is a finite set and since there exist $\dot\gamma(s+)$, $\dot\gamma(s-)$ at each $s \in \gamma^{-1}(p)$, we can find a small half-cone C with vertex at p, in the direction of v, with $C \subseteq \mathbf{R}^2 \setminus E$, in such a way that $C \cap (\gamma)$ contains only the small arcs of (γ) having $\mathbf{R}v$ as the tangent line at p (which are in an odd number by assumption). Choose two points z_1, z_2 in $C \setminus (\gamma)$, one on each side of v, and close enough to v. Then (see [5, Lemma 9.2.5])

$$|I(\gamma, z_1) - I(\gamma, z_2)| \equiv 1 \pmod 2,$$

and this contradicts (9.3). Similarly, one can prove that $N(\tau_1)$ and $N(\tau_2)$ are odd. Hence the claim is proven.

Set $\gamma^{-1}(p) = \{s_1, \dots, s_k\}$, $k \geq 1$, and define the unit vectors $v_1, \dots, v_{2k}$ by

$$v_i = \dot\gamma(s_i+), \qquad v_{k+i} = -\dot\gamma(s_i-), \qquad \forall i = 1, \dots, k. \tag{9.10}$$

Observe that, for any $v \in \mathbf{S}^1$ there is a one to one correspondence between A_v and the set $\{i \in \{1, \dots, 2k\} : v_i = v\}$.

We want to construct a finite subfamily $\{v_{\alpha_1}, \dots, v_{\alpha_h}, v_{\beta_1}, \dots, v_{\beta_h}\}$ of $\{v_1, \dots, v_{2k}\}$, with $1 \leq h \leq k$, such that

$$\begin{aligned}
&\alpha_1, \dots, \alpha_h, \beta_1, \dots, \beta_h \text{ are distinct indices between 1 and } 2k;\\
&v_{\alpha_1} = \tau_1, \quad v_{\beta_h} = \tau_2,\\
&\forall n = 1, \dots, h, \text{ if } \alpha_n \leq k \text{ then } \beta_n = \alpha_n + k,\\
&\forall n = 1, \dots, h, \text{ if } \alpha_n > k \text{ then } \beta_n = \alpha_n - k;\\
&v_{\alpha_n} = v_{\beta_{n-1}} \qquad \forall n = 2, \dots, h.
\end{aligned}$$

We shall suppose $\tau_1 \neq \tau_2$, since the case $\tau_1 = \tau_2$ can be treated similarly. We shall reason inductively. As $N(\tau_1)$ is odd, there exists $l \in \{1, \dots, 2k\}$ such that $v_l = \tau_1$; set $\alpha_1 = l$. If $\alpha_1 \leq k$, set $\beta_1 = \alpha_1 + k$. If $\alpha_1 > k$, set $\beta_1 = \alpha_1 - k$. If $v_{\beta_1} = \tau_1$, then, as the set $\{l \in \{1, \dots, 2k\} : v_l = \tau_1\}$ has an odd number of elements, there exists $\alpha_2 \in \{l \in \{1, \dots, 2k\} : v_l = \tau_1\}$, with $\alpha_2 \neq \alpha_1, \beta_1$, such that $v_{\alpha_2} = \tau_1$. In this case we delete the indices α_1, β_1, and we restart the algorithm by defining $\alpha_2 = \alpha_1$. Therefore we shall assume that $v_{\beta_1} \neq \tau_1$. If $v_{\beta_1} = \tau_2$ then the algorithm stops. Otherwise, if $v_{\beta_1} \neq \tau_2$, as the set $\{l \in \{1, \dots, 2k\} : v_l = v_{\beta_1}\}$ has an even number of elements, there exists $\alpha_2 \in \{l \in \{1, \dots, 2k\} : v_l = v_{\beta_1}\}$, with $\alpha_2 \neq \alpha_1, \beta_1$, such that $v_{\alpha_2} = v_{\beta_1}$. Assume that the sequence $\alpha_1, \dots, \alpha_q, \beta_1, \dots, \beta_{q-1}$ of indices has been constructed, with the required

properties. Set $\beta_q = \alpha_q + k$ if $\alpha_q \leq k$, and $\beta_q = \alpha_q - k$ if $\alpha_q > k$. If $v_{\beta_q} = \tau_2$ then the algorithm stops, thus verifying all required properties. If $v_{\beta_1} \neq \tau_2$, since $N(v_{\beta_q})$ is even if $v_{\beta_q} \neq \tau_1$, and is odd if $v_{\beta_q} = \tau_1$, there exists an index $\alpha_{q+1} \neq \alpha_1, \ldots, \alpha_q, \beta_1, \ldots, \beta_q$ such that $v_{\alpha_{q+1}} = v_{\beta_q}$. The procedure stops after h steps, for some integer h.

Note that the indices $\alpha_1, \ldots, \alpha_h, \beta_1, \ldots, \beta_h$ are not necessarily ordered.

Let $\theta_1, \theta_2 \in \mathbf{R}/2\pi$; we set $d(\theta_1, \theta_2) = \min(|\theta_1 - \theta_2|, 2\pi - |\theta_1 - \theta_2|)$. Then $0 \leq d(\theta_1, \theta_2) \leq \pi$, and by (c) of Lemma 3.1 we get

$$d(\Theta(s-), \Theta(s+)) = |\Theta(s+) - \Theta(s-)| \qquad \text{for any } s \in S_\Theta. \tag{9.11}$$

Furthermore, set $\delta(\theta_1, \theta_2) = \min\{d(\theta_1, \theta_2), d(\theta_1, \theta_2 + \pi)\}$. Then

$$\delta(\theta_1, \theta_2) = \delta(\theta_1 + \pi, \theta_2) = \delta(\theta_1, \theta_2 + \pi) = \delta(\theta_1 + \pi, \theta_2 + \pi), \tag{9.12}$$

and

$$\delta(\theta_1, \theta_2) \leq d(\theta_1, \theta_2). \tag{9.13}$$

It is readily verified that δ is a distance on $\mathbf{R}/2\pi$. If n_1, n_2 are unit vectors, by identifying n_i with the corresponding angle θ_i (hence $-n_i$ corresponds to $\theta_i + \pi$), we shall use the notation $d(n_1, n_2)$ and $\delta(n_1, n_2)$ in place of $d(\theta_1, \theta_2)$ and $\delta(\theta_1, \theta_2)$. In particular

$$\delta(n_1, n_2) = \delta(-n_1, n_2) = \delta(n_1, -n_2). \tag{9.14}$$

Observe that

$$\delta(\tau_1, \tau_2) = \min(\alpha, 2\pi - \alpha, |\pi - \alpha|). \tag{9.15}$$

Then, using the fact that $\{v_{\alpha_1}, \ldots, v_{\alpha_h}, v_{\beta_1}, \ldots, v_{\beta_h}\} \subseteq T_{(\gamma)}(p)$, recalling (9.10) we deduce

$$\begin{aligned} c_2^{-1} F_\psi(\gamma, \{p\}) &= \sum_{s \in \gamma^{-1}(p)} |\Theta(s+) - \Theta(s-)| = \\ \sum_{i=1}^{k} |\Theta(s_i+) - \Theta(s_i-)| &\geq \sum_{q=1}^{h} d(v_{\alpha_q}, -v_{\beta_q}). \end{aligned} \tag{9.16}$$

Consequently, using (9.13), (9.14), the triangular property of δ, and (9.15), we have

$$c_2^{-1} F_\psi(\gamma, \{p\}) \geq \sum_{q=1}^{h} \delta(v_{\alpha_q}, -v_{\beta_q}) = \sum_{q=1}^{h} \delta(v_{\alpha_q}, v_{\beta_q}) \geq \delta(\tau_1, \tau_2). \tag{9.17}$$

This, in view of (9.15), shows (9.8) (recall that $m = 1$), and concludes the proof of the theorem. □

Note that, if in the proof of Theorem 9.1 one has $T_{(\gamma)}(p) \subseteq \{\tau_1, \tau_2\}$ then

$$F_\psi(\gamma, \{p\}) = c_2 \sum_{s \in \gamma^{-1}(p)} |\Theta(s+) - \Theta(s-)| \geq c_2 d(\tau_1, -\tau_2) = c_2 |\pi - \alpha|. \tag{9.18}$$

COROLLARY 9.1. *Let ψ be defined as in (9.1), let E be the set of Figure 1.1, and assume that*

$$\frac{\pi}{2} \leq \alpha \leq \frac{3\pi}{2}.$$

Then

$$\overline{F_\psi}(E) = \mathcal{H}^1(\partial E) + \int_{\partial_r E} \psi(|\kappa|)\, d\mathcal{H}^1(z) + c_2 |\pi - \alpha|.$$

COROLLARY 9.2. *Let ψ be defined as in (9.1), let E be the disjoint union of a finite number of curvilinear polygons $P_1, \ldots, P_q$. For any $i = 1, \ldots, q$ denote by $\alpha_i^1, \ldots, \alpha_i^{h(i)}$ the angles of the polygon P_i. Then*

$$\mathcal{H}^1(\partial E) + \int_{\partial_r E} \psi(|\kappa|)\, d\mathcal{H}^1(z) + c_2 \sum_{i=1}^{q} \sum_{j=i}^{h(i)} \min(\alpha_i^j, 2\pi - \alpha_i^j, |\pi - \alpha_i^j|) \leq \overline{F_\psi}(E) \leq$$

$$\leq \mathcal{H}^1(\partial E) + \int_{\partial_r E} \psi(|\kappa|)\, d\mathcal{H}^1(z) + c_2 \sum_{i=1}^{q} \sum_{j=i}^{h(i)} |\pi - \alpha_i^j|,$$

where $\partial_r E$ denotes ∂E without the vertices, and κ denotes the curvature of $\partial_r E$. If in particular $\frac{\pi}{2} \leq \alpha_i^j \leq \frac{3\pi}{2}$ for any $i = 1, \ldots, q$ and any $j = 1, \ldots, h(i)$, then

$$\overline{F_\psi}(E) = \mathcal{H}^1(\partial E) + \int_{\partial_r E} \psi(|\kappa|)\, d\mathcal{H}^1(z) + c_2 \sum_{i=1}^{q} \sum_{j=i}^{h(i)} |\pi - \alpha_i^j|. \tag{9.19}$$

Reasoning as in Theorem 9.1 one can prove the following result.

PROPOSITION 9.1. *Let $E \subseteq \mathbf{R}^2$ be a bounded open set of class C^2 up to a countable set of points $\{p_i\}_{i=1}^{\infty}$. Assume that each p_i is a corner point of ∂E with angle α_i such that $0 < c \leq \alpha_i \leq 2\pi - c < 2\pi$ for any i. Then $\overline{F_\psi}(E) = +\infty$.*

Let us prove now the following result, which shows that equality (9.19) does not hold in general if $\alpha_i^j < \frac{\pi}{2}$ or $\alpha_i^j > \frac{3\pi}{2}$ for some i, j, and shows also that the functional $\overline{F_\psi}(E, \cdot)$, if considered as a set function, is not a measure and, in particular, cannot be represented as an integral of the form (6.1) (see [3]).

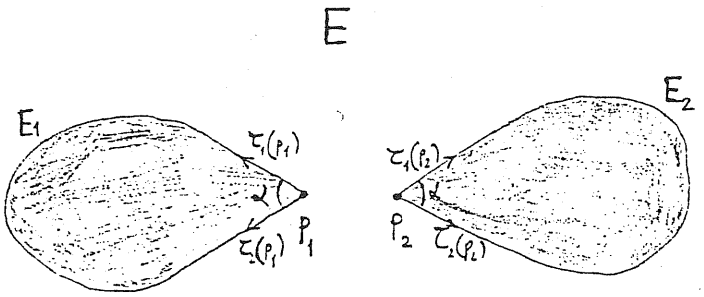

Fig. 9.1: The set E.

THEOREM 9.2. *Let $E = E_1 \cup E_2$ be the set of Figure 9.1, and assume that*

$$0 < \alpha < \frac{\pi}{2}.$$

Then if the distance D betweeen p_1 and p_2 is sufficiently small one has

$$\overline{F_\psi}(E) = \mathcal{H}^1(\partial E) + \int_{\partial_r E} \psi(|\kappa|)\, d\mathcal{H}^1(z) + 2D + 2c_2\alpha,$$

where $\partial_r E = \partial E \setminus \{p_1, p_2\}$.

In particular, if D is sufficiently small,

$$\overline{F_\psi}(E) < \mathcal{H}^1(\partial E) + \int_{\partial_r E} \psi(|\kappa|)\, d\mathcal{H}^1(z) + 2c_2(\pi - \alpha).$$

PROOF. It is not difficult to show that the sequence $\{E_h\}_h \subseteq \mathcal{C}_b^2(\mathbf{R}^2)$ of Figure 9.2 converges to E in $L^1(\mathbf{R}^2)$ and

$$\lim_{h\to+\infty} F_\psi(E_h) = \mathcal{H}^1(\partial E) + \int_{\partial_r E} \psi(|\kappa|)\, d\mathcal{H}^1(z) + 2D + 2c_2\alpha.$$

Consequently

$$\overline{F_\psi}(E) \le \mathcal{H}^1(\partial E) + \int_{\partial_r E} \psi(|\kappa|)\, d\mathcal{H}^1(z) + 2D + 2c_2\alpha. \tag{9.20}$$

Let us prove the opposite inequality. Let $\{E_h\}_h \subseteq \mathcal{C}_b^2(\mathbf{R}^2)$ be a sequence which converges to E in $L^1(\mathbf{R}^2)$ and such that $\lim_{h\to+\infty} F_\psi(E_h) = \overline{F_\psi}(E) < +\infty$. Let $\Gamma = \{\gamma^1, \dots, \gamma^m\}$

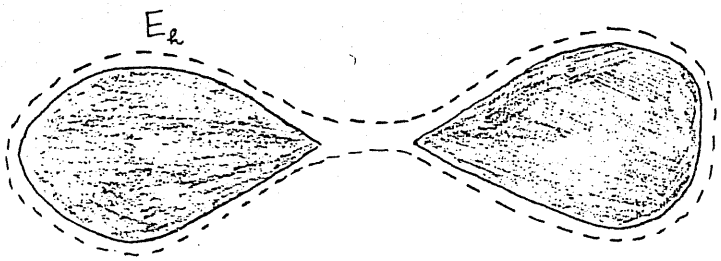

Fig. 9.2: An approximation of E.

be the limit system of curves of class $\mathcal{B}$ given by Theorem 6.2. Reasoning as in (9.4) and (9.5) we deduce

$$\overline{F_\psi}(E) \geq \mathcal{H}^1(\partial E) + \int_{\partial_r E} \psi(|\kappa|)\, d\mathcal{H}^1(z) + l(\Gamma_E) + K^a_\psi(\Gamma_E) + K^s_\psi(\Gamma), \tag{9.21}$$

where $\Gamma_E = \{z \in \mathbf{R}^2 : z \notin \partial E, z \in (\Gamma)\}$. It will be enough to show that, if D is sufficiently small, then

$$l(\Gamma_E) + K^a_\psi(\Gamma_E) + K^s_\psi(\Gamma) \geq 2D + 2c_2\alpha. \tag{9.22}$$

Let $\tau_1(p_i)$, $\tau_2(p_i)$ be the two tangent unit vectors of ∂E at p_i, $i = 1, 2$, chosen as in Figure 9.1. We claim that there is no loss of generality if we suppose that $T_{(\Gamma)}(p_i)$ strictly contains $\{\tau_1(p_i), \tau_2(p_i)\}$, for $i = 1, 2$. Assume by contradiction that $\{\tau_1(p_1), \tau_2(p_1)\} \supseteq T_{(\Gamma)}(p_1)$. Then by (9.18) we have

$$F_\psi(\Gamma, \{p_1\}) \geq c_2|\pi - \alpha| = c_2(\pi - \alpha).$$

Moreover (9.17) yields $F_\psi(\Gamma, \{p_2\}) \geq c_2\alpha$, so that $K^s_\psi(\Gamma) \geq c_2\pi$. Hence, from (9.21) we deduce

$$\overline{F_\psi}(E) \geq \mathcal{H}^1(\partial E) + \int_{\partial_r E} \psi(|\kappa|)\, d\mathcal{H}^1(z) + K^s_\psi(\Gamma) \geq \mathcal{H}^1(\partial E) + \int_{\partial_r E} \psi(|\kappa|)\, d\mathcal{H}^1(z) + c_2\pi.$$

Then (9.20) yields $2D + 2c_2\alpha \geq c_2\pi$, which is a contradiction, provided D is sufficiently small (recall that $0 < \alpha < \frac{\pi}{2}$). Obviously, the same argument can be applied by interchanging p_1 with p_2, and the claim is proven. Therefore, using the parity of the elements of $T_{(\Gamma)}(p_i)$ (see the proof of Theorem 9.1) there are at least two vectors belonging to $T_{(\Gamma)}(p_i)$ which do not belong to $\{\tau_1(p_i), \tau_2(p_i)\}$ for $i = 1, 2$.

Let us show that at least two of the elements of $T_{(\Gamma)}(p_i)$ are contained in a cone C having direction $p_2 - p_1$ with vertex at p_i and of amplitude $\alpha + o(D)$, for $i = 1, 2$. Indeed by (9.17) we have $F_\psi(\Gamma, \{p_2\}) \geq c_2\alpha$ (respectively $F_\psi(\Gamma, \{p_1\}) \geq c_2\alpha$), so that (9.20) forces $F_\psi(\Gamma, \{p_1\}) \leq c_2(\alpha + 2D)$ (respectively $F_\psi(\Gamma, \{p_2\}) \leq c_2(\alpha + 2D)$). The assertion follows from (9.16) with γ replaced by Γ.

Let $v \in T_{(\Gamma)}(p_1)$, $v \neq \tau_1(p_1), \tau_2(p_1)$, be one element of $T_{(\Gamma)}(p_1)$ which is contained in the cone C. Without loss of generality, we can assume that v points towards E_2. Let $i \in \{1, \ldots, m\}$ and $s_0 \in [0, l(\gamma^i)]$ be such that $v = \dot{\gamma}^i(s_0+)$. There are two possibilities, either $(\gamma^i) \cap \partial E_2 = \emptyset$, or $(\gamma^i) \cap \partial E_2 \neq \emptyset$. Suppose that $(\gamma^i) \cap \partial E_2 = \emptyset$, and denote by γ^i_1 the x-component of the curve γ^i. Let $\sigma \in [0, l(\gamma^i)]$ be such that $\gamma^i_1(\sigma) = \max\{\gamma^i_1(s) : s \in [0, l(\gamma^i)],\ \gamma^i(]s_0, \sigma]) \cap \partial E_1 = \emptyset\}$. If $\dot{\gamma}^i_1(\sigma+) = \dot{\gamma}^i_1(\sigma-)$, then $\dot{\gamma}^i(\sigma)$ is vertical. Therefore

$$
\begin{aligned}
l(\Gamma_E) + K_\psi(\Gamma_E) &\geq l({\gamma^i}_{|]s_0,\sigma[}) + K_\psi({\gamma^i}_{|]s_0,\sigma[}) \\
&\geq \omega \int_{s_0}^{\sigma} |\dot{\Theta}_i| \geq \omega |\Theta_i(s_0+) - \Theta_i(\sigma)| > 2D,
\end{aligned}
$$

where ω is defined in (9.9), and D is sufficiently small (recall that v is contained in the cone C).

If $\dot{\gamma}^i_1(\sigma+) \neq \dot{\gamma}^i_1(\sigma-)$, then using the same arguments we have

$$
l(\Gamma_E) + K_\psi(\Gamma_E) \geq \omega |\Theta_i(s_0+) - \Theta_i(\sigma+)| > 2D.
$$

Assertion (9.22) then follows by recalling that $K^s_\psi(\Gamma) \geq F_\psi(\Gamma, \{p_1\}) + F_\psi(\Gamma, \{p_2\}) \geq 2c_2\alpha$.

On the other hand, if $(\gamma^i) \cap \partial E_2 \neq \emptyset$, then $F_\psi(\Gamma_E) \geq 2D$, and this concludes the proof. □

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CHAPTER 2, SECTION 2.1

QUASI-OPTIMAL ERROR ESTIMATES FOR THE MEAN CURVATURE FLOW WITH A FORCING TERM*

1. Introduction. It is known that the singularly perturbed reaction-diffusion equation, with the quartic double equal well potential $\Psi(s) = (1-s^2)^2$ and forcing term g

$$\partial_t u_\varepsilon - \Delta u_\varepsilon + \tfrac{1}{2\varepsilon^2}\Psi'(u_\varepsilon) = \tfrac{c_0}{2\varepsilon} g \qquad \text{in } \mathbf{R}^n \times (0,T),$$

where $c_0 = \int_{-1}^{1} \sqrt{\Psi(s)}ds$, provides an approximation for an interface $\Sigma(t)$ evolving by the law

$$V = \kappa + g,$$

where V is the normal velocity of the interface $\Sigma(t)$ and κ the sum of its principal curvatures [1,2,3,4,6,7,8,9,12,13]. Such an equation was introduced by Allen and Cahn [1] in order to describe the motion of antiphase boundaries in crystalline solids, thus showing its relevance in phase transitions. It was independently suggested by De Giorgi [6] as a variational approach to the mean curvature flow. Such a connection has been rigorously established by Evans, Soner, and Souganidis [9] and Barles, Soner, and Souganidis [2], who proved convergence of the zero level set of u_ε to the generalized motion by mean curvature [10], even beyond the onset of singularities, provided the limit interface does not develop interior; see also [4,12]. Asymptotic analyses were carried out prior to those convergence results, but apply only to smooth evolutions [3,7,8,13].

The goal of this paper is to prove a quasi-optimal error estimate, valid before the onset of singularities, for the Hausdorff distance between the flow $\Sigma(t)$ and the approximate

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* The content of this section will be published on Diff. and Int. Eq., in a joint paper with M. Paolini.

interface $\Sigma_\varepsilon(t)=\{\mathbf{x}\in\Omega : u_\varepsilon(\mathbf{x},t)=0\}$. In fact, assuming that $\Sigma=\{(\mathbf{x},t)\in\Omega\times[0,T] : \mathbf{x}\in\Sigma(t)\}$ is sufficiently regular (see (2.4)) and $u_\varepsilon(\cdot,0)$ has the correct shape (see (6.6)), we prove that

$$\Sigma_\varepsilon(t)\subseteq\{\mathbf{x}\in\Omega : \mathrm{dist}(\mathbf{x},\Sigma(t)))\le C\varepsilon^2|\log\varepsilon|^2\}\qquad\forall t\in[0,T],$$
$$\Sigma(t)\subseteq\{\mathbf{x}\in\Omega : \mathrm{dist}(\mathbf{x},\Sigma_\varepsilon(t)))\le C\varepsilon^2|\log\varepsilon|^2\}\qquad\forall t\in[0,T],$$

where C is a constant depending on T but independent of ε. We cannot expect estimates of order higher than two in view of the formal asymptotics and the results of [15]. Our estimate improves the results obtained by Chen [4], who shows a first order error estimate via comparison arguments. If Ψ is replaced by a non smooth potential giving raise to a double obstacle problem, similar results have been established in [5,15,16,14].

The order of the interface error estimate is a consequence of the Maximum Principle and the explicit construction of sub and supersolutions, which in turn are inspired by, and indeed rely on, the formal asymptotics developped in [17]. The use of the asymptotic expansion, as in [7,8], would also lead to a rate of convergence for interfaces via nondegeneracy properties of both the continuous and truncated solutions. In this light, a nontrivial by-product of the rigorous asymptotic analysis of De Mottoni and Schatzman [7,8] would give an $\mathcal{O}(\varepsilon^2)$ interface error estimate for the special case $g=0$, but under higher regularity restrictions on Σ than in the present discussion. Regardless of convergence, the formal asymptotics of §4 still provides valuable information on the shape of u_ε, which, together with a modified distance function combined with a vertical shift and the nondegeneracy property of sub and supersolutions, plays a fundamental role in our subsequent rigorous analysis; see §6. Furthermore, the functions appearing in the definition of the sub and super solutions need to be constant far from the interface, thus requiring proper shape corrections.

The outline of the paper is as follows. In §2 we introduce some notation. §3 is devoted to the analysis of suitable limit problems (suggested by the formal asymptotics of §4) and accurate decay estimates at infinity of its solutions. These solutions, properly combined, provide the sub and supersolutions of §6. For the sake of completeness, a simple but crucial comparison result is proved in §5. The sub and supersolutions are fully examined in §6, and then used to derive interface error estimates.

2. Notations.

In what follows $\Omega\subseteq\mathbf{R}^n$ will be a bounded open set with smooth boundary, and ν will denote the outward unit normal vector to $\partial\Omega$.

Given $T>0$ and a function $v\in H^2(\Omega\times(0,T))$, we denote by ∇v and Δv the gradient and the laplacian of v with respect to the spatial variable $\mathbf{x}\in\Omega$, and by $\mathcal{H}v$ the heat operator of v, i.e., $\mathcal{H}v=\partial_t v-\Delta v$.

Let

$$g(\cdot,t)\in W^{3,\infty}(\Omega),\qquad \partial_t g\in W^{1,\infty}(\Omega\times(0,T)); \tag{2.1}$$

for any $t \in [0,T]$ we indicate by $\Sigma(t)$ a mean curvature flow with forcing term g (see (2.2)). We shall assume that, for any $t \in [0,T]$, $\Sigma(t)$ is a smooth closed manifold of dimension $n-1$, oriented by the inward unit normal vector $\mathbf{n}(\mathbf{x},t)$ to $\Sigma(t)$ at $\mathbf{x} \in \Sigma(t)$. Furthermore, we suppose that $\Sigma(t) \subset\subset \Omega$ for any $t \in [0,T]$. The precise regularity requirements on $\Sigma(t)$ are listed in (2.4).

We set $O(t) =$ outside of $\Sigma(t)$, $I(t) =$ inside of $\Sigma(t)$, and $\Sigma = \bigcup_{t\in[0,T]} \Sigma(t) \times \{t\}$. We denote by $\kappa_1(\mathbf{x},t), \dots, \kappa_{n-1}(\mathbf{x},t)$ the principal curvatures of $\Sigma(t)$ at $\mathbf{x} \in \Sigma(t)$, and we set

$$\kappa(\mathbf{x},t) = \sum_{i=1}^{n-1} \kappa_i(\mathbf{x},t), \qquad h(\mathbf{x},t) = \sum_{i=1}^{n-1} \kappa_i^2(\mathbf{x},t).$$

Note that $\kappa(\mathbf{x},t)$ equals $(n-1)$-times the mean curvature of $\Sigma(t)$ at $\mathbf{x} \in \Sigma(t)$. The evolution of Σ is then defined by

$$V(\mathbf{x},t) = \kappa(\mathbf{x},t) + g(\mathbf{x},t) \qquad \forall (\mathbf{x},t) \in \Sigma, \tag{2.2}$$

where, for any $t \in [0,T]$, $V(\cdot,t)$ denotes the normal velocity of $\Sigma(t)$, with the positive sign in the direction of $\mathbf{n}(\cdot,t)$.

The signed distance function $d : \mathbf{R}^n \times [0,T] \to \mathbf{R}$ from $\Sigma(t)$ is defined by

$$d(\mathbf{x},t) = \begin{cases} \operatorname{dist}(\mathbf{x},\Sigma(t)) & \text{if } t \in [0,T] \text{ and } \mathbf{x} \in O(t) \\ 0 & \text{if } t \in [0,T] \text{ and } \mathbf{x} \in \Sigma(t) \\ -\operatorname{dist}(\mathbf{x},\Sigma(t)) & \text{if } t \in [0,T] \text{ and } \mathbf{x} \in I(t) \end{cases}.$$

We then have

$$\nabla d(\mathbf{x},t) = -\mathbf{n}(\mathbf{x},t) \qquad \forall t \in [0,T],\ \forall \mathbf{x} \in \Sigma(t).$$

We fix a positive number D in such a way that, for any $t \in [0,T]$, the tubular neighbourhood $\mathcal{T}(t)$ of $\Sigma(t)$ defined by

$$\mathcal{T}(t) = \{\mathbf{x} \in \Omega : |d(\mathbf{x},t)| \le D\} \tag{2.3}$$

is relatively compact in Ω. We set $\mathcal{T} = \bigcup_{t\in[0,T]} \mathcal{T}(t) \times \{t\}$.

The interface error estimate in §6 will be proved under the assumptions

$$d, \partial_t d, \partial_t \partial_{\mathbf{xx}} d, \partial_{\mathbf{x}}^i d \in C^0(\overline{\mathcal{T}}) \qquad \forall i = 1,2,3,4. \tag{2.4}$$

If D is sufficiently small, from (2.4) it follows that any point $(\mathbf{x},t) \in \mathcal{T}$ has a unique projection $\mathbf{s}(\mathbf{x},t) \in \Sigma(t)$ such that

$$\operatorname{dist}(\mathbf{s}(\mathbf{x},t),\mathbf{x}) = |d(\mathbf{x},t)|.$$

Given a scalar or vector function f defined on Σ, we indicate with $\overline{f}(\mathbf{x},t)$ the composite function $f\big(\mathbf{s}(\mathbf{x},t),t\big)$, which is defined on $\mathcal{T}$. Hence, if f is scalar, we have $\nabla d\cdot\nabla\overline{f}=0$ on $\mathcal{T}$. We point out that (2.4) yields

$$\|\overline{h}\|_{L^\infty(\mathcal{T})},\|\partial_t\overline{h}\|_{L^\infty(\mathcal{T})},\|\nabla\overline{h}\|_{L^\infty(\mathcal{T})},\|\Delta\overline{h}\|_{L^\infty(\mathcal{T})}<+\infty. \tag{2.5}$$

The following property holds for all $(\mathbf{x},t)\in\mathcal{T}$ [11, 14.6]:

$$\Delta d(\mathbf{x},t)=\sum_{i=1}^{n-1}\frac{\overline{\kappa}_i(\mathbf{x},t)}{1-d(\mathbf{x},t)\overline{\kappa}_i(\mathbf{x},t)}=\overline{\kappa}(\mathbf{x},t)+d(\mathbf{x},t)\overline{h}(\mathbf{x},t)+\mathcal{O}(d^2(\mathbf{x},t)).$$

Hence, as $\partial_t d(\mathbf{x},t)=\overline{V}(\mathbf{x},t)=\overline{\kappa}(\mathbf{x},t)+\overline{g}(\mathbf{x},t)$ for any $\mathbf{x}\in\Sigma(t)$ (recall (2.2)), we have

$$\mathcal{H}d(\mathbf{x},t)=\overline{g}(\mathbf{x},t)+d(\mathbf{x},t)\overline{h}(\mathbf{x},t)+\mathcal{O}(d^2(\mathbf{x},t))\qquad\forall(\mathbf{x},t)\in\mathcal{T}. \tag{2.6}$$

We denote by $\Psi:\mathbf{R}\to[0,+\infty[$ the double equal well potential $\Psi(s)=(1-s^2)^2$, and we set

$$\psi=\tfrac{1}{2}\Psi',\qquad c_0=\int_{-1}^{1}\sqrt{\Psi(s)}ds=\frac{4}{3}, \tag{2.7}$$

$$\alpha=\psi'(1)=\psi'(-1)=4,\qquad\beta=\psi''(1)=-\psi''(-1)=12. \tag{2.8}$$

Finally, for any $\varepsilon>0$ we denote by u_ε the classical solution of the problem

$$\begin{aligned}&\mathcal{H}u_\varepsilon+\varepsilon^{-2}\psi(u_\varepsilon)-\varepsilon^{-1}\frac{c_0}{2}g=0 && \text{in }\Omega\times(0,T),\\ &u_\varepsilon(\cdot,0)=u_\varepsilon^0(\cdot)\in\mathcal{C}^2(\Omega)\cap L^\infty(\Omega) && \text{on }\Omega,\\ &\frac{\partial u_\varepsilon}{\partial\nu}=0 && \text{on }\partial\Omega\times(0,T).\end{aligned} \tag{2.9}$$

for a given initial datum u_ε^0 which will be specified later on (see (6.6)). Such a solution u_ε by classical methods (see, for instance, [18, p.98])).

3. Decay estimates.

Any absolute minimizer γ of the functional

$$\int_{\mathbf{R}}\big(|\zeta'(x)|^2+\Psi(\zeta(x))\big)dx \tag{3.1}$$

defined on $\{\zeta\in H^1_{\mathrm{loc}}(\mathbf{R}):\lim_{x\to\pm\infty}\zeta(x)=\pm1\}$ is a solution of the problem

$$\gamma''(x)-\psi(\gamma(x))=0\qquad\forall x\in\mathbf{R}. \tag{3.2}$$

One can show that such a minimizer γ must be nondecreasing. Since the functional in (3.1) is greater than or equal to $2\int_{\mathbf{R}} \zeta'(x)\sqrt{\Psi(\zeta(x))}\,dx = 2c_0$, it follows that γ satisfies

$$\gamma'(x) = \sqrt{\Psi(\gamma(x))} \qquad \forall x \in \mathbf{R}. \tag{3.3}$$

It turns out that imposing the condition $\gamma(0) = 0$, the unique nondecreasing solution γ of (3.2) is given by $\gamma(x) = \mathrm{tgh}(x)$, $x \in \mathbf{R}$.

Note that there exists a positive constant c such that

$$|1 - \gamma(x)| \le c\gamma'(x) \qquad \forall x \in]0, +\infty[. \tag{3.4}$$

3.1. Some remarks on the operator $\mathcal{A}$. Let $\mathcal{A} : H^1(\mathbf{R}) \to H^{-1}(\mathbf{R})$ be the linear operator defined by $\mathcal{A}\zeta = \zeta'' - \psi'(\gamma)\zeta$. One can to verify that $\mathcal{A}$ is selfadjoint and, by (3.2), that $\gamma' \in \mathrm{Ker}(\mathcal{A}) \subseteq H^2_{\mathrm{loc}}(\mathbf{R})$. Let us show that $\mathrm{Ker}(\mathcal{A}) = \mathrm{span}\{\gamma'\}$. Denote by $\widetilde{\mathcal{A}} : H^2_{\mathrm{loc}}(\mathbf{R}) \to L^2_{\mathrm{loc}}(\mathbf{R})$ the operator $\widetilde{\mathcal{A}}\zeta = \zeta'' - \psi'(\gamma)\zeta$. Take $\zeta_0, \zeta_1 \in]0, +\infty[$, and let $x_0 > 0$ be such that $\psi'(\gamma(x_0)) > 0$. Let $\zeta \in H^2_{\mathrm{loc}}(\mathbf{R})$ be the unique solution of the backward and forward Cauchy problem

$$\zeta'' - \psi'(\gamma)\zeta = 0, \qquad \zeta(x_0) = \zeta_0,\ \zeta'(x_0) = \zeta_1.$$

It is not difficult to see that $\lim_{x\to+\infty} \zeta(x) = +\infty$, so that $\zeta \in \mathrm{Ker}(\widetilde{\mathcal{A}}) \setminus \mathrm{Ker}(\mathcal{A})$. Since obviously $\gamma' \in \mathrm{Ker}(\widetilde{\mathcal{A}})$ and $\mathrm{Ker}(\widetilde{\mathcal{A}})$ is a two dimensional linear space, it follows that $\mathrm{Ker}(\mathcal{A})$ must be one dimensional, which implies that $\mathrm{Ker}(\mathcal{A}) = \mathrm{span}\{\gamma'\}$.

Let $\mathcal{R} : H^{-1}(\mathbf{R}) \to H^1(\mathbf{R})$ be the isometric linear operator given by the Riesz Representation Theorem on $H^1(\mathbf{R})$ endowed with the scalar product $\langle f, g\rangle = \int_{\mathbf{R}} f'g'\,dx + \alpha\int_{\mathbf{R}} fg\,dx$, where α is defined in (2.8). Let $\mathcal{B} : H^1(\mathbf{R}) \to H^{-1}(\mathbf{R})$ be the linear operator defined by $\mathcal{B}\zeta = -\zeta'' + \alpha\zeta$ for any $\zeta \in H^1(\mathbf{R})$. Then $\mathcal{R}\mathcal{A} : H^1(\mathbf{R}) \to H^1(\mathbf{R})$, and, as $\mathcal{R}\mathcal{B} = \mathrm{Id}$ on $H^1(\mathbf{R})$, we have $\mathcal{R}\mathcal{A} = -\mathrm{Id} + \mathcal{R}(\mathcal{A} + \mathcal{B})$. Now $(\mathcal{A} + \mathcal{B})\zeta = (\alpha - \psi'(\gamma))\zeta$ for any $\zeta \in H^1(\mathbf{R})$; hence, as $\alpha - \psi'(\gamma)$ decreases exponentially to zero as $|x| \to +\infty$, the operator $\mathcal{A} + \mathcal{B} : H^1(\mathbf{R}) \to H^{-1}(\mathbf{R})$ is compact. As $\mathcal{R}$ is an isometry, $\mathcal{R}(\mathcal{A} + \mathcal{B})$ is also compact. It follows that the composite operator $\mathcal{R}\mathcal{A}$ is a Fredholm operator, so that $\mathcal{A}\zeta = f$ has a solution $\zeta \in H^2(\mathbf{R})$ for $f \in L^2(\mathbf{R})$ if and only if $\int_{\mathbf{R}} f\gamma'\,dx = 0$.

For our purposes, we need an extension of this result. Denote by $\mathbf{P}^-$ (respectively by $\mathbf{P}^+$) the ring of all polynomials with real coefficients defined on $]-\infty, -1[$ (respectively on $]1, +\infty[$).

LEMMA 3.1. *Let $f \in L^2_{\mathrm{loc}}(\mathbf{R})$ be such that there exist $p_- \in \mathbf{P}^-$, $p_+ \in \mathbf{P}^+$ with*

$$f_{|]-\infty,-1[} - p_- \in L^2(-\infty, -1), \qquad f_{|]1,+\infty[} - p_+ \in L^2(1, +\infty). \tag{3.5}$$

If $\int_{\mathbf{R}} f\gamma'\,dx = 0$ then there exist a function $\zeta \in H^2_{\mathrm{loc}}(\mathbf{R})$ and two polynomials $q_- \in \mathbf{P}^-$, $q_+ \in \mathbf{P}^+$ such that

$$\zeta - q_- \in H^2(-\infty, -1), \quad \zeta - q_+ \in H^2(1, +\infty), \qquad \widetilde{\mathcal{A}}\zeta = f. \tag{3.6}$$

Moreover, such a function ζ is unique up to an addition of a real multiple of γ'.

Proof. Define $q_- \in \mathbf{P}^-$ (respectively $q_+ \in \mathbf{P}^+$) as the unique polynomial solution of $v'' - \psi'(-1)v = p_-$ on $]-\infty,-1[$ (respectively of $v'' - \psi'(1)v = p_+$ on $]1,+\infty[$). Let $q \in H^2_{\mathrm{loc}}(\mathbf{R})$ be an arbitrary function such that $q = q_-$ on $]-\infty,-1[$ and $q = q_+$ on $]1,+\infty[$. We claim that $f - \widetilde{\mathcal{A}}q \in L^2(\mathbf{R})$. Indeed on $]1,+\infty[$ we have $\widetilde{\mathcal{A}}q = q_+'' - \psi'(\gamma)q_+ = p_+ + (\psi'(1) - \psi'(\gamma))q_+$, so that, using (3.5),

$$f - \widetilde{\mathcal{A}}q = f - p_+ - (\psi'(1) - \psi'(\gamma))q_+ \in L^2(1,+\infty)$$

(recall that $\psi'(\gamma)$ tends exponentially to $\psi'(1)$ at $+\infty$). The claim then follows by a similar argument on $]-\infty,-1[$.

Observe now that, by the exponential decreasing of γ' at infinity and recalling that $\gamma' \in \mathrm{Ker}(\mathcal{A})$, we get

$$\int_{\mathbf{R}} (f - \widetilde{\mathcal{A}}q)\gamma'\,dx = \int_{\mathbf{R}} f\gamma'\,dx - \int_{\mathbf{R}} \widetilde{\mathcal{A}}q\gamma'\,dx = \int_{\mathbf{R}} f\gamma'\,dx - \int_{\mathbf{R}} q\mathcal{A}\gamma'\,dx = \int_{\mathbf{R}} f\gamma'\,dx.$$

Therefore, if $\int_{\mathbf{R}} f\gamma'\,dx = 0$, by the Fredholm Alternative there exists a function $v \in H^2(\mathbf{R})$ (which is unique up to an addition of a real multiple of γ') such that $\mathcal{A}v = f - \widetilde{\mathcal{A}}q$, so that the function $\zeta = v + q$ satisfies properties (3.6).

As $\gamma'(0) \neq 0$, by adding to ζ a suitable real multiple of γ', we can always obtain $\zeta(0) = 0$, so that ζ is uniquely determined. □

It can be checked directly that if f is odd, then the solution ζ through the origin is itself odd. Also, if f is even, ζ is itself even, as it follows by considering the function $\frac{1}{2}(\zeta(x) + \zeta(-x)))$, which is still a solution of the problem.

LEMMA 3.2. *Let $\zeta \in H^2(\mathbf{R})$ and assume*

$$|\mathcal{A}\zeta| \leq c(1 + |x|^m)\gamma' \qquad \forall x \in \mathbf{R} \tag{3.7}$$

for some constant $c > 0$ and some $m \in \mathbf{N}$. Then there exists a positive constant C such that

$$|\zeta| \leq C(1 + |x|^{m+1})\gamma' \qquad \forall x \in \mathbf{R}. \tag{3.8}$$

If in addition $\zeta \in H^3(\mathbf{R})$ and

$$|(\mathcal{A}\zeta)'| \leq c(1 + |x|^m)\gamma' \qquad \forall x \in \mathbf{R}, \tag{3.9}$$

then

$$|\zeta'| \leq C(1 + |x|^{m+1})\gamma' \qquad \forall x \in \mathbf{R}. \tag{3.10}$$

Proof. As $\gamma' = \sqrt{\Psi(\gamma)}$ and $\gamma'' = \psi(\gamma)$ (see (3.3) and (3.2)), we have $(\frac{\gamma''}{\gamma'})^2 = \frac{\psi(\gamma)^2}{\Psi(\gamma)}$, so that, with the notation introduced in (2.7),

$$\lim_{x\to+\infty} \frac{\gamma''}{\gamma'} = \lim_{x\to+\infty} -\sqrt{\frac{\psi(\gamma)\psi'(\gamma)}{\psi(\gamma)}} = -\sqrt{\alpha} = -2. \tag{3.11}$$

Therefore we can choose $x_0 \geq m+1$ large enough in such a way that

$$\gamma''(x) < -\gamma'(x) \qquad \forall x \geq x_0, \tag{3.12}$$

and

$$\psi'(\gamma(x)) \geq \delta > 0 \qquad \forall x \geq x_0. \tag{3.13}$$

Let $k \geq 1$ be such that $|\zeta(x_0)| \leq k x_0^{m+1}\gamma'(x_0)$, and let $z(x) = kx^{m+1}\gamma'(x)$ for any $x \geq x_0$. Recalling that $\mathcal{A}\gamma' = 0$ and using (3.12), we have

$$\begin{aligned}\mathcal{A}z &= kx^{m+1}\mathcal{A}\gamma' + 2(m+1)kx^m\gamma'' + m(m+1)kx^{m-1}\gamma' = \\ &= 2(m+1)kx^m\gamma'' + m(m+1)kx^{m-1}\gamma' \leq -k\gamma'(m+1)(2x^m - mx^{m-1}).\end{aligned}$$

Then, using the assumption (3.7),

$$-\mathcal{A}(-z-\zeta) = \mathcal{A}z + \mathcal{A}\zeta \leq -k\gamma'(m+1)(2x^m - mx^{m-1}) + |\mathcal{A}\zeta| \leq 0,$$

for any $x \geq x_0$, provided that k is large enough (depending on c). Since $-z(x_0)-\zeta(x_0) \leq 0$, by the Maximum Principle (recall (3.13)) we have $-z(x) - \zeta(x) \leq 0$ for any $x \geq x_0$, i.e.,

$$-kx^{m+1}\gamma'(x) \leq \zeta(x) \qquad \forall x \geq x_0. \tag{3.14}$$

In a similar way, one can prove that $-\mathcal{A}(\zeta - z) \leq 0$, so that

$$\zeta(x) \leq kx^{m+1}\gamma'(x) \qquad \forall x \geq x_0. \tag{3.15}$$

Since analogous results are valid on $]-\infty, -x_0]$, assertion (3.8) follows from (3.14), (3.15), and the compactness of $[0, x_0]$, provided that C is large enough.

Let us prove (3.10). Recalling the definition of $\mathcal{A}$ it follows

$$|\mathcal{A}(\zeta')| \leq |(\mathcal{A}\zeta)'| + |\psi''(\gamma)\gamma'\zeta|.$$

Since $\psi''(\gamma) = 12\gamma \in L^\infty(\mathbf{R})$ and $\zeta \in L^\infty(\mathbf{R})$ by (3.8), using assumption (3.9) we get

$$|\mathcal{A}(\zeta')| \leq c(1 + |x|^m)\gamma' \qquad \forall x \in \mathbf{R}.$$

The result then follows from the first part of the Lemma. □

3.2. Estimates on the functions ξ, η, ω, π. We denote by $\xi \in H^2(\mathbf{R})$ the solution of the problem

$$\mathcal{A}\xi = x\gamma', \qquad \xi(0) = 0,$$

which exists by the Fredholm Alternative, since $x\gamma'$ is odd. Furthermore ξ is odd. By Lemma 3.2 we have

$$|\xi|, |\xi'| \leq c(1+|x|^2)\gamma' \qquad \forall x \in \mathbf{R}. \tag{3.16}$$

We denote by $\eta \in H^2_{\mathrm{loc}}(\mathbf{R})$ the polynomially increasing solution of the problem

$$\widetilde{\mathcal{A}}\eta = \gamma' - \frac{c_0}{2}, \qquad \eta(0) = 0, \tag{3.17}$$

where c_0 is defined in (2.7). Such a solution exists by Lemma 3.1 applied with $f = \gamma' - \frac{c_0}{2}$, $p_- = p_+ = -\frac{c_0}{2}$ (an easy calculation yields $\int_{\mathbf{R}}(\gamma' - \frac{c_0}{2})\gamma'\, dx = 0$). Furthermore η is even, since $\gamma' - \frac{c_0}{2}$ is even. With the notation of Lemma 3.1, we find $q_- = q_+ = \frac{c_0}{2\alpha}$, where α is defined in (2.8). Hence

$$\lim_{x\to\pm\infty} \eta(x) = \frac{c_0}{2\alpha} = \eta_\infty.$$

Now

$$\mathcal{A}(\eta - \frac{c_0}{2\alpha}) = \gamma' - \frac{c_0}{2\alpha}(\alpha - \psi'(\gamma)),$$

and

$$\lim_{x\to\pm\infty} \frac{\alpha - \psi'(\gamma)}{\gamma'} = \lim_{x\to\pm\infty} -\frac{\psi''(\gamma)\gamma'}{\gamma''} = \frac{1}{2}\psi''(\pm 1) \tag{3.18}$$

(see (3.11)). It follows that there exists a constant $c > 0$ such that

$$|\mathcal{A}(\eta - \frac{c_0}{2\alpha})| \leq c\gamma' \qquad \forall x \in \mathbf{R}, \tag{3.19}$$

and hence, by Lemma 3.2,

$$|\eta - \frac{c_0}{2\alpha}| \leq c(1+|x|)\gamma' \qquad \forall x \in \mathbf{R}. \tag{3.20}$$

We denote by $\omega \in H^2(\mathbf{R})$ the solution of the problem

$$\mathcal{A}\omega = \eta', \qquad \omega(0) = 0. \tag{3.21}$$

Such a solution exists by the Fredholm Alternative, since η is even and hence η' is odd. Furthermore ω is odd. Using (3.17) we have

$$|(\mathcal{A}(\eta - \frac{c_0}{2\alpha}))'| = |(\gamma' - \frac{c_0}{2} + \frac{c_0}{2\alpha}\psi'(\gamma))'| = |\gamma'' + \frac{c_0}{2\alpha}\psi''(\gamma)\gamma'| \leq c\gamma'$$

for a suitable positive constant c, so that, by (3.19), using Lemma 3.2, we have

$$|\mathcal{A}\omega| = |\eta'| \le c(1+|x|)\gamma' \qquad \forall x \in \mathbf{R}. \tag{3.22}$$

Using again Lemma 3.2 and (3.21), it follows

$$|\omega| \le c(1+|x|^2)\gamma' \qquad \forall x \in \mathbf{R}. \tag{3.23}$$

In addition

$$(\mathcal{A}\omega)' = \eta'' = \widetilde{\mathcal{A}}\eta + \psi'(\gamma)\eta = \gamma' - \frac{c_0}{2} + \psi'(\gamma)\eta = \gamma' + \psi'(\gamma)(\eta - \frac{c_0}{2\alpha}) + \frac{c_0}{2\alpha}(\psi'(\gamma) - \alpha),$$

so that, by (3.18) and (3.20), we have $|(\mathcal{A}\omega)'| \le c(1+|x|)\gamma'$ on $\mathbf{R}$, which, together with (3.22) and Lemma 3.2, gives

$$|\omega'| \le c(1+|x|^2)\gamma' \qquad \forall x \in \mathbf{R}. \tag{3.24}$$

Finally, we denote by $\pi \in H^2_{\mathrm{loc}}(\mathbf{R})$ the polynomially increasing solution of the problem

$$\widetilde{\mathcal{A}}\pi = \frac{1}{2}\eta^2\psi''(\gamma), \qquad \pi(0) = 0. \tag{3.25}$$

Such a solution exists by Lemma 3.1 applied with $f = \frac{1}{2}\eta^2\psi''(\gamma)$, $p_- = -p_+ = -\frac{1}{2}(\frac{c_0}{2\alpha})^2\beta$, (recall that $\frac{1}{2}\eta^2\psi''(\gamma)$ is odd), where β is defined in (2.8). Furthermore π is odd, since $\frac{1}{2}\eta^2\psi''(\gamma)$ is odd. With the notation of Lemma 3.1, we find $q_- = -q_+ = \frac{c_0^2\beta}{8\alpha^3}$. Hence

$$\lim_{x\to\pm\infty} \pi(x) = \mp\frac{c_0^2\beta}{8\alpha^3} = \pm\pi_\infty.$$

Now, setting q as in the proof of Lemma 3.1, we have

$$\mathcal{A}(\pi - q) = \begin{cases} \frac{1}{2}\eta^2\psi''(\gamma) - \frac{c_0^2\beta}{8\alpha^3}\psi'(\gamma)\gamma + \frac{c_0^2\beta}{8\alpha^3}\gamma'' & \text{on }]-\infty,-1], \\ \frac{1}{2}\eta^2\psi''(\gamma) + \frac{c_0^2\beta}{8\alpha^3}\psi'(\gamma)\gamma - \frac{c_0^2\beta}{8\alpha^3}\gamma'' & \text{on } [1,+\infty[, \end{cases}$$

and

$$\lim_{x\to\pm\infty} \frac{\eta^2\psi''(\gamma) \mp \frac{c_0^2\beta}{4\alpha^3}\psi'(\gamma)\gamma}{x\gamma'} \le \lim_{x\to\pm\infty} \frac{\eta^2\psi'''(\gamma)\gamma' \mp \frac{c_0^2\beta}{4\alpha^3}\psi''(\gamma)\gamma'\gamma \mp \frac{c_0^2\beta}{4\alpha^3}\psi'(\gamma)\gamma''}{\gamma' + x\gamma''} +$$
$$\limsup_{x\to\pm\infty} \frac{2\eta\eta'\psi''(\gamma)}{\gamma' + x\gamma''}.$$

It is easy to see that the first limit is zero, and, by (3.22),

$$\limsup_{x\to\pm\infty} \frac{2\eta\eta'\psi''(\gamma)}{\gamma' + x\gamma''} \le \frac{c(1+|x|)\gamma'\eta\psi''(\gamma)}{\gamma' + x\gamma''} \le c,$$

for a suitable positive constant c. It follows that there exists a constant $c > 0$ such that

$$|\mathcal{A}(\pi - q)| \leq c(1 + |x|)\gamma' \qquad \forall x \in \mathbf{R},$$

and hence, by Lemma 3.2,

$$|\pi - q| \leq c(1 + |x|^2)\gamma' \qquad \forall x \in \mathbf{R}. \tag{3.26}$$

Since by (3.25) and (3.22) we have

$$|(\widetilde{\mathcal{A}}\pi)'| \leq |\eta\eta'\psi''(\gamma)| + \frac{1}{2}\eta^2\psi'''(\gamma)\gamma' \leq c(1 + |x|)\gamma',$$

using Lemma 3.2 we conclude

$$|\pi'| \leq c(1 + |x|^2)\gamma' \qquad \forall x \in \mathbf{R}. \tag{3.27}$$

4. Formal asymptotics. The shape of the subsolution is suggested by a formal asymptotic expansion, that can be obtained following the ideas in [17]. Since how such an expansion is obtained is not important in our context, we simply give the final result.

