LEVI-TANAKA ALGEBRAS AND CR MANIFOLDS

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Thesis submitted for the degree of "Doctor Philosophiæ"

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Il presente lavoro costituisce la tesi presentata dal Dott. Costantino Medori, sotto la direzione del Prof. Mauro Nacinovich, al fine di ottenere il diploma di *"Doctor Philosophiæ"* presso la Scuola Internazionale Superiore di Studi Avanzati, Classe di Matematica, Settore di Analisi Funzionale ed Applicazioni.

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Ai miei genitori

Contents

Introduction

The subject of this dissertation is the study of the automorphism groups of Cauchy-Riemann manifolds. The general notion of a CR manifold generalizes that of a smooth real submanifold M of a complex manifold X, having the property that the dimension of the analytic tangent space H_xM is constant for $x \in M$. The automorphisms we are looking for generalize the diffeomorphisms of M that are restrictions of biholomorphic maps of X onto itself.

A simple example of this situation is the sphere $S^3 \subset \mathbb{C}^2$. The CR automorphisms of S^3 are obtained by embedding S^3 into $\mathbb{CP}^2$ and considering the projectivities of $\mathbb{CP}^2$ which leave S^3 invariant: we obtain in this case a Lie group of transformations which is isomorphic to $\mathbf{SU}(1,2)/\{\pm I\}$. The general case of a strictly pseudoconvex hypersurface of $\mathbb{C}^2$ was fully discussed by E. Cartan (cf. [8] and [9]) in 1932. The extension of Cartan's results to hypersurfaces in complex manifolds of higher dimension has been a very interesting and impressive achievement in complex differential geometry (cf. [39], [11], [41]).

In my thesis I am mainly concerned with the case of CR manifolds of higher codimension. Indeed, interesting examples of CR manifolds of arbitrary codimension also arise as orbits of holomorphic group action (see [4]). CR manifolds have been one of the main subjects in complex analysis and geometry over the last decades, both from the analytic and the geometrical point of view. I pursue the geometrical point of view, continuing the investigations started by N. Tanaka (cf. [39] and [40]) in the context of the generalized contact structures. Extending the method of $\mathbf{G}$-structures, we are led to consider special Lie algebras, that we call of *Levi-Tanaka*, which parametrize the infinitesimal CR automorphisms of homogeneous CR manifolds. In this work I describe the general properties of the Levi-Tanaka algebras and, after classifying all semisimple ones, I discuss the homogeneous CR manifolds which have the largest Lie group of CR automorphisms for a given (higher order) Levi form.

The thesis is organized as follows. In the first chapter I collect the general definitions and preliminary notions related to CR manifolds.

In chapter 2 it is described how to each point of a CR manifold M one can attach the main invariant for discussing CR transformations: the Levi-Tanaka algebra. This is defined as the canonical prolongation (i.e. the max-

imal pseudocomplex transitive one) of the fundamental graded Lie algebra $\oplus_{p<0}\mathfrak{g}_p$ associated to the filtration of the tangent bundle TM induced by the distribution $HM \subset TM$ of the holomorphic tangent vectors. Such a prolongation $\mathfrak{g} = \oplus_{p\in\mathbb{Z}}\mathfrak{g}_p$ is unique up to isomorphisms of graded Lie algebras (see Theorem 2.3.1). The remaining part of the chapter contains a proof of a criterion for the finiteness of the dimension of the prolongation which is based on a result of Serre (see [13]) and had been proved by N. Tanaka in [40]. In the case of pseudocomplex algebras this criterion boils down to a nondegeneracy condition, namely to: $\{X \in \mathfrak{g}_{-1} \mid [X, \mathfrak{g}_{-1}] = 0\} = (0)$. According to a classical theorem of Kobayashi (see, for instance, [20]), when this condition holds at all points of M, the group of CR automorphisms is a finite dimensional Lie group. If the Levi-Tanaka algebra is semisimple, it is possible to associate to the manifold a principal bundle endowed with a Cartan connection, in such a way that CR transformations are lifted to diffeomorphisms of the principal bundle preserving the Cartan connection.

Chapter 3 is devoted to a general study of the Levi-Tanaka algebras (and, more generally, of graded Lie algebras $\mathfrak{g} = \oplus_{p\in\mathbb{Z}}\mathfrak{g}_p$ endowed with a characteristic element, i.e. an element such that the subspaces $\mathfrak{g}_p$ are the eigenspaces of its adjoint representation, associated to the eigenvalues $p \in \mathbb{Z}$). Next we prove that a Levi-Tanaka algebra admits an appropriate Levi-Malčev decomposition in which the semisimple part is the semidirect sum of two ideals, one of which is still a Levi-Tanaka algebra while the other one is a semisimple algebra of derivations of the radical. From this decomposition we deduce criteria to investigate the structure of the Levi-Tanaka algebras based on the adjoint representation of $\mathfrak{g}_0$ on $\mathfrak{g}_{-1}$ or $\mathfrak{g}_{-2}$. Then a suitable version of the Cartan decomposition for semisimple Levi-Tanaka algebras is given. The chapter ends with a section collecting significant examples. It is worth of noticing that part of these examples were actually suggested by the previous discussion of the general properties of the Levi-Tanaka algebras.

Chapter 4 is devoted to a complete classification of the semisimple Levi-Tanaka algebras. This is also important because of the connection of these algebras to compact homogeneous CR manifolds, that will be stressed in the last chapter. The key to this classification is the fact that the pseudocomplex structure is defined in this case by a 0-degree inner derivation and it uses the results on the classification of semisimple graded Lie algebras which are related to the construction of the weighted Satake diagrams (due to Djoković [12]). After reducing the problem to that of classifying simple Levi-Tanaka

algebras, I discuss separately simple graded Lie algebras of the complex and of the real type, first giving general criteria on their weighted Dynkin and Satake diagrams in order that they admit a structure of Levi-Tanaka algebras and next applying these criteria to the different classes. In the classical cases we give matrix representations, while for the exceptional ones we give in the appendix a complete list, which was obtained by symbolic calculus, using a software that I produced for this purpose.

Chapter 5 is devoted to Levi-Tanaka algebras of the second kind. Up to isomorphisms they are classified by the subspaces of the space of Hermitian symmetric forms modulo conjunctivity. This is the reason why the results are further restricted to the CR codimension 2 case: there is indeed a complete theory of the canonical form of a pair of Hermitian symmetric matrices. The complete classification obtained in this special situation provides a rich set of examples of Levi-Tanaka algebras which are not semisimple.

In chapter 6 we associate to each Levi-Tanaka algebra $\mathfrak{g}$ a real universal homogeneous space: the standard CR manifold $S_{\mathfrak{g}}$. These simply connected CR manifolds have at each point $x \in S_{\mathfrak{g}}$ a Levi-Tanaka algebra $\mathfrak{g}(x)$ isomorphic to $\mathfrak{g}$ and a maximal group of global automorphisms. The CR automorphisms of these manifolds satisfy also a localization property. Moreover $S_{\mathfrak{g}}$ is embedded in a complex homogeneous space $X_{\mathfrak{g}}$ in such a way that $\mathfrak{g}$ is the Levi-Tanaka algebra associated to the complex structure induced on $S_{\mathfrak{g}}$ by the embedding. We consider also the case of projective immersions and give some criteria for the compactness of $S_{\mathfrak{g}}$.

The results proved in Chapters 3, 4, 5 and 6 are original and are partially published in [24], [25], and [26]. The remaining of the original part of this thesis, in particular the results of Chapter 5, is the subject of a forthcoming publication.

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Chapter 1

Preliminaries

1.1 Partial complex structures and CR manifolds

Let M be a smooth real manifold of dimension m, countable at infinity. Let n, k be nonnegative integers with $2n + k = m$. A *partial almost-complex structure of type* (n, k) on M is the pair (HM, J) consisting of a real vector subbundle HM of rank $2n$ of the tangent bundle TM and a smooth fiber preserving bundle isomorphism $J : HM \to HM$ with

$$J^2 = -Id : HM \to HM$$

and such that

$$[X, Y] - [JX, JY] \in \Gamma(M, HM) \quad \forall X, Y \in \Gamma(M, HM). \tag{1.1}$$

Here we use Γ to indicate smooth sections of a fiber bundle.

The triple $\mathrm{M} = (M, HM, J)$, where (HM, J) is a partial almost-complex structure of type (n, k) on M, is then called an *almost-CR manifold of type* (n, k). The number n is called the *CR-dimension* of M and k the *CR-codimension* of M.

We say that the partial almost-complex structure (HM, J) on M is a *partial complex structure* (or *CR structure*) if it is *formally-integrable*, i.e. if

$$\mathcal{N}(X, Y) := [JX, Y] + [X, JY] - J([X, Y] - [JX, JY]) = 0 \tag{1.2}$$

for every $X, Y \in \Gamma(M, HM)$. When (HM, J) is a partial complex structure of type (n, k), we say that the triple $\mathbf{M} = (M, HM, J)$ is a *CR manifold of type* (n, k).
Note that $-J$ is also a partial complex structure of the same type and with the same properties of integrability as J. It is called *the conjugated structure.*

The integrability conditions (1.1) and (1.2) can be expressed in another equivalent formulation. Let

$$T^{1,0}M = \{X - \sqrt{-1}JX \mid X \in HM\}, \quad T^{0,1}M = \{X + \sqrt{-1}JX \mid X \in HM\}$$

be the complex vector subbundles of the complexification $\mathbb{C} \otimes_{\mathbb{R}} HM$ of HM, corresponding to the eigenvalues $\sqrt{-1}$ and $-\sqrt{-1}$ of J. Then (1.1) and (1.2) are equivalent to each of the following:

$$\begin{aligned} [\Gamma(M, T^{1,0}M), \Gamma(M, T^{1,0}M)] &\subset \Gamma(M, T^{1,0}M), \\ [\Gamma(M, T^{0,1}M), \Gamma(M, T^{0,1}M)] &\subset \Gamma(M, T^{0,1}M). \end{aligned}$$

Note that a complex subbundle HTM of the complexification $\mathbb{C} \otimes_{\mathbb{R}} TM$ of the tangent bundle TM such that:

$$HTM \cap \overline{HTN} = \{0\} \quad \text{and} \tag{1.3}$$

$$[\Gamma(HTM), \Gamma(HTM)] \subset \Gamma(HTM), \tag{1.4}$$

defines a CR structure (HM, J) given by

$$\begin{aligned} &HM = \{X \mid X + \sqrt{-1}Y \in HTM\} \quad \text{and} \\ &J(X) = Y \quad \text{if } X + \sqrt{-1}Y \in HTM. \end{aligned}$$

This is a good definition because two elements in HTM with the same real part are equal by (1.3). This partial complex structure is formally-integrable by (1.4).

Note that $\overline{HTM}$ defines the conjugate CR structure $(HM, -J)$.

1.1.1 The CR structure of a complex manifold

Let M be a complex manifold of complex dimension m. If $(z^1, \ldots, z^m)$ are holomorphic coordinates in a neighborhood of a point $x \in M$, we call

holomorphic and antiholomorphic tangent space of M at x the subspaces of $\mathbb{C} \otimes_{\mathbb{R}} TM$:

$$T^{1,0}M_x = < \frac{\partial}{\partial z^1}(x), \dots, \frac{\partial}{\partial z^m}(x) >_{\mathbb{C}} \text{ and} \tag{1.5}$$

$$T^{0,1}M_x = < \frac{\partial}{\partial \bar{z}^1}(x), \dots, \frac{\partial}{\partial \bar{z}^m}(x) >_{\mathbb{C}}, \tag{1.6}$$

respectively, where $< \cdot >_{\mathbb{C}}$ denotes the complex linear span of the vectors included in the brackets.

The holomorphic tangent bundle $T^{1,0}M = \bigcup_{x \in M} T^{1,0}M_x$ defines a CR structure on M of type $(m, 0)$ as:

$$T^{1,0}M_x \cap \overline{T^{1,0}M_x} = \{0\} \tag{1.7}$$

$$T^{0,1}M_x \oplus \overline{T^{0,1}M_x} = \mathbb{C} \otimes_{\mathbb{R}} TM_x. \tag{1.8}$$

This structure is obviously integrable.

The CR structure of a complex manifold M can be given in an equivalent way by considering the pair (TM, J) where the partial almost-complex structure $J : TM \to TM$ is given for every x in M by

(1.9)

$$J\left(\frac{\partial}{\partial x^j}(x)\right) = \frac{\partial}{\partial y^j}(x) \quad , \quad J\left(\frac{\partial}{\partial y^j}(x)\right) = -\frac{\partial}{\partial x^j}(x) \qquad \text{for } j = 1, \dots, m$$

where $(z^1, \dots, z^m) = (x^1 + \sqrt{-1}y^1, \dots, x^m + \sqrt{-1}y^m)$ are holomorphic coordinates in a neighborhood of x.

A partial almost-complex structure of type $(n, 0)$ on a real $2n$-dimensional manifold (i.e. of CR-codimension $k = 0$) is called an almost-complex structure. The following theorem is due to Newlander and Nirenberg (see [31] and also [46]).

THEOREM 1.1.1 *Every manifold with a formally-integrable almost-complex structure (i.e. a CR structure of CR-codimension $k = 0$) is a complex manifold in a natural way.*

1.2 CR maps and immersions

Let $\mathrm{M}_1 = (M_1, HM_1, J_1)$ and $\mathrm{M}_2 = (M_2, HM_2, J_2)$ be two almost-CR manifolds. A differentiable map $f : M_1 \to M_2$ is a *CR map* if

1. $f_*(HM_1) \subset HM_2$ and

2. $f_*(J_1 X_x) = J_2 f_*(X_x) \quad \forall x \in M_1\,, \forall X_x \in H_x M_1.$

When $\mathbf{M}_2$ is $\mathbb{C}$ with the complex structure of a CR manifold of type $(1,0)$ given by $T^{0,1}\mathbb{C}$, a CR map from M_1 to $\mathbb{C}$ is called a *CR function.*

A differomorphism $f : M_1 \to M_2$ is called a *CR diffeomorphism* if f and $f^{-1} : M_2 \to M_1$ are both CR maps. Two CR diffeomorphic almost-CR manifolds are necessarily of the same type.

A *CR immersion* (respectively *CR embedding*) of a CR manifold $\mathbf{M}$ of type (n,k) is the datum (X,ϕ) of a complex manifold X and an immersion (resp. embedding) of manifolds $\phi : M \to X$ which is a CR map (considering on X the CR structure given from $T^{0,1}X$).

It is called *generic* if $\phi(M)$ is a generic submanifold of X, that is if $\dim_{\mathbb{C}} X = n + k$. A generic embedding is also called a *complexification.* A CR manifold is called embeddable if it admits a CR embedding (X,ϕ).

Every real-analytic CR manifold admits a complexification (cf. [2]). General results on the complexification of smooth CR manifolds have been obtained only in the CR-codimension one case (cf. [1], [47] and [10]), while counterexamples to the embeddability have also been obtained for higher codimension (see, for instance, [32], [17], [33], [7], [34], and [16]).

1.3 The form of Levi-Tanaka

We begin by an easy proposition from linear algebra, that will be useful in the sequel.

PROPOSITION 1.3.1 *Let V be a real vector space, of even dimension $2n$, on which a complex structure $J \in \mathrm{Hom}_{\mathbb{R}}(V,V)$, with $J^2 = -Id$, is given. Then:*

1. *For every alternating $\mathbb{R}$-bilinear form $\mathfrak{a} : V \times V \to \mathbb{R}^k$ such that $\mathfrak{a}(Jv, Jw) = \mathfrak{a}(v,w)$ for every $v, w \in V$ there is a unique Hermitian symmetric form $\mathfrak{f} : V \times V \to \mathbb{C}^k$ such that*

$$\Im\mathfrak{f}(v,w) = \mathfrak{a}(v,w) \quad \forall v, w \in V.$$

It is given by

$$\mathfrak{f}(v,w) = \mathfrak{a}(Jv,v) + \sqrt{-1}\mathfrak{a}(v,w) \quad \forall v,w \in V$$

and is Hermitian symmetric for the real form $\mathbb{R}^k$ *of* $\mathbb{C}^k$.

2. *If* $\mathfrak{a}$ *and* $\mathfrak{f}$ *are as in* (1), $A \in \mathrm{Hom}_{\mathbb{C}}(V,V)$ *and* $B \in \mathrm{Hom}_{\mathbb{R}}(\mathbb{R}^k,\mathbb{R}^k)$, *the following are equivalent:*

   (*i*) $$\mathfrak{a}(Av,Aw) = B\mathfrak{a}(v,w) \quad \forall v,w \in V,$$

   (*ii*) $$\mathfrak{f}(Av,Av) = B\mathfrak{f}(v,v) \quad \forall v \in V.$$

3. *If* $\mathfrak{a}$ *and* $\mathfrak{f}$ *are as in* (1), $A \in \mathrm{Hom}_{\mathbb{C}}(V,V)$ *and* $B \in \mathrm{Hom}_{\mathbb{R}}(\mathbb{R}^k,\mathbb{R}^k)$, *the following are equivalent:*

   (*iii*) $$\mathfrak{a}(Av,w) + \mathfrak{a}(v,Aw) = B\mathfrak{a}(v,w) \quad \forall v,w \in V,$$

   (*iv*) $$\mathfrak{f}(Av,v) + \mathfrak{f}(v,Av) = B\mathfrak{f}(v,v) \quad \forall v \in V.$$

Moreover, (*ii*) *and* (*iv*) *are respectively equivalent to:*

(*ii'*) $$\mathfrak{f}(Av,Aw) = B\mathfrak{f}(v,w) \quad \forall v,w \in V,$$

(*iv'*) $$\mathfrak{f}(Av,w) + \mathfrak{f}(v,Aw) = B\mathfrak{f}(v,w) \quad \forall v,w \in V,$$

for the complexification, still denoted by B, *of the real linear map* B.

Let now $\mathrm{M} = (M, HM, J)$ be an almost-CR manifold of type (n,k), denote by QM the quotient bundle TM/HM and let $\pi : TM \to QM$ be the projection onto the quotient. Given two sections $X, Y \in \Gamma(M, HM)$ and a point $x \in M$, the value $\pi([X,Y]_x) \in Q_xM$ only depends on the values X_x, Y_x at x of X and Y. Thus we obtain an alternating bilinear form

$$\mathfrak{l}_x : H_xM \times H_xM \ni (X_x, Y_x) \to \pi([X,Y]_x) \in Q_xM$$

which is called the *Levi-Tanaka form* of M at x. Clearly the assignement $M \ni x \to \mathfrak{l}_x \in \Lambda^2(HM, QM)$ is smooth.

By condition (1.1) this form is J-invariant:

$$\mathfrak{l}_x(JX_x, JY_x) = \mathfrak{l}_x(X_x, Y_x) \quad \forall x \in M, \quad \forall X_x, Y_x \in H_xM.$$

By applying the proposition above, we obtain for every $x \in M$ a unique Hermitian symmetric form $\mathfrak{f}_x$ for the complex structure of H_xM such that

$$\mathfrak{l}_x(X_x, Y_x) = \Im\mathfrak{f}_x(X_x, Y_x) \quad \forall X_x, Y_x \in H_xM.$$

It is given by

$$\mathfrak{f}_x(X_x, Y_x) = \mathfrak{l}_x(JX_x, Y_x) + \sqrt{-1}\mathfrak{l}_x(X_x, Y_x)$$

and therefore smoothly depends on x. The corresponding Hermitian quadratic form

$$H_xM \ni X_x \to \mathfrak{f}_x(X_x, X_x) \in Q_xM$$

is often referred to as the *(vector valued) Levi form.*

1.3.1 Pseudoconvexity and pseudoconcavity

Let $\mathbf{M} = (M, HM, J)$ be an almost-CR manifold of type (n, k). We define the *characteristic bundle* of $\mathbf{M}$ as the smooth linear subbundle H^0M of the cotangent bundle T^*M of M whose fiber H^0_xM at the point $x \in M$ is the annihilator of $H_xM \subset T_xM$:

$$H^0_xM = \{\xi_x \in T^*_xM \mid \langle X_x, \xi_x\rangle = 0 \quad \forall X_x \in H_xM\}.$$

We define the *(scalar) Levi form* at $\xi_x \in H^0_xM$ by

$$\mathcal{L}(\xi_x, X_x) = \langle \mathfrak{f}_x(X_x, X_x), \xi_x\rangle \quad \text{for} \quad X_x \in H_xM.$$

This is a real valued Hermitian form for the complex structure of H_xM.

We say that $\mathbf{M}$ is *q-pseudoconvex* at $x \in M$ if we can find $\xi_x \in H^0_xM$ such that the Hermitian form $\mathcal{L}(\xi_x, \cdot)$ has at least $(n-q)$ positive eigenvalues.

We say that $\mathbf{M}$ is *q-pseudoconcave* at $x \in M$ if for every $\xi_x \in H^0_xM$ with $\xi_x \neq 0$ the Hermitian form $\mathcal{L}(\xi_x, \cdot)$ has at least q negative eigenvalues.

Pseudoconvexity and pseudoconcavity are related to the local properties of the CR complexes (see, for instance, [30] and [27]).

Chapter 2

Prolongations of fundamental graded Lie algebras

2.1 Graded Lie algebras

A $\mathbb{Z}$-*graduation* (or briefly *graduation*) of a Lie algebra $\mathfrak{g}$ over a field $\mathbb{K}$ is a decomposition of $\mathfrak{g}$ into a direct sum of finite dimensional $\mathbb{K}$-linear subspaces $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ such that

$$[\mathfrak{g}_p,\, \mathfrak{g}_q] \subset \mathfrak{g}_{p+q} \quad \forall p, q \in \mathbb{Z}.$$

A Lie algebra $\mathfrak{g}$ with a given graduation is called *graded.* We say that $\mathfrak{g}$ is *of finite kind* μ, for a nonnegative integer μ, if $\mathfrak{g}_p = 0$ for $p < -\mu$ and $\mathfrak{g}_{-\mu} \neq 0$. In this case we call the dimension k of $\oplus_{p<-1}\mathfrak{g}_p$ the *codimension* of $\mathfrak{g}$.

LEMMA 2.1.1 *Let $\mathfrak{g} = \oplus_{-\mu\leq p\leq \nu}\mathfrak{g}_p$ be a finite dimensional graded Lie algebra. Let $\kappa_{\mathfrak{g}}$ denote the Killing form of $\mathfrak{g}$. Then*

$$\kappa_{\mathfrak{g}}(\mathfrak{g}_p, \mathfrak{g}_q) \neq 0 \implies p + q = 0\,.$$

Assume now that $p \neq 0$ and let $X \in \mathfrak{g}_p$ and $Y \in \mathfrak{g}_{-p}$. Then

$$\kappa_{\mathfrak{g}}(X, Y) \neq 0 \implies [X, Y] \neq 0\,.$$

Proof. If $X \in \mathfrak{g}_p$ and $Y \in \mathfrak{g}_q$ with $p+q \neq 0$, then the linear operator $\mathrm{ad}_{\mathfrak{g}}(X) \circ \mathrm{ad}_{\mathfrak{g}}(Y) : \mathfrak{g} \to \mathfrak{g}$ is nilpotent as $\mathrm{ad}_{\mathfrak{g}}(X) \circ \mathrm{ad}_{\mathfrak{g}}(Y)(\mathfrak{g}_h) \subset \mathfrak{g}_{h+p+q}$, and so $\kappa_{\mathfrak{g}}(X,Y) = 0$.

Let now $X \in \mathfrak{g}_p$, $Y \in \mathfrak{g}_{-p}$ with $p \neq 0$ and assume that $[X,Y] = 0$. Then $\mathrm{ad}_{\mathfrak{g}}(X)$ and $\mathrm{ad}_{\mathfrak{g}}(Y)$ are commuting nilpotent operator and so $\mathrm{ad}_{\mathfrak{g}}(X) \circ \mathrm{ad}_{\mathfrak{g}}(Y)$, being nilpotent, has trace $\kappa_{\mathfrak{g}}(X,Y) = 0$. □

We note that $\mathfrak{g}_0$ is a Lie subalgebra of $\mathfrak{g}$ and, for every $p \in \mathbb{Z}$, the map

$$\rho_p : \mathfrak{g}_0 \to \mathrm{Hom}_{\mathbb{K}}(\mathfrak{g}_p, \mathfrak{g}_p),$$

defined by

$$\rho_p(X_0)(X_p) = [X_0, X_p] \quad \text{for} \quad X_0 \in \mathfrak{g}_0,\ X_p \in \mathfrak{g}_p\,, \tag{2.1}$$

is a linear representation of the Lie algebra $\mathfrak{g}_0$ in $\mathfrak{g}_p$.

A graded Lie algebra $\mathfrak{g}$ is said to be:

1. *fundamental* if $\mathfrak{g}_p = 0$ for $p \geq 0$ and $[\mathfrak{g}_p,\ \mathfrak{g}_{-1}] = \mathfrak{g}_{p-1}$ for $p < 0$, i.e. $\mathfrak{g}_{-1}$ generates $\mathfrak{g}$;
2. *nondegenerate* if $[X,\ \mathfrak{g}_{-1}] \neq 0$ when $0 \neq X \in \mathfrak{g}_{-1}$;
3. *irreducible* (respectively *completely reducible*) if the representation ρ_{-1} of $\mathfrak{g}_0$ in $\mathfrak{g}_{-1}$ is irreducible (resp. completely reducible);
4. *transitive* if $[X,\ \mathfrak{g}_{-1}] \neq 0$ when $p \geq 0$ and $0 \neq X \in \mathfrak{g}_p$; in particular, the representation ρ_{-1} of $\mathfrak{g}_0$ in $\mathfrak{g}_{-1}$ is faithful.

Note that if $\mathfrak{g}$ is a nontrivial transitive graded Lie algebra, then $\mu \geq 1$, and if $\mathfrak{g}$ is also nondegenerate, then $\mu \geq 2$.

Let $\mathfrak{g} = \oplus_{p \in \mathbb{Z}} \mathfrak{g}_p$ be a graded Lie algebra over $\mathbb{R}$. A *partial complex structure* on $\mathfrak{g}$ is an $\mathbb{R}$-linear map $J : \mathfrak{g}_{-1} \to \mathfrak{g}_{-1}$ which satisfies

$$\begin{cases} J^2 = -\mathrm{Id}_{\mathfrak{g}_{-1}} \\ [JX, JY] = [X,Y] \quad \forall X, Y \in \mathfrak{g}_{-1}. \end{cases} \tag{2.2}$$

In this case the real dimension of $\mathfrak{g}_{-1}$ is even, and we say that $\mathfrak{g}$ is *of type* (n,k) if $\dim_{\mathbb{R}} = 2n$ and $\dim_{\mathbb{R}} \oplus_{p \leq -2} \mathfrak{g}_p = k$. The integers n and k will be also referred to, respectively, as the CR-dimension and CR-codimension of $\mathfrak{g}$.

Note that $-J$ is also a partial complex structure, of the same type of J. It is called *the conjugated structure.*

A graded Lie algebra $\mathfrak{g}$ over $\mathbb{R}$ with a partial pseudocomplex structure J is said to be *pseudocomplex* if the elements of $\rho_{-1}(\mathfrak{g}_0)$ commute with J on $\mathfrak{g}_{-1}$.

For a graded Lie algebra $\mathfrak{g} = \oplus_{p\in\mathbb{Z}}\mathfrak{g}_p$ over $\mathbb{K}$, the $\mathbb{K}$-linear map $e : \mathfrak{g} \to \mathfrak{g}$ defined by

$$e(X) = pX \quad \text{for} \quad X \in \mathfrak{g}_p,\ p \in \mathbb{Z} \tag{2.3}$$

is a 0-degree derivation. If $e = \mathrm{ad}_{\mathfrak{g}}(E)$ for some $E \in \mathfrak{g}$ is an inner derivation, we call such an element $E \in \mathfrak{g}_0$ a *characteristic element* of the graded Lie algebra $\mathfrak{g}$. Conversely, if a finite dimensional Lie algebra $\mathfrak{g}$ over $\mathbb{K}$ contains a semisimple element E such that $\mathrm{ad}_{\mathfrak{g}}(E) : \mathfrak{g} \to \mathfrak{g}$ has integral eigenvalues, then the subspaces

$$\mathfrak{g}_p = \{X \in \mathfrak{g} \,|\, [E, X] = pX\} \qquad \text{for } p \in \mathbb{Z} \tag{2.4}$$

give a graduation of $\mathfrak{g}$ for which E is a characteristic element.

Note that a characteristic element is contained in the center $\mathfrak{z}(\mathfrak{g}_0)$ of $\mathfrak{g}_0$.

The characteristic element is uniquely determined if and only if the center $\mathfrak{z}(\mathfrak{g})$ of $\mathfrak{g}$ is trivial. In this case we will call E *the characteristic element* of $\mathfrak{g}$.

An *isomorphism* of graded Lie algebras is a map $\phi : \mathfrak{g} = \oplus_{p\in\mathbb{Z}}\mathfrak{g}_p \to \mathfrak{g}' = \oplus_{p\in\mathbb{Z}}\mathfrak{g}'_p$ between graded Lie algebras over the same field $\mathbb{K}$ such that:

(i) ϕ is a Lie algebras isomorphism;

(ii) $\phi(\mathfrak{g}_p) = \mathfrak{g}'_p$ for all $p \in \mathbb{Z}$.

For graded Lie algebras having unique characteristic elements E and E' respectively, condition (ii) is equivalent to $\phi(E) = E'$.

Remark 2.1.2 *Every graded Lie algebra is contained as a graded Lie ideal in a graded Lie algebra with a characteristic element.*

LEMMA 2.1.3 *The center $\mathfrak{z}(\mathfrak{g})$ of a graded Lie algebra $\mathfrak{g} = \oplus_{p\in\mathbb{Z}}\mathfrak{g}_p$ with a characteristic element E is contained in $\mathfrak{g}_0$.*

If $\mathfrak{g}$ is also transitive, then the center $\mathfrak{z}(\mathfrak{g})$ of $\mathfrak{g}$ is trivial; in particular, E is the unique characteristic element of $\mathfrak{g}$.

