Scuola Internazionale Superiore di Studi Avanzati - Trieste

TOPICS ON STRING FIELD THEORY AND SOME SOLUTIONS

Thesis submitted for the degree of
Doctor Philosophiæ

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September 2006

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Acknowledgments

I would like to thank my supervisor Prof. Loriano Bonora for giving me the opportunity to work with him, for his very kind guidance, for creating a very free and friendly working environment and for his very kind behavior in general.
Next I thank my collaborators Carlo Maccaferri, Ricardo Jose Scherer Santos and Alexander Sorin for having worked with me and for many discussions we had.
I would also like to thank Serguey Petcov (Head of elementary particle sector at SISSA) for his encouraging words any time we meet.
I thank my sweet wife Kokeb Addis without whom I can not imagine life in Trieste and her mother Alemtsehay Gessese, her father Addis Alemu for making my life in Trieste more than pleasant. My thanks also goes to my little baby Sara Driba for bringing lots of happiness to my life.
I would like to thank my mother Ẹjigu Degefa for her love and for her courage to send me to school under a very unfavorable environment to do so.

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Chapter 1

Introduction

Bosonic string theory, though not realistic, as it does not have fermions and also unstable due to the presence of tachyon in its spectrum, it gives a framework to which one can add the additional complication of fermions and supersymmetry to obtain the 5 perturbative superstring models ($IIA, IIB, TypeI, Hetrotic-SO(32), E_8 \times E_8$). These are theories of Open and Closed strings where, like in the case of of bosonic string theory, the vibration of the strings create infinite tower of particles with the tachyonic ones are projected out by the GSO mechanism, and hence are all realistic as they contain also fermions by construction. In their spectrum these theories contain massless particle arranged in different multiplets of the corresponding supersymmetry. The list of massless particles contains, the dilaton, the graviton, the antisymmetric NS 2–form, various RR p–forms with $p = -1, 0, 1, ...9$, gauge particles in the case of Type I and Hetrotic and the fermionic superparteners of each of them. The effective field theory of these massless particles is a supergravity theory in ten dimension coupled to super Yang Mills in the case of Type I and Hetrotic.

In these supergravity theories there are black-hole (black p–brane) solutions, which are extended in space-time, have definite tension (mass per unit volume) and carry a p–form charge where p is the space dimension along which they are extended. An important result in the last decade showed that there is an alternative description for these solution, where they represent Dirichlet branes (D–branes). By definition a Dp–brane is a $(p+1)$-dimensional hyperplane in space–time where open strings can end. D–branes are not just boundary conditions, they are genuine dynamical objects whose dynamics is given by the excitations of the open strings ending on them, or equivalently are physical sources of closed strings by worldsheet open–closed duality. In particular they carry the RR p–form charge exactly like the black p–brane solution in supergravity.

Being the boundaries of open strings, D–branes exist not only in the supersymmetric string theory but also in the bosonic ones. For instance, the 26 dimensional open bosonic string theory can be regarded as a theory in which the open strings are constrained to end on the space filling D25–brane. However, as we don't have

the RR charge in the bosonic string theory, these D–branes are not charged and are unstable, signaling the presence of tachyons in the theory. From a point particle point of view tachyons are particles with negative $mass^2$. On the other hand, in field theory (second quantized theory), we know that the concept of mass just arises from the quadratic part of the scalar potential. If the quadratic term is positive then, by quantizing the theory, we get a massive particle, if the quadratic term is negative we get a tachyon that simply indicates that we are quantizing the theory around an unstable vacuum. For second quantized theory this does not mean any inconsistency, it means perturbation theory breaks down and some phase transition takes place, driving the theory to a new stable vacuum, with different perturbative spectrum.

In perturbative string theory we have many unstable vacua signaled by the presence of tachyons in the string spectrum. The above mentioned 26 dimensional open bosonic string theory is the simplest example. Open string theory is the theory of D–branes as the dynamic of the later is described by the excitation of the former. Therefore, if open string theory contains a tachyon this can only mean the corresponding D–brane is unstable. In this sense the 26 dimensional open string tachyon is the signal for the instability of the space filling D25–brane. In bosonic string theory there is still an other simple but most mysterious tachyon: the closed bosonic string tachyon. It signals the instability of the 26 dimensional space time itself and it is not clear that if some decaying process can bring it to a new stable space time.

We just mentioned two very simple examples of unstable vacua in string theory, the bosonic ones. However, there are other unstable vacua in supersymmetric string theory as well. On the worldsheet of type II string, the left moving degrees of freedom and the right moving degrees freedom carry separate space time charges. Since the open string boundary condition (Dp–brane) identifies the left and the right movers, the Dp–brane breaks at least half of the space time supercharges. In Type–IIA/IIB theories, precisely one half of the supersymmetry if p is even/odd is preserved. These theories also contain RR $p+1$–form charges by which the Dp–branes are coupled to the $p+1$–forms and hence are stable. The RR p–form charges assign an orientation to the D–branes. A D–brane of opposite orientation is just a D–brane with opposite RR charge, an anti–D–brane. Now if a D–brane and an anti–D–brane are placed parallel to each other at a distance less than the fundamental string length, there is a tachyon corresponding to the lowest state of open strings stretched between the two branes. This arises because the GSO projection operator for such string is opposite to the one of two parallel D–branes and that will leave the tachyon in the spectrum. This tachyon is a signal that a D–brane and anti–D–brane pair is unstable as it doesn't contain a global RR charge.

In supestring theories there are also single D–branes which are unstable, these are the D–branes of wrong dimensionality (odd for type IIA and even for type IIB). The instability is due to the fact that these branes are not charged as there are no appropriate RR p–forms in the closed string sector that can couple to them.

The meanings so far we have associated to different types of tachyons in string

theory are in principle must be based on the second quantized formulation of the theory. However, the conventional formulation of string theory uses a first quantized formalism in which the spectrum of a single particle state in the theory are obtained by quantizing the vibrational modes of a single string. Each such state is characterized by its energy E and momentum p and other quantum numbers. In that case one can also find states for which $m^2 = E^2 - p^2 < 0$ i.e. they are tachyons. One might suspect that this tachyons may have the same origin as in the second quantized theory , i.e. we may be carrying out perturbation expansion around an unstable vacuum, and that the tachyon may be removed once we expand the theory about a stable vacuum. Unfortunately, the first quantized description of string theory does not allow us to test this hypothesis. It is only in the framework of the second quantized string theory that one can have a full control of the vacua of the theory and how such vacua are dynamically connected via nonperturbative effects.

In the theory of particles we see that the right language to describe physics is to promote every particle with a corresponding space–time field and then proceed with the quantization, in which case the existence of tachyon has a clear physical interpretation (instability of the system). This interpretation also suggests a natural remedy of the problem, we simply need to expand around a new point in the field space where the potential is minimum, and carry out perturbative quantization of the theory around this point. Can that be possible in string theory? The first quantization of a single string give infinite tower of particles. Second quantization of string theory is then a quantum field theory of infinite space time fields, which seems unpractical, as one has to consider infinite interactions for a given physical process. However, string theory is not just a theory of infinite interacting particles, there is an order inside. In string theory the woeldsheet conformal symmetry gives us a consistent and unique procedure to calculate physical processes. In this sense first quantized string theory contains informations about the full non perturbative theory. The question is how to make use of this order to get the second quantized formulation of string theory.

The first task in this regard is the identification of the vacuum of string theory. It corresponds to a background in which the string propagates in such a way that the corresponding worldsheet theory is conformal. The string spectrum is then a perturbation of the vacuum, which is infinitesimal deformation of the underlying conformal field theory. These deformations are not generic, however, are marginal, which preserve the conformal symmetry. This understanding seems equivalent to the second quantization of string theory, where the conformal field theory corresponds to the classical string field and the marginal deformations are the excitations of the string field. In this sense based on the underlying conformal field theory we can think of formulating string field theory for all the brands of string theories known to us. However, so far the only well established string field theory is the bosonic open string field theory, while the bosonic closed string field theory has a complicated non polynomial form that has proven to be resistant to any kind of systematic treatment,

and there is no complete formulation of superstring field theory.

Open string field theory describing the dynamics of a D–brane is a field theory which has to have gauge inequivalent solutions of its linearized equation of motion that are in one to one correspondence with the physical states of open string theory and the S–matrix computed using its Feynman rules reproduces the S–matrix computed using the Polyakov prescription to all orders in the perturbation theory. This theory like its first quantized version contains many features of its supersymmetric generalization (yet to be formulated). For instance, when applied to tachyon condensation and related matters, it has shown itself capable of describing non-perturbative objects in string theory. It was pointed out that open string field theory can precisely determine a new stable vacuum for open string, namely one in which the D25–brane is annihilated through the condensation of the tachyonic unstable mode. The theory around this new vacuum is therefore do not contain any quantum fluctuation connected to the original unstable vacuum, which is given by the classical solution of the new theory. This new theory is named Vacuum String Field theory as it is conjectured to be a theory at the true vacuum.

In chapter 2 of this thesis we will discuss briefly the formulation of the original string field theory at the perturbative vacuum (cubic string field theory) giving few details about different kinds of operations used to construct it. In the same chapter we will put forward an important idea of integrability properties of cubic string field theory as well as the bosonic sector of light cone string field theory (LCSFT), with the detail of the construction of LCSFT is left to the references. We will show that this integrability property is a general characteristic of a 3–string vertex, which also holds true in nontrivial background. Before concluding the second chapter we also give a brief summery of the recently found Schnabl's solution to open string field theory equation of motion and on how they are used to prove Sen's first and second conjectures. Chapter 3 deals with VSFT where we will give a brief introduction of it and construct some static solution to its equation of motion. We then generalize these solution to the case where we have a constant B field in the background. These solution are then used to construct an infinite family of solution which surprisingly mimic the BPS states one meet in the AdS/CFT duality. In Chapter 4 we will construct time dependent solutions, which are solutions interpolating between the peturbative and the nonperturbative vacua, to VSFT with trivial and nontrivial backgrounds. Chapter 5 will be a general discussion.

Chapter 2

String Field Theory at the Perturbative Vacuum and Integrability Properties

2.1 Cubic String Field theory: basics

Witten's cubic open string field theory [1] is described by a string field Ψ which can be expanded in terms of all the states in the Fock space of the first quantized open string theory. The most general of such an expansion is

$$|\Psi\rangle = \big(\phi(x) + A_\mu(x)\alpha^\mu_{-1} + \alpha(x)b_-c_0 + B_\mu(x)\alpha^\mu_{-2} + B_{\mu\nu}(x)\alpha^\mu_{-1}\alpha^\mu_{-1}$$

$$+\beta_0(x)b_{-2}c_0 + \beta_1(x)b_{-1}c_{-1} + k_\mu(x)\alpha^\mu_{-1}b_{-1}c_0 + ...\big)\,|\dot{0}\rangle \qquad (2.1)$$

where x is the center of mass coordinate and $|\dot{0}\rangle = c_1|0\rangle$ with $|0\rangle$ the $SL(2,\mathbf{R})$ invariant vacuum, which is assigned a ghost number zero. In the Siegle gauge ($b_0\Psi = 0$) this will reduce to

$$|\Psi\rangle = \big(\phi(x) + A_\mu(x)\alpha^\mu_{-1} + B_\mu(x)\alpha^\mu_{-2} + B_{\mu\nu}(x)\alpha^\mu_{-1}\alpha^\mu_{-1} + \beta_1(x)b_{-1}c_{-1} + ...\big)\,|\dot{0}\rangle \quad (2.2)$$

In this infinite series, the space–time dependent coefficient functions infront of the Fock space states of the first quantized string theory are considered to be the space–time particle fields.

Next we give some operations on a $\mathbf{Z}_2$ graded Hilbert space $\mathcal{H}$ of the string field, which are important to build the string field theory action. The first is the star product ($\star$) which is a mapping from $\mathcal{H}\otimes\mathcal{H}$ to $\mathcal{H}$ and under it the $\mathbf{Z}_2$ degree G of the string fields is additive ($G_{\Psi\star\Phi} = G_\Psi + G_\Phi$). The second is the kinetic operator Q which is a mapping from $\mathcal{H}$ to $\mathcal{H}$ with degree $G_{Q\Psi} = 1 + G_\Psi$. The third is the integral $\int$ which is a mapping from $\mathcal{H}$ to $\mathbf{C}$ and vanishes for all Ψ's with degree $G_\Psi \neq 3$. These operations are also assumed to satisfy the following properties.

- Q is nilpotent: $Q^2\Psi = 0$, $\forall \Psi \in \mathcal{H}$

- Q is the derivative operator of the star product: $Q(\Psi \star \Phi) = (Q\Psi) \star \Phi + (-1)^{G_\Psi} \Psi \star (Q\Phi)$, $\forall \Psi, \Phi \in \mathcal{H}$

- The star product inside the integral is non commutative: $\int \Psi \star \Phi = (-1)^{G_\Psi G_\Phi} \int \Psi \star \Phi$, $\forall \Psi, \Phi \in \mathcal{H}$, whereas $\Psi \star \Phi$ and $\Phi \star \Psi$ are unrelated objects.

- The star product is associative: $(\Psi \star \Phi) \star \Xi = \Psi \star (\Phi \star \Xi)$, $\forall \Psi, \Phi, \Xi \in \mathcal{H}$

Having said this much about the string field and its Hilbert space we are now ready to write the action for Witten's cubic string field theory [1]:

$$S = -\frac{1}{2} \int \Psi \star Q\Psi - \frac{g}{3} \int \Psi \star \Psi \star \Psi \tag{2.3}$$

where g is the open string coupling constant. When g is taken to be zero the equation of motion derived from (2.3), which is $Q\Psi = 0$, is nothing but the physicality condition for the first quantized string theory if the kinetic operator Q is chosen to be the BRST operator Q_B, the degree G of the $\mathbf{Z}_2$ grading is identified with the ghost number, so that $G_\Psi = \#_{gh}(\Psi) = 1$, $G_{Q_B} = \#_{gh}(Q_B) = 1$ and $G_\star = \#_{gh}(\star) = 0$, the star product acts on string functionals Ψ, Φ by gluing the right half of the string represented by Ψ to the left half of the one represented by Φ, and the integration still glues the remaining halfs of the strings.

The action 2.3 is invariant under the gauge transformation

$$\delta\Psi = Q\Lambda + g(\Psi \star \Lambda - \Lambda \star \Psi) \tag{2.4}$$

for the gauge parameter $\Lambda \in \mathcal{H}$ is of ghost number zero. Again for $g = 0$ the gauge transformation reduces to $\delta\Psi = Q\Lambda$ which is the cohomology of $Q = Q_B$.

In Witten's original formulation [1], the gluing is made using delta functions. Let $X_\Psi(\sigma)$, $X_\Phi(\sigma)$ and $X_{\Psi\star\Phi}(\sigma)$ with $(0 \leq \sigma \leq \pi)$ are the embeddings of the strings Ψ, Φ and $\Psi \star \Phi$, respectively. Then the matter part of the $\star$ product is written as:

$$\begin{aligned} \Psi \star \Phi(X_{\Psi\star\Phi}(\sigma)) &= \int \mathcal{D}X_\Psi(\sigma)\mathcal{D}X_\Phi(\sigma)\Psi(X_\Psi(\sigma))\Phi(X_\Phi(\sigma)) \\ &\times \prod_{0\leq\sigma\leq\frac{\pi}{2}} \delta(X_\Phi(\sigma) - X_\Psi(\pi-\sigma))\delta(X_\Psi(\sigma) - X_{\Psi\star\Phi}(\sigma)) \\ &\times \delta(X_{\Psi\star\Phi}(\pi-\sigma) - X_\Phi(\pi-\sigma)). \end{aligned} \tag{2.5}$$

In this product the first delta function glues the left half of Ψ with the right half of Φ, the second delta function impose the condition that the right half of Ψ is the right

half of $\Psi \star \Phi$ and the third delta function impose the condition that the left half of Φ is the left half of $\Psi \star \Phi$. In this notation the integration is also written as

$$\int \Psi = \int \mathcal{D}X_\Psi(\sigma)\Psi(X_\Psi(\sigma)) \prod_{0\leq\sigma\leq\frac{\pi}{2}} \delta(X_\Psi(\sigma) - X_\Psi(\pi-\sigma)), \tag{2.6}$$

in which the delta function takes a string and fold its left half on to the right to give back a number. We can now combine (2.5) and (2.6) to rewrite the two terms in (2.3) in this new notation as:

$$\begin{aligned}\int \Psi \star Q\Psi &= \int \mathcal{D}X_\Psi(\sigma)\mathcal{D}X_{Q\Psi}(\sigma)\Psi(X_\Psi(\sigma))Q\Psi(X_{Q\Psi}(\sigma)) \\ &\times \prod_{0\leq\sigma\leq\frac{\pi}{2}} \delta(X_{Q\Psi}(\sigma) - X_\Psi(\pi-\sigma))\delta(X_\Psi(\sigma) - X_{Q\Psi}(\pi-\sigma))\end{aligned} \tag{2.7}$$

and

$$\begin{aligned}\int \Psi \star \Psi \star \Psi &= \int \mathcal{D}X_1(\sigma)\mathcal{D}X_2(\sigma)\mathcal{D}X_3(\sigma)\Psi(X_1(\sigma))\Psi(X_2(\sigma))\Psi(X_3(\sigma)) \\ &\times \prod_{0\leq\sigma\leq\frac{\pi}{2}} \delta(X_2(\sigma) - X_1(\pi-\sigma))\delta(X_3(\sigma) - X_2(\pi-\sigma)) \\ &\times \delta(X_1(\sigma) - X_3(\pi-\sigma)).\end{aligned} \tag{2.8}$$

The action (2.3) can also be written a as

$$S = -\frac{1}{2}\langle\Psi, Q\Psi\rangle - \frac{g}{3}\langle\Psi, \Psi\star\Psi\rangle, \tag{2.9}$$

where $\langle .,. \rangle$ is a bilinear inner product written in terms of the $\star$ product and the integration as:

$$\langle\Psi,\Phi\rangle = \int \Psi\star\Phi, \qquad \langle\Psi,\Phi\star\Xi\rangle = \int \Psi\star\Phi\star\Xi \tag{2.10}$$

This form is more convenient for us to relate the above rather geometrical construction of string field theory (SFT) action to the more familiar ones, the Operator formulation and the Conformal Field Theory(CFT) formulation, which will be seen in more detail in the following sections.

The ghost part of all the operations in the theory can also be defined in a similar fashion, using either the fermionic coordinates $c(\sigma)$ and $b(\sigma)$ or a bosonized scalar field $\phi(\sigma)$ and imposing similar overlap conditions as $X(\sigma)$. In the bosonized language, the $\star$ product in the ghost sector is given by Eq. (2.5) with $X(\sigma)$ replaced by $\phi(\sigma)$ and an extra insertion of $\exp\left(\frac{3i}{2}\phi(\pi/2)\right)$ inside the integral, while the integration is given by Eq. (2.6) with an extra insertion of $\exp\left(-\frac{3i}{2}\phi(\pi/2)\right)$ inside the integral.

Before concluding this preliminary section we will mention the fact that the star algebra may have an identity element. If it exists the identity element which has to be of ghost number zero satisfies

$$\mathcal{I} \star \Psi = \Psi \star \mathcal{I} = \Psi, \forall \Psi \in \mathcal{H} \tag{2.11}$$

and the kinetic operator Q or any other derivative D of the star algebra must annihilate $\mathcal{I}$, i.e

$$D(\mathcal{I} \star \Psi) = (D\mathcal{I}) \star \Psi + \mathcal{I} \star (D\Psi) = (D\mathcal{I}) \star \Psi + (D\Psi), \tag{2.12}$$

since $D(\mathcal{I} \star \Psi)$=$D\Psi$, we conclude that $(D\mathcal{I}) \star \Psi = 0, \forall \Psi$ so that $D\mathcal{I} = 0$.

2.2 $\star$ Product and Construction of Vertex Operators

In the previous section the definitions of the $\star$ product, the integration and the identity operator are too symbolic which are not suitable to do some practical computations using the SFT action. In this section our main objective is to give the operator representation [77] as well as the CFT formulation [79] of the Witten's cubic string field theory $\star$ product. In the operator formulation the first quantized string theory Fock space creation and annihilation operators are used to represent the basic operations of the theory while the CFT approach involves conformal mappings and calculation of correlation functions on a unit disk or on the upper half plane.

2.2.1 The CFT approach

Using Eq. (2.10) one can represent the quadratic part of the OSFT action as the Fock space inner product while there is no such translation for the cubic term $\int \Psi \star \Psi \star \Psi$. However, by definition $\star$ product followed by integration is simply a complete gluing of the constituent strings. In CFT language we have a very elegant representation for this process.

Let us consider a process in which three strings each propagating from infinite past ($\tau \to -\infty$, $z = e^{\tau - i\sigma} = 0$), denoted by P_r in figure 2.1, to reach the interaction point ($\tau = 0$, $|z| = 1$). In CFT language the emerging of a string in infinite past is an insertion of a corresponding vertex operator at $z = 0$ and the individual propagating string is denoted by an upper half unit disk in its local coordinate z_r, $r = 1, 2, 3$. The idea is to map these three upper half unit disks to the interior of a unit disk in a conformal plane of global coordinate ξ. We require this mapping to take the interaction points $Q(z_r)$ to the center $\xi = 0$ of the unit disk and the boundaries of the strings, the line segment $-1 \leq Re(z_r) \leq 1$ in figure 2.1 are mapped to the boundary of the unit disk. The transformation which does these two is

$$z_r \mapsto w_r = \frac{1 + iz_r}{1 - iz_r} \tag{2.13}$$

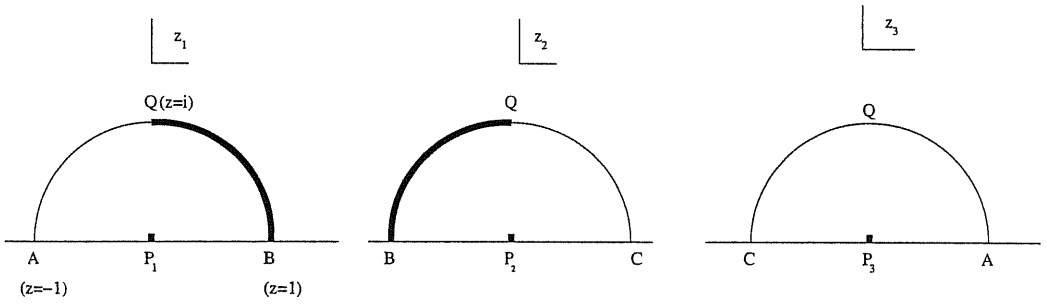


Figure 2.1: *Three strings propagating from infinity to interaction time.*

which maps the upper half unit disk to right half unit disk (figure 2.2). But to fit the

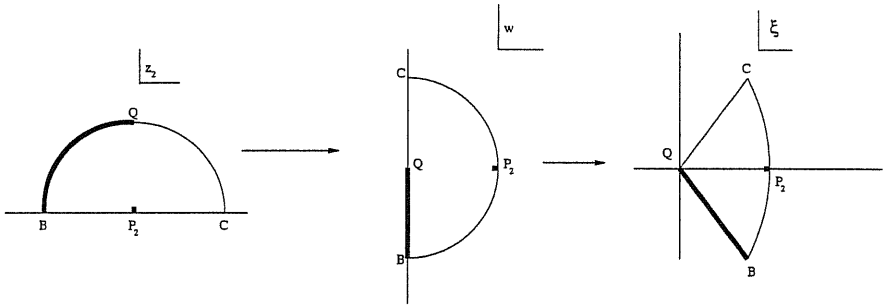


Figure 2.2: *Mapping from an upper half disk to one third of a unit disk.*

images of the three upper half unit disks in to the interior of the a unit disk in the ξ plane (as in figure 2.3) we also need to shrink w_r by a factor of $2/3$ and rotate it by $\frac{2\pi}{3}(r-1)$ so that the overall transformations are

$$f_r(z_r) = e^{\frac{2\pi i}{3}(r-1)} \left(\frac{1+iz_r}{1-iz_r}\right)^{\frac{2}{3}} \tag{2.14}$$

Using these mappings we can give the CFT representation for the cubic term of open string field theory (OSFT) action as

$$\int \Psi \star \Psi \star \Psi = \langle f_1 \circ \Psi(0) f_2 \circ \Psi(0) f_3 \circ \Psi(0) \rangle \tag{2.15}$$

where $\langle ... \rangle$ is the correlator on the global unit disk constructed above and $f_i \circ \Psi(0)$ mean the conformal transformation of of $\Psi(0)$ by f_i, which for Ψ a primary field of conformal weight h is given by

$$f_r \circ \Psi(0) = (f_r'(0))^h \Psi(f_r(0)) \tag{2.16}$$

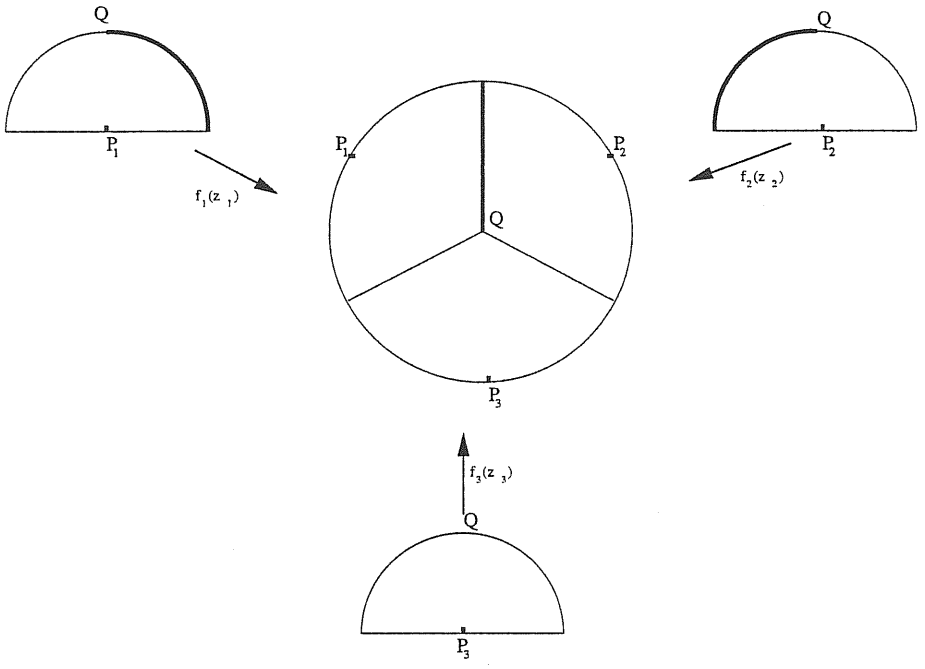


Figure 2.3: *The 3-string vertex.*

Actually, one can choose a more general conformal transformation (for example we can map to some unit wedge of angle ($\theta < 2\pi$) and reproduce the three strings gluing. However, the above conformal transformations are the one that satisfy cyclicity of the gluing $\int \Psi_1 \star \Psi_2 \star \Psi_3 = \int \Psi_2 \star \Psi_3 \star \Psi_1$, up to an $SL(2, \mathbf{C})$ transformation.

This can be also be generalized to n string gluing as

$$f_r(z_r) = e^{\frac{2\pi i}{n}(r-1)} \left(\frac{1+iz_r}{1-iz_r}\right)^{\frac{2}{n}}, \quad 1 \leq r \leq n, \tag{2.17}$$

in which each f_r maps an upper half disk to a $2\pi/n$ wedge, n such wedges are collected and rotated accordingly to make a unit disk. The above generalization also include the quadratic term ($n = 2$) for which the conformal transformations are explicitly written as

$$f_1(z_1) = \left(\frac{1+iz_1}{1-iz_1}\right), \qquad f_2(z_2) = -\left(\frac{1+iz_2}{1-iz_2}\right). \tag{2.18}$$

With this the Kinetic term of the OSFT action is written as

$$\int \Psi \star Q\Psi = \langle f_1 \circ \Psi(0) f_2 \circ Q\Psi(0) \rangle \tag{2.19}$$

still the correlator is on the global unit disk.

As we mentioned before the above conformal transformations are unique up to $SL(2,\mathbf{C})$ transformation, that means we can do an $SL(2,\mathbf{C})$ transformation to get many other possible representations that give the same amplitudes. The one which interest us is the one that takes the unit disk to the whole of the upper half plane. This is achieved by the map

$$h^{-1} : \xi \mapsto z = -i\frac{\xi - 1}{\xi + 1}. \tag{2.20}$$

Using the combined action of the h^{-1} and the f_i's we can write the OSFT action as

$$S = -\frac{1}{2}\langle \mathcal{I} \circ \Psi(0) Q\Psi(0)\rangle - \frac{g}{3}\langle h^{-1} \circ f_1 \circ \Psi(0) h^{-1} \circ f_2 \circ \Psi(0) h^{-1} \circ f_3 \circ \Psi(0)\rangle \tag{2.21}$$

where one can easily see that for the $n = 2$ case $h^{-1} \circ f_1(z_1) = z_1$ and $h^{-1} \circ f_2(z_2) = \mathcal{I}(z_2) = -\frac{1}{z_2}$. In this case the correlators are on the upper half plane.

2.2.2 Operator formulation

We begin by defining two important sates (the 2-string vertex and the 3-string vertex), whose explicit forms will be given later

$${}_{12}\langle V_2| \in \mathcal{H}_1^* \otimes \mathcal{H}_2^* \quad \text{and} \quad {}_{123}\langle V_3| \in \mathcal{H}_1^* \otimes \mathcal{H}_2^* \otimes \mathcal{H}_3^* \tag{2.22}$$

where the subscripts on the bra vectors are introduced to distinguish different copies of the string Fock space referred to different strings. For any state $|\Psi\rangle$, $\langle\Psi|$ is defined to be $(\text{bpz})|\Psi\rangle$ and $\mathcal{H}^*$ is the bpz dual of $\mathcal{H}$. The bpz of a primary field $\phi(z)$ of conformal dimension d, which creates a state $|\phi\rangle$ in the far past ($\tau \to -\infty, z = e^{\tau - i\sigma} \to 0$) by operator state correspondence, is defined as

$$\langle \text{bpz}(\phi)| = \langle 0| \lim_{z\to 0} \phi\left(-\frac{1}{z}\right) \tag{2.23}$$

On the modes

$$\phi_n = \oint \frac{dz}{2\pi} z^{n+d-1}\phi(z) \tag{2.24}$$

the bpz acts as

$$\text{bpz}(\phi_n) = (-1)^{-n+d}\phi_{-n}. \tag{2.25}$$

Using the 3-string vertex, the $\star$ product and the integration of string fields are written as[77, 80]

$$|\Psi \star \Phi\rangle_1 = {}_2\langle\Psi|_3\langle\Phi|V_3\rangle_{123} \tag{2.26}$$

$$\int \Psi \star \Phi = {}_{12}\langle V_2|\Psi\rangle_1|\Phi\rangle_2, \qquad \int \Psi \star \Phi \star \Xi = {}_{123}\langle V_3|\Psi\rangle_1|\Phi\rangle_2|\Xi\rangle_3 \tag{2.27}$$

where $|\mathcal{A}\rangle = \mathcal{A}|0\rangle$, for $\mathcal{A}$ a string field, $|\ \rangle_r$ denotes a state in the $r-th$ string Hilbert space. The $\langle\mathcal{A}|$ is the bpz of $|\mathcal{A}\rangle$ which is written in terms of the 2-string vertex as

$${}_1\langle\mathcal{A}| = {}_{12}\langle V_2|\mathcal{A}\rangle_2 \tag{2.28}$$

In this language the action (2.3) will be

$$S = -\frac{1}{2}{}_{12}\langle V_2|\Psi\rangle_1|Q\Psi\rangle_2 - \frac{g}{3}{}_{123}\langle V_3|\Psi\rangle_1|\Psi\rangle_2|\Psi\rangle_3 \tag{2.29}$$

Our task is now to determine the form of the 2 and the 3-string vertices. The natural way of doing this is to follow the original formulation of [77] in which the delta function gluing conditions given in the previous section, written in terms of the oscillators of the first quantized theory are used. For the 2-string vertex this method is rather simple while for the 3-string vertex it is a bit involved. Here we would like to use this method to determine $|V_2\rangle$ and the method of [79], which is more convenient to determine $|V_3\rangle$.

As can be seen from (2.27), $|V_2\rangle$ does a complete gluing of two strings in opposite orientation. The matter gluing condition for this process then read as $X_1(\sigma) = X_2(\pi-\sigma)$ for $0 \leq \sigma \leq \pi$, while the momentum conservation condition is $p_1(\sigma)+p_2(\pi-\sigma) = 0$ for $0 \leq \sigma \leq \pi$, where p_r is the momentum conjugate to X_r. It is convenient to define new coordinates and momenta as

$$\begin{aligned} Q_1 &= \sqrt{\frac{1}{2}}(X_1 + X_2), \qquad Q_2 = \sqrt{\frac{1}{2}}(X_1 - X_2) \\ P_1 &= \sqrt{\frac{1}{2}}(p_1 + p_2), \qquad P_2 = \sqrt{\frac{1}{2}}(p_1 - p_2) \end{aligned} \tag{2.30}$$

with the corresponding oscillators

$$A_1 = \sqrt{\frac{1}{2}}(a_1 + a_2), \qquad A_2 = \sqrt{\frac{1}{2}}(a_1 - a_2), \tag{2.31}$$

where a_r stands for $\{a_{r,n} = \alpha_{r,n}/\sqrt{n}\}$, $n \geq 1$ with $\alpha_{r,n}$ are the Fourier modes of X_r. In terms of these coordinates the gluing and the conservation conditions are

$$\begin{aligned} (1-C)_{nm}Q_{1,m} &= (1+C)_{nm}Q_{2,m} = 0 \\ (1+C)_{nm}P_{1,m} &= (1-C)_{nm}P_{2,m} = 0, \end{aligned} \tag{2.32}$$

$Q_{r,m}$ and $P_{r,m}$ are the Fourier modes of Q_r and P_r, the matrix $C_{nm} = (-1)^n\delta_{nm}$ is called twist matrix. An ansatz for $|V_2\rangle$ is

$$|V_2\rangle_{12} = \exp\left(-\frac{1}{2}A_1^\dagger\ C\ A_1^\dagger + \frac{1}{2}A_2^\dagger\ C\ A_2^\dagger\right)|0\rangle_{12} = \exp(-a_1^\dagger\ C\ a_2^\dagger)|0\rangle_{12}. \tag{2.33}$$

This can be proved to be right using the fact that

$$\langle \Psi, \mathcal{I} \rangle = {}_{12}\langle V_2 | \Psi \rangle_1 | \mathcal{I} \rangle_2 = \langle \Psi | \tag{2.34}$$

where $|\mathcal{I}\rangle = \exp(-\frac{1}{2} a^\dagger \; C \; a^\dagger)|0\rangle$ is the identity state corresponding to the $\star$ product identity operator.

The ghost part can also be calculated using the same steps as above and using the overlap conditions for the ghost and anti-ghost fields which are conjugate to one another and have to be considered as a coordinate and conjugate momentum. This gives

$${}_{12}\langle V_2 |_g = {}_{12}\langle \dot{0} | (c_0^{(1)} + c_0^{(2)}) \exp\left[- \sum_{n,m \geq 1} \left(c_n^{(1)} C_{nm} b_m^{(2)} + c_n^{(2)} C_{nm} b_m^{(1)} \right) \right] \tag{2.35}$$

The procedure for obtaining $|V_3\rangle$ involves the calculation of some correlation function using both the language of Operator formulation and the CFT formulation and compare the results. We Begin by giving an ansatz for this vertex

$$\begin{aligned} \langle V_3 | \;\; = \;\; & \int d^{26}p_{(1)} d^{26}p_{(2)} d^{26}p_{(3)} \langle \hat{0}, p |_1 \otimes \langle \hat{0}, p |_2 \otimes \langle \hat{0}, p |_3 \\ & \times \;\; \exp\left(-\frac{1}{2} \sum_{r,s} \sum_{m,n \geq 0} \eta_{\mu\nu} \alpha_m^{(r)\mu} \; N_{mn}^{rs} \; \alpha_n^{(s)\nu} \right) \\ & \times \;\; \exp\left(\sum_{r,s} \sum_{m \geq 0, n \geq 1} b_m^{(r)} \; X_{mn}^{rs} \; c_n^{(s)} \right) \end{aligned} \tag{2.36}$$

where $\langle \hat{0}, p |_r = \langle p |_r \otimes \langle 0 |_r c_0^{(r)} c_{-1}^{(r)}$. Our task now is to determine the quantities N_{mn}^{rs} and X_{mn}^{rs}, which are called the Neumann coefficients, in terms of the conformal transformations f_r.

We begin with the matter sector. Lets consider the case in which we have translational invariance so that we can set the momenta to zero so that we will have only the non zero modes in the matter Neumann coefficients $(m, n > 0)$. Now let us compute the following bracket:

$$M = {}_{123}\langle V_3 | i\partial X^{(r)}(z) i\partial X^{(s)}(w) | \dot{0} \rangle_1 | \dot{0} \rangle_2 | \dot{0} \rangle_3 \tag{2.37}$$

using two ways. The first one is to put

$$i\partial X(z) = \sum \frac{\alpha_n}{z^{n+1}} \tag{2.38}$$

an using $[\alpha_m, \alpha_n] = m\delta_{n+m,0}$ to obtain

$$M = {}_{123}\langle V_3 | \left(\sum_{m,n} \frac{1}{z^{-m+1}} \frac{1}{w^{-n+1}} \alpha_{-m}^{(r)} \alpha_{-n}^{(s)} + \frac{\delta^{rs}}{(z-w)^2} \right) | \dot{0} \rangle_1 | \dot{0} \rangle_2 | \dot{0} \rangle_3 \tag{2.39}$$

Applying the matter part of the ansatz for the 3-string vertex this will reduce to

$$M = -\mathcal{N}\sum_{m,n}\frac{1}{z^{-m+1}}\frac{1}{w^{-n+1}}mnN^{rs}_{mn} + \mathcal{N}\frac{\delta^{rs}}{(z-w)^2} \tag{2.40}$$

where

$$\mathcal{N} = {}_{123}\langle V_3|_g|\dot{0}\rangle_1|\dot{0}\rangle_2|\dot{0}\rangle_3. \tag{2.41}$$

For the second evaluation of M we write it as:

$$M = {}_{123}\langle V_3|i\partial X^{(r)}(z)i\partial X^{(s)}(w)c^{(1)}(0)c^{(2)}(0)c^{(3)}(0)|0\rangle_1|0\rangle_2|0\rangle_3. \tag{2.42}$$

Using the CFT formulation of inner product given in the previous section we can re-write it as

$$M = \langle f_r \circ i\partial X(z) f_s \circ i\partial X(w) f_1 \circ c(0) f_2 \circ c(0) f_3 \circ c(0)\rangle. \tag{2.43}$$

Recalling that $i\partial X(z)$ is a primary field of conformal weight one and using the OPE of the corresponding CFT, this gives

$$M = \mathcal{N}\frac{f_r'(z)f_s'(w)}{(f_r(z)-f_s(w))^2} \tag{2.44}$$

where $\mathcal{N} = \langle f_1 \circ c(0) f_2 \circ c(0) f_3 \circ c(0)\rangle$ is the ghost contribution. Equating the two results for M we see that the non zero mode Neumann coefficients are given by

$$N^{rs}_{mn} = -\frac{1}{mn}\oint_0 \frac{dz}{2\pi i}z^{-m}\oint_0 \frac{dw}{2\pi i}w^{-n}\frac{f_r'(z)f_s'(w)}{(f_r(z)-f_s(w))^2}. \tag{2.45}$$

To calculate the zero modes of the matter Neumann coefficients we consider the case in which the momenta are not set to zero and compute

$$M_{00} = {}_{123}\langle V_3|e^{ip_r.X(0)}e^{ip_s.X(0)}|\dot{0}\rangle_1|\dot{0}\rangle_2|\dot{0}\rangle_3 \tag{2.46}$$

with $e^{ip_r.X(0)}$ a primary field of conformal weight $\frac{p^2}{2}$, using the two ways. This yields

$$N^{rs}_{00} = \begin{cases} \log|f_r'(0)| & \text{if } r = s; \\ \log|f_r(0)-f_s(0)| + \frac{1}{2}\log|f_r'(0)f_s'(0)| & \text{if } r \neq s. \end{cases}$$

Similarly we can calculate

$$M_{0m} = {}_{123}\langle V_3|e^{ip_r.X(0)}i\partial X^{(s)}(w)|\dot{0}\rangle_1|\dot{0}\rangle_2|\dot{0}\rangle_3 \tag{2.47}$$

to obtain

$$N^{rs}_{0m} = -\frac{1}{m}\oint_0 \frac{dw}{2\pi i}w^{-m}\frac{\log|f_r'(0)|^{1/2}f_s'(w)}{(f_r(0)-f_s(w))} \tag{2.48}$$

The ghost Neumann coefficients can also be obtained with a similar procedure. This time we compute

$$G = {}_{123}\langle V_3|b^{(s)}(z)c^{(r)}(w)|\dot{0}\rangle_1|\dot{0}\rangle_2|\dot{0}\rangle_3 \tag{2.49}$$

using mode expansion of the ghost and anti-ghost fields, and then the explicit expression of the 3-string vertex. This gives

$$X^{rs}_{mn} = \oint_0 \frac{dz}{2\pi i} z^{-n+1} \oint_0 \frac{dw}{2\pi i} w^{-m-2} \frac{(f_s'(z))^2}{f_r'(w)} \frac{1}{f_s(z) - f(w)} \frac{\prod_{I=1}^3 (f_r(w) - f_I(0))}{\prod_{J=1}^3 (f_s(z) - f_J(0))}. \tag{2.50}$$

Once we determine the 2-string vertex and all the Neumann coefficients and hence the 3-string vertex, calculating the Kinetic and the interaction terms of OSFT involves purely an algebraic manipulation, though a bit tiring as we have to expand the exponentials and pick out all the terms which do not commute with the oscillators in $|\Psi\rangle_r$ and then use the commutation relations to put all the creation operators to the left of all the annihilation operators. For this reason the CFT formulation is more simple and preferred over the operator approach in many cases.

As they appear, the Neumann coefficients look all independent and in total we have nine infinite matrices. However, it is not hard to check that

$$N^{rs}_{mn} = N^{sr}_{nm}, \quad N^{rs}_{mn} = (-1)^{n+m} N^{sr}_{mn}, \quad N^{rs}_{mn} = N^{r+1,s+1}_{mn}. \tag{2.51}$$

The second and the third relations also hold for the ghost Neumann coefficients. The relations among the Neumann coefficients are not just these ones, rather a remarkable integrability property underlies the Witten's string field theory which manifests itself in the Neumann coefficients[15]. It is also true that[16] these integrability properties seem to be a general characteristic of the three strings vertex so that the Neumann coefficients of the 3-string vertex in light cone string field theory satisfy similar equation as those of the cubic SFT. Our main concern in the coming sections will be these integrability properties of the Neumann coefficients.

2.3 Integrability Properties of Neumann Coefficients (Cubic SFT)

The Neumann coefficients that define the three string vertex, a structure which is at the core of Witten's SFT since it describes the basic interaction and the star product can be shown to satisfies an integrable hierarchy which is the dispersionless Toda lattice (dTL) hierarchy. Following an inspiring suggestion of [22], in this section we show in a straightforward way that the three string Neumann coefficients satisfy the Hirota bilinear equations for the dTL hierarchy. These results are simple verifications that some identities hold for Neumann coefficients in SFT, but they may have important

implications. For it is naturally tempting to conjecture that the full SFT is based on the (dispersive) TL hierarchy. And since the latter is the hierarchy underlying two–matrix models, one is lead to conjecture that at the basis of SFT there may lie a two–matrix model.

2.3.1 dKP and dTL Hirota equations

The essence of integrable hierarchies is contained in the Hirota bilinear equations [7]. We present them here for the dKP and dTL hierarchies [8, 9] . To this end let us introduce the flow parameters $t_0, t_k, \bar{t}_k$ with $k = 1, 2, .., \infty$ and the differential operators

$$D(z) = \sum_{k=1}^{\infty} \frac{1}{kz^k} \frac{\partial}{\partial t_k}, \qquad \bar{D}(\bar{z}) = \sum_{k=1}^{\infty} \frac{1}{k\bar{z}^k} \frac{\partial}{\partial \bar{t}_k}. \tag{2.52}$$

Next we need the τ–function (free energy) $F = F(\{t_0, t_k, \bar{t}_k\})$. For the dKP F depends only on $\{t_k\}$, $k \geq 1$. Then the Hirota equations for the dKP hierarchy can be written in the following compact elegant form [10, 11]:

$$(z_1 - z_2)e^{D(z_1)D(z_2)F} + (z_2 - z_3)e^{D(z_2)D(z_3)F} + (z_3 - z_1)e^{D(z_3)D(z_1)F} = 0. \tag{2.53}$$

The dispersionless Toda Lattice hierarchy depends instead on all the parameters $\{t_0, t_k, \bar{t}_k\}$. The Hirota equations can still be written in compact form

$$(z_1 - z_2)e^{D(z_1)D(z_2)F} = z_1 e^{-\partial_{t_0} D(z_1)F} - z_2 e^{-\partial_{t_0} D(z_2)F}, \tag{2.54}$$

$$z_1 \bar{z}_2 \left(1 - e^{-D(z_1)\bar{D}(\bar{z}_2)F}\right) = e^{\partial_{t_0}(\partial_{t_0} + D(z_1) + \bar{D}(\bar{z}_2))F} \tag{2.55}$$

and their bar-counterparts.

Our task now is to identify the exponents in the above equations (derivatives of F) with suitable generating functions of Neumann coefficients and verify that the relevant Hirota equations are identically satisfied.