Let u_ε be a classical solution of problem (2.9). Introduce the vector $\widetilde{\mathbf{n}}$, the stretched variable $\widetilde{y} = \frac{\widetilde{d}(\mathbf{x},t)}{\varepsilon}$, and the projection $\widetilde{\mathbf{s}}$, as in §2 with $\Sigma(t)$ replaced by the set $\{\mathbf{x} \in \Omega : u_\varepsilon(\mathbf{x}, t) = 0\}$, and define

$$U_\varepsilon(\widetilde{y}, t) = u_\varepsilon\big(\widetilde{\mathbf{s}}(\mathbf{x}, t) - \varepsilon\widetilde{y}\ \widetilde{\mathbf{n}}(\mathbf{x}, t), t\big).$$

Then U_ε can be expressed formally in terms of ε (inner expansion) as follows:

$$U_\varepsilon(\widetilde{y}, t) = \sum_{i=0}^{2} \varepsilon^i U_i(\widetilde{y}, t) + \mathcal{O}(\varepsilon^3),$$

where

$$\begin{aligned}
U_0(\widetilde{y}, t) &= \gamma(\widetilde{y}),\\
U_1(\widetilde{y}, t) &= g(\mathbf{x}, t)\eta(\widetilde{y}),\\
U_2(\widetilde{y}, t) &= (\overline{h}(\mathbf{x}, t) - 2\nabla\widetilde{d}(\mathbf{x}, t)\cdot\nabla g(\mathbf{x}, t))\xi(\widetilde{y}) + (g^2(\mathbf{x}, t) - 2\nabla\widetilde{d}(\mathbf{x}, t)\cdot\nabla g(\mathbf{x}, t))\omega(\widetilde{y})\\
&\quad + g^2(\mathbf{x}, t)\pi(\widetilde{y}).
\end{aligned}$$

5. Comparison Lemma.

In this section we prove an elementary Comparison Lemma for subsolutions similar to that in [5] (see also [18, p.98]).

LEMMA 5.1. *Fix $\varepsilon > 0$, let $u, v \in L^2(0,T;H^2(\Omega)) \cap H^1(0,T;L^2(\Omega))$, and consider the following three conditions:*

$$
\begin{aligned}
&\mathcal{H}u + \varepsilon^{-2}\psi(u) - \varepsilon^{-1}\frac{c_0}{2}g \geq \mathcal{H}v + \varepsilon^{-2}\psi(v) - \varepsilon^{-1}\frac{c_0}{2}g \qquad \text{a.e. in } \Omega\times(0,T),\\
&u(\mathbf{x},0) \geq v(\mathbf{x},0) \qquad \text{for a.e. } \mathbf{x}\in\Omega,\\
&\frac{\partial u}{\partial \nu} \geq \frac{\partial v}{\partial \nu} \qquad \text{a.e. on } \partial\Omega\times(0,T).
\end{aligned}
\tag{5.1}
$$

Then

$$
u \geq v \qquad \text{a.e. in } \Omega\times(0,T). \tag{5.2}
$$

Proof. Set $e = \max(v-u,0)$. Multiply (5.1) by e, integrate over Ω and subtract. We get

$$
\frac{d}{dt}\|e(\cdot,t)\|^2_{L^2(\Omega)} \leq 2\varepsilon^{-2}\langle \psi(u)-\psi(v), e\rangle_{L^2(\Omega)} \qquad \text{for a.e. } t\in[0,T]. \tag{5.3}
$$

Let $t \in (0,T)$ be a point where (5.3) holds, and write $\psi = \psi_l + \psi_i$, where ψ_l is Lipschitz continuous on $\mathbf{R}$, and ψ_i non decreasing on $\mathbf{R}$. Integrating (5.3) on $(0,t)$ and recalling that by assumption $e(\mathbf{x},0) = 0$ for almost every $\mathbf{x}\in\Omega$, we get

$$
\begin{aligned}
\|e(\cdot,t)\|^2_{L^2(\Omega)} &\leq 2\varepsilon^{-2}\int_0^t |\langle \psi_l(u)-\psi_l(v), e(\cdot,\tau)\rangle_{L^2(\Omega)}|\, d\tau\\
&\leq 2\mathrm{lip}(\psi_l)\varepsilon^{-2}\int_0^t \|e(\cdot,\tau)\|^2_{L^2(\Omega)}\, d\tau.
\end{aligned}
$$

Then for almost every $t\in[0,T]$ Gronwall's Lemma implies that $\|e(\cdot,t)\|_{L^2(\Omega)} = 0$ and thus, by Fubini's Theorem, $e = 0$ almost everywhere in $\Omega\times(0,T)$, i.e., (5.2). □

6. Subsolution and Supersolution. In this section we construct a sub and supersolution for problem (2.9). Such functions will be used to derive the desired interface error estimate.

Let $\delta \geq 3$ be a fixed natural number; for any $\varepsilon > 0$ let $x_\varepsilon = \delta|\log\varepsilon|$. Note that $\gamma(x_\varepsilon) = 1 - 2\varepsilon^{2\delta}(1+\varepsilon^{2\delta})^{-1} = 1 - \mathcal{O}(\varepsilon^{2\delta})$, and $\gamma'(x_\varepsilon) = 1-\gamma^2(x_\varepsilon) = \mathcal{O}(\varepsilon^{2\delta})$. As a consequence, using (3.20), (3.22), (3.16), (3.23), (3.24), (3.26) and (3.27), we get

$$
\begin{aligned}
&|\gamma(x_\varepsilon)-1|, |\gamma'(x_\varepsilon)| = \mathcal{O}(\varepsilon^{2\delta}),\\
&|\eta(x_\varepsilon)-\eta_\infty|, |\eta'(x_\varepsilon)| = |\log\varepsilon|\mathcal{O}(\varepsilon^{2\delta}),\\
&|\xi(x_\varepsilon)|, |\xi'(x_\varepsilon)|, |\omega(x_\varepsilon)|, |\omega'(x_\varepsilon)|, |\pi(x_\varepsilon)-\pi_\infty|, |\pi'(x_\varepsilon)| = |\log\varepsilon|^2\mathcal{O}(\varepsilon^{2\delta}).
\end{aligned}
$$

We construct five functions

$$
\gamma_\varepsilon, \xi_\varepsilon, \eta_\varepsilon, \omega_\varepsilon, \pi_\varepsilon \in C^{1,1}(\mathbf{R}) \cap C^\infty(\mathbf{R}\setminus\{\pm x_\varepsilon, \pm 2x_\varepsilon\}),
$$

which coincide, respectively, with $\gamma, \xi, \eta, \omega, \pi$ on $[-x_\varepsilon, x_\varepsilon]$ and are constant outside the interval $[-2x_\varepsilon, 2x_\varepsilon]$, as follows:

$$\gamma_\varepsilon(x) = \begin{cases} \gamma(x) & 0 \le x < x_\varepsilon \\ P_\gamma(x) & x_\varepsilon \le x \le 2x_\varepsilon \\ 1 & x > 2x_\varepsilon \\ -\gamma_\varepsilon(-x) & x < 0 \end{cases}, \qquad \xi_\varepsilon(x) = \begin{cases} \xi(x) & 0 \le x < x_\varepsilon \\ P_\xi(x) & x_\varepsilon \le x \le 2x_\varepsilon \\ 0 & x > 2x_\varepsilon \\ -\xi_\varepsilon(-x) & x < 0 \end{cases},$$

$$\eta_\varepsilon(x) = \begin{cases} \eta(x) & 0 \le x < x_\varepsilon \\ P_\eta(x) & x_\varepsilon \le x \le 2x_\varepsilon \\ \frac{c_0}{2\alpha} & x > 2x_\varepsilon \\ \eta_\varepsilon(-x) & x < 0 \end{cases}, \qquad \omega_\varepsilon(x) = \begin{cases} \omega(x) & 0 \le x < x_\varepsilon \\ P_\omega(x) & x_\varepsilon \le x \le 2x_\varepsilon \\ 0 & x > 2x_\varepsilon \\ -\omega_\varepsilon(-x) & x < 0 \end{cases},$$

$$\pi_\varepsilon(x) = \begin{cases} \pi(x) & 0 \le x < x_\varepsilon \\ P_\pi(x) & x_\varepsilon \le x \le 2x_\varepsilon \\ -\frac{c_0^2\beta}{8\alpha^3} & x > 2x_\varepsilon \\ -\pi_\varepsilon(-x) & x < 0. \end{cases}.$$

Here $P_\gamma, P_\xi, P_\eta, P_\omega, P_\pi : [x_\varepsilon, 2x_\varepsilon] \to \mathbf{R}$ are the five polynomials of degree 3 such that $\gamma_\varepsilon, \xi_\varepsilon, \eta_\varepsilon, \omega_\varepsilon, \pi_\varepsilon \in C^{1,1}(\mathbf{R})$. For example, for any $x \in [x_\varepsilon, 2x_\varepsilon]$

$$P_\gamma(x) = \Big(\frac{2x_\varepsilon - x}{x_\varepsilon}\Big)^3 [2(1-\gamma(x_\varepsilon)) - x_\varepsilon\gamma'(x_\varepsilon)] + \Big(\frac{2x_\varepsilon - x}{x_\varepsilon}\Big)^2 [x_\varepsilon\gamma'(x_\varepsilon) - 3(1-\gamma(x_\varepsilon))] + 1,$$
$$P_\xi(x) = \Big(\frac{2x_\varepsilon - x}{x_\varepsilon}\Big)^3 [-2\xi(x_\varepsilon) - x_\varepsilon\xi'(x_\varepsilon)] + \Big(\frac{2x_\varepsilon - x}{x_\varepsilon}\Big)^2 [x_\varepsilon\xi'(x_\varepsilon) + 3\xi(x_\varepsilon)].$$

Note that these polynomials are not necessarily monotone (for instance P_γ is decreasing at the point $2x_\varepsilon$).

One can check by direct computation that

$$\|P_\zeta - \zeta_\infty\|_{L^\infty(x_\varepsilon, 2x_\varepsilon)} \le C(|\zeta(x_\varepsilon) - \zeta_\infty| + x_\varepsilon|\zeta'(x_\varepsilon)|),$$
$$\|P'_\zeta\|_{L^\infty(x_\varepsilon, 2x_\varepsilon)} \le \frac{C}{x_\varepsilon}(|\zeta(x_\varepsilon) - \zeta_\infty| + x_\varepsilon|\zeta'(x_\varepsilon)|),$$
$$\|P''_\zeta\|_{L^\infty(x_\varepsilon, 2x_\varepsilon)} \le \frac{C}{x_\varepsilon^2}(|\zeta(x_\varepsilon) - \zeta_\infty| + x_\varepsilon|\zeta'(x_\varepsilon)|),$$

where ζ stands for any one of the functions $\gamma, \xi, \eta, \omega, \pi$, $\zeta_\infty = \lim_{x\to+\infty}\zeta(x)$, and C is a

suitable positive constant. As a consequence, we get the following estimates:

$$
\begin{aligned}
&\|P_\gamma - 1\|_{L^\infty(x_\varepsilon,2x_\varepsilon)} = |\log\varepsilon|\mathcal{O}(\varepsilon^{2\delta}) = o(\varepsilon^{2\delta-1}),\\
&\|P'_\gamma\|_{L^\infty(x_\varepsilon,2x_\varepsilon)} = \mathcal{O}(\varepsilon^{2\delta}), \qquad \|P''_\gamma\|_{L^\infty(x_\varepsilon,2x_\varepsilon)} = |\log\varepsilon|^{-1}\mathcal{O}(\varepsilon^{2\delta}) = o(\varepsilon^{2\delta}),\\
&\|P_\eta - \eta_\infty\|_{L^\infty(x_\varepsilon,2x_\varepsilon)} = |\log\varepsilon|^2\mathcal{O}(\varepsilon^{2\delta}) = o(\varepsilon^{2\delta-1}),\\
&\|P'_\eta\|_{L^\infty(x_\varepsilon,2x_\varepsilon)} = |\log\varepsilon|\mathcal{O}(\varepsilon^{2\delta}) = o(\varepsilon^{2\delta-1}), \qquad \|P''_\eta\|_{L^\infty(x_\varepsilon,2x_\varepsilon)} = \mathcal{O}(\varepsilon^{2\delta}),\\
&\|P_\xi\|_{L^\infty(x_\varepsilon,2x_\varepsilon)}, \|P_\omega\|_{L^\infty(x_\varepsilon,2x_\varepsilon)}, \|P_\pi - \pi_\infty\|_{L^\infty(x_\varepsilon,2x_\varepsilon)} = |\log\varepsilon|^3\mathcal{O}(\varepsilon^{2\delta}) = o(\varepsilon^{2\delta-1}),\\
&\|P'_\xi\|_{L^\infty(x_\varepsilon,2x_\varepsilon)}, \|P'_\omega\|_{L^\infty(x_\varepsilon,2x_\varepsilon)}, \|P'_\pi\|_{L^\infty(x_\varepsilon,2x_\varepsilon)} = |\log\varepsilon|^2\mathcal{O}(\varepsilon^{2\delta}) = o(\varepsilon^{2\delta-1}),\\
&\|P''_\xi\|_{L^\infty(x_\varepsilon,2x_\varepsilon)}, \|P''_\omega\|_{L^\infty(x_\varepsilon,2x_\varepsilon)}, \|P''_\pi\|_{L^\infty(x_\varepsilon,2x_\varepsilon)} = |\log\varepsilon|\mathcal{O}(\varepsilon^{2\delta}) = o(\varepsilon^{2\delta-1}).
\end{aligned}
\tag{6.1}
$$

Finally, from (6.1), it follows

$$
\|\gamma_\varepsilon - \gamma\|_{L^\infty(\mathbf{R})} = \|\gamma_\varepsilon - \gamma\|_{L^\infty(x_\varepsilon,2x_\varepsilon)} = o(\varepsilon^{2\delta-1}),
\tag{6.2}
$$

and similarly

$$
\|\xi_\varepsilon - \xi\|_{L^\infty(\mathbf{R})}, \|\eta_\varepsilon - \eta\|_{L^\infty(\mathbf{R})}, \|\omega_\varepsilon - \omega\|_{L^\infty(\mathbf{R})}, \|\pi_\varepsilon - \pi\|_{L^\infty(\mathbf{R})} = o(\varepsilon^{2\delta-1}).
\tag{6.3}
$$

6.1. Subsolution. For any $\varepsilon > 0$ and any $(\mathbf{x}, t) \in \Omega \times [0,T]$ we define the modified distance function $d_\varepsilon^-(\mathbf{x}, t)$ as

$$
d_\varepsilon^-(\mathbf{x}, t) = d(\mathbf{x}, t) - c_1(t)\varepsilon^2|\log\varepsilon|^2,
$$

where $c_1 : [0,T] \to]0, +\infty[$ is a continuous (exponentially increasing) function to be determined later on independently of ε (see (6.16)). For any $t \in [0,T]$, let

$$
\mathcal{T}_\varepsilon^-(t) = \{\mathbf{x} \in \Omega : |d_\varepsilon^-(\mathbf{x}, t)| < 2\delta\varepsilon|\log\varepsilon|\}, \qquad \mathcal{T}_\varepsilon^- = \bigcup_{t\in[0,T]} \mathcal{T}_\varepsilon^-(t) \times \{t\}.
$$

For convenience, we remove the superscript $^-$, thus denoting $d_\varepsilon = d_\varepsilon^-$, $\mathcal{T}_\varepsilon^-(t) = \mathcal{T}_\varepsilon(t)$, and $\mathcal{T}_\varepsilon^- = \mathcal{T}_\varepsilon$.

Observe that there exists $\varepsilon_0 > 0$ (depending on $\delta, c_1(\cdot), D$) such that $\mathcal{T}_\varepsilon(t) \subseteq \mathcal{T}(t)$ for any $0 < \varepsilon \le \varepsilon_0$ and any $t \in [0,T]$, so that $\mathcal{T}_\varepsilon \subseteq \mathcal{T}$ for any $0 < \varepsilon \le \varepsilon_0$. Moreover

$$
d(\mathbf{x}, t) = \mathcal{O}(\varepsilon|\log\varepsilon|) \qquad \forall (\mathbf{x}, t) \in \mathcal{T}_\varepsilon.
\tag{6.4}
$$

Note that $\nabla d_\varepsilon = \nabla d$ and $(\nabla d_\varepsilon, \nabla\overline{h}) = 0$ on $\mathcal{T}_\varepsilon$ (see §2). In addition, on $\mathcal{T}_\varepsilon$, using (2.6), (2.5), and (6.4), we have

$$
\mathcal{H}d_\varepsilon = \mathcal{H}d - c_1'\varepsilon^2|\log\varepsilon|^2 = \overline{g} + d_\varepsilon\overline{h} + \varepsilon^2|\log\varepsilon|^2\big(c_1\overline{h} - c_1'\big) + \mathcal{O}(\varepsilon^2|\log\varepsilon|^2).
\tag{6.5}
$$

Our aim is to introduce a subsolution v_ε^- for problem (2.9). As we shall see, the definition of v_ε^- is suggested by the formal asymptotics of §4. Define the real stretched variable $y = y(\mathbf{x},t) = \varepsilon^{-1} d_\varepsilon(\mathbf{x},t)$, set

$$\mathbf{b} = (\overline{h} - \nabla d \cdot \nabla g, g^2 - 2\nabla d \cdot \nabla g, g^2), \qquad \mathbf{p}_\varepsilon = (\xi_\varepsilon, \omega_\varepsilon, \pi_\varepsilon),$$

and define on $\Omega \times [0,T]$

$$v_\varepsilon^-(\mathbf{x},t) = \begin{cases} \gamma_\varepsilon(y) + \varepsilon g(\mathbf{x},t)\eta_\varepsilon(y) + \varepsilon^2 \mathbf{b}(\mathbf{x},t)\cdot \mathbf{p}_\varepsilon(y) - c_2\varepsilon^3 |\log\varepsilon|^2 & \text{on } \mathcal{T}_\varepsilon, \\ 1 + \varepsilon\eta_\infty g(\mathbf{x},t) + \varepsilon^2 \pi_\infty g^2(\mathbf{x},t) - c_2\varepsilon^3|\log\varepsilon|^2 & \text{on } \{d_\varepsilon \geq 2\delta\varepsilon|\log\varepsilon|\}, \\ -1 + \varepsilon\eta_\infty g(\mathbf{x},t) - \varepsilon^2\pi_\infty g^2(\mathbf{x},t) - c_2\varepsilon^3|\log\varepsilon|^2 & \text{on } \{d_\varepsilon \leq -2\delta\varepsilon|\log\varepsilon|\}, \end{cases}$$

where $c_2 > 0$ is a constant to be determined later on independently of ε. For simplicity, we will use the notation $v_\varepsilon = v_\varepsilon^-$. It is clear, from the regularity of d_ε and the properties of the functions $\gamma_\varepsilon, \xi_\varepsilon, \eta_\varepsilon, \omega_\varepsilon, \pi_\varepsilon$, that v_ε belongs to $L^2(0,T;H^2(\Omega)) \cap H^1(0,T;L^2(\Omega))$.

Note that, in view of the definition of d_ε, the term $\gamma_\varepsilon(y)$ corresponds to a right shift of the function γ_ε of order ε, which is natural when looking for a subsolution. On the other hand, the term $\varepsilon g\eta_\varepsilon + \varepsilon^2 \mathbf{b}\cdot\mathbf{p}_\varepsilon$ is a shape correction suggested by the formal asymptotic expansion given in §4; one can prove that this correction is of higher order with respect to the previous right translation. Finally, the term $c_2\varepsilon^3|\log\varepsilon|^2$ provides a further downward translation, and it is necessary both for the comparison with the initial datum and to provide control of "bad" terms far from the interface.

6.2. Comparison with the initial datum. Fix $\varepsilon > 0$; inspired by the asymptotics in §4, we assume

$$u_\varepsilon(\mathbf{x},0) = \gamma(\varepsilon^{-1} d(\mathbf{x},0)) + \varepsilon \frac{c_0}{2\alpha} g(\mathbf{x},0) - \varepsilon^2 \frac{c_0^2\beta}{8\alpha^3} g^2(\mathbf{x},0)\gamma(\varepsilon^{-1}d(\mathbf{x},0)). \tag{6.6}$$

Let $z = \varepsilon^{-1} d(\mathbf{x},0)$, and, as before, let $y = \varepsilon^{-1} d_\varepsilon(\mathbf{x},0)$. Note that $z = y + c_1(0)\varepsilon|\log\varepsilon|^2 > y$. We have to check

$$v_\varepsilon(\mathbf{x},0) \leq u_\varepsilon(\mathbf{x},0) \qquad \forall \mathbf{x} \in \Omega. \tag{6.7}$$

Define $\mathbf{p} = (\xi,\omega,\pi)$, and

$$w_\varepsilon(\mathbf{x},t) = \gamma(y(\mathbf{x},t)) + \varepsilon g(\mathbf{x},t)\eta(y(\mathbf{x},t)) + \varepsilon^2 \mathbf{b}(\mathbf{x},t)\cdot\mathbf{p}(y(\mathbf{x},t)) - \frac{c_2}{2}\varepsilon^3|\log\varepsilon|^2.$$

To prove (6.7) it will be enough to show

$$w_\varepsilon(\mathbf{x},0) \leq u_\varepsilon(\mathbf{x},0) \qquad \forall \mathbf{x} \in \Omega. \tag{6.8}$$

Indeed, by (2.1), (6.2) and (6.3) we have

$$|v_\varepsilon - w_\varepsilon + \frac{c_2}{2}\varepsilon^3|\log\varepsilon|^2| \le o(\varepsilon^{2\delta-1}),$$

so that, if (6.8) is true,

$$v_\varepsilon = (v_\varepsilon - w_\varepsilon) + w_\varepsilon \le w_\varepsilon - \frac{c_2}{2}\varepsilon^3|\log\varepsilon|^2 + o(\varepsilon^{2\delta-1}) \le$$
$$u_\varepsilon(\mathbf{x},0) - \frac{c_2}{2}\varepsilon^3|\log\varepsilon|^2 + o(\varepsilon^{2\delta-1}) \le u_\varepsilon(\mathbf{x},0),$$

as $\varepsilon \to 0$.

Let us show (6.8). We have

$$w_\varepsilon(\mathbf{x},0) - u_\varepsilon(\mathbf{x},0) = (\gamma(y) - \gamma(z)) + \varepsilon g(\mathbf{x},0)(\eta(y) - \frac{c_0}{2\alpha}) + \varepsilon^2(\mathbf{b}(\mathbf{x},0)\cdot\mathbf{p}(y)$$
$$+ \frac{c_0^2\beta}{8\alpha^3}g^2(\mathbf{x},0)\gamma(z)) - \frac{c_2}{2}\varepsilon^3|\log\varepsilon|^2 =: \mathrm{I} + \mathrm{II} + \mathrm{III} - \frac{c_2}{2}\varepsilon^3|\log\varepsilon|^2.$$

We need the following result.

LEMMA 6.1. *Let $z = \varepsilon^{-1}d(\mathbf{x},0)$, $y = \varepsilon^{-1}d_\varepsilon(\mathbf{x},0) = z - c_1(0)\varepsilon|\log\varepsilon|^2$. We have*

$$\frac{1}{2}\gamma'(y) \le \gamma'(t) \le 2\gamma'(y) \tag{6.9}$$

for all $t \in [y,z]$ and ε sufficiently small.

Proof. Observe that $(\log(\gamma'))' = \frac{\gamma''}{\gamma'} = -2\gamma \in L^\infty(\mathbf{R})$, so that $\log(\gamma')$ is Lipschitz continous with Lipschitz constant 2. Therefore

$$|\log(\frac{\gamma'(t_1)}{\gamma'(t_2)})| = |\log(\gamma'(t_1)) - \log(\gamma'(t_2))| \le 2|t_2 - t_1|.$$

Hence, if $|t_2 - t_1| \le c_1(0)\varepsilon|\log\varepsilon|^2$, we have

$$\frac{\gamma'(t_1)}{\gamma'(t_2)} \ge \exp(-2|t_2 - t_1|) \ge \exp(-2c_1(0)\varepsilon|\log\varepsilon|^2) \to 1$$

as $\varepsilon \to 0$, and the left inequality of (6.9) follows. Similarly,

$$\frac{\gamma'(t_1)}{\gamma'(t_2)} \le \exp(2|t_2 - t_1|) \le \exp(2c_1(0)\varepsilon|\log\varepsilon|^2) \to 1$$

as $\varepsilon \to 0$, and the Lemma follows. □

Using Lemma 6.1, we find

$$\mathrm{I} \le -\frac{1}{2}\gamma'(y)c_1(0)\varepsilon|\log\varepsilon|^2; \tag{6.10}$$

moreover, by (2.1) and (3.20), there exists a positive constant c such that

$$|\mathrm{II}| \le c\varepsilon(1+|y|)\gamma'(y). \tag{6.11}$$

In addition, by (2.1), the fact that $\mathbf{b}(\cdot,0) \in L^\infty(\Omega)$, and (3.16), (3.23), (3.26), (3.4), and Lemma 6.1, we have

$$|\mathrm{III}| \le \varepsilon^2|\mathbf{b}(\mathbf{x},0)\cdot\mathbf{p}(y) + \frac{c_0^2\beta}{8\alpha^3}g^2(\mathbf{x},0)\gamma(y)| + \varepsilon^2\frac{c_0^2\beta}{8\alpha^3}g^2(\mathbf{x},0)|\gamma(z)-\gamma(y)| \le$$
$$\varepsilon^2|\mathbf{b}(\mathbf{x},0)\cdot\mathbf{p}(y) - g^2(\mathbf{x},0)q| + \varepsilon^2 g^2(\mathbf{x},0)|\frac{c_0^2\beta}{8\alpha^3}\gamma(y) - q| +$$
$$\varepsilon^2\frac{c_0^2\beta}{8\alpha^3}g^2(\mathbf{x},0)|\gamma(z)-\gamma(y)| \le c\varepsilon^2(1+|y|^2)\gamma'(y),$$

where q is as in the proof of Lemma 3.1 with $q_- = q_+ = \frac{c_0^2\beta}{8\alpha^3}$.

Let us distinguish two different cases. If $|y| > 2|\log\varepsilon|$, by direct computation we have $\gamma'(y) < c\varepsilon^4$ for some positive constant c, hence the terms II and III can be controlled by the negative term $-\frac{c_2}{2}\varepsilon^3|\log\varepsilon|^2$. On the contrary, if $|y| \le 2|\log\varepsilon|$, they are easily controlled by the term I. This proves (6.8), and so the proof of (6.7) is concluded.

6.3. v_ε is a subsolution. In order to apply the Comparison Lemma 5.1 we must show

$$\mathcal{H}v_\varepsilon + \varepsilon^{-2}\psi(v_\varepsilon) - \varepsilon^{-1}\frac{c_0}{2}g \le 0 \qquad \text{for a.e. } (\mathbf{x},t) \in \Omega\times(0,T). \tag{6.12}$$

We first restrict ourselves to points $(\mathbf{x},t) \in \mathcal{T}_\varepsilon$. Direct computations yield

$$\nabla v_\varepsilon = \varepsilon^{-1}\gamma_\varepsilon'(y)\nabla d_\varepsilon + g\eta_\varepsilon'\nabla d_\varepsilon + \varepsilon\mathbf{b}\cdot\mathbf{p}_\varepsilon'\nabla d_\varepsilon + \varepsilon\eta_\varepsilon\nabla g + \varepsilon^2\mathbf{p}_\varepsilon\cdot\nabla\mathbf{b} \qquad \text{in } \mathcal{T}_\varepsilon,$$

where, if $\mathbf{b} = (b_1,b_2,b_3)$, then $\nabla\mathbf{b} = (\nabla b_1, \nabla b_2, \nabla b_3)$. Enforcing the equality $|\nabla d_\varepsilon| = 1$ and (2.1), we get

$$\partial_t v_\varepsilon = \varepsilon^{-1}\gamma_\varepsilon'\partial_t d_\varepsilon + g\eta_\varepsilon'\partial_t d_\varepsilon + \varepsilon\mathbf{b}\cdot\mathbf{p}_\varepsilon'\partial_t d_\varepsilon + \varepsilon\eta_\varepsilon\partial_t g + \varepsilon^2\mathbf{p}_\varepsilon\cdot\partial_t\mathbf{b},$$
$$\Delta v_\varepsilon = \varepsilon^{-2}\gamma_\varepsilon'' + \varepsilon^{-1}g\eta_\varepsilon'' + \varepsilon^{-1}\gamma_\varepsilon'\Delta d_\varepsilon + g\eta_\varepsilon'\Delta d_\varepsilon + \mathbf{b}\cdot\mathbf{p}_\varepsilon'' + \varepsilon\mathbf{b}\cdot\mathbf{p}_\varepsilon'\Delta d_\varepsilon + 2\eta_\varepsilon'\nabla g\cdot\nabla d_\varepsilon + \mathcal{O}(\varepsilon).$$

Hence, using (6.5), in the layer $\mathcal{T}_\varepsilon$ we have

$$\mathcal{H}v_\varepsilon = -\varepsilon^{-2}\gamma_\varepsilon'' - \varepsilon^{-1}g\eta_\varepsilon'' - \mathbf{b}\cdot\mathbf{p}_\varepsilon'' + (\varepsilon^{-1}\gamma_\varepsilon' + g\eta_\varepsilon')\mathcal{H}d_\varepsilon - 2\eta_\varepsilon'\nabla g\cdot\nabla d_\varepsilon + \mathcal{O}(\varepsilon) =$$
$$-\varepsilon^{-2}\gamma_\varepsilon'' - \varepsilon^{-1}g\eta_\varepsilon'' + \varepsilon^{-1}\gamma_\varepsilon'\overline{g} - \mathbf{b}\cdot\mathbf{p}_\varepsilon'' - 2\eta_\varepsilon'\nabla g\cdot\nabla d_\varepsilon + g\overline{g}\eta_\varepsilon' + y\gamma_\varepsilon'\overline{h} +$$
$$c_1\varepsilon|\log\varepsilon|^2\gamma_\varepsilon'\overline{h} - c_1'\varepsilon|\log\varepsilon|^2\gamma_\varepsilon' + \mathcal{O}(\varepsilon|\log\varepsilon|^2).$$

Now observe that, on $\mathcal{T}_\varepsilon$,

$$g(\mathbf{x},t) = g(\mathbf{s}(\mathbf{x},t) + d\nabla d, t) =$$
$$\overline{g} + d\nabla d \cdot \overline{\nabla g} + \mathcal{O}(d^2) = \overline{g} + \varepsilon y \nabla d_\varepsilon \cdot \overline{\nabla g} + c_1\varepsilon^2 |\log\varepsilon|^2 \nabla d_\varepsilon \cdot \overline{\nabla g} + \mathcal{O}(\varepsilon^2|\log\varepsilon|^2),$$

so that

$$g\overline{g}\eta_\varepsilon' = g^2\eta_\varepsilon' + \mathcal{O}(\varepsilon|\log\varepsilon|), \tag{6.13}$$

and

$$\varepsilon^{-1}\gamma_\varepsilon'\overline{g} = \varepsilon^{-1}\gamma_\varepsilon' g - y\gamma_\varepsilon'\nabla d_\varepsilon \cdot \overline{\nabla g} - c_1\varepsilon|\log\varepsilon|^2\gamma_\varepsilon'\nabla d_\varepsilon \cdot \overline{\nabla g} + \mathcal{O}(\varepsilon|\log\varepsilon|^2). \tag{6.14}$$

Inserting (6.13) and (6.14) in the previous expression of $\mathcal{H}v_\varepsilon$, we get

$$\mathcal{H}v_\varepsilon = -\varepsilon^{-2}\gamma_\varepsilon'' - \varepsilon^{-1}g\eta_\varepsilon'' + \varepsilon^{-1}\gamma_\varepsilon' g - y\gamma_\varepsilon'\nabla d_\varepsilon \cdot \overline{\nabla g} - c_1\varepsilon|\log\varepsilon|^2\gamma_\varepsilon'\nabla d_\varepsilon \cdot \overline{\nabla g} -$$
$$- \mathbf{b}\cdot\mathbf{p}_\varepsilon'' - 2\eta_\varepsilon'\nabla g\cdot\nabla d_\varepsilon + g^2\eta_\varepsilon' + y\gamma_\varepsilon'\overline{h} + c_1\varepsilon|\log\varepsilon|^2\gamma_\varepsilon'\overline{h} - c_1'\varepsilon|\log\varepsilon|^2\gamma_\varepsilon' + \mathcal{O}(\varepsilon|\log\varepsilon|^2).$$

Enforcing the formula

$$\varepsilon^{-2}\psi(v_\varepsilon) = \varepsilon^{-2}\psi(\gamma_\varepsilon) + \varepsilon^{-1}g\eta_\varepsilon\psi'(\gamma_\varepsilon) + \mathbf{b}\cdot\mathbf{p}_\varepsilon\psi'(\gamma_\varepsilon) + \frac{1}{2}g^2\eta_\varepsilon^2\psi''(\gamma_\varepsilon) - c_2\varepsilon|\log\varepsilon|^2\psi'(\gamma_\varepsilon) + \mathcal{O}(\varepsilon),$$

we finally get that, on $\mathcal{T}_\varepsilon$,

$$\mathcal{H}v_\varepsilon + \varepsilon^{-2}\psi(v_\varepsilon) - \varepsilon^{-1}\frac{c_0}{2}g = \mathrm{I}_\varepsilon + \mathrm{II}_\varepsilon + \mathrm{III}_\varepsilon + \mathrm{IV}_\varepsilon + \mathcal{O}(\varepsilon|\log\varepsilon|^2),$$

where, recalling also the expression of $\mathbf{b}$ and $\mathbf{p}_\varepsilon$ and the equality $\nabla d_\varepsilon = \nabla d$,

$$\mathrm{I}_\varepsilon = -\varepsilon^{-2}(\gamma_\varepsilon'' - \psi(\gamma_\varepsilon)),$$
$$\mathrm{II}_\varepsilon = -\varepsilon^{-1}g[\eta_\varepsilon'' - \eta_\varepsilon\psi'(\gamma_\varepsilon) + \frac{c_0}{2} - \gamma_\varepsilon'],$$
$$\mathrm{III}_\varepsilon = -\mathbf{b}\cdot\mathbf{p}_\varepsilon'' + \mathbf{b}\cdot\mathbf{p}_\varepsilon\psi'(\gamma_\varepsilon) - 2\eta_\varepsilon'\nabla g\cdot\nabla d + g^2\eta_\varepsilon' + y\gamma_\varepsilon'\overline{h} + \frac{1}{2}g^2\eta_\varepsilon^2\psi''(\gamma_\varepsilon) - y\gamma_\varepsilon'\nabla d\cdot\overline{\nabla g}$$
$$= -(\overline{h} - \nabla d\cdot\nabla g)(\xi_\varepsilon'' - \psi'(\gamma_\varepsilon)\xi_\varepsilon - y\gamma_\varepsilon') + y\gamma_\varepsilon'\nabla d\cdot(\nabla g - \overline{\nabla g})$$
$$- (g^2 - 2\nabla d\cdot\nabla g)(\omega_\varepsilon'' - \psi'(\gamma_\varepsilon)\omega_\varepsilon - \eta_\varepsilon') - g^2(\pi_\varepsilon'' - \psi'(\gamma_\varepsilon)\pi_\varepsilon - \frac{1}{2}\eta_\varepsilon^2\psi''(\gamma_\varepsilon)),$$
$$\mathrm{IV}_\varepsilon = c_1\varepsilon|\log\varepsilon|^2\gamma_\varepsilon'\overline{h} - c_1'\varepsilon|\log\varepsilon|^2\gamma_\varepsilon' - c_2\varepsilon|\log\varepsilon|^2\psi'(\gamma_\varepsilon) - c_1\varepsilon|\log\varepsilon|^2\gamma_\varepsilon'\nabla d\cdot\overline{\nabla g} =$$
$$\mathrm{IV}_\varepsilon^1 + \mathrm{IV}_\varepsilon^2 + \mathrm{IV}_\varepsilon^3 + \mathrm{IV}_\varepsilon^4.$$

Observe now that

$$\gamma_\varepsilon'' - \psi(\gamma_\varepsilon) = o(\varepsilon^{2\delta-1}) \qquad \forall y \in \mathbf{R}\setminus\{\pm x_\varepsilon, \pm 2x_\varepsilon\}. \tag{6.15}$$

Indeed $\gamma_\varepsilon'' - \psi(\gamma_\varepsilon) = 0$ on the set $]-\infty, -2x_\varepsilon[\cup]-x_\varepsilon, x_\varepsilon[\cup]2x_\varepsilon, +\infty[$. Let now $y \in]-2x_\varepsilon, -x_\varepsilon[\cup]x_\varepsilon, 2x_\varepsilon[$. By the definition of γ_ε and the second relation in (6.1), one has $\gamma_\varepsilon''(y) = o(\varepsilon^{2\delta})$; in addition, by the first relation in (6.1),

$$\psi(\gamma_\varepsilon(y)) = o(\varepsilon^{2\delta-1}),$$

and this gives (6.15). Similarly, the definitions of $\xi_\varepsilon, \eta_\varepsilon, \omega_\varepsilon, \pi_\varepsilon$ and iterated applications of formulae (6.1) yield

$$\begin{aligned}
&\eta_\varepsilon'' - \psi'(\gamma_\varepsilon)\eta_\varepsilon + \frac{c_0}{2} - \gamma_\varepsilon' = o(\varepsilon^{2\delta-1}),\\
&\xi_\varepsilon'' - \psi'(\gamma_\varepsilon)\xi_\varepsilon - y\gamma_\varepsilon' = \omega_\varepsilon'' - \psi'(\gamma_\varepsilon)\omega_\varepsilon - \eta_\varepsilon' =\\
&\pi_\varepsilon'' - \psi'(\gamma_\varepsilon)\pi_\varepsilon - \frac{1}{2}\eta_\varepsilon^2\psi''(\gamma_\varepsilon) = o(\varepsilon^{2\delta-1}),
\end{aligned}$$

for any $y \in \mathbf{R} \setminus \{\pm x_\varepsilon, \pm 2x_\varepsilon\}$. Noting that $y\gamma_\varepsilon'\nabla d \cdot (\nabla g - \overline{\nabla g}) = \mathcal{O}(\varepsilon|\log\varepsilon|)$, the previous estimates imply that, on $\mathcal{T}_\varepsilon$,

$$\mathrm{I}_\varepsilon = \mathrm{II}_\varepsilon = \mathrm{III}_\varepsilon = o(\varepsilon^{2\delta-3}) + \mathcal{O}(\varepsilon|\log\varepsilon|).$$

Take

$$c_1(t) = c\exp((1+K)t), \tag{6.16}$$

where c and K are positive constants. Then we have

$$\mathrm{IV}_\varepsilon^1 + \mathrm{IV}_\varepsilon^2 + \mathrm{IV}_\varepsilon^4 \le -c_1\varepsilon|\log\varepsilon|^2\gamma_\varepsilon'.$$

Choose now $K = \|h\|_{L^\infty(\Sigma)} + \|\nabla g\|_{L^\infty(\Omega\times(0,T))}$. As $c_3\gamma_\varepsilon' + \psi'(\gamma_\varepsilon)$ is uniformly positive for a proper choice of the positive constant c_3, we realize that, if c and c_2 are large enough (independently of ε), then

$$\mathcal{H}v_\varepsilon + \varepsilon^{-2}\psi(v_\varepsilon) - \varepsilon^{-1}\frac{c_0}{2}g \le 0 \qquad \text{on } \mathcal{T}_\varepsilon,$$

for ε sufficiently small (depending on c_1 and c_2).

On the other hand, outside the transition layer $\mathcal{T}_\varepsilon$, we have

$$v_\varepsilon = \pm 1 + \varepsilon\frac{c_0}{2\alpha}g \mp \varepsilon^2\frac{c_0^2\beta}{8\alpha^3}g^2 - c_2\varepsilon^3|\log\varepsilon|^2,$$

so that, by the assumption (2.1), one easily gets

$$\partial_t v_\varepsilon = \mathcal{O}(\varepsilon), \qquad \Delta v_\varepsilon = \mathcal{O}(\varepsilon).$$

Moreover,

$$\varepsilon^{-2}\psi(v_\varepsilon) = \varepsilon^{-2}\psi(\pm 1) + \varepsilon^{-2}\alpha(\varepsilon\frac{c_0}{2\alpha}g \mp \varepsilon^2\frac{c_0^2\beta}{8\alpha^3}g^2 - c_2\varepsilon^3|\log\varepsilon|^2)$$
$$\pm\frac{1}{2}\varepsilon^{-2}(\varepsilon\frac{c_0}{2\alpha}g + \mathcal{O}(\varepsilon^2))^2\beta + \mathcal{O}(\varepsilon) = \varepsilon^{-1}\frac{c_0}{2}g - c_2\alpha\varepsilon|\log\varepsilon|^2 + \mathcal{O}(\varepsilon).$$

Therefore

$$\mathcal{H}v_\varepsilon + \varepsilon^{-2}\psi(v_\varepsilon) - \varepsilon^{-1}\frac{c_0}{2}g = -c_2\alpha\varepsilon|\log\varepsilon|^2 + \mathcal{O}(\varepsilon),$$

so that $\mathcal{H}v_\varepsilon + \varepsilon^{-2}\psi(v_\varepsilon) - \varepsilon^{-1}\frac{c_0}{2}g \le 0$ as $\varepsilon \to 0$ for any choice of the positive constant c_2.

Hence, applying the Comparison Lemma 5.1 with u replaced by the solution u_ε of problem (2.9) with initial datum (6.6) and v replaced by v_ε, we conclude

$$v_\varepsilon^-(\mathbf{x},t) \le u_\varepsilon(\mathbf{x},t) \qquad \forall\,(\mathbf{x},t) \in \Omega\times[0,T].$$

If $(\mathbf{x},t) \in \Omega\times(0,T)$, the construction of a supersolution

$$v_\varepsilon^+(\mathbf{x},t) = \begin{cases} \gamma_\varepsilon(y) + \varepsilon g(\mathbf{x},t)\eta_\varepsilon(y) + \varepsilon^2\mathbf{b}(\mathbf{x},t)\cdot\mathbf{p}_\varepsilon(y) + c_2\varepsilon^3|\log\varepsilon|^2 & \text{on } \mathcal{T}_\varepsilon^+, \\ 1 + \varepsilon\eta_\infty g(\mathbf{x},t) + \varepsilon^2\pi_\infty g^2(\mathbf{x},t) + c_2\varepsilon^3|\log\varepsilon|^2 & \text{on } \{d_\varepsilon^+ \ge 2\delta\varepsilon|\log\varepsilon|\}, \\ -1 + \varepsilon\eta_\infty g(\mathbf{x},t) - \varepsilon^2\pi_\infty g^2(\mathbf{x},t) + c_2\varepsilon^3|\log\varepsilon|^2 & \text{on } \{d_\varepsilon^+ \le -2\delta\varepsilon|\log\varepsilon|\}, \end{cases}$$

where $y = \frac{d_\varepsilon^+(\mathbf{x},t)}{\varepsilon}$, and

$$d_\varepsilon^+(\mathbf{x},t) = d(\mathbf{x},t) + c_1(t)\varepsilon^2|\log\varepsilon|^2,$$

is similar, and thus omitted.

Using the subsolution v_ε^- and supersolution v_ε^+ defined in §6, it is now possible to deduce the following error estimates, valid before the onset of singularities, for interfaces evolving by mean curvature with a forcing term g satisfying (2.1).

THEOREM 6.1. *Let $\Sigma(t)$ be a mean curvature flow with forcing term g which satisfies (2.4). For any $\varepsilon > 0$ let u_ε be a solution of problem (2.9) with initial datum (6.6). Let $\Sigma_\varepsilon(t) = \{\mathbf{x} \in \Omega : u_\varepsilon(\mathbf{x},t) = 0\}$. Then there exist $0 < \varepsilon_0 < 1$ and a constant C depending on Σ, g, and T such that for all $0 < \varepsilon \le \varepsilon_0$*

$$\Sigma_\varepsilon(t) \subseteq \{\mathbf{x} \in \Omega : \mathrm{dist}(\mathbf{x},\Sigma(t))) \le C\varepsilon^2|\log\varepsilon|^2\} \qquad \forall t \in [0,T], \tag{6.17}$$

$$\Sigma(t) \subseteq \{\mathbf{x} \in \Omega : \mathrm{dist}(\mathbf{x},\Sigma_\varepsilon(t))) \le C\varepsilon^2|\log\varepsilon|^2\} \qquad \forall t \in [0,T]. \tag{6.18}$$

Proof. Let us prove (6.17). Fix $t \in [0,T]$, and let $\mathbf{x} \in \Sigma_\varepsilon(t)$. We claim that $\mathbf{x} \in \mathcal{T}(t)$ (see (2.3)). Indeed, if $\mathbf{x} \in \Sigma_\varepsilon(t)$, then $u_\varepsilon(\mathbf{x},t) = 0$, so that

$$v_\varepsilon^-(\mathbf{x},t) \le u_\varepsilon(\mathbf{x},t) = 0 \le v_\varepsilon^+(\mathbf{x},t). \tag{6.19}$$

If by contradiction $\mathbf{x} \notin \mathcal{T}(t)$, by the inclusions $\mathcal{T}_\varepsilon^\mp(t) \subseteq \mathcal{T}(t)$ we have $\mathbf{x} \notin \mathcal{T}_\varepsilon^-(t)$ and $\mathbf{x} \notin \mathcal{T}_\varepsilon^+(t)$. Recalling the definitions of $v_\varepsilon^\mp$, we deduce that, for ε sufficiently small, $v_\varepsilon^-(\mathbf{x},t)$ and $v_\varepsilon^+(\mathbf{x},t)$ have the same sign, a contradiction with (6.19).