Proof. Let $X = \sum_{p\in\mathbb{Z}} X_p$ be an element of the center of $\mathfrak{g}$, decomposed into the sum of its homogeneous components. From $\mathrm{ad}_{\mathfrak{g}}(X)(E) = -\sum_{p\in\mathbb{Z}} pX_p = 0$ we deduce that $X_p = 0$ for every $p \neq 0$, and so $X = X_0 \in \mathfrak{g}_0$.

Moreover, as $\mathrm{ad}_{\mathfrak{g}}(X_0)(\mathfrak{g}_{-1}) = \rho_{-1}(X_0)(\mathfrak{g}_{-1}) = 0$, if $\mathfrak{g}$ is transitive, then $X_0 = 0$. □

If $\mathfrak{l}$ is any subset of a graded Lie algebra $\mathfrak{g} = \oplus_{p\in\mathbb{Z}}\mathfrak{g}_p$, we will use in the following the notation $\mathfrak{l}_p$ for the set $\mathfrak{l}\cap\mathfrak{g}_p$ of its elements that are homogeneous of degree p. We say that $\mathfrak{l}$ is *graded* if $\mathfrak{l} = \oplus_{p\in\mathbb{Z}}\mathfrak{l}_p$.

LEMMA 2.1.4 *Every ideal of a graded Lie algebra $\mathfrak{g} = \oplus_{p\in\mathbb{Z}}\mathfrak{g}_p$ with a characteristic element E is graded.*

Proof. Let $X = X_{-\mu} + X_{1-\mu} + \ldots + X_\nu$ be an element of an ideal $\mathfrak{i}$ of $\mathfrak{g}$, decomposed as a sum of its homogeneous components. Then $\mathfrak{i}$ contains all elements $\mathrm{ad}_{\mathfrak{g}}(E)^\ell(X)$, where E is a characteristic element of $\mathfrak{g}$ and ℓ is any positive integer. Therefore $\mathfrak{i}$ contains:

$$\begin{array}{ccccccc} X_{-\mu} & + & X_{1-\mu} & + & \ldots & + & X_\nu \\ -\mu X_{-\mu} & + & (1-\mu)X_{1-\mu} & + & \ldots & + & \nu X_\nu \\ \ldots & & & & & & \\ (-\mu)^\ell X_{-\mu} & + & (1-\mu)^\ell X_{1-\mu} & + & \ldots & + & \nu^\ell X_\nu \end{array}$$

from which it follows that the ideal $\mathfrak{i}$ contains all the homogeneous components of X. □

2.2 Fundamental graded Lie algebras associated to vector distributions

Graded Lie algebras were considered by Tanaka in [39] in order to investigate canonical forms of vector distributions and CR manifolds. We rehearse here the relevant construction.

Let $D \subset TM$ be a rank r linear subbundle of the tangent bundle of a smooth differentiable manifold M of dimension m. We set

$$\mathcal{D}_{-1} = \Gamma(M, D)$$

and define by recurrence

$$\mathcal{D}_p = [\mathcal{D}_{p+1}, \mathcal{D}_{-1}] + \mathcal{D}_{p+1} \quad \text{for} \quad p < -1.$$

Then we have an increasing sequence of $\mathcal{E}(M)$-modules of vector fields

$$\mathcal{D}_{-1} \subset \mathcal{D}_{-2} \subset \ldots \subset \Gamma(M, TM).$$

For every $x \in M$ and $p < 0$ we set

$$(\mathcal{D}_p)_x = \{X_x \in T_xM \mid X \in \mathcal{D}_p\}$$

Note that:

(i) $\quad [\mathcal{D}_p, \mathcal{D}_q] \subset \mathcal{D}_{p+q} \quad \forall p, q < 0;$

(ii) $\quad$ if $p, q < 0$, $X \in \mathcal{D}_p$, $Y \in \mathcal{D}_q$, $f, g \in \mathcal{E}(M)$, then

$$[fX, gY] - fg[X, Y] \in \mathcal{D}_{p+q+1}.$$

Let us define then, for every fixed $x \in M$,

$$\begin{cases} \mathfrak{g}_{-1}(x) = (\mathcal{D}_{-1})_x \\ \mathfrak{g}_p(x) = \frac{(\mathcal{D}_p)_x}{(\mathcal{D}_{p+1})_x} \end{cases} \quad \text{for} \quad p < -1.$$

By conditions (i) and (ii), the commutator of vector fields in $\mathcal{D}_p$ and $\mathcal{D}_q$, composed with the projection onto the quotient $(\mathcal{D}_{p+q})_x \to \mathfrak{g}_{p+q}(x)$, defines on

$$\mathfrak{g}(x) = \bigoplus_{p<0} \mathfrak{g}_p(x)$$

the structure of a real fundamental graded Lie algebra.

We say that D is *regular* if, for every $p < 0$, $\mathcal{D}_p$ is a vector distribution of constant rank in M, i.e. if

$$\dim_{\mathbb{R}} \mathfrak{g}_p(x) = \dim_{\mathbb{R}} \mathfrak{g}_p(y) \quad \forall p < 0, \quad \forall x, y \in M.$$

In this case there is a smallest positive integer μ such that

$$\mathcal{D}_p = \mathcal{D}_{-\mu} \quad \forall p < -\mu$$

and $\mathcal{D}_{-\mu}$ is the smallest formally integrable vector distribution in M containing $\mathcal{D}_{-1} = \Gamma(M, D)$. By the classical Frobenius theorem M is locally foliated by integral leaves of $\mathcal{D}_{-\mu}$.

In particular we can apply the construction above to the linear vector subbundle HM of TM for a given almost CR manifold $\mathrm{M} = (M, HM, J)$. We say that M is *contact regular* if HM is regular.

We shall denote by $\mathfrak{m}(x)$ the fundamental graded Lie algebra associated to HM at the point $x \in M$. It is pseudocomplex with respect to the complex structure J on $H_xM = \mathfrak{m}_{-1}(x)$. We note that $\mathfrak{m}(x)$ is nondegenerate if and only if the Levi form is nondegenerate at x.

A CR diffeomorphism induces isomorphisms of the pseudocomplex fundamental graded Lie algebras associated to the partial almost-complex structures at the corresponding points. In particular the algebras $\mathfrak{m}(x)$ are pseudoconformal invariants of the CR manifolds.

The fundamental graded Lie algebra $\mathfrak{m}(x)$ takes into account also the higher order Levi forms (see, for instance, [38]). However, for the study of the local CR invariants of M, it is convenient to extend $\mathfrak{m}(x)$ to a larger graded Lie algebra $\mathfrak{g}(x)$, via a canonical prolongation. This $\mathfrak{g}(x)$ will be called the *Levi-Tanaka algebra* of M at x.

2.3 Canonical prolongations of fundamental graded Lie algebras

Given a finite dimensional graded Lie algebra $\mathfrak{a} = \oplus_{-\mu \leq p \leq \nu} \mathfrak{a}_p$, we say that a graded Lie algebra $\mathfrak{b} = \oplus_{-\mu \leq p} \mathfrak{b}_p$ is a *prolongation* of $\mathfrak{a}$ if there is a monomorphism of graded Lie algebras $\mathfrak{a} \to \mathfrak{b}$ inducing an isomorphism of $\mathfrak{a}$ onto $\mathfrak{b}_{\leq \nu} = \oplus_{-\mu \leq p \leq \nu} \mathfrak{b}_p$.

In this section we sketch the proof of the fundamental theorem about the existence of canonical prolongations of fundamental graded Lie algebras (see [40]).

THEOREM 2.3.1 *Let $\mathfrak{m} = \bigoplus_{-\mu \leq p < 0} \mathfrak{m}_p$ be a fundamental graded Lie algebra over $\mathbb{R}$. Then we can construct a graded Lie algebra $\mathfrak{g} = \bigoplus_{p \geq -\mu} \mathfrak{g}_p$, unique up to isomorphisms, which is maximal between the transitive graded*

Lie algebras $\mathfrak{g}$ for which there is a graded Lie algebras isomorphism:

$$\mathfrak{g}_- = \bigoplus_{-\mu \le p < 0} \mathfrak{g}_p \to \mathfrak{m}.$$

Such a transitive graded Lie algebra $\mathfrak{g}$ will be called the *canonical prolongation* of $\mathfrak{m}$.

Proof. The existence of a canonical prolongation trivially follows from the Zorn lemma. However, the explicit construction also shows uniqueness and gives a method for computing the canonical prolongation.

To construct a canonical prolongation $\mathfrak{g}$, we first set

$$\mathfrak{g}_p = \mathfrak{m}_p \quad \text{for} \quad p < 0.$$

The linear spaces $\mathfrak{g}_p$ for $p \ge 0$ will be defined below recursively as linear subspaces of $\mathrm{Hom}_{\mathbb{R}}(\mathfrak{m}, \oplus_{q<p}\mathfrak{g}_q)$. For $X_p \in \mathrm{Hom}_{\mathbb{R}}(\mathfrak{m}, \oplus_{q<p}\mathfrak{g}_q)$ we write:

$$X_p(Y) = [X_p, Y] = -[Y, X_p] \quad \forall Y \in \mathfrak{m}.$$

With this notation we set:

$$\begin{aligned} \mathfrak{g}_p \;=\; & \{X_p \in \mathrm{Hom}_{\mathbb{R}}(\mathfrak{m}, \oplus_{q<p}\mathfrak{g}_q) \mid X_p(\mathfrak{g}_h) \subset \mathfrak{g}_{p+h}\ \forall h < 0, \quad \text{and} \\ & [X_p(Y), Z] - [X_p(Z), Y] = X_p([Y, Z])\ \forall Y, Z \in \mathfrak{m}\}. \end{aligned}$$

We note that $\mathfrak{g}_0 \subset \mathrm{Hom}_{\mathbb{R}}(\mathfrak{m}, \mathfrak{m})$ is the Lie algebra of derivations of degree 0 of the graded Lie algebra $\mathfrak{m}$.

By this construction, the Lie product $[X, Y]$ in $\mathfrak{g}$ is defined when one of the two elements $X, Y \in \mathfrak{g}$ belongs to $\mathfrak{m}$ and the Jacobi identity

$$[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$$

holds when $X, Y, Z \in \mathfrak{g}$ and two of them belong to $\mathfrak{m}$.

Moreover, because $\mathfrak{m}$ is fundamental, we have

$$[X_p, \mathfrak{g}_{-1}] \ne 0 \quad \text{if} \quad p \ge 0 \quad \text{and} \quad X_p \in \mathfrak{g}_p, \quad X_p \ne 0.$$

To define the Lie product of arbitrary elements of $\mathfrak{g}$ and to show that the Jacobi identity holds true, we argue by induction on the degrees.

It suffices to consider products of homogeneous elements $[X_p, Y_q]$ for $X_p \in \mathfrak{g}_p$ and $Y_q \in \mathfrak{g}_q$ with $p, q \geq 0$. When $p = q = 0$ we consider $\mathfrak{g}_0$ as a Lie subalgebra of $\mathfrak{gl}(\mathfrak{m})$ and set

$$[X_0, Y_0] = X_0 \circ Y_0 - Y_0 \circ X_0.$$

Since the derivations of a Lie algebra form a Lie algebra, we have $[X_0, Y_0] \in \mathfrak{g}_0$ and moreover the Jacobi identity

$$[[X_p, Y_q], Z_r] + [[Y_q, Z_r], X_p] + [[Z_r, X_p], Y_q] = 0 \tag{2.5}$$

for $X_p \in \mathfrak{g}_p$, $Y_q \in \mathfrak{g}_q$ and $Z_r \in \mathfrak{g}_r$ holds true when $p, q, r \leq 0$. This we call a Jacobi identity homogeneous of degree $p + q + r$.

Assume now that $\ell > 0$ is fixed and $[X_p, Y_q]$ has been defined when $p + q < \ell$, $p, q \geq 0$, in such a way that the Jacobi identity (2.5) holds true when $r < 0$. For $p, q \geq 0$, $p + q = \ell$ and $Z \in \mathfrak{m}$ we define then

$$[X_p, Y_q](Z) = [[X_p, Z], Y_q] + [X_p, [Y_q, Z]].$$

One easily verifies that this yields $[X_p, Y_q] \in \mathfrak{g}_\ell$ and that the Jacobi identity (2.5) is satisfied when $r < 0$.

To show that $\mathfrak{g}$ is a Lie algebra we only need now to show that the Jacobi identity (2.5) is true without any assumption on p, q, r. The cases where $p + q + r \leq 0$ are obvious. We can argue then by induction on $p + q + r \geq 0$. Taking into account that the left hand side of (2.5) is an element of $\mathfrak{g}$ and that $\mathfrak{m}$ is fundamental, we reduce to the easy verification that the Lie product of this left hand side and of every element of $\mathfrak{g}_{-1}$ is zero. In this way we reduce to Jacobi identities homogeneous of degree $< p + q + r$.

The uniqueness is also clear, as for every transitive prolongation $\tilde{\mathfrak{g}}$ of $\mathfrak{m}$ an element $\tilde{X}_p \in \tilde{\mathfrak{g}}_p$ with $p \geq 0$ is completely determined by the homomomorphism

$$\mathfrak{m} \ni Z \to [\tilde{X}_p, Z] \in \bigoplus_{q<p} \tilde{\mathfrak{g}}_q$$

it defines. Arguing recursively on the degree, we obtain an injective morphism of Lie algebras

$$\tilde{\mathfrak{g}} \to \mathfrak{g}$$

which is an isomorphism by maximality. □

2.4 Definition of Levi-Tanaka algebras

Let $\mathfrak{m}$ be a fundamental graded Lie algebra of kind μ and let $\tilde{\mathfrak{m}}$ be its canonical prolongation, as constructed in the proof of Theorem 2.3.1. We fix a Lie subalgebra $\mathfrak{g}_0$ of the algebra $\tilde{\mathfrak{m}}_0$ of all derivations of degree 0 of $\mathfrak{m}$. Then we define the *canonical prolongation* of $\mathfrak{m} \oplus \mathfrak{g}_0$ setting by recurrence:

$$\mathfrak{g}_p = \{X_p \in \tilde{\mathfrak{m}}_p \mid [X_p, \mathfrak{g}_{-1}] \subset \mathfrak{g}_{p-1}\}.$$

This is a graded Lie subalgebra of $\tilde{\mathfrak{m}}$ and hence a transitive graded Lie algebra, maximal between the graded Lie algebras $\mathfrak{a}$ which are transitive and satisfy

$$\mathfrak{m} \oplus \mathfrak{g}_0 \simeq \bigoplus_{p \leq 0} \mathfrak{a}_p \quad \text{as graded Lie algebras.}$$

When $\mathfrak{m}$ is a pseudocomplex fundamental graded Lie algebra, we say that a prolongation $\mathfrak{a} = \oplus_{p \geq -\mu} \mathfrak{a}_p$ of $\mathfrak{m}$ is *pseudocomplex* if the elements of $\mathfrak{a}_0$ define derivations of degree 0 of $\mathfrak{m}$ which are $\mathbb{C}$-linear on $\mathfrak{m}_{-1}$ for the complex structure induced by J.

If we define $\mathfrak{g}_0$ to be the space of all 0-degree derivations of a pseudocomplex fundamental graded Lie algebra $\mathfrak{m}$ which are $\mathbb{C}$-linear on $\mathfrak{m}_{-1}$, we call the canonical prolongation of $\mathfrak{m} \oplus \mathfrak{g}_0$ the *canonical pseudocomplex prolongation* of $\mathfrak{m}$.

A graded Lie algebra $\mathfrak{g} = \oplus_{p \geq -\mu} \mathfrak{g}_p$ such that

(*i*) $\mathfrak{m} = \oplus_{-\mu \leq p < 0} \mathfrak{g}_p$ is a fundamental Lie algebra with a partial almost-complex structure J,

(*ii*) $\mathfrak{g}$ is the canonical pseudocomplex prolongation of $\mathfrak{m}$,

will be called a *Levi-Tanaka algebra.*

An *isomorphism* of Levi-Tanaka algebras $\mathfrak{g} = \oplus_{p \in \mathbb{Z}} \mathfrak{g}_p$ and $\mathfrak{g}' = \oplus_{p \in \mathbb{Z}} \mathfrak{g}'_p$, whose partial complex structures are denoted by J and J' respectively, is an isomorphism $\phi : \mathfrak{g} \to \mathfrak{g}'$ of graded Lie algebras such that $\phi(JX) = J'\phi(X)$ for every $X \in \mathfrak{g}_{-1}$.

We recall that, when $\mathfrak{m} = \mathfrak{m}(x)$ is the pseudocomplex fundamental graded Lie algebra associated to a point $x \in M$ of an almost CR manifold $\mathrm{M} = (M, HM, J)$, its canonical pseudocomplex prolongation $\mathfrak{g}(x)$ is called the *Levi-Tanaka algebra* of M at x.

We note that CR diffeomorphisms induce isomorphisms of the Levi-Tanaka algebras at corresponding points. In particular the Levi-Tanaka algebras -modulo isomorphisms- are pseudoconformal invariants.

2.5 Finiteness of the canonical prolongation

In this section we give a complete proof of a criterion for the finiteness of the canonical prolongation, which was given by Tanaka in [40], using a result of Serre contained in [13].

For this it is convenient to consider a slightly more general setting.

Let $\mathfrak{m} = \oplus_{-\mu \leq j < 0} \mathfrak{g}_j$ be a fundamental graded Lie algebra over a field $\mathbb{K}$ of characteristic 0. A *right* $\mathfrak{m}$-*module* is the datum of a $\mathbb{K}$-linear space E and of a bilinear map

$$E \times \mathfrak{m} \ni (v, X) \to vX \in E$$

such that

$$(vX)Y - (vY)X = v[X, Y] \quad \forall v \in E, \quad \forall X, Y \in \mathfrak{m}.$$

Note that a right $\mathfrak{m}$-module is in a natural way a right $\mathfrak{A}_{\mathfrak{m}}$-module, where $\mathfrak{A}_{\mathfrak{m}}$ is the universal enveloping algebra of $\mathfrak{m}$.

An $\mathfrak{m}$-graduation of E is a decomposition of E into a direct sum of $\mathbb{K}$-linear subspaces $E = \oplus_{p \in \mathbb{Z}} E_p$ such that

$$E_p \mathfrak{g}_j \subset E_{p+j} \qquad \forall p \in \mathbb{Z},\ -\mu \leq j < 0.$$

Given any subset F of a graded right $\mathfrak{m}$-module $E = \oplus_{p \in \mathbb{Z}} E_p$ we denote by F_p the subset $F \cap E_p$ of all elements of F which are homogeneous of degree p. If F is a $\mathbb{K}$-linear subspace of E, then all F_p are $\mathbb{K}$-linear subspaces of E.

An $\mathfrak{m}$-submodule F of E is called a graded $\mathfrak{m}$-submodule of E if $F = \oplus_{p \in \mathbb{Z}} F_p$.

An $\mathfrak{m}$-graduation on E is *good* if the subspaces E_p in the direct sum decomposition are all finite dimensional.

We say that a well graded right $\mathfrak{m}$-module E satisfies condition (C) if

$$(C) \qquad \begin{cases} i) & E_p = 0 \quad \text{for} \quad p < -\mu\,; \\ ii) & \text{if} \quad v \in E_p\,, \text{with}\, p \geq 0\,, \quad \text{and} \quad v\mathfrak{g}_{-1} = 0, \quad \text{then} \quad v = 0. \end{cases}$$

We associate to an element $v \in E_p$ for $p > 0$ the multilinear map:

$$\sigma(v) : \mathfrak{g}_{-1}^p \ni (X_1, \dots, X_p) \to vX_1 \cdots X_p \in E_0.$$

Then (ii) in condition (C) implies that the map $v \to \sigma(v)$ is injective from E_p into the space $\mathfrak{M}^p(\mathfrak{g}_{-1}, E_0)$ of $\mathbb{K}$-multilinear maps $\mathfrak{g}_{-1}^p \to E_0$.

Remark 2.5.1 *If $E = \oplus_{p\in\mathbb{Z}} E_p$ is a well graded right $\mathfrak{m}$-module satisfying condition (C), and, for some $p \geq 0$ we have $E_p = 0$, then $E_q = 0$ for all $q \geq p$.*

Remark 2.5.2 *If $E = \oplus_{p\in\mathbb{Z}} E_p$ is a graded right $\mathfrak{m}$-module, then the truncated $E_{|p_0} = \oplus_{p\leq p_0} E_p$ is also a graded right $\mathfrak{m}$-module for every $p_0 \in \mathbb{Z}$. It satisfies condition (C) when E satisfies condition (C).*

Remark 2.5.3 *The quotient of a well graded right $\mathfrak{m}$-module by a graded right $\mathfrak{m}$-submodule is, in a natural way, a well graded right $\mathfrak{m}$-module. In particular, this is the case for $E/E_{|p_0}$ with $p_0 \in \mathbb{Z}$. These $\mathfrak{m}$-modules will play an important role in the following discussion. However, it will be convenient, in order that condition (C) be kept while passing to the quotient, to redefine the graduation of these modules by a shift in the natural grading, as explained below.*

Remark 2.5.4 Definition of the right $\mathfrak{m}$-modules $E(q)$.

Let $E = \oplus_{p\in\mathbb{Z}} E_p$ be a graded right $\mathfrak{m}$-module. Let us fix an integer $q \geq 0$. We denote by $E(q)$ the right $\mathfrak{m}$-module $E/E_{|q-\mu-1}$ endowed with the following graduation: if $\pi : E \to E(q)$ is the projection onto the quotient, we set $E(q)_p = \pi(E_{q+p})$. The $\mathbb{K}$-linear isomorphism $\oplus_{p\geq q-\mu} E_p \to E(q)$ identifies E_{q+p} to $E(q)_p$ for $p \geq -\mu$. With this identification in mind, we can define the right $\mathfrak{m}$-module structure of $E(q)$ by

$$vX = \begin{cases} vX & \text{if } v \in E_h,\ X \in \mathfrak{g}_p,\ h \geq q-\mu,\ -\mu \leq p < 0,\ h+p \geq q-\mu \\ 0 & \text{if } v \in E_h,\ X \in \mathfrak{g}_p,\ h \geq q-\mu,\ -\mu \leq p < 0,\ h+p < q-\mu. \end{cases}$$

We note that $E(q)_p = 0$ for $p < -\mu$ and $E(q)$ is well graded when E is well graded. Moreover, $E(q)$ satisfies condition (C) if E satisfies condition (C).

A $\mathbb{K}$-linear vector space M, together with a bilinear map $\mathfrak{m} \times M \ni (X, a) \to Xa \in M$ such that

$$X(Ya) - Y(Xa) = [X, Y]a \quad \forall X, Y \in \mathfrak{m}, \quad \forall a \in M$$

is called a *left* $\mathfrak{m}$*-module.*

An $\mathfrak{m}$-graduation of a left $\mathfrak{m}$-module M is a decomposition: $M = \oplus_{p \in \mathbb{Z}} M^p$ of M into a direct sum of $\mathbb{K}$-linear subspaces with the property that

$$\mathfrak{g}_j M^p \subset M^{p-j} \quad \text{for} \quad -\mu \leq j < 0, \quad p \in \mathbb{Z}.$$

The $\mathfrak{m}$-graduation is *good* if the subspaces M^p are finite dimensional. Dual to condition (C) for right $\mathfrak{m}$-modules, we introduce condition (C') for well graded left $\mathfrak{m}$-modules:

$$(C') \qquad \begin{cases} i)' & M^p = 0 & \text{for } p < -\mu \\ ii)' & M^p = \mathfrak{g}_{-1} M^{p-1} & \text{for } p \geq 0. \end{cases}$$

We note that every left $\mathfrak{m}$-module M can also be considered as a left $\mathfrak{A}_{\mathfrak{m}}$-module. Condition (C') implies that M is a left $\mathfrak{A}_{\mathfrak{m}}$-module M of finite type.

Let $M = \oplus_{p \in \mathbb{Z}} M^p$ be a graded left $\mathfrak{m}$-module and let $q \geq 0$ be fixed. We define $M(q)^p = M^{p+q}$. Then

$$M(q) = \oplus_{p \geq -\mu} M(q)^p$$

is a graded left $\mathfrak{m}$-module, which is obviously well graded when M is, and satisfies condition (C') when M does.

The relationship between right and left $\mathfrak{m}$-modules is explained by the following:

LEMMA 2.5.5 *Let $E = \oplus_{p \in \mathbb{Z}} E_p$ be a well graded right $\mathfrak{m}$-module. Denote by M the subspace of its algebraic dual E' consisting of all linear functionals $a : E \to \mathbb{K}$ such that $a(E_p) = 0$ for $|p|$ sufficiently large. Then M has a unique structure of well graded left $\mathfrak{m}$-module $M = \oplus_{p \in \mathbb{Z}} M^p$ such that*

$$\langle vX, a \rangle = \langle v, Xa \rangle \quad \forall v \in E,\ X \in \mathfrak{m},\ a \in M$$

and

$$\langle v, a \rangle = 0 \quad \textit{for} \quad v \in E_p,\ a \in M^q \quad \textit{and} \quad p \neq q.$$

Moreover, E satisfies condition (C) if and only if M satisfies condition (C').

We call M the dual graded left $\mathfrak{m}$-module of the graded right $\mathfrak{m}$-module E. For every integer $q \geq 0$, $M(q)$ is the dual left $\mathfrak{m}$-module of the right $\mathfrak{m}$-module $E(q)$.
The verification of the statements of the lemma is straightforward.

We will prove the following:

THEOREM 2.5.6 *Let $\mathfrak{m} = \oplus_{-\mu \leq p < 0} \mathfrak{g}_p$ be a fundamental graded Lie algebra over a field $\mathbb{K}$ of characteristic 0 and let $E = \oplus_{p \geq -\mu} E_p$ be a well graded right $\mathfrak{m}$-module satisfying condition (C). Let:*

$$H(E) = \{v \in E \mid vX = 0 \quad \forall X \in \oplus_{p<-1} \mathfrak{g}_p\}.$$

Then a necessary and sufficient condition in order that E be a finite dimensional $\mathbb{K}$-linear space is that $H(E)$ be finite dimensional.

The proof of this theorem requires several lemmas. It is based upon an inductive argument on the kind μ of the fundamental Lie algebra $\mathfrak{m}$. Therefore we start by considering the structure of a well graded right $\mathfrak{m}$-module satisfying condition (C) in the case where $\mathfrak{m}$ is of the first kind. Then $\mathfrak{m}$ is a finite dimensional vector space of finite dimension n over $\mathbb{K}$, with the trivial structure of a commutative Lie algebra and its universal enveloping algebra $\mathfrak{A}_{\mathfrak{m}}$ is isomorphic to the ring $\mathcal{P} = \mathbb{K}[z_1, \ldots, z_n]$ of polynomials in n indeterminates over the field $\mathbb{K}$, with the natural graduation which identifies the elements of $\mathfrak{m}$ to the homogeneous polynomials of degree 1 in $\mathcal{P}$. The following lemma, due to Serre (cf. [13]), deals with the Koszul complex of commutative algebra (see, for instance, [36] or [22]).

LEMMA 2.5.7 *Let $\mathfrak{m}$ be a fundamental Lie algebra of the first kind. Let $E = \oplus_{p \in \mathbb{Z}} E_p$ be a well graded right $\mathfrak{m}$-module satisfying condition (C). Then we can find $p_0 \geq 0$ such that*

1. *$E(p_0)$ is the classical prolongation of $(E(p_0)_{-1} \oplus E(p_0)_0)$: this means that, for every $p > 0$, the map*

   $$E(p_0)_p \ni v \to \sigma(v) \in \mathfrak{M}^p(\mathfrak{m}, E(p_0)_0)$$

   defines an isomorphism of $E(p_0)_p$ onto the space $E(p_0)^{(p)}$ of symmetric multilinear maps $s : \mathfrak{m}^p \to E(p_0)_0$ such that

   $$\mathfrak{m}^{p+1} \ni (Y_0, Y_1, \ldots, Y_p) \to s(Y_1, \ldots, Y_p)Y_0 \in E(p_0)_{-1}$$

   is also symmetric.

2. *We can find a basis* $X_1, \dots, X_n$ *of* $\mathfrak{m}$ *such that, setting*

$$E_p{}^j = \{A \in E_p \,|\, AX_h = 0 \quad \textit{for} \quad h = 1, \dots, j\}$$

the maps $E_p{}^j \ni A \to AX_{j+1} \in E_{p-1}{}^j$ *are surjective for* $p > p_0$ *and* $j = 0, 1, \dots, n-1$.

Proof. Denote by $\tilde{\mathfrak{m}}$ the ideal of $\mathcal{P}$ generated by $\mathfrak{m}$ and let M be the dual left $\mathfrak{m}$-module of E, with the structure of well graded left $\mathfrak{m}$-module satisfying (C') given by Lemma 2.5.5.

Then M is in a natural way a left graded $\mathcal{P}$-module. By condition (C') it is a $\mathcal{P}$-module of finite type. We consider the Koszul complex associated to the $\mathcal{P}$-module M:

$$0 \to M^{(n)} \to M^{(n-1)} \to \dots \to M^{(1)} \to M^{(0)} \to 0.$$

Its homology groups $H_i(M)$ are isomorphic to $Tor_i^{\mathcal{P}}(M, \mathcal{P}/\tilde{\mathfrak{m}})$.

They are therefore finite dimensional vector spaces over $\mathbb{K}$.