2.3.2 Neumann coefficients for three string vertex

The ket of the three string vertex of the Open String Field Theory, Eq.(2.36) can be written as:

$$|V_3\rangle = \int d^{26}p_{(1)} d^{26}p_{(2)} d^{26}p_{(3)} \delta^{26}(p_{(1)} + p_{(2)} + p_{(3)}) \exp(-E)\, |0, p\rangle_{123} \tag{2.56}$$

where

$$E = \sum_{r,s=1}^{3} \left(\frac{1}{2} \sum_{m,n \geq 1} \eta_{\mu\nu} a_m^{(r)\mu\dagger} V_{mn}^{rs} a_n^{(s)\nu\dagger} + \sum_{n \geq 1} \eta_{\mu\nu} p_{(r)}^{\mu} V_{0n}^{rs} a_n^{(s)\nu\dagger} + \frac{1}{2} \eta_{\mu\nu} p_{(r)}^{\mu} V_{00}^{rs} p_{(s)}^{\nu} \right) \tag{2.57}$$

where

$$V_{mn}^{rs} = (-1)^{n+m}\sqrt{nm}N_{mn}^{rs}, \quad V_{0m}^{rs} = \sqrt{2m}\left(\frac{2i^m}{3m} - N_{0m}^{rs}\right), \quad V_{00}^{rs} = \log\frac{27}{16} \tag{2.58}$$

and the operators $\left(a_m^{(r)\mu\dagger} = \frac{1}{\sqrt{m}}\alpha_{-m}^{\mu}, \ m > 0\right)$ denote the non–zero modes matter oscillators of the r-th string ($r = 1, 2, 3$). $p_{(r)}^{\mu}$ are the center of mass momenta of the r-th string. $|p_{(r)}\rangle$ is annihilated by the annihilation operators $a_m^{(r)\mu}$, which can be defined in the same way as the creation operators, and is eigenstate of the momentum operator $\hat{p}_{(r)}^{\mu}$ with eigenvalue $p_{(r)}^{\mu}$. In Eq. (2.57) the zero-zero Neumann coefficient is completely arbitrary due to momentum conservation. The choice given in Eq.(2.58) is motivated by the integrability property of SFT as we will see in the present section. We also recall that in the definition of V_{0n}^{rs} there is a gauge freedom, since, due to momentum conservation we are allowed to make the redefinition

$$V_{0n}^{rs} \to V_{0n}^{rs} + A_n^s. \tag{2.59}$$

We will see below that the Hirota equations which in our case are the ones characteristic of the dispersionless Toda Lattice hierarchy, involve only gauge invariant combinations of them.

The strategy is to first define a more efficient method to calculate the Neumann coefficient, by introducing suitable generating functions. In our case we have three functions involved

$$\begin{aligned} f_1(z_1) &= e^{\frac{2\pi i}{3}}\left(\frac{1+iz_1}{1-iz_1}\right)^{\frac{2}{3}}, \\ f_2(z_2) &= \left(\frac{1+iz_2}{1-iz_2}\right)^{\frac{2}{3}}, \\ f_3(z_3) &= e^{-\frac{2\pi i}{3}}\left(\frac{1+iz_3}{1-iz_3}\right)^{\frac{2}{3}}. \end{aligned} \tag{2.60}$$

We define

$$N^{rs}(z_1, z_2) = \sum_{n,m=1}^{\infty} \frac{1}{z_1^n}\frac{1}{z_2^m}N_{nm}^{rs}, \tag{2.61}$$

$$N_0^{rs}(z) = \sum_{n=1}^{\infty} \frac{1}{z^n}N_{0n}^{rs}. \tag{2.62}$$

Now, we insert in these definitions the formulas for N_{nm}^{rs} and N_{0n}^{rs} and then we extract the corresponding generating functions. The task is easy for N^{rr},

$$N^{rr}(z_1, z_2) = \ln\left(\frac{f_r'(0)}{z_1 - z_2}\frac{f_r(z_1^{-1}) - f_r(z_2^{-1})}{(f_r(0) - f_r(z_2^{-1}))(f_r(z_1^{-1}) - f_r(0))}\right). \tag{2.63}$$

For $r \neq s$ we start by rewriting N^{rs}_{nm} as

$$N^{rs}_{nm} = -\frac{1}{nm}\oint_0 \frac{dz}{2\pi i}\frac{1}{z^n}\oint_0 \frac{dw}{2\pi i}\frac{1}{w^m}\partial_z\partial_w \ln\left(f_r(z) - f_s(w)\right). \tag{2.64}$$

Next we insert this formula in (2.61). We notice that, since $r \neq s$ implies $f_r(0) \neq f_s(0)$, so the method works straightforward away. The result is

$$N^{rs}(z_1, z_2) = -\ln\left(\frac{(f_r(z_1^{-1}) - f_s(z_2^{-1}))(f_r(0) - f_s(0))}{(f_r(0) - f_s(z_2^{-1}))(f_r(z_1^{-1}) - f_s(0))}\right). \tag{2.65}$$

As for $N^{rs}_0(z)$ we start again by rewriting it as

$$N^{rs}_{0n} = \frac{1}{m}\oint_0 \frac{dz}{2\pi i}\frac{1}{z^n}\partial_z \ln(f_r(0) - f_s(z)), \tag{2.66}$$

and plug it into eq. (2.62). Once again we have to distinguish two cases $r = s$ and $r \neq s$. The result is

$$N^{rr}_0(z) = \ln\left(z\frac{f_r(0) - f_r(z^{-1})}{-f_r'(0)}\right), \tag{2.67}$$

while, for the case $r \neq s$, we obtain

$$N^{rs}_0(z) = \ln\left(\frac{f_r(0) - f_s(z^{-1})}{f_r(0) - f_s(0)}\right). \tag{2.68}$$

Now, having all the generating functions at hand, we want to show that they satisfy the Hirota equations (2.54–2.55). To this end we proceed to identify the exponents of these equations with the previous generating functions. When zero modes are not involved things are straightforward. We have, for instance,

$$D(z_1)D(z_2)F = N^{11}(z_1, z_2), \tag{2.69}$$
$$D(z_1)\bar{D}(\bar{z}_2)F = N^{12}(z_1, \bar{z}_2). \tag{2.70}$$

The identification for the generating functions involving zero modes is more complicated and can only be by trial and error, because of the gauge freedom (2.59) mentioned above. After some attempts we came to the following identification:

$$\partial_{t_0} D(z)F = N^{11}_0(z) - N^{21}_0(z), \tag{2.71}$$
$$\partial_{t_0} \bar{D}(\bar{z})F = N^{22}_0(\bar{z}) - N^{12}_0(\bar{z}). \tag{2.72}$$

We see that the combinations in the RHS are gauge invariant!

Now we can proceed to verify that our generating functions satisfy the Hirota equations. Substituting (2.69) in the LHS of (2.54) and (2.71) on the RHS, it is trivial to verify that the equations identically satisfied. Similarly, for eq. (2.55) we

have to insert (2.70) in its LHS and (2.71–2.72) in its RHS. However, in this case, we have an indeterminate: $F_{00} := \partial_{t_0}^2 F$. After some elementary algebra one finds that eq. (2.55) is identically satisfied provided

$$e^{F_{00}} = \frac{f_1'(0) f_2'(0)}{(f_1(0) - f_2(0))^2} = \frac{16}{27}. \tag{2.73}$$

We see that if we identify F_{00} with the standard coefficient $-V_{00}^{rr}$ the Hirota equation fixes the later as we mentioned before.

As we mentioned before the Neumann coefficients that one can extract by expanding the generating function (2.63) and (2.65) coincide, up to normalization, with the standard Neumann coefficients V_{nm}^{rs} of Eq. (2.57) [12]. For instance, for $r \neq s$, let us set $N^{12}(1/x, 1/y) \equiv H(x, y)$. Then

$$V_{nm}^{12} = -\frac{\sqrt{nm}}{n!m!} \partial_x^n \partial_y^m H(0,0). \tag{2.74}$$

Here are some examples

$$V_{11}^{12} = -\frac{16}{27}, \quad V_{12}^{12} = -V_{21}^{12} = -\frac{32}{81}\sqrt{\frac{2}{3}}, \qquad V_{31}^{12} = \frac{16}{\sqrt{3}\ 3^5}, \qquad V_{23}^{12} = \sqrt{2}\frac{160}{3^6}.$$

More generally, on the basis of the above equations, we have the following identifications among second derivatives of F, standard Neumann coefficients and the Neumann coefficients N_{nm}^{rs}:

$$F_{t_n t_m} \equiv \frac{\partial}{\partial t_n}\frac{\partial}{\partial t_m} F = -\sqrt{nm} V_{nm}^{11} = nm N_{nm}^{11}, \tag{2.75}$$

$$F_{t_n \bar{t}_m} \equiv \frac{\partial}{\partial t_n}\frac{\partial}{\partial \bar{t}_m} F = -\sqrt{nm} V_{nm}^{12} = nm N_{nm}^{12}, \tag{2.76}$$

$$F_{t_0 \bar{t}_n} \equiv \frac{\partial}{\partial t_0}\frac{\partial}{\partial \bar{t}_n} F = \sqrt{\frac{n}{2}} \left(V_{0n}^{12} - V_{0n}^{22}\right) = n \left(N_{0n}^{22} - N_{0n}^{12}\right), \tag{2.77}$$

$$F_{t_0 t_n} \equiv \frac{\partial}{\partial t_0}\frac{\partial}{\partial t_n} F = \sqrt{\frac{n}{2}} \left(V_{0n}^{21} - V_{0n}^{11}\right) = n \left(N_{0n}^{11} - N_{0n}^{21}\right). \tag{2.78}$$

The Neumann coefficients involving zero modes and computed by expanding (2.67) and (2.68), do not coincide in general with the standard ones. This is no surprise since the latter were calculated with a definite gauge choice. As far as the N_{0n}^{rs} coefficients are concerned we see once again that the Hirota equations only constrain gauge invariant combinations of them.

The Hirota equations generate infinite many relations among the coefficients N_{nm}^{rs}, N_{0n}^{rs} and F_{00}, relations which one can derive by expanding (2.54–2.55) in powers of $z_1, z_2, \bar{z}_2$. For instance, from eq. (2.54) we get

$$F_{t_1 t_1} = \frac{1}{2} F_{t_0 t_2} - \frac{1}{2}(F_{t_0 t_1})^2, \tag{2.79}$$

$$\frac{1}{2}F_{t_1t_2} = \frac{1}{3}F_{t_0t_3} - \frac{1}{2}F_{t_0t_1}F_{t_0t_2} + \frac{1}{6}(F_{t_0t_1})^3, \tag{2.80}$$

$$\tag{2.81}$$

and, from (2.125),

$$e^{F_{00}} = F_{t_1\bar{t}_1}, \tag{2.82}$$

$$e^{F_{00}}F_{t_0t_1} = \frac{1}{2}F_{\bar{t}_1t_2}, \tag{2.83}$$

and so on. All these equations are identically satisfied.

2.4 Integrability Properties of Neumann Coefficients (LCSFT)

In the previous section it was shown that the Neumann coefficients in the case of Witten's covariant open bosonic string field theory (OSFT) [1], satisfy the Hirota equations [7], of the *dispersionless* Toda lattice hierarchy [9, 11, 10]. This is to be traced back to the existence of a conformal mapping [13, 22] underlying the three strings vertex. What we would like to show in this section is that the above integrability properties seem to be a general characteristic of the three strings vertex. Indeed, below, we will prove that it holds for the light–cone string field theory (LCSFT) [24] in flat background. But, what is more important, we will present evidence that it may also hold for a nontrivial background. We will specifically examine the Neumann coefficients of the three strings vertex in a maximally supersymmetric pp–wave background and expand them in terms of the 'string mass' μ. We will show that, up to second order in this expansion, they satisfy the (correspondingly expanded) Hirota equations for the *dispersive* Toda lattice hierarchy (i.e. for the full Toda lattice hierarchy), [14]. This leads us to the conjecture that the Neumann coefficients in this background satisfy the Hirota equations of the full Toda lattice hierarchy.

Once the generating function of the Neumann coefficients of the 3-string vertex in LCSFT are written in terms of conformal mappings (see below), they are expected to satisfy some kind of integrability condition, [22]. That this is true also for the three strings vertex in the pp–wave background may be seen as the consequence of the solvability of string theory on such background. However the main value of our results is that they point towards the existence of an integrable model (of which the Hirota equations are a signal) underlying the SFT structure. The fact that this is probably true also for a nontrivial background, such as the pp–wave one, may suggest a way to approach the problem of defining SFT on more general backgrounds.

2.4.1 The conformal maps for the light–cone SFT

In this section we work out the conformal properties of the three strings vertex in light–cone SFT. We introduce first the relevant Neumann coefficients and then we

show that, like in Witten's OSFT, they can be defined in terms of conformal mappings and determine them explicitly. We show how our formulas are related to the ones of Mandelstam, [23]. Finally we discuss the general features of these conformal mappings and describe an explicit example.

In LCSFT the three superstrings vertex is determined entirely in terms of the Neumann coefficients of the bosonic part. Therefore we will concentrate on the latter, which can be written in the form

$$|V_3\rangle = \int \prod_{s=1}^{3} d\alpha_s \prod_{I=1}^{8} dp^I_{(r)} \delta(\sum_{r=1}^{3} \alpha_r) \delta(\sum_{r=1}^{3} p^I_{(r)}) \exp(-\Delta_B) \, |0,p\rangle_{123} \tag{2.84}$$

where $I = 1, \ldots, 8$ label the transverse directions, $\alpha_r = 2p^+_{(r)}$ and

$$\Delta_B = \sum_{r,s=1}^{3} \delta_{IJ} \left(\sum_{m,n\geq 1} a^{(r)I\dagger}_m V^{rs}_{mn} a^{(s)J\dagger}_n + \sum_{n\geq 1} p^I_{(r)} V^{rs}_{0n} a^{(s)J\dagger}_n + p^I_{(r)} V^{rs}_{00} p^J_{(s)} \right) \tag{2.85}$$

where summation over I and J is understood. The operators $a^{(s)J}_n, a^{(s)J\dagger}_n$ are the non–zero mode transverse oscillators of the s–th string. They satisfy

$$[a^{(r)I}_m, a^{(s)J\dagger}_n] = \delta_{mn} \delta^{IJ} \delta^{rs}$$

$p_{(r)}$ is the transverse momentum of the r–th string and $|0,p\rangle_{123} \equiv |p_{(1)}\rangle \otimes |p_{(2)}\rangle \otimes |p_{(3)}\rangle$ is the tensor product of the Fock vacua relative to the three strings. $|p_{(r)}\rangle$ is annihilated by $a^{(r)}_n$ and is the eigenstate of the operator $\hat{p}^I_{(r)}$ with eigenvalue $p^I_{(r)}$. The vertex coefficients V_{nm} are more conveniently expressed in terms of the Neumann coefficients N_{nm} as follows:

$$V^{rs}_{mn} = -\sqrt{mn} N^{rs}_{mn}, \qquad V^{rs}_{m0} = \frac{\sqrt{m}}{6} \epsilon^{stu} \frac{\alpha_t - \alpha_u}{\alpha_r} N^r_m \equiv \frac{1}{\sqrt{m}} N^{rs}_{m0} \tag{2.86}$$

$$V^{rr}_{00} = \frac{1}{3}(\alpha^2_{r-1} + \alpha^2_{r+1}) \frac{\tau_0}{\alpha}, \qquad V^{rs}_{00} = -\frac{2}{3} \alpha_r \alpha_s \frac{\tau_0}{\alpha}, \qquad r \neq s \, . \tag{2.87}$$

In these equations and in the sequel $r, s = 1, 2, 3$ modulo 3 as in the previous section. In eq.(2.86) ϵ denotes the completely antisymmetric tensor with $\epsilon^{123} = 1$ and the indices t, u are summed over. In these equations the N's are related to the Mandelstam $\bar{N}$ by $N^{rs}_{mn} = \bar{N}^{rs}_{mn}$ and $N^r_m = \alpha_r \bar{N}^r_m$, with, see [23, 24],

$$N^r_m = \frac{\Gamma(-\frac{\alpha_{r+1}}{\alpha_r} \, m)}{m! \, \Gamma(-\frac{\alpha_{r+1}}{\alpha_r} \, m - m + 1)} \, e^{\frac{\tau_0}{\alpha_r} m}, \tag{2.88}$$

$$N^{rs}_{mn} = -\frac{\alpha \, m \, n}{\alpha_r \alpha_s (m \, \alpha_s + n \, \alpha_r)} N^r_m N^s_n \tag{2.89}$$

and

$$\alpha = \alpha_1 \alpha_2 \alpha_3, \qquad \tau_0 = \sum_{r=1}^{3} \alpha_r \, \ln |\alpha_r|, \qquad \sum_{r=1}^{3} \alpha_r = 0 \, . \tag{2.90}$$

The three strings vertex V^{rs}_{m0} and V^{rs}_{00} are not exactly the same as in [24]. We have symmetrized them using momentum conservation.

The Neumann coefficients (2.88–2.89) are homogeneous functions of the parameters α_r, so they actually depend on a single parameter β

$$\beta_r := \frac{\alpha_{r+1}}{\alpha_r}, \quad \beta_1\beta_2\beta_3 = 1, \quad \beta_r(\beta_{r+1}+1) = -1, \tag{2.91}$$

$$\beta_1 = \beta, \quad \beta_2 = -\frac{\beta+1}{\beta}, \quad \beta_3 = -\frac{1}{\beta+1}, \tag{2.92}$$

$$-\infty < \beta_r \leq -1, \quad -1 \leq \beta_{r+1} \leq 0, \quad 0 \leq \beta_{r+2} < \infty,$$

$$e^{\frac{\tau_0}{\alpha_1}} = \frac{|\beta|^\beta}{|\beta+1|^{\beta+1}}, \quad e^{\frac{\tau_0}{\alpha_2}} = \frac{|\beta|}{|\beta+1|^{\frac{\beta+1}{\beta}}}, \quad e^{\frac{\tau_0}{\alpha_3}} = \frac{|\beta+1|}{|\beta|^{\frac{\beta}{\beta+1}}}. \tag{2.93}$$

We can organize the Neumann coefficients by means of generating functions

$$N^r(z) := \sum_{n=1}^{\infty} \frac{1}{z^n} N^r_n, \qquad N^{rs}(z_1,z_2) := \sum_{n,m=1}^{\infty} \frac{1}{z_1^n}\frac{1}{z_2^m} N^{rs}_{nm}. \tag{2.94}$$

Our first purpose is to express these functions in terms of conformal mapping from the unit semidisk to the complex plane, in analogy with the case of covariant OSFT, [15].

We write the conformal mappings for LCSFT as follows

$$f_r(z^{-1}) = f_{r+2}(0) - \frac{(f_{r+2}(0)-f_r(0))(f_{r+1}(0)-f_{r+2}(0))}{(f_{r+1}(0)-f_r(0))\varphi_r(z^{-1}) + f_r(0) - f_{r+2}(0)} \tag{2.95}$$

with $f_1(0) \neq f_2(0) \neq f_3(0)$. The functions $\varphi_r(z^{-1}) \equiv \varphi_r$ are solutions to the equations

$$\varphi_r^{\beta_r}(\varphi_r - 1) = \frac{1}{z} e^{\frac{\tau_0}{\alpha_r}} := x_r \,. \tag{2.96}$$

We remark that, as a consequence of (2.95–2.96), we have the following identities

$$f_r'(0) = e^{\frac{\tau_0}{\alpha_r}} \frac{(f_{r+1}(0)-f_r(0))(f_{r+2}(0)-f_r(0))}{f_{r+1}(0)-f_{r+2}(0)},$$

$$\frac{f_r'(0)f_s'(0)}{(f_r(0)-f_s(0))^2} = e^{\tau_0(\frac{1}{\alpha_r}+\frac{1}{\alpha_s})}, \quad r \neq s \,. \tag{2.97}$$

By means of the conformal mappings we now define the generating functions

$$N^r(z) = -\beta_{r+2} \ln\left(-\frac{f_r'(0)}{z}\left(\frac{1}{f_r(0)-f_r(z^{-1})} + \frac{1}{f_{r+1}(0)-f_r(0)}\right)\right), \tag{2.98}$$

$$N^{rr}(z_1,z_2) = \ln\left(\frac{f_r'(0)}{z_1-z_2}\left(\frac{1}{f_r(0)-f_r(z_2^{-1})} - \frac{1}{f_r(0)-f_r(z_1^{-1})}\right)\right), \tag{2.99}$$

$$N^{rs}(z_1,z_2) = \ln\left(\frac{(f_r(z_1^{-1})-f_s(z_2^{-1}))(f_r(0)-f_s(0))}{(f_r(0)-f_s(z_2^{-1}))(f_r(z_1^{-1})-f_s(0))}\right), \quad r \neq s \,. \tag{2.100}$$

We notice that these definitions are very close to those in Witten's SFT, [15].

Let us analyze eq. (2.96). First of all we notice that any solution will have a branch that can be expanded around the value 1 as $1 + x_r + \ldots$

$$\begin{aligned}\varphi_r &\equiv \varphi_r(z^{-1}) \equiv \varphi_r(x_r) = 1 + x_r + \sum_{k=2}^{\infty} a_k x_r^k \\ &= 1 + x_r - \beta_r x_r^2 + \frac{1}{2}\beta_r(1+3\beta_r)x_r^3 - \frac{1}{3}\beta_r(1+2\beta_r)(1+4\beta_r)x_r^4 \\ &+ \frac{1}{24}\beta_r(1+5\beta_r)(2+5\beta_r)(3+5\beta_r)x_r^5 + \ldots \quad . \end{aligned} \tag{2.101}$$

Using these expansions inside the definitions (2.98–2.100) one can make a direct comparison and verify that they do generate the Neumann coefficients of Mandelstam (2.88–2.89).

It must be clarified that the three functions $\varphi_1, \varphi_2, \varphi_3$ are not distinct. With very simple manipulations one can see that the solutions to eqs.(2.96) with different r are related by the following transformations:

$$\varphi_{r+1}(x_{r+1}) = \frac{1}{1 - \varphi_r\left(-x_{r+1}^{\beta_r}\right)} = 1 - \frac{1}{\varphi_{r-1}\left((-x_{r+1})^{\frac{1}{\beta_{r+1}}}\right)}. \tag{2.102}$$

Therefore the three equations (2.96) give rise to a unique solution (which describes a Riemann surface, see below).

The importance of these transformations should not be underestimated. The three strings vertex represents the fusion of two strings that come together and give rise to a third one. In Witten's covariant OSFT this process is very symmetric: two strings evolving from $\tau = -\infty$ come together at $\tau = 0$ in such a way that the right half of one string overlap with the left half of the other; the three strings vertex describes this process which ends with the emergence of the third string. The three strings vertex in the present case (LCSFT), as we shall see, has a different stringy/geometric interpretation. It is however still characterized by precise overlapping conditions that are made possible by the above equations.

Let us analyze the gluing conditions for the mappings $f_r(z_r^{-1})$ (2.95), i.e. let us see how we can satisfy the conditions

$$f_{r+1}(z_{r+1}^{-1}) = f_r(z_r^{-1}) \ . \tag{2.103}$$

They lead to

$$\varphi_{r+1}(x_{r+1}) = \frac{1}{1 - \varphi_r(x_r)} \tag{2.104}$$

which can easily be solved if one compares eqs. (2.104) and (2.102)

$$x_r = -x_{r+1}^{\beta_r}. \tag{2.105}$$

Equivalently, we have the following relations for the string coordinates:

$$z_r = -z_{r+1}^{\beta_r} \ . \tag{2.106}$$

For later use we record that on the unit circle these conditions become

$$z_r = e^{i\theta_r}, \quad \theta_r = \beta_r \ \theta_{r+1} + \pi. \tag{2.107}$$

Starting from the definitions (2.98–2.100) and using (2.91), (2.95) and (2.96), one can derive the following explicit representations

$$\begin{aligned} N^r(z) &= \frac{1}{\beta_r} \ln\left(-\frac{f_r'(0)}{z}\left(\frac{1}{f_r(0)-f_r(z^{-1})}+\frac{1}{f_{r+2}(0)-f_r(0)}\right)\right) \\ &\equiv \ln\varphi_r(z^{-1}) \equiv -\beta_{r+2}\ln\left(-\frac{e^{\frac{\tau_0}{\alpha_r}}}{z}\left(\frac{1}{1-\varphi_r(z^{-1})}-1\right)\right) \\ &\equiv \frac{1}{\beta_r}\ln\left(-\frac{e^{\frac{\tau_0}{\alpha_r}}}{z}\frac{1}{1-\varphi_r(z^{-1})}\right), \end{aligned} \tag{2.108}$$

$$\begin{aligned} N^{rr}(z_1,z_2) &= \ln\left(\frac{1}{z_1-z_2}\left(z_1\varphi_r^{\beta_r}(z_1^{-1})-z_2\varphi_r^{\beta_r}(z_2^{-1})\right)\right) \\ &\equiv \ln\left(\frac{e^{\frac{\tau_0}{\alpha_r}}}{z_1-z_2}\left(\frac{1}{1-\varphi_r(z_2^{-1})}-\frac{1}{1-\varphi_r(z_1^{-1})}\right)\right), \end{aligned} \tag{2.109}$$

$$\begin{aligned} N^{rr+1}(z_1,z_2) &= \ln\left(\frac{1}{\varphi_r(z_1^{-1})}+\varphi_{r+1}(z_2^{-1})-\frac{\varphi_{r+1}(z_2^{-1})}{\varphi_r(z_1^{-1})}\right) \\ &\equiv \ln\left(1+\frac{(1-\varphi_r(z_1^{-1}))(1-\varphi_{r+1}(z_2^{-1}))}{\varphi_r(z_1^{-1})}\right), \\ N^{rs}(z_1,z_2) &= N^{sr}(z_2,z_1) \end{aligned} \tag{2.110}$$

which actually do not depend on the values of the conformal mappings (2.95) at the origin, $f_r(0)$, as long as they are distinct, i.e. $f_1(0) \neq f_2(0) \neq f_3(0)$. Therefore the parameters $f_r(0)$ in (2.95) are inessential gauge parameters.

Eqs. (2.96) are related to the ones that appear in the derivation of open string tree amplitudes by means of the Neumann function method. At the tree level the interaction of open strings consists of the joining of two strings by the endpoints to form a unique string or the splitting of a string into two. These two processes can geometrically be described, [23, 25, 26], in terms of cutting and pasting string world–sheet strips. In turn such a geometry can be nicely represented in the complex plane by means of logarithmic conformal mappings. This leads to the explicit evaluation of the Neumann coefficients for the interaction of three strings that are precisely those

introduced at the beginning of section 2. For instance, a suitable logarithmic map for the r–th three–string configuration is

$$\rho_r = \alpha_{r+1}\ln(z'-1) + \alpha_{r-1}\ln z' \tag{2.111}$$

where z' takes values in the upper half plane. This can be transformed into, see [25, 26],

$$y_r = -\beta_r \ln(1 + x_r e^{y_r}) \ . \tag{2.112}$$

This equation can easily be reduced to the form (2.96) if we set $\varphi_r = \exp(-\beta_{r+1}\beta_{r+2}y_r)$ and make the identification

$$x_r \equiv \frac{e^{\frac{\tau_0}{\alpha_r}}}{z} = -z'^{\frac{1}{\beta_{r-1}}}(z'-1)^{\beta_r} \ . \tag{2.113}$$

This means that our variable z in (2.96) is a 'uniformizing' variable for the three equations. To clarify this issue, next we work out an explicit example.

Let us study in detail a particular case, specified by values of α_r for which the parameter $\beta = 1$. This case turns out to be particularly simple and can be analyzed in full detail. In this case $\beta_1 = 1, \beta_2 = -2, \beta_3 = -\frac{1}{2}$ and eqs. (2.96) can easily be solved. For each φ_r we have two branches

$$\varphi_1^{\pm}(z^{-1}) = \frac{1 \pm \sqrt{1+\frac{1}{z}}}{2}, \tag{2.114}$$

$$\varphi_2^{\pm}(z^{-1}) = \frac{2}{1 \pm \sqrt{1-\frac{1}{z}}}, \tag{2.115}$$

$$\varphi_3^{\pm}(z^{-1}) = \left(\frac{1}{z} \pm \sqrt{1+\frac{1}{z^2}}\right)^2 \ . \tag{2.116}$$

The $^{(+)}$ branch is the one for which the expansion for small x_r is of the form $\varphi_r = 1 + x_r + \ldots$. We can now define the corresponding conformal mappings. To specify the latter we need to fix the values they take at the origin. We recall that these values can be arbitrary as long as they are distinct. Therefore we choose them simply as

$$f_1(0) = -1, \qquad f_2(0) = 0, \qquad f_3(0) = 1 \ .$$

With this choice we get

$$f_1^{(\pm)}(1/z) = -\frac{1 \pm \sqrt{1+1/z}}{3 \mp \sqrt{1+1/z}} \ , \tag{2.117}$$

$$f_2^{(\pm)}(1/z) = \frac{\sqrt{1-1/z} \mp 1}{\sqrt{1-1/z} \pm 3} \ , \tag{2.118}$$

$$f_3^{(\pm)}(1/z) = -\frac{1}{1 - 2(1/z \pm \sqrt{1+(1/z)^2})^2} \ . \tag{2.119}$$

Each couple of functions $f_r^{(\pm)}(1/z)$ defines a Riemann surface represented by two sheets joined through a cut. In the first case the cut runs from -1 to ∞, in the second it runs from 1 to ∞ and in the third between i and $-i$. These three Riemann surfaces however are related by the maps (2.102).

In the $\beta = 1$ case it is rather easy to see in detail the gluing conditions that characterize the three strings vertex. We consider three unit semidisks S_1, S_2 and S_3 cut out in the complex $\zeta_1 = 1/z_1$, $\zeta_2 = 1/z_2$ and $\zeta_3 = 1/z_3$ upper half planes. Each function $f_r^{(\pm)}$ maps the semidisk into a region $D_r^{(\pm)}$ in the image complex plane. $D_1^{(\pm)}$ and $D_2^{(\pm)}$ have the form of lobes, while $D_3^{(\pm)}$ are the outer part of a compact bi–lobed domain in the lower (+) and upper (−) half plane, respectively (see Fig. 2.4–2.6, where the $D_r^{(+)}$'s are shown, the $D_r^{(-)}$'s being the same regions reflected with respect to the real axis).

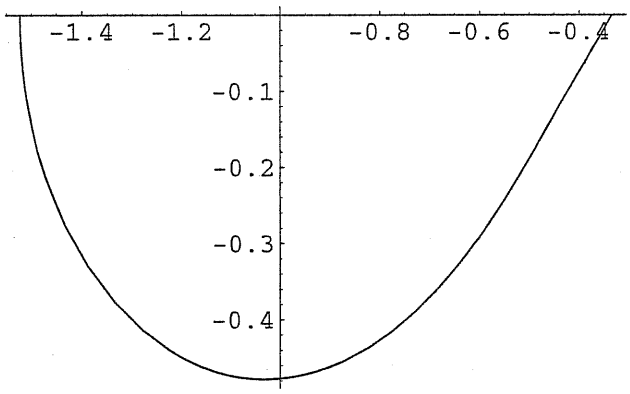


Figure 2.4: *The region $D_1^{(+)}$ is contained between the curve and the real axis.*

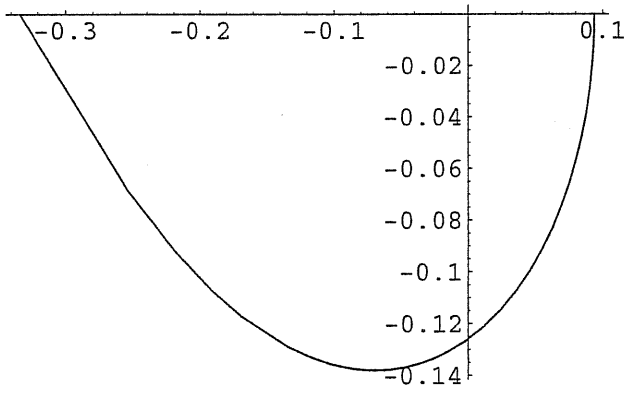


Figure 2.5: *The region $D_2^{(+)}$ is contained between the curve and the real axis.*

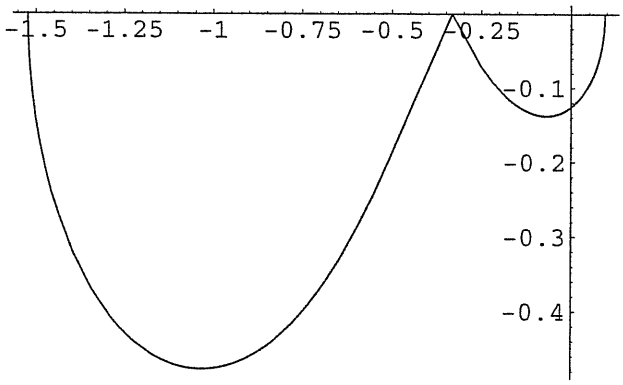


Figure 2.6: *The region $D_3^{(+)}$ is the portion of the lower half plane external to the curve.*

These domains are glued together along the borderlines in the following way. To start with, $f_1^{(+)}(\zeta_1) = f_2^{(-)}(\zeta_2)$ if $\zeta_1 = -\zeta_2$. This means that S_1 and S_2 are glued together along the real axis with opposite orientation. The image common boundary in $D_1^{(+)}$ and $D_2^{(-)}$ stretches also along the real axis. The second overlap condition we consider is between $D_1^{(+)}$ and $D_3^{(+)}$: $f_1^{(+)}(\zeta_1) = f_3^{(+)}(\zeta_3)$ if $\zeta_3 = -1/\sqrt{\zeta_1}$. This means the absolute values of ζ_1 and ζ_3 are 1, while their phases θ_1 and θ_3 are related by $\theta_3 = \pi - \frac{\theta_1}{2}$. So, while θ_1 runs between 0 and π, θ_3 spans the interval between π and $\pi/2$. These two angular intervals are mapped to the same curve in the target complex plane: the curved part of the lobe boundary in $D_1^{(+)}$ and a piece of the curved boundary of $D_3^{(+)}$. Finally $f_2^{(+)}(\zeta_2) = f_3^{(+)}(\zeta_3)$ if $\zeta_2 = -1/\zeta_3^2$. That means the moduli of ζ_2 and ζ_3 are 1, while their phases θ_2 and θ_3 are related by $\theta_2 = \pi - 2\theta_3$. So, while θ_2 runs between 0 and π, θ_3 spans the interval between $\pi/2$ and 0. Again these two angular intervals are mapped to the same curve in the target complex plane: the curved boundary of $D_2^{(+)}$ and the other piece of the curved boundary of $D_3^{(+)}$. In a similar way one can proceed with the remaining three correspondences.

So far the discussion has been purely mathematical, without any concern for the string process we want to describe. From the point of view of string theory the 'physical' branches are the ones where the expansion for small x_r is of the form $\varphi_r = 1 + x_r + \ldots$, i.e. the + branches in (2.114–2.116). So we have to glue together the images of the semidisks S_1, S_2, S_3 by $f_1^{(+)}, f_2^{(+)}, f_3^{(+)}$ respectively, according to the maps (2.106), identifying the common boundaries. This means that we have to glue the inner part of the lobe $D_1^{(+)}$ and $D_2^{(+)}$ with $D_3^{(+)}$, i.e. with the outer part of the bi–lobe. The result is the entire lower half plane. The process described is the joining of the strings 1 and 2 by two endpoints to form the third string at the interaction point $\zeta_1 = -1, \zeta_2 = 1$ and $\zeta_3 = i$. The picture obtained in this way allows us to make a comparison with the three string interaction in covariant SFT. As one can see there are remarkable differences. In particular the midpoints of the three strings

never overlap (at variance with covariant OSFT); on the contrary, the LCSFT vertex preserves its characteristic perturbative geometry where the strings interact by the endpoints.

For completeness we write down the explicit expressions of the generating functions for the Neumann coefficients. In these definitions we always use the + branch of φ_r

$$\begin{aligned} N^1(z) &= \ln\frac{1+\sqrt{1+\frac{1}{z}}}{2}, \qquad N^2(z) = -N^1(-z), \\ N^3(z) &= 2\ln\left(\frac{1}{z}+\sqrt{1+\frac{1}{z^2}}\right) \end{aligned} \tag{2.120}$$

and

$$\begin{aligned} N^{11}(z_1,z_2) &= \ln\left(\frac{1}{2(z_1-z_2)}\left(\frac{1}{1-\sqrt{1+\frac{1}{z_2}}}-\frac{1}{1-\sqrt{1+\frac{1}{z_1}}}\right)\right), \\ N^{22}(z_1,z_2) &= N^{11}(-z_1,-z_2), \\ N^{33}(z_1,z_2) &= \ln\left(\frac{1}{z_1-z_2}\left(\sqrt{1+z_1^2}-\sqrt{1+z_2^2}\right)\right) \end{aligned} \tag{2.121}$$

as well as

$$\begin{aligned} N^{12}(z_1,z_2) &= -N^{11}(z_1,-z_2), \\ N^{23}(z_1,z_2) &= \ln\left(\frac{\sqrt{1-\frac{1}{z_1}}-\sqrt{1+z_2^2}}{1-\sqrt{1+z_2^2}}\right), \\ N^{13}(z_1,z_2) &= \ln\left(\frac{\sqrt{1+\frac{1}{z_1}}+\sqrt{1+z_2^2}}{1+\sqrt{1+z_2^2}}\right), \\ N^{rs}(z_1,z_2) &= N^{sr}(z_2,z_1). \end{aligned} \tag{2.122}$$

Finally let us remark that, for $\beta \neq 1$, the conformal mappings f_r may be far more complicated than in the above example. With β rational they have a finite number of branches that describe a Riemann surface. Like in the $\beta = 1$ case, only one of them will define the correct Neumann coefficients. When β is irrational, as shown in (2.101), this branch can always be determined, but the Riemann surface interpretation is lost.

2.4.2 The dTL Hirota equations

Let us introduce the flow parameters $t_0^{(r)}, t_k^{(r)}, \bar{t}_k^{(r)}$ with $k = 1, 2, .., \infty$ and the differential operators

$$D_r(z) = \sum_{k=1}^{\infty} \frac{1}{kz^k} \frac{\partial}{\partial t_k^{(r)}}, \quad \bar{D}_r(\bar{z}) = \sum_{k=1}^{\infty} \frac{1}{k\bar{z}^{\,k}} \frac{\partial}{\partial \bar{t}_k^{\,(r)}}, \quad r, s = 1, 2, 3 \text{ modulo } 3 \quad (2.123)$$

which are generalizations of the corresponding operators in section (2.3.1). The Hirota equations for the three decoupled copies of the dispersionless Toda lattice hierarchies fit for the present case are (for more details, see [10] and references therein)

$$(z_1 - z_2) e^{D_r(z_1) D_r(z_2) F} = z_1 e^{-\partial_{t_0^{(r)}} D_r(z_1) F} - z_2 e^{-\partial_{t_0^{(r)}} D_r(z_2) F}, \quad (2.124)$$

$$(\bar{z}_1 - \bar{z}_2) e^{\bar{D}_r(\bar{z}_1) \bar{D}_r(\bar{z}_2) F} = \bar{z}_1 e^{-\partial_{t_0^{(r)}} \bar{D}_r(\bar{z}_1) F} - \bar{z}_2 e^{-\partial_{t_0^{(r)}} \bar{D}_r(\bar{z}_2) F}, \quad (2.125)$$

$$-z_1 \bar{z}_2 \left(1 - e^{-D_r(z_1) \bar{D}_r(\bar{z}_2) F}\right) = e^{\partial_{t_0^{(r)}} (\partial_{t_0^{(r)}} + D_r(z_1) + \bar{D}_r(\bar{z}_2)) F}, \quad (2.126)$$

where $F \equiv F(\{t_0^{(r)}, t_k^{(r)}, \bar{t}_k^{(r)}\})$ is the τ–function (free energy) of the hierarchy. The minus sign on the l.h.s. of eq. (2.126) can be replaced by the standard plus sign via the transformations $\{t_n^r,\ \bar{t}_n{}^r\} \Rightarrow \{t_n^r,\ (-1)^n \bar{t}_n{}^r\}$ or $\{t_n^r,\ \bar{t}_n{}^r\} \Rightarrow \{(i)^n t_n^r,\ (i)^n \bar{t}_n{}^r\}$.

We remark that eqs. (2.124–2.126) form three distinct sets of equations, each one being formally the same as in Witten's OSFT, [15]. The fact is that in the latter case these sets of equations collapse to the same set.

Now, it is elementary to verify that the generating functions $N^r(z)$, $N^{rs}(z_1, z_2)$ (2.98–2.100) of the Neumann coefficients satisfy the following equations:

$$\begin{aligned} (z_1 - z_2) e^{N^{rr}(z_1, z_2)} &= z_1 e^{-\beta_{r+1}\beta_r N^r(z_1)} - z_2 e^{-\beta_{r+1}\beta_r N^r(z_2)} \\ &= z_1 e^{\beta_r N^r(z_1)} - z_2 e^{\beta_r N^r(z_2)} \end{aligned} \quad (2.127)$$

and

$$-z_1 \bar{z}_2 \left(1 - e^{N^{rr+1}(z_1, \bar{z}_2)}\right) = e^{\tau_0 (\frac{1}{\alpha_r} + \frac{1}{\alpha_{r+1}}) + \beta_{r+1}\beta_r N^r(z_1) - \beta_{r+1} N^{r+1}(\bar{z}_2)}. \quad (2.128)$$

These are easily seen to reproduce the Hirota equations (2.124–2.126) provided we suitably identify the generating functions with the second derivatives of F as follows:

$$\begin{aligned} &D_r(z_1) D_r(z_2) F = N^{rr}(z_1, z_2), \\ &\bar{D}_r(\bar{z}_1) \bar{D}_r(\bar{z}_2) F = N^{r+1r+1}(\bar{z}_1, \bar{z}_2), \\ &D_r(z_1) \bar{D}_r(\bar{z}_2) F = -N^{rr+1}(z_1, \bar{z}_2), \\ &\partial_{t_0^{(r)}} D_r(z) F = \beta_{r+1}\beta_r N^r(z), \\ &\partial_{t_0^{(r)}} \bar{D}_r(\bar{z}) F = -\beta_{r+1} N^{r+1}(\bar{z}), \\ &\partial_{t_0^{(r)}} \partial_{t_0^{(r)}} F = \tau_0 (\frac{1}{\alpha_r} + \frac{1}{\alpha_{r+1}}) \,. \end{aligned} \quad (2.129)$$

However we remark that we can also define a consistent reduction by:

$$\frac{\partial}{\partial \bar{t}_n^{(r)}} F = \pm \frac{\partial}{\partial t_n^{(r+1)}} F. \tag{2.130}$$

This allows us to define new identifications as follows

$$\begin{aligned}
&D_r(z_1)D_r(z_2)F = N^{rr}(z_1,z_2),\\
&D_r(z_1)D_{r+1}(z_2)F = \mp N^{rr+1}(z_1,z_2),\\
&\partial_{t_0^{(r)}} D_r(z)F = \beta_{r+1}\beta_r N^r(z),\\
&\partial_{t_0^{(r)}} D_{r+1}(z)F = \mp\beta_{r+1} N^{r+1}(z),\\
&\partial_{t_0^{(r)}}\partial_{t_0^{(r)}} F = \tau_0(\frac{1}{\alpha_r}+\frac{1}{\alpha_{r+1}}) \ .
\end{aligned} \tag{2.131}$$

The corresponding equations for the tau function are

$$\begin{aligned}
(z_1-z_2)e^{D_r(z_1)D_r(z_2)F} &= z_1 e^{-\partial_{t_0^{(r)}} D_r(z_1)F} - z_2 e^{-\partial_{t_0^{(r)}} D_r(z_2)F}\\
&= z_1 e^{\mp\partial_{t_0^{(r-1)}} D_r(z_1)F} - z_2 e^{\mp\partial_{t_0^{(r-1)}} D_r(z_2)F},\\
-z_1 z_2\left(1-e^{\mp D_r(z_1)D_{r+1}(z_2)F}\right) &= e^{\partial_{t_0^{(r)}}(\partial_{t_0^{(r)}} + D_r(z_1) \pm D_{r+1}(z_2))F}.
\end{aligned} \tag{2.132}$$

These Hirota equations refer to the dispersionless hierarchy produced by the cyclic (coupled) reduction (2.130) of the three copies of the dispersionless Toda lattice hierarchies. Our conjecture is that the resulting hierarchy can be related to the 3-punctures Whitham hierarchy [8] and the dispersionless limit of the 3-component KP hierarchy.

2.4.3 Hirota equations for more general correlators

As we will see in the next section, in order to verify the validity of the full Hirota equations we need to know correlators with more than two entries. The dispersionless Hirota equations for 'n–point functions' are obtained by differentiating $n-2$ times the equations (2.124,2.125) and (2.126). These derived Hirota equations, like the original ones, do not uniquely determine their solutions. In order to be able to write down the latter we have to provide some additional information. In fact what we are looking for are n–point correlators that are compatible with the Neumann coefficients of the previous section and satisfy the full Hirota equation (see next section). Although we don't have a proof of it, we believe these two requirements completely determine the full series of n–point functions.

In this section we would like to concentrate on three– and four–point functions. On the basis of what we have just said, in order to determine them we have to rely on plausibility arguments and verify the results *a posteriori*. We notice first that

the Neumann coefficients for the three strings vertex can be interpreted in terms of two–point correlators of a 1D system. For instance

$$\langle V_3|a_m^{(r)I\dagger}a_n^{(s)J\dagger}|0\rangle = \delta^{IJ}\,V_{mn}^{rs} = -\sqrt{nm}\,N_{mn}^{rs}\,\delta^{IJ}\ . \tag{2.133}$$

Analogous relations hold for two–point functions involving zero modes. In the same way we can of course consider correlators with more insertions. It is obvious that all the correlators with an odd number of insertions identically vanish. We take these correlators as a model in order to calculate three– and four–point functions of a quantum system that satisfies the full Hirota equations. We will see below that the correct answer is not the three– and four–point correlators obtained by inserting one and two creation operators in (2.133), respectively, but combinations of them. We will refer to the underlying model, based on the two–point correlators (2.133) and satisfying the Hirota equation, as the *associated quantum system.* It must be clear that we do not know yet what this system is, we simply postulate its existence. In particular, as a working hypothesis, we assume that its three–point functions identically vanish.