We have $v_\varepsilon^-(\mathbf{x},t) = \gamma_\varepsilon(\frac{d_\varepsilon^-(\mathbf{x},t)}{\varepsilon}) + \mathcal{O}(\varepsilon) \leq 0$, so that $\gamma_\varepsilon(\frac{d_\varepsilon^-(\mathbf{x},t)}{\varepsilon}) \leq \mathcal{O}(\varepsilon)$. As $\gamma'(0) = 1$, we deduce that $\frac{d_\varepsilon^-(\mathbf{x},t)}{\varepsilon} \leq \mathcal{O}(\varepsilon)$; hence, recalling the definition of d_ε^-, we get $d(\mathbf{x},t) \leq \mathcal{O}(\varepsilon^2|\log\varepsilon|^2)$.

A similar argument applied with v_ε^- replaced by v_ε^+ gives $d(\mathbf{x},t) \geq \mathcal{O}(\varepsilon^2|\log\varepsilon|^2)$. We conclude that $|d(\mathbf{x},t)| \leq \mathcal{O}(\varepsilon^2|\log\varepsilon|^2)$, which proves (6.17).

Let us prove (6.18). Fix $t \in [0,T]$, and let $\mathbf{x} \in \Sigma(t)$. We indicate by I the connected component of the intersection between $\mathcal{T}(t)$ and the normal line to $\Sigma(t)$ at $\mathbf{x}$ which contains $\mathbf{x}$. Set $\{\mathbf{x}^-,\mathbf{x}^+\} = I \cap \partial\mathcal{T}(t)$, where $v_\varepsilon^-(\mathbf{x}^+) > 0$ (see the definition of v_ε^-).

Set $\Sigma_\varepsilon^\mp(t) = \{\mathbf{x} \in \Omega : v_\varepsilon^\mp(\mathbf{x},t) = 0\}$. We claim that there exist $\mathbf{x}_\varepsilon^\mp \in I \cap \Sigma_\varepsilon^\mp(t)$ such that $|\mathbf{x}_\varepsilon^\mp - \mathbf{x}| \leq C\varepsilon^2|\log\varepsilon|^2$, for a suitable absolute positive constant C and ε small enough. Indeed, we have $v_\varepsilon^-(\mathbf{x},t) = \gamma_\varepsilon(-c_1(t)\varepsilon|\log\varepsilon|^2) + \mathcal{O}(\varepsilon) < 0$, and $v_\varepsilon^-(\mathbf{x}^+,t) > 0$, so that there exists a points $\mathbf{x}_\varepsilon \in I$ lying between $\mathbf{x}$ and $\mathbf{x}^+$ such that $v_\varepsilon(\mathbf{x}_\varepsilon^-,t) = 0$, i.e., $\mathbf{x}_\varepsilon^- \in I \cap \Sigma_\varepsilon^-(t)$. Moreover,

$$|v_\varepsilon^-(\mathbf{x}_\varepsilon^-,t) - v_\varepsilon^-(\mathbf{x},t)| = |v_\varepsilon^-(\mathbf{x},t)| \leq C\varepsilon|\log\varepsilon|^2, \tag{6.20}$$

and, in view of the non degeneracy property of v_ε, namely $|\nabla v_\varepsilon^- \cdot \mathbf{n}| = \frac{1}{\varepsilon}$, we have

$$|v_\varepsilon^-(\mathbf{x}_\varepsilon^-,t) - v_\varepsilon^-(\mathbf{x},t)| = |\mathbf{x}_\varepsilon^- - \mathbf{x}||\nabla v_\varepsilon^-(\xi,t)\cdot\mathbf{n}| \geq C\frac{|\mathbf{x}_\varepsilon^- - \mathbf{x}|}{\varepsilon}, \tag{6.21}$$

for a suitable absolute positive constant C and a suitable ξ between $\mathbf{x}_\varepsilon^-$ and $\mathbf{x}$. Then (6.20) and (6.21) yield $|\mathbf{x}_\varepsilon^- - \mathbf{x}| \leq C\varepsilon^2|\log\varepsilon|^2$.

A similar argument applied with v_ε^- replaced by v_ε^+ concludes the proof of the claim.

As $\mathbf{x}_\varepsilon^\mp \in \Sigma_\varepsilon^\mp(t)$, we have $u_\varepsilon(\mathbf{x}_\varepsilon^-,t) \geq v_\varepsilon(\mathbf{x}_\varepsilon^-,t) = 0$, and $u_\varepsilon(\mathbf{x}_\varepsilon^+,t) \leq v_\varepsilon(\mathbf{x}_\varepsilon^-,t) = 0$, so that there exists a point $\mathbf{z} \in I$ between $\mathbf{x}_\varepsilon^-$ and $\mathbf{x}_\varepsilon^+$ such that $u_\varepsilon(\mathbf{z},t) = 0$. Therefore

$$\mathrm{dist}(\mathbf{x},\Sigma_\varepsilon(t)) \leq |\mathbf{z} - \mathbf{x}| \leq C\varepsilon^2|\log\varepsilon|^2,$$

and this gives (6.18), and concludes the proof of the theorem. □

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CHAPTER 3, SECTION 3.1

Γ-CONVERGENCE OF DICRETE APPROXIMATIONS

TO INTERFACES WITH PRESCRIBED MEAN CURVATURE*

0. INTRODUCTION

Recently, E. De Giorgi has drawn attention to the numerical approximation of problems in the calculus of variations that fall within the general setting given by him during the last few years [8,2,5]. The discretization of such problems is very difficult, because of the lack of convexity and regularity properties of the functionals involved. This paper addresses the questions formulated by De Giorgi for a particular functional relevant in a number of different contexts involving surface tension, such as fluid phase transition theory, capillarity theory, and image segmentation in computer vision theory [1,3,7,11,13].

We present here a numerical approximation to a functional whose minimizers give rise to interfaces with prescribed mean curvature. Such a functional is first regularized following an idea of L. Modica and S. Mortola [12], who proved convergence of the relaxed functionals; see also [4,11,14]. It is next discretized by means of piecewise linear finite elements with numerical quadrature, thus allowing the actual implementation on a computer. We demonstrate the convergence of the discrete minimizers to a solution of the continuous problem.

More specifically, given an open bounded convex set $\Omega \subset \mathbf{R}^n$ $(n \geq 2)$ with piecewise C^2 boundary, a function $\kappa \in L^\infty(\Omega)$, and $\theta \in \mathbf{R}$, we consider the minimum problem

$$(0.1) \qquad \min_{A \subseteq \Omega} \tilde{\mathcal{F}}(A), \qquad \text{where} \qquad \tilde{\mathcal{F}}(A) := P_\Omega(A) + \cos(\theta)\mathcal{H}^{n-1}(\partial A \cap \partial\Omega) - \int_A \kappa \, dx.$$

Here $P_\Omega(A)$ denotes the perimeter of A in Ω [15] and $\mathcal{H}^{n-1}$ the $(n-1)$-dimensional Hausdorff measure. The solution to this problem is a measurable set $A \subseteq \Omega$ whose boundary has mean curvature κ and contact angle θ at the intersection of ∂A with $\partial\Omega$ [10].

In order to introduce a rigorous formulation of (0.1) we need further notations. Let $BV(\Omega;\{-1,1\})$ be the space of the functions of bounded variation with values in $\{-1,1\}$ and denote by S_v and $\mathrm{tr}(v)$ the *jump* set and the trace on $\partial\Omega$, respectively, of the BV function v [15]. Set $\mu := \cos(\theta)$ and define the following energy functional in $L^1(\Omega)$

$$\mathcal{F}(v) := \begin{cases} 2\mathcal{H}^{n-1}(S_v) + \int_{\partial\Omega} |\mathrm{tr}(v) + \mu| \, d\mathcal{H}^{n-1}(x) - \int_\Omega \kappa v \, dx, & \text{if } v \in D(\mathcal{F}), \\ +\infty, & \text{if } v \in L^1(\Omega) \setminus D(\mathcal{F}), \end{cases}$$

where $D(\mathcal{F}) := BV(\Omega;\{-1,1\})$. It is well known [10] that $\mathcal{F}$ admits at least a minimizer u so that the set $A_u := \{x \in \Omega : \ u(x) = +1\}$ is a solution to problem (0.1). In addition,

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we have $\mathcal{F}(v) = 2\tilde{\mathcal{F}}(A_v) + (1-\mu)\mathcal{H}^{n-1}(\partial\Omega) + \int_\Omega \kappa$, for all $v \in D(\mathcal{F})$. As a generalization, from now on we will consider μ to be a piecewise constant function $\mu \in BV(\partial\Omega;[-1,1])$.

We first approximate $\mathcal{F}$ with a family $\{\mathcal{F}_\epsilon\}_{\epsilon>0}$ of regular nonconvex functionals defined in $H^1(\Omega)$ which, in turn, will be discretized by finite elements; ϵ is the *relaxation* parameter and h is the *meshsize*. In order to define the relaxed functionals $\mathcal{F}_\epsilon$ we need some preparations. Let $\omega : \mathbf{R} \to \mathbf{R}^+ \cup \{+\infty\}$ be defined by

$$\omega(t) = \begin{cases} 1-t^2, & \text{if } |t| \le 1, \\ +\infty, & \text{if } |t| > 1; \end{cases} \tag{0.2}$$

different choices for ω are discussed in Remark 2.1. In addition, set

$$\varphi(t) := \int_0^t \sqrt{\omega(s)}\, ds, \qquad \forall\, t \in [-1,1], \tag{0.3}$$

$$\delta(t_1,t_2) := \left| \int_{t_1}^{t_2} \sqrt{\omega(t)}\, dt \right| = |\varphi(t_2) - \varphi(t_1)|, \qquad \forall\, t_1, t_2 \in [-1,1], \tag{0.4}$$

$$c_0 := \int_{-1}^{1} \sqrt{\omega(t)}\, dt = \delta(-1,1) = 2\varphi(1) = -2\varphi(-1) = \frac{\pi}{2}. \tag{0.5}$$

Since φ is an (odd) strictly increasing function, the following piecewise constant function $g \in BV(\partial\Omega;[-1,1])$ is well defined

$$g(x) := \varphi^{-1}\left(-\frac{c_0}{2}\mu(x)\right), \qquad \forall\, x \in \partial\Omega. \tag{0.6}$$

Let $\{g_1,\dots,g_m\}$ be the values of g and set $\Gamma_i := \{x \in \partial\Omega :\ g(x) = g_i\}$, for all $1 \le i \le m$. Note that $\mathcal{H}^{n-2}(S_g) < +\infty$. For any $\epsilon > 0$, let $g_\epsilon \in C^{0,1}(\partial\Omega;[-1,1])$ approximate g in the sense that

$$g_\epsilon(x) = g(x), \ \text{ if } \operatorname{dist}(x,S_g) \ge \epsilon, \qquad \operatorname{Lip}(g_\epsilon) \le \frac{L_g}{\epsilon}, \tag{0.7}$$

where L_g is a constant independent of ϵ. It is obvious that $g_\epsilon \to g$ in $L^1(\partial\Omega)$, as $\epsilon \to 0$.

We are now in a position to define the relaxed functionals as

$$\mathcal{F}_\epsilon(v) := \begin{cases} \int_\Omega \left[\epsilon|\nabla v|^2 + \frac{1}{\epsilon}\omega(v) - c_0\kappa v\right] dx, & \text{if } v \in D(\mathcal{F}_\epsilon), \\ +\infty, & \text{if } v \in L^1(\Omega) \setminus D(\mathcal{F}_\epsilon), \end{cases}$$

where $D(\mathcal{F}_\epsilon) := \{v \in H^1(\Omega;[-1,1]) :\ v = g_\epsilon \text{ on } \partial\Omega\}$. The existence of a solution u_ϵ to the minimum problem for $\mathcal{F}_\epsilon$ can be proven by direct methods. We have that $\mathcal{F}_\epsilon \xrightarrow{\Gamma} c_0\mathcal{F}$, as $\epsilon \to 0$ (we refer to [9] for basic issues about Γ-convergence). In fact, it is known [14] that

$$\mathcal{F}_\epsilon \xrightarrow[\epsilon\to 0]{\Gamma} \hat{\mathcal{F}}, \ \text{ where } \hat{\mathcal{F}}(v) := 2c_0\mathcal{H}^{n-1}(S_v) + 2\int_{\partial\Omega} \delta(\operatorname{tr}(v)(x), g(x))\, d\mathcal{H}^{n-1}(x) - c_0\int_\Omega \kappa v\, dx,$$

which, on using (0.4), (0.5), and (0.6), yields the asserted Γ-convergence result because

$$\delta(\mathrm{tr}(v), g) = |\varphi(\mathrm{tr}(v)) - \varphi(g)| = \frac{c_0}{2}|\mathrm{tr}(v) + \mu|, \qquad \forall\, v \in D(\mathcal{F}). \tag{0.8}$$

At this stage, we can introduce a discretization of the relaxed functional $\mathcal{F}_\epsilon$ by piecewise linear finite elements. Let us denote by $\{\mathcal{S}_h\}_{h>0}$ a regular family of partitions of Ω into simplices [6, p. 132]. For the sake of simplicity we shall assume that Ω is a polyhedron, so that $\bar{\Omega} \equiv \Omega_h := \cup_{S\in\mathcal{S}_h} S$, for all $h > 0$. This, as well as the convexity of the domain, is just assumed to avoid technical difficulties. Set $\partial_a\Omega := \{x \in \partial\Omega :\ \partial\Omega \notin C^2 \text{ in } x\}$ and note that $\mathcal{H}^{n-2}(\partial_a\Omega) < +\infty$. Let $\mathbf{V}_h \subset H^1(\Omega)$ indicate the usual piecewise linear finite element space over $\mathcal{S}_h$ and $\Pi_h : C^0(\bar{\Omega}) \to \mathbf{V}_h$ the Lagrange interpolation operator.

Finally, for any $\epsilon > 0$, let $\kappa_\epsilon \in C^{0,1}(\bar{\Omega})$ approximate the *curvature* function κ so that

$$\|\kappa_\epsilon\|_{L^\infty(\Omega)} \le C, \qquad \mathrm{Lip}(\kappa_\epsilon) \le \frac{L_\kappa}{\epsilon}, \qquad \kappa_\epsilon \underset{\epsilon\to 0}{\to} \kappa \text{ in } L^1(\Omega), \tag{0.9}$$

where C and L_κ are constants independent of ϵ.

We are now in a position to introduce the discrete functionals

$$\mathcal{F}_{\epsilon,h}(v) := \begin{cases} \int_\Omega \left[\epsilon|\nabla v|^2 + \frac{1}{\epsilon}\Pi_h\omega(v) - c_0\Pi_h(\kappa_\epsilon v)\right] dx, & \text{if } v \in D(\mathcal{F}_{\epsilon,h}), \\ +\infty, & \text{if } v \in L^1(\Omega) \setminus D(\mathcal{F}_{\epsilon,h}), \end{cases} \tag{0.10}$$

where $D(\mathcal{F}_{\epsilon,h}) := \{v \in \mathbf{V}_h :\ |v| \le 1 \text{ in } \Omega,\ v = \Pi_h g_\epsilon \text{ on } \partial\Omega\}$. The existence of a solution $u_{\epsilon,h}$ to the minimum problem for $\mathcal{F}_{\epsilon,h}$ is trivial. The integrals in (0.10) can be evaluated via the vertex quadrature rule, which is exact for piecewise linear functions. On the other hand, the interpolation operator Π_h in (0.10) will introduce extra difficulties in proving the main Γ-convergence result in our paper, which reads as follows

THEOREM 0.1. *Let $h = o(\epsilon)$. Then the sequence $\{\mathcal{F}_{\epsilon,h}\}_{\epsilon,h}$ Γ-converges to $c_0\mathcal{F}$, as $\epsilon \to 0$. Moreover, any family $\{u_{\epsilon,h}\}_{\epsilon,h}$ of absolute minimizers of $\{\mathcal{F}_{\epsilon,h}\}_{\epsilon,h}$ is relatively compact in $L^1(\Omega)$, and any limit point u minimizes $\mathcal{F}$.*

The proof of this Theorem will be given in full detail in §2 whereas, in §1, we study the functional $\mathcal{F}_\epsilon$ in a one dimensional domain with $\kappa = 0$.

1. ONE DIMENSIONAL MINIMIZERS

Our present purpose is to calculate an absolute minimizer u_ϵ of the functional $\mathcal{F}_\epsilon$ in the one dimensional domain $\Omega := (-l, l)$ with curvature $\kappa = 0$ and boundary conditions $g(-l) = -1$, $g(l) = 1$, namely

$$\mathcal{G}_\epsilon(v) := \int_{-l}^{l} \left[\epsilon v'^2 + \frac{1}{\epsilon}\omega(v)\right] dx, \qquad \text{if } v \in W^{2,\infty}(\Omega) :\ v(-l) = -1,\ v(l) = 1. \tag{1.1}$$

It is easy to check that any absolute minimizer of $\mathcal{G}_\epsilon$ has to be a monotone function. Since the following inequality holds for any nondecreasing function v such that $v(-l) = -1$ and $v(l) = 1$ (use the Young inequality and (0.5)),

$$\mathcal{G}_\epsilon(v) \geq 2\int_{-l}^{l} v'(x)\sqrt{\omega(v(x))}\,dx = 2\int_{-1}^{1}\sqrt{\omega(t)}\,dt = 2c_0 = \pi,$$

then any monotone function u_ϵ satisfying the imposed boundary conditions and equality

$$\epsilon^2 {u_\epsilon}'(x)^2 = \omega(u_\epsilon(x)), \qquad \forall\, x \in \Omega, \tag{1.2}$$

is an absolute minimizer of $\mathcal{G}_\epsilon$. In addition, the minimal energy is

$$\mathcal{G}_\epsilon(u_\epsilon) = 2c_0 = \pi. \tag{1.3}$$

Thus, for all ϵ sufficiently small and a such that $(a - \epsilon\frac{\pi}{2}, a + \epsilon\frac{\pi}{2}) \subset \Omega$, we have

$$u_\epsilon(x) = \begin{cases} -1, & \text{if } x \in [-l, a - \epsilon\frac{\pi}{2}], \\ \sin\left(\frac{x-a}{\epsilon}\right), & \text{if } x \in (a - \epsilon\frac{\pi}{2}, a + \epsilon\frac{\pi}{2}), \\ 1, & \text{if } x \in [a + \epsilon\frac{\pi}{2}, l]. \end{cases}$$

2. Proof of the main result

Our goal in this section is to demonstrate Theorem 0.1 by proving first the Γ-convergence of the functionals, and next the $L^1(\Omega)$-compactness of any sequence of minimizers.

Theorem 2.1. *If $h = o(\epsilon)$, then $\Gamma\text{-}\lim_{\epsilon\to 0}\mathcal{F}_{\epsilon,h} = c_0\mathcal{F}$. More specifically, the two following properties hold:*

(i) For any $v_0 \in L^1(\Omega)$ and any sequence $\{v_{\epsilon,h}\}_{\epsilon,h} \in L^1(\Omega)$ converging to v_0 in $L^1(\Omega)$, we have $c_0\mathcal{F}(v_0) \leq \liminf_{\epsilon\to 0}\mathcal{F}_{\epsilon,h}(v_{\epsilon,h})$.

(ii) For any $v_0 \in L^1(\Omega)$, there exists a sequence $\{v_{\epsilon,h}\}_{\epsilon,h} \in L^1(\Omega)$ converging to v_0 in $L^1(\Omega)$ such that $c_0\mathcal{F}(v_0) \geq \limsup_{\epsilon\to 0}\mathcal{F}_{\epsilon,h}(v_{\epsilon,h})$.

Proof. Let us split the functionals $\mathcal{F}$ and $\mathcal{F}_{\epsilon,h}$, with obvious notation, as follows

$$\mathcal{F}(v) = \left[2\mathcal{H}^{n-1}(S_v) + \int_{\partial\Omega}|\mathrm{tr}(v) + \mu|\,d\mathcal{H}^{n-1}\right] - \int_\Omega kv =: \mathcal{F}^1(v) + \mathcal{F}^2(v), \ \forall\, v \in D(\mathcal{F}),$$

$$\mathcal{F}_{\epsilon,h}(v) = \int_\Omega\left[\epsilon|\nabla v|^2 + \frac{1}{\epsilon}\omega(v)\right] - c_0\int_\Omega \Pi_h(\kappa_\epsilon v) + \int_\Omega\frac{1}{\epsilon}[\Pi_h\omega(v) - \omega(v)]$$
$$=: \mathcal{F}^1_{\epsilon,h}(v) + c_0\mathcal{F}^2_{\epsilon,h}(v) + \mathcal{F}^3_{\epsilon,h}(v), \quad \forall\, v \in D(\mathcal{F}_{\epsilon,h}).$$

PROOF OF *(i)*. Let $v_0 \in L^1(\Omega)$ and $v_{\epsilon,h} \to v_0$ in $L^1(\Omega)$, as $\epsilon \to 0$. We can suppose, possibly taking a subsequence, that $v_{\epsilon,h} \in D(\mathcal{F}_{\epsilon,h})$ and $\lim_{\epsilon\to 0} \mathcal{F}_{\epsilon,h}(v_{\epsilon,h}) < +\infty$, otherwise *(i)* is obvious. Then, by virtue of the properties $|v_{\epsilon,h}| \le 1$ and (0.9), we have that $\{\int_\Omega \Pi_h(\kappa_\epsilon v_{\epsilon,h})dx\}_{\epsilon,h}$ is equibounded with respect to ϵ; hence, both $\{\int_\Omega \frac{1}{\epsilon}\Pi_h\omega(v_{\epsilon,h})dx\}_{\epsilon,h}$ and $\{\int_\Omega \epsilon|\nabla v_{\epsilon,h}|^2 dx\}_{\epsilon,h}$ are equibounded. In particular, the latter property reads as

$$\|\nabla v_{\epsilon,h}\|_{L^2(\Omega)} \le C\epsilon^{-1/2}. \tag{2.1}$$

Now we deal with each functional $\mathcal{F}^1_{\epsilon,h}$, $\mathcal{F}^2_{\epsilon,h}$, and $\mathcal{F}^3_{\epsilon,h}$, separately.

STEP 1: BEHAVIOUR OF $\mathcal{F}^3_{\epsilon,h}$. Using well known properties of the interpolation operator Π_h and (2.1), we get

$$\begin{aligned}|\mathcal{F}^3_{\epsilon,h}(v_{\epsilon,h})| &\le \frac{1}{\epsilon}\sum_{S\in\mathcal{S}_h}\int_S |\Pi_h\omega(v_{\epsilon,h}) - \omega(v_{\epsilon,h})|dx \le C\frac{h^2}{\epsilon}\sum_{S\in\mathcal{S}_h}\|D^2\omega(v_{\epsilon,h})\|_{L^1(S)}\\ &\le C\frac{h^2}{\epsilon}\mathrm{Lip}(\omega')\sum_{S\in\mathcal{S}_h}\|\nabla v_{\epsilon,h}\|^2_{L^2(S)} = C\frac{h^2}{\epsilon}\mathrm{Lip}(\omega')\|\nabla v_{\epsilon,h}\|^2_{L^2(\Omega)} \le C\frac{h^2}{\epsilon^2}.\end{aligned}$$

Thus, enforcing the relation $h = o(\epsilon)$, we obtain

$$\lim_{\epsilon\to 0}\mathcal{F}^3_{\epsilon,h}(v_{\epsilon,h}) = 0. \tag{2.2}$$

STEP 2: BEHAVIOUR OF $\mathcal{F}^1_{\epsilon,h}$. We closely follow [14, Lemma 1]. *Step 1* guarantees that both $\{\mathcal{F}^1_{\epsilon,h}(v_{\epsilon,h})\}_{\epsilon,h}$ and $\{\int_\Omega \frac{1}{\epsilon}\omega(v_{\epsilon,h})dx\}_{\epsilon,h}$ are equibounded with respect to ϵ. In particular, the latter property implies $|v_0| = 1$ a.e. in Ω. Note that *step 2* is actually independent of *step 1* because, at this stage, the equiboundedness of $\{\mathcal{F}^1_{\epsilon,h}(v_{\epsilon,h})\}_{\epsilon,h}$ could be assumed; otherwise the assertion (2.3) is obvious. As we shall see, here both parameters ϵ and h could go to 0 independently. Since Ω is a convex set, for any $x \in \mathbf{R}^n\backslash\Omega$, there exists a unique point $r(x) \in \partial\Omega$ so that $\mathrm{dist}(x, r(x)) = \mathrm{dist}(x, \partial\Omega)$. For any $\sigma > 0$, let $E_\sigma := \{x \in \mathbf{R}^n \setminus \Omega : \mathrm{dist}(x,\partial\Omega) < \sigma\}$ and $S_\sigma := \{x \in E_\sigma : r(x) \in \partial_a\Omega \cap S_g\}$. We extend the functions v_0 and $v_{\epsilon,h}$ on $\Omega_\sigma := \Omega \cup E_\sigma$ as follows

$$\hat{v}_0(x) := \begin{cases} v_0(x), & \text{if } x \in \Omega,\\ g(r(x)), & \text{if } x \in E_\sigma \setminus S_\sigma,\\ g_\epsilon(r(x)), & \text{if } x \in S_\sigma,\end{cases} \qquad \hat{v}_{\epsilon,h}(x) := \begin{cases} v_{\epsilon,h}(x), & \text{if } x \in \Omega,\\ (\Pi_h g_\epsilon)(r(x)), & \text{if } x \in E_\sigma.\end{cases}$$

Note that, for all $x \in S_g$, $g_\epsilon(x)$ can be supposed independent of ϵ; hence $\hat{v}_0$ does not depend on ϵ. Since $\mathcal{H}^{n-2}(S_g \cup \partial_a\Omega) < +\infty$, it is easy to check that $\hat{v}_0 \in L^1(\Omega_\sigma;[-1,1])$ and $\hat{v}_{\epsilon,h} \in H^1(\Omega_\sigma;[-1,1])$. In addition, using (0.7), we have that $\hat{v}_{\epsilon,h} \to \hat{v}_0$ in $L^1(\Omega_\sigma)$, as $\epsilon \to 0$, because $\hat{v}_0(x) = \hat{v}_{\epsilon,h}(x)$, for all $x \in E_\sigma \setminus E_{\sigma,\epsilon}$, where $E_{\sigma,\epsilon} := \{x \in E_\sigma : \mathrm{dist}(r(x), S_g) <$

$\epsilon\} \setminus S_\sigma$, and $\mathcal{H}^n(E_{\sigma,\epsilon}) = O(\epsilon\sigma)$. Now, since $v_{\epsilon,h} \in D(\mathcal{F}_{\epsilon,h})$, using the Young inequality and definition (0.3), we get

$$\frac{1}{2}\mathcal{F}^1_{\epsilon,h}(v_{\epsilon,h}) \geq \int_\Omega |\nabla v_{\epsilon,h}|\sqrt{\omega(v_{\epsilon,h})}\, dx = \int_{\Omega_\sigma} |\nabla\varphi(\hat{v}_{\epsilon,h})|dx - \int_{E_\sigma} |\nabla\varphi(\hat{v}_{\epsilon,h})|dx =: I^2 + II^2.$$

Since $\hat{v}_{\epsilon,h}$ is locally constant on $E_\sigma \setminus E_{\sigma,\epsilon}$, using (0.7) we get $|II^2| \leq C\sigma$. For any $v \in BV(\Omega)$, let $\int_\Omega |Dv|$ denote the total variation of the Radon measure Dv on Ω. Since $\varphi(\hat{v}_{\epsilon,h}) \to \varphi(\hat{v}_0)$ in $L^1(\Omega_\sigma)$, as $\epsilon \to 0$, the equiboundedness of $\{\mathcal{F}^1_{\epsilon,h}(v_{\epsilon,h})\}_{\epsilon,h}$ and the semicontinuity of the total variation [15] give $\varphi(\hat{v}_0) \in BV(\Omega_\sigma)$ (hence $v_0 \in D(\mathcal{F})$) and $\liminf_{\epsilon\to 0} I^2 \geq \int_{\Omega_\sigma} |D\varphi(\hat{v}_0)|$, respectively. Now, (0.5) yields $\varphi(v_0) = \frac{c_0}{2}v_0$; hence, using (0.8) we have

$$\int_{\Omega_\sigma} |D\varphi(\hat{v}_0)| = \frac{c_0}{2}\int_\Omega |Dv_0| + \int_{\Omega_\sigma\setminus\bar{\Omega}} |D\varphi(\hat{v}_0)| + \int_{\partial\Omega} |\varphi(\mathrm{tr}(v_0)) - \varphi(g)|\, d\mathcal{H}^{n-1} \geq \frac{c_0}{2}\mathcal{F}^1(v_0).$$

Thus, letting $\sigma \to 0$, we conclude that

$$c_0\mathcal{F}^1(v_0) \leq \liminf_{\epsilon\to 0} \mathcal{F}^1_{\epsilon,h}(v_{\epsilon,h}). \tag{2.3}$$

STEP 3: BEHAVIOUR OF $\mathcal{F}^2_{\epsilon,h}$. We first split $\mathcal{F}^2_{\epsilon,h}(v_{\epsilon,h}) - \mathcal{F}^2(v_0)$ as follows

$$\mathcal{F}^2_{\epsilon,h}(v_{\epsilon,h}) - \mathcal{F}^2(v_0) = \int_\Omega [\kappa_\epsilon v_{\epsilon,h} - \Pi_h(\kappa_\epsilon v_{\epsilon,h})]dx + \int_\Omega [\kappa v_0 - \kappa_\epsilon v_{\epsilon,h}]dx =: I^3 + II^3,$$

Then, using well known properties of the interpolation operator Π_h, $|v_{\epsilon,h}| \leq 1$, (0.9), and (2.1), we can estimate I^3 as follows

$$|I^3| \leq Ch\left[\|\nabla v_{\epsilon,h}\|_{L^1(\Omega)} + \|\nabla\kappa_\epsilon\|_{L^1(\Omega)}\right] \leq C\frac{h}{\epsilon}.$$

On the other hand, we have $\lim_{\epsilon\to 0} II^3 = 0$. Thus, imposing the relation $h = o(\epsilon)$, we conclude that

$$\lim_{\epsilon\to 0} \mathcal{F}^2_{\epsilon,h}(v_{\epsilon,h}) = \mathcal{F}^2(v_0). \tag{2.4}$$

The assertion *(i)* is thus proven in view of (2.2)-(2.4).

PROOF OF *(ii)*. Let $v_0 \in L^1(\Omega)$. We can suppose $v_0 \in D(\mathcal{F})$, otherwise *(ii)* is obvious. Hence, the approximating sequence $\{v_{\epsilon,h}\}_{\epsilon,h}$ has to satisfy $v_{\epsilon,h} \in D(\mathcal{F}_{\epsilon,h})$. First we construct $v_\epsilon \in D(\mathcal{F}_\epsilon)$, closely following the ideas of [11, Prop. 2]. We need some preparations. Since $v_0 \in D(\mathcal{F}) = BV(\Omega;\{-1,1\})$, it is well known that v_0 is the characteristic function of a set A of finite perimeter in Ω, namely, for a.e. $x \in \Omega$,

$$v_0(x) = \chi_A(x) := \begin{cases} +1, & \forall\, x \in A, \\ -1, & \forall\, x \in \Omega \setminus A. \end{cases} \tag{2.5}$$

Set $\partial_a A := \{x \in \partial A \cap \Omega : \ \partial A \notin C^2 \text{ in } x\}$. Using a diagonal argument ([3, Appendix]), we can suppose that A is a polyhedron; hence $\mathcal{H}^{n-2}(\partial_a A) < +\infty$. Let β be the minimum angle between the faces of $\partial\Omega$ and those of $\partial A \cap \Omega$, and $\eta := \cot\frac{\beta}{2}$. Employing the symbol Q to denote either A or Ω and symbol ∂Q for $\partial A \cap \Omega$ or $\partial\Omega$, let us introduce the following notations

$$d_Q(x) := \begin{cases} \operatorname{dist}(x, \partial Q), & \text{if } x \in Q, \\ -\operatorname{dist}(x, \partial Q), & \text{if } x \in \Omega \setminus Q, \end{cases} \qquad \mathbf{r}_Q(x) := \{y \in \partial Q : \ \operatorname{dist}(y, x) = d_Q(x)\}, \quad \text{a.e. } x \in \Omega,$$

$$Q_t := \{x \in \bar{\Omega} : \ d_Q(x) = t\}, \quad \forall\, t \in \mathbf{R},$$

$$L_\epsilon^A := \left\{x \in \Omega : \ |d_A(x)| \le \frac{\pi}{2}\epsilon\right\}, \quad T_\epsilon^A := \left\{x \in L_\epsilon^A : \ \operatorname{dist}(\mathbf{r}_A(x), \partial_a A) \le \frac{1}{2}\pi\eta\epsilon\right\},$$

$$L_\epsilon^\Omega := \{x \in \Omega : \ d_\Omega(x) \le \pi\epsilon\}, \quad T_\epsilon^\Omega := \{x \in L_\epsilon^\Omega : \ \operatorname{dist}(\mathbf{r}_\Omega(x), \partial_a \Omega) \le \pi\eta\epsilon\},$$

$$L_\epsilon^{\Gamma_i} := \{x \in L_\epsilon^\Omega \setminus T_\epsilon^\Omega : \ r_\Omega(x) \in \Gamma_i\}, \ 1 \le i \le m,$$

$$T_\epsilon^g := \{x \in L_\epsilon^\Omega \setminus T_\epsilon^\Omega : \ \operatorname{dist}(r_\Omega(x), S_g) \le \epsilon\},$$

$$T_\epsilon^c := L_\epsilon^A \cap L_\epsilon^\Omega, \qquad L_\epsilon := L_\epsilon^A \cup L_\epsilon^\Omega, \qquad T_\epsilon := T_\epsilon^g \cup T_\epsilon^\Omega \cup T_\epsilon^c \cup T_\epsilon^A,$$

$$L_{\epsilon,h} := \cup\{S \in \mathcal{S}_h : \ S \cap L_\epsilon \neq \emptyset\}, \qquad T_{\epsilon,h} := \cup\{S \in \mathcal{S}_h : \ S \cap T_\epsilon \neq \emptyset\}.$$

Note that, for all $x \in L_\epsilon^Q \setminus T_\epsilon^Q$, $\mathbf{r}_Q(x)$ is a single valued function, that is $\mathbf{r}_Q(x) =: \{r_Q(x)\}$. In addition, it is easy to check that

$$|\nabla d_Q(x)| = 1, \quad \text{a.e. } x \in \Omega, \qquad \lim_{t\to 0} \mathcal{H}^{n-1}(Q_t) = \mathcal{H}^{n-1}(\partial Q), \tag{2.6}$$

$$\mathcal{H}^n(L_\epsilon),\ \mathcal{H}^n(L_{\epsilon,h}) = O(\epsilon), \qquad \mathcal{H}^n(T_\epsilon),\ \mathcal{H}^n(T_{\epsilon,h}) = O(\epsilon^2). \tag{2.7}$$

Now we introduce the following function $\gamma : \mathbf{R} \to [-1, 1]$

$$\gamma(t) := \begin{cases} -1, & \text{if } t < -\frac{\pi}{2}, \\ \sin t, & \text{if } t \in [-\frac{\pi}{2}, \frac{\pi}{2}], \\ +1, & \text{if } t > \frac{\pi}{2}, \end{cases} \tag{2.8}$$

and define $\gamma_\epsilon(t) := \gamma(\frac{t}{\epsilon})$. This function is a minimizer of the one dimensional functional $\mathcal{G}_\epsilon$ defined in (1.1). Obviously we have

$$|\gamma_\epsilon'(t)| \le \frac{1}{\epsilon}, \qquad |\gamma_\epsilon''(t)| \le \frac{1}{\epsilon^2}, \qquad \text{a.e. } t \in \mathbf{R}. \tag{2.9}$$

Using γ_ϵ we can construct the approximating sequence $\{v_{\epsilon,h}\}_{\epsilon,h}$. First, we define a function v_ϵ on $\Omega \setminus T_\epsilon$ as follows

$$v_\epsilon(x) = \begin{cases} v_0(x), & \text{if } x \in \Omega \setminus L_\epsilon, \\ \gamma_\epsilon(d_A(x)), & \text{if } x \in L_\epsilon^A \setminus T_\epsilon, \\ \gamma_\epsilon\left(\epsilon \arcsin(g_i) + d_\Omega(x)\right), & \text{if } x \in (L_\epsilon^{\Gamma_i} \cap A) \setminus T_\epsilon, \quad 1 \le i \le m, \\ \gamma_\epsilon\left(\epsilon \arcsin(g_i) - d_\Omega(x)\right), & \text{if } x \in (L_\epsilon^{\Gamma_i} \setminus A) \setminus T_\epsilon, \quad 1 \le i \le m. \end{cases} \tag{2.10}$$

Because of the particular shape of each connected component of T_ϵ, using a standard extension theorem, v_ϵ can be extended on the whole Ω as a Lipschitz continuous function (with Lipschitz constant $\frac{C}{\epsilon}$) so that $v_\epsilon \in D(\mathcal{F}_\epsilon)$. In addition, $v_\epsilon \to v_0$ in $L^1(\Omega)$, as $\epsilon \to 0$, because $\int_\Omega |v_\epsilon - v_0|\, dx = \int_{L_\epsilon} |v_\epsilon - v_0|\, dx \le 2\mathcal{H}^n(L_\epsilon) = O(\epsilon)$ (use (2.7)). Then, we define $\{v_{\epsilon,h}\}_{\epsilon,h}$ as follows

$$v_{\epsilon,h} := \Pi_h v_\epsilon \in \mathbb{V}_h.$$

Obviously we have $v_{\epsilon,h} \in D(\mathcal{F}_{\epsilon,h})$ and

$$|\nabla v_{\epsilon,h}| \le |\nabla v_\epsilon| \le \frac{C}{\epsilon}, \qquad \text{a.e. in } \Omega. \tag{2.11}$$

In addition, by virtue of the assumption $h = o(\epsilon)$, we have $v_{\epsilon,h} \to v_0$ in $L^1(\Omega)$, as $\epsilon \to 0$.

STEP 4: BEHAVIOUR OF $\mathcal{F}^1_{\epsilon,h}$. Note that, by virtue of (2.5) and (0.8), $\mathcal{F}^1(v_0)$ can be represented as follows

$$\mathcal{F}^1(v_0) = 2\mathcal{H}^{n-1}(\partial A \cap \Omega) + \frac{2}{c_0}\sum_{i=1}^m \left[\delta(-1,g_i)\mathcal{H}^{n-1}(\Gamma_i \setminus \partial A) + \delta(1,g_i)\mathcal{H}^{n-1}(\Gamma_i \cap \partial A)\right]. \tag{2.12}$$

Since $v_{\epsilon,h} \in D(\mathcal{F}_{\epsilon,h})$, we can split $\mathcal{F}^1_{\epsilon,h}(v_{\epsilon,h})$ as follows

$$\begin{aligned}\mathcal{F}^1_{\epsilon,h}(v_{\epsilon,h}) &= \int_\Omega \left[\epsilon|\nabla v_\epsilon|^2 + \frac{1}{\epsilon}\omega(v_\epsilon)\right] dx \\ &\quad + \int_\Omega \epsilon\left[|\nabla v_{\epsilon,h}|^2 - |\nabla v_\epsilon|^2\right] dx + \int_\Omega \frac{1}{\epsilon}\left[\omega(v_{\epsilon,h}) - \omega(v_\epsilon)\right] dx =: I^4 + II^4 + III^4.\end{aligned}$$

First, using that $v_\epsilon \in D(\mathcal{F}_\epsilon)$, we work to prove the limsup estimate for I^4. This term will be further decomposed. Let Ω be represented as $\Omega = (\Omega \setminus L_\epsilon) \cup (L^A_\epsilon \setminus T_\epsilon) \cup (\cup_{i=1}^m (L^{\Gamma_i}_\epsilon \cap A) \setminus T_\epsilon) \cup (\cup_{i=1}^m (L^{\Gamma_i}_\epsilon \setminus A) \setminus T_\epsilon) \cup T_\epsilon$. We deal with each term separately. Set $[\star] := [\epsilon|\nabla v_\epsilon|^2 + \frac{1}{\epsilon}\omega(v_\epsilon)]$. Since $v_\epsilon = v_0$ on $\Omega \setminus L_\epsilon$ is locally constant and $|v_0| = 1$, we have

$$I^4_0 := \int_{\Omega \setminus L_\epsilon} [\star]\, dx = 0. \tag{2.13}$$

In view of (2.10) and (2.6), the next term can be written in the form

$$I^4_1 := \int_{L^A_\epsilon \setminus T_\epsilon} [\star]\, dx = \int_{L^A_\epsilon \setminus T_\epsilon} \left[\epsilon|\gamma_\epsilon'(d_A(x))|^2 + \frac{1}{\epsilon}\omega(\gamma_\epsilon(d_A(x))\right] |\nabla d_A(x)| dx.$$

Then, using the coarea formula and (1.3), we have

$$\begin{aligned} I^4_1 &= \int_{\mathbf{R}} \left[\epsilon|\gamma_\epsilon'(t)|^2 + \frac{1}{\epsilon}\omega(\gamma_\epsilon(t))\right] \mathcal{H}^{n-1}(A_t \setminus T_\epsilon)\, dt \\ &\le [\mathcal{H}^{n-1}(\partial A \setminus T_\epsilon) + o(1)] \int_{-\epsilon\frac{\pi}{2}}^{\epsilon\frac{\pi}{2}} \left[\epsilon|\gamma_\epsilon'(t)|^2 + \frac{1}{\epsilon}\omega(\gamma_\epsilon(t))\right] dt = 2c_0[\mathcal{H}^{n-1}(\partial A \setminus T_\epsilon) + o(1)].\end{aligned}$$

Hence

$$\limsup_{\epsilon\to 0} I_1^4 \le 2c_0\mathcal{H}^{n-1}(\partial A\cap\Omega). \tag{2.14}$$

Similarly, for $I_{2,i}^4 := \int_{(L_\epsilon^{\Gamma_i}\cap A)\setminus T_\epsilon}[\star]\,dx$ we have

$$I_{2,i}^4 = \int_0^\infty \left[\epsilon|\gamma_\epsilon'(t+\epsilon\arcsin(g_i))|^2 + \frac{1}{\epsilon}\omega(\gamma_\epsilon(t+\epsilon\arcsin(g_i)))\right]\mathcal{H}^{n-1}((\Omega_t\cap L_\epsilon^{\Gamma_i}\cap A)\setminus T_\epsilon)dt$$
$$\le [\mathcal{H}^{n-1}((\Gamma_i\cap\bar{A})\setminus T_\epsilon)+o(1)]\int_{\epsilon\arcsin(g_i)}^{\epsilon\frac{\pi}{2}}\left[\epsilon|\gamma_\epsilon'(t)|^2+\frac{1}{\epsilon}\omega(\gamma_\epsilon(t))\right]dt.$$

With the following change of variables, $\gamma_\epsilon(t)=z$, using (2.8), (0.2), and (0.4), we obtain

$$\int_{\epsilon\arcsin(g_i)}^{\epsilon\frac{\pi}{2}}\left[\epsilon|\gamma_\epsilon'(t)|^2+\frac{1}{\epsilon}\omega(\gamma_\epsilon(t))\right]dt = 2\int_{g_i}^1\sqrt{1-z^2}\,dz = 2\delta(g_i,1),$$

hence

$$\limsup_{\epsilon\to 0} I_{2,i}^4 \le 2\delta(1,g_i)\mathcal{H}^{n-1}(\Gamma_i\cap\partial A). \tag{2.15}$$

Arguing on $I_{3,i}^4 := \int_{(L_\epsilon^{\Gamma_i}\setminus A)\setminus T_\epsilon}[\star]\,dx$ as before, we obtain

$$\limsup_{\epsilon\to 0} I_{3,i}^4 \le 2\delta(-1,g_i)\mathcal{H}^{n-1}(\Gamma_i\setminus\partial A). \tag{2.16}$$

Finally (2.11) and (2.7) easily lead to

$$I_4^4 := \int_{T_\epsilon}[\star]\,dx = O(\epsilon). \tag{2.17}$$

Collecting (2.13)-(2.17) and using (2.12) we thus obtain

$$\limsup_{\epsilon\to 0} I^4 = \limsup_{\epsilon\to 0}(I_0^4+I_1^4+\sum_{i=1}^m(I_{2,i}^4+I_{3,i}^4)+I_4^4) \le c_0\mathcal{F}^1(v_0). \tag{2.18}$$

Our next task is to prove that terms II^4 and III^4 vanish. Since $v_{\epsilon,h}=v_\epsilon$ on $\Omega\setminus L_{\epsilon,h}$, using (2.11) we get $|II^4|\le\int_{L_{\epsilon,h}}\epsilon|\nabla(v_\epsilon+v_{\epsilon,h})|\,|\nabla(v_\epsilon-v_{\epsilon,h})|dx \le C\int_{L_{\epsilon,h}}|\nabla(v_\epsilon-v_{\epsilon,h})|dx$. Using again (2.11), in conjunction with (2.6), (2.7), (2.9), and well known properties of the interpolation operator Π_h, we obtain

$$\begin{aligned}|II^4| &\le \int_{T_{\epsilon,h}}|\nabla(v_\epsilon-v_{\epsilon,h})|\,dx+\int_{L_{\epsilon,h}\setminus T_{\epsilon,h}}|\nabla(v_\epsilon-v_{\epsilon,h})|\,dx\\ &\le \frac{C}{\epsilon}\mathcal{H}^n(T_{\epsilon,h})+Ch\mathcal{H}^n(L_{\epsilon,h}\setminus T_{\epsilon,h})\|D^2v_\epsilon\|_{L^\infty(L_\epsilon\setminus T_\epsilon)} \le C\left[\epsilon+\frac{h}{\epsilon}\right],\end{aligned} \tag{2.19}$$

because $D^2 v_\epsilon = \gamma_\epsilon''(d_A)|\nabla d_A|^2 + \gamma_\epsilon'(d_A) D^2 d_A = \gamma_\epsilon''(d_A)$ on $L_\epsilon^A \setminus T_\epsilon$ and similarly on $L_\epsilon^\Omega \setminus T_\epsilon$. Finally, using (2.11) and (2.7), we get

$$|III^4| = \left| \int_{L_{\epsilon,h}} \frac{1}{\epsilon}[\omega(v_\epsilon) - \omega(v_{\epsilon,h})]dx \right| \leq C\frac{h}{\epsilon}\mathrm{Lip}(\omega)\mathcal{H}^n(L_{\epsilon,h})\|\nabla v_\epsilon\|_{L^\infty(\Omega)} \leq C\frac{h}{\epsilon}. \tag{2.20}$$

Collecting (2.18)-(2.20) and enforcing the relation $h = o(\epsilon)$, we conclude that

$$\limsup_{\epsilon\to 0} \mathcal{F}^1_{\epsilon,h}(v_{\epsilon,h}) \leq c_0 \mathcal{F}^1(v_0). \tag{2.21}$$

Since (2.21) guarantees that (2.1) holds, *step 1* and *step 3* are valid. This leads to the assertion *(ii)* and concludes the proof of the Theorem. □

It is an easier task to obtain the following compactness result which, in turn, achieves the proof of Theorem 0.1, in view of basic properties of Γ-convergence.