Indeed they could also have been computed starting from a Hilbert resolution of the $\mathcal{P}$-module M, of minimal lenght $d \leq n$:

$$0 \to L_d \to L_{d-1} \to \dots \to L_1 \to L_0 \to M \to 0,$$

where the L_j's are free $\mathcal{P}$-modules of finite type: the groups $H_i(M)$ are therefore isomorphic to the homology groups of the complex:

$$0 \to \frac{\mathcal{P}}{\tilde{\mathfrak{m}}} \otimes_{\mathcal{P}} L_d \to \frac{\mathcal{P}}{\tilde{\mathfrak{m}}} \otimes_{\mathcal{P}} L_{d-1} \to \dots \to \frac{\mathcal{P}}{\tilde{\mathfrak{m}}} \otimes_{\mathcal{P}} L_1 \to \frac{\mathcal{P}}{\tilde{\mathfrak{m}}} \otimes_{\mathcal{P}} L_0 \to 0.$$

Since $\mathcal{P}/\tilde{\mathfrak{m}} \simeq \mathbb{K}$, this one is a complex of finite dimensional linear spaces over $\mathbb{K}$ and $\mathbb{K}$-linear maps. In particular the groups $H_i(M)$ are finite dimensional vector spaces and are 0 when $i > d$.

The graduation of M as a $\mathcal{P}$-module yields a graduation of the groups $H_i(M)$. For every integer q we denote by $H_i(M)^q$ the set of homology classes that are homogeneous of degree q. Since $\oplus_{0 \leq i \leq n} H_i(M)$ is a finite dimensional vector space, we can find $q_0 \in \mathbb{Z}$ such that

$$H_i(M)^q = 0 \quad \text{for every} \quad i \quad \text{and} \quad q \geq q_0.$$

Let $M(q) = \oplus_{p\geq -1} M^{q+p}$. By Lemma 2.5.5, $M(q)$, with the graduation $M(q)^p = M^{q+p}$, is also a well graded left $\mathfrak{m}$-module satisfying condition (C'). Since the differentials in the Koszul complex are homogeneous maps of degree 1, one obtains

$$H_i(M(q))^1 = H_i(M)^{q+1} \quad \text{for every} \quad i \quad \text{and} \quad q \geq 0.$$

We already noted that $M(q)$ is the dual $\mathfrak{m}$-module of $E(q)$.

In the statement of the lemma, we can as well substitute the $\mathfrak{m}$-module E by anyone of the modules $E(q)$ for $q \geq 0$. Thus in the proof we can assume for simplicity that $M = M(q_0)$, so that

$$H_i(M)^+ := \bigoplus_{q\geq 0} H_i(M)^q = 0.$$

From $H_n(M)^+ = 0$ we obtain that the set of ideals of $\mathcal{P}$

$$\{\mathrm{Ann}(a) \mid a \in M^+ = \oplus_{q\geq 0} M^q\}$$

does not contain $\tilde{\mathfrak{m}}$. This set of ideals is finite, because M^+ is a $\mathcal{P}$-module of finite type. Therefore, by choosing $X_1 \in \mathfrak{m}$ outside the union of a finite set of linear subspaces of $\mathfrak{m}$, the map

$$M(0)^+ \ni a \to X_1 a \in M(1)^+$$

is injective. Let W be a linear complement of $\ker(X_1 : M^{-1} \to M^0)$ in M^{-1}. Then $\tilde{M} = W \oplus M^+$ is a graded $\mathcal{P}$-module and $X_1\tilde{M} = X_1 M$, so that we have a short exact sequence of $\mathcal{P}$-modules:

$$0 \to \tilde{M} \xrightarrow{X_1 \cdot} M \to M/X_1 M \to 0,$$

from which we deduce the long exact sequence:

$$0 \to H_n(\tilde{M}) \to H_n(M) \to H_n(M/X_1 M) \to H_{n-1}(\tilde{M}) \to \dots$$

From the assumptions on M we obtain:

$$H_n(\tilde{M})^+ \simeq H_n(M)^+ = 0 \quad \text{and} \quad H_{i+1}(M/X_1 M)^+ \simeq H_i(\tilde{M})^+$$

for $0 \leq i \leq n-1$.

Next we show that $\tilde{M}$ satisfies condition (C'). To this aim, it suffices to verify that $\mathfrak{m}\tilde{M}^{-1} = M^0$. Let us complete X_1 to a basis $X_1, X_2, \ldots, X_n$ of $\mathfrak{m}$ as a linear space over $\mathbb{K}$. Given $a \in M^0$, by condition (C') for M there are $b_1, b_2, \ldots, b_n \in M^{-1}$ such that

$$a = X_1 b_1 + X_2 b_2 + \ldots + X_n b_n.$$

We set $b_i = w_i + z_i$ with $w_i \in W$ and $z_i \in \ker(X_1 : M^{-1} \to M^0)$. Then we obtain

$$a = X_1 w_1 + X_2 w_2 + \ldots + X_n w_n + X_2 z_2 + \ldots + X_n z_n.$$

But $X_2 z_2 + \ldots + X_n z_n \in M^0$ and $X_1(X_2 z_2 + \ldots + X_n z_n) = 0$ shows that $X_2 z_2 + \ldots + X_n z_n = 0$ and therefore proves our contention. By substituting $M(q)$ to M for some positive q we obtain then that also $H_i(\tilde{M})^+ = 0$ for every i. In particular, $H_n(M/X_1 M)^+ = 0$. Note now that $M/X_1 M$ is in a natural way a well graded left $\mathfrak{m}'$-module satisfying condition (C') for the fundamental graded Lie algebra of the first kind μ' generated by $X_2, \ldots, X_n$. Arguing by recurrence, we obtain then a basis $X_1, \ldots, X_n$ of $\mathfrak{m}$ (with perhaps different $X_2, \ldots, X_n$) such that, when $M = M(q)$ for some positive q, the maps

$$\frac{M^+}{(X_1, \ldots, X_j)M} \ni \alpha \to X_{j+1}\alpha \in \frac{M^+}{(X_1, \ldots, X_j)M}$$

are injective for $j = 0, \ldots, n-1$. This yields by duality the second statement of the lemma. From this we deduce the first.

Condition (C) tells us that the correspondence $\sigma : E_p \to E^{(p)}$ is injective when $p > 0$. To prove that it is also surjective, we argue by induction. When $p = 1$, an element $s \in E^{(1)}$ defines n elements $A_i = s(X_i) \in E_0$ with the property:

$$A_i X_j = A_j X_i \quad \forall 1 \leq i, j \leq n.$$

By property 2, we can find $B_1 \in E_1$ such that

$$A_1 = B_1 X_1.$$

Then $A'_i = A_i - B_1 X_i \in {E_0}^1$ and

$$A'_i X_j - A'_j X_i = A_i X_j - A_j X_i - B_1 X_i X_j + B_1 X_i X_j = 0 \quad \text{for } 1 \leq i, j \leq n.$$

In particular, we can find $B_2 \in {E_1}^1$ with

$$A_2' = B_2 X_2.$$

Repeating this argument, we find $B_i \in {E_1}^{i-1}$ for $i = 1, \ldots, n$ such that

$$A_i - (B_1 + \ldots + B_{i-1})X_i = B_i X_i \quad \text{for} \quad i = 1, \ldots, n.$$

Taking into account that $B_i X_j = 0$ when $j < i$, for $B = B_1 + \ldots + B_n$ we obtain

$$A_i = BX_i \quad \text{for} \quad i = 1, \ldots, n$$

and hence $s = \sigma(B)$. Assume now that $p > 1$ and $\sigma : E_{p-1} \to E^{(p-1)}$ is surjective. Let $s \in E^{(p)}$. Then we obtain n maps

$$s_i : \mathfrak{m}^{p-1} \to E_0$$

by

$$s_i(Y_2, \ldots, Y_p) = s(X_i, Y_2, \ldots, Y_p) \quad \text{for} \quad Y_2, \ldots, Y_p \in \mathfrak{m}.$$

Clearly $s_i \in E^{(p-1)}$ for $i = 1, \ldots, n$. By the inductive assumption we can find $A_1, \ldots, A_n \in E_{p-1}$ such that $s_i = \sigma(A_i)$ for $i = 1, \ldots, n$. By condition (C) we have

$$A_i X_j = A_j X_i \quad \text{for} \quad 1 \leq i, j \leq n.$$

The same argument used in the case $p = 1$ gives then that

$$A_i = BX_i \quad \text{for} \quad i = 1, \ldots, n$$

for some $B \in E_p$. Then

$$s(Y_1, \ldots, Y_p) = BY_1 \cdots Y_p \quad \text{for} \quad Y_1, \ldots, Y_p \in \mathfrak{m}$$

and $s = \sigma(B)$, which completes the proof. □

LEMMA 2.5.8 *Let $\mathfrak{m} = \oplus_{-\mu \leq p < 0} \mathfrak{m}_p$ be a fundamental Lie algebra of kind $\mu > 1$ and $M = \oplus_{p \in \mathbb{Z}} M^p$ a well graded left $\mathfrak{m}$-module satisfying condition (C'). Let $U = [V, W] \in \mathfrak{g}_{-\mu}$ for $V \in \mathfrak{g}_{-1}$ and $W \in \mathfrak{g}_{-\mu+1}$ be such that the maps*

$$M^p \ni a \to Ua \in M^{p+\mu}$$

be injective for $p \geq 0$. Then we can find $t \in \mathbb{K}$ such that

$$M^p \ni a \to (V^{\mu-1} + tW)a \in M^{p+\mu-1}$$

is injective for $p \geq 0$.

Proof. Note that either the maps $V^{\mu-1}+tW$ are never injective, or the set of $t \in \mathbb{K}$ for which they are not is finite. Indeed, U commutes with every element of $\mathfrak{m}$ and therefore, with all linear operators on M corresponding to elements of the universal enveloping algebra $\mathfrak{A}_{\mathfrak{m}}$ of $\mathfrak{m}$. Therefore it suffices to prove the injectivity of $M^p \ni a \to (V^{\mu-1}+tW)a \in M^{p+\mu-1}$ when $0 \leq p \leq \mu-1$. For simplicity, we can restrict to the proof of the injectivity in the case $p=0$.

Assuming by contradiction that

$$V^{\mu-1}+tW : M^0 \to M^{\mu-1}$$

is never injective for $t \in \mathbb{K}$, we can find a polynomial

$$f(t) = a_0 + a_1 t + \ldots + a_m t^m, \quad \text{with } a_0, \ldots, a_m \in M^0 \quad \text{and} \quad a_0,\, a_m \neq 0,$$

such that

$$(V^{\mu-1}+tW)f(t) = 0 \quad \forall t \in \mathbb{K}.$$

Indeed, if the map

$$(V^{\mu-1}+tW) : \mathbb{K}[t] \otimes M^0 \to \mathbb{K}[t] \otimes M^{\mu-1}$$

is injective, then $V^{\mu-1}+tW : M^0 \to M^{\mu-1}$ is injective for all, but a finite number of $t \in \mathbb{K}$.

With $a_{-1} = a_{m+1} = 0$ we obtain the set of equalities:

$$V^{\mu-1}a_i + Wa_{i-1} = 0 \quad \text{for} \quad i = 0, \ldots, m+1.$$

We have

$$V^j \circ W = W \circ V^j + j\,U \circ V^{j-1} \quad \forall j \geq 1.$$

Indeed, for $j=1$ this is the definition of U. Assuming the formula holds true for some fixed $j \geq 1$, we obtain:

$$\begin{aligned} V^{j+1} \circ W &= V \circ (W \circ V^j + jU \circ V^{j-1}) \\ &= U \circ V^j + W \circ V^{j+1} + jV \circ U \circ V^{j-1} \\ &= W \circ V^{j+1} + (j+1)U \circ V^j \end{aligned}$$

since U and V commute as linear operators on M.

Moreover, we have

$$V^{(j+1)\mu-1}a_j = 0 \qquad \forall j = 0, \dots, m.$$

This follows directly from the equalities above for $j = 0$, while the formula of recurrence:

$$\begin{aligned} 0 &= V^{j\mu}(V^{\mu-1}a_j + Wa_{j-1}) \\ &= V^{(j+1)\mu-1}a_j + W \circ V^{j\mu}a_{j-1} + j\mu U \circ V^{j\mu-1}a_{j-1} \end{aligned}$$

shows that it holds for every $j = 0, \dots, m$. Therefore, there is a minimum integer $h \geq 0$ such that $V^h a_m = 0$. If $h > 0$, we have

$$hU \circ V^{h-1}a_m = V^h \circ W\, a_m - W \circ V^h\, a_m = V^h \circ W\, a_m = 0.$$

Therefore $h = 0$ and hence $a_m = 0$, contrary to the assumptions. □

LEMMA 2.5.9 *Let* $\mathfrak{m}$, $U = [V, W]$ *and* M *be as in the statement of Lemma 2.5.8. If moreover* $U : M^p \to M^{p+\mu}$ *is an isomorphism for every* $p \geq 0$, *then* $M^p = 0$ *for every* $p \geq 0$.

Proof. By Lemma 2.5.8, we obtain

$$\dim_{\mathbb{K}} M^p \leq \dim_{\mathbb{K}} M^{p+\mu-1} \quad \text{for} \quad p \geq 0\,,$$

while by the assumption

$$\dim_{\mathbb{K}} M^p = \dim_{\mathbb{K}} M^{p+\mu} \quad \text{for} \quad p \geq 0\,.$$

Hence we have:

$$\dim_{\mathbb{K}} M^0 \leq \dim_{\mathbb{K}} M^{\mu-1} \leq \dim_{\mathbb{K}} M^{2\mu-2} = \dim_{\mathbb{K}} M^{\mu-2} \leq \dots \leq \dim_{\mathbb{K}} M^0$$

and this shows that $\dim_{\mathbb{K}} M^p = d$ is constant for $p \geq 0$. We write $U^{(p)}$ for the linear isomorphism $M^p \to M^{p+\mu}$ defined by U (for $p \geq 0$). Analogously we write $V^{(p)}$ for the linear map $M^p \to M^{p+1}$ defined by V and $W^{(p)}$ for the

linear map $M^p \to M^{p+\mu-1}$ defined by W. For $p \geq 0$ we identify $M^{p+\mu}$ to M^p by the linear isomorphism $(U^{(p)})^{-1}$. We consider then the maps

$$\begin{cases} \tilde{W}^{(0)} = W^{(0)} : M^0 \to M^{\mu-1} \\ \tilde{W}^{(1)} = (U^{(0)})^{-1} \circ W^{(1)} : M^1 \to M^0 \\ \dots \\ \tilde{W}^{(\mu-1)} = (U^{(\mu-2)})^{-1} \circ W^{(\mu-1)} : M^{\mu-1} \to M^{\mu-2} \end{cases}$$

and

$$\begin{cases} \tilde{V}^{(p)} = V^{(p)} : M^p \to M^{p+1} \quad \text{for} \quad 0 \leq p \leq \mu - 2 \\ \tilde{V}^{(\mu-1)} = (U^{(0)})^{-1} \circ V^{(\mu-1)} : M^{\mu-1} \to M^0. \end{cases}$$

Since U commutes with V and W, the following relations hold true:

$$\begin{gathered} \tilde{V}^{(\mu-1)} \circ \tilde{W}^{(0)} - \tilde{W}^{(1)} \circ \tilde{V}^{(0)} = I_{M^0} \\ \vdots \\ \tilde{V}^{(j-1)} \circ \tilde{W}^{(j)} - \tilde{W}^{(j+1)} \circ \tilde{V}^{(j)} = I_{M^j} \\ \vdots \\ \tilde{V}^{(\mu-2)} \circ \tilde{W}^{(\mu-1)} - \tilde{W}^{(0)} \circ \tilde{V}^{(\mu-1)} = I_{M^{\mu-1}}. \end{gathered}$$

Let $\tilde{M} = M^0 \oplus \ldots \oplus M^{\mu-1}$ and

$$\begin{gathered} \tilde{V} = \tilde{V}^{(0)} \oplus \ldots \oplus \tilde{V}^{(\mu-1)} : \tilde{M} \to \tilde{M} \\ \tilde{W} = \tilde{W}^{(0)} \oplus \ldots \oplus \tilde{W}^{(\mu-1)} : \tilde{M} \to \tilde{M}. \end{gathered}$$

Then the relations above can be written in the form:

$$[\tilde{V}, \tilde{W}] = \tilde{V} \circ \tilde{W} - \tilde{W} \circ \tilde{V} = I_{\tilde{M}}.$$

The trace of the endomorphism in the left hand side is 0, while the trace of the one in the right hand side is $\mu\, d$. Thus $d = 0$ and the lemma is proved. □

From this we derive the dual statement:

LEMMA 2.5.10 *Let $E = \oplus_{p \geq -\mu} E_p$ be a well graded right $\mathfrak{m}$-module satisfying condition (C) and let $U = [V, W] \in \mathfrak{g}_{-\mu}$ with $V \in \mathfrak{g}_{-1}$, $W \in \mathfrak{g}_{1-\mu}$ be such that the linear maps:*

$$U^{(p)} : E_p \ni v \to vU \in E_{p-\mu}$$

be isomorphisms for $p \geq \mu$. Then $E_p = 0$ for every $p \geq 0$.

Proof (of Theorem 2.5.6). The condition that $H(E)$ be finite dimensional is obviously necessary in order that E be finite dimensional.

To prove the sufficiency, we argue by induction on μ. When $\mu = 1$, then $E = H(E)$ and there is nothing to prove. Assume then that $\mu > 1$ and that the statement has already been proved for fundamental graded Lie algebras of kind $< \mu$. Set

$$E' = \{v \in E \mid v\mathfrak{g}_{-\mu} = 0\}/E_{-\mu}.$$

Then E' is a graded right $\mathfrak{m}/\mathfrak{g}_{-\mu}$-module for the fundamental graded Lie algebra $\mathfrak{m}/\mathfrak{g}_{-\mu}$ of kind $\mu-1$, satisfying condition (C). Because $H(E') = H(E)$ as vector spaces, it follows that E' is finite dimensional by the inductive assumption. In particular, we can find an integer q such that $E'_p = 0$ for $p \geq q$.

Substituting $E(q)$ for E, we can as well assume that $E'_p = 0$ for $p \geq 0$. Then we split E into a direct sum of right $\mathfrak{g}_{-\mu}$-modules:

$$E^{(j)} = \oplus_{p \geq -1} E_{j+p\mu} \quad \text{for} \quad j = 0, \ldots, \mu - 1.$$

We consider $\mathfrak{g}_{-\mu}$ as a fundamental graded Lie algebra of the first kind. By the way they are defined, the $E^{(j)}$'s satisfy condition (C).

By Lemma 2.4, we can find $p_0 \in \mathbb{Z}$ and a basis $U_1, \ldots, U_s$ of $\mathfrak{g}_{-\mu}$ such that, for

$$\tilde{E}^h_p = \{v \in E_p \mid vU_1 = \ldots = vU_h = 0\}, \quad p \in \mathbb{Z}, \quad h = 0, \ldots, s-1$$

the maps

$$\tilde{E}^h_p \ni v \to vU_{h+1} \in \tilde{E}^h_{p-\mu}$$

are surjective for $p \geq p_0$ and $h = 0, 1, \ldots, s-1$. Again we can assume for simplicity that $p_0 = 0$ (this corresponds to substitute E by $E(q)$ with a larger q).

Then we first apply Lemma 2.5.10 to $\tilde{E}^{s-1} = \oplus_{p \geq -\mu} \tilde{E}^{s-1}_p$. Since by the definition of a fundamental graded Lie algebra $\mathfrak{g}_{-\mu} = [\mathfrak{g}_{-1}, \mathfrak{g}_{1-\mu}]$, we can find $V \in \mathfrak{g}_{-1}$ and $W \in \mathfrak{g}_{1-\mu}$ such that $U = [V, W]$ is equal to U_s modulo an element of the subspace of $\mathfrak{g}_{-\mu}$ generated by $U_1, \ldots, U_{s-1}$. Then $U = U_s$ as a linear operator on $\tilde{E}^{s-1}$ defines isomorphisms $\tilde{E}^{s-1}_p \to \tilde{E}^{s-1}_{p-\mu}$ for $p \geq 0$. Lemma 2.5.10 implies that $\tilde{E}^{s-1}_p = 0$ for $p \geq 0$. This shows that U_{s-1} is injective as a linear operator $\tilde{E}^{s-2}_p \to \tilde{E}^{s-2}_{p-\mu}$ for $p \geq 0$. By the choice of

$U_1, \ldots, U_s$ it is then an isomorphism for $p \geq 0$ and then Lemma 2.5.10 yields $\tilde{E}_p^{s-2} = 0$ for $p \geq 0$. By recurrence we obtain the thesis. $\square$

As a consequence of Theorem 2.5.6 we obtain the following criterion:

THEOREM 2.5.11 *Let $\mathfrak{g} = \oplus_{p \geq -\mu} \mathfrak{g}_p$ be a transitive prolongation of a fundamental graded Lie algebra of kind μ and let*

$$H(\mathfrak{g}) = \{X \in \mathfrak{g} \,|\, [X, Y] = 0 \quad \forall Y \in \oplus_{p<-1} \mathfrak{g}_p\}.$$

Then $\mathfrak{g}$ is finite dimensional if and only if $H(\mathfrak{g})$ is finite dimensional.

Chapter 3

Levi-Tanaka algebras

3.1 Canonical pseudocomplex prolongations

The finiteness criterion given by Theorem 2.5.11 yields in the pseudocomplex case:

THEOREM 3.1.1 *Let* $\mathfrak{m} = \oplus_{-\mu \leq p \leq -1} \mathfrak{m}_p$ *be a pseudocomplex fundamental graded Lie algebra. The canonical pseudocomplex prolongation* $\mathfrak{g} = \oplus_{p \geq -\mu} \mathfrak{g}_p$ *of* $\mathfrak{m}$ *is finite dimensional if and only if* $\mathfrak{m}$ *is nondegenerate, i.e.*

$$\{X \in \mathfrak{g}_{-1} \,|\, [X, \mathfrak{g}_{-1}] = 0\} = 0.$$

A necessary and sufficient condition in order that $\mathfrak{g}$ *be finite dimensional is that*

$$\{X \in \mathfrak{g}_1 \,|\, [X, Y] = 0 \quad \forall Y \in \oplus_{p<-1} \mathfrak{g}_p\} = 0. \tag{3.1}$$

Proof. Let $\mathfrak{n} = \oplus_{p<-1} \mathfrak{g}_p$ and let $\mathfrak{h}$ denote the graded Lie subalgebra of $\mathfrak{g}$ defined by $\mathfrak{h} = \{X \in \mathfrak{g} \,|\, [X, \mathfrak{n}] = 0\}$.

Assume that $\mathfrak{m}$ is degenerate, i.e. there is $0 \neq X \in \mathfrak{g}_{-1}$ such that $[X, Y] = 0$ for every $Y \in \mathfrak{g}_{-1}$. Let $\mathfrak{g}'_{-1}$ be a J-invariant subspace of $\mathfrak{g}_{-1}$ complementary to the subspace $\mathfrak{g}''_{-1}$ generated by X and JX. Then we define $Y_0 \in \mathfrak{h}_0 = \mathfrak{h} \cap \mathfrak{g}_0 \subset \mathfrak{g}_0$ by

$$\begin{cases} [Y_0, Z] = 0 & \text{for} \quad Z \in \mathfrak{g}'_{-1} \oplus \mathfrak{n} \\ [Y_0, Z] = Z & \text{for} \quad Z \in \mathfrak{g}''_{-1}. \end{cases}$$

We note that $Y_0 \in \mathfrak{h}_0$ and that also the element $\tilde{Y}_0$, defined by

$$\begin{cases} [\tilde{Y}_0, Z] = 0 \quad \text{for} \quad Z \in \mathfrak{g}'_{-1} \oplus \mathfrak{n} \\ [\tilde{Y}_0, Z] = JZ \quad \text{for} \quad Z \in \mathfrak{g}''_{-1} \end{cases}$$

belongs to $\mathfrak{h}_0 \subset \mathfrak{g}_0$. By recurrence we can define sequences $\{Y_p\}_{p\geq 0}$, $\{\tilde{Y}_p\}_{p\geq 0}$, with $0 \neq Y_p, \tilde{Y}_p \in \mathfrak{h}_p \subset \mathfrak{g}_p$ by setting, for $p \geq 1$,

$$\begin{cases} [Y_p, Z] = 0 \quad \text{for} \quad Z \in \mathfrak{g}'_{-1} \oplus \mathfrak{n} = 0 \\ [Y_p, X] = Y_{p-1} \\ [Y_p, JX] = \tilde{Y}_{p-1} \end{cases}$$

$$\begin{cases} [\tilde{Y}_p, Z] = 0 \quad \text{for} \quad Z \in \mathfrak{g}'_{-1} \oplus \mathfrak{n} \\ [\tilde{Y}_p, X] = \tilde{Y}_{p-1} \\ [\tilde{Y}_p, JX] = -Y_{p-1}\,. \end{cases}$$

This shows that $\mathfrak{g}$ is infinite dimensional.

Conversely, when $\mathfrak{m}$ is nondegenerate, we prove that $\mathfrak{h}_1 = \mathfrak{h} \cap \mathfrak{g}_1 = 0$ and the criterion applies. Indeed, let us consider the Hermitian symmetric $\mathbb{C} \otimes \mathfrak{g}_2$-valued form

$$(X|Y) = [JX, Y] + \sqrt{-1}[X, Y] \quad \text{for} \quad X, Y \in \mathfrak{g}_{-1}.$$

Let $\xi \in \mathfrak{g}_1$ such that $[\xi, \mathfrak{g}_{-2}] = [\xi, \mathfrak{g}_{-3}] = 0$, and denote by $B : \mathfrak{g}_{-1} \to \mathfrak{g}_0$ the corresponding $\mathbb{R}$-linear map. Then we have

$$\begin{cases} B(X)Y = B(Y)X \quad \forall X, Y \in \mathfrak{g}_{-1} \\ (B(X)Y|Z) + (Y|B(X)Z) = 0 \quad \forall X, Y, Z \in \mathfrak{g}_{-1}. \end{cases}$$

From these we obtain:

$$\begin{aligned} (B(X)Y|Z) &= (B(Y)X|Z) = -(X|B(Y)Z) = -(X|B(Z)Y) \\ &= (B(Z)X|Y) = (B(X)Z|Y) = -(Z|B(X)Y) \\ &= -\overline{(B(X)Y|Z)} \quad \forall X, Y, Z \in \mathfrak{g}_{-1}. \end{aligned}$$

This shows that

$$\Re(B(X)Y|Z) = 0 \quad \forall X, Y, Z \in \mathfrak{g}_{-1}$$

and hence $B = 0$, which gives $\xi = 0$. □

Remark 3.1.2 *In the proof of the previous theorem we showed that condition (3.1) is equivalent to*

$$\{X \in \mathfrak{g}_1 \mid [X, \mathfrak{g}_{-2}] = [X, \mathfrak{g}_{-3}] = 0\} = 0. \tag{3.2}$$

PROPOSITION 3.1.3 *Assume that the pseudocomplex graded Lie algebra* $\mathfrak{m} = \oplus_{-\mu \leq p < 0} \mathfrak{m}_p$ *is the direct sum of two pseudocomplex graded ideals* $\mathfrak{a} = \oplus_{\mu \leq p < 0} \mathfrak{a}_p$ *and* $\mathfrak{b} = \oplus_{-\mu \leq p < 0} \mathfrak{b}_p$. *Then:*

(i) $\mathfrak{m}$ *is fundamental if and only if* $\mathfrak{a}$ *and* $\mathfrak{b}$ *are both fundamental;*

(ii) $\mathfrak{m}$ *is nondegenerate if and only if* $\mathfrak{a}$ *and* $\mathfrak{b}$ *are both nondegenerate;*

(iii) if $\mathfrak{m}$ *is fundamental and nondegenerate, then its canonical pseudocomplex prolongation* $\mathfrak{g}(\mathfrak{m})$ *is isomorphic to the direct sum of the canonical pseudocomplex prolongations* $\mathfrak{g}(\mathfrak{a})$ *and* $\mathfrak{g}(\mathfrak{b})$ *of* $\mathfrak{a}$ *and* $\mathfrak{b}$ *respectively.*

In particular, the direct sum of two finite dimensional Levi-Tanaka algebras is a Levi-Tanaka algebra.