The dispersionless Hirota equations for four–point functions are obtained by differentiating twice the equations (2.124,2.125) and (2.126). Of course we have the possibility of applying $\partial_{t_0^{(r)}}$ or $D_r(z)$. Therefore from (2.124), for instance, we obtain

$$(z_1-z_2)e^{D_r(z_1)D_r(z_2)F}\prod_{i=1}^{4}D_r(z_i)F = \tag{2.134}$$
$$z_2e^{-\partial_{t_0^{(r)}}D_r(z_2)F}\partial_{t_0^{(r)}}D_r(z_2)\prod_{i=3}^{4}D_r(z_i)F - z_1e^{-\partial_{t_0^{(r)}}D_r(z_1)F}\partial_{t_0^{(r)}}D_r(z_1)\prod_{i=3}^{4}D_r(z_i)F,$$

$$(z_1-z_2)e^{D_r(z_1)D_r(z_2)F}\partial_{t_0^{(r)}}\prod_{i=1}^{3}D_r(z_i)F = \tag{2.135}$$
$$z_2e^{-\partial_{t_0^{(r)}}D_r(z_2)F}\partial^2_{t_0^{(r)}}D_r(z_2)D_r(z_3)F - z_1e^{-\partial_{t_0^{(r)}}D_r(z_1)F}\partial^2_{t_0^{(r)}}D_r(z_1)D_r(z_3)F,$$

$$(z_1-z_2)e^{D_r(z_1)D_r(z_2)F}\partial^2_{t_0^{(r)}}D_r(z_2)D_r(z_1)F = \tag{2.136}$$
$$z_2e^{-\partial_{t_0^{(r)}}D_r(z_2)F}\partial^3_{t_0^{(r)}}D_r(z_2)F - z_1e^{-\partial_{t_0^{(r)}}D_r(z_1)F}\partial^3_{t_0^{(r)}}D_r(z_1)F$$

where we have assumed that three–point correlators vanish.

After expanding the above equations in powers of z_1 and z_2, and collecting terms of the form $z_1^m z_2^n$ we obtain equations involving two and four derivatives of F. We already know how to identify the two–derivative terms in terms of the Neumann coefficients of LCSFT (see previous section). The task now is to try to make the corresponding identifications for the four–derivative ones. The solutions to eqs. (2.134–2.136) are not uniquely defined (see below). The solutions we are interested in are as

follows:

$$
\begin{aligned}
F_{t_0^{(r)}t_0^{(r)}t_0^{(r)}t_m^{(r)}} &= -3(\alpha_r)^2\frac{1+\beta_r}{\beta_r}m(N_{m0}^{rr+1}-N_{m0}^{rr}),\\
F_{t_0^{(r)}t_0^{(r)}t_m^{(r)}t_n^{(r)}} &= -(\alpha_r)^2mn\left[\frac{1+\beta_r}{\beta_r}(n+m)N_{mn}^{rr}+2(N_{m0}^{rr+1}-N_{m0}^{rr})(N_{n0}^{rr+1}-N_{n0}^{rr})\right],\\
F_{t_0^{(r)}t_m^{(r)}t_n^{(r)}t_l^{(r)}} &= -(\alpha_r)^2mnl[(n+l)(N_{m0}^{rr+1}-N_{m0}^{rr})N_{nl}^{rr}+(n+m)(N_{l0}^{rr+1}-N_{l0}^{rr})N_{mn}^{rr}\\
&\quad +(m+l)(N_{n0}^{rr+1}-N_{n0}^{rr})N_{ml}^{rr}],\\
F_{t_m^{(r)}t_n^{(r)}t_l^{(r)}t_k^{(r)}} &= -(\alpha_r)^2mnlk[(m+n)(l+k)N_{mn}^{rr}N_{lk}^{rr}+(m+l)(n+k)N_{ml}^{rr}N_{nk}^{rr}\\
&\quad +(m+k)(n+l)N_{mk}^{rr}N_{nl}^{rr}],
\end{aligned}
\tag{2.137}
$$

where N_{m0}^{rs} are defined in eqs. (2.86). Multiplying these by the appropriate monomials of z variables and summing, leads to the following compact expressions (which will be used later on):

$$
\begin{aligned}
\partial_{t_0^{(r)}}^3 D_r(z)F &= \frac{3\alpha_{r-1}^2}{\beta_r}\frac{\varphi_r(z^{-1})-1}{(1+\beta_r)\varphi_r(z^{-1})-\beta_r},\\
\partial_{t_0^{(r)}}^2\prod_{i=1}^{2}D_r(z_i)F &= -3\alpha_{r-1}^2\prod_{i=1}^{2}\frac{\varphi_r(z_i^{-1})-1}{(1+\beta_r)\varphi_r(z_i^{-1})-\beta_r},\\
\partial_{t_0^{(r)}}\prod_{i=1}^{3}D_r(z_i)F &= 3\alpha_{r-1}^2\beta_r\prod_{i=1}^{3}\frac{\varphi_r(z_i^{-1})-1}{(1+\beta_r)\varphi_r(z_i^{-1})-\beta_r},\\
\prod_{i=1}^{4}D_r(z_i)F &= -3\alpha_{r-1}^2\beta_r^2\prod_{i=1}^{4}\frac{\varphi_r(z_i^{-1})-1}{(1+\beta_r)\varphi_r(z_i^{-1})-\beta_r}.
\end{aligned}
\tag{2.138}
$$

Using these expressions one can easily verify that they satisfy equations (2.134), (2.135) and (2.136).

A comment is in order concerning the results we have just written down. We have already said that eqs. (2.138) are not unique solutions to the Hirota equations for four–point functions. We have single them out because they are compatible with the dispersive Hirota equations (see next section) and they will allow us to find a solution of the latter coherent with the dispersionless Neumann coefficients. We recall that they are also compatible with vanishing three–point functions. As was mentioned at the beginning of this section, at first sight it would seem that the two–point functions, three–point functions and four–point functions (i.e. the derivatives of F with respect to two, three and four t_n parameters) of our system are simply given by

$$
\langle V_3|a_{n_1}^{r_1\dagger}\dots a_{n_k}^{r_k\dagger}|0\rangle. \tag{2.139}
$$

While this is true for two– and three–point functions, it is not quite true for the four–point ones, as one can see by comparing (2.139) with (2.137). It is evident that the

four–point functions given by (2.139) has the right form, but must be suitably combined in order to coincide with (2.137), which satisfy the Hirota equations. Therefore the correlators of the associated quantum system that underlies the Neumann coefficients of LCSFT are made of suitable combinations of the k–point functions (2.139). In conclusion: we have found a solution to the Hirota equations for the four–point functions, which is compatible with the Mandelstam's three strings vertex of the LCSFT (and with its deformations up to second order in the expansion parameter, see next section), however we do not have a proof that this solution is unique.

Next we would like say a little more about the general solution to eqs. (2.134–2.136). Using Hirota equations (2.134–2.136) one can easily express the four–point functions $\prod_{i=1}^{4} D_r(z_i)F$, $\partial_{t_0^{(r)}} \prod_{i=1}^{3} D_r(z_i)F$ and $\partial^2_{t_0^{(r)}} \prod_{i=1}^{2} D_r(z_i)F$ in terms of the two–point functions $\prod_{i=1}^{2} D_r(z_i)F$ and $\partial_{t_0^{(r)}} D_r(z)F$ as well as the four–point function $\partial^3_{t_0^{(r)}} D_r(z)F$. The two–point functions were already identified with the corresponding generating functions of the Neumann coefficients (see, eqs. (2.129)), so one is left with the problem of first finding a suitable identification for the four–point function $\partial^3_{t_0^{(r)}} D_r(z)F$, which leads to the straightforward derivation of all other. It is interesting that if one additionally requires that the four–point functions admit a factorizable form as functions of z_i, then the identification of $\partial^3_{t_0^{(r)}} D_r(z)F$ is uniquely fixed modulo three arbitrary $SL(2)$-group parameters a_r, b_r, c_r and d_r ($b_r c_r - a_r d_r = 1$) as follows

$$\partial^3_{t_0^{(r)}} D_r(z)F = \frac{\varphi_r(z^{-1}) - 1}{\varphi_r(z^{-1})} \, \frac{a_r \varphi_r(z^{-1}) + b_r}{c_r \varphi_r(z^{-1}) + d_r} \, . \tag{2.140}$$

Using this and eqs. (2.134–2.136), one can derive the corresponding explicit expressions for the remaining four–point functions

$$\partial^2_{t_0^{(r)}} \prod_{i=1}^{2} D_r(z_i)F = - \prod_{i=1}^{2} \frac{\varphi_r(z_i^{-1}) - 1}{c_r \varphi_r(z_i^{-1}) + d_r} \, , \tag{2.141}$$

$$\partial_{t_0^{(r)}} \prod_{i=1}^{3} D_r(z_i)F = -d_r \prod_{i=1}^{3} \frac{\varphi_r(z_i^{-1}) - 1}{c_r \varphi_r(z_i^{-1}) + d_r} \, , \tag{2.142}$$

$$\prod_{i=1}^{4} D_r(z_i)F = -d_r^2 \prod_{i=1}^{4} \frac{\varphi_r(z_i^{-1}) - 1}{c_r \varphi_r(z_i^{-1}) + d_r} \, . \tag{2.143}$$

The four–point functions (2.138) are a particular case of the latter for

$$d_r = -\frac{1}{a_r} = -\frac{\beta_r}{\sqrt{3}\ \alpha_{r-1}}, \quad b_r = 0, \quad c_r = -\frac{1}{\sqrt{3}\ \alpha_r}. \tag{2.144}$$

Let us remark that if one rescales the parameters in eqs. (2.140–2.143) as follows

$$\{a_r,\ b_r,\ c_r,\ d_r\} \quad \Rightarrow \quad \frac{1}{\epsilon} \{a_r,\ b_r,\ c_r,\ d_r\} \tag{2.145}$$

and consider the limit $\epsilon \to 0$, then the four–point function $\partial^3_{t_0^{(r)}} D_r(z)F$ preserves its form (2.140) with the parameters satisfying $b_r c_r - a_r d_r = 0$, so it becomes

$$\partial^3_{t_0^{(r)}} D_r(z)F = \frac{\varphi_r(z^{-1}) - 1}{\varphi_r(z^{-1})} \frac{a_r}{c_r} , \tag{2.146}$$

but all the other four–point functions degenerate and become equal to zero, $\partial^2_{t_0^{(r)}} \prod_{i=1}^2 D_r(z_i)F = \partial_{t_0^{(r)}} \prod_{i=1}^3 D_r(z_i)F = \prod_{i=1}^4 D_r(z_i)F = 0$.

2.4.4 PP–wave SFT and the dispersive Hirota equations

String theory on a maximally supersymmetric pp–wave background (plane wave limit of $AdS_5 \times S^5$) [28, 29], is exactly solvable [30]. Building on this it has been recently possible to construct the exact three strings vertex for the LCSFT on this background, i.e. to completely specify the relevant Neumann coefficients. They depend on the 'string mass' μ (determined by the five form flux of type IIB superstring theory). When $\mu \to 0$ one recovers Mandelstam's Neumann coefficients discussed in the previous sections. In other words the nontrivial string background deforms the Neumann coefficients. It is interesting to see whether the Hirota equations discussed in the previous sections get deformed accordingly in such a way as to preserve integrability. This is our guess and this is what we would like to provide evidence for in the remaining part of this chapter.

Let us start from the expression of the μ–deformed three strings vertex, [31, 32, 33, 34, 35, 36, 37, 38, 39, 40]. The bosonic part, to which we limit ourselves[1], is defined by

$$\Delta'_B = \frac{1}{2}\delta_{IJ} \left(\sum_{r,s=1}^{3} \sum_{m,n=-\infty}^{\infty} a_m^{(r)I\dagger} \mathcal{V}^{rs}_{mn} a_n^{(s)J\dagger} \right) . \tag{2.147}$$

The commutation relations are as in section 2, but $a_m^{(r)I\dagger} \neq a_{-m}^{(r)I}$ and $a_m^{(r)I}|0\rangle = 0$ for $m \in \mathbb{Z}$. The coefficients with negative n, m labels can be obtained from those with positive ones[2]. In this section we consider only the $\mathcal{V}^{rs}_{mn}$ with $m, n \geq 0$. We set

$$\mathcal{V}^{rs}_{mn} = -\sqrt{mn}\, \mathcal{N}^{rs}_{mn} . \tag{2.148}$$

[1]For the problems connected with the supersymmetric completion of the vertex and its prefactor see the reviews [41, 42, 43] and references therein.

[2]There is a subtle point here. The expressions of Δ'_B and Δ_B in eq.(2.85) for the three strings vertex refer to two different bases: Δ_B is expressed in terms of the transverse momenta p_r^I (momentum basis) while Δ'_B contains the zero mode oscillators $a_0^{(r)I}$ (oscillator basis). This means that starting from (2.84) we have passed from the former basis to the latter by explicitly integrating over the transverse momenta (see, for instance, [44]). This operation modifies in a well–known way the vertex coefficients. However, in the present case, it turns out that in the limit $\mu \to 0$ the coefficients $\mathcal{V}^{rs}_{mn}$ tend to the corresponding coefficients V^{rs}_{mn} in (2.85) at least for $m, n \geq 1$.

The Neumann coefficients $\mathcal{N}^{rs}_{mn}$ with $m, n \geq 1$ have been calculated in an explicit way in [39, 40]:

$$\mathcal{N}^{rs}_{mn} = -\frac{mn\alpha}{\alpha_s\omega_{r,m} + \alpha_r\omega_{s,n}} \frac{\mathcal{N}^r_m \mathcal{N}^s_n}{\alpha_r\alpha_s} \tag{2.149}$$

where $\omega_{r,m} = \sqrt{m^2 + \alpha_r^2\mu^2}$ and

$$\mathcal{N}^r_m = \sqrt{\frac{\omega_{r,m}}{m}} \frac{\omega_{r,m} + \alpha_r\mu}{m} f^{(r)}_m \ . \tag{2.150}$$

In these formulas

$$f^{(r)}_m = -\frac{e^{\tau_0\omega_{\frac{m}{\alpha_r}}}}{m(\alpha_r + \alpha_{r+1})\omega_{\frac{m}{\alpha_r}}} \frac{\Gamma^{(r+1)}_\mu\left(-\frac{m}{\alpha_r}\right)}{\Gamma^{(r)}_\mu\left(\frac{m}{\alpha_r}\right)\Gamma^{(r-1)}_\mu\left(\frac{m}{\alpha_r}\right)} \tag{2.151}$$

and $\omega_z = \mathrm{sgn}(z)\sqrt{\mu^2 + z^2}$. The μ–deformed Γ functions are defined by

$$\Gamma^{(r)}_\mu(z) = \frac{e^{-\gamma\alpha_r\omega_z}}{\alpha_r z} \prod_{n=1}^{\infty} \left(\frac{n}{\omega_{r,n} + \alpha_r\omega_z} e^{\frac{\alpha_r\omega_z}{n}}\right) \tag{2.152}$$

where γ is the Euler–Mascheroni constant. It must be remarked that the above coefficients are not written in the usual form, we have simplified them by dropping some intermediate inessential factors.

Here we present some results which are going to be needed in the next sections. Expanding (2.150) and (2.149) in powers of μ up to order μ^2 we obtain

$$\mathcal{N}^r_m = N^r_m \left[1 + \frac{\alpha_r}{m}\mu + \left(\frac{\alpha_r^2}{4m^2} + \frac{\alpha_r}{2m}\tau_0\right)\mu^2 + ...\right] \tag{2.153}$$

and

$$\mathcal{N}^{rs}_{mn} = N^{rs}_{mn} \left[1 + \left(\frac{\alpha_r}{m} + \frac{\alpha_s}{n}\right)\mu + \frac{1}{4}\left(\frac{\alpha_r}{m} + \frac{\alpha_s}{n}\right)\left(\frac{\alpha_r}{m} + \frac{\alpha_s}{n} + 2\tau_0\right)\mu^2 + ...\right] \tag{2.154}$$

where N^r_m and N^{rs}_{mn} are Mandelstam's Neumann coefficients. For reasons that will become clear in the next section, it is convenient to rewrite these expansions in powers of λ related to μ as: $\mu = \lambda - \frac{\tau_0}{2}\lambda^2 + ...$

$$\mathcal{N}^r_m = N^r_m \left[1 + \frac{\alpha_r}{m}\lambda + \left(\frac{\alpha_r^2}{4m^2}\right)\lambda^2 + ...\right] \equiv \sum_{j=0}^{\infty} \mathcal{N}^r_{j,m}\lambda^j, \tag{2.155}$$

$$\mathcal{N}^{rs}_{mn} = N^{rs}_{mn} \left[1 + \left(\frac{\alpha_r}{m} + \frac{\alpha_s}{n}\right)\lambda + \frac{1}{4}\left(\frac{\alpha_r}{m} + \frac{\alpha_s}{n}\right)^2\lambda^2 + ...\right] \equiv \sum_{j=0}^{\infty} \mathcal{N}^{rs}_{j,mn}\lambda^j \tag{2.156}$$

where the subscript j refers to the order of expansion in λ.

Again, for later use, it is convenient to organize these Neumann coefficients by means of associated generating functions

$$\mathcal{N}_j^r(z) := \sum_{m=1}^{\infty} \frac{1}{z^m}\, m^{2j}\, \mathcal{N}_{j,m}^r \,, \tag{2.157}$$

$$\mathcal{N}_j^{rs}(z_1,z_2) := \sum_{m,n=1}^{\infty} \frac{1}{z_1^m}\frac{1}{z_2^n}\, (mn)^j\, \mathcal{N}_{j,mn}^{rs} \,, \tag{2.158}$$

for $j = 0, 1, 2$. At each order the summation can easily be carried out to give

$$\mathcal{N}_1^r(z) = \alpha_r \frac{\varphi_r(z^{-1}) - 1}{(1+\beta_r)\varphi_r(z^{-1}) - \beta_r}, \tag{2.159}$$

$$\mathcal{N}_2^r(z) = \frac{\alpha_r^2}{4}\frac{\varphi_r(z^{-1})(\varphi_r(z^{-1}) - 1)}{((1+\beta_r)\varphi_r(z^{-1}) - \beta_r)^3}, \tag{2.160}$$

$$\mathcal{N}_1^{rs}(z_1,z_2) = -\frac{\alpha}{\alpha_r\alpha_s}\, \frac{\varphi_r(z_1^{-1}) - 1}{(1+\beta_r)\varphi_r(z_1^{-1}) - \beta_r}\, \frac{\varphi_s(z_2^{-1}) - 1}{(1+\beta_s)\varphi_s(z_2^{-1}) - \beta_s}, \tag{2.161}$$

$$\begin{aligned}\mathcal{N}_2^{rs}(z_1,z_2) = &-\frac{\alpha}{4\alpha_r\alpha_s}\frac{(\varphi_r(z_1^{-1}) - 1)(\varphi_s(z_2^{-1}) - 1)}{[(1+\beta_r)\varphi_r(z_1^{-1}) - \beta_r][(1+\beta_s)\varphi_s(z_2^{-1}) - \beta_s]} \\ &\times \left[\frac{\alpha_s\varphi_r(z_1^{-1})}{[(1+\beta_r)\varphi_r(z_1^{-1}) - \beta_r]^2} + \frac{\alpha_r\varphi_s(z_2^{-1})}{[(1+\beta_s)\varphi_s(z_2^{-1}) - \beta_s]^2}\right] .\end{aligned} \tag{2.162}$$

We notice that in order to get these compact generating functions it is necessary to insert the m^{2j} and $(mn)^j$ factors in (2.157) and (2.158), respectively.

2.4.5 The dispersive Hirota equations

Here, we consider the full (dispersive) Hirota equations for the Toda lattice hierarchy (see, e.g. ref. [10] and references therein). For the sake of brevity we deal in the sequel with half of the story, namely only with those equations that involve unbarred z variables (corresponding to eq.(2.124))

$$\begin{aligned}&z_1\left(e^{\lambda\left(\partial_{t_0} - D_r(z_1)\right)}\tau_\lambda\right)\left(e^{-\lambda D_r(z_2)}\tau_\lambda\right) - z_2\left(e^{\lambda\left(\partial_{t_0} - D_r(z_2)\right)}\tau_\lambda\right)\left(e^{-\lambda D_r(z_1)}\tau_\lambda\right) \\ &= (z_1 - z_2)\left(e^{-\lambda(D_r(z_1) + D_r(z_2))}\tau_\lambda\right)\left(e^{\lambda\partial_{t_0}}\tau_\lambda\right)\end{aligned} \tag{2.163}$$

where λ is a deformation parameter and τ_λ is the full *tau function* of the Toda lattice (KP) hierarchy:

$$\tau_\lambda = \exp(\mathcal{F}_\lambda), \qquad \mathcal{F}_\lambda = \frac{1}{\lambda^2}F_0 + \frac{1}{\lambda}F_1 + F_2 + \dots \quad . \tag{2.164}$$

In order to find a solution to (2.163) it is useful to proceed in two steps, and solve first the Hirota equation that does not involve t_0 derivatives (corresponding to the KP hierarchy), that is

$$(z_1 - z_2)\left(e^{-\lambda(D_r(z_1)+D_r(z_2))}\tau_\lambda\right)\left(e^{-\lambda D(z_3)}\tau_\lambda\right) + \text{cycl. perms. of} \quad 1,2,3 = 0. \quad (2.165)$$

Our conjecture is that these equations are obeyed by the Neumann coefficients $\mathcal{N}^{rs}_{mn}$ introduced above, provided we identify λ with a suitable function $f(\mu)$ of μ. At present we are not in the condition to prove this conjecture in a non–perturbative way. But we can expand in powers of λ and try to prove it order by order in λ. We will be able to do it up to second order in λ, with the identification $\lambda = \mu + \frac{\tau_0}{2}\mu^2 + \ldots$.

Expanding (2.163) in powers of λ we obtain an infinite set of equations that constrain the correlators at the different orders of approximation. To order 0 we get the dispersionless Hirota equation. Our task is therefore to prove that the next order equations hold.

Expanding (2.165) to first order in λ and identifying λ with μ we find

$$(z_1 - z_2)e^{D_r(z_1)D_r(z_2)F_0}D_r(z_1)D_r(z_2)F_1 + \text{cycl. perms. of} \quad 1,2,3 = 0 \quad (2.166)$$

while, expanding (2.163), we get

$$\begin{aligned} -z_1 e^{-\partial_{t_0^{(r)}}D_r(z_1)F_0}\partial_{t_0^{(r)}}D_r(z_1)F_1 + z_2 e^{-\partial_{t_0^{(r)}}D_r(z_2)F_0}\partial_{t_0^{(r)}}D_r(z_2)F_1 \\ = (z_1 - z_2)e^{D_r(z_1)D_r(z_2)F_0}D_r(z_1)D_r(z_2)F_1. \end{aligned} \quad (2.167)$$

In deriving these equations we have used the information that all third order derivatives of F_0 (three–point functions) vanish (see previous section). It is easy to see that if we make the following identifications:

$$D_r(z_1)D_r(z_2)F_1 = \mathcal{N}_1^{rr}(z_1, z_2) \quad (2.168)$$

with $D_r(z_1)D_r(z_2)F_0$ as in (2.129), eq.(2.166) is satisfied. If in addition we identify

$$\partial_{t_0^{(r)}}D_r(z)F_1 = \beta_{r+1}\beta_r\mathcal{N}_1^r(z), \quad (2.169)$$

with $\partial_{t_0^{(r)}}D_r(z)F_0$ as in (2.129), the more general eq.(2.167) is satisfied as well.

Expanding (2.165) to order λ^2 and identifying λ with $\mu + \frac{\tau_0}{2}\mu^2 + \ldots$ we find

$$\begin{aligned} & (z_1 - z_2)e^{D_r(z_1)D_r(z_2)F_0}\left(D_r(z_1)D_r(z_2)F_2 + \frac{1}{2}(D_r(z_1)D_r(z_2)F_1)^2\right. \\ + \quad & \left.\frac{1}{6}D_r(z_1)D_r(z_2)^3F_0 + \frac{1}{4}D_r(z_1)^2D_r(z_2)^2F_0 + \frac{1}{6}D_r(z_1)^3D_r(z_2)F_0\right) \\ & + \text{cycl. perms. of } 1,2,3 = 0 \end{aligned} \quad (2.170)$$

while expanding (2.163) to the same order we get

$$
\begin{aligned}
& z_1 e^{-\partial_{t_0^{(r)}} D_r(z_1) F_0} \left(-\partial_{t_0^{(r)}} D_r(z_1) F_2 + \frac{1}{2} (\partial_{t_0^{(r)}} D_r(z_1) F_1)^2 + \frac{1}{4} \partial^2_{t_0^{(r)}} D_r(z_1)^2 F_0 \right. \\
- \; & \left. \frac{1}{6} \partial_{t_0^{(r)}} D_r(z_1)^3 F_0 - \frac{1}{6} \partial^3_{t_0^{(r)}} D_r(z_1) F_0 \right) \\
- \; & z_2 e^{-\partial_{t_0^{(r)}} D_r(z_2) F_0} \left(-\partial_{t_0^{(r)}} D_r(z_2) F_2 + \frac{1}{2} (\partial_{t_0^{(r)}} D_r(z_2) F_1)^2 \right. \\
+ \; & \left. \frac{1}{4} \partial^2_{t_0^{(r)}} D_r(z_2)^2 F_0 - \frac{1}{6} \partial_{t_0^{(r)}} D_r(z_2)^3 F_0 - \frac{1}{6} \partial^3_{t_0^{(r)}} D_r(z_2) F_0 \right) \\
= \; & (z_1 - z_2) e^{D_r(z_1) D_r(z_2) F_0} \left(D_r(z_1) D_r(z_2) F_2 + \frac{1}{2} \left(D_r(z_1) D_r(z_2) F_1 \right)^2 \right. \\
+ \; & \left. \frac{1}{6} D_r(z_1) D_r(z_2)^3 F_0 + \frac{1}{4} D_r(z_1)^2 D_r(z_2)^2 F_0 + \frac{1}{6} D_r(z_1)^3 D_r(z_2) F_0 \right) . \qquad (2.171)
\end{aligned}
$$

In deriving this equation we have used once again the information that all odd order derivatives of F_0 and F_1 vanish. Eq.(2.170) is satisfied provided we make the following identifications:

$$
D_r(z_1) D_r(z_2) F_2 = \mathcal{N}_2^{rr}(z_1, z_2) \qquad (2.172)
$$

with four-derivatives as defined in (2.138). If, in addition, we identify

$$
\partial_{t_0^{(r)}} D_r(z) F_2 = \beta_{r+1} \beta_r \mathcal{N}_2^r(z) \qquad (2.173)
$$

the more general equation (2.171) is satisfied.

Taking into account eqs. (2.129), (2.168–2.169) and (2.172–2.173) it is plausible to expect that the following universal identification is valid in general:

$$
\begin{aligned}
D_r(z_1) D_r(z_2) F_j &= \mathcal{N}_j^{rr}(z_1, z_2), \\
\partial_{t_0^{(r)}} D_r(z) F_j &= \beta_{r+1} \beta_r \mathcal{N}_j^r(z), j = 0, 1, 2, 3, ... \qquad (2.174)
\end{aligned}
$$

although in this case one may have to modify the definitions of the generating functions for $j > 2$.

In this section we are able to show that the Hirota equations for the full Toda lattice hierarchy expanded in powers of the deformation parameter are satisfied by the Neumann coefficients of the 3-string vertex of the LCSFT in a maximally supersymmetric pp-wave background, expanded in powers of the μ parameter, up to second order. The calculations beyond second order become technically very challenging and it is clear that a non–perturbative approach is needed for a satisfactory proof that LCSFT three–strings vertex obey the Hirota equations for the full Toda lattice hierarchy.

The type of hierarchy underlying the three strings vertex of LCSFT as we have mentioned is the Toda lattice hierarchy, but one should be more precise. Looking at section 2.4.2, it is evident that one has to do not with one Toda lattice hierarchy but rather with three coupled copies of the Toda lattice hierarchy. As we mentioned in the same section, there is the possibility of a consistent reduction. A precise formulation and identification of the relevant reduced hierarchies is a task we have not tackled here.

As we have pointed, the four–point functions we have found are not the unique solutions to the Hirota equations for four–point functions. The one that we have found however are likely to be the unique solution that are consistent with the two–point functions of order 0, 1 and 2 in the μ expansion (although we have not been able to exclude other solutions). This question is intertwined with two related problems: on the one hand the question of defining in terms of matter oscillators $a_n^{(r)}$ the 1D associated quantum system whose two–point correlators are given by $\langle V_3|a_m^{(r)\dagger}a_n^{(s)\dagger}|0\rangle$ and underlies the LCSFT three strings vertex; on the other hand defining an integrable system where all these correlators can explicitly be calculated. A successful search of such an integrable model is also likely to lead us to a natural explanation of why the somewhat mysterious factors m^{2j} and $(nm)^j$ need to be inserted in the generating functions (2.157–2.158) in order to square matters.

2.5 Analytic solution for Witten's OSFT and the proofs of Sen's first and second conjectures

Despite a large effort made towards constructing analytic solutions for Witten's cubic OSFT, such solutions were lacking and tests of the Sen's conjectures were made either numerically or using VSFT, where some of the conjectures were true by construction. However, it has been lately found that (see [85] and [18, 19] for simplified versions) the Witten's OSFT equation of motion can be solved analytically and the solution are proved to satisfy the first two of the Sen's three conjectures [18, 17, 85]. The complexity of the equation of motion were overcome by using an intelligent choice of worldsheet complex coordinates which simplify the star product and a novel gauge choice, which also simplify the kinetic term. Before Schnabl came up with this new gauge the most popular gauge choice in use was the Siegel gauge for which the OSFT equation of motion is

$$\Psi + \frac{b_0}{L_0}(\Psi \star \Psi) = 0. \tag{2.175}$$

It seems natural to look for an analytic solution of this equation that is made up of wedge states(which are surface states obeying a simple multiplication rule $|m\rangle \star |n\rangle = |m+n-1\rangle$) with extra ghost insertions to obtain ghost number one string field. However this was not possible because, even though the star multiplication

give back a state in the same family, the application of the operator $\frac{b_0}{L_0}$ will take the state out of this family. Schnabl's main observation [85] is to choose instead a gauge $\mathcal{B}_0\Psi = 0$, where $\mathcal{B}_0$ is the zero mode of the b ghost in a new conformal frame $z = f(\xi) = \arctan(\xi)$ with ξ is a local coordinate of the upper half disc with puncture at $\xi = 0$.

$$\mathcal{B}_0 = \oint \frac{dz}{2\pi i} z\, b(z) = \oint \frac{d\xi}{2\pi i}(1+\xi^2)\arctan(\xi) b(\xi) = b_0 + \sum_{k=1}^{\infty} \frac{2(-1)^{k+1}}{4k^2-1} b_{2k} \quad (2.176)$$

Similarly the L_0 operator has a corresponding operator $\mathcal{L}_0$ in this new frame given by

$$\mathcal{L}_0 = \oint \frac{dz}{2\pi i} z\, T(z) = \oint \frac{d\xi}{2\pi i}(1+\xi^2)\arctan(\xi) T(\xi) = L_0 + \sum_{k=1}^{\infty} \frac{2(-1)^{k+1}}{4k^2-1} L_{2k} \quad (2.177)$$

Under the map $\xi \to f(\xi)$ the upper half plane is mapped to semi–infinite strip of width π, while the upper half unit disk is mapped to a semi–infinite strip of width $\pi/2$ (see figure 2.7). In this frame star product can be described in a very simple

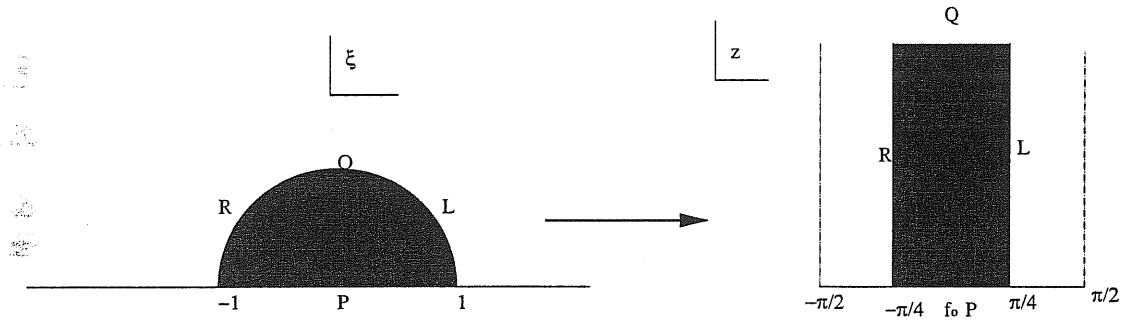


Figure 2.7: *A string field as seen by two coordinate systems*

manner as gluing of the strips representing each string state to get a strip with width the sum of the widths of the individual strips (see figure 2.8). A wedge state $|n\rangle$ is a state represented by a strip of width $(n-1)\pi/2$ and the star multiplication rule for these states is obvious in this frame. In particular the states $n = 2$ and $n = 1$ are the $SL(2,\mathbf{R})$ invariant vacuum and the identity state, respectively, while $n \to \infty$ is the sliver. It is also possible to write

$$|n\rangle = (|0\rangle_\star)^{n-1} \quad (2.178)$$

which means we obtain any wedge state by star multiplying the vacuum $(n-1)$ times.

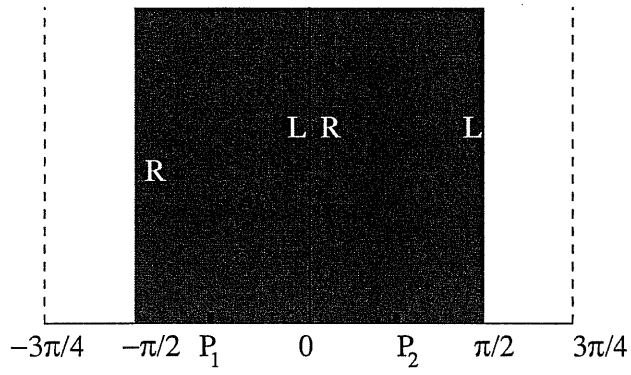


Figure 2.8: *The star product of string states as it is done in the new frame*

The following operators are also needed in the sequel

$$\begin{aligned} B_1^{R,L} &= \int_{C_{R,L}} \frac{d\xi}{2\pi i}(1+\xi^2)b(\xi) = (-,+)B = (-,+)\int_{i\infty}^{-i\infty} \frac{dz}{2\pi i} b(z) \\ K_1^{R,L} &= \int_{C_{R,L}} \frac{d\xi}{2\pi i}(1+\xi^2)T(\xi) = (-,+)K = (-,+)\int_{i\infty}^{-i\infty} \frac{dz}{2\pi i} T(z) \end{aligned} \quad (2.179)$$

where R stands for contour integration over the right half of a unit circle and L stands for contour integration over the left half of a unit circle. T is energy momentum tensor.

2.5.1 The tachyon vacuum solution

Let us start by defining a state ψ_n for $n > 0$ as

$$\psi_n = \frac{1}{\pi} c_1|0\rangle \star (B_1^L - B_1^R)|n\rangle \star c_1|0\rangle. \quad (2.180)$$

We note that the all the states that make up ψ_n are wedge sates with a c or b ghost insertion. From their representation in the z frame it is easy to see that $B_1^L|n\rangle = -B_1^R|n\rangle = B|n\rangle$ (similar relation holds for the K's) for any wedge state $|n\rangle$ so that we can write

$$\psi_n = \frac{2}{\pi} c_1|0\rangle \star B_1^L|n\rangle \star c_1|0\rangle \quad (2.181)$$

By definition an inner product $\langle\ ,\ \rangle$ is the star product followed by integration Eq.(2.10). In the z frame this corresponds to the identification of the two sides of the strip representing the product state. Hence, for any Fock space state $|\phi\rangle = \phi|0\rangle$ (it is a strip of width $\pi/2$ with $f \circ \phi(0)$ insertion at $z = 0$), the inner product $\langle\phi, \psi_n\rangle$ is

given by

$$\langle \phi, \psi_n \rangle = \frac{2}{\pi} \left\langle f \circ \phi(0) c(\pi/2) Bc \left(\frac{\pi(n+1)}{2} \right) \right\rangle_{C_{\frac{\pi(n+2)}{2}}} \tag{2.182}$$

where $C_{\frac{\pi(n+2)}{2}}$ is a cylinder obtained when we identify the two free sides of the strip representing $\phi|0\rangle \star |\psi_n\rangle$ (see figure 2.9). Since the contour of the integral in B can be

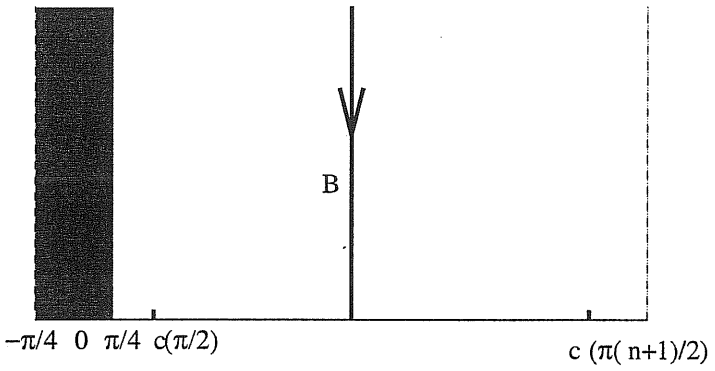


Figure 2.9: *A representation of the inner product in the new frame*

deformed as long as it passes between $\pi/2$ and $\frac{\pi(n+1)}{2}$ we can also write

$$\psi_n = \frac{2}{\pi} c_1|0\rangle \star |n\rangle \star B_1^L c_1|0\rangle. \tag{2.183}$$

The wedge state $|n\rangle$ for $n = 1$ is the identity state of the star product i.e.

$$\psi_1 = \frac{2}{\pi} c_1|0\rangle \star B_1^L c_1|0\rangle. \tag{2.184}$$

For $n \leq 0$ the wedge state is singular but the state ψ_0 can be defined by a limit and is given by

$$\psi_0 = \frac{2}{\pi} c_1|0\rangle \tag{2.185}$$

In terms of the operators K_1^L, K_1^R the wedge state is written as [21, 85]

$$|n\rangle = e^{\frac{\pi(n-2)}{4}(K_1^L - K_1^R)}|0\rangle = e^{\frac{\pi(n-2)}{2} K_1^L}|0\rangle. \tag{2.186}$$

Treating n as a continuous parameter we can now write the derivative of ψ_n as

$$\psi_n' = c_1|0\rangle \star K_1^L|n\rangle \star B_1^L c_1|0\rangle. \tag{2.187}$$

The inner product of this state with a Fock space state $|\phi\rangle$ is given by

$$\langle \phi, \psi'_n \rangle = \left\langle f \circ \phi(0) c(\pi/2) BKc\left(\frac{\pi(n+1)}{2}\right)\right\rangle_{C_{\frac{\pi(n+2)}{2}}} \tag{2.188}$$

which is exactly like in figure 3.9 with additional arrow insertion for the operator K any where between $\pi/2$ and $\pi(n+1)/2$. K commutes with B and its contour of integral can be deformed in any way between the c insertions so that ψ'_n can be written as

$$\psi'_n = c_1|0\rangle \star |n\rangle \star B_1^L K_1^L c_1|0\rangle. \tag{2.189}$$

Again the cases $n = 0, 1$ are special and theses states take the form

$$\psi'_1 = c_1|0\rangle \star B_1^L K_1^L c_1|0\rangle, \qquad \psi'_0 = K_1^L c_1|0\rangle + B_1^L c_0 c_1|0\rangle. \tag{2.190}$$

Now let us define a string field Ψ_λ as

$$\Psi_\lambda = \sum_{n=0}^{\infty} \lambda^{n+1} \psi'_n \tag{2.191}$$

for any real λ. If the two conditions

$$Q_B \psi'_0 = 0, \qquad Q_B \psi'_{n+1} = -\sum_{m=0}^{n} \psi'_m \star \psi'_{n-m} \tag{2.192}$$

are satisfied then Ψ_λ is the solution to the OSFT equation of motion. Our task next is then to prove these two conditions. We start by writing the BRST transformations of the operators $c(z)$ and B, which are

$$Q_B.c(z) = c\partial c(z), \qquad Q_B.B = K. \tag{2.193}$$

Since $Q_B^2 = 0$ the BRST transformations of the operators $c\partial c(z)$ and K are zero. For any Fock space state ϕ it then follows

$$\begin{aligned} \langle \phi, Q_B \psi'_0 \rangle &= \left\langle f \circ \phi(0) Q_B.\left(K_1^L c(\frac{\pi}{2})\right)\right\rangle_{C_\pi} \\ &- \left\langle f \circ \phi(0) Q_B.\left(B_1^L c\partial c(\frac{\pi}{2})\right)\right\rangle_{C_\pi} = 0. \end{aligned} \tag{2.194}$$

This proves the first condition. Similarly the left hand side of the second condition contracted with ϕ takes the form

$$\langle \phi, Q_B \psi'_n \rangle = \left\langle f \circ \phi(0) Q_B.\left(c(\frac{\pi}{2}) BKc\left(\frac{\pi(n+1)}{2}\right)\right)\right\rangle_{C_{\frac{\pi(n+2)}{2}}}$$

$$
\begin{aligned}
&= \left\langle f\circ\phi(0)c\partial c(\frac{\pi}{2})BKc\left(\frac{\pi(n+1)}{2}\right)\right\rangle_{C_{\frac{\pi(n+2)}{2}}} \\
&- \left\langle f\circ\phi(0)c(\frac{\pi}{2})K^2c\left(\frac{\pi(n+1)}{2}\right)\right\rangle_{C_{\frac{\pi(n+2)}{2}}} \\
&+ \left\langle f\circ\phi(0)c(\frac{\pi}{2})BKc\partial c\left(\frac{\pi(n+1)}{2}\right)\right\rangle_{C_{\frac{\pi(n+2)}{2}}} . \qquad (2.195)
\end{aligned}
$$

The star product $\psi'_m \star \psi'_{n-m}$ is done by gluing the strips representing each string field and displacing the insertions accordingly. Doing that we can write the inner product of ϕ with the left hand side of the second condition as

$$
\begin{aligned}
&\langle \phi, \psi'_m \star \psi'_{n-m}\rangle = \\
&\left\langle f\circ\phi(0)c(\frac{\pi}{2})KBc\left(\frac{\pi(m+1)}{2}\right)c\left(\frac{\pi(m+2)}{2}\right)BKc\left(\frac{\pi(n+2)}{2}\right)\right\rangle_{C_{\frac{\pi(n+3)}{2}}} \qquad (2.196)
\end{aligned}
$$

Using the fact that $\{B, c(z)\} = 1$ and $B^2 = 0$ it is trivial to show $Bc(z_1)c(z_2)B = Bc(z_1) - Bc(z_2)$, for $z_1 < z_2$, which leads to

$$
\begin{aligned}
&\langle \phi, \psi'_m \star \psi'_{n-m}\rangle \\
&= \left\langle f\circ\phi(0)c(\frac{\pi}{2})KBc\left(\frac{\pi(m+1)}{2}\right)Kc\left(\frac{\pi(n+2)}{2}\right)\right\rangle_{C_{\frac{\pi(n+3)}{2}}} \\
&- \left\langle f\circ\phi(0)c(\frac{\pi}{2})KBc\left(\frac{\pi(m+2)}{2}\right)Kc\left(\frac{\pi(n+2)}{2}\right)\right\rangle_{C_{\frac{\pi(n+3)}{2}}} . \qquad (2.197)
\end{aligned}
$$

When we make the summation over m from 0 to n every term cancel between the first and the second term except the $m = 0$ of the first and $m = n$ of the second, and hence we get

$$
\begin{aligned}
\sum_{m=0}^{n}\langle \phi, \psi'_m \star \psi'_{n-m}\rangle = &- \left\langle f\circ\phi(0)c\partial c(\frac{\pi}{2})BKc\left(\frac{\pi(n+1)}{2}\right)\right\rangle_{C_{\frac{\pi(n+2)}{2}}} \\
&+ \left\langle f\circ\phi(0)c(\frac{\pi}{2})K^2c\left(\frac{\pi(n+1)}{2}\right)\right\rangle_{C_{\frac{\pi(n+2)}{2}}} \\
&- \left\langle f\circ\phi(0)c(\frac{\pi}{2})BKc\partial c\left(\frac{\pi(n+1)}{2}\right)\right\rangle_{C_{\frac{\pi(n+2)}{2}}} . \qquad (2.198)
\end{aligned}
$$

This is $(-)$ the right hand side of (2.195) with $n \to (n+1)$ and hence

$$
\langle \phi, Q_B\psi'_{n+1}\rangle = -\sum_{m=0}^{n}\langle \phi, \psi'_m \star \psi'_{n-m}\rangle . \qquad (2.199)
$$

This is what we wanted to show.

The Schnabl's solution consists of two pieces, the piece we have just written (2.191) with $\lambda = 1$ and another piece given by $(\lim_{N\to\infty} \psi_N)$. This second piece was proved to play no role in solving the equation of motion but play an important role (see below) in proving first and second of the Sen's three conjectures, which are:

- The energy difference between the non–perturbative vacuum, given by the Schnabl's solution (also called tachyon vacuum) and the original perturbative vacuum exactly matches with the D–25 brane energy, hence the new vacuum (non-perturbative vacuum) represent a configuration with no perturbative open string degrees of freedom,

- The cohomology of the BRST operator of vacuum string field theory (a string field theory at the tachyon vacuum) is trivial, indicating that there are no open string degrees of freedom at the non-perturbative vacuum,

- The action of vacuum string field theory must have classical solutions representing the original D–25 brane configuration as well as lump solutions representing D-branes of lower dimension.

In [85], solutions with $\lambda \neq 1$ were referred to as pure-gauge solution.