THEOREM 2.2. *If $h = o(\epsilon)$, then any family $\{u_{\epsilon,h}\}_{\epsilon,h}$ of absolute minimizers of $\{\mathcal{F}_{\epsilon,h}\}_{\epsilon,h}$ is relatively compact in $L^1(\Omega)$.*

Proof. We follow a standard approach (see, e.g., [11, Prop. 3]). Given a family $\{u_{\epsilon,h}\}_{\epsilon,h}$ of absolute minimizers of $\{\mathcal{F}_{\epsilon,h}\}_{\epsilon,h}$, it is sufficient to prove that $\{\varphi(u_{\epsilon,h})\}_{\epsilon,h}$ is relatively compact in $L^1(\Omega)$, where φ is defined in (0.3). We stick with the notation of Theorem 2.1 and use (0.3) and the Young inequality to get

$$\begin{aligned} 2\int_\Omega |\nabla\varphi(u_{\epsilon,h})|\,dx &= 2\int_\Omega \sqrt{\omega(u_{\epsilon,h})}|\nabla u_{\epsilon,h}|\,dx \leq \int_\Omega \left[\epsilon|\nabla u_{\epsilon,h}|^2 + \frac{1}{\epsilon}\omega(u_{\epsilon,h})\right] dx \\ &= \mathcal{F}_{\epsilon,h}(u_{\epsilon,h}) - c_0\mathcal{F}^2_{\epsilon,h}(u_{\epsilon,h}) - \mathcal{F}^3_{\epsilon,h}(u_{\epsilon,h}). \end{aligned} \tag{2.22}$$

Now, let $u \in D(\mathcal{F})$ be a minimizer of $\mathcal{F}$ and $\{v_{\epsilon,h}\}_{\epsilon,h}$ a sequence converging to u, as of assertion *(ii)* of Theorem 2.1; then $c_0\mathcal{F}(u) \geq \limsup_{\epsilon\to 0} \mathcal{F}_{\epsilon,h}(v_{\epsilon,h}) \geq \limsup_{\epsilon\to 0} \mathcal{F}_{\epsilon,h}(u_{\epsilon,h})$, because $u_{\epsilon,h}$ minimizes $\mathcal{F}_{\epsilon,h}$. In addition, we have $|\mathcal{F}^2_{\epsilon,h}(u_{\epsilon,h})| \leq C$ (use (0.9) and $|u_{\epsilon,h}| \leq 1$), and *step 1* of Theorem 2.1 gives $\lim_{\epsilon\to 0} \mathcal{F}^3_{\epsilon,h}(u_{\epsilon,h}) = 0$ (note that $\{\int_\Omega \epsilon|\nabla u_{\epsilon,h}|^2 dx\}_{\epsilon,h}$ is equibounded with respect to ϵ). Hence, inserting these estimates in (2.22), leads to the equiboundedness of $\{\int_\Omega |\nabla\varphi(u_{\epsilon,h})|dx\}_{\epsilon,h}$, so that $\|\varphi(u_{\epsilon,h})\|_{BV(\Omega)} \leq C$, constant independent on ϵ. Hence the compactness theorem in BV [15] gives the assertion. □

REMARK 2.1. Basically, Theorems 2.1 and 2.2 still hold for any graph ω such that

$$\begin{aligned} &\omega_{|[-1,1]} \text{ is a even positive } C^{1,1} \text{ function with two isolated zeros in } -1, 1, \\ &\omega = +\infty, \text{ in } \mathbf{R} \setminus [-1,1]. \end{aligned} \tag{2.23}$$

In particular, if the following property holds: $\int_{-1}^1 \frac{dt}{\sqrt{\omega(t)}} < +\infty$, as of (0.2), then the (new) absolute minimizer γ_ϵ in (2.8) of the one dimensional functional $\mathcal{G}_\epsilon$ in (1.1) ranges

from -1 to 1 in a finite interval $O(\epsilon)$-wide, whereas it is locally constant outside. Hence, even the proof of Theorems 2.1 and 2.2 remains unchanged. On the contrary, if ω verifies (2.23), but $\int_{-1}^{1} \frac{dt}{\sqrt{\omega(t)}} = +\infty$, then there is unicontinuation, that is the solution of (1.2) cannot assume the values -1 and 1 in $\mathbf{R}$. As an example consider the classical function $\omega(t) := (1-t^2)^2$; then the solutions of (1.2) are $v(x) = \tanh(\frac{x-a}{\epsilon})$. In order to prove the contructive part *(ii)* of Theorem 2.1, we can define the $W^{2,\infty}$ even function $\gamma_\epsilon(t)$ in (2.8) as follows: $\gamma_\epsilon(t) := v(t)$, if $t \in [0, \epsilon|\log\epsilon|]$ (set $a = 0$); $\gamma_\epsilon(t) = 1$, if $t \geq 2\epsilon|\log\epsilon|$; $\gamma_\epsilon \in \mathbf{P}^3(\epsilon|\log\epsilon|, 2\epsilon|\log\epsilon|)$; $\gamma_\epsilon(t) = -\gamma_\epsilon(-t)$ if $t < 0$. It is easy to check that the property (2.9) still holds and that $\mathcal{G}_\epsilon(\gamma_\epsilon) \leq \mathcal{G}_\epsilon(u_\epsilon) + C\epsilon^4$. Since the *transition* interval of γ_ϵ has now size $4\epsilon|\log\epsilon|$, the definition of the sets L_ϵ^A, L_ϵ^Ω, and T_ϵ has to be modified, as well. Assertion *(ii)* of Theorem 2.1 then follows with the fairly stronger assumption $h = o(\epsilon|\log\epsilon|^{-1})$.

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CHAPTER 3, SECTION 3.2

NUMERICAL MINIMIZATION OF GEOMETRICAL TYPE PROBLEMS RELATED TO CALCULUS OF VARIATIONS*

1. Introduction. In this paper we consider the numerical minimization of the following functional:

$$(1.1)\quad \mathcal{G}(\mathbf{v}) := \mathcal{H}(S_{\mathbf{v}}) + \int_{\partial\Omega} \mathbf{m}\cdot \mathrm{tr}(\mathbf{v})\, d\mathcal{H}(x) - \int_{\Omega} \mathbf{k}\cdot \mathbf{v}\, dx, \quad \forall\, \mathbf{v}\in D(\mathcal{G}) := BV(\Omega;\mathcal{E}),$$

where $\Omega \subset \mathbf{R}^2$ is an open bounded set with Lipschitz continuous and piecewise C^2 boundary, $\mathcal{H}$ is the one-dimensional Hausdorff measure, $\mathcal{E} := \{\mathbf{e}_1,...,\mathbf{e}_d\}$ is the canonical basis of $\mathbf{R}^d$ $(d \geq 2)$, $BV(\Omega;\mathcal{E})$ is the space of bounded variation functions on Ω with values in $\mathcal{E}$, $S_{\mathbf{v}}$ and $\mathrm{tr}(\mathbf{v})$ are the *jump* set and the trace on $\partial\Omega$ of the vector valued BV function $\mathbf{v} = (v_1,...,v_d)$ [14,15]. In addition, $\mathbf{m} = (m_1,...,m_d) \in BV(\partial\Omega;\mathbf{R}^d)$ is a piecewise constant vector valued BV function, such that $|m_i - m_j| \leq 1$, a.e. on $\partial\Omega$, for all $1 \leq i,j \leq d$, and $\mathbf{k} = (k_1,...,k_d) \in C^{0,1}(\bar{\Omega};\mathbf{R}^d)$ is a vector valued Lipschitz continuous function. Note that a shift of $\mathbf{m}$ by $\lambda_{\mathbf{m}}\mathbf{e}_0$ (resp. of $\mathbf{k}$ by $\lambda_{\mathbf{k}}\mathbf{e}_0$), where $\mathbf{e}_0 := (1,...,1)$ and $\lambda_{\mathbf{m}} \in L^1(\partial\Omega)$ (resp. $\lambda_{\mathbf{k}} \in L^1(\Omega)$), corresponds to a translation of $\mathcal{G}$ by $\int_{\partial\Omega}\lambda_{\mathbf{m}}$ (resp. $\int_{\Omega}\lambda_{\mathbf{k}}$), so that the minima of the functional are not affected. Hence, we can assume that $\sum_{i=1}^{d} k_i = 0$. Moreover, the Lipschitz continuity of $\mathbf{k}$ is assumed just for the sake of simplicity; if $\mathbf{k} \in L^{\infty}(\Omega;\mathbf{R}^d)$, a regularizing procedure can be used. Finally, note that any function $\mathbf{v} \in D(\mathcal{G})$ can be associated to a partition $\Omega_{\mathbf{v}} := \{\Omega_1^{\mathbf{v}},...,\Omega_d^{\mathbf{v}}\}$ of Ω, where $\Omega_i^{\mathbf{v}} := \{x \in \Omega : \mathbf{v}(x) = \mathbf{e}_i\}$.

The minimum problem for $\mathcal{G}$ is related to various geometrical type problems in calculus of variations [4], such as the minimal partition of a set, the segmentation of images, and the search for sets with prescribed curvature. We stress that the uniqueness of the solution cannot be expected, because of the nonconvexity of the functional $\mathcal{G}$.

a. *Minimal Partitions Problem* (MPP): [1] Given a partition $\{\Gamma_1,...,\Gamma_d\}$ of $\partial\Omega$, find a partition $\Omega_{\min} = \{\Omega_1,...,\Omega_d\}$ of Ω which minimizes

$$\sum_{i=1}^{d}\left[\frac{1}{2}\mathcal{H}(\partial\Omega_i \cap \Omega) + \mathcal{H}(\partial\Omega \cap (\partial\Omega_i \backslash \Gamma_i))\right].$$

We have $\Omega_{\min} = \Omega_{\mathbf{u}}$, where $\mathbf{u}$ is a minimum of the functional $\mathcal{G}$ defined in (1.1) with

$$(1.2)\qquad \mathbf{k} := 0 \qquad \text{and} \qquad m_i(x) := \begin{cases} -1 & \text{if } x \in \Gamma_i, \\ 0 & \text{if } x \in \partial\Omega\backslash\Gamma_i, \end{cases} \qquad \forall\, 1 \leq i \leq d.$$

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b. *Image Segmentation Problem* (ISP): [12] Given an image $r \in L^\infty(\Omega)$ ($r(x)$ is the grey level at $x \in \Omega$), find a cartoon $R \in BV(\Omega; \{r_1, ..., r_d\})$ which minimizes

$$\mathcal{H}(S_R) + \alpha \|R - r\|^2_{L^2(\Omega)},$$

where S_R is the jump set of R, $r_i \in \mathbf{R}$ for all $1 \leq i \leq d$, and $\alpha \in \mathbf{R}^+$. Choosing, in (1.1),

$$\text{(1.3)} \qquad \mathbf{m} := 0 \qquad \text{and} \qquad k_i(x) := \alpha(k_0(x) - |r_i - r(x)|^2), \text{ a.e. } x \in \Omega, \ \forall\, 1 \leq i \leq d,$$

where $k_0(x) := \frac{1}{d} \sum_{i=1}^d |r_i - r(x)|^2$, the locally constant function R can be associated to a minimum $\Omega_{\mathbf{u}}$ of $\mathcal{G}$ as follows: $R(x) = r_i$ if $x \in \Omega_i^{\mathbf{u}}$, for all $1 \leq i \leq d$.

c. *Prescribed Curvature Problem* (PCP): Find a set $A \subseteq \Omega$ whose boundary has prescribed mean curvature κ and contact angle θ at the intersection with $\partial\Omega$. It is well known [8] that A is a minimum of the functional: $P_\Omega(A) + \cos(\theta)\mathcal{H}(\partial A \cap \partial\Omega) - \int_A \kappa dx$. Hence, choosing, in (1.1),

$$\text{(1.4)} \qquad d := 2, \qquad \mathbf{k} := \frac{\kappa}{2}\,\mathbf{p}, \qquad \text{and} \qquad \mathbf{m} := \frac{\cos(\theta)}{2}\,\mathbf{p},$$

where $\mathbf{p} := \mathbf{e}_1 - \mathbf{e}_2$, the set A can be associated to a minimum of $\mathcal{G}$ as follows: $A = \Omega_1^{\mathbf{u}}$.

The numerical minimization of the functional $\mathcal{G}$ is very difficult because of the lack of regularity and convexity of the problem at hand. Then, we first approximate $\mathcal{G}$ with a family $\{\mathcal{G}_\epsilon\}_{\epsilon>0}$ of regular nonconvex functionals defined in $H^1(\Omega; \mathbf{R}^d)$ which, in turn, will be discretized by piecewise linear finite elements with numerical quadratures, thus allowing their actual minimization on a computer; ϵ is the *relaxation* parameter and h is the *meshsize*. The Γ-convergence of both the relaxed and discrete functionals as well as the compactness of any sequence of approximate minimizers has been proved in [1,10,11,13] and [2], respectively.

The relaxed functionals are defined in §2, whereas in §3 we describe the numerical algorithms. In §4 we discuss an adaptive domain and mesh modification technique, which takes advantage of the structure of the problem, since singularities are concentrated near the jump sets. The paper concludes in §5 with various numerical experiments.

2. Relaxed Problems. In order to define the relaxed functionals $\mathcal{G}_\epsilon$ we need to introduce a double well potential and a suitable boundary datum. The following notations will be extensively used through the paper:

$$\text{(2.1)} \qquad \widetilde{\mathcal{P}} := \{\mathbf{s} \in \mathbf{R}^d : \sigma(\mathbf{s}) = 1\}: \quad \text{affine hyperplane generated by } \mathcal{E},$$

$$\text{(2.2)} \qquad \mathcal{P} := \{\mathbf{s} \in \mathbf{R}^d : \sigma(\mathbf{s}) = 0\}: \quad \text{hyperplane through the origin, parallel to } \widetilde{\mathcal{P}},$$

$$\text{(2.3)} \qquad \mathcal{T} := \{\mathbf{s} \in \widetilde{\mathcal{P}} : s_i \geq 0,\ 1 \leq i \leq d\}: \quad \text{convex hull of } \mathcal{E},$$

where

$$\sigma(\mathbf{s}) := \sum_{i=1}^{d} s_i, \qquad \forall\, \mathbf{s} \in \mathbf{R}^d. \tag{2.4}$$

2.1. Double Well Potential. Let $\omega : \mathbf{R}^d \to \mathbf{R}^+$ and $\mathbf{F} := (F_1, ..., F_d) : \mathbf{R}^d \to \mathbf{R}^d$ be defined by

$$\omega(\mathbf{s}) := \prod_{i=1}^{d} |\mathbf{s} - \mathbf{e}_i|^2, \qquad F_i(\mathbf{s}) := \frac{\partial \omega(\mathbf{s})}{\partial s_i}, \qquad \forall\, \mathbf{s} \in \mathbf{R}^d, \tag{2.5}$$

where $|\cdot|$ stands for the euclidean norm of any finite dimensional vector. A different choice for the potential ω is discussed in Remark 2.2. Let us then introduce the following homogeneous riemannian metric in $\mathbf{R}^d$:

$$\delta(\mathbf{s}_1, \mathbf{s}_2) := \inf_{\mathbf{l} \in C^1([a,b];\mathbf{R}^d):\mathbf{l}(a)=\mathbf{s}_1, \mathbf{l}(b)=\mathbf{s}_2} \left(\int_a^b \sqrt{\omega(\mathbf{l}(t))}\ |\mathbf{l}'(t)|\ dt \right), \quad \forall\, \mathbf{s}_1, \mathbf{s}_2 \in \mathbf{R}^d. \tag{2.6}$$

Let $\mathbf{b} := \frac{1}{d}\mathbf{e}_0$ be the barycentre of $\mathcal{T}$. Since ω is invariant with respect to the isometries associated to permutations in $\mathcal{E}$, then $\delta(\mathbf{e}_i, \mathbf{e}_j)$ and $\delta(\mathbf{e}_i, \mathbf{b})$ are independent of i and j, namely,

$$\delta(\mathbf{e}_i, \mathbf{e}_j) =: c_0, \quad \forall\, i \neq j; \qquad \delta(\mathbf{e}_i, \mathbf{b}) =: c_0\mu, \quad \forall\, 1 \leq i \leq d. \tag{2.7}$$

Letting the curves $\mathbf{l}$ in (2.6), connecting $\mathbf{s}_1$ to $\mathbf{s}_2$, be parametrized by arc-length (i.e., $|\mathbf{l}'(t)| = 1$ for all $t \in (a,b)$), it can be shown that the minimizing curve $\mathbf{l}$ does satisfy the following geodesic equation:

$$2\omega\big(\mathbf{l}(t)\big)\ \mathbf{l}''(t) = \mathbf{F}(\mathbf{l}(t)) - \big(\mathbf{F}(\mathbf{l}(t)) \cdot \mathbf{l}'(t)\big)\mathbf{l}'(t), \qquad \forall\, t \in (a,b). \tag{2.8}$$

It turns out that the explicit computation of $\delta(\mathbf{s}_1, \mathbf{s}_2)$, for generic vectors $\mathbf{s}_1, \mathbf{s}_2 \in \mathbf{R}^d$, is rather difficult. Since ω exhibits symmetry properties with respect to $\widetilde{\mathcal{P}}$ (see Lemma 2.1), the relaxed functionals can be formulated on $\widetilde{\mathcal{P}}$ rather than in the whole $\mathbf{R}^d$ (see Remark 2.1). Within this frame, we note that the optimal walk $\mathbf{l}$ connecting any $\mathbf{s}_1, \mathbf{s}_2 \in \widetilde{\mathcal{P}}$ lies in $\widetilde{\mathcal{P}}$, because the absolute minimum of the function ω restricted to any straight line orthogonal to $\widetilde{\mathcal{P}}$ lies in $\widetilde{\mathcal{P}}$. This property is obviously in agreement with (2.8), because its solution belongs to $\widetilde{\mathcal{P}}$, if the initial data $\mathbf{l}(a) \in \widetilde{\mathcal{P}}$, $\mathbf{l}'(a) \in \mathcal{P}$ (use Lemma 2.1). Despite this remark, the integration of (2.8) is trivial only in the case $d = 2$, which includes PCP, whereas it is still difficult for $d \geq 3$. In fact, for $d = 2$, we have that $\widetilde{\mathcal{P}}$ is a straight line, so that the optimal curve in (2.6) is simply the segment $\mathbf{l}(t) := \mathbf{s}_1 + t(\mathbf{s}_2 - \mathbf{s}_1)$ connecting $\mathbf{s}_1, \mathbf{s}_2 \in \widetilde{\mathcal{P}}$; hence

$$\delta(\mathbf{s}_1, \mathbf{s}_2) = |\mathbf{s}_2 - \mathbf{s}_1| \int_0^1 \sqrt{\omega(\mathbf{l}(t))}\ dt, \qquad \forall\, \mathbf{s}_1, \mathbf{s}_2 \in \widetilde{\mathcal{P}}. \tag{2.9}$$

In §3.2 we shall describe a numerical procedure for the actual computation of $\delta(\mathbf{e},\mathbf{s})$, for $\mathbf{e}\in\mathcal{E}$ and $\mathbf{s}\in\widetilde{\mathcal{P}}$, in the general case $d\geq 3$. However, we shall see in §§2.2,2.3 that the computation of the metric δ is not necessary for MPP, because of (1.2) and (2.7).

2.2. Boundary Data. We now introduce a boundary datum $\mathbf{g}$, which, in turn, will be regularized and used for the relaxed functionals. The function $\mathbf{g}\in BV(\partial\Omega;\mathbf{R}^d)$ must satisfy the following compatibility property with respect to the given function $\mathbf{m}\in BV(\partial\Omega;\mathbf{R}^d)$:

$$\text{(2.10)}\qquad \mathbf{g}\in BV(\partial\Omega;\mathcal{T}):\qquad \delta(\mathbf{e}_i,\mathbf{g})=c_0(m_i+\lambda_{\mathbf{m}}),\qquad \text{a.e. on }\partial\Omega,\quad \forall\, 1\leq i\leq d,$$

for some $\lambda_{\mathbf{m}}\in L^1(\partial\Omega)$. For any $\epsilon>0$, $\mathbf{g}$ is then regularized by means of a function $\mathbf{g}_\epsilon\in C^{0,1}(\partial\Omega;\mathcal{T})$ so that

$$\text{(2.11)}\qquad \begin{cases}\lim_{\epsilon\to 0}\mathbf{g}_\epsilon=\mathbf{g}, & \text{in } L^1(\partial\Omega;\mathbf{R}^d),\\ \mathbf{g}_\epsilon(x)=\mathbf{g}(x), & \forall\, x\in\partial\Omega:\ \operatorname{dist}(x,S_{\mathbf{g}})\geq c_1\epsilon;\qquad \operatorname{Lip}(\mathbf{g}_\epsilon)\leq\frac{c_2}{\epsilon},\end{cases}$$

where c_1 and c_2 are positive constants and $S_{\mathbf{g}}$ is the jump set of $\mathbf{g}$.

As a matter of fact, property (2.10) has to be viewed as a restriction on the admissible data $\mathbf{m}$ of the functional $\mathcal{G}$. We stress that the definition of $\mathbf{g}$ and $\lambda_{\mathbf{m}}$ in terms of $\mathbf{m}$ can be given easily for MPP, ISP, and PCP whereas, in general, we must use a numerical procedure (see §3.2).

a. (MPP): Using (1.2) and (2.7), Eq. (2.10) holds for

$$\text{(2.12)}\qquad \mathbf{g}:=-\mathbf{m}=\mathbf{e}_i \text{ on } \Gamma_i,\ \forall\, 1\leq i\leq d,\qquad \text{and}\qquad \lambda_{\mathbf{m}}:=1.$$

b. (ISP): Using (1.3) and (2.7), Eq. (2.10) holds for

$$\text{(2.13)}\qquad \mathbf{g}:=\mathbf{b}\qquad \text{and}\qquad \lambda_{\mathbf{m}}:=\mu.$$

c. (PCP): In the case $d=2$, which includes PCP, (2.9) entails that $\varphi(s):=\delta(\mathbf{e}_1,\mathbf{e}_1-s\mathbf{p}):[0,1]\to[0,c_0]$ is a strictly increasing function and verifies $\delta(\mathbf{e}_2,\mathbf{e}_1-s\mathbf{p})=c_0-\varphi(s)$. Hence, recalling that $|m_1-m_2|\leq 1$, Eq. (2.10) holds if we define

$$\mathbf{g}:=\mathbf{e}_1-\varphi^{-1}\left(c_0\frac{1+m_1-m_2}{2}\right)\mathbf{p}\qquad \text{and}\qquad \lambda_{\mathbf{m}}:=\frac{1-m_1-m_2}{2}.$$

Recalling (1.4), for PCP then we have

$$\text{(2.14)}\qquad \mathbf{g}:=\mathbf{e}_1-\varphi^{-1}\left(c_0\frac{1+\cos(\theta)}{2}\right)\mathbf{p}\qquad \text{and}\qquad \lambda_{\mathbf{m}}:=\frac{1}{2}.$$

2.3. Relaxed Functionals. We are now in a position to define the relaxed functionals:

(2.15)
$$\mathcal{G}_\epsilon(\mathbf{v}) := \int_\Omega \left[\epsilon|\nabla \mathbf{v}|^2 + \frac{1}{\epsilon}\omega(\mathbf{v}) - 2c_0\mathbf{k}\cdot\mathbf{v}\right] dx, \quad \forall\, \mathbf{v}\in D(\mathcal{G}_\epsilon) := \{\mathbf{v}\in H^1(\Omega;\mathbf{R}^d) : \mathbf{v}_{|\partial\Omega} = \mathbf{g}_\epsilon\},$$

where $\mathbf{v}_{|\partial\Omega}$ denotes the trace on $\partial\Omega$ in the sense of Sobolev spaces and

$$|\nabla \mathbf{v}|^2 := \sum_{\substack{n=1,2\\ i=1,\dots,d}} \left|\frac{\partial v_i}{\partial x_n}\right|^2, \qquad x := (x_1, x_2) \in \mathbf{R}^2.$$

The following result holds [1, Theorem 1.1; 10,11,13]: *For all $\epsilon > 0$, the functional $\mathcal{G}_\epsilon$ defined above admits at least a minimizer $\mathbf{u}_\epsilon \in W^{2,p}(\Omega;\mathbf{R}^d)$, $\forall\ p < \infty$. Moreover, each family $\{\mathbf{u}_\epsilon\}_{\epsilon>0}$ of such minimizers has at least one L^1-limit point $\mathbf{u} \in D(\mathcal{G})$, which minimizes the functional $\mathcal{G}$.* More precisely, it has been proved that the sequence $\{\mathcal{G}_\epsilon\}_\epsilon$ Γ-converges to the following functional:

$$\sum_{i\neq j} \delta(\mathbf{e}_i, \mathbf{e}_j)\mathcal{H}(\partial\Omega_i^{\mathbf{v}} \cap \partial\Omega_j^{\mathbf{v}}) + 2\int_{\partial\Omega} \sum_{i=1}^{d} \delta(\mathbf{e}_i, \mathbf{g})\mathrm{tr}(v_i)d\mathcal{H}(x) - 2c_0\int_\Omega \mathbf{k}\cdot\mathbf{v}\, dx,$$

i.e., using (2.7) and (2.10), to $2c_0(\mathcal{G}(\mathbf{v}) + \int_{\partial\Omega}\lambda_{\mathbf{m}})$. We refer to [6] for basic issues about Γ-convergence. In particular, we stress that the convergence of any sequence of relaxed minima to a minimum of $\mathcal{G}$ is a straightforward consequence of the Γ-convergence of the $\mathcal{G}_\epsilon$'s. The uniqueness of the relaxed solution cannot be expected because of the nonconvexity of the functional $\mathcal{G}_\epsilon$.

The Euler-Lagrange equation of $\mathcal{G}_\epsilon$ corresponds to the following system of d elliptic differential equations:

$$\Delta\mathbf{u}_\epsilon = \frac{1}{2\epsilon^2}\,\mathbf{F}(\mathbf{u}_\epsilon) - \frac{c_0}{\epsilon}\mathbf{k}, \quad \text{in } H^{-1}(\Omega;\mathbf{R}^d), \quad \mathbf{u}_\epsilon \in D(\mathcal{G}_\epsilon), \tag{2.16}$$

where $\Delta\mathbf{v} := (\Delta v_1, \dots, \Delta v_d)$, for all $\mathbf{v} \in H^1(\Omega;\mathbf{R}^d)$.

Now we prove some basic symmetry properties of the relaxed solution, under the assumptions $\mathbf{g}_\epsilon \in BV(\partial\Omega;\mathcal{T})$ and $\mathbf{k} \in C^{0,1}(\bar{\Omega};\mathcal{P})$.

LEMMA 2.1. *Let ω and $\mathbf{F}$ be defined in* (2.5). *Then*

$$\sigma(\mathbf{F}(\mathbf{s})) = 0, \qquad \forall\, \mathbf{s} \in \widetilde{\mathcal{P}}, \tag{2.17}$$

$$\sigma(\mathbf{F}(\mathbf{s}))[\sigma(\mathbf{s}) - 1] > 0, \qquad \forall\, \mathbf{s} \in \mathbf{R}^d \setminus \widetilde{\mathcal{P}}, \tag{2.18}$$

$$F_l(\mathbf{s}) \le 2(d-1)^{2(1-d)}s_l, \qquad \forall\, \mathbf{s} \in \widetilde{\mathcal{P}}:\ s_l \le 0. \tag{2.19}$$

In view of (2.1)-(2.4), property (2.17) simply means that the gradient of $\omega(\mathbf{s})$ belongs to $\mathcal{P}$ if $\mathbf{s} \in \widetilde{\mathcal{P}}$, whereas property (2.18) means that the gradient of $\omega(\mathbf{s})$ *moves far from* $\widetilde{\mathcal{P}}$ if $\mathbf{s} \notin \widetilde{\mathcal{P}}$.

Proof. If we split any $\mathbf{s} \in \mathbf{R}^d$ as $\mathbf{s} = \mathbf{s}^0 + \lambda_{\mathbf{s}} \mathbf{e}_0$, where $\mathbf{s}^0 \in \widetilde{\mathcal{P}}$ stands for the orthogonal projection of $\mathbf{s}$ onto $\widetilde{\mathcal{P}}$ and $\lambda_{\mathbf{s}} \in \mathbf{R}$, then we get

$$\sigma(\mathbf{F}(\mathbf{s})) = \mathbf{F}(\mathbf{s}) \cdot \mathbf{e}_0 = \frac{d\omega(\mathbf{s})}{d\mathbf{e}_0} = \left[\frac{d\omega(\mathbf{s}^0 + \lambda \mathbf{e}_0)}{d\lambda}\right]_{|\lambda=\lambda_{\mathbf{s}}}.$$

Now we observe that ω is symmetric with respect to $\widetilde{\mathcal{P}}$, i.e., $\omega(\mathbf{s}^0 + \lambda \mathbf{e}_0) = \omega(\mathbf{s}^0 - \lambda \mathbf{e}_0)$, because we have

$$\omega(\mathbf{s}^0 + \lambda \mathbf{e}_0) = \prod_{i=1}^{d} \left[|\mathbf{s}^0 - \mathbf{e}_i|^2 + \lambda^2 d + 2\lambda(\mathbf{s}^0 - \mathbf{e}_i) \cdot \mathbf{e}_0\right] = \prod_{i=1}^{d} \left[|\mathbf{s}^0 - \mathbf{e}_i|^2 + \lambda^2 d\right],$$

by virtue of the equalities $(\mathbf{s}^0 - \mathbf{e}_i) \cdot \mathbf{e}_0 = \sigma(\mathbf{s}^0) - 1 = 0$. Then

$$\sigma(\mathbf{F}(\mathbf{s})) = \left[2\lambda d \sum_{j=1}^{d} \prod_{i \neq j} (|\mathbf{s}^0 - \mathbf{e}_i|^2 + \lambda^2 d)\right]_{\lambda=\lambda_s},$$

which leads to the first two assertions, since $\lambda_s = 0$ if $\mathbf{s} \in \widetilde{\mathcal{P}}$ and $\lambda_s > 0$ (resp. < 0) if $\sigma(\mathbf{s}) > 1$ (resp. < 1). Similarly, if we decompose any $\mathbf{s} \in \mathbf{R}^d$ as $\mathbf{s} = \sum_{i \neq l} s_i \mathbf{e}_i + s_l \mathbf{e}_l =: \mathbf{s}^l + s_l \mathbf{e}_l$, we obtain

$$F_l(\mathbf{s}) = \left[\frac{d\omega(\mathbf{s}^l + \lambda \mathbf{e}_l)}{d\lambda}\right]_{|\lambda=s_l},$$

and

$$\omega(\mathbf{s}^l + \lambda \mathbf{e}_l) = \prod_{i=1}^{d} (|\mathbf{s}^l - \mathbf{e}_i|^2 + \lambda^2 - 2\lambda \delta_{i,l}),$$

where $\delta_{i,l}$ is the Kronecker's symbol. If $\mathbf{s} \in \widetilde{\mathcal{P}}$ and $s_l \leq 0$, then there exists $r \neq l$ such that $s_r \geq \frac{1 - s_l}{d-1} \geq \frac{1}{d-1}$, so that $\prod_{i \neq r} |\mathbf{s}^l - \mathbf{e}_i|^2 \geq (d-1)^{2(1-d)}$. Hence,

$$F_l(\mathbf{s}) = \sum_{j=1}^{d} \left[2(s_l - \delta_{j,l}) \prod_{i \neq j} \left(|\mathbf{s}^l - \mathbf{e}_i|^2 + s_l^2 - 2 s_l \delta_{i,l}\right)\right] \leq 2 s_l \prod_{i \neq r} |\mathbf{s}^l - \mathbf{e}_i|^2,$$

which leads to the third assertion. □

We introduce the further notation:

$$\mathcal{T}_\epsilon := \{\mathbf{s} \in \widetilde{\mathcal{P}} : s_i \geq -\eta_\epsilon, \ \forall\, 1 \leq i \leq d\}, \tag{2.20}$$

where $\eta_\epsilon := K c_0 (d-1)^{2(d-1)} \epsilon$ and $K := \max_{x \in \bar{\Omega}, 1 \leq i \leq d} |k_i(x)|$, and prove the following theorem.

THEOREM 2.1. *Let* $\mathbf{u}_\epsilon$ *be a solution of* (2.16). *Let* $\mathbf{g}_\epsilon(x) \in \mathcal{T}$ *for all* $x \in \partial\Omega$ *and* $\mathbf{k}(x) \in \mathcal{P}$ *for all* $x \in \Omega$. *Then* $\mathbf{u}_\epsilon(x) \in \mathcal{T}_\epsilon$ *for all* $x \in \Omega$.

Proof. Using the assumptions on $\mathbf{g}_\epsilon$ and $\mathbf{k}$ and definition (2.4), from (2.16) we get

$$\Delta\sigma(\mathbf{u}_\epsilon) = \frac{1}{2\epsilon^2}\sigma(\mathbf{F}(\mathbf{u}_\epsilon)), \qquad \sigma(\mathbf{u}_\epsilon)_{|\partial\Omega} = \sigma(\mathbf{g}_\epsilon) = 1.$$

Suppose by contradiction that there exists $x \in \Omega$ such that $\mathbf{u}_\epsilon(x) \notin \mathcal{T}_\epsilon$. Let us first suppose that $\mathbf{u}_\epsilon(x) \notin \widetilde{\mathcal{P}}$, i.e. $\sigma(\mathbf{u}_\epsilon(x)) \neq 1$. Let $\sigma(\mathbf{u}_\epsilon(x)) > 1$ and let $\sigma(\mathbf{u}_\epsilon)$ attain its maximum in $\bar{x} \in \Omega$, namely $\mathbf{u}_\epsilon(\bar{x})$ lies above $\widetilde{\mathcal{P}}$. By virtue of (2.18), $\sigma(\mathbf{F}(\mathbf{u}_\epsilon(\bar{x}))) > 0$ hence we have $\Delta\sigma(\mathbf{u}_\epsilon(\bar{x})) > 0$ which is in contradiction with the maximum principle for σ. Clearly, the same reasoning gives a contradiction if one supposes $\sigma(\mathbf{u}_\epsilon(x)) < 1$. Hence $\mathbf{u}_\epsilon(x) \in \widetilde{\mathcal{P}} \setminus \mathcal{T}_\epsilon$, i.e., there exists an index l such that $u_{\epsilon,l}(x) < -\eta_\epsilon$. Let $u_{\epsilon,l}$ attain its minimum in $\bar{x} \in \Omega$. Then property (2.19) gives a contradiction relatively to the l-th component of (2.16). □

REMARK 2.1. As observed in §1, the condition $\mathbf{k}(x) \in \mathcal{P}$ for all $x \in \Omega$ can be assumed. Hence, as a consequence of (2.10) and Theorem 2.1, Eq. (2.16) can be stated in $\widetilde{\mathcal{P}}$ rather than in the whole $\mathbf{R}^d$. In fact, looking for solutions of the form $\mathbf{u}_\epsilon = (u_{\epsilon,1}, ..., u_{\epsilon,d-1}, 1 - \sum_{i=1}^{d-1} u_{\epsilon,i})$, the d-th component of (2.16) linearly depends on the first $(d-1)$ ones. In other words, the set $\mathcal{E}$ could be replaced by a set of d mutually equidistant vectors in $\mathbf{R}^{d-1}$. The same remark applies to the discrete problem in §3. In particular, the relaxed PCP then reads as follows:

$$\min_{\{v \in H^1(\Omega): v_{|\partial\Omega} = g\}} \left(\int_\Omega \left\{ 3\left[\epsilon|\nabla v|^2 + \frac{1}{\epsilon} v^2(1-v)^2 \right] - \kappa v \right\} dx \right). \tag{2.21}$$

In fact, we have $c_0 = \frac{\sqrt{2}}{3}$ and $\varphi(s) = c_0(-2s^3 + 3s^2)$, so that $g := 1 - r$, where $0 \leq r \leq 1$ is the solution of the scalar equation $-2r^3 + 3r^2 = \frac{1+\cos(\theta)}{2}$ (note that: $g = 0$ for $\theta = 0$, $g = 0.5$ for $\theta = \frac{\pi}{2}$, and $g = 1$ for $\theta = \pi$). Then, (2.21) follows from (2.15) rescaling ϵ as $\tilde{\epsilon} := \frac{\epsilon}{\sqrt{2}}$. The Euler-Lagrange equation corresponding to (2.21) is

$$\Delta u_\epsilon = \frac{1}{\epsilon^2}\, u_\epsilon(1-u_\epsilon)(1-2u_\epsilon) - \frac{1}{6\epsilon}\kappa, \quad \text{in } H^{-1}(\Omega), \qquad u_{\epsilon|\partial\Omega} = g. \quad \square \tag{2.22}$$

REMARK 2.2. *A Different Choice of* ω. We stress that, for any relaxed solution $\mathbf{u}_\epsilon$ corresponding to the ω defined in (2.5), there exists no $x \in \Omega$ so that $\mathbf{u}_\epsilon(x) \in \mathcal{E}$ (unless $\mathbf{u}_\epsilon \equiv$ constant). This can be seen explicitly in 1-D, for instance, [2, Remark 2.1]. On the other hand, the following choice of ω allows the corresponding relaxed solution $\mathbf{u}_\epsilon$ coincide with some $\mathbf{e}_i$ in open subsets of Ω [2, §2]:

$$\omega(\mathbf{s}) := \omega_0(\mathbf{s}) + J_{\mathcal{T}}(\mathbf{s}), \qquad F_i(\mathbf{s}) := \frac{\partial\omega_0(\mathbf{s})}{\partial s_i}, \qquad \forall\, \mathbf{s} \in \mathbf{R}^d, \tag{2.23}$$

where

$$\omega_0(\mathbf{s}) := \frac{d-1}{d} - \sum_{i=1}^{d} \left(s_i - \frac{1}{d}\right)^2, \qquad \forall\, \mathbf{s} \in \mathbf{R}^d, \tag{2.24}$$

and

$$J_{\mathcal{T}}(\mathbf{s}) = \begin{cases} 0, & \text{if } \mathbf{s} \in \mathcal{T}, \\ +\infty, & \text{if } \mathbf{s} \notin \mathcal{T}. \end{cases} \tag{2.25}$$

Observe that $\omega(\mathbf{e}_i) = 0$ and $\omega(\mathbf{s}) > 0$ for all $\mathbf{s} \in \mathcal{T} \setminus \mathcal{E}$. In addition, ω_0 is symmetric with respect to $\widetilde{\mathcal{P}}$. The term $J_{\mathcal{T}}$ in (2.23) makes the Euler-Lagrange equation (2.16) be replaced by the following variational inequality:

(2.26)

$$\langle \nabla \mathbf{u}_\epsilon, \nabla(\mathbf{v}-\mathbf{u}_\epsilon)\rangle + \frac{1}{2\epsilon^2}\langle \mathbf{F}(\mathbf{u}_\epsilon), \mathbf{v}-\mathbf{u}_\epsilon\rangle - \frac{c_0}{\epsilon}\langle \mathbf{k}, \mathbf{v}-\mathbf{u}_\epsilon\rangle \geq 0, \ \ \forall\, \mathbf{v} \in D(\mathcal{G}_\epsilon;\mathcal{T}), \ \ \mathbf{u}_\epsilon \in D(\mathcal{G}_\epsilon;\mathcal{T}),$$

where $D(\mathcal{G}_\epsilon;\mathcal{T}) := \{\mathbf{v} \in H^1(\Omega;\mathcal{T}) : \mathbf{v}_{|\partial\Omega} = \mathbf{g}_\epsilon\}$ and $\langle\cdot,\cdot\rangle$ stands for the inner product in $L^2(\Omega)$. Now, let us introduce the following notations:

$$\mathring{\mathcal{T}} := \{\mathbf{s} \in \mathcal{T} : \ s_i > 0, \ 1 \leq i \leq d\}, \tag{2.27}$$

$$\mathcal{U}(x) := \{\mathbf{s} \in \mathcal{P} : \ s_i \geq 0 \text{ if } u_{\epsilon,i}(x) = 0\}. \tag{2.28}$$

It is easy to see that, for a.e. $x \in \Omega$, we have

$$\left[-\Delta \mathbf{u}_\epsilon(x) + \frac{1}{2\epsilon^2}\mathbf{F}(\mathbf{u}_\epsilon(x)) - \frac{c_0}{\epsilon}\mathbf{k}(x)\right] \cdot \mathbf{s} \geq 0, \qquad \forall\, \mathbf{s} \in \mathcal{U}(x). \tag{2.29}$$

Let $\mathbf{w} := -\Delta \mathbf{u}_\epsilon + \frac{1}{2\epsilon^2}\mathbf{F}(\mathbf{u}_\epsilon) - \frac{c_0}{\epsilon}\mathbf{k}$. If $\mathbf{u}_\epsilon(x) \in \mathring{\mathcal{T}}$, then $\mathcal{U}(x) = \mathcal{P}$; hence, we can replace $\mathbf{s}$ with $-\mathbf{s}$ in (2.29), obtaining an orthogonality condition for $\mathbf{w}(x)$ with respect to $\mathcal{P}$. On the other hand, if $u_{\epsilon,i}(x) = 0$ for all $i \in I \subseteq \{1,...,d\}$, then $\mathcal{U}(x) = \mathcal{P} \cap \{\mathbf{s} \in \mathbf{R}^d : \ s_i \geq 0, \ \forall\, i \in I\}$; hence we get an orthogonality condition for $\mathbf{w}(x)$ with respect to the linear space $\mathcal{P}_I := \mathcal{P} \cap \{\mathbf{s} \in \mathbf{R}^d : \ s_i = 0, \ \forall\, i \in I\}$. We stress that the assumption $\mathbf{k} \in \mathcal{P}$ and the symmetry of ω_0 with respect to $\widetilde{\mathcal{P}}$ give $\mathbf{w}(x) \in \mathcal{P}$ for all $x \in \Omega$; hence, the orthogonality condition with respect to $\mathcal{P}$ entails $\mathbf{w}(x) = 0$, namely, Eq. (2.16) holds for a.e. $x \in \Omega$ such that $\mathbf{u}_\epsilon(x) \in \mathring{\mathcal{T}}$.

Eventually, note that, arguing as in Remark 2.1, we can write the relaxed PCP in scalar form as follows:

$$\min_{\{v \in H^1(\Omega):\, 0 \leq v \leq 1,\, v_{|\partial\Omega} = g\}} \left(\int_\Omega \left[\epsilon |\nabla v|^2 + \frac{1}{\epsilon}(v - v^2) - c_0 \kappa v \right] dx \right),$$

where $c_0 = \frac{\pi}{4}$ and g is computed using $\varphi(s) = c_0(\frac{1}{2} + \frac{\arcsin(2s-1)+2(2s-1)\sqrt{s-s^2}}{\pi})$. □

3. Discrete Problems. In this section we introduce the finite element approximations of both (2.16) and (2.26), and discuss various convergence results. Whenever no confusion is possible, we will omit the subscript ϵ occurring in the discrete functions.

3.1. Notations. For any $h > 0$, let $\mathcal{S}_h$ denote a partition of Ω into triangles; $\{\mathcal{S}_h\}_h$ is assumed to be weakly acute and regular. The first condition means that for any pair of adjacent triangles the sum of the opposite angles with respect to the common side does not exceed π. Given a triangle $S \in \mathcal{S}_h$, h_S stands for its size and verifies $h_S \leq h$, whereas ρ_S denotes the diameter of the biggest ball contained in S. The second condition above is then equivalent to requiring $\rho_S \geq c h_S$ for all $S \in \mathcal{S}_h$ and $h > 0$, where $0 < c < 1$ is a fixed constant [3, p. 132]. For the sake of simplicity we shall assume that the discrete domain $\Omega_h := \cup_{S \in \mathcal{S}_h} S$ coincides with $\bar{\Omega}$.

Let $\mathcal{L}_h \subset H^1(\Omega)$ indicate the usual piecewise linear finite element space over $\mathcal{S}_h$ and $\Pi_h : C^0(\bar{\Omega}) \to \mathcal{L}_h$ the Lagrange interpolation operator. Let $\{x_j\}_{j=1}^J$ denote the nodes of $\mathcal{S}_h$ and $\{\chi_j\}_{j=1}^J$ the canonical basis of $\mathcal{L}_h$; we assume that $\{x_j\}_{j=1}^{J_0}$ are the internal nodes. From now on we will identify any discrete function $\mathbf{V} := (V_1, ..., V_d) \in (\mathcal{L}_h)^d$ with the vector of its nodal values $\mathbf{V}(x_j)$, $j = 1, ..., J$. Let us introduce the following matrices:

$$
\begin{aligned}
&\mathbf{M} := \{\langle \chi_j, \chi_l \rangle_h\}_{j,l=1,...,J_0} =: \{m_{jl}\}, \quad \text{(diagonalized mass matrix)},\\
&\widehat{\mathbf{A}} := \{\langle \nabla\chi_j, \nabla\chi_l \rangle\}_{\substack{j=1,...,J_0\\ l=1,...,J}} =: \{a_{jl}\}, \quad \text{(stiffness matrix)}, \quad \mathbf{A} := \{a_{jl}\}_{j,l=1,...,J_0},\\
&\mathbf{D} := \text{diagonal of } \mathbf{A}, \qquad \hat{\mathbf{I}} := \{\delta_{jl}\}_{\substack{j=1,...,J_0\\ l=1,...,J}}, \qquad \widehat{\mathbf{B}} := \hat{\mathbf{I}} - \mathbf{D}^{-1}\widehat{\mathbf{A}},\\
&\mathbf{R}\mathbf{R}^T := \text{incomplete Cholesky factorization of } \mathbf{A}.
\end{aligned}
\tag{3.1}
$$

Here $\langle \cdot, \cdot \rangle_h$ is defined by

$$
\langle \varphi, \chi \rangle_h := \int_\Omega \Pi_h(\varphi\chi) dx, \qquad \forall\, \varphi, \chi \in C^0(\bar{\Omega}).
\tag{3.2}
$$

This integral can be easily evaluated element-by-element via the vertex quadrature rule, which is exact for piecewise linear functions [3, p. 182]. Note that

$$
m_{jj} = \frac{1}{3}\,\text{meas}(\text{supp}\,\chi_j), \qquad \sum_{l=1}^{J} a_{jl} = 0, \qquad \forall\, 1 \leq j \leq J_0.
\tag{3.3}
$$

Moreover, it is well known and easy to prove that the weakly acuteness of $\mathcal{S}_h$ entails

$$
a_{jl} \leq 0, \qquad \forall\, j \neq l.
\tag{3.4}
$$

Hence, the following Discrete Maximum Principle holds: *let $\chi \in \mathcal{L}_h$ attain its maximum at the internal node x_j and let $\chi_j \in \mathcal{L}_h$ be the corresponding basis function; then*

$$
\langle \nabla\chi, \nabla\chi_j \rangle \geq 0.
\tag{3.5}
$$

In view of (3.3) and (3.4), we have

$$b_{jj} = 0, \qquad b_{jl} \geq 0, \qquad \sum_{l=1}^{J} b_{jl} = 1, \qquad \forall\, 1 \leq j \leq J_0. \tag{3.6}$$

Hence, setting

$$\chi^\star := \widehat{\mathbf{B}}\,\chi, \qquad \forall\, \chi \in \mathcal{L}_h, \tag{3.7}$$

$\chi^\star(x_j) = \sum_{l=1}^{J} b_{jl}\chi(a_l)$ is a convex combination of the values of χ on the adjacent nodes.

Finally, the following properties are well known and easy to prove:

$$\|\chi\|^2_{L^2(\Omega)} \leq \langle \chi, \chi \rangle_h \leq C\|\chi\|^2_{L^2(\Omega)}, \qquad \forall\, \chi \in \mathcal{L}_h, \tag{3.8}$$

$$|\langle \chi, \varphi \rangle - \langle \chi, \varphi \rangle_h| \leq Ch\|\chi\|_{L^2(\Omega)}\|\nabla\varphi\|_{L^2(\Omega)}, \qquad \forall\, \chi, \varphi \in \mathcal{L}_h, \tag{3.9}$$

$$\|\chi - \chi^\star\|_{L^2(\Omega)} \leq Ch\|\nabla\chi\|_{L^2(\Omega)}, \qquad \forall\, \chi \in \mathcal{L}_h. \tag{3.10}$$

3.2. Approximation of the Boundary Data. Let us briefly describe the numerical evaluation of both the energy constant c_0 in (2.7) and the boundary datum $\mathbf{g}$ in (2.10), in the event that the metric δ is not explicitly evaluable (see §2.1). First, we discuss how to compute $\delta(\mathbf{e}, \mathbf{s})$, for any $\mathbf{e} \in \mathcal{E}$ and $\mathbf{s} \in \widetilde{\mathcal{P}}$. Let us define the following functionals in one space dimension:

$$\mathcal{I}(\mathbf{v}) := 2c_0\,\mathrm{card}(S_{\mathbf{v}}) + 2\delta(\mathbf{e}, \mathrm{tr}(\mathbf{v}(0))) + 2\delta(\mathbf{s}, \mathrm{tr}(\mathbf{v}(1))), \qquad \forall\, \mathbf{v} \in BV(I; \mathcal{E}), \tag{3.11}$$

$$\mathcal{I}_\epsilon(\mathbf{v}) := \int_0^1 \epsilon|\mathbf{v}'|^2 + \frac{1}{\epsilon}\omega(\mathbf{v}), \qquad \forall\, \mathbf{v} \in H^1(I; \widetilde{\mathcal{P}}) : \mathbf{v}(0) = \mathbf{e}, \mathbf{v}(1) = \mathbf{s}, \tag{3.12}$$

where $I := (0,1)$ and $S_{\mathbf{v}} := \{x_1, ..., x_N\}$ is the jump set of $\mathbf{v}$. Let $x_0 := 0$ and $x_{N+1} := 1$; denoting by ψ_n the characteristic function of the interval (x_n, x_{n+1}), $\mathbf{v}$ can be written as $\mathbf{v} = \sum_{n=0}^{N} \mathbf{v}_n\psi_n$, where $\mathbf{v}_n \in \mathcal{E}$; then, using (2.7) and the triangular inequality, we get

$$\mathcal{I}(\mathbf{v}) = 2\left[\delta(\mathbf{e}, \mathbf{v}_0) + \sum_{n=1}^{N-1} \delta(\mathbf{v}_n, \mathbf{v}_{n+1}) + \delta(\mathbf{v}_N, \mathbf{s})\right] \geq 2\delta(\mathbf{e}, \mathbf{s}), \qquad \forall\, \mathbf{v} \in BV(I; \mathcal{E}).$$

Hence, the function $\mathbf{u} \equiv \mathbf{e}$ is an absolute minimum of $\mathcal{I}$, because $\mathcal{I}(\mathbf{u}) = 2\delta(\mathbf{e}, \mathbf{s})$. On the other hand, denoting by $\mathbf{u}_\epsilon$ a minimum of $\mathcal{I}_\epsilon$, using well-known properties of Γ-convergence, we have [1, Theorem 1.1]

$$\lim_{\epsilon \to 0} \mathcal{I}_\epsilon(\mathbf{u}_\epsilon) = \mathcal{I}(\mathbf{u}) = 2\delta(\mathbf{e}, \mathbf{s}).$$

The numerical approximation of both $\mathbf{u}_\epsilon$ and $\mathcal{I}_\epsilon(\mathbf{u}_\epsilon)$ then can be easily carried out using the methods of §3.3 in 1-D. The efficient evaluation of $\delta(\mathbf{e}, \mathbf{s})$ is important for the numerical approximation of the boundary datum $\mathbf{g}$ associated, through the compatibility condition

(2.10), to any $\mathbf{m} \in BV(\partial\Omega;\mathcal{P})$ sufficiently small, which corresponds to a "quasi-normal" contact angle of $S_{\mathbf{u}}$ with $\partial\Omega$. Note that $\mathbf{m}=0$ corresponds to $\mathbf{g}=\mathbf{b}$. Since we can assume $\mathbf{m}(x) \in \mathcal{P}$ for all $x \in \partial\Omega$ (see §1 and (2.10)), the compatibility condition (2.10) can be rephrased as $\Pi(\psi(\mathbf{g}))=\mathbf{m}$ where $\psi(\mathbf{g}) := \frac{1}{c_0}(\delta(\mathbf{g},\mathbf{e}_1),...,\delta(\mathbf{g},\mathbf{e}_d))$ and Π is the orthogonal projection on $\mathcal{P}$. The Jacobian $\mathcal{J}_{\Pi\circ\psi}$ of the map $(\Pi\circ\psi):\mathcal{T}\to\mathcal{P}$ can be actually computed in $\mathbf{b}$, that is,

$$\mathcal{J}_{\Pi\circ\psi}(\mathbf{b}) = -\frac{1}{c_0}\sqrt{\omega(\mathbf{b})\frac{d}{d-1}}\; I_{|\mathcal{P}},$$

where $I_{|\mathcal{P}}$ denotes the identity map on $\mathcal{P}$. Hence, it is quite natural to use $\mathcal{J}_{\Pi\circ\psi}(\mathbf{b})$ as a fixed approximation of the Jacobian of $(\Pi\circ\psi)$ in a Newton-type iterative scheme for the equation $(\Pi\circ\psi)(\mathbf{g})-\mathbf{m}=0$, as follows:

$$\mathbf{g}_n := \begin{cases} \mathbf{b}, & \text{if } n=0,\\ \mathbf{g}_{n-1} - \mathcal{J}_{\Pi\circ\psi}(\mathbf{b})^{-1}\left[(\Pi\circ\psi)(\mathbf{g}_{n-1})-\mathbf{m}\right] =: \Phi(\mathbf{g}_{n-1}), & \text{if } n>0. \end{cases} \tag{3.13}$$

Note that $\Phi(\widetilde{\mathcal{P}})\subset\widetilde{\mathcal{P}}$. Since $\mathcal{J}_\Phi(\mathbf{b})=0$, there exists a constant $\rho>0$ such that $\Phi_{|B_\rho(\mathbf{b})}$ is a contraction with Lipschitz constant $\frac{1}{2}$. Here $B_\rho(\mathbf{b}) := \{\mathbf{s}\in\mathcal{T} : |\mathbf{s}-\mathbf{b}|<\rho\}$. Suppose now that $|\mathbf{m}| < \frac{\rho}{2|\mathcal{J}_{\Pi\circ\psi}(\mathbf{b})^{-1}|}$; then $|\mathbf{g}_1-\mathbf{b}|\le\frac{\rho}{2}$ and $|\mathbf{g}_n-\mathbf{b}|\le 2|\mathbf{g}_1-\mathbf{b}|\le\rho$, which implies that $\mathbf{g}_n\in B_\rho(\mathbf{b})$, for all $n\ge 0$, and hence the convergence of (3.13).