Proof. Statements *(i)* and *(ii)* are trivial, as $[\mathfrak{a}, \mathfrak{b}] = 0$. Let us prove *(iii)*. Let $\mathfrak{A}$ and $\mathfrak{B}$ be respectively the canonical pseudocomplex prolongations of $\mathfrak{a}$ and $\mathfrak{b}$. We note that the direct sum of Lie algebras $\mathfrak{A} \oplus \mathfrak{B}$ is a pseudocomplex prolongation of $\mathfrak{m}$ which is transitive and therefore there is a Lie algebras monomorphism

$$\mathfrak{A} \oplus \mathfrak{B} \to \mathfrak{g},$$

where $\mathfrak{g}$ is the canonical pseudocomplex prolongation of $\mathfrak{m}$. We can assume therefore that $\mathfrak{A} \oplus \mathfrak{B} \subset \mathfrak{g}$. To prove equality, we show by induction on $p \in \mathbb{Z}$, that

$$\mathfrak{A}_j \oplus \mathfrak{B}_j = \mathfrak{g}_j \quad \text{for} \quad j < p \tag{3.3}$$

holds for every $p \in \mathbb{Z}$. Indeed this is true for $p \leq 0$ by assumption. Assuming that (3.3) holds for some $p \geq 0$, let us prove that it also holds for $p + 1$. Denote by $\pi_{\mathfrak{a}} : \oplus_{j<p} \mathfrak{g}_j \to \oplus_{j<p} \mathfrak{a}_j$ and $\pi_{\mathfrak{b}} : \oplus_{j<p} \mathfrak{g}_j \to \oplus_{j<p} \mathfrak{b}_j$ the projections corresponding to the direct sum decomposition

$$\oplus_{j<p} \mathfrak{g}_j = \oplus_{j<p} \mathfrak{a}_j \bigoplus \oplus_{j<p} \mathfrak{b}_j.$$

For $X \in \mathfrak{g}_p$ we consider the induced homomorphism

$$\tilde{X} : \mathfrak{m} \to \oplus_{j<p}\mathfrak{g}_j$$

defined by

$$\tilde{X}(Y) = \mathrm{ad}(X)Y = [X, Y] \quad \forall Y \in \mathfrak{m}.$$

Then $\tilde{X}$ splits into the sum

$$\tilde{X} = \pi_{\mathfrak{a}} \circ \tilde{X} \circ \pi_{\mathfrak{a}}|_{\mathfrak{m}} + \pi_{\mathfrak{a}} \circ \tilde{X} \circ \pi_{\mathfrak{b}}|_{\mathfrak{m}} + \pi_{\mathfrak{b}} \circ \tilde{X} \circ \pi_{\mathfrak{a}}|_{\mathfrak{m}} + \pi_{\mathfrak{b}} \circ \tilde{X} \circ \pi_{\mathfrak{b}}|_{\mathfrak{m}}.$$

An easy computation shows that $\pi_{\mathfrak{a}} \circ \tilde{X} \circ \pi_{\mathfrak{a}}|_{\mathfrak{m}}$ and $\pi_{\mathfrak{b}} \circ \tilde{X} \circ \pi_{\mathfrak{b}}|_{\mathfrak{m}}$ actually define elements of $\mathfrak{g}_p$, belonging respectively to $\mathfrak{A}_p$ and to $\mathfrak{B}_p$. It follows that $\pi_{\mathfrak{a}} \circ \tilde{X} \circ \pi_{\mathfrak{b}}|_{\mathfrak{m}} + \pi_{\mathfrak{b}} \circ \tilde{X} \circ \pi_{\mathfrak{a}}|_{\mathfrak{m}}$ defines an element Z of $\mathfrak{g}_p$. We want to prove that $Z = 0$, using the fact that $\mathfrak{g}$ is transitive. We note that

$$[Z, U] \in \begin{cases} \mathfrak{B} & \text{for} \quad U \in \mathfrak{a} \\ \mathfrak{A} & \text{for} \quad U \in \mathfrak{b}. \end{cases}$$

Then for $U \in \mathfrak{a}_{-1}$ we obtain

$$[[Z, U], V] = 0 \qquad \text{for } V \in \mathfrak{a}_{-1}$$

because $[Z, U] \in \mathfrak{B}$,

$$[[Z, U], V] = -[U, [Z, V]] + [Z, [U, V]] = -[U, [Z, V]] = 0 \qquad \text{for } V \in \mathfrak{b}_{-1}$$

because $[[Z, U], V] \in \mathfrak{B}$, while $[U, [Z, V]] \in \mathfrak{A}$. This shows that $[Z, U] = 0$ for all $U \in \mathfrak{a}_{-1}$. When $p = 0$, this follows from the assumption that $\mathfrak{m}$ is nondegenerate and when $p > 0$ from the fact that $\mathfrak{g}$ is transitive. Analogously we prove that $[Z, \mathfrak{b}_{-1}] = 0$ and this implies $Z = 0$ because $\mathfrak{g}$ is transitive. $\square$

Given a pseudocomplex graded Lie algebra $\mathfrak{g} = \oplus_{p \geq -\mu}\mathfrak{g}_p$ with partial complex structure $J : \mathfrak{g}_{-1} \to \mathfrak{g}_{-1}$, we consider its complexification $\mathfrak{g}^{\mathbb{C}} = \oplus_{p \geq -\mu}\mathfrak{g}_p^{\mathbb{C}}$. The complexification of the partial complex structure J is a partial complex structure $J^{\mathbb{C}} = \mathbb{H} \otimes J : \mathfrak{g}_{-1}^{\mathbb{C}} \to \mathfrak{g}_{-1}^{\mathbb{C}}$. In this way we obtain that $\mathfrak{g}^{\mathbb{C}}$ is a pseudocomplex graded Lie algebra by considering $\mathfrak{g}^{\mathbb{C}}$ as a graded real Lie algebra endowed with the pseudocomplex structure $J^{\mathbb{C}}$. We have the following:

PROPOSITION 3.1.4 *A necessary and sufficient condition in order that the complexification of a pseudocomplex graded Lie algebra $\mathfrak{g}$ be a Levi-Tanaka algebra is that $\mathfrak{g}$ is a Levi-Tanaka algebra.*

Proof. First we note that $\mathfrak{m}^{\mathbb{C}} = \oplus_{p<0}\mathfrak{g}_p^{\mathbb{C}}$ is fundamental (resp. nondegenerate) if and only if $\mathfrak{m} = \oplus_{p<0}\mathfrak{g}_p$ is fundamental (resp. nondegenerate) and that $\mathfrak{g}^{\mathbb{C}}$ is transitive if and only if $\mathfrak{g}$ is transitive. Next we consider the canonical pseudocomplex prolongation $\mathfrak{a}$ of $\mathfrak{m}^{\mathbb{C}}$ and we prove by recurrence that the anti-$\mathbb{C}$-linear part of the map $\mathfrak{g}_{-1}^{\mathbb{C}} \to \mathfrak{g}_{p-1}^{\mathbb{C}}$ defined by any $X \in \mathfrak{a}_p$ (for $p \geq 0$) is 0. This implies our contention. □

3.2 Properties of Levi-Tanaka algebras

LEMMA 3.2.1 *If $\mathfrak{g} = \oplus_{p\geq -\mu}\mathfrak{g}_p$ is a Levi-Tanaka algebra, then there is a unique element $E \in \mathfrak{g}_0$ such that*

$$[E, X_p] = pX_p \quad \forall p \in \mathbb{Z}, \forall X_p \in \mathfrak{g}_p.$$

Proof. The $\mathbb{R}$-linear map $\tilde{E} : \mathfrak{m} \to \mathfrak{m}$ defined by

$$\tilde{E}(X_p) = pX_p \qquad \text{for} \quad p < 0 \quad \text{and} \quad X_p \in \mathfrak{g}_p$$

is a derivation of order zero of $\mathfrak{m}$, which commutes with J on $\mathfrak{g}_{-1}$ and therefore defines an element $E \in \mathfrak{g}_0$. We have to show that $[E, X_p] = pX_p$ when $p \geq 0$ and $X_p \in \mathfrak{g}_p$. This is certainly true when $p = 0$, because $\rho_{-1}(E)$ commutes with all endomorphisms in $\mathrm{Hom}_{\mathbb{R}}(\mathfrak{g}_{-1}, \mathfrak{g}_{-1})$. Assuming it is true for some $p \geq 0$, we have for $X_{p+1} \in \mathfrak{g}_{p+1}$ and $Y_{-1} \in \mathfrak{g}_{-1}$:

$$\begin{aligned} [[E, X_{p+1}], Y_{-1}] &= [E, [X_{p+1}, Y_{-1}]] + [X_{p+1}, [Y_{-1}, E]] \\ &= (p+1)\,[X_{p+1}, Y_{-1}]. \end{aligned}$$

Since $\mathfrak{g}$ is transitive, this implies that $[E, X_{p+1}] = (p+1)X_{p+1}$. □

We recall that the element E described in the previous lemma is called the characteristic element of the graded Lie algebra $\mathfrak{g}$. The following two corollaries are consequence of Lemma 2.1.3 and Lemma 2.1.4.

COROLLARY 3.2.2 *The center of a Levi-Tanaka algebra $\mathfrak{g}$ is trivial. In particular, the adjoint representation $\mathfrak{g} \ni X \to ad_{\mathfrak{g}}(X) \in \mathfrak{gl}(\mathfrak{g})$ is faithful.*

COROLLARY 3.2.3 *Every ideal of a Levi-Tanaka algebra is graded.*

Using Corollary 3.2.2, we will often identify $\mathfrak{g}$ with the Lie subalgebra of $\mathfrak{gl}(\mathfrak{g})$ which is the image of $\mathfrak{g}$ by the adjoint representation.

Let V be a vector space over a field $\mathbb{K}$ of characteristic 0. A Lie subalgebra of $\mathfrak{gl}_{\mathbb{K}}(V)$ is *splittable* if it contains the semisimple and nilpotent component of each of its elements. A Lie algebra $\mathfrak{l}$ with trivial center is called splittable if its image by the adjoint representation is splittable. In the same way we call an element X of $\mathfrak{l}$ *nilpotent* (resp. *semisimple*) if its image $\mathrm{ad}_{\mathfrak{l}}(X)$ by the adjoint representation is nilpotent (resp. semisimple) as an element of $\mathfrak{gl}(\mathfrak{l})$.

LEMMA 3.2.4 *The Lie subalgebra $\mathfrak{g}_0$ of a Levi-Tanaka algebra $\mathfrak{g} = \oplus_{p\in\mathbb{Z}}\mathfrak{g}_p$, considered as a subalgebra of the Lie algebra $\mathfrak{gl}(\mathfrak{m})$ of the endomorphisms of $\mathfrak{m} = \oplus_{p<0}\mathfrak{g}_p$, is splittable.*

Assume that $\mathfrak{m}$ is nondegenerate, so that $\mathfrak{g}$ is finite dimensional. Then:

(*i*) *if S and $N \in \mathfrak{g}_0$ are the semisimple and nilpotent components of $A \in \mathfrak{g}_0$ as endomorphisms of $\mathfrak{m}$, then $ad_{\mathfrak{g}}(S)$ and $ad_{\mathfrak{g}}(N)$ are respectively semisimple and nilpotent in $\mathfrak{gl}(\mathfrak{g})$;*

(*ii*) *the algebra $\mathfrak{g}$ is splittable as a Lie subalgebra of $\mathfrak{gl}(\mathfrak{g})$.*

Proof. Every element $A \in \mathfrak{g}_0$ defines a derivation of the fundamental Lie algebra $\mathfrak{m}$. The semisimple component S and the nilpotent component N of A in $\mathfrak{gl}(\mathfrak{m})$ are still derivations of $\mathfrak{m}$ (cf. [6] Ch.7 §1 Proposition 4(ii)). Moreover, since S and N are polynomials of A, we have $S(\mathfrak{g}_p) \subset \mathfrak{g}_p$ and $N(\mathfrak{g}_p) \subset \mathfrak{g}_p$ for all $p < 0$ and S and N define $\mathbb{C}$-linear endomorphisms of $\mathfrak{g}_{-1}$. This shows that $S, N \in \mathfrak{g}_0$.

Let us assume now that $\mathfrak{g}$ is finite dimensional. First we note that the elements of $\mathfrak{g}_0$ are splittable as endomorphisms of $\mathfrak{g}$. This follows by the same argument given above: if $A \in \mathfrak{g}_0$, then $\mathrm{ad}_{\mathfrak{g}}(A)$ is a 0-degree derivation of $\mathfrak{g}$ which defines a $\mathbb{C}$-linear endomorphism of $\mathfrak{g}_{-1}$. Then the semisimple and nilpotent components $\tilde{S}$ and $\tilde{N}$ of $\mathrm{ad}_{\mathfrak{g}}(A)$ in $\mathfrak{gl}(\mathfrak{g})$ are 0-degree derivations of $\mathfrak{g}$ which define $\mathbb{C}$-linear endomorphisms of $\mathfrak{g}_{-1}$. Their restrictions to $\mathfrak{m}$ are

commuting semisimple and nilpotent endomorphisms of $\mathfrak{m}$ and thus are the semisimple and nilpotent components S and N of the representation of A in $\mathfrak{gl}(\mathfrak{m})$. This shows that $\mathrm{ad}_{\mathfrak{g}}(S)$ and $\mathrm{ad}_{\mathfrak{g}}(N)$ are still semisimple and nilpotent respectively. Indeed they coincide with $\tilde{S}$ and $\tilde{N}$ because by the construction of the canonical prolongation 0-degree derivations of $\mathfrak{m}$ uniquely extend to 0-degree derivations of $\mathfrak{g}$. This proves (i).

To complete the proof, we observe that when $\mathfrak{g}$ is finite dimensional the elements of $\cup_{p\neq 0}\mathfrak{g}_p$ are all nilpotent. It follows from (i) that $\mathfrak{g}$, considered as a Lie subalgebra of $\mathfrak{gl}(\mathfrak{g})$, is generated by its semisimple and nilpotent elements. This implies that $\mathrm{ad}_{\mathfrak{g}}(\mathfrak{g}) \subset \mathfrak{gl}(\mathfrak{g})$ is splittable (see [6] Ch.VII § 5 Theorem 1). □

LEMMA 3.2.5 *Let $\mathfrak{g} = \oplus_{p\in\mathbb{Z}}\mathfrak{g}_p$ be a finite dimensional graded Lie algebra. We assume that $\mathfrak{g}$ contains a characteristic element E and is splittable. Then we can find a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ contained in $\mathfrak{g}_0$.*

All Cartan subalgebras of the Lie algebra $\mathfrak{g}_0$ are Cartan subalgebras of $\mathfrak{g}$ and contain the element E. In particular, $\mathfrak{g}_0$ contains regular elements of $\mathfrak{g}$.

Proof. Let $\mathcal{S}$ denote the set of semisimple elements of $\mathfrak{g}$ and $\mathcal{T}$ the set of all commutative Lie subalgebras of $\mathfrak{g}$ contained in $\mathcal{S}$. Let $\mathcal{T}_1$ denote the set of maximal (with respect to $\subset$) elements of $\mathcal{T}$. As $\mathfrak{g}$ is splittable the centralizer $C_{\mathfrak{g}}(\mathfrak{t}) = \{X \in \mathfrak{g} \,|\, [X,\mathfrak{t}] = 0\}$ in $\mathfrak{g}$ of every $\mathfrak{t} \in \mathcal{T}_1$ is a Cartan subalgebra of $\mathfrak{g}$ (see [6] Ch.VII §5 Proposition 6). A characteristic element E of $\mathfrak{g}$ is semisimple, hence it can be included in a Lie subalgebra $\mathfrak{t} \in \mathcal{T}_1$. Its centralizer $\mathfrak{h}$ in $\mathfrak{g}$ is a Cartan subalgebra of $\mathfrak{g}$ and, if $X \in \mathfrak{h}$ and $X = \sum_p X_p$ is its decomposition into the sum of its homogeneous components, we have $0 = [E, X] = \sum_{p\neq 0} p\, X_p$, and hence $X = X_0 \in \mathfrak{g}_0$.

A characteristic element E belongs to the center of $\mathfrak{g}_0$ and hence to all Cartan subalgebras of $\mathfrak{g}_0$. Hence the normalizer in $\mathfrak{g}$ of a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}_0$ coincides with its normalizer in $\mathfrak{g}_0$: it is equal to $\mathfrak{h}$ and $\mathfrak{h}$ is therefore a Cartan subalgebra of $\mathfrak{g}$.

The last statement follows from [6] Ch.VII §3 Proposition 3. □

Remark 3.2.6 *Because of Lemma 3.2.1 and Lemma 3.2.4, the conclusions of Lemma 3.2.5 apply in particular to Levi-Tanaka algebras.*

3.2.1 The (J) property

LEMMA 3.2.7 *Let $\mathfrak{m}$ be a pseudocomplex fundamental graded Lie algebra of the second kind and let $\mathfrak{g} = \oplus_{p\geq -2}\mathfrak{g}_p$ be its canonical pseudocomplex prolongation. Then there is a unique element $\tilde{J} \in \mathfrak{g}_0$ such that*

$$[\tilde{J}, X] = JX \quad \forall X \in \mathfrak{g}_{-1}. \tag{3.4}$$

Proof. When $\mathfrak{m}$ is of kind 2, the elements of $\mathfrak{g}_0$ can be identified to the space of $\mathbb{C}$-linear maps $A : \mathfrak{g}_{-1} \to \mathfrak{g}_{-1}$ for which there is an $\mathbb{R}$-linear map $B : \mathfrak{g}_{-2} \to \mathfrak{g}_{-2}$ such that

$$[AX, Y] + [X, AY] = B([X, Y]) \quad \forall X, Y \in \mathfrak{g}_{-1}.$$

From the definition of a pseudocomplex graded Lie algebra, this relation holds true for $A = J$ and $B = 0$. □

We will see below that the existence of such an element $\tilde{J}$ is not guaranteed when the kind μ of $\mathfrak{m}$ is greater than 2. We say in general that a pseudocomplex graded Lie algebra $\mathfrak{g} = \oplus_{p\geq -\mu}\mathfrak{g}_p$ *has the (J) property* if there is an element $\tilde{J} \in \mathfrak{g}_0$ for which (3.4) holds true. In this case we denote by J_p the representation $\rho_p(\tilde{J})$ of $\tilde{J}$ in $\mathfrak{g}_p$. Note that $J_{-1} = J$ is the complex structure of $\mathfrak{g}_{-1}$.

By Lemma 3.2.7, Levi-Tanaka algebras of the second kind have the (J) property. Later on we will show that all finite dimensional semisimple Levi-Tanaka algebras have the (J) property.

LEMMA 3.2.8 *Let $\mathfrak{g} = \oplus_{p\geq -\mu}\mathfrak{g}_p$ be a canonical pseudocomplex prolongation of a pseudocomplex fundamental graded Lie algebra $\mathfrak{m}$ of kind $\mu \geq 2$. If $\mathfrak{g}$ has the (J) property, then:*

(i) *J_p defines a complex structure on $\mathfrak{g}_p$ for $p = -3, -1, 1$;*

(ii) *$J_p = 0$ for $p = -2, 0$.*

When $\mu = 2$, and $\mathfrak{m}$ is nondegenerate, J_p is a complex structure on $\mathfrak{g}_p$ for p odd and 0 for p even.

Proof. The statement is certainly true when $p = -2, -1, 0$.

Let us consider the case $p = -3$. The elements $[X, T]$, for $X \in \mathfrak{g}_{-1}$ and $T \in \mathfrak{g}_{-2}$ are a set of generators of $\mathfrak{g}_{-3}$ because $\mathfrak{m}$ is fundamental. Since $\tilde{J}$ is a 0-degree derivation of $\mathfrak{m}$ we have

$$\begin{aligned} J_{-3}([X,T]) &= [\tilde{J},[X,T]] \\ &= [J_{-1}X,T]+[X,J_{-2}T] \\ &= [JX,T] \quad \forall X \in \mathfrak{g}_{-1},\, \forall T \in \mathfrak{g}_{-2}. \end{aligned}$$

Then we obtain

$$\begin{aligned} J_{-3}^2([X,T]) &= J_{-3}([JX,T]) \\ &= [J^2X,T] \\ &= -[X,T] \quad \forall X \in \mathfrak{g}_{-1},\, \forall T \in \mathfrak{g}_{-2}, \end{aligned}$$

from which we deduce that

$$J_{-3}^2 Y = -Y \quad \forall Y \in \mathfrak{g}_{-3}$$

because this relation holds true on a set of generators of $\mathfrak{g}_{-3}$.

In general, the argument above shows that, if $p < 0$ and $J_p = 0$, then J_{p-1} is a complex structure on $\mathfrak{g}_{p-1}$.

Let us turn now to the case $p = 1$. Let $X \in \mathfrak{g}_1$. Then we have

$$0 = J_0([X,Y]) = [J_1X,Y]+[X,JY] \quad \forall Y \in \mathfrak{g}_{-1}.$$

This yields

$$[J_1X,Y] = -[X,JY] \quad \forall X \in \mathfrak{g}_1,\, \forall Y \in \mathfrak{g}_{-1}\,,$$

from which we obtain

$$\begin{aligned} [J_1^2X,Y] &= -[J_1X,JY] = [X,J^2Y] \\ &= -[X,Y] \quad \forall X \in \mathfrak{g}_1,\, \forall Y \in \mathfrak{g}_{-1}\,. \end{aligned}$$

Since $\mathfrak{g}$ is transitive, this shows that J_1 is a complex structure on $\mathfrak{g}_1$.

More in general, this argument shows that, if $J_p = 0$ for some $p \geq 0$, then J_{p+1} is a complex structure on $\mathfrak{g}_{p+1}$.

Let us turn now to the case where $\mathfrak{m}$ is of kind 2. Then, assuming that $J_p = 0$ for some $p \geq 0$, we have

$$\begin{aligned}[J_{p+2}X_{p+2}, Y_{-2}] &= J_p[X_{p+2}, Y_{-2}] - [X_{p+2}, J_{-2}Y_{-2}] \\ &= 0 \qquad \forall X_{p+2} \in \mathfrak{g}_{p+2},\ \forall Y_{-2} \in \mathfrak{g}_{-2}.\end{aligned}$$

This implies that $J_{p+2}X_{p+2} \in \mathfrak{h}_{p+2}$, where $\mathfrak{h} = \{Z \in \mathfrak{g} \mid [Z, \mathfrak{g}_{-2}] = 0\}$. But we proved (see Theorem 3.1.1) that $\mathfrak{h}_p = 0$ for $p > 0$ when $\mathfrak{m}$ is nondegenerate. Then we obtain by recurrence that $J_p = 0$ for every p even.

By the previous remarks, this completes the proof of the lemma. □

PROPOSITION 3.2.9 *Assume that the Levi-Tanaka algebra* $\mathfrak{g} = \oplus_{p \geq -\mu}\mathfrak{g}_p$ *contains an element* $\tilde{J} \in \mathfrak{g}_0$ *such that*

$$[\tilde{J}, X] = \tilde{J}X \qquad \forall X \in \mathfrak{g}_{-1}.$$

Then $ad(\tilde{J})$ *is semisimple. Denote by* $\sigma_p(\tilde{J})$ *the set of eigenvalues of* $\rho_p(\tilde{J})$, *for* $p \in \mathbb{Z}$. *Then we have:*

$$(3.5)\ \sigma_p(\tilde{J}) \subset \{(p+2)\sqrt{-1}, (p+4)\sqrt{-1}, \ldots, -(p+2)\sqrt{-1}\} \quad \textit{for } p < -1$$
$$(3.6)\ \sigma_p(\tilde{J}) \subset \{-p\sqrt{-1}, (2-p)\sqrt{-1}, \ldots, p\sqrt{-1}\} \quad \textit{for } p \geq 0$$

Proof. Using Lemma 3.2.4, we can decompose $\tilde{J}$ as $\tilde{J} = S + N$ with $S, N \in \mathfrak{g}_0$, $[S, N] = 0$ and $\mathrm{ad}(S)$ semisimple, $\mathrm{ad}(N)$ nilpotent. Since J is semisimple on $\mathfrak{g}_{-1}$, we have $\rho_{-1}(\tilde{J}) = \rho_{-1}(S)$ and therefore $\tilde{J} = S$ because $\mathfrak{g}$ is transitive.

Denote by $\mathfrak{g}^{\mathbb{C}}$ the complexification of $\mathfrak{g}$ and let $X_1, \ldots, X_{2n}$ be a basis of $\mathfrak{g}^{\mathbb{C}}_{-1}$ whose elements are eigenvectors of J. We have $[\tilde{J}, X_h] = \lambda_h X_h$ with $\lambda_h = \pm\sqrt{-1}$.

Formula (3.5) is valid for $p = -2$. To prove it holds also when $p < -2$, we observe that in this case the space $\mathfrak{g}^{\mathbb{C}}_p$ is generated by the vectors

$$Y_{i_1, i_2, \ldots, i_{1-p}, i_{-p}} = [X_{i_1}, [X_{i_2}, [\ldots, [X_{i_{1-p}}, X_{i_{-p}}], \ldots]]]$$

for $1 \leq i_1, i_2, \ldots, i_{1-p}, i_{-p} \leq 2n$. These are eigenvectors for $\rho_p(\tilde{J})$, corresponding to the eigenvalues

$$\lambda_{i_1} + \lambda_{i_2} + \ldots + \lambda_{i_{1-p}} + \lambda_{i_{-p}}.$$

Since $\rho_{-2}(\tilde{J}) = 0$, we need that $\lambda_{i_{1-p}} + \lambda_{i_{-p}} = 0$ in order that $Y_{i_1, i_2, \dots, i_{1-p}, i_{-p}} \neq 0$. This shows that the elements of $\sigma_p(\tilde{J})$ are the sum of $2-p$ terms, each equal to $\pm\sqrt{-1}$ and so (3.5) is proved.

To prove (3.6), we note that it holds true when $p = 0$. Let $p > 0$ and $Y \in \mathfrak{g}_p^{\mathbb{C}} \setminus \{0\}$ be an eigenvector corresponding to an eigenvalue λ of $\rho_p(\tilde{J})$. Then, since $\mathfrak{g}$ is transitive, there exists $1 \leq h \leq 2n$ such that $[Y, X_h] \neq 0$. This is an eigenvector corresponding to the eigenvalue $\lambda + \lambda_h \in \sigma_{p-1}(\tilde{J})$. Therefore we have

$$\sigma_p(\tilde{J}) \subset \{\lambda + \sqrt{-1} \mid \lambda \in \sigma_{p-1}(\tilde{J})\} \cup \{\lambda - \sqrt{-1} \mid \lambda \in \sigma_{p-1}(\tilde{J})\}.$$

Together with $\sigma_0(\tilde{J}) = \{0\}$, this inclusion implies (3.6). □

Remark 3.2.10 *Proposition 3.2.9 remains valid under weaker assumptions on $\mathfrak{g}$: namely it suffices that the pseudocomplex graded Lie algebra $\mathfrak{g} = \oplus_{p \geq -\mu} \mathfrak{g}_p$ satisfies:*

(i) the adjoint representation $\mathfrak{g} \to ad(\mathfrak{g}) \subset \mathfrak{gl}_{\mathbb{R}}(\mathfrak{g})$ is faithful;

(ii) $ad(\mathfrak{g})$ is a splittable subalgebra of $\mathfrak{gl}_{\mathbb{R}}(\mathfrak{g})$;

(iii) $\mathfrak{g}$ is transitive;

(iv) $\mathfrak{m} = \oplus_{p<0} \mathfrak{g}_p$ is fundamental;

(v) there exists $\tilde{J} \in \mathfrak{g}_0$ such that $[\tilde{J}, X] = JX$ for every $X \in \mathfrak{g}_{-1}$.

We note in particular that (i) and (ii) are valid when $\mathfrak{g}$ is semisimple and finite dimensional.

Remark 3.2.11 *Since $[\tilde{J}, A] = 0$ for every $A \in \mathfrak{g}_0$, the linear endomorphism $\rho_p(A)$ is $\mathbb{C}$-linear in $\mathfrak{g}_p$ whenever J_p defines a complex structure on $\mathfrak{g}_p$.*

Remark 3.2.12 *From the lemma above we obtain that $\dim_{\mathbb{R}} \mathfrak{g}_{-3}$ must be even when $\mathfrak{g}$ has the (J) property. In particular we cannot expect (J) to hold for the Tanaka algebras of a CR manifold M of type $(1,2)$ in which the vector fields in $\Gamma(M, HM)$ generate the Lie algebra of tangent vector fields to M.*

3.3 Semisimple prolongations

We first recall a lemma on the structure of finite dimensional semisimple graded Lie algebras (see, for instance, [42]).

LEMMA 3.3.1 *Let* $\mathfrak{s} = \oplus_{-\mu \le p \le \nu} \mathfrak{s}_p$ *be a finite dimensional semisimple graded Lie algebra over* $\mathbb{R}$. *Then:*

(*i*) *the algebra* $\mathfrak{s}$ *contains a characteristic element* $E \in \mathfrak{s}_0$ *such that*

$$[E, X_p] = pX_p \quad \forall -\mu \le p \le \nu, \quad \forall X_p \in \mathfrak{s}_p;$$

this element is unique as the center of $\mathfrak{s}$ *is trivial;*

(*ii*) *every ideal* $\mathfrak{i}$ *of* $\mathfrak{s}$ *is graded, i.e.* $\mathfrak{i} = \oplus_{p \in \mathbb{Z}} (\mathfrak{i} \cap \mathfrak{s}_p)$;

(*iii*) *the Killing form* $\kappa_{\mathfrak{s}}$ *of* $\mathfrak{s}$ *defines a duality pairing between* $\mathfrak{s}_p$ *and* $\mathfrak{s}_{-p}$: *in particular* $\nu = \mu$ *and* $\dim_{\mathbb{R}} \mathfrak{s}_p = \dim_{\mathbb{R}} \mathfrak{s}_{-p}$ *for* $0 \le p \le \mu$;

(*iv*) *the Lie algebra* $\mathfrak{g}_0$ *is reductive, i.e. decomposes into the direct sum of a semisimple and an abelian ideal;*

(*v*) *if* $\mu > 0$, *then* $\mathfrak{s}$ *is of the noncompact type.*

Proof. (*i*). The linear operator defined as in (2.3) is a derivation of order zero of $\mathfrak{s}$ and hence, because $\mathfrak{s}$ is real semisimple, it defines an element E of $\mathfrak{s}_0$.

(*ii*) is a consequence of (*i*) and Lemma 2.1.4.

(*iii*) is a consequence of Remark 2.1.1, because $\kappa_{\mathfrak{s}}$ is nondegenerate on $\mathfrak{s}$.

Statement (*iv*) follows because the restriction to $\mathfrak{s}_0$ of the Killing form $\kappa_{\mathfrak{s}}$, which is nondegenerate by (*iii*), is the invariant bilinear form in $\mathfrak{s}_0$ induced by the adjoint representation. Then we apply [6] Ch.I §6 Proposition 5(d). The last statement is a trivial remark, as by (*iii*) the Witt index of the Killing form is larger than or equal to $\dim_{\mathbb{R}} \oplus_{-\mu \le p < 0} \mathfrak{s}_p$. □

LEMMA 3.3.2 *Let* $\mathfrak{s} = \oplus_{-\mu \le p \le \mu} \mathfrak{s}_p$ *be a semisimple graded Lie algebra over* $\mathbb{R}$. *Assume that* $\mathfrak{m} = \oplus_{p<0} \mathfrak{s}_p$ *is fundamental. Then a necessary and sufficient condition in order that* $\mathfrak{s}$ *be transitive is that*

$$\{X \in \mathfrak{s}_0 \mid [X, \mathfrak{s}_{-1}] = 0\} = 0\,. \tag{3.7}$$

Proof. The condition (3.7) is trivially necessary. Let us prove sufficiency. First we show that, for $X \in \mathfrak{s}$,

$$[X, \mathfrak{s}_{-1}] = 0 \Rightarrow [X, \mathfrak{s}_p] = 0 \quad \forall p < 0.$$

This follows by recurrence: indeed $[\mathfrak{s}_p, \mathfrak{s}_{-1}] = \mathfrak{s}_{p-1}$ for $p < 0$ because $\mathfrak{m}$ is fundamental; then

$$[X, \mathfrak{s}_{p-1}] = [[X, \mathfrak{s}_p], \mathfrak{s}_{-1}] + [\mathfrak{s}_p, [X, \mathfrak{s}_{-1}]]$$

shows that $[X, \mathfrak{s}_{p-1}] = 0$ when $[X, \mathfrak{s}_p] = [X, \mathfrak{s}_{-1}] = 0$.