2.5.2 Proofs of the Sen's first and second conjectures

The gauge nontrivial tachyon vacuum solution we will use is

$$\Psi_0 = \lim_{N\to\infty} \left(\psi_N - \sum_{n=0}^{N} \psi'_n \right). \tag{2.200}$$

The Sen's first conjecture states, the energy density of the tachyon vacuum should be equal to minus the tension of the D25–brane $\left(T_{D25} = -\frac{1}{2\pi^2 g^2}\right)$. The energy density of the tachyon vacuum is minus the action evaluated at the classical solution, so we want to show

$$T_{D25} = \frac{1}{g^2} \left(\frac{1}{2} \langle \Psi_0, Q_B \Psi_0 \rangle + \frac{1}{3} \langle \Psi_0, \Psi_0 \star \Psi_0 \rangle \right). \tag{2.201}$$

Since Ψ_0 is the solution to the equation of motion what we have to show is

$$\begin{aligned} \langle \Psi_0, Q_B \Psi_0 \rangle &= \lim_{N\to\infty} \left(\langle \psi_N, Q_B \psi_N \rangle - 2 \sum_{n=0}^{N} \langle \psi_N, Q_B \psi'_n \rangle + \sum_{n=0}^{N} \sum_{m=0}^{N} \langle \psi'_n, Q_B \psi'_m \rangle \right) \\ &= -\frac{3}{\pi^2}. \end{aligned} \tag{2.202}$$

The inner product $\langle\psi_n, Q_B\psi_m\rangle$ was calculated in [85] and is given by

$$\begin{aligned}\langle\psi_n, Q_B\psi_m\rangle &= \\ &\frac{1}{\pi^2}\left(1+\cos\frac{\pi(m-n)}{m+n+2}\right)\left(-1+\frac{m+n+2}{\pi}\sin\frac{2\pi}{m+n+2}\right) \\ &+ 2\sin^2\frac{\pi}{m+n+2}\left(-\frac{m+n+1}{\pi^2}+\frac{mn}{\pi^2}\cos\frac{\pi(m-n)}{m+n+2}\right. \\ &+ \left.\frac{(m+n+2)(m-n)}{2\pi^3}\sin\frac{\pi(m-n)}{m+n+2}\right) \qquad (2.203)\end{aligned}$$

When the sum $m+n$ is large, this inner product will reduce to

$$\begin{aligned}\lim_{m+n\to\infty}\langle\psi_n, Q_B\psi_m\rangle &= \frac{1}{\pi^2}\left(1+\cos\frac{\pi(m-n)}{m+n}\right)+\frac{2mn}{(m+n)^2}\cos\frac{\pi(m-n)}{m+n} \\ &+ \frac{m-n}{\pi(m+n)}\sin\frac{\pi(m-n)}{m+n}. \qquad (2.204)\end{aligned}$$

Let us introduce a function

$$\begin{aligned}\mathcal{F}(x,y) &= \frac{1}{\pi^2}\left(1+\cos\frac{\pi(x-y)}{x+y}\right)+\frac{2xy}{(x+y)^2}\cos\frac{\pi(x-y)}{x+y} \\ &+ \frac{x-y}{\pi(x+y)}\sin\frac{\pi(x-y)}{x+y} \qquad (2.205)\end{aligned}$$

which satisfies $\mathcal{F}(x,0)=\mathcal{F}(0,y)=0$ and $\mathcal{F}(ax,ay)=\mathcal{F}(x,y)$ for any $a\neq 0$. Using these relations we can write

$$\lim_{m+n\to\infty}\langle\psi_n, Q_B\psi_m\rangle = \mathcal{F}(an,am), \qquad a\neq 0. \qquad (2.206)$$

In particular choosing a to be $\frac{1}{N}$ this can be written as

$$\lim_{m+n\to\infty}\langle\psi_n, Q_B\psi_m\rangle = \mathcal{F}\left(\frac{n}{N},\frac{m}{N}\right). \qquad (2.207)$$

The first term in (2.202) is then

$$\lim_{N\to\infty}\langle\psi_N, Q_B\psi_N\rangle = \mathcal{F}(1,1) = \frac{2}{\pi^2}+\frac{1}{2}, \qquad (2.208)$$

while the second term can be written as

$$\lim_{N\to\infty}\sum_{n=0}^{N}\langle\psi_n', Q_B\psi_N\rangle = \lim_{N\to\infty}\sum_{n=0}^{N}\frac{\partial}{\partial n}\mathcal{F}\left(\frac{n}{N},1\right) = \lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N}\frac{\partial}{\partial x}\mathcal{F}(x,1)\mid_{x=\frac{n}{N}} \qquad (2.209)$$

Using the formula

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N}\frac{\partial}{\partial x}f\left(\frac{n}{N}\right)=\int_0^1 dx f(x) \tag{2.210}$$

we obtain

$$\lim_{N\to\infty}\sum_{n=0}^{N}\langle\psi_n',Q_B\psi_N\rangle=\int_0^1 dx\frac{\partial}{\partial n}\mathcal{F}\left(\frac{n}{N},1\right)=\mathcal{F}(1,1)-\mathcal{F}(0,1)=\frac{2}{\pi^2}+\frac{1}{2}. \tag{2.211}$$

The computation of the third term needs the decomposition of the double sum as follows

$$\begin{aligned}\sum_{n=0}^{N}\sum_{m=0}^{N}\langle\psi_n',Q_B\psi_m'\rangle&=\sum_{m=0}^{N}\sum_{n=0}^{N-m}\langle\psi_n',Q_B\psi_m'\rangle+\sum_{m=1}^{N}\sum_{n=N-m+1}^{N}\langle\psi_n',Q_B\psi_m'\rangle\\&=\sum_{n=0}^{N}\sum_{m=0}^{n}\langle\psi_m',Q_B\psi_{n-m}'\rangle+\sum_{m=1}^{N}\sum_{n=N-m+1}^{N}\langle\psi_n',Q_B\psi_m'\rangle.\end{aligned} \tag{2.212}$$

Using (2.203) is can be shown that the first term in the last line is is vanishing [85] so that we can write

$$\begin{aligned}\lim_{N\to\infty}\sum_{n=0}^{N}\sum_{m=0}^{N}\langle\psi_n',Q_B\psi_m'\rangle&=\lim_{N\to\infty}\sum_{m=1}^{N}\sum_{n=N-m+1}^{N}\langle\psi_n',Q_B\psi_m'\rangle\\&=\lim_{N\to\infty}\sum_{m=1}^{N}\sum_{n=N-m+1}^{N}\frac{\partial}{\partial n}\frac{\partial}{\partial m}\mathcal{F}\left(\frac{n}{N},\frac{m}{N}\right)\\&=\lim_{N\to\infty}\frac{1}{N^2}\sum_{m=1}^{N}\sum_{n=N-m+1}^{N}\frac{\partial}{\partial x}\frac{\partial}{\partial y}\mathcal{F}(x,y)\mid_{x=\frac{n}{N},y=\frac{m}{N}}\\&=\int_0^1 dy\int_{1-y}^1 dx\frac{\partial}{\partial x}\frac{\partial}{\partial y}\mathcal{F}(x,y)=\frac{1}{2}-\frac{1}{\pi^2}.\end{aligned} \tag{2.213}$$

Putting back equation (2.208, 2.211, 2.213) into (2.202) we obtain

$$\langle\Psi_0,Q_B\Psi_0\rangle=-\frac{3}{\pi^2} \tag{2.214}$$

which completes the proof of Sen's first conjecture.

Sen's second conjecture states that the new kinetic operator $\hat{Q}$ given by (3.2) has a vanishing cohomology so that there are no open string states at the tachyon vacuum. This has been proved in [17] and we summarize it here. It is trivial to proof that (see below) $\hat{Q}$ has a vanishing cohomology if and only if there exist a state A such

that $\hat{Q}A = \mathcal{I}$ where $\mathcal{I}$ is the identity of the star algebra. Hence, the the problem of showing that $\hat{Q}$ has vanishing cohomology reduces to showing if such state A exists.

The proof of the former statement goes as follows: Assume $\hat{Q}$ has a vanishing cohomology. Then (3.2) gives

$$\hat{Q}\mathcal{I} = Q_B\mathcal{I} + \Psi_0 \star \mathcal{I} - \mathcal{I} \star \Psi_0 = Q_B\mathcal{I}. \tag{2.215}$$

We already know that (2.12) $Q_B\mathcal{I} = 0$ it follows that $\hat{Q}\mathcal{I} = 0$ i.e. $\mathcal{I}$ is $\hat{Q}$ closed. Since $\hat{Q}$ has a vanishing cohomology this implies that there must exist A such that $\mathcal{I} = \hat{Q}A$ and that is half of the poof of the statement. The other half goes as follows: Assume we have a state A such that $\hat{Q}A = \mathcal{I}$ and a $\hat{Q}$ closed state Λ ($\hat{Q}\Lambda = 0$), then

$$\hat{Q}(A \star \Lambda) = \hat{Q}(A) \star \Lambda = \mathcal{I} \star \Lambda = \Lambda \tag{2.216}$$

so that Λ is $\hat{Q}$ exact. Therefore, as any $\hat{Q}$ closed state Λ is $\hat{Q}$ exact it follows that $\hat{Q}$ has vanishing cohomology.

Now to conclude the proof of Sen's second conjecture what remains to proof is the existence of the state A with the prescribed property. In [20] while trying to show that the perturbative vacuum kinetic operator Q_B has a vanishing cohomology using the same trick as we are using here, it was found that the state which plays the same role as the state A in Siegel gauge, has an approximate form of

$$A_S \sim \frac{1}{L_0} b_0 \mathcal{I}. \tag{2.217}$$

In that case however, $Q_B A_S = \mathcal{I} - |0\rangle$ so that the cohomology of Q_B is non-vanishing.

A natural guess for Schnabl's gauge is to take the same form of A but with b_0 and L_0 replaced by $\mathcal{B}_0$ and $\mathcal{L}_0$, respectively.

$$A = \frac{1}{\mathcal{L}_0} \mathcal{B}_0 \mathcal{I}. \tag{2.218}$$

It is a property of the identity state that for any operator $\mathcal{O}$ and its BPZ conjugate $\mathcal{O}^\star$

$$\mathcal{O}\mathcal{I} = \mathcal{O}^\star\mathcal{I} = \frac{1}{2}(\mathcal{O} + \mathcal{O}^\star)\mathcal{I}, \tag{2.219}$$

so that we can write

$$A = \frac{1}{2\mathcal{L}_0}(\mathcal{B}_0 + \mathcal{B}_0^\star)\mathcal{I}. \tag{2.220}$$

An alternative way of writing a wedge state is give in [21] which is

$$|n\rangle = e^{\frac{2-n}{2}(\mathcal{L}_0 + \mathcal{L}_0^\star)}|0\rangle \tag{2.221}$$

so that

$$|\mathcal{I}\rangle = e^{\frac{1}{2}(\mathcal{L}_0+\mathcal{L}_0^\star)}|0\rangle \tag{2.222}$$

Using this in (2.220) we obtain

$$\begin{aligned} A &= \frac{1}{2\mathcal{L}_0}(\mathcal{B}_0+\mathcal{B}_0^\star)\sum_{n=0}^{\infty}\frac{1}{2^n n!}(\mathcal{L}_0+\mathcal{L}_0^\star)^n|0\rangle = \sum_{n=0}^{\infty}(\mathcal{B}_0+\mathcal{B}_0^\star)\frac{1}{2^n(n+1)!}(\mathcal{L}_0+\mathcal{L}_0^\star)^n|0\rangle \\ &= \frac{1}{2}(\mathcal{B}_0+\mathcal{B}_0^\star)\frac{e^{(\mathcal{L}_0+\mathcal{L}_0^\star)/2}-1}{(\mathcal{L}_0+\mathcal{L}_0^\star)/2}|0\rangle = \frac{1}{2}(\mathcal{B}_0+\mathcal{B}_0^\star)\int_1^2 dr e^{\frac{2-r}{2}(\mathcal{L}_0+\mathcal{L}_0^\star)}|0\rangle \\ &= \frac{1}{2}(\mathcal{B}_0+\mathcal{B}_0^\star)\int_1^2 dr|r\rangle = \frac{\pi}{2}B_1^L\int_1^2 dr|r\rangle \end{aligned} \tag{2.223}$$

Next we will show that this state satisfies the property $\hat{Q}A = \mathcal{I}$. The first term in this expression is $Q_B A$ which is given by

$$Q_B A = \frac{\pi}{2}K_1^L\int_1^2 dr|r\rangle = -\int_1^2 dr\partial_r|r\rangle = \mathcal{I}-|0\rangle \tag{2.224}$$

where we have used $Q_B.B_1^L = K_1^L$ and the representation (2.186) of the wedge state $|r\rangle$. Next we have to compute $\Psi_0 \star A$ and $A \star \Psi_0$. We begin by evaluating $\psi_n \star A$, which can be written as

$$\psi_n \star A = \int_1^2 dr c_1|0\rangle \star B_1^L\left(|n\rangle \star c_1|0\rangle\right) \star B_1^L|r\rangle. \tag{2.225}$$

Using the very known facts [4, 17] $B_1^L|n\rangle = -B_1^R|n\rangle$ for any wedge state $|n\rangle$, $B_1^L(\Phi \star \Psi) = B_1^L(\Phi)\star\Psi$ and $B_1^R(\Phi\star\Psi) = (-1)^{\#_{gh}(\Phi)}(\Phi)\star(B_1^R\Psi)$ we can rewrite (2.225) as

$$\psi_n \star A = \int_1^2 dr c_1|0\rangle \star B_1^L B_1^R(|n\rangle \star c_1|0\rangle \star |r\rangle). \tag{2.226}$$

Since $(B_1^L)^2 = 0$ and $B_1 c_1|0\rangle = |0\rangle$ this can be further simplified to obtain

$$\begin{aligned} \psi_n \star A &= \int_1^2 dr c_1|0\rangle \star B_1^L(|n\rangle \star |0\rangle \star |r\rangle) \\ &= \int_1^2 dr c_1|0\rangle \star B_1^L|n+r\rangle = \int_1^2 dr B_1^R(c_1|0\rangle \star |n+r\rangle). \end{aligned} \tag{2.227}$$

We can also repeat the same steps to get

$$A\star\psi_n = \int_1^2 dr B_1^L(|n+r\rangle \star c_1|0\rangle). \tag{2.228}$$

We notice that in both 2.227 and 2.228 the n and the r appear only in the combination $n+r$ so that we can replace the derivative ∂_n by ∂_r and write the second term in $\Psi_0 \star A$ as

$$\begin{aligned} -\sum_{n=0}^{N} \partial_n \psi_n \star A &= -\sum_{n=0}^{N} \int_1^2 dr B_1^R (c_1|0\rangle \star \partial_r |n+r\rangle) \\ &= \sum_{n=0}^{N} B_1^R (c_1|0\rangle \star [|n+1\rangle - |n+2\rangle]). \end{aligned} \tag{2.229}$$

This sum can easily be evaluated as there is a cancellation between the first and the second term in the square bracket except the $n=0$ of the first and $n=N$ of the second, so that

$$-\sum_{n=0}^{N} \partial_n \psi_n \star A = B_1^R c_1|0\rangle \star (\mathcal{I} - |N+2\rangle) \tag{2.230}$$

Similarly

$$-A \star \sum_{n=0}^{N} \partial_n \psi_n = (\mathcal{I} - |N+2\rangle) \star B_1^L c_1|0\rangle. \tag{2.231}$$

Using (2.227,2.228,2.230,2.231) and taking the limit $N \to \infty$ we easily obtain

$$\Psi_0 \star A + A \star \Psi_0 = |0\rangle \tag{2.232}$$

which together with (2.224) gives $\hat{Q}A = \mathcal{I}$. This proves that the cohomology of the kinetic operator $\hat{Q}$ is empty.

Every computation which has been carried out in this section was based on the CFT formalism which is not very much convenient to carry out similar computations in non trivial background, for instance, when a constant B field is turned on. In that case one needs to use the operator approach where one should use the modified 3–string vertex operator (3.41) to do the star and inner product in a constant B field background. Unfortunately, it proves to be very complicated even to repeat the computations in this section with trivial background because of the subtlety of the star product in the ghost sector. Had we be able to do that it could have been very easy to obtain Schnabl's solutions representing the decay of lower dimensional branes and hence could be able to prove the Sen's third conjecture.

Chapter 3

Vacuum String Field Theory and half–BPS State Solutions

The Witten's open string field theory is formulated around the D–25 brane vacuum and is instable due to the presence of the open string tachyon. Such instability is understood as the instability of the D–25 brane itself. Experience tell us that such instabilities should not be worrying as they just signals the fact that we are expanding around the wrong vacuum and no more exists once we get to the right vacuum, which ofcourse should be stable. The search for such a stable vacuum in principle involves the determination of the classical solution to the equation of motion SFT Ψ_0 and then expand the SFT action setting $\Psi = \Psi_0 + \tilde{\Psi}$ where $\tilde{\Psi}$ is the fluctuation field. In practice this has not been simple to carry out, as it was lacking a closed form of Ψ_0 before the seminal work of Martin Schnabl [85] and all the results were obtained using the method of level truncation [3], which involves a tedious numerical computation. This problem has been also analyzed from a different angle. Instead of trying to construct the classical solution Ψ_0 in [112] a guess about the form of SFT action expanded around the tachyon vacuum was put forward under the name Vacuum String Field Theory (VSFT) and proved to satisfy various consistency requirements which are similar to those of the original cubic SFT [4, 5, 99, 100, 121, 122, 124] . Beside these consistency requirements the action is also constrained by the Sen's Conjectures. In the next section we will brief on VSFT while focusing on the things we need most in the following sections.

3.1 Vacuum String Field Theory: basics

Let us suppose we know Ψ_0 (actually we know it[85]). The cubic SFT action expressed in terms of the shifted field $\tilde{\Psi} = \Psi - \Psi_0$ has the form

$$S(\Psi_0 + \tilde{\Psi}) = S(\Psi_0) - \frac{1}{g^2}\left(\frac{1}{2}\langle\tilde{\Psi}, \hat{Q}\tilde{\Psi}\rangle + \frac{1}{3}\langle\tilde{\Psi}, \tilde{\Psi}\star\tilde{\Psi}\rangle\right) \tag{3.1}$$

where $S(\Psi_0)$ is a constant equals the mass of the D–brane and negative of the potential energy $V(\Psi_0)$ associated to the string field configuration, by the Sen's first conjecture. The kinetic operator $\hat{Q}$ is related to the original kinetic operator Q as:

$$\hat{Q}\mathcal{A} = Q\mathcal{A} + \Psi_0 \star \mathcal{A} - (-1)^{G_\mathcal{A}} \mathcal{A} \star \Psi_0 \tag{3.2}$$

for arbitrary string field $\mathcal{A}$ of degree $G_\mathcal{A}$. The consistency of this action is guaranteed by the consistency of the cubic SFT action and can be readily checked. Like the cubic SFT action this action is invariant under a gauge transformation $\delta\tilde{\Psi} = \hat{Q}\Lambda + \tilde{\Psi}\star\Lambda - \Lambda\star\tilde{\Psi}$ for any Grassmann-even ghost number zero string field Λ

Since the energy density of the brane represents a positive cosmological constant, it is natural to add the constant $-M = -S(\Psi_0)$ to the cubic SFT action, so as to cancel the term $S(\Psi_0)$ in (3.1). This will make manifest the expected zero energy in the new vacuum without D–branes, and the action will be

$$S(\tilde{\Psi}) = -\frac{1}{g^2}\left(\frac{1}{2}\langle\tilde{\Psi}, \hat{Q}\tilde{\Psi}\rangle + \frac{1}{3}\langle\tilde{\Psi}, \tilde{\Psi}\star\tilde{\Psi}\rangle\right) \tag{3.3}$$

Knowing Ψ_0 the problem of formulating SFT around the non-perturbative vacuum is significantly simplified, as we have to only understand the properties of the kinetic operator $\hat{Q}$. This may not be the most convenient way of constructing the VSFT action. Typically a non trivial field redefinition, which leave cubic interaction term the same and transform the kinetic operator $\hat{Q}$ to a simpler operator $\mathcal{Q}$ that has a vanishing cohomology, is necessary to bring the shifted SFT action to the canonical form representing the new vacuum. We will consider a field redefinition of the kind

$$\tilde{\Psi} = e^K \psi \tag{3.4}$$

where K is a ghost number zero Grassmann even operator, which satisfies

$$\begin{aligned} K(\psi\star\phi) &= (K\psi)\star\phi + \psi\star(K\phi) \\ \langle K\psi, \phi\rangle &= -\langle\psi, K\phi\rangle \end{aligned} \tag{3.5}$$

for arbitrary string fields ψ and ϕ. These properties guarantee that the form of the cubic term doesn't change and the action after the field redefinition will be:

$$S(\psi) = -\frac{1}{g^2}\left(\frac{1}{2}\langle\psi, \mathcal{Q}\psi\rangle + \frac{1}{3}\langle\psi, \psi\star\psi\rangle\right) \tag{3.6}$$

with $\mathcal{Q} = e^{-K}\hat{Q}e^K$. Gauge invariance requires

$$\begin{aligned} \mathcal{Q}^2 &= 0 \\ \mathcal{Q}(\psi\star\phi) &= (\mathcal{Q}\psi)\star\phi + (-1)^{G_\psi}\psi\star(\mathcal{Q}\phi) \\ \langle\mathcal{Q}\psi, \phi\rangle &= -(-1)^{G_\psi}\langle\psi, \mathcal{Q}\phi\rangle. \end{aligned} \tag{3.7}$$

We will proceed by postulating a $\mathcal{Q}$ that satisfies this conditions and of ghost number one, has a vanishing cohomology and is universal, means doesn't have any reference to the original brane conformal field theory. These requirements can be satisfied by choosing $\mathcal{Q}$ to be purely ghost. In particular the ghost number one operator

$$\mathcal{C}_n = c_n + (-1)^n c_{-n}, \qquad n = 0, 1, 2, ... \tag{3.8}$$

satisfies all the properties in Eq. (3.7). In addition, each of the operator $\mathcal{C}_n$ has a vanishing cohomology since for each n the operator $\mathcal{B}_n = \frac{1}{2}(b_n + (-1)^n b_{-n})$ satisfies $\{\mathcal{C}_n, \mathcal{B}_n\} = 1$. It then follows that whenever $\mathcal{C}_n\psi = 0$ we have $\psi = \{\mathcal{C}_n, \mathcal{B}_n\}\psi = \mathcal{C}_n(\mathcal{B}_n\psi)$, showing that ψ is $\mathcal{C}_n$ trivial. Finally, since they are built from ghost oscillators $\mathcal{C}_n$'s are manifestly universal.

The conditions in Eq. (3.7) are also satisfied by a more general choice

$$\mathcal{Q} = \sum_0^\infty a_n \mathcal{C}_n \tag{3.9}$$

where the a_n's are constant coefficients. Here we made one choice for the kinetic operator satisfying all the requirements stated above. There may be other choices but future analysis will not require the knowledge of the detailed form of $\mathcal{Q}$, as long as it doesn't involve any matter operator. For such a kinetic operator the equation of motion of VSFT,

$$\mathcal{Q}\psi = -\psi \star \psi \tag{3.10}$$

should have a solution of the form

$$\psi = \psi_m \otimes \psi_g \tag{3.11}$$

where the subscripts m and g stand for matter and ghost. Both the space–time independent solution, describing D–25 barane, and lump solutions, describing lower dimensional branes, should exist. Factorizing the $\star$ product also as $\star^m$ and $\star^g$ the equation of motion factorizes as

$$\mathcal{Q}\psi_g = -\psi_g \star^g \psi_g \quad \text{and} \quad \psi_m = \psi_m \star^m \psi_m \tag{3.12}$$

In looking for solutions describing D–branes of various dimensions we shall assume that the ψ_g remains the same, whereas ψ_m is different for different D–branes. We note that the matter equation of motion is a projector equation and we already know one answer for that which is the identity state $|I\rangle$. In the following section we will construct other solutions to the matter equation of motion, first the space–time independent solutions representing the D–25 brane, and then lump solutions representing lower dimensional Dp–branes.

3.2 Static solutions: $\star_m$ Projectors

3.2.1 The space filling D25–barane solution

Following [2, 6, 12, 81, 82] we will construct the solution to $\psi_m = \psi_m *^m \psi_m$ in the form of a squeezed state: an exponential of a bilinears of the string creation operators acting on the vacuum. We will use the operator formulation to construct these solutions. Let us start by doing some rewriting of the Neumann coefficients of the 3-string vertex Eq.(2.57).

Ref.[77] introduced a matrix U with indices running over $m, n \geq 1$, in terms of which the matrix V^{rs}_{mn} is decomposed as:

$$V^{rs} = \frac{1}{3}(C + \omega^{s-r}U + \omega^{r-s}\bar{U}) \tag{3.13}$$

where $\omega = e^{2\pi i/3}$, and $C_{mn} = (-1)^m \delta_{mn}, \quad m, n \geq 1$ is the twist matrix. The matrix U satisfies

$$\bar{U} = U^* = CUC, \quad U^2 = \bar{U}^2 = 1, \quad U^\dagger = U, \quad \bar{U}^\dagger = \bar{U}. \tag{3.14}$$

With this properties one can show

$$[CV^{rs}, CV^{r's'}] = 0, \ \ \forall r, \ s, \ r', \ s' \tag{3.15}$$

and

$$\begin{aligned} &CV^{12}CV^{21} = CV^{21}CV^{12} = (CV^{11})^2 - CV^{11} \\ &(CV^{12})^3 + (CV^{21})^3 = 2(CV^{11})^3 - 3(CV^{11})^2 + 1. \end{aligned} \tag{3.16}$$

The strategy to find the space–time independent solution is by taking a trial solution of the form

$$|\psi_m\rangle = \mathcal{N}^{26} \exp\left(-\frac{1}{2}\eta_{\mu\nu} \sum_{m,n\geq 1} S_{mn} a_m^{\mu\dagger} a_n^{\nu\dagger}\right) |0\rangle, \tag{3.17}$$

where $|0\rangle$ is the $SL(2,R)$ invariant vacuum of the matter CFT, $\mathcal{N}$ is a normalization constant and S_{mn} is an infinite dimensional matrix with indices running from 1 to ∞. We will assume S_{mn} is twist invariant, that is,

$$CSC = S \tag{3.18}$$

so that the string state $|\psi_m\rangle$ is hermitian.

Now, we will do the $\star_m$ multiplication of $|\psi_m\rangle$ with itself using the 3-string vertex Eq. (2.57). Since we are looking for the space filling D25–brane which is space–time

translational invariant, the 3-string vertex we will be using reduces to the momentum independent part of (2.57), i.e.

$$E = \frac{1}{2}\sum_{r,s}\sum_{m,n\geq 1}\eta_{\mu\nu}a_m^{(r)\mu\dagger}V_{mn}^{rs}a_n^{(s)\nu\dagger}. \tag{3.19}$$

Define

$$\Sigma = \begin{pmatrix} S & 0 \\ 0 & S \end{pmatrix}, \qquad \mathcal{V} = \begin{pmatrix} V^{11} & V^{12} \\ V^{21} & V^{22} \end{pmatrix}, \tag{3.20}$$

and

$$\chi^{\mu T} = (a^{(3)\mu\dagger}V^{31},\ a^{(3)\mu\dagger}V^{32}), \qquad \xi^{\mu} = \begin{pmatrix} V^{13}a^{(3)\mu\dagger} \\ V^{23}a^{(3)\mu\dagger} \end{pmatrix} \tag{3.21}$$

Using the general formula [2]

$$\begin{aligned} &\langle 0|\exp\left(\lambda_i a_i - \frac{1}{2}P_{ij}a_i a_j\right)\exp\left((\mu_i a_i^{\dagger} - \frac{1}{2}Q_{ij}a_i^{\dagger}a_j^{\dagger}\right)|0\rangle \\ &= \det(K)^{-1/2}\exp\left(\mu^T K^{-1}\lambda - \frac{1}{2}\lambda^T Q K^{-1}\lambda - \frac{1}{2}\mu^T K^{-1}P\mu\right), \\ &K = 1 - PQ \end{aligned} \tag{3.22}$$

in which $a_i = (a_m^{(1)}, a_m^{(2)})$, $m \geq 1$ and in our case the matrices P and Q are identified with Σ and $\mathcal{V}$, and the vectors μ and λ are identified with χ and zero, respectively, we obtain

$$\begin{aligned} |\psi_m \star \psi_m\rangle_3 &= \mathcal{N}^{52}\det[(1-\Sigma\mathcal{V})^{-1/2}] \\ &\times \exp\left[\frac{1}{2}\eta_{\mu\nu}\{\chi^{\mu T}[(1-\Sigma\mathcal{V})^{-1}\Sigma]\chi^{\nu} + a^{(3)\mu\dagger}.V^{33}.a^{(3)\nu\dagger}\}\right]|0\rangle_3 \end{aligned} \tag{3.23}$$

Demanding that the exponents in $|\psi_m\rangle$ and $|\psi_m \star \psi_m\rangle$ matches we obtain

$$S = V^{11} + (V^{12}, V^{21})(1-\Sigma\mathcal{V})^{-1}\begin{pmatrix} V^{21} \\ V^{12} \end{pmatrix}. \tag{3.24}$$

Multiplying this last equation by C on the left it can be written as

$$T = X + (X_+, X_-)(1-\Sigma\mathcal{V})^{-1}\begin{pmatrix} TX_- \\ TX_+ \end{pmatrix}, \tag{3.25}$$

where we have introduced

$$T \equiv CS = SC, \quad X_+ = CV^{12}, \qquad X_- = CV^{21} \quad X = CV^{11}. \tag{3.26}$$

Assuming

$$[CS, CV^{rs}] = 0 \ \ \forall \ \ r, s \tag{3.27}$$

and because of (3.15) we can manipulate the equations as if T and $X_\pm$ are numbers rather than infinite dimensional matrices. With this assumption (3.25) will reduce to

$$(T-1)(XT^2 - (1+X)T + X) = 0 \tag{3.28}$$

where we have eliminated $X_\pm$ in favor of X using (3.16). This gives three solutions for S: the first one is $T = 1$ gives the identity state $|I\rangle$ which we already mentioned, and the non-trivial ones are

$$T_\pm = \frac{1}{2X}(1 + X \pm \sqrt{(1+3X)(1-X)}). \tag{3.29}$$

Of these two, T_- has a diverging eigenvalue and we will keep its discussion to the next chapter, and concentrate here on the T_+. Since CS is a function of $X \equiv CV^{11}$, which commutes with CV^{rs}, CS also commutes with CV^{rs}. Furthermore, since V^{11}, is twist invariant, so is X. It then follows that the inverse of X and any polynomial in X are twist invariant. Therefore, T and S are twist invariant, in agreement with the assumption (3.18).

Demanding the normalization factors in $|\psi_m\rangle$ and $|\psi_m \star \psi_m\rangle$ matches gives

$$\mathcal{N} = \det(1 - \Sigma\mathcal{V})^{1/2} = (\det(1-X)\det(1+T))^{1/2}, \tag{3.30}$$

where we have used (3.28) to simplify things. Thus the solution is given by

$$|\psi_m\rangle = [(\det(1-X)\det(1+T))^{1/2}]^{26} \exp\left(-\frac{1}{2}\eta_{\mu\nu}\sum_{m,n\geq 1} S_{mn} a_m^{\mu\dagger} a_n^{\nu\dagger}\right)|0\rangle. \tag{3.31}$$

It has been shown by Rastelli Sen and Zwiebach [112] that this solution can be identified with a particular surface state called the sliver. In general we have a family of surface states corresponding to once punctured disks with special kind of local coordinates. These states are called wedge states because the half–disk representing the local coordinate can be viewed as a wedge of the full unit disk. The puncture is on the boundary and the wedge in general has an angle $2\pi/n$ at the origin, where n is an integer. The sliver state then corresponds to $n = \infty$.

3.2.2 Lower dimensional brane solutions

The solution (3.31) representing the D25–brane has a factorized form of 26 factors, each involving the oscillators associated with a given space–time direction. In order to construct a solution of codimension k representing $D(25-k)$–branes, we need to replace k of the factors associated with the directions transverse to the D–brane with

a different set of solutions, but the factors associated with the directions tangential to the D–brane remains the same. Let us denote by $x^\mu (0 \leq \mu \leq (25-k))$ the directions tangential to the brane and by $x^\alpha ((26-k) \leq \alpha \leq 25)$ the directions transverse to the brane. We now use the representation of the vertex in the zero mode oscillator basis for the transverse direction related to the center of mass coordinate and momenta as

$$a_0^\alpha = \frac{1}{2}\sqrt{b}p^\alpha - \frac{i}{\sqrt{b}}x_0^\alpha, \qquad a_0^{\alpha\dagger} = \frac{1}{2}\sqrt{b}p^\alpha + \frac{i}{\sqrt{b}}x_0^\alpha, \tag{3.32}$$

where b is an arbitrary parameter. We also denote by $|\Omega_b\rangle$ the normalized vacuum state annihilated by all the creation operators a^α and by $|\Omega_b\rangle_{123}$ the direct product of the vacuum $|\Omega_b\rangle$ for each of the three strings.

For each string the relation between the momentum basis and the oscillator basis is given by

$$|p^\alpha\rangle = \left(\frac{2\pi}{b}\right)^{-k/4} \exp\left(-\frac{b}{4}p^\alpha p^\alpha + \sqrt{b}a_0^{\alpha\dagger}p^\alpha - \frac{1}{2}a_0^{\alpha\dagger}a_0^{\alpha\dagger}\right)|\Omega_b\rangle. \tag{3.33}$$

Substituting into Eq. (2.57) and integrating over $p_{(r)}^\alpha$ we can express the string vertex as

$$\begin{aligned}
|V_3\rangle \;=\; & \int d^{26-k}p_{(1)}d^{26-k}p_{(2)}d^{26-k}p_{(3)}\delta^{26-k}(p_{(1)}+p_{(2)}+p_{(3)}) \\
& \exp\left(-\frac{1}{2}\sum_{m,n\geq 1,r,s}\eta_{\mu\nu}a_m^{(r)\mu\dagger}V_{mn}^{rs}a_n^{(s)\nu\dagger} - \sum_{n\geq 1,r,s}\eta_{\mu\nu}p_{(r)}^\mu V_{0n}^{rs}a_n^{(s)\nu\dagger}\right. \\
& \left. -\;\frac{1}{2}\eta_{\mu\nu}p_{(r)}^\mu V_{00}^{rr}p_{(r)}^\nu\right)|0,p\rangle_{123} \otimes \left(\frac{\sqrt{3}}{(2\pi b^3)^{1/4}}(V_{00}^{rr}+\frac{b}{2}\right)^{-k} \\
& \exp\left[-\frac{1}{2}\sum_{r,s}\sum_{M,N\geq 0}a_M^{(r)\alpha\dagger}V_{MN}^{\prime rs}a_N^{(s)\alpha\dagger}\right]|\Omega_b\rangle_{123}
\end{aligned} \tag{3.34}$$

In this expression the sums over μ, ν run from 0 to $(25-k)$ and sum over α runs from $(26-k)$ to 25 while the capital M, N indice means including the zero modes. The coefficients $V_{MN}^{\prime rs}$ are given in terms of the V_{mn}^{rs} in appendix B of [44].

A decomposition similar to (3.13) can be made for $V_{MN}^{\prime rs}$ as follows

$$V^{\prime rs} = \frac{1}{3}(C' + \omega^{s-r}U' + \omega^{r-s}\bar{U}') \tag{3.35}$$

where $C_{MN} = (-1)^M\delta_{MN}$ with indice M, N running from 0 to ∞, and $U', \bar{U}' = U'^*$ viewed as matrices with $M, N \geq 0$ satisfying the relations

$$\bar{U}' = U'^* = C'U'C', \quad U'^2 = \bar{U}'^2 = 1, \quad U'^\dagger = U'. \tag{3.36}$$

We note that this is in complete analogy with (3.14) and it also follows the matrice V' together with C' satisfy equations exactly analogous to (3.15) and (3.16). Thus we can construct a solution to the equation of motion in an identical manner as in the previous section with the unprimed quantities replaced by the primed quantities along the direction transverse to the D–brane and the tangential factor remains the same. Taking into account the extra normalization factor appearing in the last line of (3.34) we get the following form for the solution

$$\begin{aligned} |\psi'_m\rangle &= [(\det(1-X)\det(1+T))^{1/2}]^{26-k}\exp\left(-\frac{1}{2}\eta_{\mu\nu}\sum_{m,n\geq 1} S_{mn}a_m^{\mu\dagger}a_n^{\nu\dagger}\right)|0\rangle \\ &\otimes \left(\frac{\sqrt{3}}{(2\pi b^3)^{1/4}}(V_{00}^{rr}+\frac{b}{2})\right)^k [(\det(1-X')\det(1+T'))^{1/2}]^k \\ &\exp\left(-\frac{1}{2}\sum_{M,N\geq 0} a_M^{(r)\alpha\dagger}S'_{MN}a_N^{(s)\alpha\dagger}\right)|\Omega_b\rangle, \end{aligned} \tag{3.37}$$

where

$$S'=C'T', \quad T'=\frac{1}{2X'}(1+X'+\sqrt{(1+3X')(1-X')}), \quad X'=C'V'^{11}. \tag{3.38}$$

An evidence for the fact that these solutions represent lower dimensional brane is given in [112]. This was done by evaluating the action of VSFT at this solution, which corresponds to the tension of the D–brane represented by the solution, and taking its ratio with the same quantity using the D25–brane solution. This ratio was proved to give the right ratio of tensions of the corresponding D–barens [115, 116].

In ref.[87] it has been shown that similar solutions exist when a constant B field is switched on along some of the transverse directions. In the following section we will use these solutions to construct a more general infinite family of solutions.

3.2.3 Lower dimensional brane solutions with B field

Now we look for solutions which are superpositions of matter projectors (stacks of D–branes) having the following characteristics: they must cover the ordinary 4D Minkowski space (parallel directions) and the remaining 22 directions will be referred to as the transverse. Out of the later two will have a special status, in that a constant B field will be switched on along them. We can imagine that all the internal dimensions are compactified on tori, but this is not strictly necessary for our argument. In the remaining part of this section we deal with the construction of the solutions.

As said above, two of the transverse directions are special, in that a constant background B field is switched on there. We denote these two directions by the labels α and β (for instance $\alpha,\beta=24,25$) and denote them simply by y_1, y_2; we take

for B the explicit form

$$B_{\alpha\beta} = \begin{pmatrix} 0 & B \\ -B & 0 \end{pmatrix} \tag{3.39}$$

Then, as is well–known [90, 93, 94], in these two directions we have a new effective metric $G_{\alpha\beta}$, the open string metric, as well as an effective antisymmetric parameter $\theta_{\alpha\beta}$, given by

$$G_{\alpha\beta} = \sqrt{\mathrm{Det}G}\delta_{\alpha\beta}, \qquad \theta^{\alpha\beta} = \epsilon^{\alpha\beta}\theta \tag{3.40}$$

where until further notice we set $(\alpha' = 1)$, $\mathrm{Det}G = (1 + (2\pi B)^2)^2$ and $\theta = -(2\pi)^2 B/\sqrt{\mathrm{Det}G}$. In (3.40) $\epsilon^{\alpha\beta}$ represents the 2×2 antisymmetric symbol with $\epsilon^1{}_2 = 1$. The transverse vertex $|V_{3,\perp}\rangle$ will become in this case $|\mathcal{V}_{3,\perp}\rangle$

$$|\mathcal{V}_{3,\perp}\rangle = |V_{3,\perp,\theta}\rangle \otimes |V'_{3,\perp}\rangle \tag{3.41}$$

where

$$|V_{3,\perp,\theta}\rangle = K_\theta\, e^{-E_\theta}|\Omega_b\rangle \tag{3.42}$$

K_θ is a suitable constant and, [87, 78],

$$E_\theta = \frac{1}{2}\sum_{r,s=1}^{3}\sum_{M,N\geq 0} a_M^{(r)\alpha\dagger}\mathcal{V}^{rs}_{\alpha\beta,MN}a_N^{(s)\beta\dagger} \tag{3.43}$$

The coefficients $\mathcal{V}^{\alpha\beta,rs}_{MN}$ are given in [87]. The new Neumann coefficients matrices $\mathcal{V}^{rs}$ satisfy the same relations as the V^{rs} ones. One introduces the matrices $\mathcal{X}^{rs} = C\mathcal{V}^{rs}$. Then the lump solution $|\mathcal{S}\rangle$ along α and β has the form (3.31) with $|0\rangle$ replaced by $|\Omega_b\rangle$ and S replaced by $\mathcal{S}$, where $\mathcal{S} = C\mathcal{T}$ and $\mathcal{T}$ has the same form as T in eq.(3.29) with X replaced by $\mathcal{X}$. It can also be be verified that the ratio of tensions for such solutions is the appropriate one for D–branes in a magnetic field, [88].

It is possible to construct a full family of such solutions which are $\star$– and bpz–orthonormal. This goes as follows, [89, 78, 92]. First we introduce two 'vectors' $\xi = \{\xi_{N\alpha}\}$ and $\zeta = \{\zeta_{N\alpha}\}$, which are chosen to satisfy the conditions

$$\rho_1\xi = 0, \qquad \rho_2\xi = \xi, \qquad \text{and} \quad \rho_1\zeta = 0, \qquad \rho_2\zeta = \zeta, \tag{3.44}$$

where ρ_1, ρ_2 are the half–string projectors [81, 82]. Moreover we define the matrix τ as $\tau = \{\tau_\alpha{}^\beta\} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Next we set

$$\mathrm{x} = (a^\dagger\tau\xi)\,(a^\dagger C\zeta) = (a_N^{\alpha\dagger}\tau_\alpha{}^\beta\xi_{N\beta})(a_N^{\alpha\dagger}C_{NM}\zeta_{M\alpha}) \tag{3.45}$$

Finally, we introduce the Laguerre polynomials $L_n(z)$, of the generic variable z, and define the sequence of states

$$|\Lambda_n\rangle = (-\kappa)^n L_n\Big(\frac{\mathrm{X}}{\kappa}\Big)|\mathcal{S}_{\perp\theta}\rangle \tag{3.46}$$

where, for simplicity, we have written down the tensorial factor involving the the y_1, y_2 directions only and understood the other directions. As part of the definition of $|\Lambda_n\rangle$ we require the following two conditions to be satisfied

$$\xi^T\tau\frac{1}{\mathbb{I}-\mathcal{T}^2}\zeta = 1, \qquad \xi^T\tau\frac{\mathcal{T}}{\mathbb{I}-\mathcal{T}^2}\zeta = \kappa \tag{3.47}$$

where κ is a real number. To guarantee Hermiticity for $|\Lambda_n\rangle$, we require

$$\zeta = \tau\xi^*. \tag{3.48}$$

The states $|\Lambda_n\rangle$ satisfy the remarkable property

$$|\Lambda_n\rangle * |\Lambda_m\rangle = \delta_{n,m}|\Lambda_n\rangle \tag{3.49}$$

$$\langle\Lambda_n|\Lambda_m\rangle = \delta_{n,m}\langle\Lambda_0|\Lambda_0\rangle \tag{3.50}$$

Therefore each Λ_n, as well as any combination of Λ_n with unit coefficients, are lump solution.

So far we have set $\alpha' = 1$. It is easy to insert back α'. In order to evaluate the low energy profile of $|\Lambda_n\rangle$ we first contract it with the eigenstate of the position operators with eigenvalues y^α: $\langle y|\Lambda_n\rangle$, and then take the limit $\alpha' \to 0$, [86, 88, 89]. The leading term in the α' expansion turns out to be

$$\langle y|\Lambda_n\rangle = \frac{1}{\pi}(-1)^n L_n\left(\frac{2\rho^2}{\theta}\right) e^{-\frac{\rho^2}{\theta}}|\Xi\rangle + \mathcal{O}(\sqrt{\alpha'}) \tag{3.51}$$

where $\rho^2 = y^\alpha y^\beta \delta_{\alpha\beta}$ and $|\Xi\rangle$ is the sliver solution.

The projectors we need in the following have this $\alpha' \to 0$ limit in the y^α directions; as for the remaining directions, they have the form of the sliver in the parallel directions and, finally, they become delta–like functions multiplied by the sliver in the remaining transverse directions, i.e. they are localized at the origin of the latter. This can be easily seen by taking the limit $\theta \to 0$ in the case $n = 0$ in (3.51)[1]. The solutions we just constructed surprisingly have some correspondence with some class of half–BPS states one meets in AdS/CFT duality, which we will clarify in the following section[27].

[1]One could easily construct projectors that are 'fat' also along other transverse directions, but we will not need them in the sequel.

3.3 Half–BPS state like VSFT solutions

The essence of this section is the observation that there is a remarkable correspondence between states one meets in the framework of the AdS/CFT duality, and solutions of VSFT. The correspondence is simply sketched and is far from exhaustive (mainly because we do not know enough about supersymmetric VSFT), but it is very suggestive and, if confirmed, it could lead to very interesting consequences.

Roughly speaking it goes as follows. In the framework of type IIB superstring theory, AdS/CFT duality establishes a correspondence between $\mathcal{N} = 4$ superconformal $U(N)$ gauge field theory on the boundary of AdS_5 and supergravity on the background $AdS_5 \times S^5$. In the strongest formulation the correspondence is between the two theories as a whole. Here we will limit our consideration to a class of 1/2–BPS states which can be formulated as composite of the $U(N)$ gauge theory scalars. In general they can be cast in the form of Schur polynomials, and thus they are in one to one correspondence with Young diagrams, represented (in the case of giant gravitons [49], for instance) by columns of maximal size (number of boxes) N. On the supergravity side they correspond to 1/2 BPS states that are solutions to the supergravity equations of motion. The latter represent localized states in the AdS geometry that wrap around S^3 cycles of S^5; they are stabilized by their angular momentum J in S^5, with the condition $J \leq N$. There are other significant 1/2–BPS states with mass $\sim N^2$, the superstars [52]. They correspond to Young diagrams represented by approximate triangles. On the supergravity side these are singular 1/2–BPS states with a naked singularity, which are regarded as solutions on the verge of developing a black-hole horizon due to the quantum corrections. Giant gravitons and superstars are two examples of a zoo of new entities that can be constructed in similar ways.

An interesting question is the following one: what is the precise relation between a state in gauge field theory and the corresponding supergravity solution? More precisely, how does the geometry that characterize the latter arise from the former (which, at first sight, is a totally ungeometrical object)? The answer seems to be coarse-graining: geometry arises from averaging details of the quantum states in the gauge theory side.