3.3. Numerical Algorithms. Let ω be defined in (2.5). We introduce two different discrete schemes (S1) and (S2), which read as follows:

(S1)-(S2). *Find* $\mathbf{U}=(U_1,...,U_d)\in(\mathcal{L}_h)^d$ *such that* $\mathbf{U}_{|\partial\Omega}=\Pi_h\mathbf{g}_\epsilon$ *and, for all* $i=1,...,d$,

$$\langle\nabla U_i,\nabla\chi\rangle + \frac{1}{2\epsilon^2}\,\langle F_i(\mathbf{Z}),\chi\rangle_h - \frac{c_0}{\epsilon}\langle k_i,\chi\rangle_h = 0, \qquad \forall\,\chi\in\mathcal{L}_h\cap H_0^1(\Omega), \tag{3.14}$$

where either

$$\mathbf{Z}:=\mathbf{U}, \quad \textit{(scheme (S1)), or} \qquad \mathbf{Z}:=\mathbf{U}^\star, \quad \textit{(scheme (S2))}. \tag{3.15}$$

The equation (3.14) can be written in matrix form as

$$\widehat{\mathbf{A}}U_i + \frac{1}{2\epsilon^2}\,\mathbf{M}F_i(\mathbf{Z}) - \frac{c_0}{\epsilon}\mathbf{M}(\Pi_h k_i) = 0, \tag{3.16}$$

where either $\mathbf{Z}:=\mathbf{U}$ or $\mathbf{Z}:=\widehat{\mathbf{B}}\mathbf{U}$, and $\mathbf{U}(x_j)=\mathbf{g}_\epsilon(x_j)$, for $j=J_0+1,...,J$. The nonlinear system (3.16) admits at least a solution (see Theorem 3.2), which can be approximated using the following iterative methods:

Given $\mathbf{U}^0$, *for any* $n\ge 0$, *compute* $\mathbf{U}^{n+1}=(U_1^{n+1},...,U_d^{n+1})$ *as*

$$\mathbf{R}\mathbf{R}^T U_i^{n+1} = (\mathbf{R}\mathbf{R}^T-\widehat{\mathbf{A}})U_i^n - \frac{1}{2\epsilon^2}\mathbf{M}F_i(\mathbf{U}^n) + \frac{c_0}{\epsilon}\mathbf{M}(\Pi_h k_i), \quad \textit{(scheme (S1))}, \tag{3.17}$$

$$U_i^{n+1} = \widehat{\mathbf{B}}U_i^n - \frac{1}{2\epsilon^2}\mathbf{D}^{-1}\mathbf{M}F_i(\widehat{\mathbf{B}}\mathbf{U}^n) + \frac{c_0}{\epsilon}\mathbf{D}^{-1}\mathbf{M}(\Pi_h k_i), \quad \textit{(scheme (S2))}. \tag{3.18}$$

REMARK 3.1. It is easy to see that Eq. (3.14), with $\mathbf{Z} = \mathbf{U}$, is the Euler-Lagrange equation corresponding to the functional

$$G_h(\mathbf{V}) := \int_\Omega \left[\epsilon|\nabla \mathbf{V}|^2 + \frac{1}{\epsilon}\Pi_h\omega(\mathbf{V}) - 2c_0\Pi_h(\mathbf{kV})\right] dx, \qquad \forall\, \mathbf{V} \in D(\mathcal{G}_h), \tag{3.19}$$

where $D(\mathcal{G}_h) := \{\mathbf{V} \in (\mathcal{L}_h)^d : \mathbf{V}_{|\partial\Omega} = \Pi_h \mathbf{g}_\epsilon\}$. On the contrary, there exists no functional $\mathcal{F}: \mathbf{R}^{J_0} \times \mathbf{R}^d \to \mathbf{R}$ for which Eq. (3.14), with $\mathbf{Z} = \mathbf{U}^\star$, is the Euler-Lagrange equation. In fact, set $\mathbf{W} := \mathbf{AU} + \frac{1}{2\epsilon^2}\mathbf{MF(BU)} - \frac{c_0}{\epsilon}\mathbf{M}(\Pi_h\mathbf{k})$ and assume that $W_\alpha = \frac{\partial \mathcal{F}}{\partial U_\alpha}$, where $\alpha := (\alpha_1, \alpha_2)$ and $\alpha_1 \in \{1, ..., J_0\}$, $\alpha_2 \in \{1, ..., d\}$. We have

$$W_\alpha = \sum_{j=1}^{J_0} a_{\alpha_1 j} U_{\alpha_2}(x_j) + \frac{1}{2\epsilon^2} m_{\alpha_1\alpha_1} F_{\alpha_2}\left(\sum_{j=1}^{J_0} b_{\alpha_1 j}\mathbf{U}(x_j)\right) - \frac{c_0}{\epsilon} m_{\alpha_1\alpha_1} k_{\alpha_2}(x_{\alpha_1})$$

and, setting $W_\alpha =: I_\alpha + \frac{1}{2\epsilon^2} II_\alpha - \frac{c_0}{\epsilon} III_\alpha$,

$$\frac{\partial I_\alpha}{\partial U_\beta} = \begin{cases} a_{\alpha_1\beta_1}, & \text{if } \alpha_2 = \beta_2, \\ 0, & \text{if } \alpha_2 \neq \beta_2, \end{cases} \qquad \frac{\partial II_\alpha}{\partial U_\beta} = m_{\alpha_1\alpha_1} b_{\alpha_1\beta_1} \frac{\partial F_{\alpha_2}(U^\star_{\alpha_1})}{\partial s_{\beta_2}}, \qquad \frac{\partial III_\alpha}{\partial U_\beta} = 0;$$

hence, the nonsymmetry of $\mathbf{B}$ makes the necessary condition $\frac{\partial W_\alpha}{\partial U_\beta} = \frac{\partial W_\beta}{\partial U_\alpha}$ fail. □

Now we prove that $\mathcal{T}_\epsilon$ (the convex set defined in (2.20)) is an invariant set for the iterative method (3.18). This is not true for the method (3.17), nor for the simpler scheme

$$U_i^{n+1} := \widehat{\mathbf{B}} U_i^n - \frac{1}{2\epsilon^2}\mathbf{D}^{-1}\mathbf{M}F_i(\mathbf{U}^n) + \frac{c_0}{\epsilon}\mathbf{D}^{-1}\mathbf{M}(\Pi_h k_i). \tag{3.20}$$

The difference between (3.18) and (3.20) consists in the argument of the function F_i.

THEOREM 3.1. *Let $\{\mathbf{U}^n\}_{n\geq 0}$ be a solution of (3.18) and $\mathbf{U}^0(x) \in \mathcal{T}_\epsilon$ for all $x \in \Omega$. Let $\mathbf{g}_\epsilon(x) \in \mathcal{T}$ for all $x \in \partial\Omega$ and $\mathbf{k}(x) \in \mathcal{P}$ for all $x \in \Omega$. Then, for all $\epsilon > 0$, there exists $h_0 = O(\epsilon) > 0$ such that, for all $0 < h \leq h_0$,*

$$\mathbf{U}^n(x) \in \mathcal{T}_\epsilon, \qquad \forall\, x \in \Omega \text{ and } n > 0. \tag{3.21}$$

Proof. First, note that the property $\mathbf{U}^n(x) \in \mathcal{T}_\epsilon$ for all $x \in \Omega$ is equivalent to $\mathbf{U}^n(x_j) \in \mathcal{T}_\epsilon$ for all nodes x_j, because of the convexity of $\mathcal{T}_\epsilon$. We argue by induction. Suppose that $\mathbf{U}^n$ assumes only values in $\mathcal{T}_\epsilon$. From (3.6) and the convexity of $\mathcal{T}_\epsilon$, it follows that $\widehat{\mathbf{B}}\mathbf{U}^n(x) \in \mathcal{T}_\epsilon$ for all $x \in \Omega$; hence property (2.17) guarantees that $\mathbf{U}^{n+1}(x) \in \widetilde{\mathcal{P}}$ for all $x \in \Omega$. In fact, for any node x_j, we have

$$\sigma(\mathbf{U}^{n+1}(x_j)) = \sigma(\widehat{\mathbf{B}}\mathbf{U}^n(x_j)) - \frac{m_{jj}}{2\epsilon^2 a_{jj}}\sigma(\mathbf{F}(\widehat{\mathbf{B}}\mathbf{U}^n(x_j))) + \frac{c_0 m_{jj}}{\epsilon a_{jj}}\sigma(\Pi_h\mathbf{k}(x_j)) = 1.$$

Hence, to prove the theorem, it remains to show that $U_i^{n+1}(x_j) \geq -\eta_\epsilon$, for any $1 \leq i \leq d$ and node x_j. Define $\mathcal{Q}_\epsilon := \{\mathbf{s} \in \mathbf{R}^d : \ -\eta_\epsilon \leq s_i \leq 1 + \eta_\epsilon, \ \forall\, 1 \leq i \leq d\}$ and set

$$M_\epsilon := \max_{\substack{\mathbf{s} \in \mathcal{Q}_\epsilon \\ 1 \leq i,l \leq d}} \left| \frac{\partial F_i}{\partial s_l}(\mathbf{s}) \right|.$$

Let $h_0 > 0$ be the biggest value of h for which $m_{jj} M_\epsilon \leq 2\epsilon^2 a_{jj}$, for all $1 \leq j \leq J_0$. Note that $h_0 = O(\epsilon)$. Let x_j be a generic node. We set $\mathbf{s} := (\widehat{\mathbf{B}}\mathbf{U}^n)(x_j) \in \mathcal{T}_\epsilon$ and, for any $1 \leq i \leq d$, define $\mathbf{s}^i \in \mathbf{R}^d$ by means of the following decomposition of $\mathbf{s}$:

$$\mathbf{s} = \sum_{l \neq i} s_l \mathbf{e}_l + s_i \mathbf{e}_i =: \mathbf{s}^i + s_i \mathbf{e}_i,$$

where $s_l = (\widehat{\mathbf{B}} U_l^n)(x_j)$, for all $1 \leq l \leq d$. Since $s_i^i = 0$, then $\mathbf{s}^i - \eta_\epsilon \mathbf{e}_i \in \mathcal{Q}_\epsilon$; hence, using the mean value theorem, we have $|F_i(\mathbf{s}) - F_i(\mathbf{s}^i - \eta_\epsilon \mathbf{e}_i)| \leq M_\epsilon(s_i + \eta_\epsilon)$ and, in particular,

$$F_i(\mathbf{s}) \leq M_\epsilon(s_i + \eta_\epsilon) + F_i(\mathbf{s}^i - \eta_\epsilon \mathbf{e}_i).$$

Moreover, from (2.19) we have $F_i(\mathbf{s}^i - \eta_\epsilon \mathbf{e}_i) \leq -2(d-1)^{2(1-d)}\eta_\epsilon = -2\epsilon c_0 K$. Hence, since $\mathbf{s} \in \mathcal{T}_\epsilon$, i.e., $s_i + \eta_\epsilon \geq 0$, we get

$$m_{jj} F_i(\mathbf{s}) \leq m_{jj} M_\epsilon(s_i + \eta_\epsilon) + m_{jj} F_i(\mathbf{s}^i - \eta_\epsilon \mathbf{e}_i) \leq 2\epsilon^2 a_{jj}(s_i + \eta_\epsilon) - 2\epsilon c_0 K m_{jj}.$$

Finally, we have

$$U_i^{n+1}(x_j) = s_i - \frac{m_{jj}}{2\epsilon^2 a_{jj}} F_i(\mathbf{s}) + \frac{c_0 m_{jj}}{\epsilon a_{jj}} k_i(x_j) \geq -\eta_\epsilon + \frac{c_0 m_{jj}}{\epsilon a_{jj}}(K + k_i(x_j)) \geq -\eta_\epsilon,$$

which leads to the assertion. □

Finally, we prove existence of solutions for both schemes (S1) and (S2). We stress that uniqueness cannot be expected.

THEOREM 3.2. *Let $\mathbf{g}_\epsilon(x) \in \mathcal{T}$ for all $x \in \partial\Omega$ and $\mathbf{k}(x) \in \mathcal{P}$ for all $x \in \Omega$. Then the discrete schemes* (S1) *and* (S2) *admit at least one solution* $\mathbf{U}$, *so that* $\mathbf{U}(x) \in \mathcal{T}_\epsilon$ *for all* $x \in \Omega$.

Proof. Scheme (S1). Eq. (3.14), with $\mathbf{Z} = \mathbf{U}$, is the Euler-Lagrange equation of the functional $\mathcal{G}_h$ defined in (3.19), which admits at least a minimizer $\mathbf{U}$. To prove that $\mathbf{U}(x) \in \mathcal{T}_\epsilon$ for all $x \in \Omega$, we argue as in Theorem 2.1 and use the Discrete Maximum Principle (see (3.5)).

Scheme (S2). The assertion follows immediately from Theorem 3.1, using the Schauder fixed point theorem. □

REMARK 3.2. *Velocity of Convergence of the Iterative Methods.* Let, just for the sake of simplicity, $d = 2$. The Jacobi iterative scheme (3.20) can be viewed as an explicit Euler time discretization of the evolution equation

$$\frac{\partial u_\epsilon}{\partial t} = \Delta u_\epsilon - \frac{1}{2\epsilon^2} F(u_\epsilon) + \frac{c_0}{\epsilon}\kappa, \tag{3.22}$$

with a time step $\tau = Ch^2$. This equivalence can be easily seen when the mesh $\mathcal{S}_h$ is made of equilateral triangles of side h. Eq. (3.22) describes the *motion by mean curvature* (see [5,7]). Hence, the interface is expected to move with a speed proportional to its mean curvature (diminished by κ) and to converge to the asymptotic configuration in a time $O(1)$. We can then expect to perform $O(h^{-2})$ time steps, i.e., $O(h^{-2})$ Jacobi iterations, to get convergence for (3.20). The same heuristic applies to (3.17) and (3.18). □

REMARK 3.3. Let ω be defined in (2.23), (2.24). The discrete version of the variational inequality (2.26) reads, in matrix form, as follows: for all $1 \le i \le d$,

$$\left[\widehat{\mathbf{A}}U_i + \frac{1}{2\epsilon^2}\,\mathbf{M}F_i(\mathbf{Z}) - \frac{c_0}{\epsilon}\mathbf{M}(\Pi_h k_i)\right]_j \cdot \mathbf{s} \ge 0, \quad \forall\, \mathbf{s} \in \mathcal{U}_j, \quad \mathbf{U}(x_j) \in \mathcal{T}, \quad \forall\, 1 \le j \le J_0, \tag{3.23}$$

where either $\mathbf{Z} := \mathbf{U}$ or $\mathbf{Z} := \mathbf{U}^\star$, $\mathbf{U}(x_j) = \mathbf{g}_\epsilon(x_j)$, for $j = J_0+1, ..., J$, and $\mathcal{U}_j := \{\mathbf{s} \in \mathcal{P} : s_i \ge 0, \text{ if } U_i(x_j) = 0\}$. The solution of (3.23) can be computed via iterative procedures like (3.17) and (3.18), modified by performing a projection on $\mathcal{T}$ after each iteration. More precisely, denoting by $\mathbf{U}^{n+\frac{1}{2}}$, instead of $\mathbf{U}^{n+1}$, the vector generated by either (3.17) or (3.18), then we compute

$$U_i^{n+1} := \Pi_{\mathcal{T}}(U_i^{n+\frac{1}{2}}). \tag{3.24}$$

Here, for any $\mathbf{s} \in \widetilde{\mathcal{P}}$, the projection $\Pi_{\mathcal{T}}(\mathbf{s}) \in \mathcal{T}$ is defined by $|\Pi_{\mathcal{T}}(\mathbf{s}) - \mathbf{s}| \le |\tilde{\mathbf{s}} - \mathbf{s}|$, for all $\tilde{\mathbf{s}} \in \mathcal{T}$, and it can be effectively computed as follows. For any $\mathbf{s} \in \widetilde{\mathcal{P}}$, let $I^+(\mathbf{s}) := \{i : s_i > 0\}$ and $I^-(\mathbf{s}) := \{i : s_i \le 0\}$. Then, set $\mathbf{s}^{(0)} := \mathbf{s}$ and, for $m \ge 0$, let $\mathbf{s}^{(m+1)} \in \widetilde{\mathcal{P}}$ be defined by

$$s_i^{(m+1)} := \begin{cases} 0, & \text{if } i \in I^-(\mathbf{s}^{(m)}), \\ s_i^{(m)} - \alpha_m, & \text{if } i \in I^+(\mathbf{s}^{(m)}), \end{cases}$$

where

$$\alpha_m := -\frac{\sum_{i \in I^-(\mathbf{s}^{(m)})} s_i^{(m)}}{\mathrm{card}(I^+(\mathbf{s}^{(m)}))}.$$

It is easy to check that $\mathbf{s}^{(M)} \in \mathcal{T}$ for some $M \le d$ and that $\Pi_{\mathcal{T}}(\mathbf{s}) = \mathbf{s}^{(M)}$. □

REMARK 3.4. *Minimal Graphs for* MPP. Let Ω be a convex set and $d \ge 3$. Then the jump set S_u of any solution of MPP can be viewed as a *minimal graph* connecting the jump

points of the boundary datum $\mathbf{m}$. Hence, various properties of minimal graphs [9] should be used to improve the accuracy of the discrete variational solution $\mathbf{U}$. For instance, the topological structure of $S_{\mathbf{U}}$ can be inferred from $\mathbf{U}$, thus allowing the formulation of a nonlinear system for the coordinates of the graph nodes lying inside Ω. The coordinates inherited by $\mathbf{U}$ seem to be a good initial guess for the Newton method to be applied for the solution of this nonlinear system. □

3.4. Convergence of Discrete Solutions. Using compactness techniques, we shall prove the convergence of any sequence of discrete solutions $\{\mathbf{U}_{\epsilon,h}\}_h$ to a relaxed solution $\mathbf{u}_\epsilon$, as $h \to 0$, for a fixed $\epsilon > 0$. On the other hand, the family of relaxed functionals $\{\mathcal{G}_\epsilon\}_\epsilon$ Γ-converges to $\mathcal{G}$, as $\epsilon \to 0$ [1,10,11,13]; this leads to the convergence of any sequence of relaxed minima to a minimum of $\mathcal{G}$. Nevertheless, the relation $h = o(\epsilon)$ is required for the Γ-convergence of the discrete family $\{G_{\epsilon,h}\}_{\epsilon,h}$ to $\mathcal{G}$, as $(\epsilon, h) \to 0$ simultaneously (see [2], where the proof is given for the scalar case corresponding to PCP). We stress that, if $h = O(\epsilon)$, then the discrete energies $\{G_{\epsilon,h}(\mathbf{U}_{\epsilon,h})\}_{\epsilon,h}$ remain bounded, but may not converge to the minimal energy $\mathcal{G}(\mathbf{u})$. Instabilities appear if this relation $h = O(\epsilon)$, at least, is not satisfied.

THEOREM 3.3. *Any family of solutions* $\{\mathbf{U}_{\epsilon,h} : \mathbf{U}_{\epsilon,h}(x) \in \mathcal{T}, \ \forall x \in \Omega\}_h$ *of* (3.14) *admits a subsequence converging to a solution* $\mathbf{u}_\epsilon$ *of* (2.16), *as* $\epsilon \to 0$.

The same result holds for any family of solutions of the discrete variational inequality (3.23).

Proof. Since $\epsilon > 0$ is fixed, we will omit the subscript ϵ. C will denote any constant independent of h (possibly dependent on ϵ). Just for the sake of simplicity, let us consider $\mathrm{k} = 0$ in the proof. Moreover we only prove the convergence for solutions of (S2), since the convergence for (S1) then can be obtained as a by-product of the present proof. Since $\mathbf{U}_h(x) \in \mathcal{T}$ for all $x \in \Omega$, it follows that

$$\|\mathbf{F}(\mathbf{U}_h)\|_{L^\infty(\Omega)} \leq C. \tag{3.25}$$

Let $\tilde{\mathbf{g}} \in D(\mathcal{G}_h)$ be defined by: $\langle \nabla \tilde{\mathbf{g}}, \nabla \chi \rangle = 0$, for all $\chi \in \mathcal{L}_h \cap H_0^1(\Omega)$. Taking $\chi = U_{h,i} - \tilde{g}_i \in \mathcal{L}_h \cap H_0^1(\Omega)$ in (3.14), we get

$$0 = \langle \nabla \mathbf{U}_h, \nabla(\mathbf{U}_h - \tilde{\mathbf{g}}) \rangle + \frac{1}{2\epsilon^2} \langle \mathbf{F}(\mathbf{U}_h^\star), \mathbf{U}_h - \tilde{\mathbf{g}} \rangle =: I + II.$$

From the Lipschitz continuity of $\mathbf{g}$, (2.11), and the continuity of the trace operator in H^1, we have $I \geq \|\nabla \mathbf{U}_h\|^2_{L^2(\Omega)} - C$. Since $|II| \leq C$, it follows that

$$\|\nabla \mathbf{U}_h\|_{L^2(\Omega)} \leq C. \tag{3.26}$$

From (3.25) and (3.26), there exist $\mathbf{u} \in H^1(\Omega;\mathbf{R}^d)$ and $\mathbf{f} \in L^\infty(\Omega;\mathbf{R}^d)$ such that, possibly taking a subsequence, as $h \to 0$,

$$\mathbf{U}_h \to \mathbf{u} \qquad \text{weakly in } H^1(\Omega;\mathbf{R}^d) \text{ and a.e. in } \Omega, \tag{3.27}$$

$$\mathbf{F}(\mathbf{U}_h) \to \mathbf{f} \qquad \text{weakly } \star \text{ in } L^\infty(\Omega;\mathbf{R}^d). \tag{3.28}$$

In order to take the limit in (3.14), we write this equation as follows:

$$\begin{aligned} &2\epsilon^2\langle \nabla U_{h,i}, \nabla\chi\rangle + \langle F_i(\mathbf{U}_h),\chi\rangle \\ &= \langle F_i(\mathbf{U}_h) - \Pi_h F_i(\mathbf{U}_h),\chi\rangle + (\langle \Pi_h F_i(\mathbf{U}_h),\chi\rangle - \langle \Pi_h F_i(\mathbf{U}_h),\chi\rangle_h) + \\ &+ (\langle \Pi_h F_i(\mathbf{U}_h),\chi\rangle_h - \langle \Pi_h F_i(\mathbf{U}_h^\star),\chi\rangle_h) = III + IV + V, \qquad \forall\,\chi \in \mathcal{L}_h \cap H_0^1(\Omega). \end{aligned} \tag{3.29}$$

Now, using well-known properties of the interpolation operator Π_h, the Lipschitz continuity of $\mathbf{F}$, and (3.26), we get

$$|III| \le Ch\|\nabla F_i(\mathbf{U}_h)\|_{L^2(\Omega)}\|\chi\|_{L^2(\Omega)} \le Ch.$$

Moreover, since $\Pi_h F_i(\mathbf{U}_h) \in \mathcal{L}_h \cap H_0^1(\Omega)$, (3.9) and (3.25) imply

$$|IV| \le Ch\|\Pi_h F_i(\mathbf{U}_h)\|_{L^2(\Omega)}\|\nabla\chi\|_{L^2(\Omega)} \le Ch,$$

whereas, using (3.8), (3.10), and (3.26), we have

$$|V| \le \|\mathbf{U}_h - \mathbf{U}_h^\star\|_{L^2(\Omega)}\|\chi\|_{L^2(\Omega)} \le Ch\|\nabla\mathbf{U}_h\|_{L^2(\Omega)}\|\chi\|_{L^2(\Omega)} \le Ch.$$

Then, taking the limit as $h \to 0$ in (3.29), from (3.27) and (3.28) we finally get $\langle \nabla u_i, \nabla\chi\rangle + \frac{1}{2\epsilon^2}\langle f_i(\mathbf{u}),\chi\rangle = 0$. Hence, the density of $(\bigcup_{h>0}\mathcal{L}_h) \cap H_0^1(\Omega)$ in $H_0^1(\Omega)$ gives

$$\langle \nabla\mathbf{u}, \nabla\varphi\rangle + \frac{1}{2\epsilon^2}\langle \mathbf{f},\varphi\rangle = 0, \qquad \forall\,\varphi \in H_0^1(\Omega). \tag{3.30}$$

From (3.27) and the continuity of $\mathbf{F}$, one has $\mathbf{F}(\mathbf{U}_h) \to \mathbf{F}(\mathbf{u})$ a.e. in Ω, so that, using the Lebesgue dominated convergence theorem (see (3.25)), it follows that

$$\mathbf{F}(\mathbf{U}_h) \to \mathbf{F}(\mathbf{u}) \qquad \text{in } L^1(\Omega),$$

which, in view of (3.28), implies $\mathbf{f} = \mathbf{F}(\mathbf{u})$. This concludes the proof. □

4. Cutting Strategy. Any solution $\mathbf{U}_{\epsilon,h}$ of either Eq. (3.16) or Eq. (3.23) qualitatively reproduces the behavior of the minima $\mathbf{u}$ of the functional $\mathcal{G}$, that is $\mathbf{U}_{\epsilon,h}$ is essentially constant, equal to the $\mathbf{e}_i$'s, in large regions, corresponding to the Ω_i's, whereas it exhibits transition layers $O(\epsilon)$-wide, corresponding to the jump sets $S_{\mathbf{u}}$. Therefore, the use of highly graded nonuniform meshes, locally refined within the jump layers, should

improve considerably the accuracy of our numerical approximation. Nevertheless, the stiffness matrix $\mathbf{A}$ corresponding to such nonuniform meshes is very ill conditioned. We overcome this inconvenient using the following *cutting strategy*. First, we compute the discrete solution $\mathbf{U}_{\epsilon,h}$ with rough values of ϵ and h. Given any threshold $\gamma > 0$, for all $1 \le i \le d$, let Ω_i^γ denote the (possibly empty) closed subset of Ω where $\mathbf{U}_{\epsilon,h} \approx \mathbf{e}_i$, namely,

$$\Omega_i^\gamma := \{x \in \Omega : \ |\mathbf{U}_{\epsilon,h}(x) - \mathbf{e}_i| \le \gamma\}. \tag{4.1}$$

We define a smaller domain

$$\Omega' := \Omega \setminus \left(\bigcup_{i=1}^{d} \Omega_i^\gamma \right), \tag{4.2}$$

and new boundary data on $\partial\Omega'$ as follows:

$$\mathbf{g}'(x) := \begin{cases} \mathbf{g}(x), & \text{if } x \in \partial\Omega' \cap \partial\Omega, \\ \mathbf{e}_i, & \text{if } x \in \partial\Omega' \cap \Omega_i^\gamma, \ \forall\, 1 \le i \le d. \end{cases} \tag{4.3}$$

Next, using Ω' and $\mathbf{g}'$, we compute the discrete solution $\mathbf{U}'_{\epsilon',h'}$, with $\epsilon' < \epsilon$ and $h' < h$; taking a new threshold $\gamma' < \gamma$, we repeat the same cutting process until ϵ meets a certain tolerance.

To justify this procedure, first we must stress that the absolute minimizers of the functional $\mathcal{G}$ do not depend on the domain, in the following sense: *Let* $\mathbf{u}$ *be an absolute minimum of* $\mathcal{G}_\Omega := \mathcal{G}$ *and* $\Omega_{\mathbf{u}} = \{\Omega_1^{\mathbf{u}}, ..., \Omega_d^{\mathbf{u}}\}$ *be the associated partition of* Ω. *Let* $\tilde{\Omega} \subset \Omega$ *satisfy* $S_{\mathbf{u}} \subset \tilde{\Omega}$ *and set*

$$\tilde{\mathbf{m}}(x) := \begin{cases} \mathbf{m}(x), & \text{if } x \in \partial\tilde{\Omega} \cap \partial\Omega, \\ -\mathbf{e}_i, & \text{if } x \in \partial\tilde{\Omega} \cap \Omega_i^{\mathbf{u}}, \ \forall\, 1 \le i \le d. \end{cases}$$

Then $\tilde{\mathbf{u}} := \mathbf{u}_{|\tilde{\Omega}}$ *is an absolute minimum of the functional*

$$\tilde{\mathcal{G}}(\tilde{\mathbf{v}}) := \mathcal{G}_{\tilde{\Omega}}(\tilde{\mathbf{v}}) := \mathcal{H}(S_{\tilde{\mathbf{v}}}) + \int_{\partial\tilde{\Omega}} \tilde{\mathbf{m}} \cdot \mathrm{tr}(\tilde{\mathbf{v}}) d\mathcal{H}(x) - \int_{\tilde{\Omega}} \mathbf{k} \cdot \tilde{\mathbf{v}} \ dx, \qquad \forall\, \tilde{\mathbf{v}} \in D(\tilde{\mathcal{G}}) := BV(\tilde{\Omega}; \mathcal{E}).$$

Let us prove this property. For all $\tilde{\mathbf{v}} \in D(\tilde{\mathcal{G}})$, let $\mathbf{v} \in D(\mathcal{G})$ be defined by

$$\mathbf{v}(x) := \begin{cases} \tilde{\mathbf{v}}(x), & \text{if } x \in \tilde{\Omega}, \\ \mathbf{u}(x), & \text{if } x \in \Omega \setminus \tilde{\Omega}. \end{cases}$$

Then, it easy to check that

$$\tilde{\mathcal{G}}(\tilde{\mathbf{v}}) = \mathcal{G}(\mathbf{v}) - C(\tilde{\Omega}, \mathbf{u}), \qquad \forall\, \tilde{\mathbf{v}} \in D(\tilde{\mathcal{G}}),$$

where $C(\tilde{\Omega}, \mathbf{u}) := \sum_{i=1}^{d} [\int_{(\partial\Omega \setminus \partial\tilde{\Omega}) \cap \partial\Omega_i^{\mathbf{u}}} m_i + \mathcal{H}(\partial\tilde{\Omega} \cap \Omega_i^{\mathbf{u}}) - \int_{\Omega_i^{\mathbf{u}} \setminus \tilde{\Omega}} k_i]$ is a constant independent of $\tilde{\mathbf{v}}$. Hence, we obtain that

$$\tilde{\mathcal{G}}(\tilde{\mathbf{u}}) = \mathcal{G}(\mathbf{u}) - C(\tilde{\Omega}, \mathbf{u}) \leq \mathcal{G}(\mathbf{v}) - C(\tilde{\Omega}, \mathbf{u}) = \tilde{\mathcal{G}}(\tilde{\mathbf{v}}), \qquad \forall\, \tilde{\mathbf{v}} \in D(\tilde{\mathcal{G}}). \quad \square$$

The domain independence of the minima does no longer hold for both the relaxed and discrete functionals. Hence, any choice of $\gamma > 0$ in the definition (4.2) of the new domain Ω' pollutes the solution of the discrete problem. The numerical experiments show that this pollution becomes more evident when the set Ω' is just a narrow strip around the transition region of the solution, which corresponds to the jump set. Nevertheless, recalling that the function ω defined in (2.23) allows the solution of both the relaxed and discrete problems to be constant in open subsets of Ω, we can choose $\gamma = 0$ in (4.1). This avoids pollution of the discrete solution in performing our cutting strategy. The main computational advantages of the choice (2.23) of ω is indeed the role played in the cutting strategy.

5. Numerical Experiments. We present various examples for the problems MPP, ISP, and PCP, introduced in §1. The numerical experiments were carried out on a VAX 8550 computer using a Tektronix terminal for graphic representations. In all our tests, we relax the functional $\mathcal{G}$ in (1.1) using the potential function ω defined in (2.23) and (2.24), and approximate the relaxed solution by algorithm (S2) using the Jacobi iterative scheme (3.18). Similar results can be obtained using either the potential ω defined in (2.5) or the scheme (S1) and (3.17). We shall present various pictures, where, whenever the exact solution is known, the dashed lines represent the jump set; the bold lines always represent the computed solutions. More precisely, denoting by $\mathbf{U}$ any discrete solution, we define

$$\Omega_i^{\mathbf{U}} := \{x \in \Omega : |\mathbf{U} - \mathbf{e}_i| \leq |\mathbf{U} - \mathbf{e}_j|,\ \forall\, 1 \leq j \leq d\};$$

the boundary of the partition $\{\Omega_1^{\mathbf{U}}, ..., \Omega_d^{\mathbf{U}}\}$ of Ω then represents the computed jump set. Finally, we use the following notations: $\mathbf{S} :=$ card$\mathcal{S}_h$, $\mathbf{J} :=$ number of nodes, $\mathbf{I} :=$ number of iterations, $\mathbf{t} :=$ CPU time in seconds.

5.1. Minimal Partitions Problem (MPP).

Example 5.1.1. Let $\Omega := (0,4) \times (0,4)$ and $\{\Gamma_1, \Gamma_2\}$ be the partition of $\partial\Omega$ defined by

$$\Gamma_1 := ([2,4] \times \{0\}) \cup (\{4\} \times [0,2]) \cup (\{0\} \times [2,4]) \cup ([0,2] \times \{4\}), \qquad \Gamma_2 := \partial\Omega \setminus \Gamma_1.$$

There exist two minimal partitions $\Omega_{\mathbf{u}}$ and $\Omega_{\mathbf{w}}$ extending $\{\Gamma_1, \Gamma_2\}$ to Ω:

$$\Omega_1^{\mathbf{u}} := \{(x_1, x_2) \in \Omega : x_2 - 2 \leq x_1 \leq x_2 + 2\}, \qquad \Omega_2^{\mathbf{u}} := \Omega \setminus \Omega_1^{\mathbf{u}},$$
$$\Omega_1^{\mathbf{w}} := \{(x_1, x_2) \in \Omega : 2 - x_1 \leq x_2 \leq 6 - x_1\}, \qquad \Omega_2^{\mathbf{w}} := \Omega \setminus \Omega_1^{\mathbf{w}}.$$

In view of (1.2) and (2.12), $\Omega_{\mathbf{u}}$ and $\Omega_{\mathbf{w}}$ correspond to minima of the functional $\mathcal{G}$ defined in (1.1) with $d = 2$, $\mathbf{k} := 0$, and boundary datum $\mathbf{g}(x) := -\mathbf{m}(x) := \mathbf{e}_i$ if $x \in \Gamma_i$, $i = 1, 2$.

Fig. 5.1.a and Fig. 5.1.b show the exact minimal partitions Ω_u and Ω_w together with the computed ones, produced on a quasi-uniform mesh; here $h = 0.1$ ($S = 2634$, $J = 1398$) and $\epsilon = 0.17$. The initial guesses are $U^0 := e_1$ ($I = 987$, $t = 237$) and $U^0 := e_2$ ($I = 1004$, $t = 247$) for Fig. 5.1.a and Fig. 5.1.b, respectively.

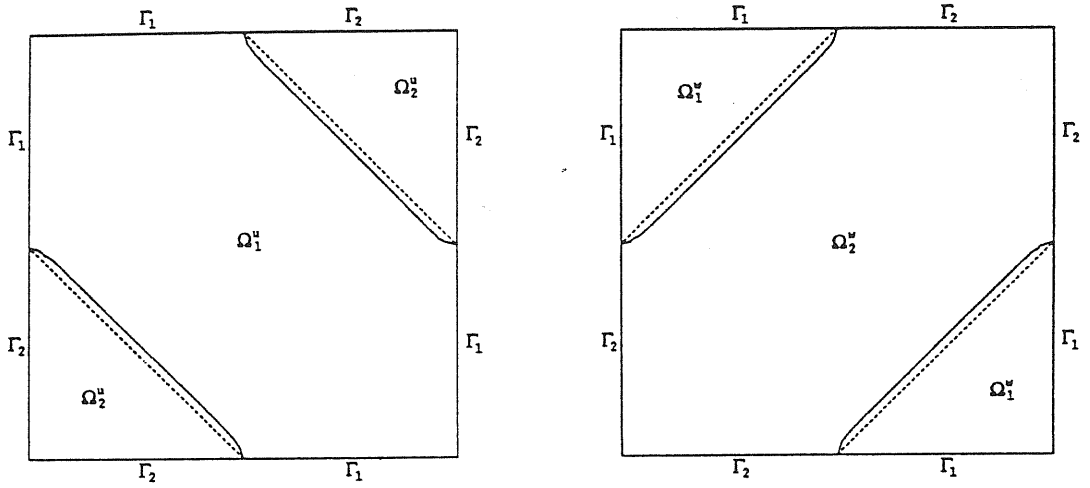

FIGURE 5.1. Ex. 5.1.1 (MPP): Fixed quasi-uniform mesh ($h = 0.1$, $\epsilon = 0.17$).

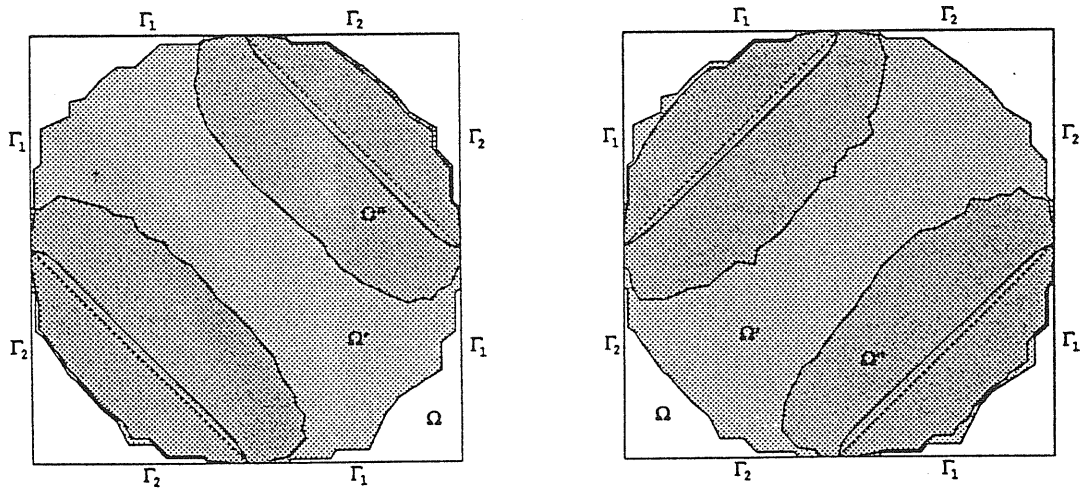

FIGURE 5.2. Ex. 5.1.1 (MPP): Cutting strategy ($h = 0.4, 0.2, 0.1$; $h = 0.7\epsilon^{1.1}$).

The cutting strategy has been used to produce Fig. 5.2.a and Fig. 5.2.b, which show the sequence of domains $\{\Omega, \Omega', \Omega''\}$ and the final computed minimal partitions. With obvious notations, we have, for Fig. 5.2.a: $h = 0.4$, $S = 162$, $J = 102$, $I = 301$, $t = 3.4$; $h' = 0.2$, $S' = 565$, $J' = 322$, $I' = 191$, $t' = 9.2$; $h'' = 0.1$, $S'' = 1255$, $J'' = 716$, $I'' = 831$, $t'' = 82.9$, and $h = 0.7\epsilon^{1.1}$. Similar data hold for Fig. 5.2.b. It is to be observed that the cutting strategy reaches the same accuracy in the approximation of the jump set using much less CPU time than the fixed mesh strategy.

Example 5.1.2. Let Ω be the circle of centre $(2,2)$ and radius 2, and $\{\Gamma_1,\Gamma_2,\Gamma_3,\Gamma_4\}$ be the arcs in the four quadrants. We have two minimal partitions $\Omega_{\mathbf{u}}$ and $\Omega_{\mathbf{w}}$ extending $\{\Gamma_1,\Gamma_2,\Gamma_3,\Gamma_4\}$ to Ω. The computed and exact jump sets are compared in Fig. 5.3.a and Fig. 5.3.b. Here we used a fixed quasi-uniform mesh with $h = 0.15$ ($\mathbf{S} = 920$, $\mathbf{J} = 503$) and $\epsilon = 0.25$. The initial guesses are $\mathbf{U}^0 := \mathbf{e}_2$ ($\mathbf{I} = 1472$, $\mathbf{t} = 246$) and $\mathbf{U}^0 := \mathbf{e}_1$ ($\mathbf{I} = 1665$, $\mathbf{t} = 283$), respectively. Observe that the jump set $S_{\mathbf{u}}$, for instance, is the minimal graph connecting the four points $(2,0),(4,2),(2,4),(0,2)$ and exhibits two Steiner's points [11].

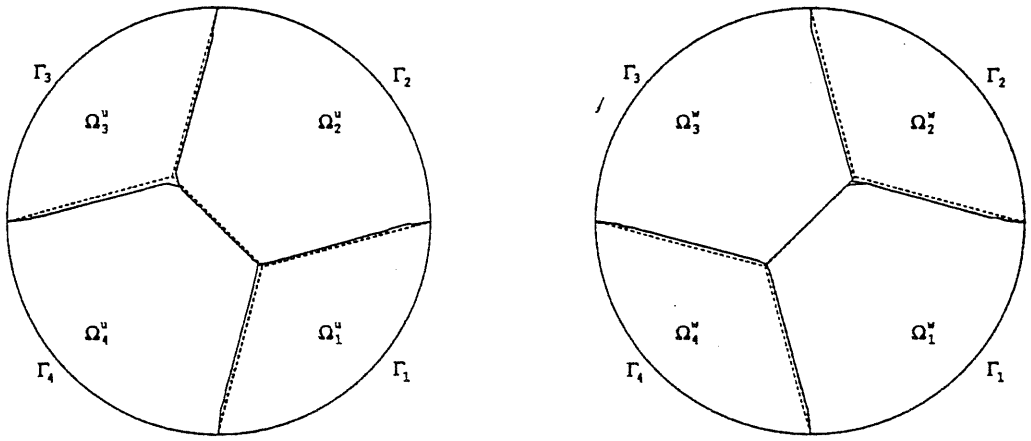

FIGURE 5.3. Ex. 5.1.2 (MPP): Fixed quasi-uniform mesh ($h = 0.15$, $\epsilon = 0.25$).

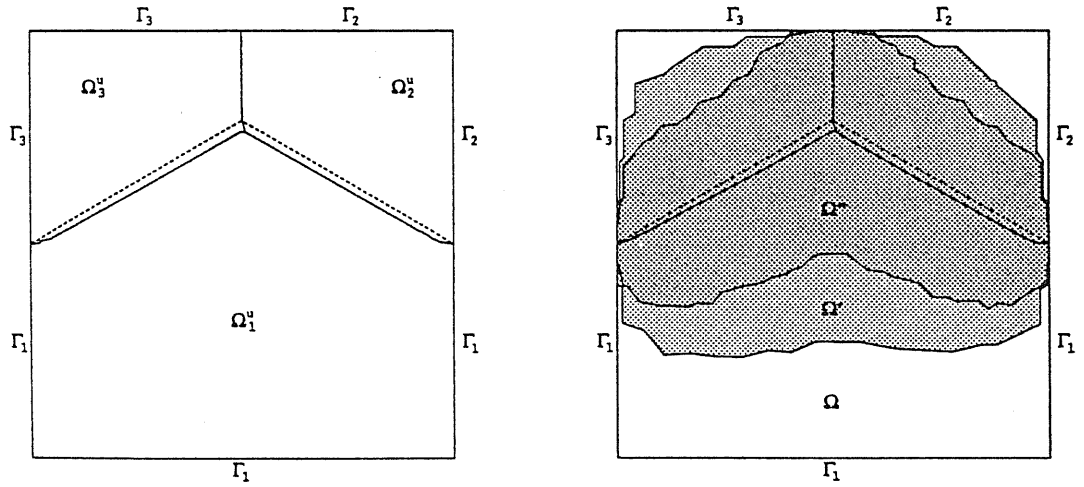

FIGURE 5.4. Ex. 5.1.3 (MPP): a. Fixed quasi-uniform mesh ($h = 0.1$, $\epsilon = 0.19$);
b. Cutting strategy ($h = 0.4, 0.2, 0.1$; $h = 0.6\epsilon^{1.1}$).

Example 5.1.3. Let $\Omega := (0,4)\times(0,4)$ and $\Gamma_1 := (\{0\}\times[0,2])\cup([0,4]\times\{0\})\cup(\{4\}\times[0,2])$, $\Gamma_2 := (\{4\}\times[2,4])\cup([2,4]\times\{4\})$, $\Gamma_3 := \partial\Omega\setminus(\Gamma_1\cup\Gamma_2)$. The exact minimal partition is shown in Fig. 5.4.a together with the solution computed using a fixed quasi-uniform mesh with $h = 0.1$ ($\mathbf{S} = 2634$, $\mathbf{J} = 1398$) and $\epsilon = 0.19$. The initial guess is $\mathbf{U}^0 := \mathbf{b}$ ($\mathbf{I} = 1910$, $\mathbf{t} = 715$). The cutting strategy is used to produce Fig. 5.4.b with $h = 0.4$,

$\mathbf{S} = 162$, $\mathbf{J} = 102$, $\mathbf{I} = 124$, $\mathbf{t} = 2.3$; $h' = 0.2$, $\mathbf{S}' = 450$, $\mathbf{J}' = 259$, $\mathbf{I}' = 430$, $\mathbf{t}' = 24$; $h'' = 0.1$, $\mathbf{S}'' = 1233$, $\mathbf{J}'' = 683$, $\mathbf{I}'' = 1034$, $\mathbf{t}'' = 178$, and $h = 0.6\epsilon^{1.1}$.

5.2. Images Segmentation Problem (ISP).

Example 5.2.1. Let $\Omega := (0,4) \times (0,4)$. The image

$$r(x_1, x_2) := \begin{cases} \frac{x_1}{4} & \text{if } |x - (2,2)| > 1, \\ 0 & \text{if } |x - (2,2)| \leq 1, \end{cases}$$

is approximated with a cartoon R taking values in $\{r_1, r_2, r_3\} := \{0, 0.5, 1\}$. The function R corresponds to a minimum of the functional $\mathcal{G}$ with $d = 3$, boundary data $\mathbf{m} := 0$ (that is $\mathbf{g} := \mathbf{b}$) and $\mathbf{k}$ defined in (1.3). From the initial guess $\mathbf{U}^0 := \mathbf{b}$ and using two different values $c_0\alpha := 25$ and $c_0\alpha := 15$, we obtain the cartoons represented in Fig. 5.5.a and Fig. 5.5.b, respectively. The dotted lines show the circle $|x - (2,2)| = 1$. Here a fixed quasi-uniform mesh with $h = 0.1$ ($\mathbf{S} = 2634$, $\mathbf{J} = 1398$) and $\epsilon = 0.17$ have been used. In addition, we have $\mathbf{I} = 713$, $\mathbf{t} = 319$ and $\mathbf{I} = 643$, $\mathbf{t} = 278$, respectively.