Let now X be a nonzero element of $\mathfrak{s}_q$ for some $q > 0$. By (*iii*) of Lemma 3.3.1 there is $Y \in \mathfrak{s}_{-q}$ such that $\kappa_{\mathfrak{s}}(X, Y) \neq 0$. Lemma 2.1.1 implies that $[X, Y] \neq 0$. Since $\mathfrak{m}$ is fundamental, the proof is complete. □

If $\mathfrak{g}$ is a Lie algebra, we define by recurrence $[X] = X$ for every element $X \in \mathfrak{g}$ and $[X_1, X_2, \dots, X_k] = [X_1, [X_2, [\dots, [X_{k-1}, X_k] \dots]]]$ for every $X_1, \dots, X_k \in \mathfrak{g}$ when $k > 1$.

For $\mathfrak{l} \subset \mathfrak{g}$ denote by $\mathfrak{l}^k$ the linear span of $[X_1, \dots, X_k]$ for $X_1, \dots, X_k \in \mathfrak{l}$.

LEMMA 3.3.3 *Let $\mathfrak{s} = \oplus_{-\mu \leq p \leq \mu} \mathfrak{s}_p$ be a simple graded Lie algebra over $\mathbb{R}$ of kind $\mu \geq 1$. Then $\mathfrak{m} = \oplus_{-\mu \leq p < 0} \mathfrak{s}_p$ is fundamental if and only if $\mathfrak{s}_{-1}^{\mu} \neq 0$ (i.e. there exist $X_1, \dots, X_\mu \in \mathfrak{s}_{-1}$ such that $[X_1, \dots, X_\mu] \neq 0$).*

Assume that $\mathfrak{m}$ is fundamental, then:

(*i*) *$\mathfrak{s}$ is transitive;*

(*ii*) *if $\mu \geq 2$, $\mathfrak{s}$ is nondegenerate.*

Proof. In the proof we shall use the following

CLAIM 3.3.4 *For every element X of a graded Lie algebra $\mathfrak{g} = \oplus_{p \in \mathbb{Z}} \mathfrak{g}_p$ the elements of the ideal $\mathfrak{i}(X)$ generated by X are linear combinations of X and elements of the form $[Z_k, \dots, Z_1, X]$ with Z_i homogeneous and $\deg Z_k \geq \dots \geq \deg Z_1$.*

(This claim can be easily obtained using induction and Jacobi's identity.) Suppose there exist $X_1, \dots, X_\mu \in \mathfrak{s}_{-1}$ such that $[X_\mu, \dots, X_1] \neq 0$. Then we

have that $Y_j = [X_j, \dots, X_1] \neq 0$ for $1 < j \leq \mu$ and the ideals $\mathfrak{i}(Y_j)$ generated by the Y_j's are not zero. Because $\mathfrak{s}$ is simple, they coincide with $\mathfrak{s}$ and $\mathfrak{i}(Y_j)_{-p} = \mathfrak{i}(Y_j) \cap \mathfrak{s}_{-p} = \mathfrak{s}_{-p}$. We will prove, by recurrence, that $\mathfrak{s}_{-q} = \mathfrak{s}_{-1}^q$ for $1 < q \leq \mu$. If $q = \mu$, then it follows from the claim that $\mathfrak{s}_{-\mu} = \mathfrak{i}(Y_\mu)_{-\mu}$ is generated by elements of the form $[Z_k, \dots, Z_1, Y_\mu]$ with $Z_i \in \mathfrak{s}_0$ for every i. Because $\mathfrak{s}_{-1}^\mu$ is invariant under the adjoint action of $\mathfrak{g}_0$, we conclude that $\mathfrak{s}_{-\mu} = \mathfrak{i}(Y_\mu)_{-\mu} = \mathfrak{s}_{-1}^\mu$. Assume now that $q < \mu$ and $\mathfrak{s}_{-p} = \mathfrak{s}_{-1}^p$ for $q < p \leq \mu$. We want to prove that $\mathfrak{s}_{-q} = \mathfrak{s}_{-1}^q$. By the claim $\mathfrak{s}_{-q} = \mathfrak{i}(Y_q)_{-q}$ is generated by linear combinations of Y_q and elements of the form $[Z_k, \dots, Z_1, Y_q]$ with Z_i homogeneous, $\deg Z_k \geq \dots \geq \deg Z_1$ and $\sum_1^k \deg Z_i = 0$. It suffices to prove that they all belong to $\mathfrak{s}_{-1}^q$. If $\deg Z_k = 0$, then $Z_1, \dots, Z_k \in \mathfrak{s}_0$ and therefore $[Z_k, \dots, Z_1, Y_q] \in \mathfrak{s}_{-1}^q$. If $\deg Z_k > 0$, then $[Z_k, \dots, Z_1, Y_q]$ is a linear combination of elements of the form $[Z_k, U_r, \dots, U_1]$ with $U_j \in \mathfrak{s}_{-1}$ and $r = q + \deg Z_k$. By repeated application of the formula $[V, V_s, \dots, V_1] = \sum_{i=1}^s [V_s, \dots, V_{i+1}, [V, V_i], V_{i-1}, \dots, V_1]$, we can show that the commutator $[Z_k, U_r, \dots, U_1]$ belongs to $\mathfrak{s}_{-1}^q$. The converse is obvious.

Suppose now that $\mathfrak{m}$ is fundamental.

(i). Let $\mathfrak{a}$ be equal to $\{A \in \mathfrak{s}_0 | [A, \mathfrak{m}] = 0\}$. By Lemma 3.3.2, it suffices to prove that $\mathfrak{a}$ is zero. Assume that $A \in \mathfrak{a}$. If $X \in \mathfrak{s}_0$, then $[[A, X], Z] = [[A, Z], X] + [A, [X, Z]] = 0$ for every $Z \in \mathfrak{m}$, so that $[\mathfrak{a}, \mathfrak{s}_0] \subset \mathfrak{a}$. If $X \in \mathfrak{s}$ is homogeneous of positive degree, then we have

$$0 = \kappa_{\mathfrak{s}}([A, Z], X) = -\kappa_{\mathfrak{s}}(Z, [A, X]) \quad \forall Z \in \mathfrak{m}$$

and, by Lemma 3.3.1, we obtain $[A, X] = 0$. It follows that $\mathfrak{a}$ is an ideal of $\mathfrak{s}$. Since it is contained in $\mathfrak{s}_0$ and $\mathfrak{s}$ is simple with $\mu \geq 2$, we have $\mathfrak{a} = 0$.

(ii). Assume $\mu \geq 2$. If $\mathfrak{s}$ were degenerate then we could find $X \in \mathfrak{s}_{-1}$ such that $[X, \mathfrak{m}] = 0$ and the ideal $\mathfrak{i}(X)$ generated by such an X would be different from zero, hence equal to $\mathfrak{s}$. On the other hand, using the claim above we obtain that $\mathfrak{s}_{-\mu} = \mathfrak{i}(X)_{-\mu} = 0$ and this gives a contradiction. □

Remark 3.3.5 *If $\mathfrak{s}$ is the Levi-Tanaka algebra at a point x of a CR manifold, the condition in the previous lemma means that the highest order Levi form is not identically zero at x (cf. [38]).*

LEMMA 3.3.6 *Let $\mathfrak{g} = \oplus_{-\mu \leq p \leq \mu} \mathfrak{g}_p$ be a semisimple transitive prolongation of a fundamental graded Lie algebra $\mathfrak{m} = \oplus_{-\mu \leq p < 0} \mathfrak{g}_p$. Then $\mathfrak{g}_0 = [\mathfrak{g}_{-1}, \mathfrak{g}_1]$.*

Proof. Setting $\mathfrak{l}_p = \mathfrak{g}_p$ for $p \neq 0$ and $\mathfrak{l}_0 = [\mathfrak{g}_{-1}, \mathfrak{g}_1]$, we obtain an ideal $\mathfrak{l} = \oplus_{p\in\mathbb{Z}}\mathfrak{l}_p$ of $\mathfrak{g}$. Then $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{l}$ for an ideal $\mathfrak{a} \subset \mathfrak{g}$. By Lemma 3.3.1 each ideal of $\mathfrak{g}$ is graded, and so $\mathfrak{a} \subset \mathfrak{g}_0$. Since $[\mathfrak{a}, \mathfrak{g}_{-1}] \subset \mathfrak{a}_{-1} = 0$, we have $\mathfrak{a} = 0$ because $\mathfrak{g}$ is transitive. □

We have

THEOREM 3.3.7 *Let $\mathfrak{m} = \oplus_{-\mu\leq p<0}\mathfrak{g}_p$ be a fundamental graded Lie algebra and let $\mathfrak{s}$ be a semisimple transitive prolongation of $\mathfrak{m}$. If $\mathfrak{g} = \oplus_{-\mu\leq p\leq\nu}\mathfrak{g}_p$ is a finite dimensional transitive prolongation of $\mathfrak{m}$ containing $\mathfrak{s}$, then $\mathfrak{g}$ coincides with $\mathfrak{s}$.*

In particular, if $\mathfrak{m}$ is also pseudocomplex and nondegenerate and if $\mathfrak{s}$ is a semisimple transitive pseudocomplex prolongation, then $\mathfrak{s}$ is isomorphic to the canonical pseudocomplex prolongation of $\mathfrak{m}$.

Proof. Assume that $\mathfrak{s}$ is a semisimple transitive prolongation of $\mathfrak{m}$. In this case we can consider $\mathfrak{s}$ as a subalgebra of $\mathfrak{g}$. If $\mathfrak{g}$ is semisimple, then $\mathfrak{g}$ and $\mathfrak{s}$ coincide. Indeed, by (*iii*) in Lemma 3.3.1, $\mathfrak{g}_p$ is equal to $\mathfrak{s}_p$ for $p \neq 0$ (because they have the same dimension as vector spaces) and, by the lemma above, $\mathfrak{g}$ coincides with $\mathfrak{s}$.

Let us prove now that $\mathfrak{g}$ is semisimple. We already know that $\mathfrak{g}$ is finite dimensional. Then it suffices to show that its radical $\mathfrak{r}$ is 0. By Corollary 3.2.3 $\mathfrak{r}$ is a graded ideal of $\mathfrak{g}$. We have $\mathfrak{r}\cap\mathfrak{s} = 0$ because $\mathfrak{s}$ is semisimple and hence $\oplus_{-\mu\leq p<0}\mathfrak{r}_p = 0$ because $\oplus_{-\mu\leq p<0}\mathfrak{g}_p \subset \mathfrak{s}$.

Let us show by recurrence that $\mathfrak{r}_p = 0$ also when $p \geq 0$. For $p = 0$ we have $[\mathfrak{r}_0, \mathfrak{g}_{-1}] \subset \mathfrak{r}_{-1} = 0$ and hence $\mathfrak{r}_0 = 0$ because $\mathfrak{g}$ is transitive. Assuming $\mathfrak{r}_p = 0$ for some $p \geq 0$, we deduce that also $\mathfrak{r}_{p+1} = 0$ from the transitivity of $\mathfrak{g}$ and the fact that $[\mathfrak{r}_{p+1}, \mathfrak{g}_{-1}] \subset \mathfrak{r}_p = 0$. □

The following is a criterion for the simplicity of the prolongation, which is close to one which was stated in [39].

THEOREM 3.3.8 *Let $\mathfrak{g}$ be the canonical pseudocomplex prolongation of a nondegenerate pseudocomplex fundamental graded Lie algebra $\mathfrak{m}$ and assume that ρ_{-1} is irreducible and $\mathfrak{g}_1 \neq 0$. Then $\mathfrak{g}$ is simple.*

In particular, if a finite dimensional fundamental simple graded Lie algebra $\mathfrak{g} = \oplus_{-\mu\leq p\leq\mu}\mathfrak{g}_p$ of kind $\mu \geq 2$ is pseudocomplex, then $\mathfrak{g}$ is a Levi-Tanaka algebra.

Proof. Let $\mathfrak{r}$ be the radical of $\mathfrak{g}$. We want to show that $\mathfrak{r} = 0$. We consider two cases.

(a) *Assume* $\mathfrak{r}_{-1} = 0$.

In this case, we claim that $\mathfrak{r}_p = 0$ for $p \geq -1$. Indeed, we argue by recurrence on $p \geq -1$. We have $\mathfrak{r}_{-1} = 0$ by assumption. If $\mathfrak{r}_p = 0$ for some $p \geq -1$, we have $[\mathfrak{r}_{p+1}, \mathfrak{g}_{-1}] \subset \mathfrak{r}_p = 0$ and hence $\mathfrak{r}_{p+1} = 0$ because $\mathfrak{g}$ is transitive. This shows that $\mathfrak{r} \subset \mathfrak{n} = \oplus_{-\mu \leq p < -1} \mathfrak{g}_p$. Let $\mathfrak{s}$ be a Levi subalgebra of $\mathfrak{g}$: $\mathfrak{s}$ is semisimple and $\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{r}$. We have $\mathfrak{s} \simeq \mathfrak{g}/\mathfrak{r}$ and, since $\mathfrak{r}_{-1} = 0$, for every $X \in \mathfrak{g}_{-1}$ the subalgebra $\mathfrak{s}$ contains an element of the form $X + Z$ with $Z \in \mathfrak{n}$. Since

$$[X_1 + Z_1, [\ldots, [X_{\mu-1} + Z_{\mu-1}, X_\mu + Z_\mu] \ldots]] = [X_1, [\ldots, [X_{\mu-1}, X_\mu] \ldots]]$$

if $X_1, X_2, \ldots, X_\mu \in \mathfrak{g}_{-1}$ and $Z_1, Z_2, \ldots, Z_\mu \in \mathfrak{n}$, we obtain $\mathfrak{g}_{-\mu} \subset \mathfrak{s}$ because $\mathfrak{m}$ is fundamental.

Repeating a similar argument we deduce that also $\mathfrak{g}_{1-\mu}, \ldots, \mathfrak{g}_{-2}$ are contained in $\mathfrak{s}$ and then $\mathfrak{g} = \mathfrak{s}$ and $\mathfrak{r} = 0$.

(b) *Assume* $\mathfrak{r}_{-1} \neq 0$.

Since $\mathfrak{r}_{-1}$ is a $\rho_{-1}(\mathfrak{g}_0)$-invariant subspace of $\mathfrak{g}_{-1}$ and by assumption ρ_{-1} is irreducible, we have in this case $\mathfrak{r}_{-1} = \mathfrak{g}_{-1}$. Let $\mathfrak{r}^{(0)} = \mathfrak{r}$ and define recursively the ideals $\mathfrak{r}^{(\ell)} = [\mathfrak{r}^{(\ell-1)}, \mathfrak{r}^{(\ell-1)}]$ for $\ell > 0$. We have $\mathfrak{r}^{(\ell)} = 0$ for $\ell > 0$ and sufficiently large because $\mathfrak{r}$ is a solvable ideal of $\mathfrak{g}$. Then there is a smallest positive integer h such that $\mathfrak{r}_{-1}^{(h)} = 0$, while $\mathfrak{r}_{-1}^{(h-1)} \neq 0$. We note that $\mathfrak{r}^{(h-1)}$ is an ideal of $\mathfrak{g}$, in particular $\mathfrak{r}_{-1}^{(h-1)}$ is a $\rho_{-1}(\mathfrak{g}_0)$-invariant subspace of $\mathfrak{g}_{-1}$. Therefore $\mathfrak{r}_{-1}^{(h-1)} = \mathfrak{g}_{-1}$.

On the other hand, arguing as in (a), we prove that $\mathfrak{r}^{(h)} \subset \mathfrak{n}$. Therefore we have

$$[\mathfrak{r}_p^{(h-1)}, \mathfrak{g}_{-1}] = [\mathfrak{r}_p^{(h-1)}, \mathfrak{r}_{-1}^{(h-1)}] \subset \mathfrak{r}_{p-1}^{(h)} = 0 \qquad \text{for} \quad p \geq 0$$

and this implies that $\mathfrak{r}_p^{(h-1)} = 0$ by the transitivity of $\mathfrak{g}$. This gives a contradiction, because

$$\mathfrak{r}_0^{(h-1)} \supset [\mathfrak{r}_{-1}^{(h-1)}, \mathfrak{g}_1] = [\mathfrak{g}_{-1}, \mathfrak{g}_1] \neq 0.$$

This shows that $\mathfrak{r}_{-1} = 0$ and then $\mathfrak{r} = 0$ by (a).

Therefore $\mathfrak{g}$ is semisimple. It is simple, because if it were the direct sum of two semisimple ideals $\mathfrak{s}'$ and $\mathfrak{s}''$, then each of the subspaces $\mathfrak{s}'_{-1}$ and $\mathfrak{s}''_{-1}$ would be $\rho_{-1}(\mathfrak{g}_0)$-invariant. One of these, say $\mathfrak{s}'_{-1}$ is then equal to $\mathfrak{g}_{-1}$ and the other is 0 by the irreducibility of ρ_{-1}. But, since $\mathfrak{m}$ is fundamental, $\mathfrak{s}'$ is then a semisimple pseudocomplex prolongation of $\mathfrak{m}$ and therefore coincides with $\mathfrak{g}$. This gives $\mathfrak{s}'' = 0$ and completes the proof of the theorem. □

Remark 3.3.9 *Vice versa, when $\mathfrak{g}$ is semisimple, then the representation ρ_{-1} is completely reducible. Indeed, $\mathfrak{g}_0$ is reductive. Then its radical $\mathfrak{r}(\mathfrak{g}_0)$ is equal to its center $\mathfrak{z}(\mathfrak{g}_0)$ and therefore is contained in every Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ which is contained in $\mathfrak{g}_0$. Hence its elements are semisimple together with their ρ_{-1} representation. Then ρ_{-1} is completely reducible (cf.[6] Ch.I §6 Theorem 4).*

3.4 Solvable prolongations

We consider in this section criteria for the solvability of the canonical pseudocomplex prolongation.

Let $\mathfrak{m} = \oplus_{-\mu \leq p < 0} \mathfrak{g}_p$ be a pseudocomplex fundamental Lie algebra. We denote by $\mathfrak{f}$ the Hermitian symmetric form $\mathfrak{f} : \mathfrak{g}_{-1} \times \mathfrak{g}_{-1} \to \mathbb{C} \otimes \mathfrak{g}_{-2}$ such that

$$[X, Y] = \Im \mathfrak{f}(X, Y) \quad \text{for} \quad X, Y \in \mathfrak{g}_{-1}.$$

Let $\mathfrak{g}^*_{-2}$ be the dual space of $\mathfrak{g}_{-2}$ and, for every $\xi \in \mathfrak{g}^*_{-2}$ denote by $\mathfrak{f}_\xi$ the Hermitian symmetric form

$$\mathfrak{g}_{-1} \times \mathfrak{g}_{-1} \ni (X, Y) \to \mathfrak{f}_\xi(X, Y) = \langle \mathfrak{f}(X, Y), \xi \rangle \in \mathbb{C}.$$

Then we have the following:

THEOREM 3.4.1 *Let $\mathfrak{m}$ be a pseudocomplex fundamental Lie algebra of kind 2 and let $\mathfrak{g} = \oplus_{p \geq -2} \mathfrak{g}_p$ be its canonical pseudocomplex prolongation. Assume that*

(*i*) $\dim_{\mathbb{R}} \mathfrak{g}_{-2} \geq 2$;

(*ii*) *there is $\xi \in \mathfrak{g}^*_{-2}$ such that the Hermitian symmetric form $\mathfrak{f}_\xi$ is nondegenerate;*

(*iii*) $\quad \rho_{-2}(\mathfrak{g}_0) = \{\lambda\, Id_{\mathfrak{g}_{-2}} \mid \lambda \in \mathbb{R}\}$.

Then $\mathfrak{g}_p = 0$ *for all* $p \geq 1$. *Moreover* $\mathfrak{g}$ *is solvable if and only if* $\mathfrak{g}_0$ *is solvable.*

Proof. Let us prove that under the assumptions (*i*), (*ii*), (*iii*) we have $\mathfrak{g}_1 = 0$. By the transitivity of $\mathfrak{g}$ this implies that $\mathfrak{g}_p = 0$ for $p > 0$.

Let $V \in \mathfrak{g}_1$ and denote by $A : \mathfrak{g}_{-1} \to \mathfrak{g}_0$ and $B : \mathfrak{g}_{-2} \to \mathfrak{g}_{-1}$ the corresponding $\mathbb{R}$-linear homomorphisms. Then the following equations are satisfied:

(3.8)
$$\begin{cases} \rho_{-1}(A(X))Y - \rho_{-1}(A(Y))X = B([X,Y]) \quad \forall X, Y \in \mathfrak{g}_{-1} \\ [B(T), X] = \rho_{-2}(A(X))T \quad \forall T \in \mathfrak{g}_{-2}, \quad \forall X \in \mathfrak{g}_{-1}. \end{cases}$$

By assumption (*ii*) we can find a basis $\xi^1, \ldots, \xi^k$ of $\mathfrak{g}_{-2}^*$ such that $\mathfrak{f}_{\xi^j}$ is nondegenerate for $j = 1, \ldots, k$. We take the dual basis $T_1, \ldots, T_k$ of $\mathfrak{g}_{-2}$ defined by the condition that

$$\langle T_j, \xi^h \rangle = \delta_j^h \quad \text{for} \quad 1 \leq j, h \leq k.$$

Then the second equation in (3.8) yields, by assumption (*iii*):

$$\mathfrak{f}_{\xi^h}(B(T_j), X) = 0 \quad \forall X \in \mathfrak{g}_{-1} \qquad \text{for} \quad h \neq j$$

and hence $B(T_1) = \ldots = B(T_k) = 0$. This shows that, with $\mathfrak{h} = \{X \in \mathfrak{g} \mid [X, \mathfrak{g}_{-2}] = 0\}$, we have $V \in \mathfrak{h} \cap \mathfrak{g}_1 = \mathfrak{h}_1$. But $\mathfrak{h}_1 = 0$ by Theorem 3.1.1. Therefore $V = 0$ and this shows that $\mathfrak{g}_1 = 0$.

In this case we have $[\mathfrak{a}, \mathfrak{a}]_0 = [\mathfrak{a}_0, \mathfrak{a}_0]$ for every ideal $\mathfrak{a}$ of $\mathfrak{g}$ and then it is clear that $\mathfrak{g}$ is solvable if and only if $\mathfrak{g}_0$ is solvable. □

In the following we will assume that $\mathfrak{g}$ is a finite dimensional Levi-Tanaka algebra and denote by $\mathcal{S}$ the set of all semisimple elements of $\mathfrak{g}$ and by $\mathfrak{n}$ the set of all nilpotent elements of $\mathfrak{g}$.

LEMMA 3.4.2 *Let* $\mathfrak{g} = \oplus_{-\mu \leq p \leq \nu} \mathfrak{g}_p$ *be a solvable graded Lie algebra. We assume that* $\mathfrak{g}$ *has trivial center and is splittable (as a subalgebra of* $\mathfrak{gl}(\mathfrak{g})$ *via the adjoint representation). Then the set* $\mathfrak{n}$ *of its nilpotent elements is the maximal nilpotent ideal of* $\mathfrak{g}$.

Let $\mathcal{T}$ be the set of commutative Lie subalgebras of $\mathfrak{g}$ contained in $\mathcal{S}$ and $\mathcal{T}_1$ the subset of maximal elements of $\mathcal{T}$. Then for every $\mathfrak{t} \in \mathcal{T}_1$ we have a decomposition of $\mathfrak{g}$ into a semidirect sum:

$$\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{n}. \tag{3.9}$$

We can find a regular element $X_0 \in \mathcal{S} \cap \mathfrak{g}_0$ such that the centralizer

$$C_{\mathfrak{g}}(X_0) = \{Y \in \mathfrak{g} \mid [X_0, Y] = 0\}$$

is a Cartan subalgebra of $\mathfrak{g}$ containing the characteristic element E and contained in $\mathfrak{g}_0$.

Proof. The fact that $\mathfrak{n}$ is an ideal of $\mathfrak{g}$ follows from [6] Ch.I §5 Corollary 7 to Theorem 1. The above decomposition is in [6] Ch.VII §5 Corollary 2 to Proposition 6.

The last statement then follows from [6] Ch.VII §2 Theorem 1(iv). Indeed (cf. [6] Ch.VII §5 Corollary 1 to Proposition 6) any Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ contains a regular element A of $\mathfrak{g}$. Then, taking a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}_0$, we find a regular element $A \in \mathfrak{g}_0$. Its semisimple component S belongs to $\mathfrak{g}_0$ by Lemma 3.2.4. Since $\mathrm{ad}_{\mathfrak{g}} S$ has the same characteristic polynomial as $\mathrm{ad}_{\mathfrak{g}} A$, it follows that S is a regular element of $\mathfrak{g}$. Moreover E and S commute and thus the centralizer of S is a Cartan subalgebra of $\mathfrak{g}$ containing E and hence contained in $\mathfrak{g}_0$. □

COROLLARY 3.4.3 *Let $\mathfrak{g} = \oplus_{-\mu \le p \le \nu} \mathfrak{g}_p$ be a solvable Levi-Tanaka algebra such that all elements of $\mathfrak{g}_0$ are semisimple. Then $\mathfrak{g}_p = 0$ for every $p \ge 1$.*

Proof. We have $\oplus_{p \ne 0} \mathfrak{g}_p \subset \mathfrak{n}$. Therefore, if $X_1 \in \mathfrak{g}_1$, then

$$[X_1, Y_{-1}] \in [\mathfrak{g}_1, \mathfrak{g}_{-1}] = [\mathfrak{n}_1, \mathfrak{n}_{-1}] \subset \mathfrak{n}_0 = 0 \quad \forall Y_{-1} \in \mathfrak{g}_{-1}.$$

This shows that $\mathfrak{g}_1 = 0$ and hence $\mathfrak{g}_p = 0$ for all $p \ge 1$ because $\mathfrak{g}$ is transitive. □

3.5 Pseudocomplex Levi-Mal'čev decomposition of Levi-Tanaka algebras

We turn in this section to the general case. First we prove

THEOREM 3.5.1 *Let $\mathfrak{g} = \oplus_{-\mu \leq p \leq \nu} \mathfrak{g}_p$ be a finite dimensional Levi-Tanaka algebra. Then its radical $\mathfrak{r}$ is pseudocomplex and $\mathfrak{g}$ contains a Levi subalgebra $\mathfrak{L}$ which is graded and pseudocomplex, i.e. a pseudocomplex semisimple graded Lie subalgebra $\mathfrak{L}$ such that*

$$\mathfrak{g} = \mathfrak{L} \oplus \mathfrak{r}. \tag{3.10}$$

Proof. As usual we denote by $\mathcal{S}$ the set of all semisimple elements of $\mathfrak{g}$ and by $\mathfrak{r}$ the radical of $\mathfrak{g}$. Being an ideal of $\mathfrak{g}$, the radical $\mathfrak{r}$ is graded. By Lemma 3.2.5 we can find a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ which is contained in $\mathfrak{g}_0$. Then, since $\mathfrak{g}$ is splittable,

$$\mathfrak{t} = \mathfrak{h} \cap \mathcal{S} \subset \mathfrak{g}_0$$

is a maximal commutative Lie subalgebra of semisimple elements of $\mathfrak{g}$ and

$$\mathfrak{h} = \{X \in \mathfrak{g} \,|\, [X, \mathfrak{t}] = 0\}.$$

Next we note that $\mathfrak{h} \cap \mathfrak{r}$ is a Cartan subalgebra of $\mathfrak{r}$ and that $\mathfrak{r}$ is also splittable. Then

$$\mathfrak{t}' = \mathfrak{h} \cap \mathfrak{r} \cap \mathcal{S} \subset \mathfrak{r}_0$$

is a maximal commutative Lie subalgebra of semisimple elements of $\mathfrak{r}$ and we have

$$\begin{aligned} \mathfrak{h} \cap \mathfrak{r} &= \{X \in \mathfrak{r} \,|\, [X, \mathfrak{t}] = 0\} \\ &= \{X \in \mathfrak{r} \,|\, [X, \mathfrak{t}'] = 0\}. \end{aligned}$$

Let $\mathfrak{n}$ be the ideal of $\mathfrak{r}$ of nilpotent elements of $\mathfrak{r}$. Then $\mathfrak{r} = \mathfrak{t}' \oplus \mathfrak{n}$.

We define the subalgebra $\mathfrak{z}$ of $\mathfrak{g}$ by setting

$$\mathfrak{z} = \{X \in \mathfrak{g} \,|\, [X, \mathfrak{t}'] = 0\}.$$

We note that $\mathfrak{z} = \oplus_{-\mu \leq p \leq \nu} \mathfrak{z}_p$ is a pseudocomplex graded Lie subalgebra of $\mathfrak{g}$. We denote by $\mathrm{rad}(\mathfrak{z})$ its radical. Then we have:

(*i*) $\mathrm{rad}(\mathfrak{z}) = \mathfrak{h} \cap \mathfrak{r} = \mathfrak{z} \cap \mathfrak{r} \subset \mathfrak{z}_0$;

(*ii*) $\mathrm{rad}(\mathfrak{z})$ is a nilpotent ideal in $\mathfrak{z}$;

(*iii*) $[\mathrm{rad}(\mathfrak{z}), \mathfrak{z}_p] = 0$ for $p \neq 0$;

(*iv*) every Levi subalgebra of $\mathfrak{z}$ is a Levi subalgebra of $\mathfrak{g}$.