The argument brought forth by [45, 46, 47] goes as follows. One remarkable aspect of the above gauge field theory states is that they can be represented also in terms of N fermionic oscillators in a harmonic potential. The correspondence can be once again established via Young diagrams: quantum systems with the same Young diagram describe the same quantum state. This lends itself to a very interesting development: to quantum systems of this type we can associate in a one–to–one way Wigner distribution functions. In general, to a point–like system in a (q, p) phase space we can associate a Wigner distribution function $W(q, p)$. This is nothing but the bosonization of the original fermion system, but $W(q, p)$ is also very close to a probability distribution in phase space. In this way we can associate a Wigner distribution to any state, such as the vacuum, "black rings", superstars, etc. Now, it so happens that

these Wigner distributions are characterized in the large N limit by (coarse–grained) profiles that can be matched to the corresponding geometry (droplets) of the 1/2 BPS supergravity solutions.

With the above premise, the point we want to make in this section is that the Wigner distributions for the above introduced states naturally appear in VSFT. More precisely, the same profiles appear as low energy space profiles of VSFT solutions spanned by Laguerre polynomials.

This correspondence leads us to a related subject: open–closed string duality as seen from the SFT point of view. A.Sen has recently conjectured, [74], that open string theory might be able to describe all the closed string physics, at least in a background where D–branes are present. In this sense VSFT should be a privileged vantage point: the non-perturbative vacuum physics can only represent closed string theory and thus VSFT should be able to describe closed string theory in the sense of [74]. The existence of the D–brane solutions mentioned above is a confirmation of this. However these D–brane solutions are expressed as squeezed (or related) states. At most, in the presence of a background B field, we can produce a space profile thereof. However it has not been known so far how to associate a corresponding geometry. The correspondence between space profiles and Wigner distributions may be the clue: by interpreting a space profile as a Wigner distribution, we can reconstruct a half–BPS state and as a consequence arrive at some definite geometry, which is the coarse–grained averaging over the corresponding fermion systems.

3.3.1 Half–BPS solutions

In the field theory side of the AdS/CFT correspondence, half–BPS multiplets of $\mathcal{N} = 4$ Yang–Mills theory fall into representations $(0, l, 0)$ of the $SO(6)$ R–symmetry group. Highest weight states can be constructed as gauge invariant polynomials of a complex scalar field X. The conformal dimension of the latter is $\Delta = 1$ and the $U(1)$ R–charge $J = 1$, where $U(1) \in SO(6)$. A highest weight therefore satisfies $\Delta = J$. Basically such states are constructed out of multiple traces of X. The most general state of this type of charge n takes the form

$$(\mathrm{tr}(X^{l_1}))^{k_1}(\mathrm{tr}(X^{l_2}))^{k_2}\dots(\mathrm{tr}(X^{l_p}))^{k_p} \tag{3.52}$$

where the integers l_i, k_i form a partition of n: $\sum_{i=1}^{p} l_i k_i = n$. A basis for these states is given by the degree n Schur polynomials of the group $U(N)$. These in turn correspond to Young tableaux of maximal column length N. Therefore we can classify these highest weight states (chiral primaries) by means of Young diagrams [53, 54, 55].

It can be shown that they can be represented in another useful way. In fact their correlators can be related, by using the canonical approach, by correlators of a suitable (time–dependent) matrix model with a quadratic potential [53, 62]. The matrix model can be solved also in another way, by diagonalizing it and producing in this way a Vandermonde determinant Jacobian factor [55]. The latter can be lifted

to the exponential giving rise to a repulsive potential among the eigenvalues. The result is that we can interpret the eigenvalues λ_i as a system of N fermionic oscillators with Hamiltonian $H = \sum_i \lambda_i^\dagger \lambda_i + \frac{1}{2}$. The energy levels of this system are given by $E_i = n_i + \frac{1}{2}$, where n_i are nonnegative integers. The corresponding wavefunctions are given in terms of Slater determinants

$$\Psi(\lambda_1, \lambda_2, \ldots, \lambda_n) \sim e^{-\sum_i \frac{\lambda_i^2}{2}} \mathrm{Det} \begin{pmatrix} H^{n_1}(\lambda_1) & H^{n_1}(\lambda_2) & \ldots & H^{n_1}(\lambda_N) \\ H^{n_2}(\lambda_1) & H^{n_2}(\lambda_2) & \ldots & H^{n_2}(\lambda_N) \\ \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ H^{n_N}(\lambda_1) & H^{n_N}(\lambda_2) & \ldots & H^{n_N}(\lambda_N) \end{pmatrix} \tag{3.53}$$

where H_n are the Hermite polynomials for a single harmonic oscillator. The ground state Ψ_0 corresponds to $n_1 = 0, n_2 = 1, \ldots n_N = N-1$. Therefore the generic excited state can be represented by means of a Young diagram with rows $(r_1, r_2, \ldots, r_N)$, with $r_i = n_i - i + 1$ not all vanishing natural numbers in decreasing order. The energy of the state above the Fermi sea is $E = J = \sum_i r_i$, which is the total number of boxes in the Young diagram (for the relation to Fermi systems, see [56, 55, 58, 60, 61, 64]).

Let us list a few states which will be considered in the sequel by means of their Young diagram representation. A giant graviton is represented by a single column Young diagram, whose maximum length is of course N. A giant graviton, [49, 50, 59] is a half–BPS state which can be described as a D3–brane wrapping around an S^3 cycles in the S^5 factor of $AdS_5 \times S^5$. Stability is guaranteed by the spinning of the brane around an axis in S^5. The angular momentum has an upper bound $J \leq N$, which is a manifestation of the stringy exclusion principle. Since $\Delta = J$, the representation by means of a Young tableau incorporates in a simple way the exclusion principle.

A dual giant graviton, i.e. a D3–brane wrapping around an S^3 cycle in AdS_5, is represented by a one–row Young diagram of arbitrary length (no bound here). A black ring is represented by a large rectangular diagram of size N (see below). A superstar is represented by a large triangular diagram of size $\sim N$. It represents a stack of giant gravitons located at the origin of AdS_5. From the supergravity viewpoint, it is a singular solution in that it has a naked singularity. It is conjectured that due to string corrections it may actually be completely regular solution.

In the last two cases the energy of the states is proportional to the area of the Young tableau and therefore $\sim N^2$. Following in particular [46], these are the states we will be mostly interested in the following and we will consider them in the large N limit. In [46] a limit shape was introduced for the corresponding Young tableaux in the continuous limit. This is a function $y(x)$, where x runs from left to right along the rows and y from bottom to top along the columns. The origin is set at the leftmost bottom box of the tableau. For instance, for the superstar ensemble we have $\Delta = NN_c/2$ and $y(x) = \frac{N_c}{N}x$, where N_c is the number of columns.

3.3.2 1/2–BPS states as supergravity solutions

In [56] a beautiful characterization of 1/2–BPS states in type IIB supergravity was found. Regular 1/2–BPS solutions with a geometry invariant under $SO(4)\times SO(4)\times R$ correspond to the following ansatz

$$\begin{aligned} ds^2 &= -h^{-2}(dt+V_i\,dx^i)^2 + h^2(dy^2+dx^i\,dx^i) + y\,e^G d\Omega_3^2 + y\,e^{-G}d\tilde{\Omega}_3^2 \\ h^{-2} &= 2y\,cosh\,G \\ y\,\partial_y V_i &= \epsilon_{ij}\,\partial_j z, \qquad y(\partial_i V_j - \partial_j V_i) = \epsilon_{ij}\,\partial_y z \\ z &= \frac{1}{2}\tanh G \end{aligned} \tag{3.54}$$

where $i,j = 1,2$ and ϵ_{ij} is the antisymmetric symbol. There are also N units of 5–form flux, with

$$F_{(5)} = F_{\mu\nu}dx^\mu\wedge dx^\nu\wedge d\Omega_3 + \tilde{F}_{\mu\nu}dx^\mu\wedge dx^\nu\wedge d\tilde{\Omega}_3$$

where $\mu,\nu = 0,...,3$ refer to t, x^1, x^2, y. As for the ansatz for F and $\tilde{F}$, see [56]. The full solution is determined in terms of a single function z, which must satisfy the equation

$$\partial_i\partial_i\, z + y\partial_y\left(\frac{\partial_y z}{y}\right) = 0 \tag{3.55}$$

One can solve this equation by remarking its analogy with the Laplace equation for an electrostatic potential. Regular solutions can exist only if at the boundary $y=0$ the function $z(0,x_1,x_2)$ takes the values $\pm\frac{1}{2}$. Therefore regular solutions correspond to boundary functions $z(0,x_1,x_2)$ that are locally constant in the x_1,x_2 plane. The region of this plane where $z=-1/2$ are called droplets and denoted by $\mathcal{D}$. Following [46] we reintroduce in the game $\hbar$ and make the identification $\hbar \leftrightarrow 2\pi\ell_p^4$, noticing that x_1,x_2 have the unusual dimension of a length square.

The area of the droplet must equal N:

$$N = \int_{\mathcal{D}} \frac{d^2x}{2\pi\hbar} \tag{3.56}$$

while the conformal dimension of the state corresponding to the droplet $\mathcal{D}$ is

$$\Delta = \int_{\mathcal{D}} \frac{d^2x}{2\pi\hbar}\,\frac{1}{2}\,\frac{x_1^2+x_2^2}{\hbar} - \frac{1}{2}\left(\int_{\mathcal{D}}\frac{d^2x}{2\pi\hbar}\right)^2 \tag{3.57}$$

In conclusion, the information about the solution is encoded in the droplet. For instance, if the droplet is a disk of radius r_0 we recover the $AdS_5\times S^5$ solution; if the droplet is the upper half plane one gets the plane wave solution. In general if the droplet is compact the solution is asymptotically $AdS_5\times S^5$. It is useful to introduce

the new notation $u(0;x_1,x_2) = \frac{1}{2} - z(0;x_1,x_2)$; u is the characteristic function of the droplet, since it equals 1 inside the droplet and 0 outside. Solutions with such (sharp) characteristic functions are regular since the boundary conditions are satisfied. Solutions characterized by a function u which is not exactly 1 or 0, are singular [71, 72] (for a connection with quantum Hall effect, see [65, 66, 63]). This is the case of the superstar solution [52].

3.3.3 The Wigner distribution

It is clearly of upmost importance to establish a dictionary between the 1/2–BPS states introduced at the beginning of this section starting from $N = 4$ SYM and the droplet solutions. This is tantamount to finding a recipe to recognize the geometry emerging from a given gauge field theory state. The clue is the free fermion representation introduced above: any state represented by a Young diagram can be interpreted as a system of N fermions with energies above the Fermi sea. To this end it is useful to rewrite the formulas (3.56) and (3.57) in the more general form

$$\Delta = \int \frac{d^2x}{2\pi\hbar}\frac{1}{2}\frac{x_1^2+x_2^2}{\hbar}u(0;x_1,x_2) - \frac{1}{2}\left(\int \frac{d^2x}{2\pi\hbar}u(0;x_1,x_2)\right)^2 \tag{3.58}$$

$$N = \int \frac{d^2x}{2\pi\hbar}u(0;x_1,x_2). \tag{3.59}$$

where the integration extends over the whole x_1, x_2 plane. These formulas suggest that u can be identified with the semiclassical limit of the quantum one-particle (q,p) phase–space distributions of the free dual fermions after the identification $(x_1,x_2) \leftrightarrow (q,p)$. A phase–space distribution may be understood as an attempt of assigning a probability distribution to the phase–space points. It is a heuristic concept and there is no unique prescription for it. The most well–known distribution is the Wigner one [68]:

$$W(q,p) = \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} dy \langle q-y|\hat{\rho}|q+y\rangle e^{2ipy/\hbar} \tag{3.60}$$

where $\hat{\rho}$ is the density matrix. In the case of a pure state ψ, $\langle q'|\hat{\rho}|q''\rangle = \psi(q')\,\psi^\star(q'')$, therefore

$$W(q,p) = \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} dy \psi^\star(q+y)\psi(q-y)e^{2ipy/\hbar} \tag{3.61}$$

In general $\hat{\rho}$ will take the form of

$$\hat{\rho}(q',q'') = \sum_{f\in\mathcal{F}} \psi_f(q')\psi_f^\star(q'')$$

$\mathcal{F}$ being a given family of pure states. We will consider family of pure states representing excited states of N (fermionic) harmonic oscillators $f_n = r_n + n - 1$, with

$n = 1, ..., N$ (where we have dropped $\hbar$). In this case $\mathcal{F}$ will be a subset of the natural numbers and

$$\psi_{f_n} = A(f_n) H_{f_n}(q/\sqrt{\hbar}) e^{-q^2/2\hbar}$$

where $A(n)$ is a normalization constant and H_n are, as above, the Hermite polynomials. Using a well–known integration formula for Hermite polynomials one gets, [68],

$$W(q,p) = \sum_{f_n \in \mathcal{F}} W_{f_n}(q,p) = \frac{1}{2\pi\hbar} e^{-(q^2+p^2)/\hbar} \sum_{f_n \in \mathcal{F}} (-1)^{f_n} L_{f_n}\left(2\frac{q^2+p^2}{\hbar}\right) \tag{3.62}$$

Thinking of W as a probability distribution is certainly a heuristic and approximate concept, because it may be negative (see the considerations in [46], where an improved always positive distribution is introduced, the Husimi distribution). However we will not need it in the following, because we will compare Wigner distributions with space profiles of VSFT (which are not probability distributions either).

Here we are interested in Wigner distributions because they represent a precise recipe to bosonize associated fermion systems: from the fermion system we easily get the Wigner distribution and from the latter we can reconstruct the former, [60, 67]. In the following we will use Wigner distributions in this sense, and will be concerned specifically with distributions relative to ensembles, in which N is supposed to be very large. The semiclassical limit will correspond to $\hbar \to 0$ keeping $\hbar N$ finite. We will use such distributions to make a comparison with the u droplet distributions, [46], and with space profiles in VSFT (for coarse–graining, see also [57, 48, 70]).

Let us consider a few significant cases. The first concerns the Fermi sea. The relevant distribution is

$$2\pi\hbar W_{FS} = 2\pi\hbar \sum_{n=0}^{N-1} W_n(q,p) \tag{3.63}$$

By using a well–known identity for Laguerre polynomials one formally obtains 1 when the summation extends to infinity with fixed $\hbar$. This would not correspond to $AdS_5 \times S^5$. However a numerical analysis shows that the limit $\hbar \to 0$ with $\hbar N$ fixed reproduces the finite disk characteristic of the latter solution (see, for instance, [69]).

The second example involves the Young diagram corresponding to a giant graviton. It has $r_n = 0$, $n < k$ and $r_n = 1$ for $k \leq n \leq N$. The distribution is

$$2\pi\hbar W_{GG} = 2\pi\hbar \left(\sum_{n=0}^{k-2} + \sum_{n=k}^{N} \right) W_n(q,p) \tag{3.64}$$

It is evident that in the large N limit with k fixed, this distribution will be indistinguishable from the Fermi sea one.

The third example is the case corresponding to a rectangular Young diagram of row length K. It represents N fermions all excited above the sea by the same amount K. There is no a priori relation between N and K, but we are interested in the limit of large N and K such that $\hbar K$ as well as $\hbar N$ are finite. The Wigner distribution is

$$2\pi\hbar W_{\text{rect}} = 2\pi\hbar \sum_{n=K}^{N+K-1} W_{K+n-1}(q,p) \tag{3.65}$$

Setting $u(0,x_1,x_2) = 2\pi\hbar W_{\text{rect}}$ this identifies a characteristic function which is (approximately) 1 in the ring $\hbar K \leq \frac{q^2+p^2}{2} \leq \hbar(N+K)$ and 0 outside, in the large N and K limit. This corresponds to the 1/2–BPS called "black ring" in [56]. It has conformal dimension $\Delta = NK \sim N^2$, since K must be some rational multiple of N.

The last example concerns Young diagrams which are approximately triangular with $\Delta = NN_c/2$ and so correspond to superstar ensembles. In this case we have $f_n = (n-1)\delta_n$, with δ_n an integer $\sim \frac{N_c}{N} + 1$. For illustrative purposes let us set $\delta_n = \delta = \frac{N_c}{N} + 1$. Then

$$2\pi\hbar W_{\text{triangle}} = 2\pi\hbar \sum_{n=0}^{N-1} W_{n\delta}(q,p) = 2e^{-\frac{2H}{\hbar}} \sum_{n=0} (-1)^{n\delta} L_{n\delta}(4H/\hbar) \tag{3.66}$$

where $H = (q^2+p^2)/2$. The result of the analysis in [46] is that in the large N limit

$$2\pi\hbar W^{\infty}_{\text{triangle}} = \frac{1}{\delta} + \text{oscillations at scale } \Delta\text{H} = \hbar$$

Therefore identifying once again $2\pi\hbar W^{\infty}_{\text{triangle}}$ with $u(0;x_1,x_2)$ we get approximately $u(0,x_1,x_2) = 1/\delta$ within a finite radius disk. This corresponds to a fractionally filled droplet and represents the superstar solution, which is singular. It is also an explicit example of the relations

$$u(0;r^2) = \frac{1}{1+y'} = g(E) \tag{3.67}$$

which was conjectured and verified in various examples in [46]. The function $g(E)$ is called *grayscale distribution* and encodes the effective behavior of coarse–grained semiclassical observables in a given quantum state.

3.3.4 A correspondence

Looking at eqs.(3.63,3.65,3.66) of section 3.3.3, one immediately notices that they can be seen (up to an overall normalization constant) as the low energy limit space profiles of combinations of the string states Λ_n introduced in section 3.2.3, with unit coefficients. Since combinations of Λ_n with unit coefficients are solutions to the equation of motion of VSFT, we can see the above Wigner distributions as the low energy

profile of VSFT solutions (up to the common $|\Xi\rangle$ factor). It is therefore tantalizing to make the following association

$$\text{Wigner distribution for an } N \text{ fermion system} \leftrightarrow \text{VSFT solution}$$

For this to work we must require the correspondence[2] $\hbar \leftrightarrow \theta$ and that the coordinates x_1, x_2 be identified with y_1, y_2. This is what we suggest and would like to motivate in this section. The previous correspondence can be read in two directions. First: one can say that to any 1/2–BPS state to which we can associate a Wigner distribution of the type (3.60), there corresponds a VSFT solution given by a combination

$$|W\rangle = \sum_{f_n \in \mathcal{F}} |\Lambda_{f_n}\rangle, \qquad 2\hbar W(q,p)|\Xi\rangle = \langle y|W\rangle \tag{3.68}$$

where (p,q) is identified with (y_1, y_2) and the latter are the eigenvalues of $|y\rangle$. Vice versa: to any VSFT solution of the type (3.68) we can associate a Wigner distribution $W(q,p)$ according to (3.60). In this way we can associate to $|W\rangle$ a Young tableau and therefore a 1/2–BPS state in the $\mathcal{N} = 4$ superconformal field theory (before taking the large N limit) and we can associate a geometry (after taking it[3]). The latter point of view is probably the most interesting one. It implies that we may be able to associate a geometry to a given VSFT solution, therefore we are in the condition to answer some of the questions posed by open–closed string duality. Here we see how geometry emerges from a VSFT solution which is entirely expressed in terms of open string creation operators.

In the following we would like to list some arguments in support of our proposal.

1) With the above association we connect a microstate corresponding to a geometry, which is a supergravity solution, to a string state which is a solution of the VSFT equation of motion. The correspondence (3.68) is one–to–one[4] (before the large N limit).

2) The droplet geometry lives in a (x_1, x_2) plane which lies in the internal (compactified) dimensions. In the same way the plane (y_1, y_2) stays in the compactified part of the bosonic target space. As pointed out above, we identify the two planes. One could phrase it by saying that the two space coordinates x_1, x_2, which had been replaced by two phase–space coordinates q, p in the intermediate argument, have returned to their natural role via the identification with y_1, y_2.

3) The correspondence (3.68) tells us how the Pauli principle gets incorporated into a bosonic setting. The numbers f_n in the LHS of (3.68) correspond to the fermion energy levels in the original fermion system. Therefore, due to the Pauli exclusion

[2]It should be recalled that on the SUGRA side we have three parameters, α', g_s and N. With the first two one forms the combination $\hbar = 2\pi g_s \alpha'^2$. On the VSFT side we have also three parameters α', θ and N.

[3]In the process of taking the large N limit one smears out many details, so that multiple states are mapped to the same geometry

[4]See on this point the remark at the end of section 5.

principle, each f_n can appear only once in the family $\mathcal{F}$. Therefore in the summation each $|\Lambda_{f_n}\rangle$ appears only once. This guarantees that $|W\rangle$ is a VSFT solution[5]. On the other hand any VSFT solution that can be written in the form $\sum_{f_n\in\mathcal{F}}|\Lambda_{f_n}\rangle$ tells us that the numbers $f_n \in \mathcal{F}$ can be interpreted as energy levels of a fermionic harmonic oscillator system, since each appears only once. This is the way the D–brane solutions of VSFT manifest their fermionic nature.

4) The VSFT solution corresponding to the Fermi sea (3.63) is represented by a stack of N (unstable) VSFT D–branes. The giant graviton solution (3.64) is represented by a D–brane missing from the stack. The superstar solution (3.66) is represented by a stack of such missing (unstable) D–branes. This is in keeping with the interpretation of superstars as stack of giant gravitons, [52]. It is worth remarking at this point that all the VSFT solutions we consider in this paper are composite of VSFT D–branes and there is no direct identification between single VSFT D–branes and single D3–branes in superstring theory.

5) There is an algebra isomorphism between Wigner distributions of the type (3.60) and VSFT solutions like $|W\rangle$, an isomorphism that was pointed out in [91, 92, 89]. It is a well–known fact that any classical function $f(q,p)$ in a (q,p) phase space can be mapped to a quantum operator $\hat{O}_f$ via the Weyl transform, and that the product for quantum operators $\hat{O}_f\hat{O}_g$ is mapped into the Moyal product $f\star g$ for functions. Under this correspondence the $(x_1,x_2)\leftrightarrow(q,p)$ plane becomes noncommutative. It is a well–known fact that, under this correspondence the classical Wigner distributions like (3.63,3.65,3.66) are mapped into projectors of the Moyal star algebra:

$$(2\pi\hbar W)\star(2\pi\hbar W) = 2\pi\hbar W. \tag{3.69}$$

Actually these distributions turn out to coincide with families of the so–called GMS solitons, [96, 95]. Let us recall the relevant construction. Define the harmonic oscillator $a = (\hat{q}+i\hat{p})/\sqrt{(2\theta)}$ and its hermitian conjugate $a^\dagger$: $[a,a^\dagger]=1$. The normalized harmonic oscillator eigenstates are: $|n\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}}|0\rangle$. Now, via the Weyl correspondence, we can map any rank one projector $|n\rangle\langle n|$ to a classical function of the coordinates x_1, x_2.

$$|n\rangle\langle n| \longleftrightarrow \psi_n(x_1,x_2) = 2\,(-1)^n L_n\Big(\frac{2r^2}{\theta}\Big)e^{-\frac{r^2}{\theta}} \tag{3.70}$$

where $r^2 = x_1^2+x_2^2$. Each of these solutions, by construction, satisfy $\psi_n\star\psi_n=\psi_n$. These are referred to as GMS solitons [96]. In the previous section we have shown that the low energy limit of $\langle y|\Lambda_n\rangle$ factorizes into the product of the sliver state and $\psi_n(y_1,y_2)$. This means that the VSFT star product factorizes into Witten's star product and the Moyal $\star$ product, [91, 92]. More precisely we can formalize the following isomorphism

$$\begin{array}{ccccc} |\Lambda_n\rangle & \longleftrightarrow & P_n & \longleftrightarrow & \psi_n(y_1,y_2) \\ |\Lambda_n\rangle * |\Lambda_{n'}\rangle & \longleftrightarrow & P_nP_{n'} & \longleftrightarrow & \psi_n\star\psi_{n'} \end{array} \tag{3.71}$$

[5]If $|\Lambda\rangle$ is some $*$–projector, $n|\Lambda\rangle$ is a $*$–projector if and only if $n=0,1$.

where $\star$ denotes the Moyal product.

This remark should not be underestimated. Let us consider $2\pi\hbar W$ and suppose it is such that we can ignore its derivatives with respect to p and q. Then eq.(3.69) becomes $(2\pi\hbar W)^2 = 2\pi\hbar W$, which is the equation of a characteristic function (it can only be either 0 or 1). This is indeed what happens in the case of the vacuum and the black ring solutions, see [46, 47]. It is not the case of the superstar distribution because in that case we cannot ignore derivatives. But this remark suggests that the property of being Moyal projectors is basic for Wigner distributions to represent 1/2–BPS states. The string state $|W\rangle$ 'inherits' this property, it is the 'continuation' of the space profile to the whole string theory. In this sense it is natural that $|W\rangle$ be a string field theory solution.

Finally one should point out that there exists a solution generating technique that allows one to produce new solutions starting from a fixed one. As an example let us consider the partial isometries introduced in [97]. They are defined as follows

$$\begin{aligned}(\mathcal{P}_+)^k|\Lambda_n\rangle &= |\Lambda_{n+k}\rangle \\ (\mathcal{P}_-)^k|\Lambda_n\rangle &= |\Lambda_{n-k}\rangle\end{aligned} \tag{3.72}$$

where we define $\Lambda_n = 0$ for $n < 0$ (see [97] for definitions of $\mathcal{P}_\pm$)). Given a state represented by a certain Young diagram, the operator $(\mathcal{P}_+)^k$ adds k boxes in each of the N rows, while its inverse $(\mathcal{P}_-)^k$ removes them. Consider for instance the Fermi sea solution corresponding to eq.(3.63) and apply to it $(\mathcal{P}_+)^K$. According the above equations one gets the solution corresponding to the rectangular Young diagram (3.65). We have seen that (3.65), in the large N limit, leads to the black ring geometry. Note that the transformations that are generated by such partial isometries are area preserving on the droplet plane (the number of fermions is left unchanged). This points to the fact that partial isometries on the open string side are mapped to topology changing transformations on the closed string side.

3.3.5 Matching observables

The basic observables in the 1/2 BPS sector are given by the total five form flux, N, and by the energy, Δ, (3.56, 3.57). It is interesting to see how these two observables are encoded in the star algebra we are dealing with. The total flux is simply given by the bpz norm of the projector corresponding to the given Young diagram. We have indeed, (3.49),

$$N = \frac{\langle W_{\mathcal{F}}|W_{\mathcal{F}}\rangle}{\langle\Lambda_0|\Lambda_0\rangle} \tag{3.73}$$

This is perfectly expected as the total flux is determined by the number of boundary branes producing it. Note that, differently from the usual open string description given by the gauge theory, this observable *is not* part of the definition of the theory but is part of a classical solution (in much the same way as it happens in gravity).

In order to understand how the Δ observable emerges from the star algebra, an extension of the (3.49) is in order. Consider the following non twist invariant states, [78]

$$|\Lambda_{n,m}\rangle = \sqrt{\frac{n!}{m!}}(-\kappa)^n Y^{m-n} L_n^{m-n}\left(\frac{\mathrm{x}}{\kappa}\right)|\mathcal{S}_\perp\rangle, \qquad n \leq m \tag{3.74}$$

$$|\Lambda_{n,m}\rangle = \sqrt{\frac{m!}{n!}}(-\kappa)^m X^{n-m} L_m^{n-m}\left(\frac{\mathrm{x}}{\kappa}\right)|\mathcal{S}_\perp\rangle, \qquad n \geq m \tag{3.75}$$

where

$$X = a^\dagger \tau \xi \qquad Y = a^\dagger C \zeta \tag{3.76}$$

so that $\mathrm{x} = XY$, and $L_n^{m-n}(z) = \sum_{k=0}^m \binom{m}{n-k}(-z)^k/k!$ are the generalized Laguerre polynomials. Note that $\Lambda_n = \Lambda_{nn}$.

These states star multiply between themselves in the following way

$$\Lambda_{nm} * \Lambda_{pq} = \delta_{mp}\Lambda_{nq} \tag{3.77}$$

In VSFT these have been shown to be the vacuum states for perturbative strings stretched between the n-th and the m-th brane, [97]. Thanks to this extended algebra we can explicitly realize the fermionic system of our concern. To this end let's define the following *inner* operators acting on the string Hilbert space

$$A_+\phi = \sum_{n=0}^{\infty}\sqrt{n+1}\,\Lambda_{n+1,n} * \phi * \Lambda_{n,n+1} \tag{3.78}$$

$$A_-\phi = \sum_{n=0}^{\infty}\sqrt{n+1}\,\Lambda_{n,n+1} * \phi * \Lambda_{n+1,n} \tag{3.79}$$

These are the (adjoint representation of) the string field oscillators defined in [101] (at *fixed* half string vector) and behave as the raising/lowering operators of an harmonic oscillators

$$A_+\Lambda_n = \sqrt{n+1}\,\Lambda_{n+1} \tag{3.80}$$

$$A_-\Lambda_n = \sqrt{n}\,\Lambda_n \tag{3.81}$$

It is then natural to consider the operator

$$H = A_+ A_- \tag{3.82}$$

which, up to zero point energy, is just the h.o. hamiltonian. For single brane states we have

$$H\,\Lambda_n = n\,\Lambda_n \tag{3.83}$$

and, more important,

$$n = \frac{\langle \Lambda_n | H | \Lambda_n \rangle}{\langle \Lambda_0 | \Lambda_0 \rangle} \tag{3.84}$$

If we evaluate this operator on the stack $W_{\mathcal{F}}$ we get

$$\frac{\langle W_{\mathcal{F}} | H | W_{\mathcal{F}} \rangle}{\langle \Lambda_0 | \Lambda_0 \rangle} = \sum_{n=0}^{N-1} f_n \tag{3.85}$$

This is nothing but the energy of the corresponding fermion ensemble given by the Young diagram $\mathcal{F}$.

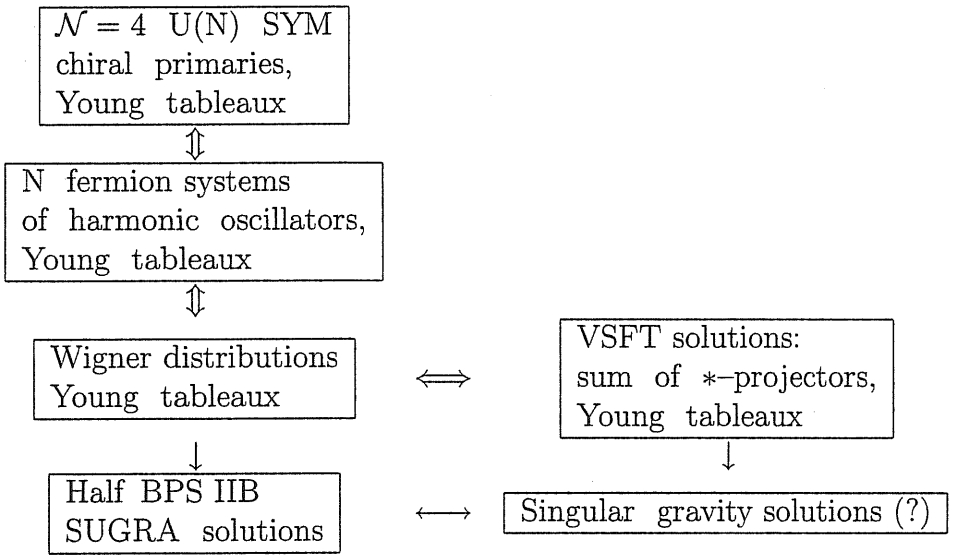
As in the gravity side the Δ observable is defined as the difference in energy between the state under consideration and the "vacuum" (empty AdS), at *fixed* five form flux N, see (3.57).

$$\Delta = \frac{\langle W_{\mathcal{F}} | H | W_{\mathcal{F}} \rangle - \langle W_{\mathcal{F}_0} | H | W_{\mathcal{F}_0} \rangle}{\langle \Lambda_0 | \Lambda_0 \rangle} = \texttt{\# of boxes of the Young Tableau} \tag{3.86}$$

where $\mathcal{F}_0 = \{0, 1, ..., N-1\}$ is the Fermi sea at given N.

3.3.6 More comments on the correspondence

The correspondence (3.68) is based on a series of facts, which have been listed above. The coincidence might be accidental, but we tend to believe it has a deeper meaning. The suggestion that comes from the previous section is summarized in the following table:



where double–line arrows represent one–to–one correspondences, simple down arrows represent the large N limit and the question mark indicates the conjectural part of our proposed correspondence. Let us describe it in more detail.

The fact that $|W\rangle$ is a VSFT solution is the strongest support of our conjectured correspondence. The weak point is that we know it is a solution of bosonic VSFT but we do not know whether it is a solution of the supersymmetric vacuum string field theory. However we would like to notice that the 1/2–BPS states considered in [45, 46, 47] in the gauge theory side, are all (very heavy) bosonic states. It is not unconceivable that the bosonic part of 1/2–BPS states is well described by solutions of the bosonic string theory. Unfortunately the study of the tachyon condensation in superstring field theory has not progressed much, see [98, 99]. From what we know nowadays it is possible that the bosonic parts of some solutions of supersymmetric VSFT take a form like $|W\rangle$, although a complete solutions has not yet been determined.

This raises a problem as to the interpretation of the lower right corner of the above table. Based on the above argument, they should represent the bosonic part of supergravity solutions. Now the Einstein equation for the latter is

$$R_{\mu\nu} \sim F_{\mu\lambda_1\lambda_2\lambda_3\lambda_4} F_\nu{}^{\lambda_1\lambda_2\lambda_3\lambda_4} \tag{3.87}$$

where F is the five–form field strength and where we have set the dilaton to 0. The contribution of the RHS is basic in the case of LLM solutions, as the latter do not satisfy the pure gravity equation $R_{\mu\nu} = 0$. As a consequence the solutions in the lower right corner above are not pure gravity solutions. This is the problem we alluded to above: from a purely bosonic theory we get, in the low energy limit, (the bosonic part of)supergravity solutions. The most likely explanation of this surprising result lies in the type of limit we have taken in VSFT: first $\alpha' \to 0$ with θ fixed, and then $\theta \to 0$ with θN fixed. This two–step limit selects the (bosonic part of the) droplet–like supergravity solutions. But, on the VSFT side, there are other possible limits. For instance, one could take the same limit, but in three steps, first $\alpha' \to 0$, then $\theta \to 0$ and finally $N \to \infty$. This shrinks the droplets to zero size and leads to singular (in the sense of delta–like) solutions, which we can identify with (singular) pure gravity solutions. However these solutions are too singular to base on them any serious discussion. So let us deal with this subject from another viewpoint.

From the above we see that the parameter θ, i.e. the B field, plays a fundamental role in the VSFT side limit. There is no B field in the IIB supergravity solutions side, there is instead a background five–form field, whose flux in suitable units equals N. Of course in the bosonic SFT side there cannot be any such background. However we have seen that $\theta \sim 1/B$ is identified with $\hbar$ and, in the large N limit, $\hbar N$ is kept finite. Therefore B and the five–form flux play a parallel role. The five–form flux supports the supergravity solution, while B supports the corresponding VSFT solution, which otherwise would collapse to a delta function. It looks like B is a surrogate of the five–form flux in the bosonic theory (see the considerations about noncommutativity in [51]). One may wonder whether this remark can be confirmed in some independent way. There actually exists a way in the gravity side, although it is very hard to verify it analytically.

The low energy equations of motion for bosonic string theory is

$$R_{\mu\nu} \sim H_{\mu\lambda\rho}H_{\nu}{}^{\lambda\rho} \tag{3.88}$$

where again we have set the dilaton to 0, and $H = dB$. Comparing this with (3.87) we see that the H^2 term might play the role of the five–form term there. There is however an obstacle. Our VSFT solutions contain a constant B–field and if we replace a constant B in (3.88) the H^2 term vanishes. Nevertheless there exists another interpretation. Let us consider for definiteness the vacuum solution, where the droplet is a disk of finite radius r_0 in the phase space. As we remarked above we can easily reproduce this solution on the VSFT side with a constant field $B = B_0$, by taking the limit $N \to \infty$ such that $\theta_0 N \sim r_0$. Now, the region $r > r_0$ corresponds to a vanishing distribution in the large N limit. This can be reproduced as well by taking first $\theta \to 0$ and *then* $N \to \infty$. In other words, in order to reproduce the vacuum solution it is not necessary to assume a constant B field everywhere. Unfortunately we do not know how to deal with VSFT in the presence of a varying background B field. But we believe it is reasonable to assume that, anyhow, well inside the droplet the VSFT solution will be described by the solution with $B = B_0$, far outside the droplet by a solution with very large B and in the intermediate region by some interpolating solution. If this is correct then our overall VSFT solution will correspond to a non constant B field: after coarse–graining the profile will be such that the H^2 term in (3.88) mimics the five–form quadratic term in (3.87). Of course one should take care also of the other gravity equations of motion: in order to satisfy them all a nontrivial dilaton might be necessary and the solution is anyhow very likely to be singular. One may object that we are here in presence of two different VSFT solutions (with constant and non constant B field) that correspond to the same space profile. This is true, but the profile we have been talking is the same only in the $\alpha' \to 0$ limit. We expect the α' corrections to remove the degeneracy between them.

In any case using bosonic string field theory to establish a correspondence with 1/2–BPS states of an $\mathcal{N} = 4$ YM theory can be taken, at this stage, only as a suggestion. However the elements we have listed above are striking. So let us suppose that our conjecture is correct at least for the class of states and solutions we are interested in. Then it is convenient to view it in the framework of open–closed string duality. One should not forget that VSFT is a version of *open* string field theory, i.e. its language is the language of open string theory. The correspondence we have established above is between 1/2–BPS states of SYM theory in 4D and *full* VSFT solutions. This suggests that the open string field theory we have been considering describes in fact the physics of open strings attached to the stack of D3–branes where the SYM is defined, that is it is the stringy completion of the latter theory. Once again, the appropriate treatment should make use of superstring field theory. But let us suppose that bosonic SFT is enough for the present purpose; then we must conclude that, using tachyon condensation, we have found a way to pass from open SFT solutions to space time geometry (via coarse–graining), which is one of the major

problems in open–closed string duality, and this with the additional bonus of the α' corrections.

We would like to add two specifications. The first concerns Chan–Paton factors one is expected to introduce in order to represent a $U(N)$ theory, and we have not. This is in fact not necessary, since it was proven in [97] that VSFT contains solutions with all type of $U(N)$ CP factors without the need to introduce them by hand. The second concerns the string critical dimension, which, in the bosonic case, is D=26, while the physics of SYM theory lives in D=4. However we have seen in section 3 that our VSFT solution spontaneously solve the problem, because we can choose them translationally invariant in 4D and of finite or zero size in the transverse dimensions, as need be.

In this paper we have only considered Wigner distributions. As pointed out in [46] there are other proposals, for instance the Husimi distribution, which is based on a convolution of the Wigner one. On the other hand there are many other solutions in VSFT, beside the family based on the sliver and the Laguerre polynomials we have considered so far, for instance the family of butterfly projectors. It would be interesting to see whether our correspondence extends to other phase–space distributions and to other VSFT solutions.

Concerning the future problems to be studied an interesting one relates to the possible utilization of the full VSFT to calculate α' corrections. It has been suggested that the superstar solution may develop a horizon due to the stringy corrections. Now, a string state like $|W\rangle$ contains the α' corrections to its low energy profile. It is therefore natural to ask whether this knowledge can be translated to the supergravity side. Related to this is the problem of counting the microstates in order to evaluate the entropy of the ensemble, [45, 46, 47]. To this end one should be able to count the distinct string fields corresponding to a given low energy profile. A problem nested into this is related to gauge equivalence. The states $|\Lambda_n\rangle$ are defined in terms of a vector ξ_n (likewise for the ghost part). These infinitely many numbers ξ_n are irrelevant in the low energy limit. This fact is understood in the sense that these numbers are likely to represent only gauge degrees of freedom (otherwise also our previous claim of one–to–one correspondence would seem to need a better phrasing). It would be interesting to find a real proof of this.

Chapter 4

Time Dependent Solutions in VSFT

The search for time–dependent solutions is one of the prominent research topics in string theory. Particularly interesting is the search for solutions describing the decay of D-branes. An archetype problem in open bosonic string theory is describing the evolution from the maximum of the tachyon potential to the (local) minimum. Such a solution known as rolling tachyon, if it exists, describes the decay of the space filling D25–brane corresponding to the unstable perturbative vacuum to the locally stable vacuum. That such a solution exists has been argued in many ways, [102, 109, 120], see also [103, 104, 105]. A natural framework where to study such a nonperturbative problem is String Field Theory (SFT). But, although there have been some attempts to describe such phenomena in a SFT framework [110], no analytical control has been achieved so far.

In this chapter following [75, 76], we will show that VSFT contain exact time–dependent solutions with the appropriate characteristic to represent S-branes, that is solitonic solutions localized in time, with trivial and non trivial bacground. We show in fact that the time profile of such solutions is dominated for large t by a factor $\exp(-at^2)$ with positive constant a. At time $t = 0$ the solution takes the form of a deformed sliver (D25-brane), the deformation being parametrized by two continuous parameters. At infinite future (and infinite past) time it becomes 0, i.e. it flows into the stable vacuum. If the initial configuration happens to coincide exactly with the sliver (no deformation present) there cannot be any time evolution. Therefore an initial deformation away from the sliver is essential for true time evolution. Needless to say this is strongly reminiscent of Sen's rolling tachyon solution, [102] or of an S–brane, [123], i.e of a state finely tuned to be poised at the initial time near the top of the tachyon potential and let free to evolve.

The technique to produce such a solution is based on double Wick–rotation, as is customary in such kind of trade. Our reference solution is obtained by picking a Euclidean lump solution with one transverse space direction (a D24-brane) and then

performing an inverse Wick–rotation along such a direction. However the important ingredient is that our lump solution is not the ordinary one. Since the spectrum of the twisted Neumann coefficient matrices of the three strings vertex nicely split into a continuous and a discrete part, we define a new solution in which the squeezed state matrix is made of a continuous part, which is the same as for the conventional lump, and a discrete part which is *inverted* with respect to the ordinary lump. We call this *unconventional lump*, see eqs.(4.76,4.34) below. After inverse–Wick–rotating it we get the desired time behavior, (4.81).

In this discussion our time is nothing but a Wick–rotated space coordinate, representing the position of the string center–of–mass, and it couples to the open string (flat) metric. We will also discuss a possible connection of such time with the closed string time (which couples to the bulk gravity metric). See [106, 107, 108] for discussions related to the definition of time in SFT.

The above discussion is mainely based on the solutions constructed on trivial background. However, the solutions can be generalized to the case when the background is non trivial, i.e. when a constant E field is turnrd on along some of the directions transverse or longitudinal to the D–brane. In the last part of the present chapter we will construct such general solutions.

4.1 Trivial background

4.1.1 Sliver, inverse sliver and lumps

In section (3.2.1) we have seen that, assuming the matrices X, X_+, X_- are mutually commuting and T commute with them as well, one can show that, the VSFT equation of motion reduces to the algebraic equation

$$XT^2 - (1+X)T + X = 0, \tag{4.1}$$

which has two non–trivial solutions, the sliver solution

$$T = \frac{1}{2X}(1 + X - \sqrt{(1+3X)(1-X)}) \tag{4.2}$$

and its inverse $1/T$. We note that (4.1) is invariant under the substitution $T \leftrightarrow 1/T$. $1/T$ is given by the RHS of eq.(4.2) with the $-$ sign replaced by the $+$ sign in front of the square root. We will call it the *inverse sliver*. This solution was previously discarded, [112], because of the bad asymptotic behaviour of the $1/T$ eigenvalues. However it is exactly this behaviour that will allow us to find interesting time–dependent solutions. Notwithstanding the divergent behaviour of the eigenvalues it is perhaps possible to associate a definite meaning to the energy density of some of these solutions. They may be interesting as solutions of VSFT also without reference to time dependence.

The sliver will be our reference solution in the following, but, in order to build our time-dependent solution, we will explicitly need the lump [112]. The lump solution is engineered to represent a lower dimensional brane, therefore it will have transverse directions along which translational invariance is broken. Exactly as in the sliver case, we can consider an *unconventional lump* solution with T' replaced by $1/T'$.

Next we quote some useful identities:

$$\begin{aligned} &X + X_+ + X_- = 1 \\ &X_+ X_- = (X)^2 - X \\ &(X_+)^2 + (X_-)^2 = 1 - (X)^2 \\ &(X_+)^3 + (X_-)^3 = 2(X)^3 - 3(X)^2 + 1 \end{aligned} \tag{4.3}$$

The same relations hold if we replace X, X_+, X_-, T by X', X'_+, X'_-, T', respectively.

4.1.2 Spectroscopy and diagonal representation

The diagonalization of the X matrix was carried out in [113], while the same analysis for X' was accomplished in [125] and [114]. Here, for later use, we summarize the results of these references. The eigenvalues of $X = X^{11}, X_+ = X^{12}, X_- = X^{21}$ and T are given, respectively, by

$$\mu^{rs}(k) = \frac{1 - 2\,\delta_{r,s} + e^{\frac{\pi k}{2}}\,\delta_{r+1,s} + e^{-\frac{\pi k}{2}}\,\delta_{r,s+1}}{1 + 2\cosh\frac{\pi k}{2}} \tag{4.4}$$

$$t(k) = -e^{-\frac{\pi|k|}{2}} \tag{4.5}$$

where $-\infty < k < \infty$. The generating function for the eigenvectors is

$$f^{(k)}(z) = \sum_{n=1}^{\infty} v_n^{(k)} \frac{z^n}{\sqrt{n}} = \frac{1}{k}(1 - e^{-k \arctan z}) \tag{4.6}$$

The completeness and orthonormality equations for the eigenfunctions are as follows

$$\sum_{n=1}^{\infty} v_n^{(k)} v_n^{(k')} = \mathcal{N}(k)\delta(k-k'), \qquad \mathcal{N}(k) = \frac{2}{k}\sinh\frac{\pi k}{2}, \qquad \int_{-\infty}^{\infty} dk\, \frac{v_n^{(k)} v_m^{(k)}}{\mathcal{N}(k)} = \delta_{nm} \tag{4.7}$$

The spectrum of X is continuous and lies in the interval $[-1/3, 0)$. It is doubly degenerate except at 0. The continuous spectrum of X' lies in the same interval, but X' in addition has a discrete spectrum. To describe it we follow [114]. We consider the decomposition

$$X'^{rs} = \frac{1}{3}(1 + \omega^{s-r} CU' + \omega^{r-s} U'C) \tag{4.8}$$

where $\omega = e^{\frac{2\pi i}{3}}$ and U' as introduced in section (3.2.2). It is convenient to express everything in terms of CU' eigenvalues and eigenvectors. It has discrete and continous eigenvalues as well.