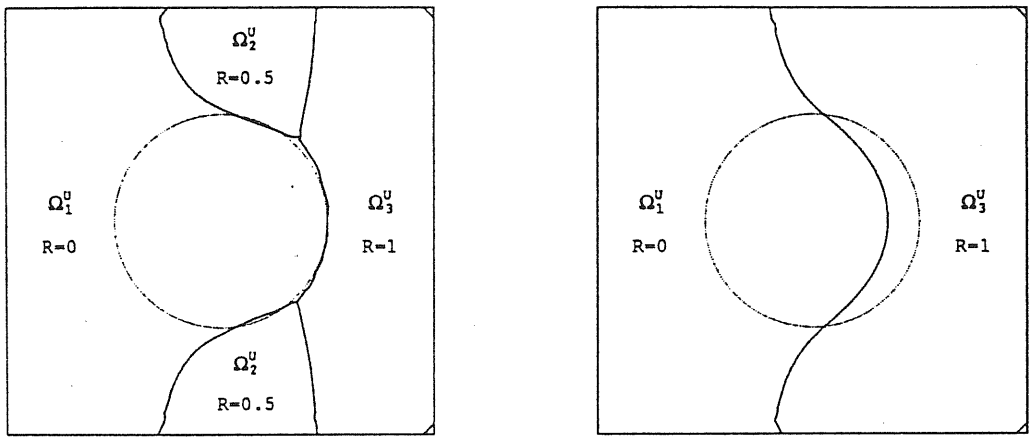

FIGURE 5.5. Ex. 5.2.1 (ISP): Fixed quasi-uniform mesh ($h = 0.1$, $\epsilon = 0.17$); a. $c_0\alpha = 25$; b. $c_0\alpha = 15$.

5.3. Prescribed Curvature Problem (PCP).

Example 5.3.1. Let $\Omega := (0,4)\times(0,4)$. In view of (1.4) and (2.14), any set $A \subseteq \Omega$ whose boundary has normal contact with $\partial\Omega$ and mean curvature $\kappa(x) := -\lambda\sqrt{x_1^2 + x_2^2} + 2\lambda + 0.5$, where $\lambda \in \mathbf{R}^+$, coincides with $\Omega_1^{\mathbf{u}}$, where $\mathbf{u}$ is a minimum of the functional $\mathcal{G}$ with $d = 2$, $\mathbf{k} := \frac{\kappa}{2}(\mathbf{e}_1 - \mathbf{e}_2)$, and boundary datum $\mathbf{m} := 0$, that is $\mathbf{g} := \frac{1}{2}(\mathbf{e}_1 + \mathbf{e}_2)$. Note that the empty set (which corresponds to $\mathbf{u} := \mathbf{e}_2$) is always a relative minimum of $\mathcal{G}$. It is easy to check that $\Omega_1^{\mathbf{u}} := \{x \in \Omega : |x| \leq 2\}$ corresponds to a relative minimum of $\mathcal{G}$ for all $\lambda > 0.25$; this minimum has energy $\mathcal{G}(\mathbf{u}) < \mathcal{G}(\mathbf{e}_2)$, if $\lambda > 0.75$ (see Fig. 5.6.a, where $\lambda = 1$). Observe that non-radial-symmetric solutions may exist, as shown in Fig. 5.6.b, where $\lambda = 0.5$. To compute the discrete solutions, we used a fixed quasi-uniform mesh with $h = 0.1$ ($\mathbf{S} = 2634$, $\mathbf{J} = 1398$) and $\epsilon = 0.17$. The dotted lines, $\{x \in \Omega : \kappa(x) = 0\}$ and

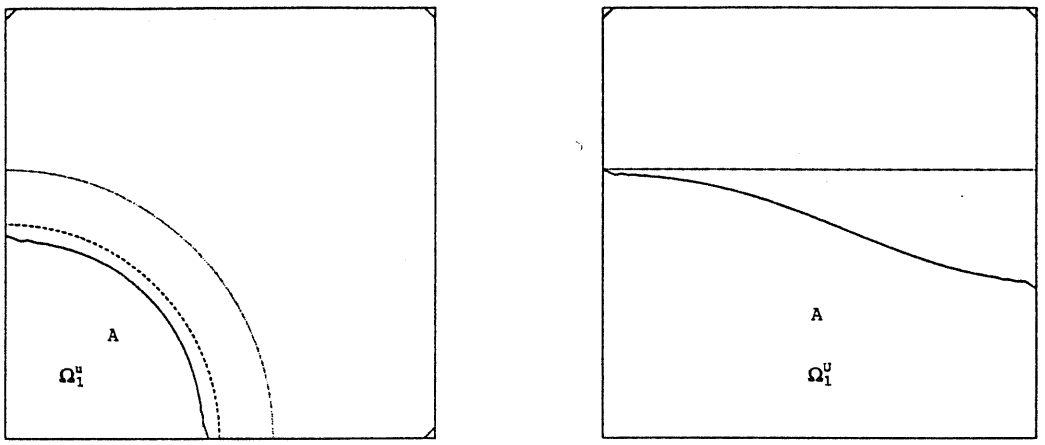

FIGURE 5.6. Ex. 5.3.1 (PCP, $\theta = \frac{\pi}{2}$): Fixed quasi-uniform mesh ($h = 0.1$, $\epsilon = 0.17$); a. $\lambda = 1$; b. $\lambda = 0.5$.

$\{x \in \Omega : x_2 = 2.5\}$, respectively, represent the jump set of the initial guesses. We have $\mathbf{I} = 1749$, $\mathbf{t} = 439$ and $\mathbf{I} = 2630$, $\mathbf{t} = 700$, respectively.

Example 5.3.2. Let either $\Omega := (0,4) \times (0,4)$ or $\Omega := ((0,4) \times (0,4)) \setminus ((2,4) \times (0,2))$, $\theta := 0$ (tangential contact with $\partial\Omega$, which corresponds to the boundary datum $\mathbf{g} := \mathbf{e}_2$) and $\kappa(x) := \lambda$, where $\lambda \in \mathbf{R}^+$. If λ is sufficiently large, Ω_1^u is given by Ω with the corner smoothed out in circular arcs of radius $\frac{1}{\lambda}$. Fig. 5.7.a ($\lambda = 1$) and Fig. 5.7.b ($\lambda = 2$) show both the exact and computed solutions. Note that the discrete minimizing sets present no contact with $\partial\Omega$; this is due to the fact that the relaxed solutions have a transition layer $O(\epsilon)$-wide along $\partial\Omega \cap \partial\Omega_1^u$. The discrete solutions were produced, from the initial guess $\mathbf{U}^0 := \mathbf{e}_1$, on a fixed quasi-uniform mesh with $h = 0.08$ ($\mathbf{S} = 4146$, $\mathbf{J} = 2174$), $\epsilon = 0.16$ and $h = 0.06$ ($\mathbf{S} = 5648$, $\mathbf{J} = 2957$), $\epsilon = 0.12$, respectively. In addition, we have $\mathbf{I} = 741$, $\mathbf{t} = 288$ and $\mathbf{I} = 447$, $\mathbf{t} = 237$, respectively.

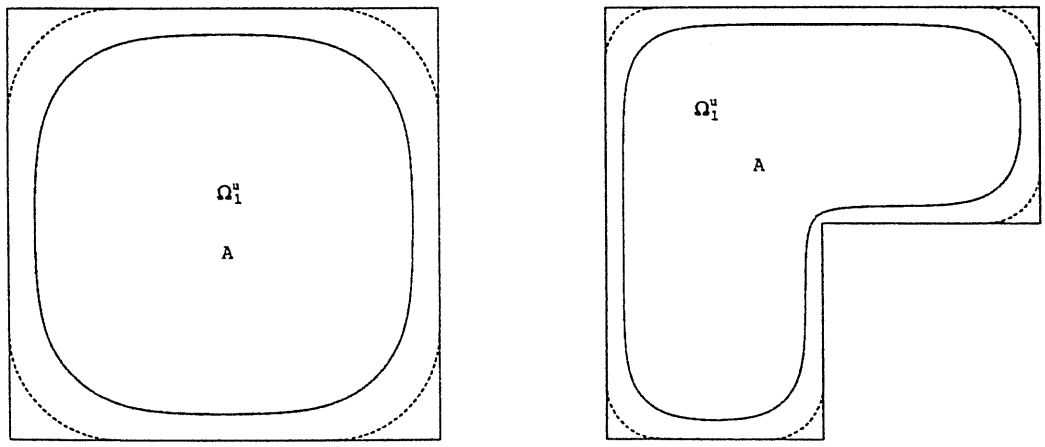

FIGURE 5.7. Ex. 5.3.2 (PCP, $\theta = 0$): Fixed quasi-uniform mesh;
a. $\lambda = 1$ ($h = 0.08$, $\epsilon = 0.16$); b. $\lambda = 2$ ($h = 0.06$, $\epsilon = 0.12$).

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CHAPTER 3, SECTION 3.3

FRONT-TRACKING AND VARIATIONAL METHODS

TO APPROXIMATE INTERFACES WITH PRESCRIBED MEAN CURVATURE*

1. Introduction. The great attention paid lately to the analysis of systems involving a surface energy on unknown interfaces (see, for instance, [3]) led us to study the numerical minimization of a functional which gives rise to interfaces with prescribed mean curvature. This problem, so-called *prescribed mean curvature* problem, can be considered as a simple but effective model in this field.

More precisely, given an open bounded convex set $\Omega \subset \mathbf{R}^d$ $(d \geq 2)$ with piecewise C^2 boundary, a function $\kappa \in L^\infty(\Omega)$, and a constant $\mu \in [-1,1]$, we consider the minimum problem for the functional $\tilde{\mathcal{F}}(A)$ defined on the measurable sets $A \subseteq \Omega$ as follows:

$$\tilde{\mathcal{F}}(A) := \mathcal{H}^{d-1}(\partial A \cap \Omega) + \mu \mathcal{H}^{d-1}(\partial A \cap \partial\Omega) - \int_A \kappa \, dx,$$

where $\mathcal{H}^{d-1}$ denotes the $(d-1)$-dimensional Hausdorff measure [4]. Any (relative or absolute) minimum of $\tilde{\mathcal{F}}$ is a set whose boundary has mean curvature κ and contact angle $\theta = \arccos(\mu)$ at its intersection with $\partial\Omega$ [5].

Denoting by $BV(\Omega;\{-1,1\})$ the space of the functions of bounded variation on Ω with values in $\{-1,1\}$, by $\int_\Omega |Dv|$ the total variation on Ω of the Radon measure Dv, and by $\mathrm{tr}(v)$ the trace on $\partial\Omega$ of the BV function v, we can also define the following energy functional on the space $D(\mathcal{F}) := BV(\Omega;\{-1,1\})$:

$$\mathcal{F}(v) := \int_\Omega |Dv| + \int_{\partial\Omega} |\mathrm{tr}(v) + \mu| \, d\mathcal{H}^{d-1}(x) - \int_\Omega \kappa v \, dx.$$

Note that $\mathcal{F}(v) = 2\tilde{\mathcal{F}}(A_v) + (1-\mu)\mathcal{H}^{d-1}(\partial\Omega) + \int_\Omega \kappa$, for all $v \in D(\mathcal{F})$, where $A_v := \{x \in \Omega : v(x) = 1\}$. As a generalization, μ should be considered to be a piecewise constant function $\mu \in BV(\partial\Omega;[-1,1])$.

It is well known [5] that $\mathcal{F}$ admits at least a minimizer u, so that the set A_u minimizes $\tilde{\mathcal{F}}$, whereas uniqueness cannot be expected. In addition, the functionals $\tilde{\mathcal{F}}$ and $\mathcal{F}$ may exhibit many relative minima (the empty set is always a relative minimum).

In this paper, we shall describe and compare two different numerical approaches to this minimization problem, namely a front-tracking algorithm (FTA) and a variational algorithm (VA). FTA minimizes the functional $\tilde{\mathcal{F}}$ on polygonal subsets of Ω, whereas VA deals with the functional $\mathcal{F}$, where the unknown interface does not appear explicitly. It is to be stressed that, in general, both methods may converge to a *relative* minimum.

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2. Front-Tracking Algorithm (FTA). Assuming that $\kappa \in C^0(\bar{\Omega})$, then any relative minimum of $\tilde{\mathcal{F}}$ has C^2 boundary; it can thus be approximated, with an L^∞-error of order $O(h^2)$, by polygons having edges $O(h)$-long. This remark leads us to minimize the functional $\tilde{\mathcal{F}}$ on the domain $D_h(\tilde{\mathcal{F}}) := \{A \subseteq \Omega : A \text{ is a polygon with edges of length } \geq h\}$.

FTA is presently implemented in 2-D. The 3-D case is considerably more complicated because of both the data structures for the representation of surfaces and the topological changes in the minimizing sequence of interfaces. An extra regularity for the curvature function, $\kappa \in C^{1,1}(\Omega)$, is required for using a Modified-Newton method [6] in the minimization process. The present version of the code does not deal with the boundary term $\mu\mathcal{H}^{d-1}(\partial A \cap \partial\Omega)$. Hence there must be no intersection of $\partial\Omega$ with the sets of the minimizing sequence. This is equivalent to require that $\Omega = \mathbf{R}^2$ and that any relative minimum of $\tilde{\mathcal{F}}$ is bounded, which is guaranteed if κ is negative outside a bounded set. The upgrading of the code to overcome this restriction is in progress. The convergence of FTA has not been proved yet; nonetheless, numerical experiments show very fast convergence and the iterative procedure seems to exhibit no oscillation phenomena, the relative minimum which is approximated just depending on the initial guess.

Let us now briefly describe the FTA code. Let $A_0 \in D_h(\tilde{\mathcal{F}})$ be the initial guess, for instance, $A_0 := \{x \in \Omega : \kappa(x) > 0\}$. We stress that any set A_0 so that $\mathrm{diam}(A_0)\sup_{x\in\Omega}\kappa(x) < 2$ evolves towards the empty set. Roughly speaking, FTA produces a minimizing sequence $\{A_n \in D_h(\tilde{\mathcal{F}})\}_n$ as follows: A_n has approximately the same number of vertices as A_{n-1} untill the value of the functional cannot be further lowered; in that case, the boundary Γ_n of A_n is enriched by adding intermediate vertices; the iterations stop when each edge of A_n has length $\simeq h$, so that A_n can approximate a C^2 curve with a prescribed error $O(h^2)$. We shall drop the subscript n whenever its presence is clear from the context. The FTA code can be summarized as follows:

(S0) Define A_0 and assign an iteration error threshold σ; $n = 0$.
(S1) $n = n+1$; define A_n, moving the vertices of A_{n-1} along the "normals" to Γ_{n-1}, and using a Modified-Newton step in order to lower $\tilde{\mathcal{F}}$.
(S2) Perform a topological consistency test to check that the new Γ_n has no self-intersections. If this test fails, Γ_n must be suitably modified by changing its topology.
(S3) Redistribute the vertices of Γ_n, in order to avoid too near and too far adjacent vertices.
(S4) If the displacement between Γ_{n-1} and Γ_n is $< \sigma$, go to (S5). Otherwise, go to (S1).
(S5) If the length of the edges of Γ_n is $\simeq h$, stop. Otherwise, double the number of vertices by adding the midpoints of the edges of Γ_n and go to (S1).

2.1. Step (S1). At this stage, the connected components of ∂A can be treated independently from each other; hence, we can suppose that A has a connected boundary Γ. Let the vertices $P_1, ..., P_I$ of Γ be given in a counter-clockwise order. For any P_i, let ν_i be the unit vector along the bisector of the inner angle of Γ at P_i; in addition, set $P_i(\lambda) := P_i + \lambda_i\nu_i$, where $\lambda_i \in \mathbf{R}$. Let $A(\lambda)$ and $\Gamma(\lambda)$ be the polygon defined by $P_1(\lambda), ..., P_I(\lambda)$ and its boundary, respectively. Clearly, $A(0) = A$ and $\Gamma(0) = \Gamma$. We are now in a position to define the finite dimensional functional $\mathcal{I} : \mathbf{R}^I \to \mathbf{R}$ as $\mathcal{I}(\lambda) := \tilde{\mathcal{F}}(A(\lambda)) - \tilde{\mathcal{F}}(A) = \mathrm{length}(\Gamma(\lambda)) - \mathrm{length}(\Gamma) - \int_{A(\lambda)\setminus A}\kappa\, dx + \int_{A\setminus A(\lambda)}\kappa\, dx$. Let

P_i^- and P_i^+ denote the previous and the next vertex to P_i, respectively. Then, using the pointwise quadrature formula to approximate the integrals of κ on triangles, we have

$$\mathcal{I}(\lambda) \approx \sum_{i=1}^{I} \left[|P_i^+(\lambda) - P_i(\lambda)| - |P_i^+ - P_i| + T(\kappa; P_i^+(\lambda), P_i(\lambda), P_i^+) + T(\kappa; P_i(\lambda), P_i, P_i^+) \right],$$

where $T(\kappa; A, B, C) := \frac{1}{3}[\kappa(A)+\kappa(B)+\kappa(C)]\mathrm{Area}(ABC)$; note that $\mathrm{Area}(ABC) < 0$ if the vertices of the triangle ABC are in clockwise order. The components of $g := \nabla\mathcal{I}(0)$ are

$$\frac{\partial \mathcal{I}}{\partial \lambda_i}(0) = -\left(\frac{P_i^- - P_i}{|P_i^- - P_i|} + \frac{P_i^+ - P_i}{|P_i^+ - P_i|} \right) \cdot \nu_i + \frac{1}{2} \left[\kappa^-(P_i^- - P_i) - \kappa^+(P_i^+ - P_i) \right] \wedge \nu_i,$$

where

$$\kappa^- := 2\int_0^1 s\kappa(sP_i + (1-s)P_i^-)\, ds, \qquad \kappa^+ := 2\int_0^1 s\kappa(sP_i + (1-s)P_i^+)\, ds,$$

and $v \wedge w := v_2 w_1 - v_1 w_2$ is the usual external vector product in $\mathbf{R}^2$. The Hessian matrix $H := (\frac{\partial^2 \mathcal{I}}{\partial\lambda_i \partial\lambda_j}(0))_{ij}$ has nonzero entries only if either $P_j = P_i$ or $P_j = P_i^+$ or $P_j = P_i^-$, namely, for adjacent vertices. Hence H is a symmetric, "almost" tridiagonal matrix (i.e., $H_{ij} = 0$ if $1 < |i-j| < I-1$). We have

$$\frac{\partial^2 \mathcal{I}}{\partial \lambda_i^2}(0) = \frac{\left[(P_i^- - P_i) \wedge \nu_i\right]^2}{|P_i^- - P_i|^3} + \frac{\left[(P_i^+ - P_i) \wedge \nu_i\right]^2}{|P_i^+ - P_i|^3} + \frac{1}{3}(\nabla\kappa^- \cdot \nu_i)\left[(P_i^- - P_i) \wedge \nu_i\right] - \frac{1}{3}(\nabla\kappa^+ \cdot \nu_i)\left[(P_i^+ - P_i) \wedge \nu_i\right],$$

where

$$\nabla\kappa^- := 3\int_0^1 s^2 \nabla\kappa(sP_i + (1-s)P_i^-)\, ds, \qquad \nabla\kappa^+ := 3\int_0^1 s^2 \nabla\kappa(sP_i + (1-s)P_i^+)\, ds,$$

and, setting $P_j := P_i^+$,

$$\frac{\partial^2 \mathcal{I}}{\partial \lambda_i \partial \lambda_j}(0) = \frac{\left[(P_j - P_i)\cdot\nu_j\right]\left[(P_j - P_i)\cdot\nu_i\right]}{|P_j - P_i|^3} - \frac{\nu_j \cdot \nu_i}{|P_j - P_i|} + \frac{1}{6}(\nu_{ij}\nabla\kappa_{ij}) \wedge (P_j - P_i) - \frac{1}{2}\kappa_{ij}\nu_j \wedge \nu_i,$$

where $\nu_{ij} := \frac{1}{2}(\nu_j \otimes \nu_i + \nu_i \otimes \nu_j)$ is a 2 by 2 matrix, and

$$\kappa_{ij} := \int_0^1 \kappa(sP_i + (1-s)P_j)\, ds, \qquad \nabla\kappa_{ij} := 6\int_0^1 s(1-s)\nabla\kappa(sP_i + (1-s)P_j)\, ds.$$

The values κ^-, κ^+, κ_{ij}, $\nabla\kappa^-$, $\nabla\kappa^+$ and $\nabla\kappa_{ij}$ are approximated by using the linear interpolants of κ and $\nabla\kappa$ along the segments $P_iP_i^-$ and $P_iP_i^+$.

Since, in general, H is *not* positive definite, the standard Newton method for $\nabla\mathcal{I}(\lambda) = 0$ with initial point $\lambda = 0$ might not lower $\tilde{\mathcal{F}}$. Hence we define the first displacement vector λ using a Modified-Newton method [6], as follows:

$$\lambda := \begin{cases} -H^{-1}g, & \text{if } H \text{ is positive definite,} \\ -(H+E)^{-1}g, & \text{otherwise,} \end{cases}$$

where E is any diagonal positive matrix such that $H + E$ is positive definite. With this definition, λ is always a descent direction for $\mathcal{I}$. The matrix E can be generated during the Cholesky decomposition of the matrix H. Hence, taking advantage of the structure of H, we can factorize H (or $H + E$) $=: LDL^T$ with an optimal memory space and time usage. Here D is a positive diagonal matrix and L is a lower triangular matrix, with unitary diagonal entries and zero entries outside the codiagonal and the last row.

The set A_n is then defined from $A_{n-1} = A$ by

$$A_n := A_{n-1}(s\lambda),$$

for an appropriate value $s \in (0, \bar{s}]$, where $\bar{s}$ is given by local topological constraints. If $\bar{s} \geq 1$, $E = 0$, and $\mathcal{I}(\lambda) < 0$, then we take $s = 1$ (real Newton step); otherwise s is determined by a very few iterations of a one dimensional minimization technique.

2.2. Step (S2). A topological consistency test is performed to check that $\Gamma := \Gamma_n$ has no self-intersections. If this test fails, Γ is suitably modified. Since this test is performed at each iteration of FTA, its computational complexity (CC) is a crucial issue for the code efficiency. We stress that all the connected components of Γ must be taken into account simultaneously, because they might intersect each other. Testing for intersection every couple of edges has CC$= O(I^2)$, which slows down the whole algorithm. On the contrary, relating the existence of self-intersections to the existence of vertices of Γ which *cross* Γ_{n-1}, so-called *bad* vertices, reduces CC to $O(I \log I)$, which is nearly optimal. These *bad* vertices can be checked as follows. Let P_i be a vertex of Γ and $r_i(t) := P_i + t\mathbf{e}_1$, $t \geq 0$, be the half straight line from P_i in the direction of the vector $\mathbf{e}_1 := (1, 0)$. For the sake of simplicity, we suppose that P_i is not a self-intersection point and that r_i does not intersect any vertex of Γ as well as any self-intersection (these restrictions can be avoided by counting the intersections $r_i \cap \Gamma$ with the appropriate multiplicity). Let α (resp. β) denotes the angle between the vectors $\mathbf{e}_1$ and $P_i^- - P_i$ (resp. $P_i^+ - P_i$); $0 \leq \alpha, \beta < 2\pi$. Finally, let m_i be the number of intersections of r_i with Γ; the vertex P_i is counted only if $\beta > \alpha$ (i.e., if r_i belongs to the inner angle of Γ at P_i). We say that P_i is *bad*, if m_i is odd. Note that the values m_i, for $1 \leq i \leq I$, can be accumulated in parallel checking, for each edge s of Γ, the vertices P_i for which r_i intersects s. This operation can be performed in logarithmic time, if the vertices of Γ are organized in a binary tree, with the x_2 coordinate as the key.

If the consistency test fails (i.e., at least one *bad* vertex is found), a change of topology must be done. Recursively, for each self-intersection, a local change of topology is performed. To this end, their position must be exactly computed. Note that a self-intersection is present along an edge whenever only one of its vertices is *bad*. As a result, one or more small connected components of Γ are produced, with the vertices ordered in clockwise order. These small *negative* components are considered as spurious, and hence they are simply deleted. The change of topology should happen only sporadically, hence its computational complexity is not a critical issue.

3. Variational Algorithm (VA). The numerical minimization of the functional $\mathcal{F}$ by fixed domain methods is very difficult because of the lack of regularity and convexity of the problem at hand. Then, we first approximate $\mathcal{F}$ with a family $\{\mathcal{F}_\epsilon\}_{\epsilon>0}$ of regular nonconvex functionals defined in $H^1(\Omega)$ which, in turn, will be discretized by finite elements; ϵ is the *relaxation* parameter and h is the *meshsize*. The Γ-convergence of both the relaxed and discrete functionals has been proved in [7] and [1], respectively. In order to define the relaxed functionals $\mathcal{F}_\epsilon$ we need some preparation. Let $\omega, \beta : [-1,1] \to \mathbf{R}$ be defined by

$$\omega(t) := 1 - t^2, \qquad \beta(t) := \omega'(t).$$

Other choices for ω can be considered as well [1,2]. In addition, set $\varphi(t) := \int_0^t \sqrt{\omega(s)}\, ds$, for all $t \in [-1,1]$, and $c_0 := \int_{-1}^1 \sqrt{\omega(t)}\, dt$. Since $\varphi : [-1,1] \to [-\frac{c_0}{2}, \frac{c_0}{2}]$ is an (odd) strictly increasing function, the following piecewise constant function $g \in BV(\partial\Omega; [-1,1])$ is well defined: $g(x) := \varphi^{-1}\left(-\frac{c_0}{2}\mu(x)\right)$, for all $x \in \partial\Omega$. For any $\epsilon > 0$, let $g_\epsilon \in C^{0,1}(\partial\Omega; [-1,1])$ approximate g in the sense that $g_\epsilon(x) = g(x)$, if $\mathrm{dist}(x, S_g) \geq \epsilon$, and $\mathrm{Lip}(g_\epsilon) \leq \frac{L_g}{\epsilon}$, where S_g denotes the *jump* set of g and L_g is a constant independent of ϵ. We are now in a position to define the relaxed functionals:

$$\mathcal{F}_\epsilon(v) := \int_\Omega \left[\epsilon|\nabla v|^2 + \frac{1}{\epsilon}\omega(v) - c_0\kappa v\right] dx, \qquad \text{if } v \in D(\mathcal{F}_\epsilon),$$

where $D(\mathcal{F}_\epsilon) := \{v \in H^1(\Omega; [-1,1]) : \ v = g_\epsilon \text{ on } \partial\Omega\}$. The existence of a solution u_ϵ to the minimum problem for $\mathcal{F}_\epsilon$ can be proven by direct methods. The nonconvexity of $\mathcal{F}_\epsilon$ allows, in general, the existence of more than one absolute or relative minima u_ϵ, which satisfy the following variational inequality (set $\langle u, v\rangle := \int_\Omega uv\, dx$):

$$2\epsilon\langle\nabla u_\epsilon, \nabla(v - u_\epsilon)\rangle + \frac{1}{\epsilon}\langle\beta(u_\epsilon), v - u_\epsilon\rangle - c_0\langle\kappa, v - u_\epsilon\rangle \geq 0, \qquad \forall\, v \in D(\mathcal{F}_\epsilon).$$

At this stage, we can introduce a discretization of the relaxed functional $\mathcal{F}_\epsilon$ by piecewise linear finite elements. Let us denote by $\{\mathcal{S}_h\}_{h>0}$ a regular and weakly acute family of partitions of Ω into simplices having diameter bounded by h. For the sake of simplicity we shall assume that Ω is a polyhedron, so that $\bar{\Omega} \equiv \Omega_h := \cup_{S\in\mathcal{S}_h} S$, for all $h > 0$. Let

$\mathbf{V}_h \subset H^1(\Omega)$ indicate the usual piecewise linear finite element space over $\mathcal{S}_h$ and $\Pi_h : C^0(\bar{\Omega}) \to \mathbf{V}_h$ the Lagrange interpolation operator. Let $\{a_j\}_{j=1}^J$ denote the nodes of $\mathcal{S}_h$ and $\{\chi_j\}_{j=1}^J$ the canonical basis of $\mathbf{V}_h$; we assume that $\{a_j\}_{j=1}^{J_0}$ are the internal nodes.

Finally, for any $\epsilon > 0$, let $\kappa_\epsilon \in C^{0,1}(\bar{\Omega})$ approximate the *curvature* function κ so that $\|\kappa_\epsilon\|_{L^\infty(\Omega)} \le \|\kappa\|_{L^\infty(\Omega)}$, $\kappa_\epsilon \to \kappa$ in $L^1(\Omega)$, as $\epsilon \to 0$, and $\mathrm{Lip}(\kappa_\epsilon) \le \frac{L_\kappa}{\epsilon}$, where L_κ is a constant independent of ϵ. We are now in a position to introduce the discrete functionals:

$$(3.1) \qquad \mathcal{F}_{\epsilon,h}(v) := \int_\Omega \left[\epsilon |\nabla v|^2 + \frac{1}{\epsilon} \Pi_h \omega(v) - c_0 \Pi_h(\kappa_\epsilon v) \right] dx, \qquad \text{if } v \in D(\mathcal{F}_{\epsilon,h}),$$

where $D(\mathcal{F}_{\epsilon,h}) := \{v \in \mathbf{V}_h : \ |v| \le 1 \text{ in } \Omega, \ v = \Pi_h g_\epsilon \text{ on } \partial\Omega\}$. The existence of a solution $u_{\epsilon,h}$ to the minimum problem for $\mathcal{F}_{\epsilon,h}$ is trivial. The integrals in (3.1) can be evaluated via the vertex quadrature rule, which is exact for piecewise linear functions, thus allowing the actual implementation of the discrete minimum problem on a computer.

A (relative) minimum $u_{\epsilon,h}$ for the functional $\mathcal{F}_{\epsilon,h}$ satisfies the variational inequality

$$(3.2) \quad 2\epsilon \langle \nabla u_{\epsilon,h}, \nabla(v - u_{\epsilon,h}) \rangle + \frac{1}{\epsilon} \langle \beta(u_{\epsilon,h}), v - u_{\epsilon,h} \rangle_h - c_0 \langle \kappa_\epsilon, v - u_{\epsilon,h} \rangle_h \ge 0, \quad \forall\, v \in D(\mathcal{F}_{\epsilon,h}),$$

where $\langle u, v \rangle_h := \int_\Omega \Pi_h(uv) dx$. Denoting by $M := \{\langle \chi_j, \chi_k \rangle_h\}_{j,k=1}^J$ the (diagonalized) mass matrix, by $A := \{\langle \nabla \chi_j, \nabla \chi_k \rangle\}_{j,k=1}^J$ the stiffness matrix, and by $D :=$ diagonal of A, $B := I - D^{-1}A$, the variational inequality (3.2) can then be written in matrix form as follows:

$$(3.3) \quad \begin{cases} [2\epsilon AU + \frac{1}{\epsilon} M\beta(U) - c_0 MK]_j \in \psi(u_j), \quad u_j \in [-1,1], & j = 1, ..., J_0, \\ u_j = g_\epsilon(a_j), & j = J_0 + 1, ..., J. \end{cases}$$

Here $U = (u_1, ..., u_J)^T$ represents the unknown solution $u_{\epsilon,h}$ (i.e., $u_j := u_{\epsilon,h}(a_j)$, $j = 1, ..., J$), $K := (\kappa_\epsilon(a_1), ..., \kappa_\epsilon(a_J))^T$, and $-\psi$ is the inverse of the *sgn* graph. In other words, $z \in \psi(u_j)$ means: $z = 0$, if $u_j \in (-1,1)$; $z \le 0$, if $u_j = 1$; $z \ge 0$, if $u_j = -1$. A simple minded method for solving the nonlinear system (3.3) is the following Jacobi iterative scheme:

$$u_j^{k+1} := \Pi_{[-1,1]} \left(BU^k - \frac{1}{2\epsilon^2} D^{-1} M\beta(U^k) + \frac{c_0}{2\epsilon} D^{-1} MK \right)_j, \qquad j = 1, ..., J_0,$$

where $\Pi_{[-1,1]}(s) := \min(1, \max(-1, s))$. A more effective iterative method can be easily obtained using an incomplete Cholesky factorization of the matrix A as a preconditioner.

Further details on VA can be found in [2].

4. Numerical Experiments. We present several numerical examples in order to show the behaviour of the algorithms FTA and VA.

4.1. Example I. Consider the cone-shaped curvature $\kappa_m(x,y) := -m\sqrt{x^2+y^2} + 2m + 0.5$. If $m > 0.25$, the circle $x^2 + y^2 = 4$ is a relative minimum (with negative energy, if $m > 0.75$).

We also stress that the location of the relative minima might be very sensitive to the relaxation procedure or, even worse, that a relative minimum might disappears; this fact is reflected in Ex. I, for $m = 0.5$.

On the other hand, FTA has a number of restrictions. The data structure to describe the unknown interface is the major responsible for the difficulties in dealing with the 3-D case as well as with the boundary term $\mu\mathcal{H}^{d-1}(\partial A \cap \partial\Omega)$. Moreover, since the topological consistency test plays an essential role in FTA, it is another critical issue, in that it necessitates specific data structures (such as binary trees) and nontrivial algorithms to keep the CPU time under reasonable limits. On the contrary VA is completely unaware of the structure of the interface, which is defined implicitly as the zero level curve.

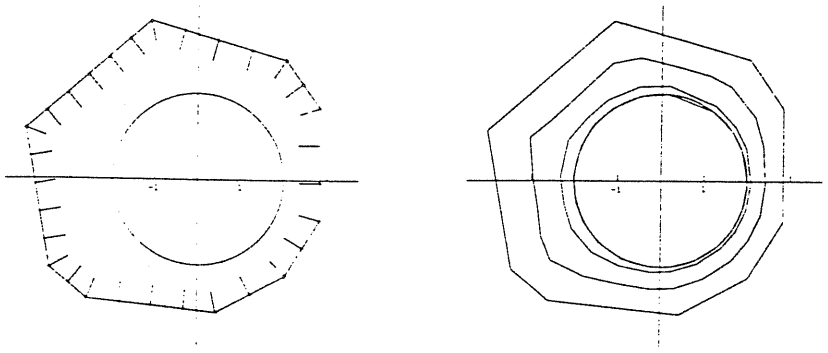

FIGURE 5.1. Ex. I: Initial guess with "normal" vectors and successive iterates (FTA).

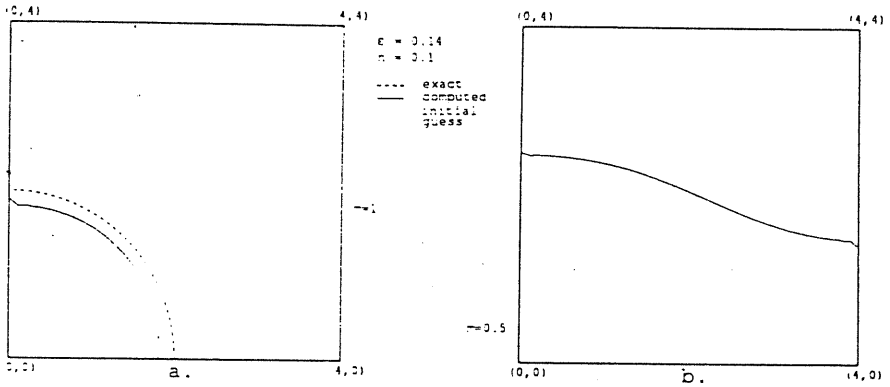

FIGURE 5.2. Ex. I: Exact and discrete relative minima (VA).

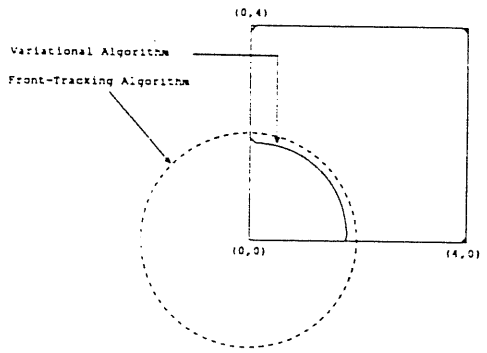

FIGURE 5.3. Ex. I: Comparison between FTA and VA.

In Fig. 5.1, we present the exact solution as well as the initial guess and the successive iterates generated by FTA, with $m = 1$. Fig. 5.2 shows two discrete interfaces obtained with VA; here $\Omega := (0,4)\times(0,4)$ and $\mu := 0$. Note the existence of a nonradially symmetric relative minimum, which is generated from the initial guess $A_0 := (0,4)\times(0,2)$. The interfaces produced by FTA and VA are compared in Fig. 5.3, whereas the performances of the two algorithms are summarized in Table 4.1 (DOF = degrees of freedom).

	L^∞-error	CPU (sec)	ϵ	h	iterates	DOF
FTA	0.002	6	—	~0.04	14	max 300
VA	0.15	735	0.14	0.1	~2000	~1500

TABLE 4.1. Comparison between FTA and VA.

4.2. Example II. Let $\kappa(x,y) := \lambda$ (positive constant) and $\mu := 1$. Note that, in this example, the boundary term plays an essential role. Fig. 5.4 illustrates the interfaces obtained with VA in two different domains Ω; the initial guess is $A_0 := \Omega$ for both.

4.3. Example III. Let $\kappa(x,y) := 2\sin x \sin y$. Fig. 5.5 shows four different relative minima of $\tilde{\mathcal{F}}$, obtained with FTA.

4.4. Example IV. Let $\kappa_t(x,y) := -40[x^2(x-1)(x+1)+y^2]+t$. Making use of FTA, we present the evolution of a relative minimum when the parameter t ranges from -1 to 3.6 (Fig. 5.6), from 3.6 to -3 (Fig. 5.7), and from -1 to 3.79 (Fig. 5.8). The results are obtained by choosing as initial guess the minimum computed for the previous value of t. Fig. 5.6 exhibits a jump at $t = 3.5$ when the two components touch, whereas Fig. 5.7 shows that, at $t = -3$, the previous relative minimum suddenly shrinks to the empty set; finally, Fig. 5.8 shows that the relative minimum becomes unstable at $t = 3.78$.

5. Comparison between FTA and VA. The two algorithms presented in this paper are based on substantially different approaches; this is reflected in their behaviour. The most evident difference between FTA and VA is their performances, as illustrated in Table 4.1 (in addition, note that, in Ex. I, VA exploits the symmetry of the problem at hand).

The slowness of VA is basically explainable as follows. In view of the Γ-convergence result proved in [1], VA requires the relation $h = o(\epsilon)$ to hold. Hence the meshsize has to be much smaller than the relaxation parameter, which then dictates the accuracy of the solution. Using a uniform mesh, we end up with $O(h^{-2})$ degrees of freedom for the nonlinear system to be solved. This concerns both the memory space and the cost of the nonlinear solver. Consequently, a possible remedy has to be be found in term of a suitable adaptive local refinement strategy, in order to concentrate the degrees of freedom near the developed interface. Moreover, as observed in [2], a number $O(h^{-2})$ of Jabobi iterations is espected for computing the discrete solution. The implementation of more efficient methods, such as nonlinear SORs, will probably reduce the number of iterations, but not its order of magnitude.

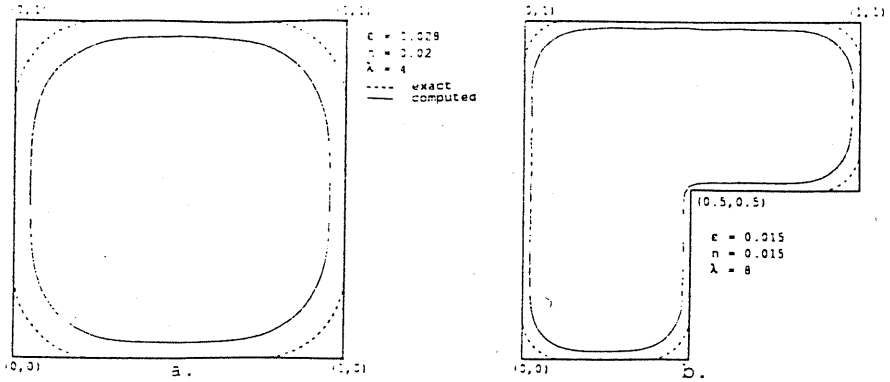

FIGURE 5.4. Ex. II: Exact and discrete relative minima (VA).

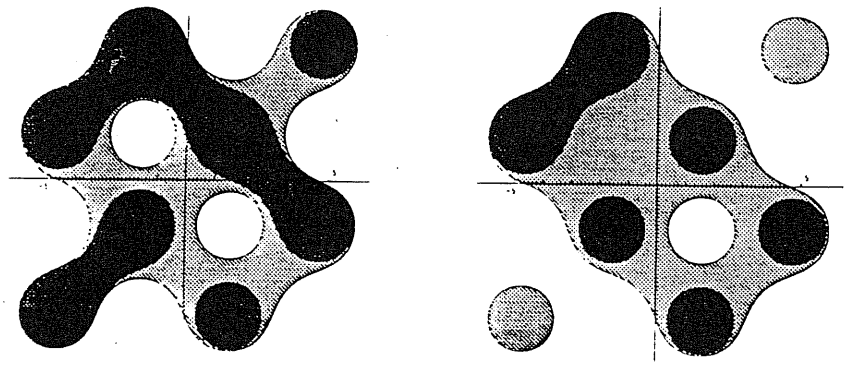

FIGURE 5.5. Ex. III: Some relative minima (FTA).

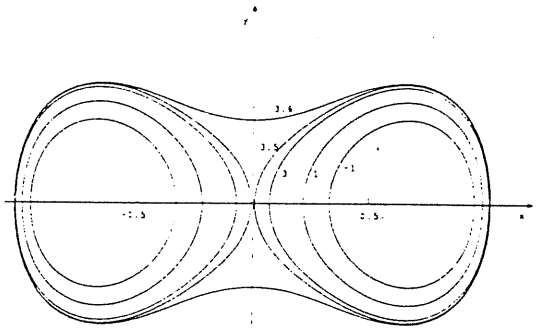

FIGURE 5.6. Ex. IV: Evolution of relative minima, $t = -1, 1, 3, 3.5, 3.6$.

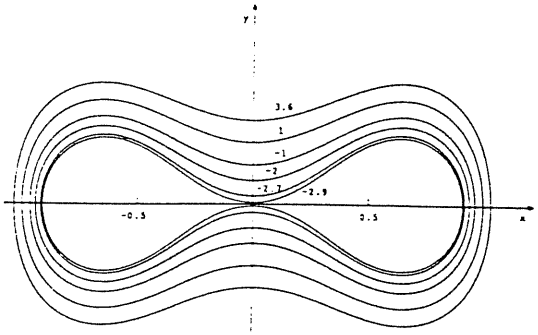

FIGURE 5.7. Ex. IV: Evolution of relative minima, $t = 3.6, 1, -1, -2, -2.7, -2.9, -3$ (empty set).

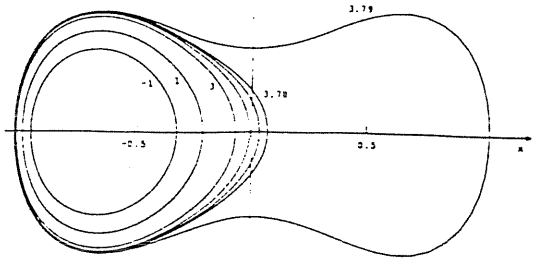

FIGURE 5.8. Ex. IV: Evolution of relative minima, $t = -1, 1, 3, 3.5, 3.7, 3.78, 3.79$.

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CHAPTER 3, SECTION 3.4

CONVEX APPROXIMATIONS OF FUNCTIONALS WITH CURVATURE*

0. INTRODUCTION

Several geometrical type problems in the calculus of variations arising, for instance, in phase transition theories [4] and computer vision theory [15], fall within the general setting proposed by E. De Giorgi [6,1]. These problems usually involve unknown interfaces, obtained as minima of functionals defined on the space $BV(\Omega;\{-1,1\})$ of the characteristic functions of sets of finite perimeter in Ω. The numerical minimization of such functionals seems quite difficult, because of the lack of convexity and regularity (see, e.g., [2]).

In this paper we address the numerical minimization of a model functional [9,11,12] via convex approximations. More precisely, given an open bounded set $\Omega \subset \mathbf{R}^n$ $(n \geq 2)$, a function $\kappa \in L^\infty(\Omega)$, and $\mu \in L^\infty(\partial\Omega;[-1,1])$, we consider the minimum problem:

$$\min_{v\in BV(\Omega;\{-1,1\})} \mathcal{F}(v), \qquad \text{where} \qquad \mathcal{F}(v) := \int_\Omega |Dv| + \int_{\partial\Omega} \mu v \; d\mathcal{H}^{n-1} - \int_\Omega \kappa v \; dx.$$

It is well known [9] that any minimum of $\mathcal{F}$ is the characteristic function of a set $A \subseteq \Omega$ whose boundary has prescribed mean curvature κ and contact angle $\arccos(\mu)$ at $\partial\Omega$.

Noting that $\mathcal{F}$ can be equivalently minimized on the larger, convex, set $BV(\Omega;[-1,1])$, the (nonstrict) convexity of $\mathcal{F}$ can be exploited for the numerical minimization via linear finite element discretizations. Since the numerical algorithms perform better for strictly convex regular functionals, $\mathcal{F}$ is preliminarly regularized by

$$\mathcal{F}_\varepsilon(v) = \int_\Omega \sqrt{\varepsilon^2 + |Dv|^2} + \int_{\partial\Omega} \mu v \; d\mathcal{H}^{n-1} - \int_\Omega \kappa v \; dx, \qquad \forall\, v \in BV(\Omega;[-1,1]),$$

which, in turn, is discretized by continuous linear finite elements.

The main result of this paper is the Γ-$L^1(\Omega)$-convergence of the discrete functionals $\{\mathcal{F}_h\}_h$ and $\{\mathcal{F}_{\varepsilon,h}\}_{\varepsilon,h}$ to $\mathcal{F}$. More specifically, we prove the following diagram of convergence:

$$\begin{array}{ccc} \mathcal{F}_\varepsilon & \xrightarrow[\varepsilon\to 0]{\text{uniformly in } BV(\Omega;[-1,1])} & \mathcal{F} \\ \Gamma\text{-}L^1(\Omega)\ \uparrow\ h\to 0 & & \Gamma\text{-}L^1(\Omega)\ \uparrow\ h\to 0 \\ \mathcal{F}_{\varepsilon,h} & \xrightarrow[\varepsilon\to 0]{\text{uniformly in } V_h\subset BV(\Omega;[-1,1]) \text{ and with respect to } h} & \mathcal{F}_h \end{array} \tag{0.1}$$

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where $\mathbf{V}_h$ is the finite element space. Hence, letting ε and h go to 0 independently, it follows that $\Gamma\text{-lim}\,\mathcal{F}_{\varepsilon,h} = \mathcal{F}$ in $L^1(\Omega)$. In view of basic properties of the Γ-convergence [7], any family $\{u_{\varepsilon,h}\}_{\varepsilon,h}$ of discrete absolute minima admits a subsequence converging to a minimum point u of $\mathcal{F}$ in $L^1(\Omega)$ and $\mathcal{F}_{\varepsilon,h}(u_{\varepsilon,h})$ converges to $\mathcal{F}(u)$. We stress that no relation between ε and h is required for the limit procedure, whereas the nonconvex approximation via double well potential, first proposed in [14], Γ-converges if $h = o(\varepsilon)$ [2].

The outline of the paper is as follows. In §1 we state precisely the functionals and recall some basic properties. For the sake of completeness, in §2 we show the semicontinuity of both $\mathcal{F}$ and $\mathcal{F}_\varepsilon$. The demonstration of the convergence results is given in §3. The paper concludes in §4 with some numerical examples.

1. The setting

Let $\Omega \subset \mathbf{R}^n$ $(n \geq 2)$ be an open bounded set with Lipschitz-continuous boundary and denote by $|\cdot|$ the n-dimensional Lebesgue measure and by $\mathcal{H}^{n-1}$ the $(n-1)$-dimensional Hausdorff measure in $\mathbf{R}^n$ [8]. Let $BV(\Omega)$ be the space of the bounded variation functions in Ω and set $\widetilde{\mathcal{K}} := BV(\Omega;\{-1,1\})$, $\mathcal{K} := BV(\Omega;[-1,1])$. Let us denote by $v \in L^1(\partial\Omega)$ the trace of $v \in BV(\Omega)$ on $\partial\Omega$. Let $\int_\Omega |Dv|$ denote the *total variation* in Ω and $\int_\Omega \sqrt{1+|Dv|^2}$ the *area* of any function $v \in BV(\Omega)$ [11, Definitions 1.1 and 14.1]. For any set $E \subseteq \Omega$, let $\chi_E(x) := 1$ if $x \in E$, $:= -1$ if $x \in \Omega \setminus E$, be its characteristic function. It is well known that $v \in \widetilde{\mathcal{K}}$ if and only if v is the characteristic function χ_E of a set $E \subseteq \Omega$ of finite perimeter in Ω, and $P(E,\Omega) := \frac{1}{2}\int_\Omega |D\chi_E|$ is the perimeter of E in Ω [11]. Finally, for any $v \in BV(\Omega)$, set $\{v > t\} := \{x \in \Omega : v(x) > t\}$ and note that $\chi_{\{v>t\}} \in \widetilde{\mathcal{K}}$, for a.e. $t \in \mathbf{R}$ [11].