To prove (*i*) and (*iv*) we use [6] Ch.VII §5 Proposition 7: for every Levi subalgebra $\mathfrak{L}$ of $\mathfrak{z}$ we have a direct sum decomposition $\mathfrak{g} = \mathfrak{L} \oplus \mathfrak{t}' \oplus \mathfrak{n}$, with $\mathfrak{r} = \mathfrak{t}' \oplus \mathfrak{n}$. This implies (*iv*). Moreover, $\mathfrak{g} = \mathfrak{z} + \mathfrak{r}$. Hence $\mathfrak{z}/\mathfrak{z} \cap \mathfrak{r} \simeq \mathfrak{g}/\mathfrak{r}$, from which (*i*) follows. Now (*ii*) is a consequence of the fact that $\mathrm{rad}(\mathfrak{z})$ is contained in the nilpotent Lie algebra $\mathfrak{h}$ and (*iii*) of the fact that the ideal $\mathrm{rad}(\mathfrak{z})$ is contained in $\mathfrak{z}_0$.

We claim that $\mathfrak{z}$ contains a graded pseudocomplex Levi subalgebra. This follows from the lemma below.

LEMMA 3.5.2 *Let $\mathfrak{g} = \oplus_{-\mu \leq p \leq \nu} \mathfrak{g}_p$ be a finite dimensional graded Lie algebra, whose radical $\mathfrak{r}$ is contained in $\mathfrak{g}_0$. Then $\mathfrak{g}$ contains a graded Levi subalgebra $\mathfrak{L} = \oplus_{-\mu \leq p \leq \nu} \mathfrak{L}_p$. If $\mathfrak{g}$ is pseudocomplex, then also $\mathfrak{L}$ is pseudocomplex.*

Proof. We argue by induction on the order of solvability of $\mathfrak{r}$, i.e. the smallest nonnegative integer h such that $\mathfrak{r}^{(h)} = 0$ (by $\mathfrak{r}^{(h)}$ we indicate the h-th term of the derived series of $\mathfrak{r}$). If $h = 0$, this means that $\mathfrak{r} = 0$ and then $\mathfrak{g}$ is semisimple and there is nothing to prove.

Assume now that $h > 0$ and that the statement of the theorem is true for graded Lie algebras with the radical composed of homogeneous terms of degree 0 and order of solvability less than h. Let $\mathfrak{L}$ be a Levi subalgebra of $\mathfrak{g}$ and set $\mathfrak{L}_0 = \mathfrak{L} \cap \mathfrak{g}_0$. Set $\mathfrak{q}_p = \mathfrak{g}_p$ for $p \neq 0$ and $\mathfrak{q}_0 = \mathfrak{L}_0 \oplus \mathfrak{r}^{(1)}$. We claim that $\mathfrak{q} = \oplus_{p \in \mathbb{Z}} \mathfrak{q}_p$ is a Lie subalgebra of $\mathfrak{g}$ with radical $\mathfrak{r}^{(1)}$.

To prove the first asset, it suffices to show that $\mathfrak{q}$ contains the Lie product $[X, Y]$ of every pair of homogeneous elements $X \in \mathfrak{q}_p$ and $Y \in \mathfrak{q}_q$. This is obviously true when $p + q \neq 0$ because in this case $[\mathfrak{q}_p, \mathfrak{q}_q] \subset [\mathfrak{g}_p, \mathfrak{g}_q] \subset \mathfrak{g}_{p+q} = \mathfrak{q}_{p+q}$. It is also obvious when $p = q = 0$ because $\mathfrak{L}_0$ is a Lie subalgebra of $\mathfrak{g}_0$ and therefore also $\mathfrak{L} \oplus \mathfrak{r}^{(1)}$, because $\mathfrak{r}^{(1)}$ is an ideal in $\mathfrak{g}$.

Then we only need to consider the case where $q = -p \neq 0$. We can find $\tilde{X}$, $\tilde{Y} \in \mathfrak{L}$ such that $\tilde{X} - X$, $\tilde{Y} - Y \in \mathfrak{r}$. Then we obtain

$$\begin{aligned}[\tilde{X}, \tilde{Y}] &= [X, Y] + [\tilde{X} - X, Y] + [X, \tilde{Y} - Y] + [\tilde{X} - X, \tilde{Y} - Y] \\ &= [X, Y] + [\tilde{X} - X, \tilde{Y} - Y] \in \mathfrak{L}_0\end{aligned}$$

because $[\mathfrak{r}, \mathfrak{g}_\ell] = 0$ if $\ell \neq 0$. Therefore

$$[X, Y] = [\tilde{X}, \tilde{Y}] - [\tilde{X} - X, \tilde{Y} - Y] \in \mathfrak{L}_0 \oplus \mathfrak{r}^{(1)}.$$

To show that $\mathfrak{r}^{(1)}$ is the radical of $\mathfrak{q}$, we observe that $\mathfrak{q}/\mathfrak{r}^{(1)}$ is isomorphic to $\mathfrak{g}/\mathfrak{r}$. Indeed the map $\mathfrak{q} \to \mathfrak{g}/\mathfrak{r}$ induced by the projection is clearly surjective and its kernel is given by $\mathfrak{q} \cap \mathfrak{r} = \mathfrak{r}^{(1)}$. From this isomorphism it also follows that every Levi subalgebra of $\mathfrak{q}$ is also a Levi subalgebra of $\mathfrak{g}$. Since $\mathfrak{r}^{(1)^{(h-1)}} = \mathfrak{r}^{(h)} = 0$, by the inductive assumption $\mathfrak{q}$ contains a graded Levi subalgebra $\mathfrak{L} = \oplus_{p \in \mathbb{Z}} \mathfrak{L}_p$, which is also a graded Levi subalgebra of $\mathfrak{g}$. We note that $\mathfrak{L}_p = \mathfrak{g}_p$ for $p \neq 0$ and therefore $\mathfrak{L}$ is pseudocomplex when $\mathfrak{g}$ is pseudocomplex. □

To complete the proof of the theorem, we consider the subalgebra $\mathrm{ad}(\mathfrak{t}')$ of $\mathfrak{gl}_{\mathbb{R}}(\mathfrak{g})$. It is a commutative algebra whose elements are semisimple. Therefore we have a direct sum decomposition

$$\mathfrak{g} = \mathfrak{z} \oplus \mathfrak{b}$$

where

$$\mathfrak{z} = \{X \in \mathfrak{g} \mid [X, \mathfrak{t}']\} = 0 \qquad \text{and} \qquad \mathfrak{b} = [\mathfrak{g}, \mathfrak{t}'].$$

Both $\mathfrak{z}$ and $\mathfrak{b}$ are graded because $\mathfrak{t}' \subset \mathfrak{g}_0$; moreover $\mathfrak{z}_{-1}$ and $\mathfrak{b}_{-1}$ are J-invariant. From $\mathfrak{z}_{-1} = \mathfrak{L}_{-1}$ and $\mathfrak{b}_{-1} \subset \mathfrak{r}_{-1}$, we obtain that $\mathfrak{r}_{-1} = \mathfrak{b}_{-1}$ and therefore $\mathfrak{r}$ is pseudocomplex. □

Remark 3.5.3 *The assignement of a maximal commutative Lie subalgebra $\mathfrak{t}'$ of $\mathfrak{r}$ contained in $\mathfrak{r}_0$, whose elements are semisimple, uniquely determines the elements of $\oplus_{p \neq 0} \mathfrak{L}_p$ and hence the semisimple subalgebra $\mathfrak{s} = \oplus_{p \neq 0} \mathfrak{L}_p + [\oplus_{p \neq 0} \mathfrak{L}_p, \oplus_{p \neq 0} \mathfrak{L}_p]$ of $\mathfrak{g}$. This subalgebra $\mathfrak{s}$ is made more precise in the statement below.*

THEOREM 3.5.4 *Let $\mathfrak{g} = \oplus_{-\mu \leq p \leq \nu} \mathfrak{g}_p$ be a finite dimensional Levi-Tanaka algebra and let $\mathfrak{L}$ be a pseudocomplex graded Levi subalgebra of $\mathfrak{g}$. Then*

$$\mathfrak{a} = \{X \in \mathfrak{L}_0 \mid [X, \mathfrak{L}_{-1}] = 0\}$$

is an ideal of $\mathfrak{L}$ and there is a pseudocomplex semisimple graded subalgebra $\mathfrak{s}$ of $\mathfrak{g}$ such that

$$\mathfrak{L} = \mathfrak{a} \oplus \mathfrak{s}.$$

Proof. First we note that $\mathfrak{a}$ is an ideal in $\mathfrak{L}$. Indeed the subalgebra $\mathfrak{m}' = \oplus_{p<0} \mathfrak{L}_p$ of $\mathfrak{m} = \oplus_{p<0} \mathfrak{g}_p$ is fundamental and therefore we obtain $[X, \mathfrak{m}'] = 0$ for all $X \in \mathfrak{a}$. Next we show that $[\mathfrak{a}, \mathfrak{L}_p] = 0$ for $p > 0$. Indeed, if $[X, Y] \neq 0$ for some $X \in \mathfrak{a}$ and some $Y \in \mathfrak{L}_p$, there is $Z \in \mathfrak{L}_{-p}$ such that $\kappa_{\mathfrak{L}}([X, Y], Z) \neq 0$, $\kappa_{\mathfrak{L}}$ being the Killing form of the semisimple Lie algebra $\mathfrak{L}$. But then

$$\kappa_{\mathfrak{L}}([X, Y], Z) = -\kappa_{\mathfrak{L}}(Y, [X, Z]) = 0$$

gives a contradiction. Finally the fact that $[\mathfrak{a}, \mathfrak{L}_0] \subset \mathfrak{a}$ follows because

$$[[X, Y], Z] = [[X, Z], Y] + [X, [Y, Z]] = 0 \quad \forall X \in \mathfrak{a},\ Y \in \mathfrak{L}_0,\ Z \in \mathfrak{L}_{-1}.$$

We note that the graded semisimple Lie algebra $\mathfrak{L}$ contains an element $E_{\mathfrak{L}} \in \mathfrak{L}_0$ such that $[E, X] = pX$ for each $p \in \mathbb{Z}$ and $X \in \mathfrak{L}_p$. Thus every ideal of $\mathfrak{L}$ is graded. We write $\mathfrak{L}$ as the direct sum of $\mathfrak{a}$ and a graded semisimple ideal $\mathfrak{s}$ of $\mathfrak{L}$. Since $\mathfrak{L}_{-1} = \mathfrak{s}_{-1}$, the ideal $\mathfrak{s}$ is pseudocomplex. □

A finite dimensional Levi-Tanaka algebra $\mathfrak{g} = \oplus_{-\mu \leq p \leq \nu} \mathfrak{g}_p$ with a graded Levi subalgebra contained in $\mathfrak{g}_0$ (resp. in $\mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$), will be called *almost-solvable* (resp. *weakly-solvable*).

Remark 3.5.5 *A Levi-Tanaka algebra is almost-solvable if and only if the characteristic element E of $\mathfrak{g}$ belongs to the radical $\mathfrak{r}$ of $\mathfrak{g}$.*

COROLLARY 3.5.6 *Let $\mathfrak{g} = \oplus_{-\mu \leq p \leq \nu} \mathfrak{g}_p$ be a finite dimensional Levi-Tanaka algebra with radical $\mathfrak{r}$. The following statements are equivalent:*

(i) $\mathfrak{g}$ *is semisimple;*
(ii) $\oplus_{p<0} \mathfrak{r}_p = 0$;
(iii) $\mathfrak{r}_{-1} = 0$;
(iv) $\mathfrak{r}_{-2} = 0$.

Proof. Clearly $(i)\Rightarrow(ii)\Rightarrow(iii)$. Let $\mathfrak{g} = \mathfrak{L} \oplus \mathfrak{r}$ be a graded Levi-Mal'čev decomposition given by Theorem 3.5.1. Then $(iii)\Rightarrow(iv)$ because $\mathfrak{m} = \oplus_{p<0}\mathfrak{g}_p$ is fundamental. Assume now $\mathfrak{r}_{-2} = 0$. We have $\mathfrak{L}_{-1} = \mathfrak{g}_{-1}$ because $\mathfrak{g}$ is nondegenerate, and therefore $\mathfrak{m} = \oplus_{p<0}\mathfrak{L}_p$ because $\mathfrak{m}$ is fundamental. By Theorem 3.3.7, the algebra $\mathfrak{g}$ coincides with $\mathfrak{L}$. This shows that $(iii)\Rightarrow(i)$. □

COROLLARY 3.5.7 *Let $\mathfrak{g} = \oplus_{-\mu\leq p\leq\nu}\mathfrak{g}_p$ be a finite dimensional Levi-Tanaka algebra. If the representation ρ_{-1} of $\mathfrak{g}_0$ in $\mathfrak{g}_{-1}$ is irreducible, then $\mathfrak{g}$ is either simple or almost-solvable.*

More precisely, it is simple when $\mathfrak{g}_1 \neq 0$ and almost-solvable when $\mathfrak{g}_1 = 0$.

Proof. By Theorem 3.3.8, if $\mathfrak{g}$ is not simple, then $\mathfrak{g}_1 = 0$ and so $\mathfrak{g}$ is almost-solvable. □

COROLLARY 3.5.8 *Let $\mathfrak{g} = \oplus_{-\mu\leq p\leq\nu}\mathfrak{g}_p$ be a finite dimensional Levi-Tanaka algebra with radical $\mathfrak{r}$. If the representation ρ_{-2} of $\mathfrak{g}_0$ in $\mathfrak{g}_{-2}$ is irreducible, then $\mathfrak{g}$ is either simple or weakly-solvable.*

Moreover, $\mathfrak{r}_1 = 0$. Hence $\mathfrak{g}$ is simple if and only if $\mathfrak{g}_2 \neq 0$, weakly-solvable if and only if $\mathfrak{g}_2 = 0$, and almost-solvable if and only if $\mathfrak{g}_1 = 0$.

Proof. By Theorem 3.5.1 we have a decomposition $\mathfrak{g} = \mathfrak{L} \oplus \mathfrak{r}$ where $\mathfrak{L}$ is a semisimple graded Lie subalgebra and $\mathfrak{r}$ is the radical of $\mathfrak{g}$. By the assumption, we have either $\mathfrak{g}_{-2} = \mathfrak{r}_{-2}$ or $\mathfrak{g}_{-2} = \mathfrak{L}_{-2}$. In the first case, we have that $\mathfrak{L}$ is contained in $\mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$, and so $\mathfrak{g}$ is weakly-solvable. In the second case, we have $\mathfrak{r}_{-2} = 0$, and hence $\mathfrak{g}$ is semisimple by Corollary 3.5.6. It is a sum of simple graded ideals by Corollary 3.2.3, which are not included in $\mathfrak{g}_0$ as $\mathfrak{g}$ is transitive. Since $\mathfrak{m} = \oplus_{p<0}\mathfrak{g}_p$ is fundamental, these ideals have a nonzero component in $\mathfrak{g}_{-1}$ and, since $\mathfrak{g}$ is nondegenerate, they have also a nonzero component in $\mathfrak{g}_{-2}$, which is an invariant subspace of $\mathfrak{g}_{-2}$ for ρ_{-2}. As ρ_{-2} is irreducible, we have that $\mathfrak{g}$ is simple.

Let $\mathfrak{n}$ be the maximal nilpotent ideal of $\mathfrak{g}$. By [6] Ch.I § 5 Corollary 7 to Theorem 1, it consists of the elements $X \in \mathfrak{r}$ such that $\mathrm{ad}_{\mathfrak{g}}X$ is nilpotent, hence it contains the subspace $\oplus_{p\neq 0}\mathfrak{r}_p$. Let ℓ be the greatest integer such that $\mathfrak{n}^\ell \cap \mathfrak{g}_{-2} \neq 0$ and $\mathfrak{n}^{\ell+1} \cap \mathfrak{g}_{-2} = 0$. Note that, as $\mathfrak{g}$ is nondegenerate, we

have $\mathfrak{n}^\ell \cap \mathfrak{g}_{-1} = 0$. Moreover, as ρ_{-2} is irreducible, the ideal $\mathfrak{n}^\ell$ contains the subspace $\mathfrak{g}_{-2}$ and then all the subalgebra $\oplus_{p\leq -2}\mathfrak{g}_p$. Consider now $X \in \mathfrak{r}_1 = \mathfrak{n}_1$. For $p = -2, -3$, we have $[X, \mathfrak{g}_p] \subset [\mathfrak{n}, \mathfrak{n}^\ell] \cap \mathfrak{g}_{p+1} = \mathfrak{n}^{\ell+1} \cap \mathfrak{g}_{p+1} = 0$, and so, by Remark 3.2, $X = 0$. □

The following theorem describes the structure of completely-reducible Levi-Tanaka algebras (see also [48]).

COROLLARY 3.5.9 *Let $\mathfrak{g} = \oplus_{-\mu\leq p\leq \nu}\mathfrak{g}_p$ be a finite dimensional Levi-Tanaka algebra with radical $\mathfrak{r} = \oplus_{p\in\mathbb{Z}}\mathfrak{r}_p$. If the representation ρ_{-1} of $\mathfrak{g}_0$ in $\mathfrak{g}_{-1}$ is completely reducible, then $\mathfrak{r}_0$ is abelian, $\mathfrak{r}_1 = 0$ and $\oplus_{p<0}\mathfrak{r}_p$ is the maximal nilpotent ideal $\mathfrak{n}$ of $\mathfrak{g}$. In particular $\mathfrak{g}$ decomposes into the direct sum*

$$\mathfrak{g} = \mathfrak{L} \oplus \mathfrak{r}_0 \oplus \mathfrak{n}$$

where $\mathfrak{L}$ is a graded pseudocomplex semisimple Lie subalgebra.

Proof. Indeed in this case $\mathfrak{r}_0$ is an abelian algebra whose elements are semisimple (see [6] Ch.VII §5 Proposition 7 (i)). But $[X, Y]$ is a nilpotent element of $\mathfrak{r}_0$ for every $X \in \mathfrak{r}_1$ and every $Y \in \mathfrak{g}_{-1}$, because $\mathfrak{r}_1$ is contained in the maximal nilpotent ideal of the adjoint representation of $\mathfrak{g}$. Therefore $[X, Y] = 0$ for every $X \in \mathfrak{r}_1$ and every $Y \in \mathfrak{g}_{-1}$ and hence $\mathfrak{r}_1 = 0$. □

3.6 Levi factors of Levi-Tanaka algebras

In this section we investigate some structural properties of the semisimple pseudocomplex graded Lie algebras that appear in the Levi-Mal'čev decomposition of a Levi-Tanaka algebra.

3.6.1 Construction of the element $\tilde{J}$

First we prove the following

LEMMA 3.6.1 *Let $\mathfrak{g} = \oplus_{-\mu\leq p\leq \mu}\mathfrak{g}_p$ be a finite dimensional semisimple pseudocomplex graded Lie algebra. Then there is a unique complex structure J_1 on $\mathfrak{g}_1$ such that:*

$$\kappa_{\mathfrak{g}}(J_1X, Y) = -\kappa_{\mathfrak{g}}(X, JY) \quad \forall X \in \mathfrak{g}_1,\ Y \in \mathfrak{g}_{-1}, \tag{3.11}$$

where $\kappa_{\mathfrak{g}}$ denotes the Killing form of $\mathfrak{g}$.

This complex structure J_1 has the properties:

(*i*) *$\rho_1(\mathfrak{g}_0)$ is a real subalgebra of the algebra $\mathfrak{gl}_{\mathbb{C}}(\mathfrak{g}_1)$ of endomorphisms of $\mathfrak{g}_1$ which are $\mathbb{C}$-linear for the complex structure defined by J_1;*

(*ii*) $[J_1X, Y] = -[X, JY] \quad \forall X \in \mathfrak{g}_1,\ Y \in \mathfrak{g}_{-1}$.

Proof. Since the Killing form $\kappa_{\mathfrak{g}}$ of $\mathfrak{g}$ defines a duality pairing between $\mathfrak{g}_1$ and $\mathfrak{g}_{-1}$, formula (3.11) uniquely defines an $\mathbb{R}$-linear map $J_1 : \mathfrak{g}_1 \to \mathfrak{g}_1$. It is a complex structure on $\mathfrak{g}_1$ because

$$\begin{aligned} \kappa_{\mathfrak{g}}(J_1^2X, Y) &= -\kappa_{\mathfrak{g}}(J_1X, JY) = \kappa_{\mathfrak{g}}(X, J^2Y) \\ &= -\kappa_{\mathfrak{g}}(X, Y) \qquad \forall X \in \mathfrak{g}_1,\ \forall Y \in \mathfrak{g}_{-1} \end{aligned}$$

implies that $J_1^2 = -Id$ on $\mathfrak{g}_1$.
(*i*) Let $A \in \mathfrak{g}_0$ and $X \in \mathfrak{g}_1$. Then, for every $Y \in \mathfrak{g}_{-1}$ we obtain

$$\begin{aligned} \kappa_{\mathfrak{g}}([A, J_1X], Y) &= -\kappa_{\mathfrak{g}}(J_1X, [A, Y]) = \kappa_{\mathfrak{g}}(X, [A, JY]) \\ &= -\kappa_{\mathfrak{g}}([A, X], JY) = \kappa_{\mathfrak{g}}(J_1[A, X], Y). \end{aligned}$$

This implies that $[A, J_1X] = J_1[A, X]$ for every $A \in \mathfrak{g}_0$ and $X \in \mathfrak{g}_1$.
(*ii*) Let $X \in \mathfrak{g}_1$, $Y \in \mathfrak{g}_{-1}$. Then for every $A \in \mathfrak{g}_0$ we obtain

$$\begin{aligned} \kappa_{\mathfrak{g}}([J_1X, Y], A) &= \kappa_{\mathfrak{g}}(J_1X, [Y, A]) = -\kappa_{\mathfrak{g}}(X, [JY, A]) \\ &= -\kappa_{\mathfrak{g}}([X, JY], A) \end{aligned}$$

and this implies (*ii*) because $\kappa_{\mathfrak{g}}$ is nondegenerate on $\mathfrak{g}_0$. □

In the following we shall write for semplicity J instead of J_1 also for the complex structure on $\mathfrak{g}_1$ defined by the lemma above.

Let $\mathfrak{g} = \oplus_{-\mu \le p \le \mu} \mathfrak{g}_p$ be a finite dimensional semisimple graded real Lie algebra. It has a unique characteristic element E and is splittable. Therefore, by Lemma 3.2.5, we can fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ containing E and contained in $\mathfrak{g}_0$. Its complexification $\mathfrak{h}^{\mathbb{C}}$ is a Cartan subalgebra of the complexification $\mathfrak{g}^{\mathbb{C}}$ of $\mathfrak{g}$. Setting $\mathfrak{g}_p^{\mathbb{C}} = \mathbb{C} \otimes_{\mathbb{R}} \mathfrak{g}_p$, we have $\mathfrak{g}^{\mathbb{C}} = \oplus_{-\mu \le p \le \mu} \mathfrak{g}_p^{\mathbb{C}}$ and $\mathfrak{g}^{\mathbb{C}}$ is a finite dimensional semisimple graded complex Lie algebra.

Let $\mathcal{R} \subset \mathrm{Hom}_{\mathbb{C}}(\mathfrak{h}^{\mathbb{C}}, \mathbb{C})$ be the set of nonzero roots of $\mathfrak{h}^{\mathbb{C}}$ and for $\alpha \in \mathcal{R}$ we denote by

$$\mathfrak{g}^{\alpha} = \{X \in \mathfrak{g}^{\mathbb{C}} \mid [H, X] = \alpha(H)X \quad \forall H \in \mathfrak{h}^{\mathbb{C}}\} \tag{3.12}$$

the eigenspace corresponding to the root α.

LEMMA 3.6.2 *Let $\mathfrak{g}$ and $\mathcal{R}$ be as above. Then, for every $\alpha \in \mathcal{R}$, we have $\alpha(E) \in \mathbb{Z}$ and $\mathfrak{g}^{\alpha} \subset \mathfrak{g}^{\mathbb{C}}_{\alpha(E)}$.*

If $\mathfrak{m} = \oplus_{p<0}\mathfrak{g}_p$ is fundamental, then there exists a basis $\mathcal{B}$ of the root system $\mathcal{R}$ such that $\alpha(E) \in \{0, -1\}$ for every $\alpha \in \mathcal{B}$.

Proof. The first assertion is a consequence of the fact that all subspaces $\mathfrak{g}^{\mathbb{C}}_p$ are invariant under $\mathrm{ad}(\mathfrak{g}^{\mathbb{C}}_0)$ and $\mathfrak{h}^{\mathbb{C}} \subset \mathfrak{g}^{\mathbb{C}}_0$. Next we note that, if $\mathfrak{m} = \oplus_{p<0}\mathfrak{g}_p$ is fundamental, then all roots α with $\alpha(E) < -1$ can be decomposed as sums of roots β's with $-1 \leq \beta(E) \leq 0$. Since $-\alpha(E) < 0$ when $\alpha(E) > 0$, it is also clear that there are simple systems $\mathcal{B}$ of roots with $\alpha(E) \leq 0$ for every $\alpha \in \mathcal{B}$. By the observation above, if α is a simple root with $\alpha(E) \leq 0$, we have $\alpha(E) \in \{0, -1\}$. □

LEMMA 3.6.3 *Let $\mathfrak{g} = \oplus_{-\mu \leq p \leq \mu}\mathfrak{g}_p$ be a finite dimensional semisimple pseudocomplex graded Lie algebra and let $\mathfrak{h}$ and $\mathcal{R}$ be as above. Assume that $\alpha \in \mathcal{R}$ and $\alpha(E) = \pm 1$. Then all vectors in $\mathfrak{g}^{\alpha}$ are either of the form $X + \sqrt{-1}JX$ with $X \in \mathfrak{g}_{\pm 1}$ or of the form $X - \sqrt{-1}JX$ with $X \in \mathfrak{g}_{\pm 1}$.*

Proof. Assume $\alpha(E) = -1$. We still denote by J the $\mathbb{C}$-linear extension of J to $\mathfrak{g}^{\mathbb{C}}_{-1}$. Since J commutes with $\rho_{-1}(H)$ for every $H \in \mathfrak{h}^{\mathbb{C}}$, the subspace $\mathfrak{g}^{\alpha}$ is J-invariant. Since $\mathfrak{g}^{\alpha}$ is 1-dimensional, we obtain, for $X + \sqrt{-1}JY \in \mathfrak{g}^{\alpha} \setminus \{0\}$,

$$J(X + \sqrt{-1}JY) = JX - \sqrt{-1}Y = \pm\sqrt{-1}(X + \sqrt{-1}JY)\,.$$

Then $Y = X$ or $Y = -X$ according to the fact that the eigenvalue of J on $\mathfrak{g}^{\alpha}$ is $-\sqrt{-1}$ or $\sqrt{-1}$.

The argument for $\alpha(E) = 1$ is similar and therefore is omitted. □

We introduce the notation:

$$\mathfrak{g}_{-1}^{(1,0)} = \{X - \sqrt{-1}JX \mid X \in \mathfrak{g}_{-1}\}, \qquad \mathfrak{g}_{1}^{(1,0)} = \{X - \sqrt{-1}JX \mid X \in \mathfrak{g}_{1}\},$$
$$\mathfrak{g}_{-1}^{(0,1)} = \{X + \sqrt{-1}JX \mid X \in \mathfrak{g}_{-1}\}, \qquad \mathfrak{g}_{1}^{(0,1)} = \{X + \sqrt{-1}JX \mid X \in \mathfrak{g}_{1}\}.$$

By (*ii*) in Lemma 3.6.1, $\mathfrak{g}_{-1}^{(0,1)} \oplus \mathfrak{g}_{1}^{(0,1)}$ and $\mathfrak{g}_{-1}^{(1,0)} \oplus \mathfrak{g}_{1}^{(1,0)}$ are commutative complex Lie subalgebras of $\mathfrak{g}^{\mathbb{C}}$. From this observation we obtain the

LEMMA 3.6.4 *Let $\mathfrak{g} = \oplus_{-\mu \le p \le \mu} \mathfrak{g}_p$ be a finite dimensional semisimple pseudocomplex graded Lie algebra. Let $\alpha \in \mathcal{R}$ be such that $\alpha(E) = -1$. Then:*

$$\mathfrak{g}^{\alpha} \subset \mathfrak{g}_{-1}^{(1,0)} \iff \mathfrak{g}^{-\alpha} \subset \mathfrak{g}_{1}^{(0,1)}$$
$$\mathfrak{g}^{\alpha} \subset \mathfrak{g}_{-1}^{(0,1)} \iff \mathfrak{g}^{-\alpha} \subset \mathfrak{g}_{1}^{(1,0)}.$$

We are now able to prove the following:

THEOREM 3.6.5 *Let $\mathfrak{g} = \oplus_{-\mu \le p \le \mu} \mathfrak{g}_p$ be a finite dimensional semisimple pseudocomplex graded Lie algebra such that $\mathfrak{m} = \oplus_{p<0} \mathfrak{g}_p$ is fundamental. Then there exists a unique element $\tilde{J}$ in the center $\mathfrak{z}(\mathfrak{g}_0)$ of $\mathfrak{g}_0$ such that*

$$[\tilde{J}, X] = JX \quad \forall X \in \mathfrak{g}_{-1}. \tag{3.13}$$

Proof. We keep all notation and conventions introduced above. In particular we fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ which is contained in $\mathfrak{g}_0$ and a basis $\mathcal{B}$ for its root system $\mathcal{R}$ such that $\alpha(E) \in \{0, -1\}$ for every $\alpha \in \mathcal{B}$.