The discrete eigenvalues, denoted by ξ and $\bar{\xi}$. Since CU' is unitary they lie on the unit circle. They are more effectively represented via the parameter η (see below), which in turn is connected to the parameter b introduced in (3.32). To each value of b there corresponds a couple of values of η with opposite sign (except for $b = 0$ which implies $\eta = 0$).

These eigenvalues ξ and $\bar{\xi}$ are determined as follows, [114]. Let

$$\xi = -\frac{2 - \cosh\eta - i\sqrt{3}\sinh\eta}{1 - 2\cosh\eta} \tag{4.9}$$

and

$$F(\eta) = \psi(\frac{1}{2} + \frac{\eta}{2\pi i}) - \psi(\frac{1}{2}), \qquad \psi(z) = \frac{d\log\Gamma(z)}{dz} \tag{4.10}$$

Then the eigenvalues ξ and $\bar{\xi}$ are the solutions of

$$\Re F(\eta) = \frac{b}{4} \tag{4.11}$$

The corresponding eigenvectors $V_n^{(\xi)}$ are defined via the generating function

$$\begin{aligned} F^{(\xi)}(z) = \quad & \sum_{n=1}^{\infty} V_n^{(\xi)} \frac{z^n}{\sqrt{n}} = -\sqrt{\frac{2}{b}} V_0^{(\xi)} \left[\frac{b}{4} + \frac{\pi}{2\sqrt{3}} \frac{\xi - 1}{\xi + 1} + \log iz \right. \\ & \left. + e^{-2i(1+\frac{\eta}{\pi i})\arctan z} \Phi(e^{-4i\arctan z}, 1, \frac{1}{2} + \frac{\eta}{2\pi i})\right] \end{aligned} \tag{4.12}$$

where $\Phi(x, 1, y) = 1/y\, {}_2F_1(1, y; y + 1; x)$, while

$$V_0^{(\xi)} = \left(\sinh\eta \frac{\partial}{\partial\eta} [\log \Re F(\eta)]\right)^{-\frac{1}{2}} \tag{4.13}$$

As for the continuous spectrum, it is spanned by the variable k, $-\infty < k < \infty$. The eigenvalues of CU' are given by

$$\nu(k) = -\frac{2 + \cosh\frac{\pi k}{2} + i\sqrt{3}\sinh\frac{\pi k}{2}}{1 + 2\cosh\frac{\pi k}{2}}$$

The generating function for the continious eigenvectors is

$$\begin{aligned} F_c^{(k)}(z) = \sum_{n=1}^{\infty} V_n^{(k)} \frac{z^n}{\sqrt{n}} = V_0^{(k)} \sqrt{\frac{2}{b}} \left[-\frac{b}{4} - \left(\Re F_c(k) - \frac{b}{4}\right) e^{-k\arctan z} - \log iz \right. \\ \left. - \left(\frac{\pi}{2\sqrt{3}} \frac{\nu(k) - 1}{\nu(k) + 1} + \frac{2i}{k}\right) + 2i\, f^{(k)}(z) - \Phi(e^{-4i\arctan z}, 1, 1 + \frac{k}{4i})\, e^{-4i\arctan z}\, e^{-k\arctan z}\right] \end{aligned} \tag{4.14}$$

where

$$F_c(k) = \psi(1 + \frac{k}{4\pi i}) - \psi(\frac{1}{2}),$$

while

$$V_0^{(k)} = \sqrt{\frac{b}{2\mathcal{N}(k)}} \left[4 + k^2 \left(\Re F_c(k) - \frac{b}{4} \right)^2 \right]^{-\frac{1}{2}}. \tag{4.15}$$

We notice that the continuous eigenvalues of X', X'_+, X'_- and T' (for the conventional lump) are given by same formulas as for the X, X_+, X_- and T, eqs(4.4,4.5). As for the discrete eigenvalues, they are given by the formulas

$$\begin{aligned} \mu_\xi^{rs} &= \frac{1 - 2\,\delta_{r,s} - e^{\eta}\,\delta_{r+1,s} - e^{-\eta}\,\delta_{r,s+1}}{1 - 2\cosh\eta} \\ t_\xi &= e^{-|\eta|}. \end{aligned} \tag{4.16}$$

The eigenvectors corresponding to the continuous spectrum $V_N^{(k)}$ $(-\infty < k < \infty)$ and the eigenvectors of the discrete spectrum $V_N^{(\xi)}$, $V_N^{(\bar{\xi})}$ form a complete basis. They will be normalized so that the completeness relation takes the form

$$\int_{-\infty}^{\infty} dk\, V_N^{(k)} V_M^{(k)} + V_N^{(\xi)} V_M^{(\xi)} + V_N^{(\bar{\xi})} V_M^{(\bar{\xi})} = \delta_{NM} \tag{4.17}$$

It has become familiar and very useful to expand all the relevant quantities in VSFT by means of this basis. To this end we define

$$a_k = \sum_{N=0}^{\infty} V_N^{(k)} a_N, \qquad a_\xi = \sum_{N=0}^{\infty} V_N^{(\xi)} a_N, \qquad a_{\bar{\xi}} = \sum_{N=0}^{\infty} V_N^{(\bar{\xi})} a_N \tag{4.18}$$

where

$$a_N = \int_{-\infty}^{\infty} dk\, V_N^{(k)}\, a_k + V_N^{(\xi)} a_\xi + V_N^{(\bar{\xi})} a_{\bar{\xi}} \tag{4.19}$$

are the usual annihilation operators including zero mode. We then introduce the even and odd twist combinations

$$e_k = \frac{a_k + C a_k}{\sqrt{2}}, \qquad e_\eta = \frac{a_\xi + C a_\xi}{\sqrt{2}}, \qquad o_k = \frac{a_k - C a_k}{i\sqrt{2}}, \qquad o_\eta = \frac{a_\xi - C a_\xi}{i\sqrt{2}}, \tag{4.20}$$

The commutation relations among them are

$$[e_k, e_{k'}^\dagger] = \delta_{k,k'}, \qquad [e_\eta, e_\eta^\dagger] = 1, \qquad [o_k, o_{k'}^\dagger] = \delta_{k,k'}, \qquad [o_\eta, o_\eta^\dagger] = 1, \tag{4.21}$$

while all the other commutators vanish. The twist properties are defined by

$$Ca_k = a_{-k}, \qquad Ca_\xi = a_{\bar{\xi}},$$

Using these combinations the three–strings vertex can be cast in diagonal form and, for instance, the exponent of the conventional lump state can be written as

$$\begin{aligned} a^\dagger S' a^\dagger &= \int_{-\infty}^{\infty} dk\, t(k)\, (a_k^\dagger, Ca_k^\dagger) + 2\, t_\xi\, (a_\xi^\dagger, Ca_\xi^\dagger) \\ &= \frac{1}{2}\int_{-\infty}^{\infty} dk\, t(k)\, (e_k^\dagger e_k^\dagger + o_k^\dagger o_k^\dagger) + t_\eta\, (e_\eta^\dagger e_\eta^\dagger + o_\eta^\dagger o_\eta^\dagger) \end{aligned} \tag{4.22}$$

where $t_\eta \equiv t_\xi = e^{-|\eta|}$. The unconventional lump is obtained by replacing t_η with its inverse $e^{|\eta|}$.

In the sequel we need the behaviour of the eigenvectors when $b \to 0$ and when $b \to \infty$. Near $b = 0$ we have

$$\begin{aligned} & b \approx 0, \qquad \eta \approx 0, \qquad \xi \approx 1 \\ & V_0^{(\xi)} = \frac{1}{\sqrt{2}} + \mathcal{O}(\eta), \qquad V_n^{(\xi)} = \mathcal{O}(\eta^2) \end{aligned} \tag{4.23}$$

The same behaviour holds for the $V^{(\bar{\xi})}$ basis.

When $b \to \infty$ we have instead

$$\begin{aligned} & b \to \infty, \qquad b \approx 4\log\eta, \qquad \xi \approx -e^{\frac{\pi i}{3}} \\ & V_0^{(\xi)} \approx e^{-\frac{\eta}{2}}\sqrt{2\eta\log\eta}, \qquad V_n^{(\xi)} \sim e^{-\frac{\eta}{2}}\sqrt{\eta} \end{aligned} \tag{4.24}$$

and the same for $V^{(\bar{\xi})}$.

If we consider the corresponding asymptotic expansions for the $V^{(k)}$ basis, we have to be more careful. The point is that the expression $(V_0^{(k)})^2$, see (4.15), would superficially seem to vanish in the limit $\eta \to \infty$, but it is in fact a representation of the Dirac delta function $\delta(k)$, see Appendix A. Therefore the result of taking the $\eta \to \infty$ limit in an integral containing $(V_0^{(k)})^2$ is to concentrate it at the point $k = 0$. This renders the generating function (4.14) very singular and, consequently, such integrals as $\int dk\, V_n^{(k)} V_m^{(k)} f(k)$ must be handled with care. As for the limit of the continuous basis when $b \to 0$, one can see that $V_0^{(k)} \to 0$, while the other eigenfunctions have a nonvanishing finite limit.

4.1.3 Time dependent solutions

In order to appreciate the very nature of the problem of finding time–localized VSFT solutions, let us examine first some obvious attempts and learn from their failure. The first thing that comes to one's mind is to start from a lump with one transverse

space direction (therefore it represents a D24-brane) and inverse–Wick–rotate it. For simplicity we denote the transverse direction coordinate, momenta and oscillators simply by x, p and a_N. The solution is written as follows:

$$|\Psi'\rangle = |\Xi\rangle_{25} \otimes |\Lambda'\rangle$$
$$|\Lambda'\rangle = \mathcal{N}' \exp\left[-\frac{1}{2}\sum_{N,M\geq 0} a_N^\dagger S'_{NM} a_M^\dagger\right] |\Omega_b\rangle \tag{4.25}$$

where $|\Xi\rangle_{25}$ is the usual sliver along the longitudinal 25 directions and

$$\mathcal{N}' = \sqrt{3}\frac{V_{00}+\frac{b}{2}}{(2\pi b^3)^{\frac{1}{4}}}\sqrt{\det(1-X')\det(1+T')} \tag{4.26}$$

In order to study the space profile of this solution in the transverse direction we contract it with the x_0–coordinate eigenstate

$$|x_0\rangle = \sqrt{\frac{2}{b\pi}}\exp\left[-\frac{1}{b}x_0^2 - \frac{2}{\sqrt{b}}ia_0^\dagger x_0 + \frac{1}{2}(a_0^\dagger)^2\right]|\Omega_b\rangle \tag{4.27}$$

The result is

$$\langle x_0|\Lambda'\rangle = \sqrt{\frac{2}{b\pi}}\frac{\mathcal{N}}{\sqrt{1+s'}}\exp\left[\frac{1}{b}\frac{s'-1}{s'+1}x_0^2 - \frac{2i}{\sqrt{b}}\frac{x_0 f_0}{1+s'} - \frac{1}{2}a^\dagger W' a^\dagger\right] \tag{4.28}$$

where the condensed notation means

$$f_0 = \sum_{n=1} S'_{0n} a_n^\dagger, \qquad a^\dagger W' a^\dagger = \sum_{n,m=1} a_n^\dagger W'_{nm} a_m^\dagger, \qquad W'_{nm} = S'_{nm} - \frac{S'_{0n}S'_{0m}}{1+s'} \tag{4.29}$$

and

$$s' = S'_{00} \tag{4.30}$$

After an inverse Wick–rotation $x_0 \to ix_0, a_n^\dagger \to ia_n^\dagger$ (4.28) becomes

$$\langle \mathrm{x}_0|\Lambda'\rangle = \sqrt{\frac{2}{b\pi}}\frac{\mathcal{N}}{\sqrt{1+s'}}\exp\left[\frac{1}{b}\frac{1-s'}{1+s'}\mathrm{x}_0^2 + \frac{2i}{\sqrt{b}}\frac{\mathrm{x}_0 f_0}{1+s'} + \frac{1}{2}a^\dagger W' a^\dagger\right] \tag{4.31}$$

We are interested in solutions localized in time. The second term in the exponent gives rise to time oscillations. Only the first term can guarantee time localization. Precisely this happens when $|s'| > 1$. However such a condition can never be achieved within the present scheme in which ordinary lump solutions are utilized. In fact it is possible to show that for such solutions $|s'| \leq 1$. Therefore with the simple–minded scheme considered so far it is impossible to achieve time localization (in this regard our negative conclusion is similar to [111]; as for the case $b \to 0$, see below).

Let us see this in more detail by showing that $|s'| \leq 1$. Using the basis of the previous section we can write

$$s' \equiv S'_{00} = \int_{-\infty}^{\infty} dk\, V_0^{(k)}(-e^{-\frac{\pi|k|}{2}})V_0^{(k)} + V_0^{(\xi)}e^{-|\eta|}V_0^{(\xi)} + V_0^{(\bar{\xi})}e^{-|\eta|}V_0^{(\bar{\xi})} \tag{4.32}$$

Using (4.15), one can see that the first term in the RHS does not contribute in the limit $b \to 0$ (i.e. $\eta \to 0$) and using the approximants (4.23) we immediately see that the remaining two terms add up to 1. Therefore when $b \to 0$, $s' \to 1$. Viceversa, in the limit $b \to \infty$, using (4.24) we see that the last two terms in the RHS of (4.32) do not contribute, while the first term contribute exactly -1. This can be also shown numerically or with the alternative analytical method of Appendix A. For generic values of b we cannot calculate s' analytically but it is easy to evaluate it numerically and to show that it is a monotonically decreasing function of b for $0 \leq b < \infty$. This in turn implies that the quantity $\frac{1-s'}{1+s'}$ is always *positive* (see figure 4.1).

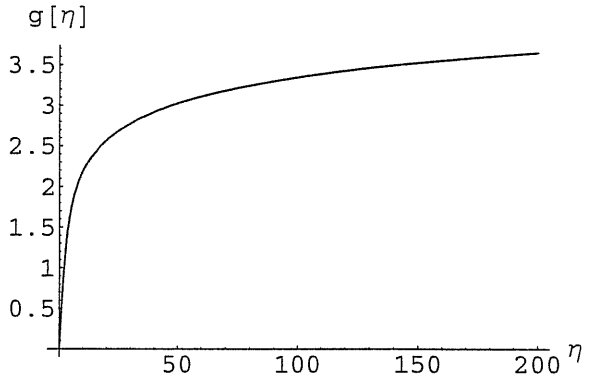


Figure 4.1: *The quantity* $g[\eta] = \frac{1-s'}{1+s'}$ *as a function of* η

Our conclusion is therefore that we cannot obtain a time–localized solution by inverse–Wick–rotating an ordinary lump solution. We will show in section (4.2) that also adding a constant B–field does not change this negative conclusion. Some drastic change has to be made in order to produce a localized time–dependent solution.

4.1.4 A Rolling Tachyon–like Solution

It is not hard to realize that if we were to replace $e^{-|\eta|}$ with $e^{|\eta|}$ in eq.(4.32) we would reverse the conclusion at the end of the previous section. In fact, see below, we would have $|s'| \geq 1$. In this section we wish to exploit this possibility. We have seen that if in the lump solution we replace T' by $1/T'$, formally, it is still a solution. Motivated

by this fact we define an unconventional lump, by replacing $|\Lambda'\rangle$ in (4.25) with

$$|\check{\Lambda}'\rangle = \mathcal{N}\exp(a^\dagger C\check{T}'a^\dagger)|\Omega_b\rangle \tag{4.33}$$

where

$$\check{T}'_{NM} = -\int_{-\infty}^{\infty} dk\, V_N^{(k)}\, V_M^{(k)} \exp\left(-\frac{\pi|k|}{2}\right) + \left(V_N^{(\xi)}\, V_M^{(\xi)} + V_N^{(\bar{\xi})}\, V_M^{(\bar{\xi})}\right) \exp|\eta| \tag{4.34}$$

Due to the fact that the star product is split into eigenspaces of the Neumann coefficients X, X_+, X_-, the projector equation split accordingly into the continuous and discrete spectrum part. Therefore we are guaranteed that (4.33) is again a projector, as one can on the other hand easily verify by direct calculation. This is the solution we propose.

Before we proceed with our analysis we would like to clarify a basic question about the solution we have just put forward. Passing from a squeezed state solution with a matrix T' to another characterized by the inverse matrix $1/T'$ may lead in general to unacceptable features of the state, such as divergent terms in the oscillator basis. However in the case at hand, in which one inverts only the discrete spectrum, such unpleasant aspects disappear. First of all the matrix $\check{T}'$ is well defined both in the oscillator and in the diagonal basis. Second, such expression as $\sqrt{\det(1-\check{T}')}$ are well–defined. This is due to the fact that, if we are allowed to factorize the discrete and continuous spectrum contribution, the former can be written as $\det(1-\check{T}')^{(d)} = (1-\exp|\eta|)^2$, so that the possible dangerous – sign under the square root disappears due to the double multiplicity of the discrete eigenvalue. Thirdly, the energy density of the (Euclidean) solution (4.33) is well–defined and equals the energy density of the ordinary lump. In fact, using the formulas of [12], the ratio between the energy densities of the two solutions reduces to

$$\sqrt{\frac{\det(1+\check{T}')}{\det(1-\check{T}')}}\Big/\sqrt{\frac{\det(1+T')}{\det(1-T')}} = \sqrt{\frac{(1+e^{|\eta|})^2}{(1-e^{|\eta|})^2}}\Big/\sqrt{\frac{(1+e^{-|\eta|})^2}{(1-e^{-|\eta|})^2}} = 1 \tag{4.35}$$

after factorization of the discrete and continuous parts of the spectrum.

After these important remarks it remains for us to show that this solution has the appropriate features to represent a rolling tachyon solution. To see if this is true we have to represent it in a more explicit way. In particular we have to extract the explicit time dependence (better, the space dependence and then inverse–Wick–rotate it). To do so, we have to choose a (coordinate) basis on which to project (4.33). There seem to be two distinguished ways to make this choice. We will work them out explicitly and then discuss them.

To start with let us define the following coordinate and momentum operator, given by the twist even and twist odd parts of the discrete spectrum,

$$\hat{x}_\eta = \frac{i}{\sqrt{2}}(e_\eta - e_\eta^\dagger) \tag{4.36}$$

$$\hat{y}_\eta = \frac{i}{\sqrt{2}}(o_\eta - o_\eta^\dagger) \tag{4.37}$$

The eigenstates of the coordinate $\hat{x}_\eta$ are given by

$$\begin{gathered} |x\rangle = \tfrac{1}{\sqrt{\pi}}\exp\left(-\tfrac{1}{2}x^2 - \sqrt{2}ie_\eta^\dagger x + \tfrac{1}{2}e_\eta^\dagger e_\eta^\dagger\right)|\Omega_{\eta_e}\rangle, \\ e_\eta|\Omega_{\eta_e}\rangle = 0 \\ \hat{x}_\eta|x\rangle = x|x\rangle \end{gathered} \tag{4.38}$$

Correspondingly the eigenstates of the momentum $\hat{y}_\eta$ are

$$\begin{gathered} |y\rangle = \tfrac{1}{\sqrt{\pi}}\exp\left(-\tfrac{1}{2}y^2 - \sqrt{2}io_\eta^\dagger y + \tfrac{1}{2}o_\eta^\dagger o_\eta^\dagger\right)|\Omega_{\eta_o}\rangle, \\ o_\eta|\Omega_{\eta_o}\rangle = 0 \\ \hat{y}_\eta|y\rangle = y|y\rangle \end{gathered} \tag{4.39}$$

In order to make the x, y dependence explicit we project our solution (4.33) into the position/momentum eigenstates (4.38, 4.39). Using standard results[1] we get

$$\langle x, y|\check{\Lambda}'\rangle = \frac{1}{\pi(1+e^{|\eta|})}\exp\left(\frac{e^{|\eta|}-1}{e^{|\eta|}+1}(x^2+y^2)\right)|\check{\Lambda}'_c\rangle \tag{4.40}$$

The state $|\check{\Lambda}'_c\rangle$ is given by (4.33), but with only oscillators from the continuous spectrum, as the contribution of the discrete spectrum is now contained in the prefactor at the rhs of (4.40) . Now we perform the inverse Wick rotation $x \to ix$, $y \to -iy$ to recover the Lorentz signature, and obtain

$$|\check{\Lambda}'(\mathrm{x};\mathrm{y})\rangle = \frac{1}{\pi(1+e^{|\eta|})}\exp\left(-\frac{e^{|\eta|}-1}{e^{|\eta|}+1}(\mathrm{x}^2+\mathrm{y}^2)\right)|\check{\Lambda}'_c\rangle^{(Wick)} \tag{4.41}$$

It is evident that for every value of η the solution is localized in the x–time coordinate. The extra coordinate y is related to internal twist odd degrees of freedom and can be interpreted as a free parameter of the representation (4.41). This solution also contains the free parameter η which is nothing but a reparametrization of b, through (4.11). Therefore it is characterized by two free parameters.

The 'time' x is not the ordinary time, i.e. the time coupled to the flat open string metric and related to the string center of mass. We will see later on a possible interpretation for x. Now, let us turn to the ordinary (open string) time, i.e. the time defined by the center of mass of the string and analyze the corresponding time profile. Despite the fact that this coordinate is not diagonal for the $*$–product we can still have complete control on the profile along it. The center of mass position operator is given by

$$\hat{x}_0 = \frac{i}{\sqrt{b}}(a_0 - a_0^\dagger) \tag{4.42}$$

[1]Here we are assuming that the vacuum factorizes into $|\Omega_{\eta_e}\rangle \otimes |\Omega_{\eta_o}\rangle \otimes |\Omega_c\rangle$ where the latter factor represents the vacuum with respect to the continuous oscillator component.

The center of mass position eigenstate is

$$|x_0\rangle = \sqrt{\frac{2}{b\pi}}\exp\left(-\frac{1}{b}x_0x_0 - \frac{2}{\sqrt{b}}ia_0^\dagger x_0 + \frac{1}{2}a_0^\dagger a_0^\dagger\right)|\Omega_b\rangle \tag{4.43}$$

Let us compute the center of mass time profile. After inverse–Wick–rotating it, it turns out to be

$$\begin{aligned} |\Lambda'(\mathrm{x}_0)\rangle &= \langle \mathrm{x}_0|\Lambda'\rangle = \qquad (4.44)\\ \sqrt{\frac{2}{b\pi}}\frac{\mathcal{N}}{\sqrt{1+\check{T}'_{00}}} &\exp\left(\frac{1}{b}\frac{1-\check{T}'_{00}}{1+\check{T}'_{00}}\mathrm{x}_0^2 + \frac{2i}{\sqrt{b}(1+\check{T}'_{00})}\mathrm{x}_0\check{T}'_{0n}a_n^\dagger + \frac{1}{2}a_n^\dagger W'_{nm}a_m^\dagger\right)|\Omega_b\rangle \\ W'_{nm} &= \check{S}'_{nm} - \frac{1}{1+\check{T}'_{00}}\check{S}'_{0n}\check{S}'_{0m} \qquad (4.45)\end{aligned}$$

The quantities $\check{S}'_{0n}$ and $\check{S}'_{nm}$ can be computed in the diagonal basis

$$\begin{aligned} \check{S}'_{0n} &= \check{T}'_{0n} \qquad (4.46)\\ &= (1+(-1)^n)\left(-\int_0^\infty V_0^{(k)}V_n^{(k)}\exp\left(-\frac{\pi k}{2}\right) + V_0^{(\xi)}\,V_n^{(\xi)}\exp|\eta|\right) \\ \check{S}'_{nm} &= (-1)^n\check{T}'_{nm} = \qquad (4.47)\\ &= ((-1)^n+(-1)^m)\left(-\int_0^\infty V_n^{(k)}V_m^{(k)}\exp\left(-\frac{\pi k}{2}\right) + V_n^{(\xi)}V_m^{(\xi)}\exp|\eta|\right)\end{aligned}$$

It is evident that the leading time dependence in (4.44), for large x_0, is contained in $\exp\left(\frac{1}{b}\frac{1-\check{T}'_{00}}{1+\check{T}'_{00}}\mathrm{x}_0^2\right)$. The number $\check{T}'_{00}$ is $b(\eta)$–dependent and can be computed via

$$\check{T}'_{00}(\eta) = -2\int_0^\infty dk\left(V_0^{(k)}(b(\eta))\right)^2\exp\left(-\frac{\pi k}{2}\right) + 2(V_0^{(\xi)})^2\exp|\eta| \tag{4.48}$$

This is the crucial quantity as far as the time profile is concerned. An analytic evaluation of it is beyond our reach. However we will later show that

$$\begin{aligned}\lim_{\eta\to 0}\check{T}'_{00} &= 1 \qquad (4.49)\\ \lim_{\eta\to\infty}\check{T}'_{00} &= \infty \qquad (4.50)\end{aligned}$$

A numerical analysis shows that this quantity is a function monotonically increasing with η within such limits. This means that the quantity $\frac{1-\check{T}'_{00}}{1+\check{T}'_{00}}$ is always *negative* (it lies in the interval $(-1,0)$, see figure 4.2) and so the profile is always localized in the center of mass time, except in the extreme case $\eta \to 0$, which corresponds to the tensionless limit.

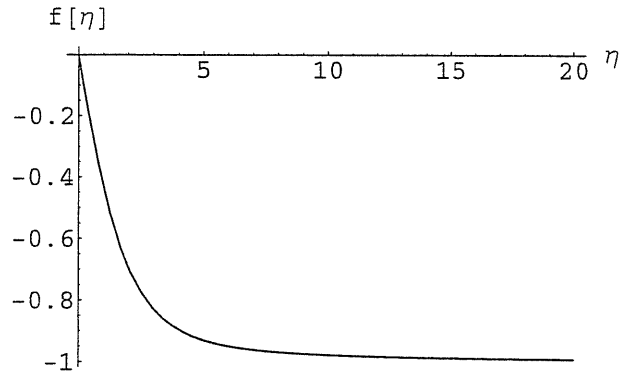


Figure 4.2: *The quantity* $f[\eta] = \frac{1-\check{T}'_{00}}{1+\check{T}'_{00}}$ *as a function of* η

This has to be compared with the usual lump solution (see previous section) for which the corresponding quantity is always positive and takes values in the interval $(0, \infty)$, allowing for localized *space* profiles but divergent along a timelike direction.

For reasons that will become clear in the next section, we extract also the free parameter y dependence, by projecting onto the corresponding twist–odd eigenstate (4.39). This operation can be done before or after the projection along the center of mass coordinate and does not interfere with it because $\hat{\mathrm{y}}$ does not contain the zero mode. We will therefore consider the following representation of our solution (inverse Wick rotation is again understood)

$$|\Lambda'(\mathrm{x}_0, y)\rangle = \langle \mathrm{x}_0, \mathrm{y}|\Xi_\eta\rangle = \sqrt{\frac{2}{b\pi}}\frac{\mathcal{N}}{\sqrt{2\pi(1+e^{|\eta|})}}\exp\left(\frac{1-e^{|\eta|}}{1+e^{|\eta|}}\mathrm{y}^2\right) \tag{4.51}$$

$$\cdot\frac{1}{\sqrt{1+\check{T}'_{00}}} \quad \exp\left(\frac{1}{b}\frac{1-\check{T}'_{00}}{1+\check{T}'_{00}}\mathrm{x}_0^2 + \frac{2i}{\sqrt{b}(1+\check{T}'_{00})}\mathrm{x}_0\check{T}'_{0n}a_n^\dagger - \frac{1}{2}a_n^\dagger W''_{nm}a_m^\dagger\right)|0\rangle$$

The quantities $\check{T}'_{00}$ and $\check{T}'_{0n}$ are the same as in (4.48, 4.46) since the momentum $\hat{y}_\eta$ is twist–odd. Some changes occur in W''_{nm}

$$W''_{nm} = \check{S}''_{nm} - \frac{1}{1+\check{T}'_{00}}\check{S}'_{0n}\check{S}'_{0m} \tag{4.52}$$

$$\check{S}''_{nm} = ((-1)^n + (-1)^m)\left(-\int_0^\infty dk\, V_n^{(k)}\, V_m^{(k)} \exp\left(-\frac{\pi k}{2}\right) + V_n^{(\eta)}V_m^{(\eta)}\exp|\eta|\right) \quad n, m \text{ even}$$

$$= -((-1)^n + (-1)^m)\int_0^\infty dk\, V_n^{(k)}\, V_m^{(k)} \exp\left(-\frac{\pi k}{2}\right) \quad n, m \text{ odd} \tag{4.53}$$

Note that $\check{S}''_{nm}$ gets contribution only from the twist–even part of the discrete spectrum.

In conclusion (4.51) provides the solution we were looking for. It represents a solution localized in x_0, with the desired profile. It depends on two free parameters y and η (or b). These are all positive features. But let us start making a closer comparison with the rolling tachyon solution (such a comparison is made with the representation (4.51)). This can be done by considering the limit $b \to \infty$, which can be derived from the eqs.(4.24). $b \to \infty$ means $\eta \to \infty$ (for simplicity from now on we take η positive) and

$$\check{T}'_{00} \approx 2\,\eta \log \eta \,(1 - \frac{\log(2\pi)}{\log \eta} + \ldots) \tag{4.54}$$

where dots denote higher order terms. Therefore we see that in this limit any time dependence in (4.51) disappears. Moreover, anticipating a result of the following section, we also have that $W''_{nm} \to S_{nm}$ (the state normalization will be discussed separately). In other words, in the limit $b \to \infty$ we obtain a static solution corresponding to the initial sliver. From this we understand that the parameter $1/b$, for large b, plays a role similar to Sen's parameter $\tilde{\lambda}$ near 0[2]. A second remark concerns the limit $y \to \infty$. In this case the first exponential factor in the RHS of (4.51) suppresses everything, so that the limit is the 0 state. In other words, we can consider this value of the parameter y as identifying the (relatively) stable vacuum state.

Although the rolling tachyon naturally compares with (4.51) rather than with (4.41), it is instructive to repeat something similar with the latter. Let us stress once more that both (4.51) and (4.41) represent the same solution, but in different bases, in particular with two different times: one, x_0, is the open string center of mass time, the other, x is related to the discrete spectrum. In the (4.41) case a parameter like b is missing. But this is something that is simply not customary and can be easily remedied. We can in fact introduce a parameter b_e in (4.36,4.37), just replacing $\sqrt{2}$ with $\sqrt{b_e}$ in those equations. Then (4.41) would become

$$|\check{\Lambda}'(x;y)\rangle = \frac{1}{b_e \pi (1+e^{\eta})} \exp\left(-\frac{1}{b_e}\frac{e^{\eta}-1}{e^{\eta}+1}(x^2+y^2)\right) |\check{\Lambda}'_c\rangle^{(Wick)} \tag{4.55}$$

and we could repeat the same argument as above and reach the same conclusion, except that in this case we have to take $b_e \to \infty$ as well as $\eta \to \infty$. The limit $y \to \infty$ plays the same role as in the (4.81) representation.

In the next section we will study the solution (4.33) in a regime we are more familiar with, the low energy regime $\alpha' \to 0$, and in the other extreme regime, $\alpha' \to \infty$, in which the solution considerably simplifies and an analytic treatment is possible. What we would like to see more closely is whether, for sufficiently small values of the parameters, the solution at time 0 is close enough to the sliver configuration, whose decay the solution is expected to describe.

[2]We recall that Sen's rolling tachyon solution depends on the parameter $\tilde{\lambda}$, which appears in the $\tilde{\lambda} \int_{\partial D} dt \cosh X^0(t)$–deformed BCFT.

4.1.5 Low energy and tensionless limits

As reviewed in appendix A, the low energy limit is obtained by performing an $\epsilon \to 0$ limit on the quantities that depend on the Neumann coefficients of the three strings vertex. ϵ is a dimensionless parameter that represents the smallness of α', [86]. As it happens, in all the expansions we consider, the parameters ϵ and b only appear through the ratio ϵ/b. Therefore, formally, the expansions for small ϵ/b are the same as the expansions for large b, i.e. $\eta \to \infty$. Therefore, in this section, when we consider the expansion in η near ∞ we really mean the expansion for ϵ/b small (i.e. ϵ small and b finite). A different attitude is required by the 'external' states like (4.27). There the rescaling of x_0 would lead to the replacement $b \to b\epsilon$. In this case we absorb ϵ into x_0 and keep b finite. In conclusion, throughout the analysis of the low energy limit, b should be considered as a finite free parameter.

Following [86], when $\alpha' \to 0$, we expect the star-algebra projector to factorize into a projector of the algebra of functions times a zero momentum projector. Let us analyze in detail what is the limit of the various quantities appearing in (4.51). First of all we have

$$\lim_{\eta\to\infty} \frac{1-\check{T}'_{00}}{1+\check{T}'_{00}} = -1 \tag{4.56}$$

This follows from (4.24) and from the discussion at the end of section (4.1.2), in particular from the property of $(V_0^{(k)})^2$ of approximating $\delta(k)$ in the limit $b \to \infty$, which implies that $\check{T}'_{00} \to \infty$ in the same limit. For the oscillating term we have

$$\lim_{\eta\to\infty} \frac{\check{T}'_{0n}}{1+\check{T}'_{00}} = \lim_{\eta\to\infty} \frac{1}{\sqrt{2\log\eta}} = 0 \tag{4.57}$$

To evaluate this limit one must evaluate $\check{T}'_{0n}$. This in turn requires knowing the asymptotic expansion of the basis $V_n^{(k)}$ for $\eta \to \infty$. This is done in Appendix A. A numerical approximation confirm the above result.

Thus, in the limit, the oscillating part completely decouples from the time dependent part. It remains for us to consider the limit of the quadratic form W''_{nm}, (4.52). When n, m are odd there are no contribution from the discrete spectrum, since the contraction with $\langle \mathrm{y}|$ has eliminated them.

$$W''_{2n-1,2m-1} = \check{S}'^{(c)}_{2n-1,2m-1}, \tag{4.58}$$

When n, m are even we have, on the contrary, potentially dangerous terms because there are divergent contributions arising from the discrete spectrum. The latter have to be carefully evaluated.

$$\begin{aligned} W''^{(d)}_{2n,2m} &= 2V^{(\xi)}_{2n}V^{(\xi)}_{2m}\left(e^{\eta} - \frac{2(V_0^{(\xi)})^2 e^{2\eta}}{2(V_0^{(\xi)})^2 e^{\eta} + \mathcal{O}(\frac{e^{-\eta}}{\eta\log\eta})}\right) \\ &= 2V^{(\xi)}_{2n}V^{(\xi)}_{2m}\,\mathcal{O}(\frac{e^{-\eta}}{\eta\log\eta}) \approx \mathcal{O}(\frac{1}{\log\eta}) \end{aligned} \tag{4.59}$$

We see that the potentially divergent contributions arising from the discrete spectrum exactly cancel when $\eta \to \infty$. Therefore, as far as $W''_{2n,2m}$ is concerned, we are left only with the contribution from the continuous spectrum. Of the two pieces that contribute to $W''^{(c)}_{2n,2m}$, see eq.(4.52) only the first survives in the limit $\eta \to \infty$, the second vanishes for the usual reasons. Therefore we can conclude that

$$W''_{nm} = \check{S}'^{(c)}_{nm} + ...$$

where dots denote subleading corrections of order at least $1/\log\eta$. At this stage we can do the calculation directly as in Appendix A, or we can resort to an indirect argument by noticing that $\check{S}'^{(c)}_{nm}$ approaches S'_{nm} in the same limit, because the discrete spectrum contribution to the latter vanishes. In both cases we conclude that

$$W''_{nm} = S_{nm} + \mathcal{O}(\epsilon/b) \tag{4.60}$$

Going back to equation (4.51) we see that, if we ignore for a moment the normalization factor, we obtain

$$\lim_{\alpha'\to 0} |\Lambda'(\mathrm{x}_0, \mathrm{y})\rangle = \mathcal{N}'(\mathrm{y})\, e^{-\frac{\mathrm{x}_0^2}{b}} |\Xi\rangle \tag{4.61}$$

where $|\Xi\rangle$ is the zero momentum sliver state. This result can be phrased as follows: in the low energy limit the solution takes the form of a time–Gaussian multiplying a sliver, the the subleading terms being proportional to ϵ/b, eq.(4.60).

Let us also briefly consider the opposite limit, that is $\alpha' \to \infty$ (tensionless limit). As in the previous case this is formally achieved by taking the $\eta \to 0$ limit in all the quantities which are related to the Neumann coefficients, but leaving b as a free parameter. This limit is well defined. Using the results of appendix A again, we get

$$\lim_{\eta\to} \frac{1 - \check{T}'_{00}}{1 + \check{T}'_{00}} = 0 \tag{4.62}$$

The oscillating term in (4.51) vanishes as well. This result implies that the Gaussian representing time dependence in (4.51) is actually completely flat: time dependence has disappeared! We believe this to be related to the fact that all strings modes become massless in this limit [117], so there are no modes to decay into. It is easy to see that the only non vanishing term in the exponent of (4.51) is the quadratic part which gets contribution only from the continuous spectrum (on the contrary in the $\eta \to \infty$ limit the discrete eigenvector has only the 0–component, while the higher components disappear like positive powers of $1/\eta$). We remark that in the tensionless limit the center of mass time and the x time are identified.

Before concluding this section lets us comment upon some loose ends which we have left aside in the course of our exposition. The first comments concerns normalization of the states we have come across. We have written down throughout

normalization factors in quite a formal way. We have already recalled the fact that the sliver state and the lump state have a vanishing normalization, but we believe these problems have to be kept separated from the normalization of our time dependent solution. As a matter of fact a normalization problem appears only for the representation (4.51) and in the low energy limit, for the coefficient $\mathcal{N}'$ in (4.61) diverges exponentially for $\eta \to \infty$ once all the contributions are taken into account (this problem does not arise for the other representation (4.41)). We remark however that, as was noticed in the discussion after eq.(4.33), the energy density of the corresponding Euclidean solution is well–defined (once the conventional lump energy density is). Therefore the exploding normalization can only be an artifact of the representation. It means that we have to use the parameters of the state to regulate the normalization, although it is not clear a priori what is the right way to do it. A possibility is to use the factor $\exp\left(\frac{1-e^{|\eta|}}{1+e^{|\eta|}}\mathrm{y}^2\right)$ in (4.51). Since this vanishes for y large, we can view y as a suitable function of η as $\eta \to \infty$. This can settle the problem. Other possibilities are connected to dressing, [83, 84].

Another important question is the number of parameters. Our solution depends on two parameters y and b. One may wonder why we extracted the y dependence from (4.33). This is indeed not a choice but a constraint. Had we not done it, we would have found a different formula (4.52) in which also the n, m odd part of S''_{nm} would have taken a contribution from the discrete spectrum (exactly as the n, m even part). However in the odd–odd part no such cancellation (4.59) as in the even–even part occurs and we would find badly divergent coefficients in W'. We gather that y is a genuine free parameter of the time–dependent state. What about b? It was argued in [12] that this parameter represents a gauge degree of freedom. This need not be in contradiction with the meaning we have attributed to it in the previous sections. We recall that in ordinary gauge theory a singular gauge transformation may convey some physical information. Now, looking at (3.32), the values $b = 0$ and $b = \infty$ may well correspond to singular gauge transformations, and therefore contain physical information. More generally the gauge nature of b may mean that using a different formulation one may be able to write the solution in terms of a single physical parameter which contains the information carried by both b and y.

The third question we would like to address is the relation between the two representations (4.41) and (4.51). The latter is expressed in terms of the open string center of mass x_0 and its interpretation is obvious. The interpretation of the former is less clear since the 'time' x does not have a clear connection with the open string center of mass time. A rather bold speculation is that x be connected with the closed string time. The closed string time couples to the closed string metric, which, in correspondence with the D–brane, must develop a singularity (it must be a solution of the effective low energy field theory associated to the closed string). So the relation between the open and the closed string time should be something like $g_c(dt_c)^2 \sim g_o(dt_o)^2$ in the field theory limit, were $g_o = 1$ and g_c becomes larger and larger near the origin. Something similar indeed occurs between x_0 and x when

$\eta \to \infty$. In fact the ratio between x_0 and the zero mode part of x decreases exponentially with η. We notice moreover that the normalization of the representation (4.41) does not need any regularization. In other words x seems to be a smoother choice of time, with respect x_0.

Next we would like to recall that [108], the role of the time coordinate represented by the midpoint $X^0\left(\frac{\pi}{2}\right)$ for causality in SFT has been emphasized. In our VSFT solution the profile along this time turns out to be highly singular: it is a constant infinite function (finite only at $X^0\left(\frac{\pi}{2}\right) = 0$), the inverse Wick–rotation of the midpoint space profile of [86].

4.2 With Constant Background Longitudinal E–field

In the previous section we have shown that it is possible to get time localized solutions of VSFT by taking an euclidean lump solution on a transverse direction (the euclidean time) and simply inverting the discrete eigenvalues of the lump Neumann matrix. This solution, preserving the same euclidean action as the conventional lump has the remarkable feature of being localized in the time coordinate identified by the twist even discrete eigenvector of the Neumann matrices and, what is more important, of being localized in the center of mass time coordinate for every value of the free parameter b.

In this and the next section we are going to study the corresponding time dependent solution in the presence of a constant E–field background on a longitudinal or transverse direction. We obtain E–field physics by first going to a euclidean signature with *imaginary* B–field, $B = iE$, and then inverse Wick rotating, in the same way as [118] for what concerns the effective target space and BCFT analysis.

One of the main differences with respect to the previous section is that when the E–field reaches its critical value $E_c = \frac{1}{2\pi\alpha'}$, the (center of mass) time dependence is lost, regardless of the b–parameter, and we get a flat non zero time profile which, along the lines of [118] should be interpreted as a static background of fundamental strings, polarized by the E–field. This result persists when $b \to 0$ if we double scale appropriately $E \to E_c$ with $b \to 0$.

Now we will analyze the case of switching the E–field along a tangential direction, i.e., along, say, the world volume of a $D25$–brane. As explained in [118], the presence of the E–field does not create non commutativity as the direction in which it is turned on is at zero momentum.

We use the double Wick rotation, that is we make space–time euclidean by sending $X^0(\sigma) \to iX^D(\sigma)$; then we construct an *unconventional* lump solution on the transverse spatial direction $X^D(\sigma)$ and inverse Wick rotate along it, $X^D(\sigma) \to -iX^0(\sigma)$. Let $\alpha, \beta = 1, D$ be the couple of directions on which the E–field is turned on. Then

E–field physics is obtained by taking an imaginary B–field

$$B_{\alpha\beta} = B\epsilon_{\alpha\beta} = iE\epsilon_{\alpha\beta}, \quad E \in \Re \tag{4.63}$$

A localized time dependent solution is easily given by straightforwardly changing the metric $\eta_{\alpha\beta}$ of the solution of [75], with the open string metric $G_{\alpha\beta}$

$$G_{\alpha\beta} = (1 - (2\pi\alpha' E)^2)\,\delta_{\alpha\beta} \tag{4.64}$$

$$G^{\alpha\beta} = \frac{1}{1 - (2\pi\alpha' E)^2}\,\delta^{\alpha\beta} \tag{4.65}$$

Note that, contrary to the case of a real B–field, a critical value shows up for the imaginary analytic continuation[3]

$$E_c = \frac{1}{2\pi\alpha'} \tag{4.66}$$

From now on all indexes (α, β) are raised/lowered with the open string metric (4.64).