1.1. The original functional. Given $\mu \in L^\infty(\partial\Omega;[-1,1])$ and $\kappa \in L^\infty(\Omega)$, let us define

$$\mathcal{F}(v) := \int_\Omega |Dv| + \int_{\partial\Omega} \mu v \, d\mathcal{H}^{n-1} - \int_\Omega \kappa v \, dx, \qquad \forall\, v \in BV(\Omega). \tag{1.1}$$

It is well known that $\mathcal{F}$ admits at least a minimum point $u \in \mathcal{K}$, because $\mathcal{F}$ is bounded from below and $L^1(\Omega)$-lower semicontinuous in $\mathcal{K}$ [13, Proposition 1.2]. We stress that,

> *if $u \in \mathcal{K}$ is a minimum point of $\mathcal{F}$ in $\mathcal{K}$, then, for a.e. $t \in [-1,1]$, the characteristic function $\chi_{\{u>t\}} \in \widetilde{\mathcal{K}}$ is a minimum point of $\mathcal{F}$ in $\widetilde{\mathcal{K}}$.*

In fact, using the coarea formula [11, Theorem 1.23] and the Cavalieri formula, we get $\mathcal{F}(v) = \frac{1}{2}\int_{-1}^1 \mathcal{F}(\chi_{\{v>t\}})dt$, that is $\int_{-1}^1 (\mathcal{F}(\chi_{\{v>t\}}) - \mathcal{F}(v))dt = 0$, for all $v \in \mathcal{K}$. Then, the minimality of u in $\mathcal{K}$ entails $\mathcal{F}(\chi_{\{u>t\}}) - \mathcal{F}(u) \geq 0$, whence $\mathcal{F}(u) = \mathcal{F}(\chi_{\{u>t\}})$, for a.e. $t \in [-1,1]$. Hence, $\min_{v\in\widetilde{\mathcal{K}}} \mathcal{F}(v) = \min_{v\in\mathcal{K}} \mathcal{F}(v)$ and the minimization of $\mathcal{F}$ on $\widetilde{\mathcal{K}}$ is equivalent to minimize $\mathcal{F}$ on the convex closed space $\mathcal{K}$, which reads as a (nonstrictly) convex problem. Note that $\mathcal{F}$ may exhibit relative minima in $\widetilde{\mathcal{K}}$; in view of the convexity of $\mathcal{K}$, they are no longer relative minima of $\mathcal{F}$ in $\mathcal{K}$. Moreover, $\mathcal{F}$ has a unique minimum point in $\widetilde{\mathcal{K}}$ if and only if $\mathcal{F}$ has a unique minimum point in $\mathcal{K}$, and they coincide.

1.2. The regularized functionals. For any $\varepsilon > 0$, the regularized functional reads:

$$(1.2) \qquad \mathcal{F}_\varepsilon(v) := \int_\Omega \sqrt{\varepsilon^2 + |Dv|^2} + \int_{\partial\Omega} \mu v \, d\mathcal{H}^{n-1} - \int_\Omega \kappa v \, dx, \qquad \forall\, v \in \mathcal{K}.$$

Since $\mathcal{F}_\varepsilon$ is bounded from below and $L^1(\Omega)$-lower semicontinuous in $\mathcal{K}$ [12, §3.8, Theorem 11], $\mathcal{F}_\varepsilon$ has a minimum point $u_\varepsilon \in \mathcal{K}$. Moreover, since $\mathcal{F}_\varepsilon$ is strictly convex in $\mathcal{K} \cap W^{1,1}_{\rm loc}(\Omega)/\mathbf{R}$, its minimum is unique up to a possible additive constant. More precisely, u_ε is unique if and only if either $\int_\Omega \kappa \neq \int_{\partial\Omega} \mu$ or $\sup_\Omega u_\varepsilon = 1$ and $\inf_\Omega u_\varepsilon = -1$. If u_ε is regular, then it satisfies the following variational inequality with Neumann boundary conditions:

$$\int_\Omega \frac{\nabla u_\varepsilon}{\sqrt{\varepsilon^2 + |\nabla u_\varepsilon|^2}} \cdot \nabla(v - u_\varepsilon) dx + \int_{\partial\Omega} \mu(v - u_\varepsilon) d\mathcal{H}^{n-1} - \int_\Omega \kappa(v - u_\varepsilon) dx \geq 0,$$

for all $v \in H^1(\Omega, [-1,1])$ (see, e.g., [11,12]). Note that, with no further assumptions (e.g., Ω convex), the minimum u_ε of $\mathcal{F}_\varepsilon$ is, in general, just a bounded variation function (see [11, Example 12.15]; if $\Omega \subset \mathbf{R}^2$ is an annulus with boundaries Γ_1 and Γ_2, $\mu = 1$ on Γ_1, $\mu = -1$ on Γ_2, and $\kappa = 0$, then $\int_{\partial\Omega} \mu v d\mathcal{H}^1 = \int_{\partial\Omega} |v + \mu| d\mathcal{H}^1 - \mathcal{H}^1(\partial\Omega)$, and our minimum problem corresponds to the Dirichlet problem for the area functional, with boundary datum $-\frac{\mu}{\varepsilon}$, suggested by Giusti as an example of nonexistence of classical solutions).

1.3. The discrete functionals. Let $\{\mathcal{S}_h\}_{h>0}$ denote a *regular* family of partitions of Ω into simplices [5, p.132]. Let $h_S \leq h$ denote the diameter of any $S \in \mathcal{S}_h$. Let $\mathbf{V}_h \subset H^1(\Omega, [-1,1]) \subset \mathcal{K}$ be the piecewise linear finite element space over $\mathcal{S}_h$ with values in $[-1,1]$ and Π_h be the usual Lagrange interpolation operator. For the sake of simplicity, we assume that $\bar{\Omega} = \cup_{S \in \mathcal{S}_h} S$. We approximate μ and κ by continuous piecewise linear functions μ_h and κ_h, respectively, so that [5]

$$(1.3) \qquad \|\mu_h\|_{L^\infty(\partial\Omega)} \leq 1, \quad \|\nabla \mu_h\|_{L^1(\partial\Omega)} = o(h^{-1}), \quad \mu_h \underset{h\to 0}{\to} \mu \text{ in } L^1(\partial\Omega),$$

$$(1.4) \qquad \|\kappa_h\|_{L^\infty(\Omega)} \leq \|\kappa\|_{L^\infty(\Omega)}, \quad \|\nabla \kappa_h\|_{L^1(\Omega)} = o(h^{-1}), \quad \kappa_h \underset{h\to 0}{\to} \kappa \text{ in } L^1(\Omega).$$

We define the discrete functionals as follows:

$$(1.5) \quad \mathcal{F}_h(v) := \int_\Omega |\nabla v| dx + \int_{\partial\Omega} \Pi_h(\mu_h v) \, d\mathcal{H}^{n-1} - \int_\Omega \Pi_h(\kappa_h v) \, dx, \qquad \forall\, v \in \mathbf{V}_h,$$

$$(1.6) \quad \mathcal{F}_{\varepsilon,h}(v) := \int_\Omega \sqrt{\varepsilon^2 + |\nabla v|^2} dx + \int_{\partial\Omega} \Pi_h(\mu_h v) d\mathcal{H}^{n-1} - \int_\Omega \Pi_h(\kappa_h v) dx, \quad \forall\, v \in \mathbf{V}_h.$$

Since $\mathcal{F}_h$ and $\mathcal{F}_{\varepsilon,h}$ are continuous over a compact subset of a finite dimensional space, they admit a minimum point. Since $\mathcal{F}_{\varepsilon,h}$ is strictly convex in $\mathbf{V}_h/\mathbf{R}$, its minimum is unique up to a possible additive constant.

The quadrature formulae in (1.5) and (1.6) allow the direct implementation on a computer of the minimization of $\mathcal{F}_h$ and $\mathcal{F}_{\varepsilon,h}$. Implementation details will appear in [3].

2. Semicontinuity

Just for the sake of completeness, we show here the lower semicontinuity of both functionals $\mathcal{F}$ and $\mathcal{F}_\varepsilon$ in $\mathcal{K}$ with respect to the $L^1(\Omega)$-topology (see also [12,13]).

We give a unified proof for both functionals $\mathcal{F}$ and $\mathcal{F}_\varepsilon$, considering $\mathcal{F}=\mathcal{F}_\varepsilon$ with $\varepsilon=0$. Hence, let $\varepsilon \geq 0$ be fixed. First, we approximate $\mu \in L^\infty(\partial\Omega;[-1,1])$ by a sequence of piecewise constant functions $\{\mu^k\}_{k\in\mathbb{N}}$, so that $\mu^k \to \mu$ in $L^1(\partial\Omega)$, as $k \to \infty$. Denoting by $\mathcal{F}_\varepsilon^k$ the functional involving μ^k, we have $|\mathcal{F}_\varepsilon^k(v)-\mathcal{F}_\varepsilon(v)| \leq \|\mu^k-\mu\|_{L^1(\partial\Omega)}$, for all $v\in\mathcal{K}$, namely, $\mathcal{F}_\varepsilon^k \to \mathcal{F}_\varepsilon$ uniformly in $\mathcal{K}$, as $k \to \infty$. The assertion is thus reduced to prove that, for any k, $\mathcal{F}_\varepsilon^k$ is semicontinuous in $\mathcal{K}$. Since no confusion is possible, we omit the superscript k. Then, let μ be a piecewise constant function with values $-1=\mu_0<\mu_1<\ldots<\mu_N=1$ and set $\nu_i := \frac{\mu_i-\mu_{i-1}}{2}$ and $G_i := \{\mu \geq \mu_i\} \subseteq \partial\Omega$, for all $1 \leq i \leq N$ ($G_1=\partial\Omega$ and $G_N=\emptyset$ are allowed). Since $\sum_{i=1}^N \nu_i = 1$ and $\mu(x) = \sum_{i=1}^N \nu_i \chi_{G_i}(x)$, for all $x \in \partial\Omega$, $\mathcal{F}_\varepsilon$ can be decomposed as a convex combination of functionals $\mathcal{F}_\varepsilon^i$ defined by:

$$\mathcal{F}_\varepsilon(v) = \sum_{i=1}^N \nu_i \left[\int_\Omega \sqrt{\varepsilon^2+|Dv|^2} + \int_{\partial\Omega} \chi_{G_i} v d\mathcal{H}^{n-1} - \int_\Omega \kappa v dx \right] =: \sum_{i=1}^N \nu_i \mathcal{F}_\varepsilon^i(v).$$

Hence, we finally have to show that, for any $1 \leq i \leq N$, $\mathcal{F}_\varepsilon^i$ is semicontinuous. Given a ball B containing $\bar\Omega$, let $\widetilde{\chi}_i \in W^{1,1}(B\setminus\bar\Omega;[-1,1])$ be a function with trace $-\chi_{G_i}$ on $\partial\Omega$ [10, Theorem 1.II; 11, Theorem 2.16]. If, for any $v\in\mathcal{K}$, we define $v_i \in BV(B;[-1,1])$ by $v_i(x) := v(x)$ if $x\in\Omega$, $:= \widetilde{\chi}_i(x)$ if $x \in B\setminus\Omega$, and set $C_i := \int_{B\setminus\bar\Omega} \sqrt{\varepsilon^2+|\nabla\widetilde{\chi}_i|^2}dx$, we have [11,§14.4] $\int_B \sqrt{\varepsilon^2+|Dv_i|^2} = \int_\Omega \sqrt{\varepsilon^2+|Dv|^2} + \int_{\partial\Omega} |v+\chi_{G_i}| d\mathcal{H}^{n-1} + C_i$. Hence, noting that $\int_{\partial\Omega} \chi_{G_i} v d\mathcal{H}^{n-1} = \int_{\partial\Omega} |v+\chi_{G_i}| d\mathcal{H}^{n-1} - \mathcal{H}^{n-1}(\partial\Omega)$, we get $\mathcal{F}_\varepsilon^i(v) = \int_B \sqrt{\varepsilon^2+|Dv_i|^2} - \int_\Omega \kappa v dx - C_i - \mathcal{H}^{n-1}(\partial\Omega)$, and the semicontinuity of $\mathcal{F}_\varepsilon^i$ follows from the $L^1(\Omega)$-lower semicontinuity of the total variation and the area in $BV(\Omega)$ [11, Theorems 1.9 and 14.2].

Remark 2.1. For any $\varepsilon \geq 0$, the functional $\mathcal{F}_\varepsilon$ is not $L^1(\Omega)$-lower semicontinuous in $\mathcal{K}$ if $\mu \notin L^\infty(\partial\Omega;[-1,1])$. In fact, let $\mu(x)>1$ for a.e. $x\in\partial\Omega\cap B$, for some ball B. Set $B_\Omega := \Omega\cap B \neq \emptyset$ and $B_{\partial\Omega} := \partial\Omega\cap B$. Let $\{B_k \subset B_\Omega\}_{k\in\mathbb{N}}$ be a sequence of sets of finite perimeter in Ω, so that $\partial B_k \cap \partial\Omega = B_{\partial\Omega}$, $\lim_{k\to\infty}|B_k|=0$, and $\lim_{k\to\infty} P(B_k,\Omega) = \mathcal{H}^{n-1}(B_{\partial\Omega})$. Let $\{v_k := -\frac{1}{2}\chi_{B_k}(\chi_{B_\Omega}+1)\}_k$ be a sequence converging to $v := \frac{1}{2}(\chi_{B_\Omega}+1)$ in $L^1(\Omega)$, as $k\to\infty$. Then, noting that $\mathcal{F}_\varepsilon(v) = P(B_\Omega,\Omega) + \varepsilon|\Omega| + \int_{B_{\partial\Omega}} \mu - \int_\Omega \kappa v$ and $\mathcal{F}_\varepsilon(v_k) = P(B_\Omega,\Omega) + \varepsilon|\Omega| + 2P(B_k,\Omega) - \int_{B_{\partial\Omega}} \mu - \int_\Omega \kappa v_k$, we have $\mathcal{F}_\varepsilon(v) > \liminf_{k\to\infty} \mathcal{F}_\varepsilon(v_k)$.

3. Convergence

We shall prove the diagram of convergence (0.1). First, we note that

$\lim_{\varepsilon\to 0}\mathcal{F}_\varepsilon=\mathcal{F}$ *uniformly in* $\mathcal{K}$, $\lim_{\varepsilon\to 0}\mathcal{F}_{\varepsilon,h}=\mathcal{F}_h$ *uniformly in* $\mathbf{V}_h$ *and with respect to* h. In fact, since $0\le\int_\Omega\sqrt{\varepsilon^2+|Dv|^2}-\int_\Omega|Dv|=\varepsilon(\int_\Omega\sqrt{1+|D(\frac{v}{\varepsilon})|^2}-\int_\Omega|D(\frac{v}{\varepsilon})|)\le\varepsilon|\Omega|$, for all $v\in BV(\Omega)$, we have

$$(3.1)\quad |\mathcal{F}_\varepsilon(v)-\mathcal{F}(v)|\le\varepsilon|\Omega|,\ \forall\, v\in\mathcal{K},\qquad\text{and}\qquad |\mathcal{F}_{\varepsilon,h}(v)-\mathcal{F}_h(v)|\le\varepsilon|\Omega|,\ \forall\, v\in\mathbf{V}_h.$$

THEOREM 3.1. $\Gamma\text{-}\lim_{h\to 0}\mathcal{F}_h=\mathcal{F}$ *and* $\Gamma\text{-}\lim_{h\to 0}\mathcal{F}_{\varepsilon,h}=\mathcal{F}_\varepsilon$, *in* $L^1(\Omega)$.

Proof. The functionals $\mathcal{F}$ and $\mathcal{F}_\varepsilon$ ($\mathcal{F}_h$ and $\mathcal{F}_{\varepsilon,h}$, respectively) are set to $+\infty$ in $L^1(\Omega)\backslash\mathcal{K}$ ($L^1(\Omega)\backslash\mathbf{V}_h$, respectively). We give a unified proof for both cases $\varepsilon>0$ and $\varepsilon=0$, considering $\mathcal{F}_h=\mathcal{F}_{\varepsilon,h}$ and $\mathcal{F}=\mathcal{F}_\varepsilon$ with $\varepsilon=0$. Hence, let $\varepsilon\ge 0$ be fixed. We prove [7] that:

(i). For any $v\in L^1(\Omega)$ and any sequence $\{v_h\in L^1(\Omega)\}_h$ converging to v in $L^1(\Omega)$, as $h\to 0$, we have $\mathcal{F}_\varepsilon(v)\le\liminf_{h\to 0}\mathcal{F}_{\varepsilon,h}(v_h)$.

(ii). For any $v\in L^1(\Omega)$ there exists a sequence $\{v_h\in L^1(\Omega)\}_h$ converging to v in $L^1(\Omega)$, as $h\to 0$, such that $\mathcal{F}_\varepsilon(v)=\lim_{h\to 0}\mathcal{F}_{\varepsilon,h}(v_h)$.

Preliminarly, we decompose $\mathcal{F}_{\varepsilon,h}(v_h)$, for all $v_h\in\mathbf{V}_h$, as follows:

$$(3.2)\qquad \mathcal{F}_{\varepsilon,h}(v_h)=\mathcal{F}_\varepsilon(v_h)+\int_{\partial\Omega}[\Pi_h(\mu_h v_h)-\mu v_h]\,d\mathcal{H}^{n-1}-\int_\Omega[\Pi_h(\kappa_h v_h)-\kappa v_h]\,dx,$$

which reads as $\mathcal{F}_{\varepsilon,h}(v_h)=:\mathcal{F}_\varepsilon(v_h)+I+II$, and split I and II as follows:

$$I=\int_{\partial\Omega}[\Pi_h(\mu_h v_h)-\mu_h v_h]\,d\mathcal{H}^{n-1}+\int_{\partial\Omega}(\mu_h-\mu)v_h d\mathcal{H}^{n-1}=:I_1+I_2,$$
$$II=\int_\Omega[\Pi_h(\kappa_h v_h)-\kappa_h v_h]\,dx+\int_\Omega(\kappa_h-\kappa)v_h dx=:II_1+II_2.$$

Since $|v_h|\le 1$ in $\bar\Omega$, we have $|I_2|\le\|\mu_h-\mu\|_{L^1(\partial\Omega)}$ and $|II_2|\le\|\kappa_h-\kappa\|_{L^1(\Omega)}$. In view of basic properties of the interpolation operator Π_h, and using the local inverse inequality $\|\nabla v_h\|_{L^\infty(T)}\le Ch_S^{-1}\|v_h\|_{L^\infty(T)}$, where either $T=\partial S$ or $T=S\in\mathcal{S}_h$ and $v_h\in\mathbf{V}_h$ [5, p.140], we get

$$\begin{aligned}|I_1|&\le C\sum_{S\in\mathcal{S}_h}h_S^2\|D^2(\mu_h v_h)\|_{L^1(\partial S\cap\partial\Omega)}\\&\le C\sum_{S\in\mathcal{S}_h}h_S^2\|\nabla\mu_h\cdot\nabla v_h\|_{L^1(\partial S\cap\partial\Omega)}\le Ch\|\nabla\mu_h\|_{L^1(\partial\Omega)}\end{aligned}$$

and, similarly, $|II_1|\le Ch\|\nabla\kappa_h\|_{L^1(\Omega)}$. Hence, using (1.3) and (1.4), for any sequence $\{v_h\in\mathbf{V}_h\}_h$, we obtain

$$(3.3)\qquad \lim_{h\to 0}[|I|+|II|]=0.$$

Proof of Step (i). Let $v \in L^1(\Omega)$ and $\{v_h \in L^1(\Omega)\}_h$ be any sequence so that $\lim_{h\to 0} v_h = v$ in $L^1(\Omega)$. We can assume that $v_h \in \mathbf{V}_h$, for any h. Then, from the lower semicontinuity of $\mathcal{F}_\varepsilon$, (3.2), and (3.3), we conclude

$$\mathcal{F}_\varepsilon(v) \le \liminf_{h\to 0} \mathcal{F}_\varepsilon(v_h) = \liminf_{h\to 0} \mathcal{F}_{\varepsilon,h}(v_h).$$

Proof of Step (ii). We can assume that $v \in \mathcal{K}$. Given a ball B containing $\bar\Omega$, let $\widetilde{v} \in W^{1,1}(B\setminus\bar\Omega;[-1,1])$ be a function with trace v on $\partial\Omega$ [10] and denote again by $v \in BV(B;[-1,1])$ the function $v(x) := v(x)$ if $x \in \Omega$, $:= \widetilde{v}(x)$ if $x \in B\setminus\Omega$. Let $\eta_h = o(h^{-1/2})$ and $\{\delta_h\}_h$ be a family of mollifiers defined by $\delta_h(x) := \eta_h^n \delta(\eta_h x)$. Set $\widehat{v}_h(x) := (v \star \delta_h)(x)$, for all $x \in B$, where v is extended to 0 outside B. It is well known [11, Proposition 1.15] that

$$\lim_{h\to 0} \|\widehat{v}_h - v\|_{L^1(\Omega)} = 0 \qquad \text{and} \qquad \lim_{h\to 0} \int_\Omega |\nabla \widehat{v}_h| dx = \int_\Omega |Dv|. \tag{3.4}$$

We claim that the sequence $\{v_h\}_h$ for Step (ii) can be defined by $v_h := \Pi_h \widehat{v}_h \in \mathbf{V}_h$. In fact, noting that $|D^2 \widehat{v}_h| \le \eta_h^2$ in Ω, using well known properties of Π_h, we have $\|v_h - \widehat{v}_h\|_{W^{1,1}(\Omega)} \le C\|D^2\widehat{v}_h\|_{L^1(\Omega)} [h^2 + h] = o(1)$. Hence, since $|\int_\Omega |\nabla v_h| dx - \int_\Omega |\nabla \widehat{v}_h| dx| \le \int_\Omega |\nabla(v_h - \widehat{v}_h)| dx$, in view of (3.4) we obtain

$$\lim_{h\to 0} \|v_h - v\|_{L^1(\Omega)} = 0 \qquad \text{and} \qquad \lim_{h\to 0} \int_\Omega |\nabla v_h| dx = \int_\Omega |Dv|.$$

This entails [11, Theorem 2.11] $\lim_{h\to 0} \int_{\partial\Omega} |v_h - v| d\mathcal{H}^{n-1} = 0$ and, using the inequality $|\int_\Omega \sqrt{\varepsilon^2 + |Dv|^2} - \int_\Omega \sqrt{\varepsilon^2 + |\nabla v_h|^2} dx| \le |\int_\Omega |Dv| - \int_\Omega |\nabla v_h| dx|$, (3.2), and (3.3), gives

$$\mathcal{F}_\varepsilon(v) = \lim_{h\to 0} \mathcal{F}_\varepsilon(v_h) = \lim_{h\to 0} \mathcal{F}_{\varepsilon,h}(v_h). \quad \square$$

A straightforward consequence of (3.1) and Theorem 3.1 is the following Γ-convergence result for $\mathcal{F}_{\varepsilon,h}$, as ε and h go to 0 independently.

COROLLARY 3.1. $\Gamma\text{-}\lim_{(\varepsilon,h)\to(0,0)} \mathcal{F}_{\varepsilon,h} = \mathcal{F}$, *in* $L^1(\Omega)$.

Proof. (i). Let $v \in L^1(\Omega)$ and $\{v_{\varepsilon,h} \in L^1(\Omega)\}_{\varepsilon,h}$ be any sequence converging to v in $L^1(\Omega)$, as $(\varepsilon,h) \to (0,0)$. Using Theorem 3.1, Step (i), for $\mathcal{F}_h$, and (3.1), we get $\mathcal{F}(v) \le \liminf \mathcal{F}_h(v_{\varepsilon,h}) = \liminf \mathcal{F}_{\varepsilon,h}(v_{\varepsilon,h})$, as $(\varepsilon,h) \to (0,0)$.
(ii). Let $v \in \mathcal{K}$ and $\{v_h \in \mathbf{V}_h\}_h$ be the sequence constructed in Step (ii) of Theorem 3.1, for $\mathcal{F}_h$. Then, using (3.1), we get $\mathcal{F}(v) = \lim_{h\to 0} \mathcal{F}_h(v_h) = \lim_{(\varepsilon,h)\to(0,0)} \mathcal{F}_{\varepsilon,h}(v_h)$. $\square$

REMARK 3.1. Let $u_{\varepsilon,h}$ be a minimum of $\mathcal{F}_{\varepsilon,h}$. We have $\mathcal{F}_{\varepsilon,h}(u_{\varepsilon,h}) \le \mathcal{F}_{\varepsilon,h}(0) = \varepsilon|\Omega|$, whence, using (1.3) and (1.4), $\int_\Omega |Du_{\varepsilon,h}| \le \int_\Omega \sqrt{\varepsilon^2 + |Du_{\varepsilon,h}|^2} \le \mathcal{H}^{n-1}(\partial\Omega) + |\Omega|(1 + \|\kappa\|_{L^\infty(\Omega)})$, for all $0 \le \varepsilon \le 1$ and $h > 0$. Then, by the compactness theorem in $BV(\Omega)$

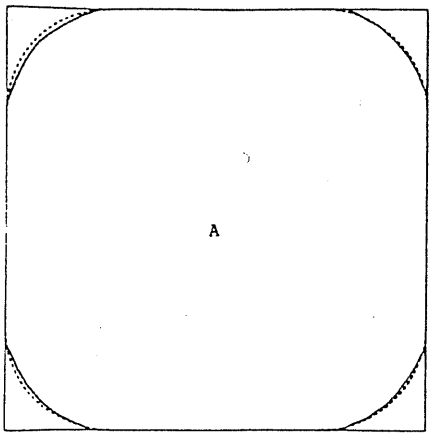

Fig. 4.1. – Ex. 1: Exact (dashed lines) and computed (solid lines) minimum.

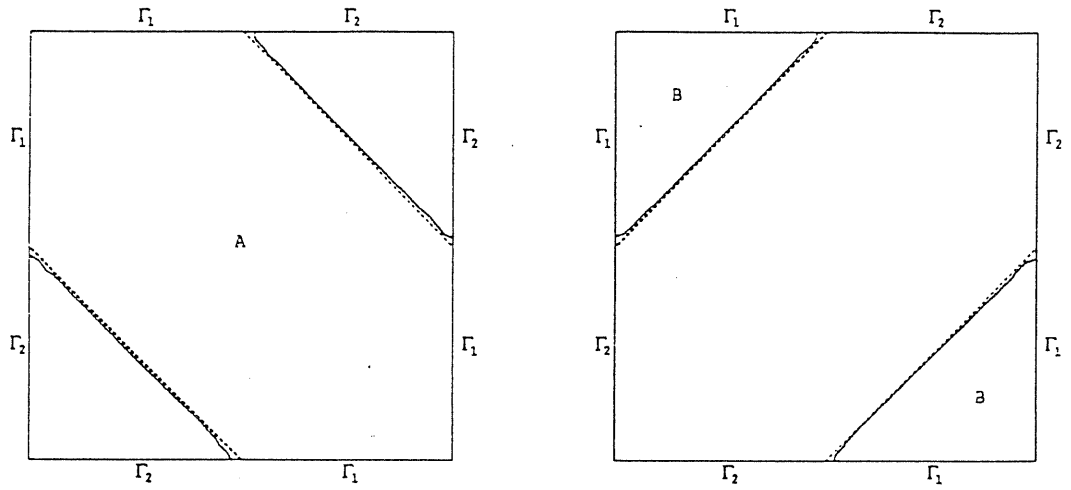

Fig. 4.2. – Ex. 2: Exact (dashed lines) and computed (solid lines) minima.

[11, Theorem 1.19], the family $\{u_{\varepsilon,h}\}_{\varepsilon,h}$ admits a subsequence converging to some $u \in \mathcal{K}$ in $L^1(\Omega)$. Corollary 3.1 entails that u is a minimum point of $\mathcal{F}$.

4. Numerical experiments

Implementation details on the minimization algorithms can be found in [3]. Here we simply present a couple of numerical examples. The unique discrete absolute minimum $u_{\varepsilon,h}$ of $\mathcal{F}_{\varepsilon,h}$ is approximated by Newton-like iterations. A quasiuniform mesh is used.

Example 1. Let $\Omega := (-2,2)^2$, $\mu := 1$ (tangential contact at $\partial\Omega$), and $\kappa := 1$. The functional $\mathcal{F}$ has one absolute minimum, $A := \{([|x_1| - 1]_+)^2 + ([|x_2| - 1]_+)^2 \leq 1\}$. Fig. 4.1 shows both the exact minimum (dashed lines) and the computed one $A_{\varepsilon,h} := \{u_{\varepsilon,h} > 0\}$ (solid lines). Here $\varepsilon = 0.2$ and $h = 0.14$; the initial guess is the empty set, which is a relative minimum of $\mathcal{F}$ in $\widetilde{\mathcal{K}}$! Note that, using the approximation via double well potential, the discrete minimizing set presents no contact with $\partial\Omega$, because the relaxed solution forms a transition layer across the interface. This effect is absent in our convex approximations

which, in turn, exhibits higher accuracy.

Example 2. Let $\Omega := (-2,2)^2$ and $\{\Gamma_1, \Gamma_2\}$ be the partition of $\partial\Omega$ defined by $\Gamma_1 := \partial\Omega \cap \{x_1 x_2 \leq 0\}$ and $\Gamma_2 := \partial\Omega \setminus \Gamma_1$. Let $\mu := -1$ on Γ_1, $:= 1$ on Γ_2, and $\kappa := 0$. Set $\mu_h := \Pi_h \mu$. The functional $\mathcal{F}$ has two absolute minima in $\widetilde{\mathcal{K}}$, A and B, shown in Fig. 4.2 (dashed lines). The computed minima are obtained from the unique discrete minimum $u_{\varepsilon,h}$ (the initial guess is the empty set) as $A_{\varepsilon,h} := \{u_{\varepsilon,h} > -0.5\}$ and $B_{\varepsilon,h} := \{u_{\varepsilon,h} > 0.5\}$ whereas, in Section 3.2, they where obtained iterating from different initial guesses. Here $\varepsilon = 0.2$ and $h = 0.14$.

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NUMERICAL MINIMIZATION OF FUNCTIONALS WITH CURVATURE BY CONVEX APPROXIMATIONS*

1. Introduction. We address the numerical minimization of the functional

$$(1.1) \qquad \mathcal{F}(v) := \int_\Omega |Dv| + \int_{\partial\Omega} \mu v \, d\mathcal{H}^{n-1} - \int_\Omega \kappa v \, dx, \qquad \forall\, v \in BV(\Omega;\{-1,1\}),$$

where $\Omega \subset \mathbf{R}^n$ $(n \geq 2)$ is a $C^{0,1}$ open bounded set, $\kappa \in L^\infty(\Omega)$, and $\mu \in L^\infty(\partial\Omega;[-1,1])$. It is well known [11] that any minimum of $\mathcal{F}$ is the characteristic function of a set $A \subseteq \Omega$ whose boundary has prescribed mean curvature κ and contact angle $\arccos(\mu)$ at $\partial\Omega$.

The minimization of the functional $\mathcal{F}$ is just viewed as a model of geometrical type problems in the calculus of variations, that involve unknown interfaces and the related surface energy. Such problems may arise in important applicative fields, such as phase transition theories [5] and computer vision theory [20] (see also [7]), and their numerical solution seems quite difficult, because of the lack of convexity and regularity of the functionals at hand (see, e.g., [1,2,4,19]).

The relaxation of the functional $\mathcal{F}$ via a double well potential has been proposed in [18]. This leads to the numerical minimization of the regular, nonconvex, functionals

$$(1.2) \qquad \mathcal{G}_\sigma(v) := \int_\Omega \left[\sigma|\nabla v|^2 + \frac{1}{\sigma}(1-v^2) - \frac{\pi}{2}\kappa v\right] dx, \qquad \forall\, v \in H^1(\Omega;[-1,1]),$$

under suitable prescribed boundary conditions. In the previous sections, we have discretized $\mathcal{G}_\sigma$ by linear finite elements and proved, under the relation $h = o(\sigma)$ between the meshsize h and the relaxation parameter σ, the Γ-convergence of the discrete functionals $\mathcal{G}_{\sigma,h}$ to $\mathcal{F}$ in $L^1(\Omega)$, as $\sigma \to 0$.

A different approximation has been investigated in [3]. Noting that the minima of $\mathcal{F}$ in $BV(\Omega;\{-1,1\})$ can be found in terms of the minima of $\mathcal{F}$ on the larger, closed, convex, space $BV(\Omega;[-1,1])$, the (nonstrict) convexity of $\mathcal{F}$ can be exploited for its numerical minimization via finite element discretizations. Since the numerical algorithms perform better for strictly convex regular functionals, $\mathcal{F}$ is preliminarly regularized by

$$(1.3) \quad \mathcal{F}_\varepsilon(v) = \int_\Omega \sqrt{\varepsilon^2 + |Dv|^2} + \int_{\partial\Omega} \mu v \, d\mathcal{H}^{n-1} - \int_\Omega \kappa v \, dx, \qquad \forall\, v \in BV(\Omega;[-1,1]),$$

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which, in turn, are discretized by continuous linear finite elements. Letting ε and h go to 0 independently, in [3] we proved that $\Gamma\text{-lim}\,\mathcal{F}_{\varepsilon,h} = \mathcal{F}$ in $L^1(\Omega)$, so that any family of discrete minima $\{u_{\varepsilon,h}\}_{\varepsilon,h}$ admits a subsequence converging to a minimum point u of $\mathcal{F}$.

We stress that no relation between ε and h is required for the limit procedure and the stability of the convex minimization algorithm, whereas the nonconvex approximation converges if $h = o(\sigma)$ and is even unstable if $h > C^\star\sigma$, where $C^\star \leq 1$ is a precise stability constant depending on the finite element mesh. Moreover, the relaxed solutions exhibit a $\mathcal{O}(\sigma)$-wide transition layer across the interfaces, which may make the numerical approximation quite inaccurate; this effect is absent in the convex approximations (see §5).

In §4 we describe the constrained Newton-like algorithm used for the minimization of the convex functionals $\mathcal{F}_{\varepsilon,h}$ and, in §5, we present several numerical experiments.

2. The setting. We state precisely the continuous and discrete functionals and recall some basic properties. Let $\mathcal{H}^{n-1}$ denote the $(n-1)$-dimensional Hausdorff measure in $\mathbf{R}^n$ [10] and $|\cdot|$ be the Lebesgue measure in $\mathbf{R}^n$. Let $BV(\Omega)$ be the space of the bounded variation functions in Ω and denote by $v \in L^1(\partial\Omega)$ the trace of $v \in BV(\Omega)$ on $\partial\Omega$. Let $\int_\Omega |Dv|$ denote the *total variation* in Ω and $\int_\Omega \sqrt{1+|Dv|^2}$ the *area* of any function $v \in BV(\Omega)$ [14, Defs. 1.1, 14.1]. For any set $E \subseteq \Omega$, let χ_E be its characteristic function, namely, $\chi_E(x) := 1$ if $x \in E$, $\chi_E(x) := -1$ if $x \in \Omega \setminus E$. Let $\widetilde{\mathcal{K}} := BV(\Omega;\{-1,1\})$ be the space of the characteristic functions of sets of finite perimeter in Ω and $\mathcal{K} := BV(\Omega;[-1,1])$. Finally, for any $v \in BV(\Omega)$, set $\{v > t\} := \{x \in \Omega : v(x) > t\}$ and note that $\chi_{\{v>t\}} \in \widetilde{\mathcal{K}}$, for a.e. $t \in \mathbf{R}$ [14].

The original functional $\mathcal{F}$, defined in (1.1) for $v \in BV(\Omega)$, admits at least a minimum point $u \in \mathcal{K}$, because $\mathcal{F}$ is bounded from below and lower semicontinuous in $\mathcal{K}$ with respect to the topology of $L^1(\Omega)$ [3,17]. We stress that $\min_{v\in\widetilde{\mathcal{K}}} \mathcal{F}(v) = \min_{v\in\mathcal{K}} \mathcal{F}(v)$ and that the minimum points of $\mathcal{F}$ in $\mathcal{K}$ give the minimum points of $\mathcal{F}$ in $\widetilde{\mathcal{K}}$, because [3]:

> *if $u \in \mathcal{K}$ is a minimum point of $\mathcal{F}$ in $\mathcal{K}$, then, for a.e. $t \in [-1,1]$, the characteristic function $\chi_{\{u>t\}} \in \widetilde{\mathcal{K}}$ is a minimum point of $\mathcal{F}$ in $\widetilde{\mathcal{K}}$ and $\mathcal{F}(u) = \mathcal{F}(\chi_{\{u>t\}})$.*

In fact, using the coarea formula [14, Thm. 1.23] and the Cavalieri formula, we get

$$\mathcal{F}(v) = \frac{1}{2}\int_{-1}^{1} \mathcal{F}(\chi_{\{v>t\}})dt, \quad \text{that is} \quad \int_{-1}^{1} (\mathcal{F}(\chi_{\{v>t\}}) - \mathcal{F}(v))dt = 0, \qquad \forall\, v \in \mathcal{K}.$$

Then, the minimality of u in $\mathcal{K}$ entails $\mathcal{F}(\chi_{\{u>t\}}) - \mathcal{F}(u) \geq 0$, whence $\mathcal{F}(u) = \mathcal{F}(\chi_{\{u>t\}})$, for a.e. $t \in [-1,1]$. In particular, $\mathcal{F}$ has a unique minimum point in $\widetilde{\mathcal{K}}$ if and only if $\mathcal{F}$ has a unique minimum in $\mathcal{K}$, and they coincide. Note that $\mathcal{F}$ may have relative minima in $\widetilde{\mathcal{K}}$: in view of the convexity of $\mathcal{F}$ in $\mathcal{K}$, they are no longer relative minima of $\mathcal{F}$ in $\mathcal{K}$.

Let us present some examples.

Example 1. In the trivial case $\mu, \kappa := 0$, $\chi_\emptyset$ and χ_Ω are the absolute minima of $\mathcal{F}$ in $\widetilde{\mathcal{K}}$ so that, for any $0 \leq \lambda \leq 1$, $u^\lambda := 2\lambda - 1$ is an absolute minimum of $\mathcal{F}$ in $\mathcal{K}$. Any "minimum diameter" of Ω corresponds to a relative minimum of $\mathcal{F}$ in $\widetilde{\mathcal{K}}$.

Example 2. Let $\Omega := (-2,2)^2$, $\mu := 0$ on $|x_2| = 2$, $\mu = 1$ on $x_1 = -2$, $\mu := -1$ on $x_1 = 2$, and $\kappa := 0$. Then, for any $-2 \le a \le 2$, $u^a := \chi_{\{x_1 \ge a\}}$ is an absolute minimum of $\mathcal{F}$ in $\widetilde{\mathcal{K}}$, whereas any x_1-nondecreasing x_2-independent function is a minimum of $\mathcal{F}$ in $\mathcal{K}$.

Example 3. Let $\Omega := (-2,2)^2$, $\mu := 1$ (tangential contact at $\partial\Omega$), and $\kappa := r \in \mathbf{R}^+$. The functional $\mathcal{F}$ has one absolute minimum, χ_{A^r}, where $A^r := \{([|x_1| - (2 - \frac{1}{r})]_+)^2 + ([|x_2| - (2 - \frac{1}{r})]_+)^2 \le \frac{1}{r^2}\}$, for any $r > \frac{2+\sqrt{\pi}}{4} =: r^\star$, and $\chi_\emptyset$, for any $0 < r < r^\star$; for $r = r^\star$, both $\chi_{A^{r^\star}}$ and $\chi_\emptyset$ are absolute minima in $\widetilde{\mathcal{K}}$, so that, for any $0 \le \lambda \le 1$, $u^\lambda := \lambda(\chi_{A^{r^\star}} + 1) - 1$ is a minimum of $\mathcal{F}$ in $\mathcal{K}$. Note that $\chi_\emptyset$ is a relative minimum of $\mathcal{F}$ in $\widetilde{\mathcal{K}}$, for any $r > r^\star$.

Example 4. Let $\Omega := (-2,2)^2$, $\mu := -1$ on $\Gamma_1 := \partial\Omega \cap \{x_1 x_2 \le 0\}$, $\mu := 1$ on $\Gamma_2 := \partial\Omega \setminus \Gamma_1$, and $\kappa := 0$. The functional $\mathcal{F}$ has two absolute minima in $\widetilde{\mathcal{K}}$, χ_A and χ_B, where $A := \Omega \cap \{|x_1 + x_2| \le 2\}$ and $B := \Omega \cap \{|x_2 - x_1| \ge 2\}$; then, for any $0 \le \lambda \le 1$, $u^\lambda := \chi_A + \lambda(\chi_B - \chi_A)$ is a minimum of $\mathcal{F}$ in $\mathcal{K}$.

Since the regularized functional $\mathcal{F}_\varepsilon$, defined in (1.3) for $v \in BV(\Omega)$, is bounded from below and $L^1(\Omega)$-lower semicontinuous in $\mathcal{K}$, $\mathcal{F}_\varepsilon$ has a minimum point $u_\varepsilon \in \mathcal{K} \cap W^{1,1}_{loc}(\Omega)$ [14,16]. Moreover, since $\mathcal{F}_\varepsilon$ is strictly convex in $(BV(\Omega) \cap W^{1,1}_{loc}(\Omega))/\mathbf{R}$, the minimum is unique up to a possible additive constant. More precisely, u_ε is unique if and only if either $\int_\Omega \kappa \ne \int_{\partial\Omega} \mu$ or $\sup_\Omega u_\varepsilon = 1$ and $\inf_\Omega u_\varepsilon = -1$. For instance, $\{u^\lambda_\varepsilon := u^\lambda = 2\lambda - 1\}_{0 \le \lambda \le 1}$ are the minima of $\mathcal{F}_\varepsilon$ in Ex. 1, whereas the minimum of $\mathcal{F}_\varepsilon$ is unique in Exs. 2,3,4, even if the identity $\int_\Omega \kappa = \int_{\partial\Omega} \mu$ holds in Exs. 2,4, and, for $r = 1$, in Ex. 3, because $\sup_\Omega u_\varepsilon = 1$ and $\inf_\Omega u_\varepsilon = -1$ (e.g., $u_\varepsilon := \frac{x_1}{2}$ is the unique absolute minimum of $\mathcal{F}_\varepsilon$ in Ex. 2).

Let $\{\mathcal{S}_h\}_{h>0}$ denote a *regular* family of partitions of Ω into simplices [6, p. 132] and $h_S \le h$ be the diameter of any $S \in \mathcal{S}_h$. For simplicity, we assume that $\bar{\Omega} = \cup_{S \in \mathcal{S}_h} S$. Let $\mathbf{V}_h \subset H^1(\Omega)$ be the piecewise linear finite element space over $\mathcal{S}_h$ and Π_h be the usual Lagrange interpolation operator. In addition, define $\mathcal{K}_h := \mathbf{V}_h \cap \mathcal{K} = \{v \in \mathbf{V}_h : |v| \le 1 \text{ in } \Omega\}$.

We approximate μ and κ by continuous, piecewise linear over $\mathcal{S}_h$, functions μ_h and κ_h, respectively, so that [6]

$$\|\mu_h\|_{L^\infty(\partial\Omega)} \le 1, \quad \|\nabla\mu_h\|_{L^1(\partial\Omega)} = o(h^{-1}), \quad \mu_h \underset{h\to 0}{\to} \mu \text{ in } L^1(\partial\Omega), \tag{2.1}$$

$$\|\kappa_h\|_{L^\infty(\Omega)} \le \|\kappa\|_{L^\infty(\Omega)}, \quad \|\nabla\kappa_h\|_{L^1(\Omega)} = o(h^{-1}), \quad \kappa_h \underset{h\to 0}{\to} \kappa \text{ in } L^1(\Omega). \tag{2.2}$$

For instance, letting $\{\delta_h\}_h$ be a family of mollifiers defined by $\delta_h(x) := \eta_h^n \delta(\eta_h x)$, where $\eta_h = o(h^{-1})$, we can define $\mu_h := \Pi_h(\mu \star \delta_h)$ and $\kappa_h := \Pi_h(\kappa \star \delta_h)$. If μ and κ are more regular, e.g., bounded variation piecewise continuous functions, then we can define $\mu_h := \Pi_h \mu$ and $\kappa_h := \Pi_h \kappa$ (and we have $\|\nabla\mu_h\|_{L^1(\partial\Omega)}, \|\nabla\kappa_h\|_{L^1(\Omega)} = \mathcal{O}(1)$).

We define the discrete functionals as follows:

$$\mathcal{F}_h(v) := \int_\Omega |\nabla v| dx + \int_{\partial\Omega} \Pi_h(\mu_h v)\, d\mathcal{H}^{n-1} - \int_\Omega \Pi_h(\kappa_h v)\, dx, \qquad \forall\, v \in \mathbf{V}_h, \tag{2.3}$$

$$\mathcal{F}_{\varepsilon,h}(v) := \int_\Omega \sqrt{\varepsilon^2 + |\nabla v|^2} dx + \int_{\partial\Omega} \Pi_h(\mu_h v) d\mathcal{H}^{n-1} - \int_\Omega \Pi_h(\kappa_h v) dx, \quad \forall\, v \in \mathbf{V}_h. \tag{2.4}$$

Since $\mathcal{F}_h$ and $\mathcal{F}_{\varepsilon,h}$ are continuous over $\mathbf{V}_h$, they admit minimum points u_h and $u_{\varepsilon,h} \in \mathcal{K}_h$, respectively, because $\mathcal{K}_h$ is a compact subset of a finite dimensional space. The minimum of $\mathcal{F}_h$ may be not unique. For instance, in Ex. 1 and, for a suitable mesh $\mathcal{S}_h$ and datum μ_h, in Exs. 2,4, $\mathcal{F}_h$ has a continuum of minima. On the other hand, since $\mathcal{F}_{\varepsilon,h}$ is strictly convex in $\mathbf{V}_h/\mathbf{R}$, the minimum $u_{\varepsilon,h}$ is unique up to a possible additive constant. More precisely, $u_{\varepsilon,h}$ is unique if and only if either $\int_\Omega \kappa_h \neq \int_{\partial\Omega} \mu_h$ or $\sup_\Omega u_{\varepsilon,h} = 1$ and $\inf_\Omega u_{\varepsilon,h} = -1$. For instance, in Exs. 2,3,4, the minimum $u_{\varepsilon,h}$ is unique; see §5.

3. Convergence. We recall the convergence results proved in [3], namely,

$$(3.1)\quad |\mathcal{F}_\varepsilon(v) - \mathcal{F}(v)| \le \varepsilon|\Omega|, \ \forall\, v \in \mathcal{K}, \qquad |\mathcal{F}_{\varepsilon,h}(v) - \mathcal{F}_h(v)| \le \varepsilon|\Omega|, \ \forall\, v \in \mathcal{K}_h, \ h>0,$$

$$(3.2)\quad |\mathcal{F}_h(v_h) - \mathcal{F}(v_h)| = o_h(1), \quad |\mathcal{F}_{\varepsilon,h}(v_h) - \mathcal{F}_\varepsilon(v_h)| = o_h(1), \quad \forall\, \{v_h \in \mathcal{K}_h\}_h, \ \varepsilon>0,$$

$$(3.3)\qquad \Gamma\text{-}\lim_{h\to 0} \mathcal{F}_h = \mathcal{F}, \qquad \Gamma\text{-}\lim_{h\to 0} \mathcal{F}_{\varepsilon,h} = \mathcal{F}_\varepsilon, \qquad \text{in } L^1(\Omega),$$

and, in particular, letting ε and h go to 0 independently,

$$(3.4)\qquad \Gamma\text{-}\lim_{(\varepsilon,h)\to(0,0)} \mathcal{F}_{\varepsilon,h} = \mathcal{F}, \qquad \text{in } L^1(\Omega).$$

The estimate (3.1) follows from the inequality [14]:

$$0 \le \int_\Omega \sqrt{\varepsilon^2 + |Dv|^2} - \int_\Omega |Dv| = \varepsilon \left(\int_\Omega \sqrt{1 + \left| D\left(\frac{v}{\varepsilon}\right)\right|^2} - \int_\Omega \left| D\left(\frac{v}{\varepsilon}\right)\right| \right) \le \varepsilon|\Omega|, \ \forall\, v \in BV(\Omega).$$

Now, considering $\mathcal{F}_h = \mathcal{F}_{\varepsilon,h}$ and $\mathcal{F} = \mathcal{F}_\varepsilon$ with $\varepsilon = 0$, for all $\varepsilon \ge 0$ and $v_h \in \mathcal{K}_h$, we have:

$$\begin{aligned}
\mathcal{F}_{\varepsilon,h}(v_h) - \mathcal{F}_\varepsilon(v_h) &= \int_{\partial\Omega} [\Pi_h(\mu_h v_h) - \mu v_h]\, d\mathcal{H}^{n-1} - \int_\Omega [\Pi_h(\kappa_h v_h) - \kappa v_h]\, dx \\
&= \int_{\partial\Omega} [\Pi_h(\mu_h v_h) - \mu_h v_h]\, d\mathcal{H}^{n-1} + \int_{\partial\Omega} (\mu_h - \mu) v_h\, d\mathcal{H}^{n-1} \\
&\quad + \int_\Omega [\Pi_h(\kappa_h v_h) - \kappa_h v_h]\, dx + \int_\Omega (\kappa_h - \kappa) v_h\, dx =: I_1 + I_2 + II_1 + II_2.
\end{aligned}$$

Since $|v_h| \le 1$ in $\bar\Omega$, we have $|I_2| \le \|\mu_h - \mu\|_{L^1(\partial\Omega)}$ and $|II_2| \le \|\kappa_h - \kappa\|_{L^1(\Omega)}$. Using a well known interpolation estimate and the local inverse inequality $\|\nabla v_h\|_{L^\infty(T)} \le C h_S^{-1} \|v_h\|_{L^\infty(T)}$, where either $T = \partial S$ or $T = S \in \mathcal{S}_h$ [6, p.140], we get

$$\begin{aligned}
|I_1| &\le C \sum_{S\in\mathcal{S}_h} h_S^2 \|D^2(\mu_h v_h)\|_{L^1(\partial S \cap \partial\Omega)} \\
&\le C \sum_{S\in\mathcal{S}_h} h_S^2 \|\nabla\mu_h \cdot \nabla v_h\|_{L^1(\partial S\cap\partial\Omega)} \le Ch \|\nabla\mu_h\|_{L^1(\partial\Omega)}
\end{aligned}$$

and, similarly, $|II_1| \le Ch\|\nabla\kappa_h\|_{L^1(\Omega)}$. Hence, (3.2) follows from (2.1) and (2.2).