For each $\alpha \in \mathcal{B}$ we fix $X_\alpha \in \mathfrak{g}^\alpha$, $H_\alpha \in \mathfrak{h}^{\mathbb{C}}$ and $X_{-\alpha} \in \mathfrak{g}^{-\alpha}$ such that

$$[X_\alpha, X_{-\alpha}] = -H_\alpha, \quad [H_\alpha, X_\alpha] = 2X_\alpha, \quad [H_\alpha, X_{-\alpha}] = -2X_{-\alpha}.$$

We obtain a generating family

$$\{(X_\alpha, H_\alpha, X_{-\alpha}) \mid \alpha \in \mathcal{B}\}$$

corresponding to the basis $\mathcal{B}$. For every real number t we obtain a new generating family

$$\{(X_\alpha(t), H_\alpha(t), X_{-\alpha}(t)) \mid \alpha \in \mathcal{B}\}$$

corresponding to the same basis $\mathcal{B}$ by setting

$$\begin{aligned}
H_\alpha(t) &= H_\alpha \quad \forall \alpha \in \mathcal{B} \\
X_\alpha(t) &= \begin{cases} X_\alpha & \text{if } \alpha \in \mathcal{B},\ \alpha(E) = 0 \\ e^{\sqrt{-1}t} X_\alpha & \text{if } \alpha \in \mathcal{B},\ \mathfrak{g}^\alpha \subset \mathfrak{g}_{-1}^{(1,0)} \\ e^{-\sqrt{-1}t} X_\alpha & \text{if } \alpha \in \mathcal{B},\ \mathfrak{g}^\alpha \subset \mathfrak{g}_{-1}^{(0,1)} \end{cases} \\
X_{-\alpha}(t) &= \begin{cases} X_{-\alpha} & \text{if } \alpha \in \mathcal{B},\ \alpha(E) = 0 \\ e^{-\sqrt{-1}t} X_{-\alpha} & \text{if } \alpha \in \mathcal{B},\ \mathfrak{g}^\alpha \subset \mathfrak{g}_{-1}^{(1,0)} \\ e^{\sqrt{-1}t} X_{-\alpha} & \text{if } \alpha \in \mathcal{B},\ \mathfrak{g}^\alpha \subset \mathfrak{g}_{-1}^{(0,1)} \end{cases}.
\end{aligned}$$

Then, by [6] Ch.VIII §4 Theorem 2, for each t there is a unique automorphism ϕ_t of $\mathfrak{g}^{\mathbb{C}}$ such that

$$\phi_t(X_\alpha) = X_\alpha(t)\,, \quad \phi_t(H_\alpha) = H_\alpha(t)\,, \quad \phi_t(X_{-\alpha}) = X_{-\alpha}(t) \qquad \forall \alpha \in \mathcal{B}\,.$$

Clearly $\mathbb{R} \ni t \to \phi_t$ is a 1-parameter group in the Lie group of automorphisms of $\mathfrak{g}^{\mathbb{C}}$. Its derivative in 0 defines a derivation $\tilde{J}$ on $\mathfrak{g}^{\mathbb{C}}$. Since $\mathfrak{g}^{\mathbb{C}}$ is semisimple, $\tilde{J}$ is an inner derivation with respect to an element of $\mathfrak{g}$, that we still denote by $\tilde{J}$. We have $\tilde{J} \in \mathfrak{g}_0^{\mathbb{C}}$ and

$$[\tilde{J}, X] = JX \quad \text{on } \mathfrak{g}_{-1}^{\mathbb{C}}\,.$$

In particular, $[\tilde{J}, \mathfrak{g}_{-1}] = J(\mathfrak{g}_{-1}) = \mathfrak{g}_{-1}$, and so

$$[\tilde{J}, \mathfrak{m}] \subset \mathfrak{m}\,.$$

The Killing form of $\mathfrak{g}^{\mathbb{C}}$ is the complexification of the Killing form of $\mathfrak{g}$. Since

$$\kappa_{\mathfrak{g}^{\mathbb{C}}}([\tilde{J}, X], Y) = -\kappa_{\mathfrak{g}^{\mathbb{C}}}(X, [\tilde{J}, Y])$$

is real for all $X \in \mathfrak{g}_p$, $Y \in \mathfrak{g}_{-p}$ with $p > 0$, we conclude that also

$$[\tilde{J}, \oplus_{p>0}\mathfrak{g}_p] \subset \oplus_{p>0}\mathfrak{g}_p\,.$$

Finally, as roots $\alpha \in \mathcal{R}$ with $\alpha(E) = 0$ are sums of roots $\beta \in \mathcal{B}$ with $\beta(E) = 0$, we have $[\tilde{J}, \mathfrak{g}_0^{\mathbb{C}}] = 0$. The restriction of $\tilde{J}$ to $\mathfrak{g}$ yields therefore the desired element.

The unicity follows from Lemma 3.6.2. □

From Theorem 3.6.5 we deduce:

PROPOSITION 3.6.6 *Let $\mathfrak{g} = \oplus_{-\mu \leq p \leq \mu} \mathfrak{g}_p$ be a finite dimensional semisimple pseudocomplex graded Lie algebra such that $\mathfrak{m} = \oplus_{p<0} \mathfrak{g}_p$ is fundamental. Then every ideal $\mathfrak{a}$ of $\mathfrak{g}$ is a finite dimensional semisimple pseudocomplex graded Lie algebra such that $\oplus_{p<0} \mathfrak{a}_p$ is fundamental.*

In particular, if $\mathfrak{g}$ is a semisimple Levi-Tanaka algebra, then every ideal of $\mathfrak{g}$ is a Levi-Tanaka algebra.

We obtain, for a Levi factor $\mathfrak{L}$ appearing in the pseudocomplex Levi-Mal'čev decomposition $\mathfrak{g} = \mathfrak{L} \oplus \mathfrak{r}$ of Theorem 3.5.1:

COROLLARY 3.6.7 *Let $\mathfrak{g} = \mathfrak{L} \oplus \mathfrak{r}$ be a Levi-Mal'čev decomposition of a finite dimensional Levi-Tanaka algebra, where $\mathfrak{r}$ is the radical of $\mathfrak{g}$ and $\mathfrak{L}$ is a semisimple graded Lie algebra of $\mathfrak{g}$ which is pseudocomplex. If*

$$\mathfrak{L} = \mathfrak{L}^1 \oplus \ldots \oplus \mathfrak{L}^m$$

is a decomposition of $\mathfrak{L}$ into a direct sum of simple ideals, each factor $\mathfrak{L}^j$ is a pseudocomplex graded Lie subalgebra $\mathfrak{L}^j = \oplus_{p \in \mathbb{Z}} \mathfrak{L}^j_p$ such that $\oplus_{p<0} \mathfrak{L}^j_p$ is fundamental.

3.6.2 Conjugation

We assume that $\mathfrak{g} = \oplus_{-\mu \leq p \leq \mu} \mathfrak{g}_p$ is a finite dimensional semisimple pseudocomplex graded Lie algebra, which is transitive and such that $\oplus_{p<0} \mathfrak{g}_p$ is fundamental. As above, we fix a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}_0$, denote by $\mathfrak{h}^{\mathbb{C}}$ its complexification and by $\mathcal{R}$ and $\mathcal{B}$ respectively the root system of $\mathfrak{h}^{\mathbb{C}}$ in $\mathfrak{g}^{\mathbb{C}} = \mathbb{C} \otimes_{\mathbb{R}} \mathfrak{g}$ and a basis for $\mathcal{R}$ such that $\alpha(E) \in \{0, -1\}$ for every $\alpha \in \mathcal{B}$.

The real form $\mathfrak{h}$ of $\mathfrak{h}^{\mathbb{C}}$ defines a real involution on the dual $\check{\mathfrak{h}}^{\mathbb{C}}$ of $\mathfrak{h}^{\mathbb{C}}$, obtained by associating to every $\mathbb{C}$-linear functional $\alpha \in \check{\mathfrak{h}}^{\mathbb{C}}$ the unique $\mathbb{C}$-linear functional $\bar{\alpha} \in \check{\mathfrak{h}}^{\mathbb{C}}$ such that

$$\bar{\alpha}(H) = \overline{\alpha(H)} \quad \forall H \in \mathfrak{h}. \tag{3.14}$$

LEMMA 3.6.8 *With the assumptions and notation above, we obtain*

$$\bar{\alpha} \in \mathcal{R} \qquad \forall \alpha \in \mathcal{R}.$$

Proof. We consider first the case of a root $\alpha \in \mathcal{R}$ with $\alpha(E) = -1$. Assume that $\mathfrak{g}^\alpha \subset \mathfrak{g}_{-1}^{(0,1)}$ and let $\tilde{X}_\alpha = X_\alpha + \sqrt{-1}JX_\alpha$ be a basis of $\mathfrak{g}^\alpha$. For every $H \in \mathfrak{h}$ we obtain

$$[H, X_\alpha - \sqrt{-1}JX_\alpha] = \overline{[H, X_\alpha + \sqrt{-1}JX_\alpha]} = \bar{\alpha}(H)(X_\alpha - \sqrt{-1}JX_\alpha),$$

where we have used complex conjugation in $\mathfrak{g}^{\mathbb{C}}$ with respect to the real form $\mathfrak{g}$. By $\mathbb{C}$-linearity this equality extends to $H \in \mathfrak{h}^{\mathbb{C}}$, showing that $\bar{\alpha} \in \mathcal{R}$ and that $\mathfrak{g}^{\bar{\alpha}}$ is the conjugated of $\mathfrak{g}^\alpha$ with respect to the real form $\mathfrak{g}$.

The case $\mathfrak{g}^\alpha \subset \mathfrak{g}_{-1}^{(1,0)}$ is analogous.

To conclude the proof of the lemma, we need only to consider the case where $\alpha \in \mathcal{R}$ and $\alpha(E) = 0$. Since $\mathfrak{g}^{\mathbb{C}}$ is transitive, there exists $\beta \in \mathcal{R}$ with $\beta(E) = -1$ such that $\alpha + \beta \in \mathcal{R}$. Then $\bar{\alpha} + \bar{\beta} \in \mathcal{R}$ and again we conclude by complex conjugation with respect to $\mathfrak{g}$ that

$$[\mathfrak{g}^{\bar{\alpha}+\bar{\beta}}, \mathfrak{g}^{-\bar{\beta}}] = \mathfrak{g}^{\bar{\alpha}} \neq 0.$$

The proof is complete. □

We note, however, that the eigenvectors in $\mathfrak{g}^{\mathbb{C}}$ corresponding to α and $\bar{\alpha} \in \mathcal{R}$, may not belong to a same simple ideal in $\mathfrak{g}^{\mathbb{C}}$. The eigenspaces $\mathfrak{g}^\alpha$ and $\mathfrak{g}^{\bar{\alpha}}$ belong to a same simple ideal when the real parts of the elements of $\mathfrak{g}^\alpha$ belong to a simple ideal of the real type, i.e. such that its complexification is still simple. We have

PROPOSITION 3.6.9 *Let $\mathfrak{g} = \oplus_{-\mu \leq p \leq \mu} \mathfrak{g}_p$ be a finite dimensional semi-simple pseudocomplex graded Lie algebra such that $\oplus_{p<0} \mathfrak{g}_p$ is fundamental. We assume that $\mu \geq 1$ and that the complexification $\mathfrak{g}^{\mathbb{C}}$ of $\mathfrak{g}$ is simple. Then $\mu \geq 2$, $\mathfrak{g}$ is a Levi-Tanaka algebra and $\mathfrak{g}^{\mathbb{C}}$ is of type A_ℓ, or D_ℓ, or E_6.*

Proof. We keep the notation introduced above. Because $\mu \geq 1$, $\mathcal{B}$ contains at least a root α with $\alpha(E) = -1$. Then also $\bar{\alpha} \in \mathcal{R}$. We note that $\bar{\alpha} \neq \alpha$ and cannot be obtained as sum of α and simple roots $\beta \in \mathcal{B}$. Indeed the eigenspaces $\mathfrak{g}^\alpha$ and $\mathfrak{g}^{\bar{\alpha}}$ belong one to $\mathfrak{g}_{-1}^{(1,0)}$ and the other to $\mathfrak{g}_{-1}^{(0,1)}$, which are disjoint and $\mathfrak{g}_0^{\mathbb{C}}$-invariant. Because we assumed that $\mathfrak{g}^{\mathbb{C}}$ is simple, the Dynkin diagram of $\mathfrak{g}^{\mathbb{C}}$ is connected and therefore $\mathcal{R}$ contains roots γ with $\gamma(E) \leq -2$. In particular, for the highest root δ we obtain $\delta(E) = -\mu \geq 2$

and we conclude that $\mathfrak{g}$ is a Levi-Tanaka algebra by Theorem 3.3.8. Finally, the conjugation map $\alpha \to \bar{\alpha}$ on the roots, described in the previous lemma, permits to define an order two automorphism of the complex Lie algebra $\mathfrak{g}^{\mathbb{C}}$, which is different from the identity. This defines an automorphism of a Weyl chamber. Hence the result follows from the classification of the automorphisms of simple complex Lie algebras (cf. [H], Ch.X). □

3.6.3 Cartan decomposition

In order to discuss the properties of standard homogeneous CR manifolds having a semisimple Levi-Tanaka algebra of infinitesimal automorphisms, it is convenient to discuss their Cartan decompositions.

PROPOSITION 3.6.10 *Every semisimple Levi-Tanaka algebra* $\mathfrak{g} = \oplus_{-\mu \leq p \leq \mu} \mathfrak{g}_p$ *admits a Cartan decomposition*

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$$

where:

(i) $\mathfrak{k}$ *is a Lie subalgebra of* $\mathfrak{g}$ *of the compact type on which the Killing form* $\kappa_{\mathfrak{g}}$ *is negative definite;*

(ii) $\mathfrak{k} = \oplus_{0 \leq p \leq \mu} \mathfrak{k}_{|p|}$ *with* $\mathfrak{k}_{|0|} = \mathfrak{k} \cap \mathfrak{g}_0$ *and* $\mathfrak{k}_{|p|} \subset \mathfrak{g}_{-p} \oplus \mathfrak{g}_p$ *for* $p > 0$*;*

(iii) $\mathfrak{p}$ *is the orthogonal complement of* $\mathfrak{k}$ *with respect to the Killing form* $\kappa_{\mathfrak{g}}$ *of* $\mathfrak{g}$ *and* $\kappa_{\mathfrak{g}}$ *is positive definite on* $\mathfrak{p}$*;*

(iv) $\mathfrak{p} = \oplus_{0 \leq p \leq \mu} \mathfrak{p}_{|p|}$ *with* $\mathfrak{p}_{|0|} = \mathfrak{p} \cap \mathfrak{g}_0$ *and* $\mathfrak{p}_{|p|} \subset \mathfrak{g}_{-p} \oplus \mathfrak{g}_p$ *for* $p > 1$*;*

(v) the natural projections $\mathfrak{k}_{|p|} \to \mathfrak{g}_{\pm p}$ *and* $\mathfrak{p}_{|p|} \to \mathfrak{g}_{\pm p}$ *are linear isomorphisms for* $p > 0$*;*

(vi) the associated Cartan involution $\theta : \mathfrak{g} \to \mathfrak{g}$ *such that* $\mathfrak{k}$ *is the set of fixed points of* θ, $\theta(X) = -X$ *for* $X \in \mathfrak{p}$*, and for which*

$$\mathfrak{g} \times \mathfrak{g} \ni (X, Y) \to -\kappa_{\mathfrak{g}}(X, \theta(Y)) \in \mathbb{R}$$

is a positive definite real symmetric form, has the properties:

$$\theta(\mathfrak{g}_p) = \mathfrak{g}_{-p} \quad \textit{for} \quad -\mu \leq p \leq \mu,$$

$$\mathfrak{g}_{-1} \ni X \to \theta(X) \in \mathfrak{g}_1 \quad \textit{and} \quad \mathfrak{g}_1 \ni X \to \theta(X) \in \mathfrak{g}_{-1}$$

are $\mathbb{C}$-linear for the complex structures of $\mathfrak{g}_{-1}$ and $\mathfrak{g}_1$ defined by J.

Proof. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ contained in $\mathfrak{g}_0$ and let $\mathfrak{h}^{\mathbb{C}}$ be the corresponding Cartan subalgebra of the complexification $\mathfrak{g}^{\mathbb{C}}$ of $\mathfrak{g}$. Let $\mathcal{R}$ be the set of nonzero roots of $\mathfrak{g}^{\mathbb{C}}$ with respect to $\mathfrak{h}^{\mathbb{C}}$ and H_α, for $\alpha \in \mathcal{R}$, the element of $\mathfrak{h}^{\mathbb{C}}$ such that

$$\kappa_{\mathfrak{g}^{\mathbb{C}}}(H, H_\alpha) = \alpha(H) \quad \forall H \in \mathfrak{h}^{\mathbb{C}}.$$

The form $\kappa_{\mathfrak{g}^{\mathbb{C}}}$ is positive definite on the real subspace $\mathfrak{h}^{\mathbb{R}}$ of $\mathfrak{h}^{\mathbb{C}}$ generated by the H_α's.

For each $\alpha \in \mathcal{R}$ we can choose a basis $\tilde{X}_\alpha$ of $\mathfrak{g}^\alpha$ in such a way that

$$[\tilde{X}_\alpha, \tilde{X}_{-\alpha}] = H_\alpha, \quad \kappa_{\mathfrak{g}^{\mathbb{C}}}(\tilde{X}_\alpha, \tilde{X}_{-\alpha}) = 1.$$

According to Lemma 3.6.3 and Lemma 3.6.4, we can split the set of roots α with $\alpha(E) = \pm 1$ into two disjoint subsets, the first $\mathcal{R}_{0,1}$ consisting of roots α for which $\tilde{X}_\alpha = X_\alpha + \sqrt{-1}JX_\alpha$, the second $\mathcal{R}_{1,0}$ consisting of roots α for which $\tilde{X}_\alpha = X_\alpha - \sqrt{-1}JX_\alpha$ with $X_\alpha \in \mathfrak{g}_{\pm 1}$.

Then we obtain a compact form $\mathfrak{u}$ by

$$\mathfrak{u} = \oplus_{p=0}^{\mu} \mathfrak{u}_{|p|}$$

where

$$\begin{aligned}
\mathfrak{u}_{|0|} &= \sqrt{-1}\,\mathfrak{h}^{\mathbb{R}} \bigoplus \sum_{\alpha(E)=0} \left(\mathbb{R}(\tilde{X}_\alpha - \tilde{X}_{-\alpha}) \oplus \sqrt{-1}\mathbb{R}(\tilde{X}_\alpha + \tilde{X}_{-\alpha})\right) \\
\mathfrak{u}_{|1|} &= \bigoplus_{\alpha \in \mathcal{R}_{0,1}} \Big(\mathbb{R}(X_\alpha - X_{-\alpha} + \sqrt{-1}J(X_\alpha + X_{-\alpha})) \\
&\quad \oplus \sqrt{-1}\mathbb{R}(X_\alpha + X_{-\alpha} + \sqrt{-1}J(X_\alpha - X_{-\alpha}))\Big) \\
\mathfrak{u}_{|p|} &= \bigoplus_{\alpha(E)=-p} \left(\mathbb{R}(\tilde{X}_\alpha - \tilde{X}_{-\alpha}) \oplus \sqrt{-1}\mathbb{R}(\tilde{X}_\alpha + \tilde{X}_{-\alpha})\right) \qquad \text{for} \quad p > 0.
\end{aligned}$$

Let us denote by $\tau : \mathfrak{g}^{\mathbb{C}} \to \mathfrak{g}^{\mathbb{C}}$ the complex conjugation in $\mathfrak{g}^{\mathbb{C}}$ with respect to the real form $\mathfrak{u}$ and by $\sigma : \mathfrak{g}^{\mathbb{C}} \to \mathfrak{g}^{\mathbb{C}}$ the complex conjugation in $\mathfrak{g}^{\mathbb{C}}$ with

respect to the real form $\mathfrak{g}$. We set

$$\begin{cases} \mathfrak{g}^{\mathbb{C}}_{|0|} = \mathfrak{g}^{\mathbb{C}}_0 \\ \mathfrak{g}^{\mathbb{C}}_{|p|} = \mathfrak{g}^{\mathbb{C}}_{-p} \oplus \mathfrak{g}^{\mathbb{C}}_p \quad \text{for} \quad p > 0. \end{cases}$$

Then we have

$$\tau(\mathfrak{g}^{\mathbb{C}}_{|p|}) = \sigma(\mathfrak{g}^{\mathbb{C}}_{|p|}) = \mathfrak{g}^{\mathbb{C}}_{|p|} \quad \text{for} \quad p = 0, \dots, \mu.$$

Moreover we note that J defines an antiinvolution on $\mathfrak{u}_{|1|}$ and therefore, by $\mathbb{C}$-linearity, also on $\sqrt{-1}\mathfrak{u}_{|1|}$. From this we derive that

$$-J \circ \tau \circ J = \tau, \quad \text{i.e.} \quad J \circ \tau = \tau \circ J \quad \text{on} \quad \mathfrak{g}^{\mathbb{C}}_{|1|}.$$

Obviously the conjugation σ commutes with J on $\mathfrak{g}^{\mathbb{C}}_{|1|}$. This property is therefore shared by the composed $\mathbb{C}$-linear automorphism $a = \sigma \circ \tau$ of $\mathfrak{g}^{\mathbb{C}}$. This is a selfadjoint map for the Hermitian scalar product

$$B_\tau : \mathfrak{g}^{\mathbb{C}} \times \mathfrak{g}^{\mathbb{C}} \ni (X, Y) \to -\kappa_{\mathfrak{g}^{\mathbb{C}}}(X, \tau Y) \in \mathbb{C}$$

and therefore a^2 is selfadjoint and positive definite for B_τ. We denote by ϕ the positive selfadjoint fourth root of a^2. This is still an automorphism of the Lie algebra $\mathfrak{g}^{\mathbb{C}}$ such that $\phi(\mathfrak{u})$ is a compact form of $\mathfrak{g}^{\mathbb{C}}$ that is invariant under σ. Moreover, by the construction,

$$\phi(\mathfrak{g}^{\mathbb{C}}_{|p|}) = \mathfrak{g}^{\mathbb{C}}_{|p|} \quad \text{and} \quad \phi \circ J = J \circ \phi \quad \text{on} \quad \mathfrak{g}^{\mathbb{C}}_{|1|}.$$

A Cartan decomposition of $\mathfrak{g}$ is obtained by setting $\mathfrak{k} = \phi(\mathfrak{u}) \cap \mathfrak{g}$ and $\mathfrak{p} = \sqrt{-1}\phi(\mathfrak{u}) \cap \mathfrak{g}$. Then the Cartan involution θ on $\mathfrak{g}$ is defined by $\phi \circ \tau \circ \phi^{-1}$ and therefore commutes with J on $\mathfrak{g}_{-1} \oplus \mathfrak{g}_1$.

We note that the positive definite symmetric real form

$$\mathfrak{g} \times \mathfrak{g} \ni (X, Y) \to g(X, Y) = -\kappa_{\mathfrak{g}}(X, \theta(Y))$$

satisfies

$$g(JX, Y) = -g(X, JY) \quad \text{for} \quad X, Y \in \mathfrak{g}_{\pm 1}$$

and therefore is on $\mathfrak{g}_{-1}$ and $\mathfrak{g}_1$ the real part of a Hermitian scalar product for the respective complex structures. $\square$

3.7 Almost-solvable and weakly-solvable radicals

Let $\mathfrak{g} = \oplus_{-\mu \le p \le \nu} \mathfrak{g}_p$ be a finite dimensional Levi-Tanaka algebra, with radical $\mathfrak{r} = \oplus_{p \ge -\mu} \mathfrak{r}_p$.

For $X \in \mathfrak{g}$, we denote by $(X)_{\mathfrak{g}}$ the ideal in $\mathfrak{g}$ generated by X. Since all ideals of $\mathfrak{g}$ are graded, $(X)_{\mathfrak{g}}$ contains all the homogeneous components of X. In particular, if $X = \sum X_p$ with $X_p \in \mathfrak{g}_p$, we have $(X_p)_{\mathfrak{g}} \subset (X)_{\mathfrak{g}}$.

For each nonnegative integer h we set

$$\mathfrak{a}^{(h)} = \{X \in \mathfrak{g} \mid (X)_{\mathfrak{g}} \cap \mathfrak{g}_{-h} \subset \mathfrak{r}_{-h}\}\,. \tag{3.15}$$

We have

PROPOSITION 3.7.1 *Let $\mathfrak{g} = \oplus_{-\mu \le p \le \nu} \mathfrak{g}_p$ be a finite dimensional Levi-Tanaka algebra of kind μ, with radical $\mathfrak{r} = \oplus_{p \ge -\mu} \mathfrak{r}_p$. For each nonnegative integer h, the set $\mathfrak{a}^{(h)}$ defined by (3.15) is a graded pseudocomplex ideal. We have*

(i) $\mathfrak{a}^{(0)} = \mathfrak{r}$ *and* $\mathfrak{a}^{(h)} = \mathfrak{g}$ *for* $h \ge \mu + 1$;

(ii) $\mathfrak{a}^{(h)} \subset \mathfrak{a}^{(h+1)}$ *for every* $h \in \mathbb{Z}, h \ge 0$.

Moreover, if $\mathfrak{L}$ is a graded pseudocomplex Levi subalgebra of $\mathfrak{g}$ and

$$\mathfrak{L} = \mathfrak{L}^1 \oplus \ldots \mathfrak{L}^m$$

its decomposition into a direct sum of simple ideals, we obtain

$$\mathfrak{a}^{(h)} = \mathfrak{r} \oplus \left(\oplus\{\mathfrak{L}^j \mid \textit{kind of } \mathfrak{L}^j < h\}\right) \tag{3.16}$$

for every nonnegative integer h.

Proof. We note that each $\mathfrak{L}^j$ in the decomposition (3.15) is graded and pseudocomplex. Therefore it is sufficient to prove (3.16). Let us fix a nonnegative integer h and denote by $\tilde{\mathfrak{r}}^{(h)}$ the right-hand side of (3.16). The inclusion $\tilde{\mathfrak{r}}^{(h)} \subset \mathfrak{a}^{(h)}$ is clear from the definition of $\mathfrak{a}^{(h)}$. To prove the opposite inclusion, we take any $X \in \mathfrak{g}$ and consider its unique decomposition

$$X = X_0 + X_1 + \ldots + X_m \quad \text{with } X_0 \in \mathfrak{r}, X_i \in \mathfrak{L}^i \quad \text{for } i = 1, \ldots, m\,.$$

Assume $1 \leq j \leq m$ and kind of $\mathfrak{L}^j \leq h$. If the $\mathfrak{L}^j$-component X_j of the decomposition above is different from 0, then $(X_j)_{\mathfrak{L}^j} = \mathfrak{L}^j$ because $\mathfrak{L}^j$ is simple. In particular, there is a Lie polynomial, i.e. a linear combination $P(T)$ of terms of the form

$$T \longrightarrow [Z_1, \dots, Z_r, T, Y_1, \dots, Y_s]$$

with $Z_1, \dots, Z_r, Y_1, \dots, Y_s \in \mathfrak{L}^j$, such that $P(X_j) \in \mathfrak{L}^j_{-h} \setminus \{0\}$.

We note that $P(X_j)$ is the $\mathfrak{L}^j$-component of the homogeneous component of degree $-h$ of $P(X)$. Note that if we had $X \in \mathfrak{a}^{(h)}$, then also $P(X)$ and all its homogeneous components would be elements of $\mathfrak{a}^{(h)}$. From this observation we deduce that $X \notin \mathfrak{a}^{(h)}$ if $X \notin \tilde{\mathfrak{r}}^{(h)}$. Thus $\mathfrak{a}^{(h)} = \tilde{\mathfrak{r}}^{(h)}$. □

The ideal $\mathfrak{a}^{(1)}$ is the sum of the radical and of a semisimple subalgebra contained in $\mathfrak{g}_0$. It is the largest ideal in $\mathfrak{g}$ having this property and will be called *the almost-solvable radical* of $\mathfrak{g}$. We note that

$$\mathfrak{a}^{(1)} = \oplus_{p \geq -\mu} \mathfrak{a}^{(1)}_p$$

with

$$\mathfrak{a}^{(1)}_p = \begin{cases} \{X \in \mathfrak{g}_0 \mid [X, \mathfrak{g}_{-1}] \subset \mathfrak{r}_{-1}\} & \text{for } p = 0 \\ \mathfrak{r}_p & \text{for } p \neq 0\,. \end{cases}$$

The ideal $\mathfrak{a}^{(2)}$ is the sum of the radical of $\mathfrak{g}$ and of a graded pseudocomplex semisimple subalgebra of kind ≤ 1 of $\mathfrak{g}$ and is the largest ideal in $\mathfrak{g}$ having this property. We call $\mathfrak{a}^{(2)}$ *the weakly-solvable radical* of $\mathfrak{g}$.

We recall that every simple graded pseudocomplex Lie algebra of kind ≥ 2 is a Levi-Tanaka algebra. In particular, if $\mathfrak{L} = \mathfrak{L}^1 \oplus \dots \oplus \mathfrak{L}^m$ is the decomposition of a pseudocomplex Levi subalgebra $\mathfrak{L}$ of $\mathfrak{g}$ into a sum of simple ideals, as in Proposition 3.7.1, the subalgebra

$$\mathfrak{s} = \oplus\{\mathfrak{L}^j \mid \text{kind of } \mathfrak{L}^j \geq 2\}$$

is a Levi-Tanaka subalgebra of $\mathfrak{g}$ and we have the decomposition

$$\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{a}^{(2)} \tag{3.17}$$

of $\mathfrak{g}$ into the direct sum of a Levi-Tanaka subalgebra which is semisimple and the weakly-solvable radical.

We recall that a finite dimensional Lie algebra $\mathfrak{l}$ is called *faithful* if the adjoint representation induces faithful representations of its Levi subalgebras in its radical. We have

THEOREM 3.7.2 *A finite dimensional Levi-Tanaka algebra* $\mathfrak{g} = \oplus_{-\mu \leq p \leq \nu} \mathfrak{g}_p$ *can be decomposed into the direct sum of two Levi-Tanaka ideals*

$$\mathfrak{g} = \mathfrak{b} \oplus \mathfrak{l}$$

with $\mathfrak{b}$ *semisimple and* $\mathfrak{l}$ *faithful.*

Proof. Let $\mathfrak{g} = \mathfrak{L} \oplus \mathfrak{r}$ be a Levi-Mal'čev decomposition with a graded pseudocomplex Levi subalgebra $\mathfrak{L}$. Let

$$\mathfrak{b} = \{X \in \mathfrak{L} \mid [X, \mathfrak{r}] = 0\}.$$

Then $\mathfrak{b}$ is a graded ideal of $\mathfrak{g}$ which is pseudocomplex. Indeed, if $X \in \mathfrak{l}_{-1}$, we have $JX \in \mathfrak{l}_{-1}$ because $\mathfrak{l}$ is an ideal in $\mathfrak{L}$ and $\mathfrak{L}$ has the (J)-property.