We have then the following commutators

$$[a_m^\alpha, a_n^{\beta\dagger}] = G^{\alpha\beta}\delta_{mn}, \qquad m, n \geq 1 \tag{4.67}$$

stating that the a^α's are canonically normalized with respect the open string metric (4.64)

We recall that, in case of a background $B_{\alpha\beta}$–field, the three string vertex is deformed to be

$$|V_3\rangle = |V_{3,\perp}\rangle \otimes |V_{3,\parallel}\rangle \tag{4.68}$$

The factor $|V_{3,\parallel}\rangle$ concerns the directions with no B–field and its expression is the usual one, [77], on the other hand $|V_{3,\perp}\rangle$ deals with the directions on which the B field is turned on[4] .

$$|V_{3,\perp}\rangle = \int d^{26}p_{(1)} d^{26}p_{(2)} d^{26}p_{(3)} \delta^{26}(p_{(1)} + p_{(2)} + p_{(3)}) \exp(-E')\,|0,p\rangle_{123} \tag{4.69}$$

The operator in the exponent is given by, [93]

$$\begin{aligned} E'_\perp = \sum_{r,s=1}^{3} \Bigg(& \frac{1}{2} \sum_{m,n\geq 1} G_{\alpha\beta} a_m^{(r)\alpha\dagger} V_{mn}^{rs} a_n^{(s)\beta\dagger} + \sum_{n\geq 1} G_{\alpha\beta} p_{(r)}^\alpha V_{0n}^{rs} a_n^{(s)\beta\dagger} \\ & + \frac{1}{2} G_{\alpha\beta} p_{(r)}^\alpha V_{00}^{rs} p_{(s)}^\beta + \frac{i}{2} \sum_{r<s} p_\alpha^{(r)} \theta^{\alpha\beta} p_\beta^{(s)} \Bigg) \end{aligned} \tag{4.70}$$

[3]In the rest of the this section and the next section we will set $\alpha' = 1$

[4]Note that in the case under consideration the symbols $\perp$ and $\parallel$ do not refer to perpendicular or transverse directions to the brane, but simply indicates directions with E–field turned on ($\perp$) or not ($\parallel$)

Note that the part giving rise to space–time non–commutativity, $\frac{i}{2}\sum_{r<s} p_\alpha^{(r)}\theta^{\alpha\beta}p_\beta^{(s)}$, does not contribute due to the zero momentum condition in the 1 spatial direction. Let's first consider the sliver solution at zero momentum along the 1 direction The three string vertex in such a direction takes the form ($p^1 = p_1 = 0$)

$$\begin{aligned} |V_3(E, p=0)\rangle &= |V_3(E=0, p=0)\rangle^{(\eta_{11}\to G(E)_{11})} \\ &= \exp\left(\frac{1}{2}\sum_{r,s=1}^{3} G_{11}a^{(r)1\dagger}\cdot V^{rs}\cdot a^{(s)1\dagger}\right)|0\rangle \end{aligned} \tag{4.71}$$

This implies that the zero momentum sliver is in this case

$$\begin{aligned} |S(E, p=0)\rangle &= |S(E=0, p=0)\rangle^{(\eta_{11}\to G(E)_{11})} \\ &= \mathcal{N}\exp\left(\frac{1}{2}G_{11}a^{1\dagger}\cdot S\cdot a^{1\dagger}\right)|0\rangle. \end{aligned} \tag{4.72}$$

On the euclidean time direction we need the full 3–string vertex in oscillator basis. This is given by

$$|V_{3,\perp}\rangle' = K\, e^{-E'}|\Omega_b\rangle \tag{4.73}$$

with

$$K = \left(\frac{\sqrt{2\pi b^3}}{3(V_{00}+b/2)^2}(1-(2\pi E)^2)^{\frac{1}{2}}\right)^{\frac{1}{2}}, \tag{4.74}$$

$$E' = \frac{1}{2}\sum_{r,s=1}^{3}\sum_{M,N\geq 0} a_M^{(r)D\dagger}V'^{rs}_{MN}a_N^{(s)D\dagger}G_{DD} \tag{4.75}$$

where M, N denote the couple of indices $\{0, m\}$ and $\{0, n\}$, respectively, and D is the (euclidean) time direction. In order to have localization in Minkowski time, we need an explosive profile in euclidean time (unconventional lump); this is explained in detail in the previous section.

$$|\check{\Lambda}'\rangle = \mathcal{N}\exp\left(\frac{1}{2}G_{DD}a^{\dagger D}C\check{T}'a^{\dagger D}\right)|\Omega_b\rangle \tag{4.76}$$

where

$$\check{T}'_{NM} = -\int_{-\infty}^{\infty} dk\, V_N^{(k)}\, V_M^{(k)}\exp\left(-\frac{\pi|k|}{2}\right) + \left(V_N^{(\xi)}\, V_M^{(\xi)} + V_N^{(\bar{\xi})}\, V_M^{(\bar{\xi})}\right)\exp|\eta| \tag{4.77}$$

We again stress that the Neumann matrix of the unconventional lump has inverted-discrete eigenvalues with respect to the ordinary lump: this is what guarantees time localization with respect to the center of mass and to the time coordinates identified by the discrete eigenvectors $V_N^{(\bar{\xi})}$, $V_N^{\xi)}$.

We get a localized time profile by projecting on the coordinates/momenta of the discrete spectrum

$$\hat{x}_\eta = \frac{i}{\sqrt{2}}\left(e_\eta - e_\eta^\dagger\right) \tag{4.78}$$

$$\hat{y}_\eta = \frac{i}{\sqrt{2}}\left(o_\eta - o_\eta^\dagger\right). \tag{4.79}$$

The profile along these coordinates is given by (inverse Wick rotation, $(x,y) \to i(\mathrm{x},-\mathrm{y})$ is assumed)

$$|\check{\Lambda}'(\mathrm{x},\mathrm{y})\rangle = \langle \mathrm{x},\mathrm{y}|\check{\Lambda}'\rangle = \frac{1}{\pi(1+e^{|\eta|})}\exp\left(-\frac{e^{|\eta|}-1}{e^{|\eta|}+1}(\mathrm{x}^2+\mathrm{y}^2)\right)|\check{\Lambda}'_c\rangle \tag{4.80}$$

where $|\check{\Lambda}'_c\rangle$ contains only continuous spectrum contributions. This profile is localized on the time coordinate x. Note however that there is no more reference to the E–field in the exponent. In order to see explicitly the presence of the E–field, we need to use the usual *open* string time, i.e. the center of mass.

Therefore we contract our solution with the center of mass euclidean time, x^D, and then inverse Wick rotate it, $x^D \to ix^0$. This is identical to previous section, so we just quote the result, paying attention to use the open string metric (4.64)

$$\begin{aligned} |\Lambda'(\mathrm{x}_0,y)\rangle \;&=\; \langle \mathrm{x}_0,\mathrm{y}|\Xi_\eta\rangle = \sqrt{\frac{2}{b\pi}}\frac{\mathcal{N}}{\sqrt{2\pi(1+e^{|\eta|})}}\exp\left(\frac{1-e^{|\eta|}}{1+e^{|\eta|}}\mathrm{y}^2\right) \\ \cdot\frac{1}{\sqrt{1+\check{T}'_{00}}} \;&\exp\left(-\mathcal{A}(x^0)^2 + \frac{2i\sqrt{1-(2\pi E)^2}}{\sqrt{b}(1+\check{T}'_{00})}x^0\check{T}'_{0n}\tilde{a}_n^\dagger - \frac{1}{2}\tilde{a}_n^\dagger W''_{nm}\tilde{a}_m^\dagger\right)|0\rangle \end{aligned} \tag{4.81}$$

The extra coordinate y is given by the twist odd contribution of the discrete spectrum, we need to project along it in order to have a well defined $b \to \infty$ limit in the oscillator part of W''_{nm}. The oscillators $\tilde{a}_n$ are canonically normalized with respect the η-metric and are given by

$$\tilde{a}_n = \sqrt{1-(2\pi E)^2}a_n \tag{4.82}$$

The quantity that give rise to time localization is then

$$\mathcal{A} = -\frac{1}{b}\frac{1-\check{T}'_{00}}{1+\check{T}'_{00}}(1-(2\pi E)^2) \tag{4.83}$$

This quantity depends on the free parameter b, as well as on the value of the E–field, through the open string metric, used to covariantize the quadratic form in time. The matrix element $\check{T}'_{00}$ is given by

$$\check{T}'_{00}(\eta) = -2\int_0^\infty dk\left(V_0^{(k)}(b(\eta))\right)^2\exp\left(-\frac{\pi k}{2}\right) + 2(V_0^{(\xi)})^2\exp|\eta|, \tag{4.84}$$

it is a monotonic increasing function of b, greater than 1: this is what ensures localization in time as opposed to the standard lump which is suited for space localization.

The life time of the brane is thus given by

$$\Delta T = \frac{1}{2}\sqrt{\frac{1}{2\mathcal{A}}} = \frac{1}{(1-(2\pi E)^2)^{\frac{1}{2}}}\,\Delta T^{(E=0)} \tag{4.85}$$

Note that for E going to the critical value $E_c = \frac{1}{2\pi}$, the life time becomes infinite. In particular we get a completely flat profile. This has to be traced back to the fact that open strings become effectively tensionless in this limit, [119], so we correctly recover the result of the previous section, that the D-brane is stable. This configuration should correspond to a background of fundamental strings stretched along the E–field direction, with closed strings completely decoupled.

4.3 With Constant Background Transverse E–field

The solitonic lump solutions in VSFT with a constant transverse B field turned on along the 24^{th} and 25^{th} directions (in what follows denoted by α and β) were determined in [87, 88] and are given by

$$|S\rangle = |S_{||}\rangle \otimes \mathcal{N} \exp\left(-\frac{1}{2}\sum_{M,N\geq 0} a_M^{\alpha\dagger}\mathcal{S}_{\alpha\beta,MN}a_N^{\beta\dagger}\right)|0\rangle \otimes |\Omega_{b,\theta}\rangle, \tag{4.86}$$

where

$$\mathcal{N} = \frac{A^2(3+4a^2)}{\sqrt{2\pi b^3}(\mathrm{Det}G)^{\frac{1}{4}}}\mathrm{Det}(\mathcal{I}-\mathcal{X})^{\frac{1}{2}}\mathrm{Det}(\mathcal{I}+\mathcal{T})^{\frac{1}{2}}, \tag{4.87}$$

and

$$A = V_{00}+\frac{b}{2}, \quad a = -\frac{\pi^2}{A}|B|, \quad \mathcal{X} = C'\mathcal{V}^{11}, \quad \mathcal{T} = C'\mathcal{S}, \quad C' = (-1)^N\delta_{NM} \tag{4.88}$$

In (4.86) $|S_{||}\rangle$ corresponds to the longitudinal part of the lump solution and it is a zero momentum sliver, $\mathcal{V}^{11}$ is as introduced in (3.43).

Exactely like the case without E–field, in order for (4.86) to satisfy the projector equation, $\mathcal{T}$ and $\mathcal{X}$ should satisfy the relation:

$$(\mathcal{T}-1)(\mathcal{X}\mathcal{T}^2-(\mathcal{I}+\mathcal{X})\mathcal{T}+\mathcal{X}) = 0. \tag{4.89}$$

In the above formulae the α, β, N, M indices are implicit. This equation is solved by $\mathcal{T}_0$, $1/\mathcal{T}_0$ and 1, where

$$\mathcal{T}_0 = \frac{1}{2\mathcal{X}}\left(1+\mathcal{X}-\sqrt{(1+3\mathcal{X})(1-\mathcal{X})}\right) \tag{4.90}$$

$\mathcal{T} = 1$ gives the identity state, whereas the first and the second solutions give the lump and the inverse lump, respectively.

4.3.1 Spectroscopy of the Neumann matrices with B–field

Next we present the computation of the eigenstates and eigenvalues of the Neumann matrix $\mathcal{X}^{\alpha}{}_{\beta}$ in the presence of **B** field, along the line of Belov[114]. A similar analysis was carried out in [125], but with no reference to the correct normalization of continuous and discrete eigenvectors; moreover the discrete eigenvectors presented in [125] does not reproduce the known ones when $B \to 0$, since the discrete spectrum is of crucial importance for our purposes we re–derive completely the whole spectroscopy following a method, [114], which is more suitable for our purposes. To avoid the degeneracy of the diagonal Neumann coefficient $\mathcal{X}^{\alpha}{}_{\beta}$, we consider the unitary matrices $C'\mathcal{U}^{\alpha}{}_{\beta}$ and $\mathcal{U}^{\alpha}{}_{\beta}C'$, which are related to $\mathcal{X}^{\alpha}{}_{\beta}$ as follows [87]:

$$\left(\mathcal{X}^{\alpha}{}_{\beta}\right)_{NM} = \frac{1}{3}\left(\delta^{\alpha}{}_{\beta} + C'\mathcal{U}^{\alpha}{}_{\beta} + C'\bar{\mathcal{U}}^{\alpha}{}_{\beta}\right)_{NM}. \tag{4.91}$$

The matrix $(C'\mathcal{U}^{\alpha}{}_{\beta})_{NM}$ can be written explicitly as:

$$C'\mathcal{U} = \begin{pmatrix} 1-3bK & 2\sqrt{3}bKa & 3\sqrt{b}K\langle W| & -2\sqrt{3b}Ka\langle W| \\ -2\sqrt{3}bKa & 1-3bK & 2\sqrt{3b}Ka\langle W| & 3\sqrt{b}K\langle W| \\ 3\sqrt{b}K|W\rangle & -2\sqrt{3b}Ka|W\rangle & CU-3K|W\rangle\langle W| & 2\sqrt{3}Ka|W\rangle\langle W| \\ 2\sqrt{3b}Ka|W\rangle & 3\sqrt{b}K|W\rangle & -2\sqrt{3}Ka|W\rangle\langle W| & CU-3K|W\rangle\langle W| \end{pmatrix}$$

where, see [75]

$$|W\rangle = -\sqrt{2}(|v_e\rangle + i|v_o\rangle), \qquad K = \frac{A^{-1}}{4a^2+3}. \tag{4.92}$$

CU is the non-zero mode analog of $C'\mathcal{U}$ without B field. We recall that, [87],

$$C'\bar{\mathcal{U}} = \tilde{\mathcal{U}}C', \tag{4.93}$$

where tilde means transposition with respect to α, β indices.

Our aim is to solve the eigenvalue equation:

$$C'\mathcal{U}|\Psi\rangle = \xi|\Psi\rangle, \qquad |\Psi\rangle = \begin{pmatrix} g_1 \\ g_2 \\ |\Lambda_1\rangle \\ |\Lambda_2\rangle \end{pmatrix}, \tag{4.94}$$

which splits into:

$$\langle W|\Lambda_1\rangle = \frac{A}{\sqrt{b}}[\xi - 1 + \frac{b}{A}]g_1 + \frac{2Aa}{\sqrt{3b}}(\xi-1)g_2 \tag{4.95}$$

$$\langle W|\Lambda_2\rangle = \frac{A}{\sqrt{b}}[\xi - 1 + \frac{b}{A}]g_2 - \frac{2Aa}{\sqrt{3b}}(\xi-1)g_1 \tag{4.96}$$

$$(CU-\xi)|\Lambda_1\rangle = \sqrt{\frac{1}{b}}g_1(\xi-1)|W\rangle \tag{4.97}$$

$$(CU-\xi)|\Lambda_2\rangle = \sqrt{\frac{1}{b}}g_2(\xi-1)|W\rangle \tag{4.98}$$

We know, [113], that CU has a continuous spectrum and the solution of (4.94) depends on whether the eigenvalue ξ is in the continuous spectrum of CU or not. So we will distinguish these two different cases and analyze each of them in detail.

Discrete spectrum

If ξ is not in the spectrum of CU, we can invert $(CU-\xi)$ in equations (4.97) and (4.98) to obtain:

$$|\Lambda_1\rangle = \sqrt{\frac{1}{b}}g_1(\xi-1)\frac{1}{(CU-\xi)}|W\rangle \tag{4.99}$$

$$|\Lambda_2\rangle = \sqrt{\frac{1}{b}}g_2(\xi-1)\frac{1}{(CU-\xi)}|W\rangle. \tag{4.100}$$

As we can see the solutions get modified w.r.t. the $B=0$ case, only via possible modifications of the eigenvalue ξ. Substitution of these solutions into equations (4.95) and (4.96) gives:

$$\sqrt{\frac{1}{2b}}(\xi-1)\langle W|\frac{1}{CU-\xi}|W\rangle g_1 - \frac{A}{\sqrt{2b}}\left(\xi-1+\frac{b}{A}\right)g_1 - \frac{2aA}{\sqrt{6b}}(\xi-1)g_2 = 0 \tag{4.101}$$

$$\sqrt{\frac{1}{2b}}(\xi-1)\langle W|\frac{1}{CU-\xi}|W\rangle g_2 - \frac{A}{\sqrt{2b}}\left(\xi-1+\frac{b}{A}\right)g_2 + \frac{2aA}{\sqrt{6b}}(\xi-1)g_1 = 0 \tag{4.102}$$

The bracket which appears here is the same as the one in [114] and is given by:

$$\langle W|\frac{1}{CU-\xi}|W\rangle = V_{00} + \frac{\xi+1}{\xi-1}2\Re F(\eta) \tag{4.103}$$

where

$$F(\eta) = \psi(\frac{1}{2}+\frac{\eta}{2\pi i}) - \psi(\frac{1}{2}), \qquad \xi = -\frac{1}{1-2\cosh\eta}[2-\cosh\eta - \mathrm{i}\sqrt{3}\sinh\eta]. \tag{4.104}$$

$\psi(x)$ is the logarithmic derivative of the Euler Γ-function.

Substitution of these in (4.101) and (4.102) gives us:

$$\left(\Re F(\eta) - \frac{b}{4}\right) g_1 - \frac{aA}{\sqrt{3}} \frac{\xi - 1}{\xi + 1} g_2 = 0, \qquad \left(\Re F(\eta) - \frac{b}{4}\right) g_2 + \frac{aA}{\sqrt{3}} \frac{\xi - 1}{\xi + 1} g_1 = 0. \quad (4.105)$$

This system of equations will have non trivial solutions for g_2 and g_1 if the determinant of the coefficient matrix is zero, i.e.

$$\frac{b}{4} = \Re F(\eta) \pm aA \text{tanh} \frac{\eta}{2}, \quad (4.106)$$

Using equations (4.105) we can show that $g_2 = \pm i g_1$. This is a constraint on g_1 and g_2 thus we cannot split the eigenstates in the two directions, choosing one of the constants to be zero. g_1 is now an overall constant, which can be chosen real and fixed by normalization completely.

The eigenstates are then

Case-1:

$$\frac{b}{4} = \Re F(\eta) + aA \text{tanh} \frac{\eta}{2}, \qquad g_2 = -i g_1 = -i g_d(\eta_1, \eta_2) \quad (4.107)$$

$$|V^{(\xi_1)}\rangle = g_d(\eta_1, \eta_2) \begin{pmatrix} 1 \\ -i \\ \sqrt{\frac{1}{b}(\xi_1 - 1)} \frac{1}{CU - \xi_1} |W\rangle \\ -i\sqrt{\frac{1}{b}(\xi_1 - 1)} \frac{1}{CU - \xi_1} |W\rangle \end{pmatrix} \quad (4.108)$$

$$|V^{(\bar{\xi}_2)}\rangle = g_d(\eta_2, \eta_1) \begin{pmatrix} 1 \\ -i \\ \sqrt{\frac{1}{b}(\bar{\xi}_2 - 1)} \frac{1}{CU - \bar{\xi}_2} |W\rangle \\ -i\sqrt{\frac{1}{b}(\bar{\xi}_2 - 1)} \frac{1}{CU - \bar{\xi}_2} |W\rangle \end{pmatrix} \quad (4.109)$$

Case-2:

$$\frac{b}{4} = \Re F(\eta) - aA \text{tanh} \frac{\eta}{2}, \qquad g_2 = i g_1 = i g_d(\eta_2, \eta_1) \quad (4.110)$$

$$|V^{(\xi_2)}\rangle = g_d(\eta_2, \eta_1) \begin{pmatrix} 1 \\ i \\ \sqrt{\frac{1}{b}(\xi_2 - 1)} \frac{1}{CU - \xi_2} |W\rangle \\ i\sqrt{\frac{1}{b}(\xi_2 - 1)} \frac{1}{CU - \xi_2} |W\rangle \end{pmatrix} \quad (4.111)$$

$$|V^{(\bar{\xi}_1)}\rangle = g_d(\eta_1, \eta_2) \begin{pmatrix} 1 \\ i \\ \sqrt{\frac{1}{b}(\bar{\xi}_1 - 1)} \frac{1}{CU - \bar{\xi}_1} |W\rangle \\ i\sqrt{\frac{1}{b}(\bar{\xi}_1 - 1)} \frac{1}{CU - \bar{\xi}_1} |W\rangle \end{pmatrix}. \quad (4.112)$$

Normalizing them in the following way

$$\bar{V}_{\alpha}^{\xi_i} V^{\xi_j,\alpha} = \delta^{ij}$$

$$\bar{V}_{\alpha}^{\bar{\xi}_i} V^{\bar{\xi}_j,\alpha} = \delta^{ij}$$

$$\bar{V}_{\alpha}^{\bar{\xi}} V^{\xi,\alpha} = 0$$

we get, use the results of [114],

$$|g_d(\eta_1,\eta_2)|^2 = \frac{1}{2\Delta}\left[(1-r(\eta_1,\eta_2)) + r(\eta_1,\eta_2)\sinh\eta_1 \frac{\partial}{\partial\eta_1}[\mathrm{Log}\Re F(\eta_1)]\right)^{-1}, \quad (4.113)$$

where

$$r(\eta_1,\eta_2) = \Re F(\eta_1)\frac{\tanh(\frac{\eta_1}{2}) + \tanh(\frac{\eta_2}{2})}{\Re F(\eta_2)\tanh(\frac{\eta_1}{2}) + \Re F(\eta_1)\tanh(\frac{\eta_2}{2})}. \quad (4.114)$$

It is important to notice that $V^{(\xi_1)}$ and $V^{(\bar{\xi}_1)}$ are degenerate eigenstates of $\mathcal{X}$, and the same holds for $V^{(\xi_2)}$ and $V^{(\bar{\xi}_2)}$.

Continuous spectrum

If ξ is in the continuous spectrum of CU ($\xi = \nu(k)$, [114]), we cannot invert the operator $(CU - \xi)$. Thus, in this case, the solution of (4.97) and (4.98) is:

$$|\Lambda_1\rangle = A_1(k)|k\rangle + \frac{1}{\sqrt{b}} g_1(\nu(k)-1)\wp\frac{1}{(CU-\nu(k))}|W\rangle \quad (4.115)$$

$$|\Lambda_2\rangle = A_2(k)|k\rangle + \frac{1}{\sqrt{b}} g_2(\nu(k)-1)\wp\frac{1}{CU-\nu(k)}|W\rangle. \quad (4.116)$$

Where $\wp$ is the principal value, [114]. Using these in (4.95) and (4.96), we get:

$$A_1(k) = g_1\sqrt{\frac{2}{b}}k\left(\Re F_c(k) - \frac{b}{4}\right) - \frac{\sqrt{2}Aa}{\sqrt{3b}}k\left(\frac{\nu(k)-1}{\nu(k)+1}\right)g_2 \quad (4.117)$$

$$A_2(k) = g_2\sqrt{\frac{2}{b}}k\left(\Re F_c(k) - \frac{b}{4}\right) + \frac{\sqrt{2}Aa}{\sqrt{3b}}k\left(\frac{\nu(k)-1}{\nu(k)+1}\right)g_1 \quad (4.118)$$

Note that in this case g_1 and g_2 are completely free and we can choose them to construct two linearly independent orthogonal vectors as follows:
Case-1: $g_2 = ig_1 = ig_c(k)$

$$V^1(k) = g_c(k)\begin{pmatrix} 1 \\ i \\ P(k)|k\rangle + \frac{1}{\sqrt{b}}(\nu(k)-1)\wp\frac{1}{CU-\nu(k)}|W\rangle - iH(k,a)|k\rangle \\ iP(k)|k\rangle + i\frac{1}{\sqrt{b}}(\nu(k)-1)\wp\frac{1}{CU-\nu(k)}|W\rangle + H(k,a)|k\rangle \end{pmatrix} \quad (4.119)$$

Case-2: $g_2 = -ig_1 = -ig_c(-k)$

$$V^2(k) = g_c(-k)\begin{pmatrix} 1 \\ -i \\ P(k)|k\rangle + \frac{1}{\sqrt{b}}(\nu(k)-1)\wp_{\frac{1}{CU-\nu(k)}}|W\rangle + iH(k,a)|k\rangle \\ -iP(k)|k\rangle - i\frac{1}{\sqrt{b}}(\nu(k)-1)\wp_{\frac{1}{CU-\nu(k)}}|W\rangle + H(k,a)|k\rangle \end{pmatrix} \quad (4.120)$$

where:

$$P(k) = \sqrt{\frac{2}{b}}k\left(\Re F_c(k) - \frac{b}{4}\right), \qquad H(k,a) = \frac{\sqrt{2}Aa}{\sqrt{3b}}k\left(\frac{\nu(k)-1}{\nu(k)+1}\right). \quad (4.121)$$

Imposing the continuous orthonormality condition

$$\bar{V}^{i,\alpha}(k)V^j_\alpha(k') = \delta^{ij}\delta(k-k') \quad (4.122)$$

we get

$$g_c(k) = \left[\frac{4\Delta}{b}N(k)\left(4 + k^2\left(\Re F_c(k) - \frac{b}{4} - \frac{Aa}{\tanh\frac{\pi k}{4}}\right)^2\right)\right]^{-1/2} \quad (4.123)$$

Sending $k \to -k$ we get the right degeneracy for $\mathcal{X}$.

4.3.2 Diagonalization of the 3-string vertex and the Lump state

We can express the oscillators $a^{(r)}_{N,\alpha}$, appearing in the 3-string vertex in terms of the oscillators of the diagonal basis as:

$$a^{(r)}_{N,\alpha} = \sum_{i=1}^{2}\left(a^{(r)}_{\xi_i}\bar{V}^{(\xi_i)}_{N,\alpha} + a^{(r)}_{\bar{\xi}_i}\bar{V}^{(\bar{\xi}_i)}_{N,\alpha} + \int_{-\infty}^{\infty} dk a^{(r)}_i(k)\bar{V}^{(i)}_{N,\alpha}(k)\right) \quad (4.124)$$

$$a^{(r)\dagger}_{N,\alpha} = \sum_{i=1}^{2}\left(a^{(r)\dagger}_{\xi_i}V^{(\xi_i)}_{N,\alpha} + a^{(r)\dagger}_{\bar{\xi}_i}V^{(\bar{\xi}_i)}_{N,\alpha} + \int_{-\infty}^{\infty} dk a^{(r)\dagger}_i(k)V^{(i)}_{N,\alpha}(k)\right). \quad (4.125)$$

Using these oscillators and the fact that $\tau\bar{V} = V$ ($\tau^\alpha_\beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$), we can rewrite the 3-string vertex as:

$$|V_3^m\rangle = N_m \exp[-\frac{1}{2}\sum_{r,s}\sum_{i=1}^{2}\left(a^{(r)\dagger}_{\xi_i}\bar{V}^{(\xi_i)}_{N,\alpha} + a^{(r)\dagger}_{\bar{\xi}_i}\bar{V}^{(\bar{\xi}_i)}_{N,\alpha} + \int_{-\infty}^{\infty} dk a^{(r)\dagger}_i(k)\bar{V}^{(i)}_{N,\alpha}(k)\right)(\tau C'\mathcal{X})^{\alpha(rs)}_{\beta,NM}$$

$$\times \sum_{j=1}^{2} \left(a_{\xi_j}^{(s)\dagger} V_M^{(\xi_j)\beta} + a_{\bar{\xi}_j}^{(s)\dagger} V_M^{(\bar{\xi}_j)\beta} + \int_{-\infty}^{\infty} dk' a_j^{(s)\dagger}(k') V_M^{(j)\beta}(k') \right)] |\Omega_{b,\theta}\rangle \tag{4.126}$$

The twist operator $\tau C'$ acts on the eigenstates of the discrete and continous spectra as follows:

$$\tau C' V^{(\xi_i)} = V^{(\bar{\xi}_i)} \qquad \tau C' V^{(i)}(k) = V^{(i+1)}(-k), \tag{4.127}$$

where $V^3(k)$ is identified with $V^1(k)$. Then (4.126) becomes

$$|V_3^m\rangle = N_m \exp \left[-\frac{1}{2} \sum_{r,s} \sum_{i=1}^{2} \left(a_{\xi_i}^{(r)\dagger} \mu^{rs}(\bar{\xi}_i) a_{\bar{\xi}_i}^{(s)\dagger} + a_{\bar{\xi}_i}^{(r)\dagger} \mu^{rs}(\xi_i) a_{\xi_i}^{(s)\dagger} + \right.\right.$$

$$\left.\left. + \int_{-\infty}^{\infty} dk a_i^{(r)\dagger}(k) \mu^{rs}(-k) a_{i+1}^{(s)\dagger}(-k) \right) \right] |\Omega_b\rangle. \tag{4.128}$$

In order to write this in an exact diagonal form, we need to introduce oscillators with definite τ–twist parity

$$e_{\xi_i}^r = \frac{a_{\xi_i}^r + a_{\bar{\xi}_i}^r}{\sqrt{2}} = \frac{a_{\xi_i}^r + \tau C' a_{\xi_i}^r}{\sqrt{2}}, \qquad o_{\xi_i}^r = -i \frac{a_{\xi_i}^r - a_{\bar{\xi}_i}^r}{\sqrt{2}} = -i \frac{a_{\xi_i}^r - \tau C' a_{\xi_i}^r}{\sqrt{2}} \tag{4.129}$$

$$e_i^r(k) = \frac{a_i^r(k) + a_{i+1}^r(-k)}{\sqrt{2}} = \frac{a_i^r(k) + \tau C' a_i^r(k)}{\sqrt{2}}$$

$$o_i^r(k) = -i \frac{a_i^r(k) - a_{i+1}^r(-k)}{\sqrt{2}} = -i \frac{a_i^r(k) - \tau C' a_i^r(k)}{\sqrt{2}} \tag{4.130}$$

These oscillators have the following BPZ conjugation property

$$\text{bpz } o_i = -o_i^\dagger \qquad \text{bpz } e_i = -e_i^\dagger \,, \tag{4.131}$$

and satisfy the commutation relations

$$[e_{\xi_i}, e_{\xi_j}^\dagger] = \delta_{ij}, \qquad [o_{\xi_i}, o_{\xi_j}^\dagger] = \delta_{ij},$$

$$[e_i(k), e_j^\dagger(k')] = \delta_{ij}\delta(k-k'), \qquad [o_i(k), o_j^\dagger(k')] = \delta_{ij}\delta(k-k'), \tag{4.132}$$

with all the other commutators vanishing. Using them into (4.128) we finally obtain

$$|V_3^m\rangle = N_m \exp \left[-\frac{1}{4} \sum_{r,s} \sum_{i=1}^{2} \left(\left[\mu^{rs}(\xi_i) + \mu^{rs}(\bar{\xi}_i) \right] \left(e_{\xi_i}^{(r)\dagger} e_{\xi_i}^{(s)\dagger} + o_{\xi_i}^{(r)\dagger} o_{\xi_i}^{(s)\dagger} \right) \right.\right.$$

$$\left.\left. -i \left[\mu^{rs}(\xi_i) - \mu^{rs}(\bar{\xi}_i) \right] \left(o_{\xi_i}^{(r)\dagger} e_{\xi_i}^{(s)\dagger} - e_{\xi_i}^{(r)\dagger} o_{\xi_i}^{(s)\dagger} \right) \right.\right.$$

$$-\int_{-\infty}^{\infty} dk\mu^{rs}(k)\left(e_i^{(r)\dagger}(k)e_i^{(s)\dagger}(k)+o_i^{(r)\dagger}(k)o_i^{(s)\dagger}(k)\right)$$

$$-i\int_{-\infty}^{\infty} dk\mu^{rs}(k)\left(e_i^{(r)\dagger}(k)o_i^{(s)\dagger}(k)-o_i^{(r)\dagger}(k)e_i^{(s)\dagger}(k)\right)\Bigg)\Bigg]|\Omega_{b,\theta}\rangle \tag{4.133}$$

This gives the diagonal representation of the 3-string interaction vertex. The same procedure gives the following diagonal representation of the transverse part of the Lump:

$$|S_\perp\rangle=\frac{A^2(3+4a^2)}{\sqrt{2\pi b^3}(\mathrm{Det}G)^{\frac{1}{4}}}\mathrm{Det}(\mathcal{I}-\mathcal{X})^{\frac{1}{2}}\mathrm{Det}(\mathcal{I}+\mathcal{T})^{\frac{1}{2}}\exp\left(-\frac{1}{2}\sum_{i=1}^{2}\left[t_d(\eta_i)\left(e_{\xi_i}^\dagger e_{\xi_i}^\dagger+o_{\xi_i}^\dagger o_{\xi_i}^\dagger\right)+\right.\right.$$

$$\left.\left.+\frac{1}{2}\int_{-\infty}^{\infty}dkt_c(k)\left(e_i^\dagger(k)e_i^\dagger(k)+o_i^\dagger(k)o_i^\dagger(k)\right)\right]\right)|\Omega_{b,\theta}\rangle \tag{4.134}$$

4.3.3 Diagonal Coordinates and Momenta

In the previous section τ–twist definite oscillators of the diagonal basis are introduced. Due to the structure of Neumann coefficients it is natural to define the twist matrix as τC, where $\tau=\sigma^3$ acts on space–time indices. In the following C–parity will be always understood as τC–parity. Now let's define the following coordinates and momenta operators in terms of the twist even and twist odd parts of the discrete spectrum

$$\hat{X}_{\xi_i}=\frac{i}{\sqrt{2}}(e_{\xi_i}-e_{\xi_i}^\dagger)\qquad \hat{Y}_{\xi_i}=\frac{i}{\sqrt{2}}(o_{\xi_i}-o_{\xi_i}^\dagger) \tag{4.135}$$

which are hermitian by definition and have the following eigenstates

$$|X_i\rangle=\frac{1}{\sqrt{\pi}}e^{-\frac{1}{2}X_i^2-\sqrt{2}iX_ie_{\xi_i}^\dagger+\frac{1}{2}e_{\xi_i}^\dagger e_{\xi_i}^\dagger}|\Omega_{e_i}\rangle \tag{4.136}$$

$$|Y_i\rangle=\frac{1}{\sqrt{\pi}}e^{-\frac{1}{2}Y_i^2-\sqrt{2}iY_io_{\xi_i}^\dagger+\frac{1}{2}o_{\xi_i}^\dagger o_{\xi_i}^\dagger}|\Omega_{o_i}\rangle. \tag{4.137}$$

We made the assumption that the vacuum factorizes as

$$|0\rangle\otimes|\Omega_{b,\theta}\rangle=\prod_{i=1}^{2}\prod_k|\Omega_i(k)\rangle\otimes|\Omega_{e_i}\rangle\otimes|\Omega_{o_i}\rangle \tag{4.138}$$

where $|\Omega_i(k)\rangle$, $|\Omega_{e_i}\rangle$ and $|\Omega_{o_i}\rangle$ are vacua with respect to the continuous, the twist even discrete and twist odd discrete oscillators, respectively.

The explicit (X_i,Y_i) dependence of the lump state (4.86) can be obtained by projecting it onto the eigenstates $|X_i,Y_i\rangle$. After re-writing (4.86) in terms of the diagonal basis oscillators and performing the projection, it follows:

$$\langle X_i,Y_i|S\rangle=\frac{1}{\pi^2[1+t_d(\eta_1)][1+t_d(\eta_2)]}\exp\frac{1}{2}\left[\frac{t_d(\eta_1)-1}{t_d(\eta_1)+1}(X_1^2+Y_1^2)\right.$$

$$+\frac{t_d(\eta_2)-1}{t_d(\eta_2)+1}(X_2^2+Y_2^2)\Big]\,|S\rangle_c\otimes|S_{||}\rangle. \qquad (4.139)$$

$|S\rangle_c$ is given by (4.134) with only continuous spectrum oscillators and $t_d(\eta_i)=e^{-|\eta_i|}$ are the discrete eigenvalues of $\mathcal{T}$ corresponding to the eigenvalue $\xi(\eta_i)$ of the operator $C'\mathcal{U}$.

In (4.139) the directions α,β are completely mixed. As a matter of fact, it is not apparent at this stage which of these variables (X_i,Y_i) contain the information about the center of mass time dependence of the lump. To make this clear let's recall the non-diagonal basis oscillators and write the coordinates and the momenta operators as

$$\hat{X}_N^\alpha=\frac{i}{\sqrt{2}}(a_N^\alpha-a_N^{\alpha\dagger}) \qquad \hat{P}_N^\alpha=\frac{1}{\sqrt{2}}(a_N^\alpha+a_N^{\alpha\dagger}). \qquad (4.140)$$

In order to get the relation between these operators and the corresponding diagonal operators we have defined above, we need to re-write the diagonal basis oscillators in terms of the non-diagonal ones. In doing so, one has to be careful about taking the complex conjugate of the eigenstates, as we are dealing with hermitian rather then symmetric matrices. Taking this fact into account and using some results of the previous section, we obtain

$$e_{\xi_i}=\frac{1}{\sqrt{2}}\sum_{N=0}^{\infty}(V_N^{(\xi_i)\alpha}+V_N^{(\bar{\xi}_i)\alpha})a_{N,\alpha} \qquad e_{\xi_i}^\dagger=\frac{1}{\sqrt{2}}\sum_{N=0}^{\infty}(\bar{V}_N^{(\xi_i)\alpha}+\bar{V}_N^{(\bar{\xi}_i)\alpha})a_{N,\alpha}^\dagger \qquad (4.141)$$

$$o_{\xi_i}=\frac{-i}{\sqrt{2}}\sum_{N=0}^{\infty}(V_N^{(\xi_i)\alpha}-V_N^{(\bar{\xi}_i)\alpha})a_{N,\alpha} \qquad o_{\xi_i}^\dagger=\frac{i}{\sqrt{2}}\sum_{N=0}^{\infty}(\bar{V}_N^{(\xi_i)\alpha}-\bar{V}_N^{(\bar{\xi}_i)\alpha})a_{N,\alpha}^\dagger \qquad (4.142)$$

and similar relations for the continuous spectrum oscillators. Hence, the diagonal coordinates and momenta can be written as:

$$\hat{X}_{\xi_i}=\sqrt{2}\sum_{N=0}^{\infty}V_{2N}^{\xi_i,1}\hat{X}_{2N}^1+V_{2N+1}^{\xi_i,2}\hat{P}_{2N+1}^2 \qquad (4.143)$$

$$\hat{Y}_{\xi_i}=\sqrt{2}\sum_{N=0}^{\infty}V_{2N+1}^{\xi_i,1}\hat{P}_{2N+1}^1-iV_{2N}^{\xi_i,2}\hat{X}_{2N}^2 \qquad (4.144)$$

Now, to make the center of mass time dependence of the solution explicit, we need to extract the zero modes from these operators. Let's write the zero mode coordinate and momentum operators by introducing the b parameter as:

$$\hat{X}_0^\alpha=\frac{i}{\sqrt{b}}(a_0^\alpha-a_0^{\alpha\dagger}) \qquad \hat{P}_0^\alpha=\frac{\sqrt{b}}{2}(a_0^\alpha+a_0^{\alpha\dagger}). \qquad (4.145)$$

This gives

$$\hat{X}_{\xi_i}=\sqrt{2}\left[V_0^{\xi_i,1}\sqrt{\frac{2}{b}}X_0^1+\sum_{n=1}^{\infty}V_{2n}^{\xi_i,1}\hat{X}_{2n}^1+V_{2n-1}^{\xi_i,2}\hat{P}_{2n-1}^2\right], \qquad (4.146)$$

$$\hat{Y}_{\xi_i} = \sqrt{2}\left[V_0^{\xi_i,2}\sqrt{\frac{2}{b}}X_0^2 + \sum_{n=1}^{\infty} V_{2n-1}^{\xi_i,1}\hat{P}_{2n-1}^1 - iV_{2n}^{\xi_i,2}\hat{X}_{2n}^2\right]. \tag{4.147}$$

Since our aim is to obtain the localization in time by making the inverse Wick rotation on direction 1, we see that it is X_{ξ_i} that contains the time coordinate, which we have to compare with the string center of mass time (see below).

4.3.4 Projection on the center of mass coordinates

In order to obtain the open string time profile of the lump solution, we need to project it onto the center of mass coordinates of the string. The center of mass position operator is given by

$$\hat{x}_{cm,\alpha} = \frac{i}{\sqrt{b}}(a_{0,\alpha} - a_{0,\alpha}^\dagger) \tag{4.148}$$

and its eigenstate is

$$|X_{CM}\rangle = \sqrt{\frac{2\Delta}{\pi b}}\, e^{-\frac{1}{b}x_\alpha x^\alpha - \frac{2}{\sqrt{b}}ix_\alpha a_0^{\alpha\,\dagger} + \frac{1}{2}a_{0,\alpha}^\dagger a_0^{\alpha\,\dagger}}|\Omega_{\theta,b}\rangle. \tag{4.149}$$

One can project the lump on this state to obtain the center of mass time profile. However, for reasons that will be clear later, we will first project on the Y_i momenta,

$$|\Lambda\rangle = \langle Y_1, Y_2|S\rangle = \frac{\mathcal{N}}{\pi\sqrt{[1+t_d(\eta_1)][1+t_d(\eta_2)]}}\exp\frac{1}{2}\left[\frac{t_d(\eta_1)-1}{t_d(\eta_1)+1}Y_1^2 + \frac{t_d(\eta_2)-1}{t_d(\eta_2)+1}Y_2^2\right]$$

$$\times\exp -\frac{1}{2}\left[e_{\xi_i}^\dagger e_{\xi_i}^\dagger t_d(\eta_i) + \int_{-\infty}^{\infty} dk a_i^\dagger(k)a_{i+1}^\dagger(-k)t_c(k)\right]|\Omega_e\rangle\otimes|\Omega_c\rangle\otimes|S_{||}\rangle. \tag{4.150}$$

Where we have used the notation:

$$|\Omega_e\rangle = \prod_{i=1}^{2}|\Omega_{e_i}\rangle, \qquad |\Omega_c\rangle = \prod_{i=1}^{2}\prod_k|\Omega_i(k)\rangle. \tag{4.151}$$

Taking equation (4.142) and the corresponding relations for the continuous spectrum oscillators, equation (4.150) can be rewritten as:

$$|\Lambda\rangle = \frac{\mathcal{N}}{\pi\sqrt{[1+t_d(\eta_1)][1+t_d(\eta_2)]}}\exp\frac{1}{2}\left[\frac{t_d(\eta_1)-1}{t_d(\eta_1)+1}Y_1^2 + \frac{t_d(\eta_2)-1}{t_d(\eta_2)+1}Y_2^2\right]$$

$$\times\exp\left[-\frac{1}{2}a_{0,\alpha}^\dagger\hat{S}_{00}^{\alpha\beta}a_{0,\beta}^\dagger - a_{0,\alpha}^\dagger S_0^\alpha - \frac{1}{2}a_{n,\alpha}^\dagger\hat{S}_{nm}^{\alpha\beta}a_{m,\beta}^\dagger\right]|\hat{\Omega}_{b,\theta}\rangle\otimes|S_{||}\rangle, \tag{4.152}$$

where $|\hat{\Omega}_{b,\theta}\rangle = |\Omega_e\rangle \otimes |\Omega_c\rangle$ and

$$\hat{S}_{00}^{\alpha\beta} = \sum_{i=1}^{2} V_0^{(\xi_i^+)\alpha} \bar{V}_0^{(\xi_i^+)\beta} t_d(\eta_i) + \int_{-\infty}^{\infty} dk t(k) V_0^{i,\alpha}(k) \bar{V}_0^{i,\beta}(k) \tag{4.153}$$

$$\begin{aligned} S_0^{\alpha} &= \sum_{i=1}^{2}\sum_{n=1} \left[V_0^{(\xi_i^+)\alpha} \bar{V}_n^{(\xi_i^+)\beta} t_d(\eta_i) + \int_{-\infty}^{\infty} dk t(k) V_0^{i,\alpha}(k) \bar{V}_n^{i,\beta}(k) \right] a_{n,\beta}^{\dagger} \\ &= \hat{S}_{0n}^{\alpha\beta} a_{n,\beta}^{\dagger} \end{aligned} \tag{4.154}$$

$$\hat{S}_{nm}^{\alpha\beta} = \sum_{i=1}^{2} (-1)^n V_n^{(\xi_i^+)\alpha} \bar{V}_m^{(\xi_i^+)\beta} t_d(\eta_i) + \int_{-\infty}^{\infty} dk t(k) (-1)^n V_n^{i,\alpha}(k) \bar{V}_m^{i,\beta}(k) \tag{4.155}$$

with $V_N^{(\xi_i^+)\alpha}$ being the twist even combination of the discrete eigenstates. Now let's project onto the center of mass coordinates

$$\begin{aligned} \langle X_{CM}|\Lambda\rangle = & \frac{\mathcal{N}}{\pi\sqrt{[1+t_d(\eta_1)][1+t_d(\eta_2)]}} \exp\frac{1}{2}\left[\frac{t_d(\eta_1)-1}{t_d(\eta_1)+1} Y_1^2 + \frac{t_d(\eta_2)-1}{t_d(\eta_2)+1} Y_2^2\right] \\ & \times\sqrt{\frac{2\Delta}{\pi b}} \frac{1}{\sqrt{[1+s_1][1+s_2]}} \exp\frac{1}{b}\left[\frac{s_1-1}{s_1+1} x_1 x^1 + \frac{s_2-1}{s_2+1} x_2 x^2 + 2i\sqrt{b}\left(\frac{S_{0,1}x^1}{1+s_1} + \frac{S_{0,2}x^2}{1+s_2}\right)\right] \\ & \times\exp\left[-\frac{1}{2} a_{n,\beta}^{\dagger}\left(\hat{S}_{nm}^{\alpha\beta} - \frac{\hat{S}_{n0,1}^{\alpha}\hat{S}_{0m}^{1\beta}}{1+s_1} - \frac{\hat{S}_{n0,2}^{\alpha}\hat{S}_{0m}^{2\beta}}{1+s_2}\right) a_{n,\beta}^{\dagger}\right] |0\rangle \otimes |S_{||}\rangle \end{aligned} \tag{4.156}$$

where

$$\begin{aligned} s_1 &= 2\Delta[g_d^2(\eta_1,\eta_2) t_d(\eta_1) + g_d^2(\eta_2,\eta_1) t_d(\eta_2)] + \Delta \int_{-\infty}^{\infty} dk\ t_c(k)[g_c^2(k) + g_c^2(-k)], \\ s_2 &= \Delta \int_{-\infty}^{\infty} dk\ t_c(k)[g_c^2(k) + g_c^2(-k)], \end{aligned} \tag{4.157}$$

$t_c(k) = -e^{-\pi|k|/2}$ is the eigenvalue of $\mathcal{T}$ in the continuous spectrum, see the previous section for the definition of the remaining terms which enter in the last two equations. The inverse Wick-rotation along direction 1 of (4.156) will give us a time-localized solution. It depends on two parameters, b and a, which can be expressed in terms of (η_1, η_2), through the eigenvalues equations (4.106) . Let's now take a look at every term in this solution and analyze it for different values of the such parameters.

In the Wick-rotated solution, to get time-localization, the term $-\frac{1}{b}\frac{s_1-1}{s_1+1}$ should be negative. We cannot achieve this using the conventional lump, since in this case $-1 < s_1 < 1$. To correct this, as anticipated, we need to invert one or two discrete eigenvalues, ($t_d(\eta_1)$ or/and $t_d(\eta_2)$). In this case one can easily show that $1 < s_1 < \infty$

and we get the desired behaviour. Given the possibility of inverting one or two eigenvalues, it might seem that there is some arbitrariness in our procedure. Actually there is none, since the cancelation of the potentially divergent terms when $b \to \infty$ (see below), requires the inversion of only one eigenvalue. In addition, time localization in small b regime requires the inversion of the eigenvalue of $\mathcal{T}$ corresponding to the greater between η_1 and η_2 (η_2 in our conventions). From now on we will then consider a solution in which $t_d(\eta_2)$ is inverted, i.e. $t_d(\eta_2) \to t_d^{-1}(\eta_2)$.