Next, we observe that $\mathcal{F}(v)$ and $\mathcal{F}_\varepsilon(v)$, for $v \in \mathcal{K}$, can be approximated by $\{\mathcal{F}(v_h)\}_h$ and $\{\mathcal{F}_\varepsilon(v_h)\}_h$, respectively, for a suitable sequence $\{v_h \in \mathcal{K}_h\}_h$, as $h \to 0$. In fact, given a ball B containing $\bar{\Omega}$, let $\widetilde{v} \in W^{1,1}(B\setminus\bar{\Omega};[-1,1])$ be a function with trace v on $\partial\Omega$ [12], and denote again by $v \in BV(\mathbf{R}^n;[-1,1])$ the function $v(x) := v(x)$ if $x \in \Omega$, $v(x) := \widetilde{v}(x)$ if $x \in B\setminus\Omega$, $v(x) := 0$ if $x \in \mathbf{R}^n \setminus B$. Set $\widehat{v}_h(x) := (v \star \delta_h)(x)$, for all $x \in B$, and $v_h := \Pi_h \widehat{v}_h \in \mathcal{K}_h$. Noting that $\|D^2\widehat{v}_h\|_{L^1(\Omega)} \le \eta_h$, using a well known interpolation estimate, we have $\|v_h - \widehat{v}_h\|_{W^{1,1}(\Omega)} \le C[h^2 + h]\|D^2\widehat{v}_h\|_{L^1(\Omega)} = o_h(1)$. Hence, since [14, Prop. 1.15]

$$\lim_{h\to 0} \|\widehat{v}_h - v\|_{L^1(\Omega)} = 0 \qquad \text{and} \qquad \lim_{h\to 0} \int_\Omega |\nabla\widehat{v}_h| dx = \int_\Omega |Dv|,$$

using also the inequality $|\int_\Omega |\nabla v_h| dx - \int_\Omega |\nabla\widehat{v}_h| dx| \le \int_\Omega |\nabla(v_h - \widehat{v}_h)| dx$, we get

$$\lim_{h\to 0} \|v_h - v\|_{L^1(\Omega)} = 0 \qquad \text{and} \qquad \lim_{h\to 0} \int_\Omega |\nabla v_h| dx = \int_\Omega |Dv|.$$

This entails [14, Thm. 2.11]

$$\lim_{h\to 0} \|v_h - v\|_{L^1(\partial\Omega)} = 0$$

and, using the inequality $|\int_\Omega \sqrt{\varepsilon^2 + |Dv|^2} - \int_\Omega \sqrt{\varepsilon^2 + |\nabla v_h|^2} dx| \le |\int_\Omega |Dv| - \int_\Omega |\nabla v_h| dx|$,

$$\mathcal{F}(v) = \lim_{h\to 0} \mathcal{F}(v_h) \qquad \text{and} \qquad \mathcal{F}_\varepsilon(v) = \lim_{h\to 0} \mathcal{F}_\varepsilon(v_h), \qquad \forall\, v \in \mathcal{K}. \tag{3.5}$$

Now, using the semicontinuity of $\mathcal{F}$ and $\mathcal{F}_\varepsilon$, (3.2), and (3.5), we get (3.3) and, using also (3.1), we get (3.4). Let us show the latter property. The functionals $\mathcal{F}$ and $\mathcal{F}_\varepsilon$ ($\mathcal{F}_h$ and $\mathcal{F}_{\varepsilon,h}$, respectively) are set to $+\infty$ in $L^1(\Omega)\setminus\mathcal{K}$ ($L^1(\Omega)\setminus\mathcal{K}_h$, respectively). We have:

(i). Let $v \in L^1(\Omega)$ and $\{v_{\varepsilon,h} \in \mathcal{K}_h\}_{\varepsilon,h}$ be any sequence converging to v in $L^1(\Omega)$, as $(\varepsilon,h) \to (0,0)$. Using the semicontinuity of $\mathcal{F}$, (3.2), and (3.1), we have

$$\mathcal{F}(v) \le \liminf_{(\varepsilon,h)\to(0,0)} \mathcal{F}(v_{\varepsilon,h}) = \liminf_{(\varepsilon,h)\to(0,0)} \mathcal{F}_h(v_{\varepsilon,h}) = \liminf_{(\varepsilon,h)\to(0,0)} \mathcal{F}_{\varepsilon,h}(v_{\varepsilon,h}).$$

(ii). Let $v \in \mathcal{K}$ and $\{v_h \in \mathcal{K}_h\}_h$ be the sequence defined in (3.5). Using (3.5), (3.2), and (3.1), we have

$$\mathcal{F}(v) = \lim_{h\to 0} \mathcal{F}(v_h) = \lim_{h\to 0} \mathcal{F}_h(v_h) = \lim_{(\varepsilon,h)\to(0,0)} \mathcal{F}_{\varepsilon,h}(v_h).$$

REMARK 3.1. Let $u_{\varepsilon,h}$ be a minimum of $\mathcal{F}_{\varepsilon,h}$. We have $\mathcal{F}_{\varepsilon,h}(u_{\varepsilon,h}) \le \mathcal{F}_{\varepsilon,h}(0) = \varepsilon|\Omega|$, whence, using (2.1) and (2.2), $\int_\Omega |Du_{\varepsilon,h}| \le \int_\Omega \sqrt{\varepsilon^2 + |Du_{\varepsilon,h}|^2} \le \mathcal{H}^{n-1}(\partial\Omega) + |\Omega|(1 + \|\kappa\|_{L^\infty(\Omega)})$, for all $0 < \varepsilon \le 1$ and $h > 0$. Then, by the compactness theorem in $BV(\Omega)$ [14, Thm. 1.19], the family $\{u_{\varepsilon,h}\}_{\varepsilon,h}$ admits a subsequence converging to some $u \in \mathcal{K}$ in $L^1(\Omega)$ and, in view of (3.4) [8], u is a minimum point of $\mathcal{F}$.

4. Discrete algorithm. We describe the constrained Newton-like algorithm [9,13] for the minimization of $\mathcal{F}_{\varepsilon,h}$ in $\mathcal{K}_h$. The condition $|v| \le 1$, for all $v \in \mathcal{K}_h$, naturally gives rise to a constrained optimization problem. On the other hand, since $\mathcal{F}_{\varepsilon,h}$ is nonstrictly convex (linear) along the directions defined by the constant functions, and so the Hessian matrix is singular at any $v \in \mathbf{V}_h$, then the Newton steps can be performed only in nonconstant directions. It turns out that the definition of the subspaces of admissible, strictly convex, directions for $\mathcal{F}_{\varepsilon,h}$, where seeking for suitable descent directions, is the most delicate part of the algorithm. We point out that the quadrature formulae in (2.4) allow the direct implementation on a computer of the minimization algorithm. For the numerical solution of the nonparametric minimal surfaces, see also [15,21] and the papers cited therein.

4.1. Notation. Let $\mathcal{N}_h := \{x_j\}_{j=1}^J$ be the nodes of $\mathcal{S}_h$ and $\{\varphi_j\}_{j=1}^J$ be the hat functions, defined by $\varphi_j(x_i) = \delta_{ij}$ (Kronecker's symbol). For all $1 \le j \le J$, set $q_j := \nabla\varphi_j$ and $q_j^S := q_{j|S}$, for all $S \in \mathcal{S}_h$. Let $\mathbf{M}$ and $\mathbf{B}$ denote the (diagonalized) mass matrices, with elements $m_{jj} := \int_\Omega \varphi_j dx$ and $b_{jj} := \int_{\partial\Omega} \varphi_j d\mathcal{H}^{n-1}$, respectively, and define

$$c_j := b_{jj}\mu_h(x_j) - m_{jj}\kappa_h(x_j), \qquad \forall\, 1 \le j \le J.$$

Any function $v \in \mathbf{V}_h$ is identified to the vector $\mathbf{v} := \{v_j := v(x_j)\}_{j=1}^J$ of its nodal values. Let $\mathbf{1} := 1$, i.e., $l_j := 1$ for all $1 \le j \le J$, and $D := \{\alpha\,\mathbf{1}\}_{\alpha\in\mathbf{R}}$ be the subspace of the constant directions. For any $v \in \mathbf{V}_h$, let us define $q_v^S := \nabla v_{|S} = \sum_{j\in S} v_j q_j^S$ ("$j \in S$" is a shorthand for "the node $x_j \in S$"), and $\gamma_v^S := \sqrt{\varepsilon^2 + |q_v^S|^2}$. Hence, we have

$$\mathcal{F}_{\varepsilon,h}(v) = \sum_{S\in\mathcal{S}_h} |S|\gamma_v^S + \sum_{j=1}^J c_j v_j, \qquad \forall\, v \in \mathbf{V}_h, \tag{4.1}$$

and, after easy calculations, we get the gradient $\nabla\mathcal{F}_{\varepsilon,h}$:

$$\frac{\partial\mathcal{F}_{\varepsilon,h}(v)}{\partial v_j} = \sum_{S\ni j} |S|\frac{q_j^S \cdot q_v^S}{\gamma_v^S} + c_j, \qquad \forall\, 1 \le j \le J, \tag{4.2}$$

and the Hessian matrix $H\mathcal{F}_{\varepsilon,h}$:

$$\frac{\partial^2\mathcal{F}_{\varepsilon,h}(v)}{\partial v_i \partial v_j} = \sum_{S\ni i,j} |S|\frac{q_j^S \cdot q_i^S - (q_j^S \cdot q_v^S)(q_i^S \cdot q_v^S)/(\gamma_v^S)^2}{\gamma_v^S}, \qquad \forall\, 1 \le i,j \le J. \tag{4.3}$$

Obviously, $\nabla\mathcal{F}_{\varepsilon,h}$ and $H\mathcal{F}_{\varepsilon,h}$ do not change along the constant directions, that is,

$$\nabla\mathcal{F}_{\varepsilon,h}(v+\alpha)=\nabla\mathcal{F}_{\varepsilon,h}(v) \quad \text{and} \quad H\mathcal{F}_{\varepsilon,h}(v+\alpha)=H\mathcal{F}_{\varepsilon,h}(v), \qquad \forall\, v\in\mathbf{V}_h,\ \alpha\in\mathbf{R}.$$

In addition, note that, for all $v\in\mathbf{V}_h$,

$$\nabla\mathcal{F}_{\varepsilon,h}(v)\cdot\mathbf{1}=\sum_{j=1}^{J}\frac{\partial\mathcal{F}_{\varepsilon,h}(v)}{\partial v_j}=\sum_{j=1}^{J}c_j=\mathbf{c}\cdot\mathbf{1}=\int_{\partial\Omega}\mu_h-\int_{\Omega}\kappa_h, \quad H\mathcal{F}_{\varepsilon,h}(v)\,\mathbf{1}=\sum_{j=1}^{J}\frac{\partial^2\mathcal{F}_{\varepsilon,h}(v)}{\partial v_i\partial v_j}=0,$$

and that $H\mathcal{F}_{\varepsilon,h}$ is a sparse matrix, because $\frac{\partial^2\mathcal{F}_{\varepsilon,h}(v)}{\partial v_i\partial v_j}=0$ for all nonadjacent nodes x_i, x_j.

4.2. Minimization algorithm. The constrained Newton-like algorithm reads as follows:

S0. Let $v^0\in\mathcal{K}_h$ be the initial guess. Set $k:=0$ (iterations count).

S1. Compute $F^k:=\mathcal{F}_{\varepsilon,h}(v^k)$, $\mathbf{g}^k:=\nabla\mathcal{F}_{\varepsilon,h}(v^k)$, and $\mathbf{E}^k:=H\mathcal{F}_{\varepsilon,h}(v^k)$.

S2. Define $\mathcal{C}^k:=\{x_j\in\mathcal{N}_h : |v_j^k|=1 \text{ and } v_j^k g_j^k\le 0\}$ and $\mathcal{W}^k:=\mathcal{N}_h\setminus\mathcal{C}^k$
(the sets of the constrained and movable nodes with respect to the $-\mathbf{g}^k$ direction),
and $W^k:=\{\mathbf{p}\in\mathbf{R}^J:\ p_j=0 \text{ for all } x_j\in\mathcal{C}^k\}$
(the subspace where a suitable admissible descent direction is sought).

S3. IF $\mathcal{W}^k=\emptyset$, THEN $u_{\varepsilon,h}:=v^k$ is an absolute minimum of $\mathcal{F}_{\varepsilon,h}$, STOP
(since $W^k=\{0\}$, there is no admissible strict descent direction for $\mathcal{F}_{\varepsilon,h}$).

S4. Define a suitable descent direction $\mathbf{p}^k\in W^k$ and a scalar displacement α_0^k as follows:

S41. IF $\mathcal{W}^k=\mathcal{N}_h$ (a Newton step cannot be performed, because $\det(\mathbf{E}^k)=0$), THEN

S411. IF $\mathbf{g}^k\cdot\mathbf{1}<0$ (or $\mathbf{g}^k\cdot\mathbf{1}>0$), THEN set $\mathbf{p}^k:=\mathbf{1}$ (or $\mathbf{p}^k:=-\mathbf{1}$), $\alpha_0^k:=+\infty$, CONTINUE TO S45
($\mathbf{p}^k=\pm\mathbf{1}$ corresponds to a "degenerate" Newton direction, because
$\mathcal{F}_{\varepsilon,h}$ is linear along $\pm\mathbf{1}$, and so v^k will be pushed against a barrier ±1).

S412. IF $\mathbf{g}^k\cdot\mathbf{1}=0$ (note that $\mathcal{F}_{\varepsilon,h}(v^k+\alpha)=F^k$, $\forall\,\alpha\in\mathbf{R}$, because $\mathbf{c}\cdot\mathbf{1}=0$), THEN
IF $g_{j_M}^k<0$, where $v_{j_M}^k=\max_{1\le j\le J}v_j^k$ (or $g_{j_m}^k>0$, where $v_{j_m}^k=\min_{1\le j\le J}v_j^k$),
THEN set $\mathbf{v}^k:=\mathbf{v}^k+(1-v_{j_M}^k)\mathbf{1}$ (or $\mathbf{v}^k:=\mathbf{v}^k-(1+v_{j_m}^k)\mathbf{1}$), GO TO S2
(the new set of the constrained nodes $\mathcal{C}^k$ will be nonempty);
ELSE set $\mathbf{p}^k:=-\mathbf{g}^k$, $\alpha_0^k:=+\infty$, CONTINUE TO S5 (see Remark *i)* below).

S42. IF $\mathcal{W}^k\neq\mathcal{N}_h$, THEN project $\mathbf{g}^k$ and $\mathbf{E}^k$ on W^k, namely,
for any $x_j\in\mathcal{C}^k$, set $g_j^k:=0$ and $e_{ij}^k=e_{ji}^k:=\delta_{ij}$, for any $1\le i\le J$
(since $\mathbf{1}\notin W^k$, $\mathcal{F}_{\varepsilon,h}$ is strictly convex along the directions in W^k,
so that the projected $H\mathcal{F}_{\varepsilon,h|W^k}=\mathbf{E}^k$ is positive definite and
$\mathbf{p}^k$ can be defined by $\mathbf{E}^k\mathbf{p}^k=-\mathbf{g}^k$ [note that $p_j^k=0$, if $x_j\in\mathcal{N}_h\setminus\mathcal{W}^k$]).

S43. Compute the incomplete Choleski factorization $\mathbf{LDL}^T$ of $\mathbf{E}^k$, where
$\mathbf{L}$ is lower triangular with unit diagonal entries and $\mathbf{D}$ is positive diagonal
(if $\mathbf{E}^k$ is almost singular, the factorization may be inaccurate or even fail);
IF the factorization fails, THEN set $\mathbf{p}^k:=-\mathbf{g}^k$, $\alpha_0^k:=+\infty$, CONTINUE TO S5.

S44. Solve the linear system $\mathbf{E}^k\mathbf{p}^k = -\mathbf{g}^k$ by the preconditioned
conjugate gradient method, performing at most $\mathcal{O}(\sqrt{J})$ iterations
($\mathbf{p}^k$ is well approximated except the factorization mildly failed);
set $\alpha_0^k := 1$.

S45. Define $\mathcal{C}^k := \{x_j \in \mathcal{W}^k : |v_j^k| = 1 \text{ and } v_j^k p_j^k > 0\}$
(note that $\mathcal{C}^k \neq \mathcal{W}^k$, because $\mathbf{p}^k \cdot \mathbf{g}^k < 0$ and, for all $x_j \in \mathcal{W}^k$, $v_j^k g_j^k > 0$ if $|v_j^k| = 1$),
update $\mathcal{W}^k := \mathcal{W}^k \setminus \mathcal{C}^k$ and set $W^k := \{\mathbf{p} \in W^k : p_j = 0 \text{ for all } x_j \in \mathcal{C}^k\}$;
IF $\mathcal{C}^k \neq \emptyset$, THEN GO TO S42.

S5. Set $v^{k+1} := v^k$.

S6. Let α_M^k be the largest scalar for which $|\mathbf{v}^{k+1} + \alpha_M^k \mathbf{p}^k|_\infty \leq 1$ and
define $\alpha^k : \min_{\alpha \in [0, \alpha_M^k]} \mathcal{F}_{\varepsilon,h}(\mathbf{v}^{k+1} + \alpha \mathbf{p}^k) = \mathcal{F}_{\varepsilon,h}(\mathbf{v}^{k+1} + \alpha^k \mathbf{p}^k)$;
the optimal $\alpha^k \in [0, \alpha_M^k]$ is computed by a "linear search" [9] with
cubic Hermite interpolant polynomials starting from 0 and $\alpha_M^k \wedge \alpha_0^k$.

S7. Set $\mathbf{v}^{k+1} := \mathbf{v}^{k+1} + \alpha^k \mathbf{p}^k$ (note that $\alpha^k = \alpha_0^k = 1$ corresponds to a Newton step).

S8. IF $\alpha^k = \alpha_M^k$, THEN define $\mathcal{C}^k := \{x_j \in \mathcal{W}^k : |v_j^{k+1}| = 1\}$ (note that $\mathcal{C}^k \neq \emptyset$),
update $\mathcal{W}^k := \mathcal{W}^k \setminus \mathcal{C}^k$, set $p_j^k := 0$ if $x_j \in \mathcal{C}^k$, and compute $\mathbf{g}^{k+1} := \nabla \mathcal{F}_{\varepsilon,h}(v^{k+1})$;
IF $\mathbf{g}^{k+1} \cdot \mathbf{p}^k < 0$, THEN set $\alpha_0^k := +\infty$, GO TO S6.

S9. IF $|\mathbf{v}^{k+1} - \mathbf{v}^k|_\infty \leq \eta$ ($\simeq 10^{-6}$), THEN $u_{\varepsilon,h} := v^{k+1}$, STOP;
ELSE $k := k+1$, GO TO S1.

REMARKS. *i)* If $\mathbf{1} \notin W^k$ (i.e., $\mathcal{W}^k \neq \mathcal{N}_h$), we have

$$\mathcal{F}_{\varepsilon,h}(\mathbf{v}^k + \mathbf{p}) = F^k + \mathbf{g}^k \cdot \mathbf{p} + \frac{1}{2}\mathbf{p}^T \mathbf{E}^k \mathbf{p} + \mathcal{O}(|\mathbf{p}|^3) =: Q^k(\mathbf{p}) + \mathcal{O}(|\mathbf{p}|^3), \quad \forall\, \mathbf{p} \in W^k.$$

In that case, the projected Hessian matrix $\mathbf{E}^k$ is positive definite, and the Newton step S44 corresponds to the exact unconstrained minimization of the quadratic part $Q^k(\mathbf{p})$ of $\mathcal{F}_{\varepsilon,h}(\mathbf{v}^k + \mathbf{p})$ on W^k. The case $W^k = \mathbf{R}^J$, item S41 (i.e., all the nodes are movable along the steepest descent $-\mathbf{g}^k$ direction), basically may take place because of either the initial guess or the nonuniqueness of the minimum of $\mathcal{F}_{\varepsilon,h}$. In that case, a Newton step cannot be performed, because $\mathbf{E}^k\mathbf{p} = 0$, for all $\mathbf{p} \in D$, and so the quadratic form $Q^k(\mathbf{p})$ can be unbounded from below. Then the algorithm may select either the constant direction $\mathbf{p}^k = \mathbf{1}$ (or $\mathbf{p}^k = -\mathbf{1}$), step S411, or the steepest descent direction $\mathbf{p}^k = -\mathbf{g}^k$, step S412, according to the following arguments. Since $\mathbf{c} \cdot \mathbf{1} = \mathbf{g}^k \cdot \mathbf{1}$, from (4.1) we get $\mathcal{F}_{\varepsilon,h}(\mathbf{v}^k + \mathbf{p}) = F^k + \mathbf{g}^k \cdot \mathbf{p}$, for all $\mathbf{p} \in D$; hence, the constant direction $\mathbf{p}^k = \mathbf{1}$ ($\mathbf{p}^k = -\mathbf{1}$, respectively) is a global descent direction if $\mathbf{g}^k \cdot \mathbf{1} < 0$ ($\mathbf{g}^k \cdot \mathbf{1} > 0$, respectively), and may be considered as a "degenerate" Newton direction. On the other hand, since $\mathbf{p}^T \mathbf{E}^k \mathbf{p} > 0$ for all $\mathbf{p} \notin D$, then $\mathcal{F}_{\varepsilon,h}(\mathbf{v}^k + \mathbf{p}) > F^k$, as $|\mathbf{p}|$ becomes large, where $\mathbf{p}$ is any nonconstant direction: thus the steepest descent step $\mathbf{p}^k = -\mathbf{g}^k$, selected if $\mathbf{g}^k \cdot \mathbf{1} = 0$, can perform only a small displacement.

In the later case, the steepest descent direction should be effectively improved, e.g., by

constraining the functional on $\mathbf{V}_h/\mathbf{R}$, i.e., projecting the direction $\mathbf{p}^k = -\mathbf{g}^k$ and the Hessian matrix $\mathbf{E}^k$ on $\{\mathbf{p} \in \mathbf{R}^J : \mathbf{p} \cdot \mathbf{1} = 0\}$. More precisely, we set $g_j^k := g_j^k - g_J^k$ and $e_{ij}^k := e_{ij}^k - e_{Jj}^k - e_{iJ}^k + e_{JJ}^k$, for all $1 \le i, j \le J-1$. The projected $(J-1)\times(J-1)$ Hessian matrix is positive definite, so that a Newton step can be performed on that subspace.

Note that, since $\nabla \mathcal{F}_{\varepsilon,h}(v^k + \alpha) = \mathbf{g}^k$, for all $\alpha \in \mathbf{R}$, the selection of either one of the constant directions, step S411, as well as the straight movement of v^k against a barrier $+1$ or -1, step S412, give rise to constrain some nodes in view of the immediately successive execution of a Newton step on W^k.

ii) For $\varepsilon = 0$, $\nabla \mathcal{F}_h$ and $H\mathcal{F}_h$ are undefined at any function $v \in \mathbf{V}_h$ which is constant on at least one triangle $S \in \mathcal{S}_h$: in that case, $\gamma_v^S = |q_v^S| = 0$, and the definitions (4.2) and (4.3) are void for $j \in S$. Moreover, the functional $\mathcal{F}_h$ may be nonstrictly convex even along nonconstant admissible descent directions (e.g., if $\mathcal{F}_h$ has not a unique minimum), so that the projected Hessian matrix $\mathbf{E}^k$ is singular. On the other hand, for $\varepsilon > 0$, the functional $\mathcal{F}_{\varepsilon,h}$ may exhibit almost flat directions in W^k, even if its minimum is unique; for instance, if $\mathcal{F}$ has a relative minimum in $\widetilde{\mathcal{K}}$ with value approximatively equal to the absolute minimum value or if $\mathcal{F}$ has not a unique minimum, then the projected Hessian matrix $\mathbf{E}^k$ is tremendously ill-conditionned for small values of ε. Hence, the Newton step S44 may be very unstable for small values of ε. To make more robust the algorithm, a dynamic reduction of ε has been implemented, namely: letting h be fixed, we start with $\varepsilon = \varepsilon_0 \approx h$ and, once the stop test in step S9 is satisfied, we reduce ε, iterating until a given threshold $\bar{\varepsilon}$ has been lowered. Thus, the algorithm above deals with very small ε's only when the iterates v^k are close to a minimum of $\mathcal{F}_h$. Moreover, if a Newton step is expected to be unstable because the incomplete Choleski factorization fails, then a steepest descent step S43 is executed just to go through the topic situation.

iii) The algorithm is convergent also in the case that $\mathcal{F}_{\varepsilon,h}$ has not a unique minimum, i.e., $\mathcal{F}_{\varepsilon,h}$ has a continuum $\{u_{\varepsilon,h}^\lambda\}_{0\le\lambda\le1}$ of minima defined up to additive constants. In that case, $\nabla \mathcal{F}_{\varepsilon,h}(u_{\varepsilon,h}^\lambda) = 0$, for any $0 \le \lambda \le 1$, and $\mathbf{c} \cdot \mathbf{1} = 0$, so that the algorithm should always select the steepest descent step S412 and oscillations among different minima should not take place. Nevertheless, in our tests we have experienced nonuniqueness of the minimum of $\mathcal{F}_{\varepsilon,h}$ only in the trivial case $\mu, \kappa = 0$ of Ex. 1 (see §§2,5).

iv) In conclusion, the numerical experiments show that the algorithm is robust and reaches the numerical minimum in a few Newton-like iterations. Likely, the inefficient steepest descent steps as well as the expensive updatings of the Newton direction, necessary if $\mathcal{C}^k \neq \emptyset$ in step S45, are performed very rarely. We also would stress that the iterations through the "linear search", performed in step S8, are extremely important, because they allow a very fast construction of the final shape of the discrete minimum, pushing v^k against the barriers $+1$ and -1, with inexpensive updatings of the descent direction $\mathbf{p}^k$. Finally, we observe that if the algorithm stops at step S3, then the found discrete minimum verifies $|u_{\varepsilon,h}| = 1$ at the nodes, namely either $u_{\varepsilon,h} = \chi_\emptyset$ or $u_{\varepsilon,h} = \chi_\Omega$ or, very rarely, $u_{\varepsilon,h}$ has a one-triangle-wide transition layer across the interface.

5. Numerical experiments. In this section we comment on various examples. Here $\varepsilon_0 := h$; $\bar{\varepsilon}$ is simply denoted by ε. Since the discrete minimum $u_{\varepsilon,h}$ of $\mathcal{F}_{\varepsilon,h}$ is unique in Exs. 2 to 6 below, the function $v^0 := \chi_\emptyset = -1$ is set as the initial guess in those examples. The discrete solutions $u_{\sigma,h}$ obtained via the relaxation approach (1.2), so-called "relaxed" minima, are taken from the previous sections. We denote by t the computing CPU time in seconds.

5.1. Example 1. See Ex. 1 in §2. Let $\Omega := (-2,2)^2 \setminus \{4x_1^2 + (2|x_2| - 7)^2 \leq 25\}$. The functional $\mathcal{F}_{\varepsilon,h}$ has a continuum $\{u_{\varepsilon,h} := 2\lambda - 1\}_{0 \leq \lambda \leq 1}$ of minima. If the initial guess v^0 is a constant, the algorithm obviously stops directly. Starting, e.g., from $v^0 := \Pi_h \chi_{\{x_1 > 0\}}$, which interpolates on $\mathcal{S}_h$ a relative minimum of $\mathcal{F}$ in $\widetilde{\mathcal{K}}$, we get convergence, with no oscillations, to a minimum of $\mathcal{F}_{\varepsilon,h}$. Here, $\varepsilon = 0.1$ and $h = 0.1$. As expected, the algorithm always selects the steepest descent direction, step S412, thus leading to a quite slow convergence, because in all iterations all the nodal values are unconstrained, $|v_j^k| < 1$.

5.2. Example 2. See Ex. 2 in §2. Let $\mu_h := \Pi_h \mu$. Fig. 5.1 shows the level lines $\{u_{\varepsilon,h} = i0.1\}$, $-10 \leq i \leq 10$, of the discrete minimum, for $\varepsilon = 0.1$ and $h = 0.1$; they are straight lines, even if the mesh $\mathcal{S}_h$ is unstructured, because the discrete minimum $u_{\varepsilon,h}$ is a linear function and coincides with the minimum $u_\varepsilon = \frac{x_1}{2}$ of $\mathcal{F}_\varepsilon$. The absolute minima $\{u^a = \chi_{\{x_1 \geq a\}}\}_{-2 \leq a \leq 2}$ of $\mathcal{F}$ in $\widetilde{\mathcal{K}}$ can thus be precisely recovered as $\{\chi_{\{u_{\varepsilon,h} > t\}}\}_{-1 \leq t \leq 1}$.

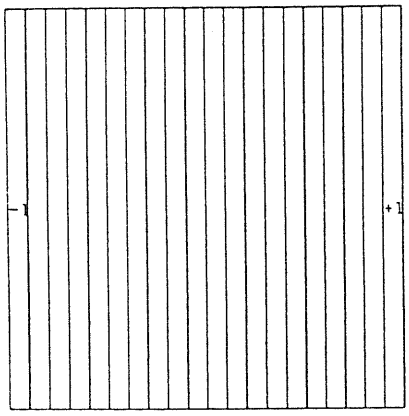

FIGURE 5.1. Ex. 2: level lines $\{u_{\varepsilon,h} = i0.1\}$, $-10 \leq i \leq 10$, of the discrete minimum [$\varepsilon = 0.1$, $h = 0.1$].

5.3. Example 3. See Ex. 3 in §2. Fig. 5.2 shows, for $r = 1$, the exact minimum A^1 (dashed lines), the relaxed minimum $A^1_{\sigma,h} := \{u_{\sigma,h} = 0\}$ (dotted line), for $h = 0.08$ and $\sigma = 0.16$ [t = 288], and the discrete minimum $A^1_{\varepsilon,h} := \{u_{\varepsilon,h} > 0\}$ (solid lines) for $h = 0.1$ and $\varepsilon = 0.05$ [t = 261]. The inaccuracy of the relaxed minimum $A^1_{\sigma,h}$ is basically due to the $\mathcal{O}(\sigma)$-wide transition layer of $u_{\sigma,h}$ across the interface, which prevents the set $A^1_{\sigma,h}$ from matching

FIGURE 5.2. Ex. 3 ($r = 1$): exact minimum A^1 (dashed lines), relaxed minimum $A^1_{\sigma,h}$ [$\sigma = 0.16$, $h = 0.08$] (dotted line), and discrete minimum $A^1_{\varepsilon,h}$ [$\varepsilon = 0.05$, $h = 0.1$] (solid lines).

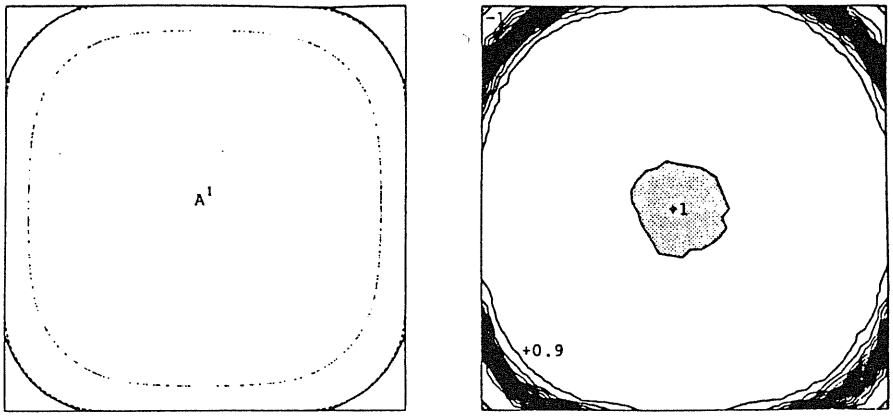

FIGURE 5.3. Ex. 3 ($r = 1$): level lines $\{u_{\varepsilon,h} = i0.1\}$, $-10 \le i \le 10$, of the discrete minimum [$\varepsilon = 0.05$, $h = 0.1$].

sharply the Dirichlet boundary condition $u_{\sigma,h} = -1$. This effect of the relaxation approach is absent in the convex approximation. In fact, the variational inequality corresponding to the minimization of $\mathcal{F}_{\varepsilon,h}$ has a Neumann condition at the boundary, namely, formally, $\nabla(\frac{v}{\varepsilon}) \cdot \nu + \mu\sqrt{1 + |\nabla(\frac{v}{\varepsilon})|^2} = 0$. Moreover, for ε sufficiently small (say $\varepsilon \le h$), the solution $u_{\varepsilon,h}$ basically does not change (see (3.1)), so that the transition layer of $u_{\varepsilon,h}$ across the interface is just a space discretization effect. Fig. 5.3 shows the level lines $\{u_{\varepsilon,h} = i0.1\}$, $-10 \le i \le 10$, of the discrete solution; we stress that most of them are bunched near $\partial A^1_{\varepsilon,h}$.

For any $r > r^\star$, the functional $\mathcal{F}$ has two minima $\chi_{A^r}, \chi_\emptyset \in \widetilde{\mathcal{K}}$, with $\mathcal{F}(\chi_{A^r}) < \mathcal{F}(\chi_\emptyset)$, whereas $\mathcal{F}(\chi_{A^{r^\star}}) = \mathcal{F}(\chi_\emptyset)$ (i.e., $\mathcal{F}$ has a flat line from $\chi_{A^{r^\star}}$ to $\chi_\emptyset$, for $r = r^\star$). The relaxed minima may approximate either the absolute minimum χ_{A^r} or the relative minimum $\chi_\emptyset$, depending on the initial guess, whereas the $\mathcal{F}_{\varepsilon,h}$'s have a unique minimum, which always converges to the absolute minimum χ_{A^r} of $\mathcal{F}$. The convex approach thus allows an accurate numerical evaluation of the critical value $r^\star$. In fact, taking for instance $h = 0.1$ and $\varepsilon = 0.05$, we see that $u_{\varepsilon,h} \simeq \chi_\emptyset$, for $r \le 0.95$ (e.g., for $r = 0.95$, $\{u_{\varepsilon,h} > -0.65\} = \emptyset$ and $\{-0.9 \le u_{\varepsilon,h} \le -0.8\}$ is a thin layer across ∂A^r), and $u_{\varepsilon,h} \simeq \chi_{A^r}$, for $r \ge 0.955$. Hence, the numerical critical value $r^\star_{\varepsilon,h} \in [0.95, 0.955]$ ($r^\star := \frac{2+\sqrt{\pi}}{4} \simeq 0.943$).

Finally, we give some data on the performance of the minimization algorithm. For $r = 1$, $h = 0.1$, and $\varepsilon = 0.05$, we get convergence in 18 Newton-like iterations (in number of 13 with $\alpha^k = 1$), with 3 updatings of the Newton direction and no steepest descent steps. Note that $u_{\varepsilon,h} = 1$ in numerous nodes (see Fig. 5.3): the iterations through the "linear search" in step S8 (in number of 114) precisely serve to push, cheapily, the iterates against

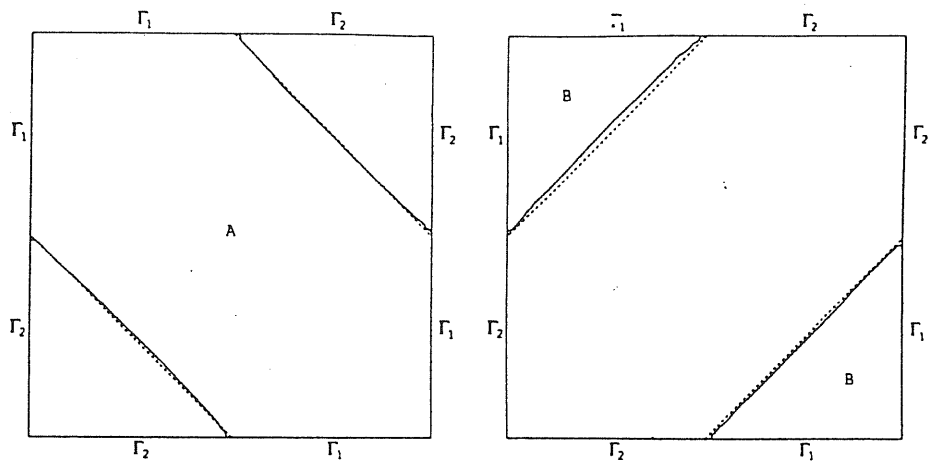

FIG. 5.4. Ex. 4: Exact A, B (dashed lines) and discrete $A_{\varepsilon,h}, B_{\varepsilon,h}$ [$\varepsilon=0.1$, $h=0.1$] (solid lines) minima.

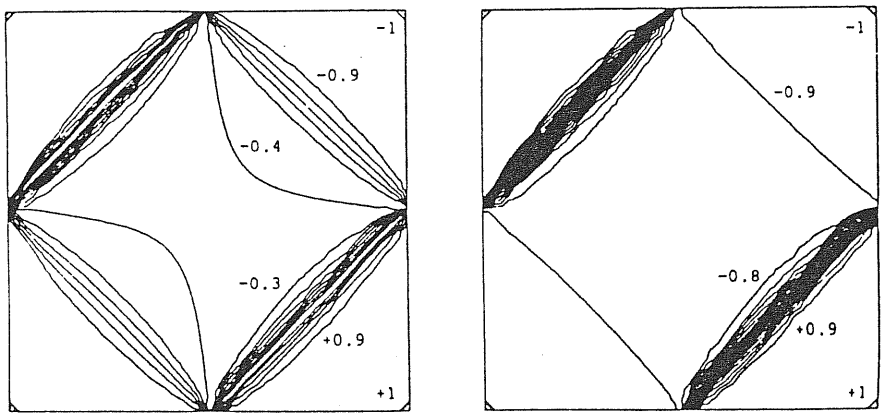

FIGURE 5.5. Ex. 4: level lines $\{u_{\varepsilon,h} = i0.1\}$, $-10 \leq i \leq 10$, of the discrete minimum [$h = 0.1$ and $\varepsilon = 0.05$, $\varepsilon = 0.01$, respectively].

the barrier $+1$, from the initial guess $v^0 = -1$.

5.4. *Example 4.* See Ex. 4 §2. The discrete boundary datum is defined by $\mu_h := \Pi_h \mu$. Figs. 5.4a,b show both the exact minima A, B (dashed lines) and the discrete ones $A_{\varepsilon,h} := \{u_{\varepsilon,h} > -0.5\}$, $B_{\varepsilon,h} := \{u_{\varepsilon,h} > 0.5\}$ (solid lines), where $u_{\varepsilon,h}$ is the unique discrete minimum of $\mathcal{F}_{\varepsilon,h}$, for $\varepsilon = 0.1$ and $h = 0.1$. We stress that, in general, $\mathcal{F}_h(A) \neq \mathcal{F}_h(B)$, because of the nonsymmetry of the mesh $\mathcal{S}_h$; hence, for ε very small, the unique discrete minimum $u_{\varepsilon,h}$ may select, independently of the initial guess, either A or B (see Figs. 5.5a,b, where $h = 0.1$ and $\varepsilon = 0.05$, $\varepsilon = 0.01$, respectively). On the contrary, in section 3.2, the relaxed minima $A_{\sigma,h}$ and $B_{\sigma,h}$ were obtained iterating from different initial guesses.

FIGURE 5.6. Ex. 5 ($r = 1$): exact minimum A (dashed line), relaxed minimum $A_{\sigma,h}$ [$\sigma = 0.17$, $h = 0.1$] (dotted line), and discrete minimum $A_{\varepsilon,h}$ [$\varepsilon = 0.02$, $h = 0.1$] (solid line).

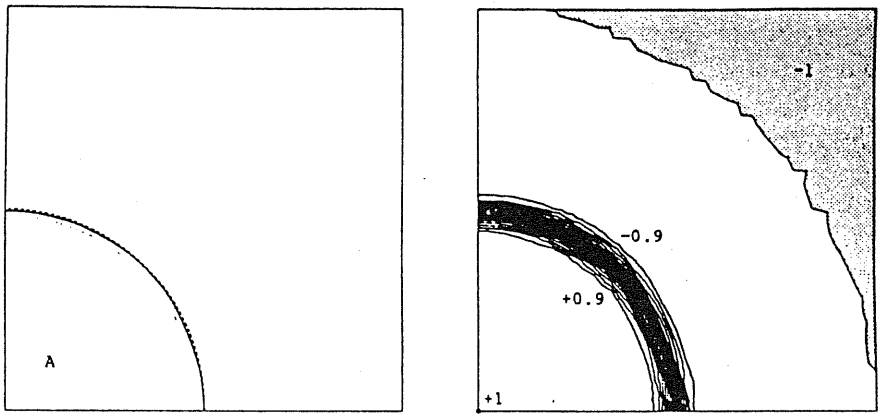

FIGURE 5.7. Ex. 5 ($r = 0.76$): level lines $\{u_{\varepsilon,h} = i0.1\}$, $-10 \le i \le 10$, of the discrete minimum [$\varepsilon = 0.005$, $h = 0.1$].

5.5. *Example 5.* Let $\Omega := (0,4)^2$, $\mu := 0$ (normal contact or reflection condition) on $x_1, x_2 = 0$, $\mu := 1$ on $x_1, x_2 = 4$, and $\kappa(x) := -r|x| + 2r + 0.5$, $r \in \mathbf{R}^+$. Define $\mu_h := \Pi_h \mu$ and $\kappa_h := \Pi_h \kappa$. The functional $\mathcal{F}$ has one absolute minimum, χ_A, where $A := \{|x| \le 2\}$, for any $r > 0.75 =: r^\star$, and $\chi_\emptyset$, for any $0 < r < r^\star$; for $r = r^\star$, both χ_A and $\chi_\emptyset$ are absolute minima in $\widetilde{\mathcal{K}}$. Note that $\chi_\emptyset$ is a relative minimum of $\mathcal{F}$ in $\widetilde{\mathcal{K}}$, for any $r > r^\star$, and χ_A is a relative minimum, for any $0.25 < r < r^\star$. Let $r = 1$. Fig. 5.6 shows A (dashed line), the relaxed minimum $A_{\sigma,h} := \{u_{\sigma,h} = 0\}$ (dotted line), for $h = 0.1$ and $\sigma = 0.17$ [$\mathbf{t} = 439$], and the discrete minimum $A_{\varepsilon,h} := \{u_{\varepsilon,h} > 0\}$ (solid line), for $h = 0.1$ and $\varepsilon = 0.02$ [$\mathbf{t} = 169$]. Note the accuracy of $A_{\varepsilon,h}$ with respect to $A_{\sigma,h}$, and compare **t**!

As in Ex. 3, also in this case the convex approach gives an accurate approximation of the critical value $r^\star$. Taking $h = 0.1$ and $\varepsilon = 0.005$, we get $u_{\varepsilon,h} \simeq \chi_\emptyset$, if $r \le 0.755$ (e.g., for $r = 0.755$, $\{u_{\varepsilon,h} > -0.8\} = \emptyset$ and $\{-0.9 \le u_{\varepsilon,h} \le -0.8\}$ is a thin layer across ∂A), and $u_{\varepsilon,h} \simeq \chi_A$, if $r \ge 0.76$ (e.g., for $r = 0.76$, Fig. 5.7 shows the level lines $\{u_{\varepsilon,h} = i0.01\}$,

FIGURE 5.8. Ex. 6 ($r = 0.4$): front-tracked absolute minimum A^1 (dashed lines), relative minimum B^1 (dotted line) and discrete minimum $A^1_{\varepsilon,h}$ [$\varepsilon = 0.02$, $h = 0.1$] (solid lines).

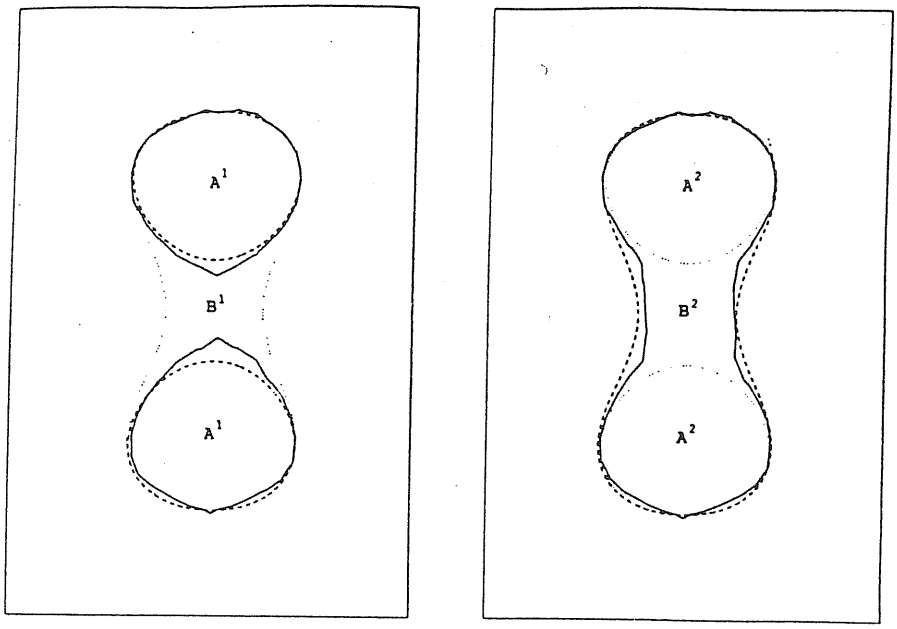

FIGURE 5.9. Ex. 6 ($r = 0.5$): front-tracked absolute minimum B^2 (dashed line), relative minimum A^2 (dotted lines) and discrete minimum $B^2_{\varepsilon,h}$ [$\varepsilon = 0.02$, $h = 0.1$] (solid line).

$-10 \le i \le 10$). Hence, the numerical critical value $r^\star_{\varepsilon,h} \in [0.755, 0.76]$.

Example 6. Let $\Omega := (-1,1)\times(-1.5,1.5)$, $\mu := 1$, and $\kappa := -40(x_1^2 + x_2^2(x_2^2 - 1)) + r$, $r \in \mathbf{R}$. Define $\kappa_h := \Pi_h \kappa$. No exact absolute minimum is known. The empty set is a relative minimum for any r. The numerical experiments show that there exists a critical value $r^\star \in (0.4, 0.5)$ for which $\mathcal{F}$ has two absolute minima. For $r < r^\star$, $\mathcal{F}$ has one absolute minimum χ_{A^r}, where A^r is the nonconnected set depicted by dashed lines in Fig. 5.8, for $r = r_1 := 0.4$ (actually, A^{r_1} has been obtained by the front-tracking method introduced in the next section); the dotted line contours a relative minimum, B^{r_1}, of $\mathcal{F}$ in $\widetilde{\mathcal{K}}$ (note that dropping one component of A^{r_1} gives a new relative minimum); the solid lines show the discrete absolute minimum $A^{r_1}_{\varepsilon,h}$, for $h = 0.1$ and $\varepsilon = 0.02$. On the other hand, for $r > r^\star$, $\mathcal{F}$ has one absolute minimum χ_{B^r}, where, for $r = r_2 := 0.5$, the front-tracked B^{r_2} is depicted by a dashed line in Fig. 5.9; the dotted lines contour the relative minimum A^{r_2}; the solid line shows the discrete absolute minimum $B^{r_2}_{\varepsilon,h}$, for $h = 0.1$ and $\varepsilon = 0.02$. For $r = r^\star$, $\chi_{A^{r^\star}}$ and $\chi_{B^{r^\star}}$ are the two absolute minima of $\mathcal{F}$ in $\widetilde{\mathcal{K}}$.

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