Next, look at the term $\frac{s_2-1}{s_2+1}x_2x^2$. Due to the $\langle Y_1, Y_2|$ projection, it gets a contribution only from the continuous spectrum, which is always negative and in the range $(-1, 0)$. As a result, this second term is always negative and gives localization in the transverse space direction.

Now we would like to point out some facts about the two parameters on which our solution depends. In [75], it has been pointed out that the inverse of the parameter b, for large b, plays the role of Sen's $\tilde{\lambda}$ near zero. Here again we can repeat the same argument. Note however that in taking b to infinity we should keep a vanishing, since we cannot overcome the critical value $|B|_c = \frac{1}{2\pi}$. For this reason the result of taking $b \to \infty$ is insensible of the value of the E-field, making this limit completely commutative.

As it is justified in Appendix A, the proper way to send b to ∞ is to take $\eta_1 \approx \eta_2 \to \infty$ keeping $\eta_1 < \eta_2$. In this case one can easily see that

$$s_1 \approx \eta_2 \log\eta_1\eta_2 + t_c(k_0 \approx 0), \qquad s_2 \approx t_c(k_0 \approx 0) \tag{4.158}$$

with k_0 as defined in Appendix A. Note that $t_c(k_0 \approx 0) = -1$. This is so because the E–field cannot scale to infinity due to existence of critical value. Then it follows

$$\lim_{\eta_1,\eta_2\to\infty} \frac{s_1 - 1}{s_1 + 1} = 1, \qquad \lim_{\eta_1,\eta_2\to\infty} \frac{s_2 - 1}{s_2 + 1} \to -\infty \tag{4.159}$$

As justified in Appendix A, in this limit $\hat{S}_{n0}^{\alpha\beta(c)} = 0$ so that the oscillating term in (4.156) receives a contribution only from the discrete part. It is also pointed out that the discrete contribution vanishes except for $\alpha = \beta = 1$, which is the only non trivial contribution to the oscillating term. Moreover, we have

$$\lim_{\eta_1,\eta_2\to\infty} \frac{\Delta\hat{S}_{n0}^{11}}{s_1 + 1} = (-1)^n \lim_{\eta_1,\eta_2\to\infty} \frac{1}{2\sqrt{\log\eta_1\eta_2}} = 0, \tag{4.160}$$

Therefore, the oscillating term in (4.156) vanishes when $b \to \infty$.

Now let's consider the non-zero mode terms, i.e, the last line in (4.156). In the $b \to \infty$ limit it is clear that $V_n^{(\xi_i^+)\alpha}$ vanishes for $\alpha = 2$ and $n \geq 1$. Therefore, the contribution of the discrete spectrum to $\hat{S}_{nm}^{\alpha\beta}$ is zero for α or $\beta = 2$ and $n, m \geq 1$. However, for $\alpha = \beta = 1$ this is not true and there are potentially divergent contributions from the discrete spectrum. We are now going to show that these

divergences cancel and the expression

$$\check{S}^{11}_{nm} = \hat{S}^{11(c)}_{nm} + \hat{S}^{11(d)}_{nm} - \frac{\hat{S}^{1(d)}_{n0,1}\hat{S}^{11(d)}_{0m}}{1+s_1}. \tag{4.161}$$

is finite when $b \to \infty$.
To this end we notice that, inverting only $t_d(\eta_2)$ but taking both η_1 and η_2 to infinity, the different terms which enter in the above expression have the following behaviours

$$\begin{aligned}
1+s_1 &\approx \Delta t_d^{-1}(\eta_2) V_0^{(\xi_2^+),1}\bar{V}_0^{(\xi_2^+),1},\\
\hat{S}^{1(d)}_{n0,1} &\approx \Delta(-1)^n t_d^{-1}(\eta_2) V_n^{(\xi_2^+),1}\bar{V}_0^{(\xi_2^+),1},\\
\hat{S}^{11(d)}_{n0} &\approx (-1)^n t_d^{-1}(\eta_2) V_n^{(\xi_2^+),1}\bar{V}_0^{(\xi_2^+),1},\\
\hat{S}^{11(d)}_{nm} &\approx (-1)^n t_d^{-1}(\eta_2) V_n^{(\xi_2^+),1}\bar{V}_m^{(\xi_2^+),1},
\end{aligned} \tag{4.162}$$

Note that $t_d^{-1}(\eta_2) = e^{|\eta_2|}$ gives a divergent contribution as $\eta_2 \to \infty$. However, using these results in eq.(4.161), it is easy to see that the divergent terms cancel and we are left with $\check{S}^{11}_{nm} = \hat{S}^{11(c)}_{nm}$. This, combined with the fact that for $\alpha = 2$ or $\beta = 2$ we have $\hat{S}^{\alpha\beta(d)}_{NM} = 0$, leads us to the conclusion that $\check{S}^{\alpha\beta}_{nm} = \hat{S}^{\alpha\beta(c)}_{nm} + O\left(\frac{1}{b}\right)$. It is also verified in Appendix A that $\hat{S}^{11(c)}_{nm} = \hat{S}^{22(c)}_{nm} = S_{nm}$.

This also show that it is not possible to invert both the discrete eigenvalues and obtain the same cancelation. Indeed, if we invert both, the term $\hat{S}^{1(d)}_{n0,1}\hat{S}^{11(d)}_{0m}$ contains mixed terms like $[t^{-1}(\eta_1)V_n^{(\xi_1^+),1}\bar{V}_0^{(\xi_1^+),1}][t^{-1}(\eta_2)V_n^{(\xi_2^+),1}\bar{V}_0^{(\xi_2^+),1}]$, for which we cannot find a counter term in $\hat{S}^{11(d)}_{nm}$ to cancel it. As a result we will not be able to get a regular time and space localized solution, since these terms diverge in the limit $\eta_1, \eta_2 \to \infty$.

After all these remarks, we can write the space-time localized solution in the $b \to \infty$ limit as

$$\lim_{b\to\infty}\langle X_{CM}|\Lambda\rangle_{Wick} = N(Y_1,Y_2)\lim_{b\to\infty} e^{-\frac{\Delta}{b}(x^0)^2}e^{-\epsilon(b)(x^2)^2}|S\rangle \tag{4.163}$$

where $|S\rangle$ is the space-time independent VSFT solution (the sliver). Note that time dependence completely disappears in this limit . A remark is in order for the quantity $\epsilon(b)$ this number is given by, see (4.156)

$$\epsilon(b) = \frac{\Delta}{b}\frac{s_2-1}{s_2+1} \tag{4.164}$$

a numerical analysis shows that this becomes vanishing as $b \to \infty$, this is so because the function $\frac{s_2(b)-1}{s_2(b)+1}$ is exploding weakly as compared to b. This in turn implies that the loss of time dependence is accompanied by loss of transverse space dependence, giving a resulting zero momentum state (the $D-25$ sliver). Therefore, taking b to

infinity is like sitting at the original unstable vacuum (the D–25 brane), which is the same situation as setting Sen's $\tilde{\lambda}$ to zero.

Another remark we would like to make is about small b limit, which we can get by taking $\eta_1 \to 0$ and keeping η_2 finite. Given that the large b limit corresponds to Sen's $\tilde{\lambda}$ near zero (i.e it represents the unstable vacuum), it is natural to think that the small b limit corresponds to $\tilde{\lambda}$ near $\frac{1}{2}$ (or the stable vacuum). As a matter of fact, taking this limit of b one gets the 0 state, which is also obtained in the $x_0 \to \infty$ limit and corresponds to the stable vacuum to which the D-brane decays. This can be seen by noting that, in this case, $V_0^\alpha(k) \to 0$, whereas $V_n^\alpha(k)$ for $n \geq 1$ have a finite nonvanishing limit. As a result s_1 do not get a contribution from the continuous spectrum and $s_2 = 0$. Then, it follows

$$-\frac{\Delta}{b}\frac{s_1-1}{s_1+1} \approx -\frac{\Delta}{\eta_1}\left(\left|\frac{s_1-1}{s_1+1}\right|\right), \qquad \frac{\Delta}{b}\frac{s_2-1}{s_2+1} \approx -\frac{\Delta}{\eta_1} \tag{4.165}$$

where we have used $(b \approx \eta_1)$, $(s_1 \approx 1 + O(\sqrt{\eta_1}))$ in the limit $\eta_1 \to 0$ and η_2 finite. These are results one can easily obtain from appendix A. For $\Delta \neq 0$ both of these terms gives a negative infinity in the exponent and suppress everything in front to give us the 0 state which corresponds to the stable vacuum. However, the case $\Delta = 0$ should be handled with care. In this case, one can send Δ and η_1 to zero, in such a way that the ratio $\frac{\Delta}{\eta_1}$ remains finite. As a result the time dependence will be lost while the solution is still space localized. One should compare this with the time independent solution obtained when we send Sen's $\tilde{\lambda}$ to $\frac{1}{2}$ and, at the same time, tune the E–field to its critical value, [118], obtaining a static fundamental strings background.

In this section and the previous section we have shown that the E–field deformed star algebra still presents time localized idempotents, which are a straightforward generalization of their $E = 0$ cousins. We have analyzed in detail the case of a longitudinal E–field (D25–brane decay) and of a transverse one (D24–brane decay). In both cases time localization is achieved by working in the Wick rotated euclidean theory with an unconventional lump having inverted discrete eigenvalues of the Neumann matrix with respect to the conventional one. In particular the decaying D–24 brane solution has the nice feature of being localized in time and on the 25-th spatial direction (although for the extreme case $b \to \infty$ the transverse space profile becomes flat). This is a consequence of the fact that we first projected the unconventional lump on the twist odd momenta of the discrete spectrum and then on the center of mass coordinates, haven't we done this we would have lost space localization in the 24–th direction.

Our solution depends on two parameter, b and E. After projecting on the center of mass time coordinate we have shown that in the extreme cases $b = 0$ and $b = \infty$ time dependence is lost and we are left respectively with the 0 state (VSFT vacuum) and the sliver state (VSFT D–brane), confirming an earlier hypothesis, [75], that, although a finite value of b can be formally changed by a gauge transformation, [12],

these two cases, at least qualitatively, seems to reproduce the $\tilde{\lambda} = \frac{1}{2}$ and $\tilde{\lambda} = 0$ of Sen's BCFT, [102]. The E–field is another free parameter in the range $0 \leq E \leq E_c = \frac{1}{2\pi\alpha'}$. For E going to its critical value, time dependence is lost and we get a flat (non zero) profile in time (effective tensionless regime), this result still persists in the $b \longrightarrow 0$ limit if we keep the ratio $\frac{\Delta}{b}$ finite. This case should correspond to the tachyon vacuum with a background of fundamental strings, prevented to decay by their polarization due to critical electric field. Of course this is a very indirect way to see these fundamental strings arising in a classical solution of VSFT, a more direct construction of such objects is given in [126].

Chapter 5

Conclusions

In this thesis we have touched on various aspects of string field theory ranging from its integrability properties to obtaining its time dependent solutions. In this conclusive chapter we will list things we think we had to mension but we left aside either because we did not had much (or any thing) to say about or we left them because they were irrelevant to our research. We will also point out some open problems and what should be the direction for further developements in string field theory in our opinion.

We start with the most important one which is the issue of superstring field theory. We mentioned in the introductory chapter that bosonic string field theory is an important play ground on which one can add the complication of supersymmetry, however it is not realistic by itself. Knowing this it is well understood that it is much more crucial to deal with the problem of formulating a complete superstring field theory than spending much time in cleaning the play ground. Then it is natural to ask why incorporating supesymmetry in string field theory is so difficult and why only little has been done in this direction. After open bosonic string field theory was constructed many attempts to apply these techniques to open superstring theory have been made, the cubic superstring field theory by Witten and the Wess–Zumino–Witten like superstring theory by Berkovist are some examples. The cubic superstring field theory was considered incomplete because of the contact term divergences which arise due to collision of two picture changing operators. The Berkovist superstring field theory, to our knowledge, does not have any inconsistency problem but too complicated to do some practical computation in it and as a result very little has been done in that case also. It seems there is some thing deep we do not understand to formulate a simple and complete superstring field theory.

Secondly, in our discussion of the string field theory algebra in the second chapter we mainly dealt with the algebra in the matter sector. Concerning the ghost part, if we limit our self to the non–zero modes of the ghost Neumann coefficients X^{rs}_{nm} then the algebra is exactly that of the matter sector. However, because of the zero modes the star product formulas are rather complicated so that we need to introduce the concept of reduced star product, which is star product of string states with no ghost

zero modes. After doing that it is possible to relate the Witten's star product to the reduced star product in a fixed gauge. It is a nice idea to see if there is a way of doing the ghost star product in a gauge independent manner, which ofcourse needs to modify the ghost part of the 3–string vertex and try to check if the ghost Neumann coefficients of the modified vertex operator obey the integrability properties obeyed by their matter counterparts, before concluding that these properties are the general characteristics of 3–string vertex.

The third point we would like to point is concerning the Schnabl's solution. In the CFT formalism these solutions were obtained by a clever man guess not through a rigorous procedure of solving the OSFT equation of motion. As a matter of this fact it seems very hard to deal with the situation when we start with a non–trivial background perturbative vacuum or when we want to deal with a decay of lower dimensional branes to the tachyon vacuum described by the same solution. A good way out is to use the operator formalism. Unfortunately, that is complicated again because of the ghost zero modes which complicate the star product and it needs much work like modifying the ghost sector of the 3–string vertex in a way, before rushing into the process of repeating the Schnabl's steps in this formalism.

Our forth comment is about the singularity of vacuum string field theory. It turns out that various consistency conditions leads to a unique choice of the kinetic operator of VSFT $\mathcal{Q}$ up to an overall constant ($\mathcal{Q} = \gamma c(i)$). The coefficient γ can be fixed by requiring that the classical solutions describing different Dp–branes not only reproduce the right ratio of tensions of the branes but also give the overall normalization of the tension correctly. Unfortunately, this leads to a singular coefficient γ. This indicates VSFT has a singular kinetic term and has to be related to the original open string field theory expanded around the tachyon vacuum by a singular field redefinition. As a result, while the original OSFT action is non-singular VSFT is singular. It also has a vanishing norm in the matter sector and infinite norm in the ghost sector. Therefore, VSFT needs to be regularized. In the literature many regularization schemes were proposed in order to solve different problems caused by this singularity. It seems every problem needs a different regularization and a complete regularization scheme has still to be found. The formulation of VSFT by a singular field redefinition was done before the the discovery of the solutions of OSFT equation of motion. At the moment these solutions are known and one can formulate a string field theory around the tachyon vacuum without any need of singular field redefinition. It is interesting to see if there is some link between this theory and vacuum string field theory or if it can shine some light on the way to regularize VSFT, otherwise the future of VSFT is not very much appealing. It is also important to note that quantization of a theory formulated about the new vacuum should be able to explain how closed string states emerge as a result of decay of the D–branes.

The last point we need to discuss is about our time dependent solutions. These solutions were constructed in the framework of VSFT as a solution interpolating between the the original perturbative vacuum and the tachyon vacuum, which in

this context was a conjectured vacuum not determined by computation. However, now we know the tachyon vacuum of OSFT (may be the same as the conjectured tachyon vacuum of VSFT) and it is interesting to construct time dependent solution interpolating between this vacuum and the perturbative vacuum and see if there is an agreement with our time dependent solutions.

Appendix A

Asymptotic Behaviours

A.1 Trivial background

A.1.1 Limits of X' and T'

In this Appendix we briefly discuss the low energy and high energy limit of X' and T' in the oscillator basis. The Neumann coefficients $V'^{(rs)}_{NM}$ we use are given in Appendix B of [12]. They explicitly depend on the b parameter. In the low energy limit the three–strings vertex can be expanded by means of a parameter ϵ (a dimensionless parameter, in fact an alias of α') , see [86]. This translates into an expansion for $V'^{(rs)}_{NM}$ triggered by the following rescalings

$$\begin{aligned} V^{(rs)}_{mn} &\to V^{(rs)}_{mn} \\ V^{(rs)}_{m0} &\to \sqrt{\epsilon} V^{(rs)}_{m0} \\ V_{00} &\to \epsilon V_{00} \end{aligned} \tag{A.1}$$

For instance X' is expanded as follows to the lowest orders of approximation

$$X' = \begin{pmatrix} -\frac{1}{3} + \frac{8}{3} V_{00} \frac{\epsilon}{b} & -\frac{4}{3}\sqrt{\frac{2\epsilon}{b}} \langle \mathrm{v}_e| \\ -\frac{4}{3}\sqrt{\frac{2\epsilon}{b}} |\mathrm{v}_e\rangle & X - \frac{8}{3}\frac{\epsilon}{b}(|\mathrm{v}_e\rangle\langle \mathrm{v}_e| - |\mathrm{v}_o\rangle\langle \mathrm{v}_o|) \end{pmatrix} \tag{A.2}$$

where

$$|\mathrm{v}_e\rangle_n = -\frac{3}{2\sqrt{2}} V^{(11)}_{0n}, \qquad |\mathrm{v}_o\rangle_n = \sqrt{\frac{3}{8}}(V^{(12)}_{0n} - V^{(21)}_{0n})$$

It is interesting to remark that the parameter ϵ appears always divided by b, so that one could just as well absorb ϵ into $1/b$ and say that the expansion is in the parameter $1/b$ for large b. However to avoid confusion it is useful to keep the two parameters distinct.

Now, it is immediate to see that

$$T' = \begin{pmatrix} -1 + \mathcal{O}(\frac{\epsilon}{b}) & \mathcal{O}(\sqrt{\frac{\epsilon}{b}}) \\ \mathcal{O}(\sqrt{\frac{\epsilon}{b}}) & T + \mathcal{O}(\frac{\epsilon}{b}) \end{pmatrix} \tag{A.3}$$

This is correct provided we can prove that the use of (A.2) to compute T' makes full sense, that is all the terms of the expansion in powers of $\sqrt{\frac{\epsilon}{b}}$ are well defined. One can actually see that a naive expansion leads to infinite coefficients. This is a well–known problem, pointed out for the first time in [86], which requires a regularization. A nice way to introduce a regulator is to switch on a constant background B field. We will not do it here, but we quote the result: in the presence of a B field the infinities disappear, and the expansion (A.3) makes full sense. From this we deduce in particular that

$$T'_{nm} = T_{nm} + \mathcal{O}(\frac{\epsilon}{b}) \tag{A.4}$$

Let us consider now another extreme expansion, that is the limit $\alpha' \to \infty$. In just the same way as above, we can introduce an alias, t ($t >> 1$) instead of ϵ. So, in particular,

$$\begin{aligned} V^{(rs)}_{mn} &\to V^{(rs)}_{mn} \\ V^{(rs)}_{m0} &\to \sqrt{t} V^{(rs)}_{m0} \\ V_{00} &\to t V_{00} \end{aligned} \tag{A.5}$$

In this case X' to the lowest orders of approximation becomes

$$X' = \begin{pmatrix} 1 + \frac{2}{3}\frac{1}{V_{00}}\frac{b}{t} & -\frac{2}{3}\sqrt{\frac{2b}{t}}\langle \mathbf{v}_e| \\ -\frac{2}{3}\sqrt{\frac{2b}{t}}|\mathbf{v}_e\rangle & X - \frac{4}{3}\frac{1}{V_{00}}(1 - \frac{1}{V_{00}}\frac{b}{2t})(|\mathbf{v}_e\rangle\langle \mathbf{v}_e| - |\mathbf{v}_o\rangle\langle \mathbf{v}_o|) \end{pmatrix} \tag{A.6}$$

The lowest order in this expansion is known as the tensionless limit [117]. Also here one must be careful about the use of this expansion in calculating T'. From eq.(A.6) one finds that

$$T'_{00} = 1 + \mathcal{O}(\frac{b}{t}) \tag{A.7}$$

A.1.2 The $\alpha' \to 0$ limit of $\check{S}'^{(c)}_{nm}$ and $\check{S}'^{(c)}_{0n}$

In this Appendix we discuss the limit of the unconventional lump matrix elements $\check{S}'^{(c)}_{nm}$ and $\check{S}'^{(c)}_{0n}$ by means of the diagonal basis. According to (4.24), we speak interchangeably of the $b \to \infty$ limit and the $\eta \to \infty$ one. When applying the results of this Appendix to section 4.1.5, we understand that $1/b$ is replaced everywhere by ϵ/b with finite b.

As a preliminary step let us prove that

$$\lim_{b\to\infty}\left(V_0^{(k)}\right)^2 = \delta(k) \tag{A.8}$$

A rather informal way to see this is as follows. Looking at (4.15) it is easy to realize that the limit always vanishes provided $k \neq 0$. Therefore the support of the limiting distribution must be at $k = 0$. We can therefore expand all the functions involved in k around $k = 0$ and keep the leading terms. Since $\Re F_c(k) \approx 1.386...$ around this point, we can disregard $\Re F_c(k)$ compared to $b/4$ in the $b \to \infty$ limit. Therefore we easily find

$$\lim_{b\to\infty}(V_0^{(k)})^2 = \lim_{b\to\infty} = \frac{\bar{b}}{\pi}\frac{1}{1+\bar{b}^2k^2}$$

where $\bar{b} = b/8$. Now defining $\bar{\epsilon} = 1/\bar{b}$, the limit becomes

$$\lim_{\bar{\epsilon}\to 0}\frac{1}{\pi}\frac{\bar{\epsilon}}{k^2+\bar{\epsilon}^2} = \delta(k) \tag{A.9}$$

according to a well–known representation of the delta function. We can also show that

$$(V_0^{(k)})^2 = \delta(k) + \mathcal{O}(1/b)$$

From now on we suppose that, in the $\int dk$ integrals , we are allowed to replace the integrands with their $1/b$ expansions, and that the results we obtain are valid at least in an asymptotic sense. This attitude is always confirmed by numerical approximations.

Limit of $\check{S}'^{(c)}_{mn}$

Let us rewrite the generating function for $V_m^{(k)}$ as follows:

$$F^{(k)}(z) = A^{(k)}f^{(k)}(z) - \frac{(1-\nu(k))V_0^{(k)}}{\sqrt{b}}B(k,z) \tag{A.10}$$

where

$$A^{(k)} = V_0^{(k)}\sqrt{\frac{2}{b}}k\left(\Re F_c(k) - \frac{b}{4}\right) \tag{A.11}$$

and

$$\begin{aligned} B(k,z) &= \frac{2}{1-\nu(k)}\left[\Re F_c(k) + \frac{\pi}{2\sqrt{3}}\frac{\nu(k)-1}{\nu(k)+1} + \frac{2i}{k} + \log(iz) - 2if^{(k)}(z)\right. \\ &+ \left.\text{LerchPhi}(e^{-4i\arctan(z)}, 1, 1+\frac{k}{4i})e^{-4i\arctan(z)}e^{-k\arctan(z)}\right] \end{aligned} \tag{A.12}$$

From (A.10) we can derive a useful expression for $V_m^{(k)}$:

$$V_m^{(k)} = A^{(k)} \frac{\sqrt{m}}{2\pi i} \oint dz \frac{f^{(k)}(z)}{z^{m+1}} - \frac{(1-\nu(k))V_0^{(k)}}{\sqrt{b}} \frac{\sqrt{m}}{2\pi i} \oint dz \frac{B(k,z)}{z^{m+1}} \tag{A.13}$$

Since $v_m^{(k)} = \frac{\sqrt{m}}{2\pi i} \oint dz \frac{f^{(k)}(z)}{z^{m+1}}$ and $S'^{(c)}_{mn} = \int_{-\infty}^{\infty} dk\ t(k) V_m^{(k)} V_n^{(-k)}$ we get:

$$\begin{aligned} \check{S}'^{(c)}_{mn} &= \int_{-\infty}^{\infty} dk\ t(k) \left[A^{(k)} A^{(-k)} v_m^{(k)} v_n^{(-k)} - A^{(k)} V_0^{(k)} v_m^{(k)} (1-\bar{\nu}(k)) \tilde{B}_n(-k) \frac{1}{\sqrt{b}} \right. \\ &- A^{(-k)} V_0^{(k)} v_n^{(-k)} (1-\nu(k)) \tilde{B}_m(k) \frac{1}{\sqrt{b}} \\ &+ \left. (V_0^{(k)})^2 (1-\bar{\nu}(k))(1-\nu(k)) \tilde{B}_m(k) \tilde{B}_n(-k) \frac{1}{b} \right] \end{aligned} \tag{A.14}$$

where

$$\tilde{B}_m(k) = \frac{\sqrt{m}}{2\pi i} \oint dz \frac{B(k,z)}{z^{m+1}}$$

Now we want to take the limit of (A.14) when $b \to \infty$. To this end we notice the following:

$$\begin{aligned} \lim_{b\to\infty} A^{(k)} A^{(-k)} &= \lim_{b\to\infty} (V_0^{(k)})^2 \left(\frac{-2k^2}{b} \right) \left(\Re F_c(k) - \frac{b}{4} \right)^2 \\ &= \lim_{x\to-\infty} \left(\frac{-k^2}{N(k)} \right) \frac{x^2}{4+k^2x^2} = \left(\frac{-k^2}{N(k)} \right) \frac{1}{k^2} = -\frac{1}{N(k)} \end{aligned} \tag{A.15}$$

where $x = \left(\Re F_c(k) - \frac{b}{4} \right)$. When k is very large $\Re F_c(k)$ tends to (slowly) diverge, but the factor $t(k)$ in the integrand of (A.14) concentrates the integral in the small k region.

We also need:

$$\begin{aligned} \lim_{b\to\infty} \frac{A^{(k)} V_0^{(k)}}{\sqrt{b}} &= \sqrt{2} k \delta(k) \left(\frac{\Re F_c(k)}{b} - \frac{1}{4} \right) \\ \lim_{b\to\infty} \frac{A^{(-k)} V_0^{(k)}}{\sqrt{b}} &= -\sqrt{2} k \delta(k) \left(\frac{\Re F_c(k)}{b} - \frac{1}{4} \right) \end{aligned}$$

Finally using these limits

$$\begin{aligned} \lim_{b\to\infty} \check{S}'^{(c)}_{mn} &= -\int_{-\infty}^{\infty} \frac{dk}{N(k)}\ t(k) v_m^{(k)} v_n^{(-k)} \\ &+ \lim_{b\to\infty} \int_{-\infty}^{\infty} dk\ t(k) \delta(k) (1-\bar{\nu}(k))(1-\nu(k)) \tilde{B}_m(k) \tilde{B}_n(-k) \frac{1}{b} \end{aligned} \tag{A.16}$$

while the other integrals vanish because they contain the factor $k\delta(k)$. Here we have used the fact that $\nu(0) = \bar{\nu}(0) = -1$ and $\tilde{B}_m(0)$ is finite. A straightforward calculation gives

$$\tilde{B}_m(0) = \frac{\sqrt{m}}{2\pi i}\oint dz \frac{\log(1+z^2)}{z^{m+1}} = \begin{cases} 0 & \text{for m odd;} \\ \frac{\sqrt{2m}}{2}(-1)^{\frac{m}{2}+1}(\frac{m}{2}+1)! & \text{for m even.} \end{cases} \tag{A.17}$$

So we are left with:

$$\lim_{b\to\infty} \check{S}'^{(c)}_{mn} = S_{mn} \tag{A.18}$$

This is the sliver. The corrections are of order $\frac{1}{b}$.

Limit of $\check{S}'_{0m}$

In the rest of this appendix we would like to justify eq.(4.57). The limit of $\check{S}'^{(c)}_{0m}$ can be computed the same way as before. We have:

$$\begin{aligned} \lim_{b\to\infty} \check{S}'^{(c)}_{0m} &= \lim_{b\to\infty} \int_{-\infty}^{\infty} dk\ t(k) V_0^{(k)} V_m^{(-k)} \\ &= \lim_{b\to\infty} \left(\int_{-\infty}^{\infty} dk\ t(k) V_0^{(k)} A^{(-k)} v_m^{(-k)} + \frac{2}{\sqrt{b}} \tilde{B}_m(0) \right) \end{aligned} \tag{A.19}$$

The last term in the RHS of course vanishes in the limit $b \to \infty$, while the first limit diverges, but, recalling (4.57), what we are really need to know is the limit of $\frac{\check{S}'_{0n}}{1+s'}$. Using the fact that $1+s' \approx 4\eta \log \eta$ when $b \to \infty$ ($b \approx 4\log \eta$) and that we can write $\check{S}'_{0m} = \check{S}'^{(c)}_{0m} + \check{S}'^{(d)}_{0m}$ (factorization into continuous and discrete parts) we have:

$$\begin{aligned} \check{S}'^{(d)}_{0m} &\approx 2\eta\sqrt{2\log\eta} \\ \check{S}'^{(c)}_{0m} &\approx \int_{-\infty}^{\infty} dk\ t(k) v_m^{(-k)} (-\sqrt{2}k)(V_0^{(k)})^2 \left(\frac{\Re F_c(k)}{4\log\eta} - \frac{1}{4} \right) 2\sqrt{\log\eta} \end{aligned}$$

Using these we get:

$$\frac{\check{S}'^{(c)}_{0m}}{1+s'} \approx \int_{-\infty}^{\infty} dk\ t(k) v_m^{(-k)} (\sqrt{2}k)\delta(k) \left(\frac{\Re F_c(k)}{4\log\eta} - \frac{1}{4} \right) \frac{1}{2\eta} = 0$$

and

$$\frac{\check{S}'^{(d)}_{0m}}{1+s'} \approx \frac{1}{\sqrt{2\log\eta}}$$

Hereby the conclusion (4.57) follows.

A.2 With constant B–field

In Section 4.3.4, we have analyzed our solution in the large and small limits of the parameter b. In this appendix we compute the relevant matrix elements in these asymptotic regimes.

A.2.1 The $b \to \infty$ Limit

From (4.106), we can write:

$$|a| = \frac{\Re F(\eta_2) - \Re F(\eta_1)}{[V_{00} + 2\Re F(\eta_2)]\tanh(\frac{\eta_1}{2}) + [V_{00} + 2\Re F(\eta_1)]\tanh(\frac{\eta_2}{2})} \tag{A.20}$$

$$\frac{b}{4} = \frac{\Re F(\eta_2)\tanh(\frac{\eta_1}{2}) + \Re F(\eta_1)\tanh(\frac{\eta_2}{2})}{\tanh(\frac{\eta_1}{2}) + \tanh(\frac{\eta_2}{2})} \tag{A.21}$$

where we take, by definition, $\eta_2 > \eta_1 > 0$. There are two ways of taking $b \to \infty$

i) $\eta_2 \to \infty$; η_1 **fixed:**

In this limit we can see that:

$$\frac{b}{4} \approx \left(\frac{\tanh(\frac{\eta_1}{2})}{1 + \tanh(\frac{\eta_1}{2})}\right)\log(\eta_2), \qquad a \approx \frac{1}{2\tanh(\frac{\eta_1}{2})} > \frac{1}{2}. \tag{A.22}$$

ii) $\eta_2 \to \infty$; $\eta_1 \to \infty$:

We can parameterize $\eta_2 = \eta^y$, $\eta_1 = \eta^x$ and then take $\eta \to \infty$, while keeping $y > x$. We then obtain:

$$\frac{b}{4} \approx \frac{1}{2}(y + x)\log(\eta), \qquad a \approx \frac{1}{2}\frac{y - x}{y + x} < \frac{1}{2} \tag{A.23}$$

We will be concerned with this second regime as it is the one connected to $a = 0$, which is a condition arising from the existence of the critical value for the E–field, when $b \to \infty$. In this second limit it can be easily seen that the discrete eigenvectors have the following behaviour

$$V_0^{\xi_i,1} = V_0^{\bar{\xi}_i,1} \approx \frac{1}{\sqrt{2\Delta}} e^{-\eta_i/2}\sqrt{\eta_i \mathrm{Log}\eta_1\eta_2},$$

$$V_0^{\xi_i,2} = -V_0^{\bar{\xi}_i,2} \approx (-1)^i \frac{i}{\sqrt{2\Delta}} e^{-\eta_i/2}\sqrt{\eta_i \mathrm{Log}\eta_1\eta_2},$$

and

$$V_n^{\xi_i,\alpha} \approx -\frac{V_0^{\xi_i,\alpha}}{\sqrt{\mathrm{Log}\eta_1\eta_2}}, \qquad V_n^{\bar{\xi}_i,\alpha} \approx -\frac{V_0^{\bar{\xi}_i,\alpha}}{\sqrt{\mathrm{Log}\eta_1\eta_2}}. \tag{A.24}$$

For the continuous spectrum the situation is more complicated and getting this limits is not easy. However, it is possible to calculate the limit of $(V_0^{i,\alpha}(k))^2$, which is enough for our purposes. We have

$$(V_0^{1,1}(k))^2 = \left[\frac{4\Delta}{b}N(k)\left(4+k^2\left(\Re F_c(k)-\frac{b}{4}-\frac{Aa}{\tanh\frac{\pi k}{4}}\right)^2\right)\right]^{-1}. \tag{A.25}$$

When $b \longrightarrow \infty$ this expression vanishes every where except at

$$k_0 \approx -\frac{4}{\pi}\text{arctanh}(2a) \tag{A.26}$$

where it diverges. Expanding around k_0 one easily gets

$$(V_0^{1,1}(k))^2 \approx \Delta^{-1}\frac{4a}{k_0(1-4a^2)N(k_0)}\frac{\bar{b}}{\pi(1+(k-k_0)^2\bar{b}^2}$$

where

$$\bar{b} = \frac{k_0\pi(1-4a^2)}{64a}b.$$

Now taking the $b \longrightarrow \infty$ limit one obtains

$$(V_0^{1,1}(k))^2 \approx \frac{1}{2\Delta}\delta(k-k_0). \tag{A.27}$$

Following the same procedure one can also show that

$$(V_0^{2,1}(k))^2 \approx \frac{1}{2\Delta}\delta(k+k_0) \tag{A.28}$$

remember that

$$|V_0^{1,2}(k)|^2 = (V_0^{1,1}(k))^2, \qquad |V_0^{2,2}(k)|^2 = (V_0^{2,1}(k))^2. \tag{A.29}$$

The non zero components, $V_m^{i,\alpha}(k)$, can be expressed in terms of a generating function. For instance, the generating function for $V_m^{1,1}(k)$ is given by

$$F^{(k)}(z) = A_1(k)f^{(k)}(z) - \frac{(1-\nu(k))V_0^{1,1}(k)}{\sqrt{b}}B(k,z) \tag{A.30}$$

where

$$A_1(k) = V_0^{1,1}(k)\sqrt{\frac{2}{b}}k\left(\Re F_c(k)-\frac{b}{4}-\frac{Aa}{\tanh(\frac{\pi k}{4})}\right),$$

$$B(k,z) = \frac{2}{1-\nu(k)}\left[\Re F_c(k)+\frac{\pi}{2\sqrt{3}}\frac{\nu(k)-1}{\nu(k)+1}+\frac{2i}{k}+\log(iz)-2if^{(k)}(z)\right]$$

$$+\frac{2}{1-\nu(k)}\left[\Phi(e^{-4i\arctan(z)},1,1+\frac{k}{4i})e^{-4i\arctan(z)}e^{-k\arctan(z)}\right] \tag{A.31}$$

where Φ is the LerchPhi function and $f^{(k)}$ is the generating function for the spectrum of the Neumann matrix without zero modes, [113]. Inverting this equation we can write $V_m^{1,1}(k)$ as

$$V_m^{1,1}(k)=A_1(k)\frac{\sqrt{m}}{2\pi i}\oint dz\frac{f^{(k)}(z)}{z^{m+1}}-\frac{(1-\nu(k))V_0^{1,1}(k)}{\sqrt{b}}\frac{\sqrt{m}}{2\pi i}\oint dz\frac{B(k,z)}{z^{m+1}} \tag{A.32}$$

With the same procedure one can also write

$$V_m^{2,1}(k)=A_1'(k)\frac{\sqrt{m}}{2\pi i}\oint dz\frac{f^{(k)}(z)}{z^{m+1}}-\frac{(1-\nu(k))V_0^{2,1}(k)}{\sqrt{b}}\frac{\sqrt{m}}{2\pi i}\oint dz\frac{B(k,z)}{z^{m+1}} \tag{A.33}$$

with

$$A_1'(k)=V_0^{2,1}(k)\sqrt{\frac{2}{b}}k\left(\Re F_c(k)-\frac{b}{4}+\frac{Aa}{\tanh(\frac{\pi k}{4})}\right). \tag{A.34}$$

The other vectors are related to these ones as

$$V_n^{1,2}(k)=iV_n^{1,1}(k),\qquad V_n^{2,2}(k)=-iV_n^{2,1}(k) \tag{A.35}$$

Limit of $\hat{S}_{mn}^{\alpha\beta(c)}$

With all these results at hand we can now calculate the continuous spectrum contribution to the non zero mode matrix elements in the limits under consideration. Recalling that spectrum of the Neumann matrix without zero modes is given by

$$v_m^{(k)}=\frac{\sqrt{m}}{2\pi i}\oint dz\frac{f^{(k)}(z)}{z^{m+1}} \tag{A.36}$$

we can write

$$\hat{S}_{nm}^{11(c)}=\int_{-\infty}^{\infty}dk\ t_c(k)(-1)^n\left[V_n^{1,1}(k)\bar{V}_m^{1,1}(k)+V_n^{2,1}(k)\bar{V}_m^{2,1}(k)\right] \tag{A.37}$$

as

$$\hat{S}_{mn}^{11(c)}=\int_{-\infty}^{\infty}dk\ t_c(k)(-1)^m[A_1(k)A_1(k)v_m^{(k)}v_n^{(k)}-A_1(k)V_0^{1,1}(k)v_m^{(k)}(1-\bar{\nu}(k))\tilde{B}_n(k)\frac{1}{\sqrt{b}}$$

$$-A_1(k)V_0^{1,1}(k)v_n^{(k)}(1-\nu(k))\tilde{B}_m(k)\frac{1}{\sqrt{b}}+(V_0^{1,1}(k))^2(1-\bar{\nu}(k))(1-\nu(k))\tilde{B}_m(k)\tilde{B}_n(k)\frac{1}{b}]$$

$$+\left[A_1(k)\to A_1'(k),V_0^{1,1}(k)\to V_0^{2,1}(k)\right] \tag{A.38}$$

where

$$\tilde{B}_m(k) = \frac{\sqrt{m}}{2\pi i}\oint dz \frac{B(k,z)}{z^{m+1}}. \tag{A.39}$$

Note that if the indices are separated by comma then the first index is the label of the vector and the second is the space time index, otherwise both are space time indices. Now we want to calculate each term in the above expression in the limit when $b \to \infty$. To this end we notice the following

$$\lim_{b\to\infty} A_1(k)A_1(k) = \lim_{b\to\infty}(V_0^{1,1}(k))^2\left(\frac{2k^2}{b}\right)\left(\Re F_c(k) - \frac{b}{4} - \frac{Aa}{\tanh(\frac{\pi k}{4})}\right)^2$$

$$= \lim_{x\to-\infty}\left(\frac{k^2}{2\Delta N(k)}\right)\frac{x^2}{4+k^2x^2} = \left(\frac{k^2}{2\Delta N(k)}\right)\frac{1}{k^2} = \frac{1}{2\Delta N(k)} \tag{A.40}$$

where $x = \left(\Re F_c(k) - \frac{b}{4} - \frac{Aa}{\tanh(\frac{\pi k}{4})}\right)$. The other terms are zero in the limit because, either they contain term like $(k-k_0)\delta(k-k_0)$ in the integral or they are of order $\frac{1}{b}$. Therefore, we are left with

$$\lim_{b\to\infty}\hat{S}^{11(c)}_{mn} = \lim_{b\to\infty}\hat{S}^{22(c)}_{mn} = \Delta^{-1}S_{mn}, \tag{A.41}$$

where

$$S_{nm} = -\int_{-\infty}^{\infty}\frac{dk\ t_c(k)}{N(k)}v_n^{(k)}v_m^{(-k)} \tag{A.42}$$

and

$$\lim_{b\to\infty}\hat{S}^{21(c)}_{mn} = \lim_{b\to\infty}\hat{S}^{12(c)}_{mn} = 0, \tag{A.43}$$

which is the sliver in each direction with corrections of order $\frac{1}{b}$.

Limit of $\hat{S}^{\alpha\beta(c)}_{0m}$

In this section we would like to justify that the contribution from the continuous spectrum to $\hat{S}^{\alpha\beta}_{0m}$ is zero in the limit. This can be computed the same way as before since we have

$$\lim_{b\to\infty}\hat{S}^{\alpha\beta(c)}_{0m} = \lim_{b\to\infty}\sum_{i=1}^{2}\int_{-\infty}^{\infty}dk\ t_c(k)V_0^{i,\alpha}(k)V_m^{i,\beta}(k). \tag{A.44}$$

For instance, lets calculate $\hat{S}^{11(c)}_{0m}$ which is given by

$$\hat{S}^{11(c)}_{0m} = \lim_{b\to\infty}\int_{-\infty}^{\infty}dk\ t_c(k)v_m^{(k)}\sqrt{\frac{2}{b}}k\left((V_0^{1,1}(k))^2\left[\Re F_c(k) - \frac{b}{4} - \frac{Aa}{\tanh(\frac{\pi k}{4})}\right]\right.$$

$$+ (V_0^{2,1}(k))^2 \left[\Re F_c(k) - \frac{b}{4} + \frac{Aa}{\tanh(\frac{\pi k}{4})} \right] \Bigg) + O(\frac{1}{\sqrt{b}}) \tag{A.45}$$

We have already verified that $\lim_{b\to\infty}(V_0^{i,\alpha}(k))^2 \approx \frac{1}{2}\delta(k \pm k_0)$. This will allow us to expand the terms in square brackets about the points $\pm k_0$ to get

$$\hat{S}_{0m}^{11(c)} = \frac{1}{\Delta} \lim_{b\to\infty} \int_{-\infty}^{\infty} dk\ t_c(k) v_m^{(k)} \left[\frac{1}{2}\delta(k-k_0)\sqrt{b}(k-k_0)k_0\pi \left(\frac{1-4a^2}{32a} \right) \right.$$

$$\left. + \frac{1}{2}\delta(k+k_0)\sqrt{b}(k+k_0)(-k_0)\pi \left(\frac{1-4a^2}{32a} \right) \right] + O(\frac{1}{\sqrt{b}}). \tag{A.46}$$

Due to the presence of the delta functions the terms $(k \pm k_0)\sqrt{b}$ are both finite in the $b \to \infty$ limit. As a matter of this fact we can safely do the integrals first and take the limits later. Since the integrals vanishes we note that

$$\hat{S}_{0m}^{11(c)} \approx 0. \tag{A.47}$$

Similar steps show that all the remaining terms of $\hat{S}_{0m}^{\alpha\beta(c)}$ are also zero.

A.2.2 The $b \to 0$ Limit

As it was mentioned before this limit can be obtained by taking $\eta_1 \to 0$. In this limit it is not hard to see that

$$b \approx 2\frac{\Re F(\eta_2)}{\tanh(\frac{\eta_2}{2})}\eta_1 \tag{A.48}$$

$$g_d(\eta_1, \eta_2) \approx \frac{1}{\sqrt{2\Delta}} \left(1 - \frac{\tanh(\frac{\eta_2}{2})}{2\Re F(\eta_2)}\eta_1 \right) \tag{A.49}$$

$$g_d(\eta_2, \eta_1) \approx \frac{1}{\sqrt{2\Delta}} \left[2\tanh(\frac{\eta_2}{2}) \left(\sinh\eta_2 \frac{\partial}{\partial\eta_2}[\mathrm{Log}\Re F(\eta_2)] - 1 \right) \right]^{-1/2} \sqrt{\eta_1}. \tag{A.50}$$

One can use these results and equations (4.108) through (4.112) to write down $V_0^{\xi_i,\alpha}$, $V_0^{\bar{\xi}_i,\alpha}$ $V_n^{\xi_i,\alpha}$ and $V_n^{\bar{\xi}_i,\alpha}$ as:

$$V_0^{\xi_1,1} = V_0^{\bar{\xi}_1,1} \approx \frac{1}{\sqrt{2\Delta}} \left(1 - \frac{\tanh(\frac{\eta_2}{2})}{2\Re F(\eta_2)}\eta_1 \right),$$

$$V_0^{\xi_1,2} = -V_0^{\bar{\xi}_1,2} \approx -i\frac{1}{\sqrt{2\Delta}} \left(1 - \frac{\tanh(\frac{\eta_2}{2})}{2\Re F(\eta_2)}\eta_1 \right), \tag{A.51}$$

$$V_0^{\xi_2,1} = V_0^{\bar{\xi}_2,1} \approx \frac{1}{\sqrt{2\Delta}} \left[2\tanh(\frac{\eta_2}{2}) \left(\sinh\eta_2 \frac{\partial}{\partial\eta_2}[\mathrm{Log}\Re F(\eta_2)] - 1 \right) \right]^{-1/2} \sqrt{\eta_1}$$

$$V_0^{\xi_2,2} = -V_0^{\bar{\xi}_2,2} \approx i\frac{1}{\sqrt{2\Delta}}\left[2\tanh(\frac{\eta_2}{2})\left(\sinh\eta_2\frac{\partial}{\partial\eta_2}[\mathrm{Log}\Re F(\eta_2)] - 1\right)\right]^{-1/2}\sqrt{\eta_1}, \tag{A.52}$$

and

$$V_n^{\xi_1,\alpha} = \pm V_n^{\bar{\xi}_1,\alpha} \approx \sqrt{\eta_1},$$

$$V_n^{\xi_2,\alpha} = \pm V_n^{\bar{\xi}_2,\alpha} \approx \frac{1}{\sqrt{2\Delta}}\left[2\tanh(\frac{\eta_2}{2})\left(\sinh\eta_2\frac{\partial}{\partial\eta_2}[\mathrm{Log}\Re F(\eta_2)] - 1\right)\right]^{-1/2} f(\eta_2). \tag{A.53}$$

The f is a regular function of η_2. On the other hand

$$g_c(k) \approx 0. \tag{A.54}$$

This shows all $V_0^{i,\alpha}(k)$ are zero, whereas $V_m^{i,\alpha}(k)$ are finite and b independent to the leading order. These results are extensively used in section 4.3.4 to calculate quantities like s_1, s_2 in the $b \to 0$ limit.

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