Denote by $\mathfrak{l}$ the orthogonal of $\mathfrak{b}$ with respect to the Killing form $\kappa_{\mathfrak{g}}$ of $\mathfrak{g}$. It is an ideal of $\mathfrak{g}$. Since $\mathfrak{b}$ is an ideal in $\mathfrak{L}$, the Killing form is nondegenerate on $\mathfrak{b}$ and we have the direct sum decomposition of ideals

$$\mathfrak{g} = \mathfrak{b} \oplus \mathfrak{l}.$$

Clearly $\mathfrak{a}^{(2)} \subset \mathfrak{l}$ and $\mathfrak{l}$ is faithful.

We need to prove that both $\mathfrak{b}$ and $\mathfrak{l}$ are Levi-Tanaka. To this aim it suffices to notice that

$$\mathfrak{m} = \oplus_{p<0} \mathfrak{g}_p = (\oplus_{p<0} \mathfrak{b}_p) \oplus (\oplus_{p<0} \mathfrak{l}_p)$$

is a decomposition of $\mathfrak{m}$ into the direct sum of two ideals which are fundamental and nondegenerate: then $\mathfrak{b}$ and $\mathfrak{l}$ correspond to the canonical pseudocomplex prolongations of their negative parts. □

3.8 Examples

Before discussing general results on the classification of Levi-Tanaka algebras, we give in this section several examples to substantiate the abstract theory developed in this chapter.

For the notation about the Levi-Tanaka algebras of kind 2 the reader is referred to § 5.1.

3.8.1 Levi-Tanaka algebras of kind 2 isomorphic to $\mathfrak{su}(p+m, q+m)$

Let m, p, q be nonnegative integers with $m > 0$ and $\ell = p + q > 0$. With I_t denoting the $t \times t$ identity matrix, we set

$$I_{p,q} = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}.$$

We consider the Hermitian symmetric matrix

$$Q = \begin{pmatrix} 0 & 0 & I_m \\ 0 & I_{p,q} & 0 \\ I_m & 0 & 0 \end{pmatrix}.$$

The Lie algebra $\mathfrak{g}$ of matrices A in $\mathfrak{sl}(\ell + 2m, \mathbb{C})$ satisfying

$$A^*Q + QA = 0$$

is isomorphic to $\mathfrak{su}(p+m, q+m)$, so it is simple. Its elements are null-trace matrices of the form

$$\begin{pmatrix} a_{11} & -a_{23}^* I_{p,q} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & -a_{21}^* I_{p,q} & -a_{11}^* \end{pmatrix}$$

with blocks a_{13}, $a_{31} \in \mathfrak{u}(m)$ and $a_{22} \in \mathfrak{u}(p, q)$. We obtain a structure of Levi-Tanaka algebra of type $(\ell m, m^2)$ by defining the elements E and $\tilde{J}$ by:

$$E = \begin{pmatrix} I_m & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -I_m \end{pmatrix} \quad \text{and} \quad \tilde{J} = \frac{\sqrt{-1}}{\ell + 2m} \begin{pmatrix} -\ell I_m & 0 & 0 \\ 0 & 2mI_\ell & 0 \\ 0 & 0 & -\ell I_m \end{pmatrix}.$$

This case generalizes the case of CR hypersurfaces, i.e. of type $(n, 1)$, with nondegenerate Levi form, that was fully discussed in [41] and [11], and corresponds to the choice $m = 1$. We note that the space of orbits of $\mathfrak{D}_1(\mathfrak{H}_s(V))$ contains only finitely many elements. In order that the canonical pseudocomplex prolongation be finite dimensional, it is necessary and sufficient to start from $\mathbb{P}L = \{[h]\}$ with h nondegenerate, i.e. of signature (p, q) with $p+q = n$. In this case $\mathfrak{g}(L)$ is isomorphic to the simple Lie algebra $\mathfrak{su}(p+1, q+1)$.

3.8.2 Levi-Tanaka algebras of kind 2 isomorphic to $\mathfrak{sl}(n, \mathbb{C})$

Let $n \geq 3$ and let us fix two positive integers m, ℓ with $2m + \ell = n$. We write a matrix $A \in \mathfrak{sl}(n, \mathbb{C})$ in the form

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \quad \text{with} \quad \begin{cases} a_{11},\ a_{13},\ a_{31},\ a_{33} \in \mathfrak{gl}(m, \mathbb{C}) \\ a_{12},\ a_{32} \quad m \times \ell \text{ complex matrices} \\ a_{21},\ a_{23} \quad \ell \times m \text{ complex matrices} \\ a_{22} \in \mathfrak{gl}(\ell, \mathbb{C}) \\ \mathrm{tr}(a_{11}) + \mathrm{tr}(a_{22}) + \mathrm{tr}(a_{33}) = 0. \end{cases}$$

We graduate the Lie algebra $\mathfrak{sl}(n, \mathbb{C})$ by setting

$$\mathfrak{g}_p = \left\{ \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \mid a_{ij} = 0 \quad \text{for} \quad j - i \neq p \right\}.$$

The elements E and $\tilde{J}$ are like in the previous example. We denote this pseudocomplex graded Lie algebra by $\mathfrak{sl}(2m + \ell, \mathbb{C})$.

We consider the $2\ell m$-dimensional complex vector space V of pairs of $\ell \times m$ complex matrices and the map

$$\mathfrak{g}_{-1} \ni \begin{pmatrix} 0 & 0 & 0 \\ a_{21} & 0 & 0 \\ 0 & a_{32} & 0 \end{pmatrix} \to (a_{21}, a_{32}^*) \in V$$

where a_{32}^* denotes the conjugated transpose of a_{32}. This map is $\mathbb{C}$-linear for the complex structure of $\mathfrak{g}_{-1}$ and the canonical complex structure of V. Identifying the space of $m \times m$ complex matrices to a $2m^2$-dimensional real space, we obtain the Levi-Tanaka form on V:

$$\Im\mathfrak{f}((v_1, v_2), (w_1, w_2)) = v_2^* w_1 - w_2^* v_1. \tag{3.18}$$

It is convenient to represent $\mathfrak{g}_{-2}$ as the direct sum of two copies of the Hermitian symmetric $m \times m$ matrices. We obtain a (vector-valued) Levi form that can be written as

$$V \ni \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \to \begin{pmatrix} (v_1^*, v_2^*) \begin{pmatrix} 0 & \sqrt{-1} I_\ell \\ -\sqrt{-1} I_\ell & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \\ (v_1^*, v_2^*) \begin{pmatrix} 0 & I_\ell \\ I_\ell & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \end{pmatrix} \in \mathfrak{H}_s(\mathbb{C}^m)^2$$

and therefore $\mathfrak{sl}(2m+\ell,\mathbb{C})$ is the Levi-Tanaka algebra of a CR manifold M of type $(2\ell m, 2m^2)$ which is ℓ-pseudoconcave. It is also $m\ell$-pseudoconvex.

Note that the algebra considered in this example can be obtained by considering the complexification of the algebra in the previous one, where $\ell = p+q$.

Remark 3.8.1 *The simple algebra $\mathfrak{sl}(n,\mathbb{C})$ admits at least one structure of Levi-Tanaka algebra of kind μ for $1<\mu<n$. Moreover, there exist several nonequivalent structures for the same μ if $1<\mu<n-1$.*

Indeed, given a partition $(n_0,\dots,n_\mu)$ of n, i.e. positive integers n_j with $0\le j\le\mu$ such that $\sum_{j=0}^{\mu} n_j = n$, we consider

$$\begin{aligned} E &= \frac{1}{2}\,diag\Big(\mu I_{n_0},\dots,(\mu-2j)I_{n_j},\dots,-\mu I_{n_\mu}\Big)+c_E I_n \\ \tilde{J} &= \frac{\sqrt{-1}}{2}\,diag\Big(I_{n_0},\dots,(-1)^j I_{n_j},\dots,(-1)^\mu I_{n_\mu}\Big)+\sqrt{-1}c_{\tilde{J}} I_n \end{aligned}$$

where $c_E, c_{\tilde{J}}\in\mathbb{R}$ are such that $E,\tilde{J}\in\mathfrak{sl}(n,\mathbb{C})$ and $diag(a_0,\dots,a_\mu)$ denotes the block-diagonal matrix of entries $a_0,\dots,a_\mu$. If we denote by $\mathfrak{g}_p$ the eigenspace of the adjoint representation of $\mathfrak{sl}(n,\mathbb{C})$ of the element E associated to the eigenvalue $-\mu\le p\le\mu$, then $\mathfrak{g}=\oplus_{-\mu\le p\le\mu}\mathfrak{g}_p$, with the pseudocomplex structure on $\mathfrak{g}_{-1}$ given by the adjoint representation of the element $\tilde{J}$, is a Levi-Tanaka algebra of kind μ.

We note that if in addition $n_j = n_{\mu-j}$ for every $0\le j\le\mu$, denoting by

$$Q=\begin{pmatrix} & & & I_{n_\mu} \\ & & \cdot & \\ & \cdot & & \\ I_{n_0} & & & \end{pmatrix},$$

we have that the algebra

$$\{A\in\mathfrak{sl}(n,\mathbb{C})\mid A^*Q+QA=0\},$$

with the graduation and the pseudocomplex structure similarly defined, is a Levi-Tanaka algebra of kind μ. If in addition μ is even, we may take in the definition of Q the matrix $I_{p,q}$ instead of $I_{\mu/2}$. These algebras are all isomorphic to $\mathfrak{su}(p,q)$ for suitable p and q.

3.8.3 Levi-Tanaka algebra of kind 2 isomorphic to $\mathfrak{so}(n+2,n)$

Let V be a complex linear space of dimension $n \geq 2$ and let W be a totally real subspace of V of real dimension n. We consider the $\frac{n(n-1)}{2}$ dimensional subspace L of $\mathfrak{H}_s(V)$ of Hermitian symmetric forms h such that $h(X,X)=0$ for all $X \in W$.

In a basis $e_1, \dots, e_n$ of V contained in W, the matrices associated to the elements of L are of the form $\sqrt{-1}\,A$ for a matrix $A \in \mathfrak{so}(n)$. We call such a subspace L of $\mathfrak{H}_s(V)$ a *skew subspace* of $\mathfrak{H}_s(V)$. Clearly, all skew subspaces of $\mathfrak{H}_s(V)$ belong to the same orbit under the action of $\mathbf{GL}_{\mathbb{C}}(V)$ and therefore define isomorphic pseudocomplex fundamental graded Lie algebras $\mathfrak{m}(L)$.

PROPOSITION 3.8.2 *The canonical pseudocomplex prolongation of a pseudocomplex fundamental graded Lie algebra $\mathfrak{m}(L)$ associated to a skew subspace L of $\mathfrak{H}_s(V)$ is a simple graded Lie algebra, isomorphic to the Lie algebra $\mathfrak{so}(n+2,n)$.*

Proof. We consider on the real vector space $\mathbb{R}^{2n+2}$ the symmetric bilinear form of signature $(n+2,n)$ defined by the Hermitian symmetric matrix

$$Q = \begin{pmatrix} 0 & 0 & I_n \\ 0 & I_2 & 0 \\ I_n & 0 & 0 \end{pmatrix},$$

where I_ℓ is the identity $\ell \times \ell$ matrix. Then $\mathfrak{so}(n+2,n)$ is identified to the space of matrices of the form

$$\begin{pmatrix} \alpha & \delta & \gamma \\ \beta & \epsilon & -{}^t\delta \\ \theta & -{}^t\beta & -{}^t\alpha \end{pmatrix} \quad \text{where} \quad \begin{cases} \alpha \in \mathfrak{gl}(n,\mathbb{R}) \\ \beta \quad \text{is a } 2\times n \text{ real matrix} \\ \delta \quad \text{is a } n \times 2 \text{ real matrix} \\ \gamma,\, \theta \in \mathfrak{so}(n) \\ \epsilon \in \mathfrak{so}(2,\mathbb{R}). \end{cases}$$

We denote by $\mathfrak{g}$ the Lie algebra of $(2n+2)\times(2n+2)$ matrices defined above. We consider the element $E \in \mathfrak{g}$:

$$E = \begin{pmatrix} I_n & 0 & 0 \\ 0 & 0_2 & 0 \\ 0 & 0 & -I_n \end{pmatrix}.$$

Then $\mathrm{ad}_{\mathfrak{g}}(E)$ is semisimple with eigenvalues -2, -1, 0, 1, 2 and we denote by $\mathfrak{g}_p$ the eigenspace corresponding to its integral eigenvalues $-2 \leq p \leq 2$. In this way $\mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$ has the structure of a simple graded Lie algebra. We note that

$$\mathfrak{g}_0 = \left\{ \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \epsilon & 0 \\ 0 & 0 & -{}^t\alpha \end{pmatrix} \mid \alpha \in \mathfrak{gl}(n, \mathbb{R}),\ \epsilon \in \mathfrak{so}(2) \right\}$$

and

$$\mathfrak{g}_{-1} = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ \beta & 0 & 0 \\ 0 & -{}^t\beta & 0 \end{pmatrix} \mid \beta \,\text{is a } 2 \times n \,\text{matrix} \right\}.$$

Let $j = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and consider

$$\tilde{J} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & j & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \mathfrak{g}_0.$$

We have $\rho_{-1}(\tilde{J})^2 = -\mathrm{Id}|_{\mathfrak{g}_{-1}}$ and $[\rho_{-1}(\tilde{J})X, \rho_{-1}(\tilde{J})Y] = [X, Y]$ for every $X, Y \in \mathfrak{g}_{-1}$, therefore $\rho_{-1}(\tilde{J})$ defines a complex structure in $\mathfrak{g}_{-1}$. If we associate to the matrix β parametrizing $\mathfrak{g}_{-1}$ the element $Z \in \mathbb{C}^n$ obtained by adding to its first row $\sqrt{-1}$ times its second row, the way the element $X_0 = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \epsilon & 0 \\ 0 & 0 & -{}^t\alpha \end{pmatrix} \in \mathfrak{g}_0$ acts on $\mathfrak{g}_{-1}$ can be described by

$$\rho_{-1}(X_0)(Z) = -\alpha\, Z + \sqrt{-1}\tau\, Z$$

if $\epsilon = \begin{pmatrix} 0 & -\tau \\ \tau & 0 \end{pmatrix}$. It is clear then that $[X_0, \mathfrak{g}_{-1}] \neq 0$ if $X_0 \in \mathfrak{g}_0$ is different from zero. Moreover, the matrices of the form $\beta\, {}^t\gamma - \gamma\, {}^t\beta$, for β, γ varying in the space of $n \times 2$ real matrices, are a basis of $\mathfrak{so}(n)$ as a real vector space and the bilinear form $(\beta, \gamma) \to \beta\, {}^t\gamma - \gamma\, {}^t\beta$ is nondegenerate. Then $\mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$ is a nondegenerate fundamental graded Lie algebra. By Lemma 3.3.2, it follows that $\mathfrak{g}$ is transitive. From $j\, {}^tj = I_2$ we obtain also that $\mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$

is pseudocomplex. By Theorem 3.3.7, it is sufficient then to establish an isomorphism between the pseudocomplex fundamental graded Lie algebras $\mathfrak{m}(L)$ and $\mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$.

To this aim we choose a basis $e_1, \ldots, e_n$ of V contained in W and associate to every vector $v \in V$ the $n \times 2$ real matrix β whose first column is the real and the second the immaginary part of the components of v in this basis. The identification of $\mathfrak{g}_{-2}$ and L^* is the standard identification of the dual of real alternating forms on W with the real alternating forms on W^*. The proof is complete. □

3.8.4 Levi-Tanaka algebras of type $(n, 2)$ with $n > 1$

Let $\mathfrak{m} = \oplus_{p \geq -2} \mathfrak{g}_p$ be a fundamental graded Lie algebra of type $(n, 2)$. Assume that $\mathfrak{m}$ is nondegenerate so that its canonical pseudocomplex prolongation is finite dimensional. The structure of $\mathfrak{m}$ can be given by a real 2-dimensional subspace L of Hermitian symmetric forms on a complex vector space V with $dim_{\mathbb{C}} V = n$. Assume that there exists a nondegenerate form belonging to L and let L_1 and L_2 be a basis of L with L_1 nondegenerate. By Theorem 4.5.19 of [15] we can choose a basis of V such that L_1 and L_2 are represented by two matrices in the diagonal form with $\ell_i \times \ell_i$ blocks A_i, respectively B_i, where

$$A_i = \epsilon_i \begin{pmatrix} & & & & 1 \\ & & & \cdot & \\ & & \cdot & & \\ & \cdot & & & \\ 1 & & & & \end{pmatrix}, \quad B_i = \epsilon_i \begin{pmatrix} & & & & \alpha_i \\ & & & \cdot & 1 \\ & & \cdot & \cdot & \\ & \cdot & \cdot & & \\ \alpha_i & 1 & & & \end{pmatrix}$$

with $\alpha_i \in \mathbb{R}$ and $\epsilon_i = \pm 1$, for $1 \leq i \leq r$, and $2\ell_i \times 2\ell_i$ blocks

$$A_i = \begin{pmatrix} & & & 1 \\ & & \cdot & \\ & \cdot & & \\ \cdot & & & \\ 1 & & & \end{pmatrix}, \qquad B_i = \begin{pmatrix} & & & & & & & & & & \alpha_i \\ & & & & & & & & & \cdot & 1 \\ & & & & & & & & \cdot & \cdot & \\ & & & & & & & \cdot & \cdot & & \\ & & & & & & \alpha_i & 1 & & & \\ & & & & & & & & & & \\ & & & & & \bar{\alpha}_i & & & & & \\ & & & & \cdot & 1 & & & & & \\ & & & \cdot & \cdot & & & & & & \\ & & \cdot & \cdot & & & & & & & \\ & \bar{\alpha}_i & 1 & & & & & & & & \end{pmatrix}$$

with $\alpha_i \in \mathbb{C} \setminus \mathbb{R}$, for $r+1 \leq i \leq r+s$.

We assume that $\ell_i = 1$ for every $1 \leq i \leq r+s$. The case $s = 0$ and $\alpha_1 = \ldots = \alpha_n$ is not possible because L has dimension 2. In the case $s = 0$ with $\alpha_1 = \ldots = \alpha_p \neq \alpha_{p+1} = \ldots = \alpha_{p+q}$, the algebra $\mathfrak{m}$ is the direct sum of two ideals and $\mathfrak{g}$ is isomorphic to $\mathfrak{su}(p+1,1) \oplus \mathfrak{su}(q+1,1)$ (see Proposition 3.1.3). When $r = 0$, n is even and if $\alpha_1 = \ldots = \alpha_{n/2}$, then $\mathfrak{g}$ is isomorphic to $\mathfrak{sl}(2+n/2, \mathbb{C})$ as in the example in 3.8.2 (using Theorem 3.3.7). In all other cases with $n \geq 3$ it can be proved that $\rho_{-2}(\mathfrak{g}_0) = \mathbb{R}Id_{\mathfrak{g}_{-2}}$ and so, by Theorem 3.4.1, we have that $\mathfrak{g}_p = 0$ for every $p > 0$.

When $n \geq 3$ and all ℓ_i's are equal to 1 and the α_i's are distinct, the corresponding standard homogeneous CR manifolds are Euclidean and are parametrized, modulo CR diffeomorphisms, by a moduli space of real dimension $n-3$. This space is indeed the quotient of the set of n-tuple of distinct points of $\mathbb{CP}^1$, symmetrical for the involution defined by $\mathbb{RP}^1 \subset \mathbb{CP}^1$, under the action of the group of automorphisms of $\mathbb{CP}^1$ which leave invariant the Poincaré half-plane. This has been shown in [28] (for the case $n \geq 7$) and by the author in [23].

For $n = 2$, assuming again that the ℓ_i's are equal to 1, we obtain Levi-Tanaka algebras isomorphic either to $\mathfrak{su}(2,1) \oplus \mathfrak{su}(2,1)$ (the pseudoconvex case) or to $\mathfrak{sl}(3, \mathbb{C})$ (the 1-pseudoconcave case).

We consider a case where $\ell_i \neq 1$ in the example below, that completes the description of all Levi-Tanaka algebras of type $(2,2)$ and kind $\mu = 2$.

3.8.5 The weakly-pseudoconcave Levi-Tanaka algebra of type $(2,2)$

Let F be the linear subspace of $\mathfrak{H}_s(\mathbb{C}^2)$ generated by the Hermitian symmetric forms associated to the matrices

$$F_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad F_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

To compute the Levi-Tanaka algebra $\mathfrak{g}(F)$ we first introduce some notation. We denote by $T_2\mathbb{C}$ the unitary associative $\mathbb{C}$-algebra of lower triangular 2×2 matrices with complex coefficients. We consider on $T_2\mathbb{C}$ the two antilinear maps $T_2\mathbb{C} \ni \alpha \to \bar{\alpha} \in T_2\mathbb{C}$ and $T_2\mathbb{C} \ni \alpha \to \tilde{\alpha} \in T_2\mathbb{C}$ associating to the matrix $\alpha = \begin{pmatrix} \alpha_{11} & 0 \\ \alpha_{21} & \alpha_{22} \end{pmatrix}$ the matrices

$$\bar{\alpha} = \begin{pmatrix} \bar{\alpha}_{11} & 0 \\ \bar{\alpha}_{21} & \bar{\alpha}_{22} \end{pmatrix}, \quad \tilde{\alpha} = \begin{pmatrix} \bar{\alpha}_{22} & 0 \\ \bar{\alpha}_{21} & \bar{\alpha}_{11} \end{pmatrix}.$$

Then we define the two subrings of $T_2\mathbb{C}$:

$$N_2\mathbb{C} = \left\{ \begin{pmatrix} z_1 & 0 \\ z_2 & z_1 \end{pmatrix} \mid z_1,\, z_2 \in \mathbb{C} \right\} = \{\alpha \in T_2\mathbb{C} \mid \bar{\alpha} = \tilde{\alpha}\},$$

$$N_2\mathbb{R} = \left\{ \begin{pmatrix} t_1 & 0 \\ t_2 & t_1 \end{pmatrix} \mid t_1, t_2 \in \mathbb{R} \right\} = \{\alpha \in T_2\mathbb{C} \mid \alpha = \bar{\alpha} = \tilde{\alpha}\}.$$

Remark 3.8.3 *We have:*

1. $\overline{\alpha\beta} = \bar{\alpha}\bar{\beta} \quad \forall \alpha,\, \beta \in T_2\mathbb{C}$;
2. $\widetilde{\alpha\beta} = \tilde{\beta}\tilde{\alpha} \quad \forall \alpha,\, \beta \in T_2\mathbb{C}$;
3. $\alpha\beta = \beta\alpha \quad \forall \alpha,\, \beta \in N_2\mathbb{C}$;
4. *if* $\alpha,\, \beta \in T_2\mathbb{C}$ *and* $\alpha\zeta = \zeta\beta \quad \forall \zeta \in N_2\mathbb{R}$, *then* $\alpha = \beta \in N_2\mathbb{C}$.

PROPOSITION 3.8.4 *The Levi-Tanaka algebra* $\mathfrak{g}(F)$ *is isomorphic to the subalgebra of* $\mathfrak{gl}(6,\mathbb{C})$ *of matrices of the form*

$$\begin{pmatrix} \alpha & \eta & \sigma \\ \zeta & \beta & \sqrt{-1}\bar{\eta} \\ \tau & -\sqrt{-1}\bar{\zeta} & -\tilde{\alpha} \end{pmatrix}, \tag{3.19}$$

where τ, $\sigma \in N_2\mathbb{R}$, ζ, $\eta \in N_2\mathbb{C}$, *and*

(3.20)
$$\alpha = \begin{pmatrix} a+\sqrt{-1}b & 0 \\ c & d+\sqrt{-1}b \end{pmatrix}, \quad \beta = \begin{pmatrix} \frac{a-d}{2} - 2\sqrt{-1}b & 0 \\ -2\sqrt{-1}\Im c & \frac{d-a}{2} - 2\sqrt{-1}b \end{pmatrix},$$

with $a, b, d \in \mathbb{R}$ *and* $c \in \mathbb{C}$.
We note that $\tilde{\beta} = -\beta$ *and that* $\alpha + \beta - \tilde{\alpha}$ *is a diagonal* 2×2 *matrix with* 0 *trace.*

The operators E, $\tilde{J} \in \mathfrak{g}_0(F)$ *are described by the matrices*

$$E = \begin{pmatrix} I_2 & 0 & 0 \\ 0 & 0_2 & 0 \\ 0 & 0 & -I_2 \end{pmatrix} \qquad \text{and} \qquad \tilde{J} = \begin{pmatrix} -\frac{\sqrt{-1}}{3}I_2 & 0 & 0 \\ 0 & \frac{2\sqrt{-1}}{3}I_2 & 0 \\ 0 & 0 & -\frac{\sqrt{-1}}{3}I_2 \end{pmatrix}.$$

We have:

$$\begin{aligned}
\mathfrak{g}_{-2}(F) &= \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \tau & 0 & 0 \end{pmatrix} \mid \tau \in N_2\mathbb{R} \right\} \simeq \mathbb{R}^2; \\
\mathfrak{g}_{-1}(F) &= \left\{ \begin{pmatrix} 0 & 0 & 0 \\ \zeta & 0 & 0 \\ 0 & -\sqrt{-1}\bar{\zeta} & 0 \end{pmatrix} \mid \zeta \in N_2\mathbb{C} \right\} \simeq \mathbb{C}^2; \\
\mathfrak{g}_0(F) &= \left\{ \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & -\tilde{\alpha} \end{pmatrix} \mid \alpha, \beta \text{ as in (3.20)} \right\}; \\
\mathfrak{g}_1(F) &= \left\{ \begin{pmatrix} 0 & \eta & 0 \\ 0 & 0 & \sqrt{-1}\bar{\eta} \\ 0 & 0 & 0 \end{pmatrix} \mid \eta \in N_2\mathbb{C} \right\} \simeq \mathbb{C}^2; \\
\mathfrak{g}_2(F) &= \left\{ \begin{pmatrix} 0 & 0 & \sigma \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mid \sigma \in N_2\mathbb{R} \right\} \simeq \mathbb{R}^2.
\end{aligned}$$

The Levi-Tanaka algebra $\mathfrak{g}(F)$ *admits a graded Levi-Mal'čev decomposition* $\mathfrak{g}(F) = \mathfrak{s} \oplus \mathfrak{r}$ *with* $\mathfrak{s} \simeq \mathfrak{su}(2,1)$ *and* $\mathfrak{r} \neq 0$.

Proof. Using the previous remark, one easily checks by direct computation that the matrix algebra defined above is a pseudocomplex prolongation of the fundamental pseudocomplex Lie algebra $\mathfrak{m}(F)$. We define $\mathfrak{r}$ as the set of matrices as in (3.19) with

$$\tau = \begin{pmatrix} 0 & 0 \\ t & 0 \end{pmatrix}, \ t \in \mathbb{R}; \quad \sigma = \begin{pmatrix} 0 & 0 \\ s & 0 \end{pmatrix}, \ s \in \mathbb{R};$$

$$\zeta = \begin{pmatrix} 0 & 0 \\ z & 0 \end{pmatrix}, \ z \in \mathbb{C}; \quad \eta = \begin{pmatrix} 0 & 0 \\ w & 0 \end{pmatrix}, \ w \in \mathbb{C};$$

$$\alpha = \begin{pmatrix} a & 0 \\ c & -a \end{pmatrix} \quad \text{and} \quad \beta = \begin{pmatrix} a & 0 \\ -2\sqrt{-1}\Im c & -a \end{pmatrix} \quad \text{for } a \in \mathbb{R},\ c \in \mathbb{C}.$$

It is easy to verify that $\mathfrak{r}$ is an ideal of $\mathfrak{g}(F)$.

Next we denote by $\mathfrak{s}$ the set of matrices of the form (3.19) with

$$\tau = \begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix}, \ t \in \mathbb{R}; \quad \sigma = \begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix}, \ s \in \mathbb{R};$$

$$\zeta = \begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix}, \ z \in \mathbb{C}; \quad \eta = \begin{pmatrix} w & 0 \\ 0 & w \end{pmatrix}, \ w \in \mathbb{C};$$

$$\alpha = \begin{pmatrix} a+\sqrt{-1}b & 0 \\ 0 & a+\sqrt{-1}b \end{pmatrix} \text{ and } \beta = \begin{pmatrix} -2\sqrt{-1}b & 0 \\ 0 & -2\sqrt{-1}b \end{pmatrix}$$

for $a, b \in \mathbb{R}$. We observe that $\mathfrak{s}$ is a Lie subalgebra of $\mathfrak{g}(F)$ which is semisimple being isomorphic to $\mathfrak{su}(2,1)$. To prove that the algebra $\mathfrak{g}(F)$ defined above is the Levi-Tanaka algebra of the second kind associated to L, we have to show that it is a maximal prolongation. First we remark that the canonical pseudocomplex prolongation of $\mathfrak{m}(F)$ is not semisimple because $\mathfrak{g}_0(F)$ is not reductive (and $\mathfrak{g}_0(F)$ is the 0-degree component of the canonical pseudocomplex prolongation). By the graded Levi-Mal'čev decomposition and the fact that the canonical prolongation is semisimple when the radical has no component of degree -1, we deduce that $\mathfrak{s}$ is the semisimple part of the canonical pseudocomplex prolongation. Knowing that a prolongation of $\mathfrak{g}(F)$ would be a prolongation of its radical, we conclude by an explicit computation that the $\mathfrak{g}(F)$ we constructed is indeed the canonical pseudocomplex prolongation of $\mathfrak{m}(F)$. □

3.8.6 Weakly-solvable Levi-Tanaka algebras

A transitive graded Lie algebra $\mathfrak{g} = \oplus_{p\in\mathbb{Z}}\mathfrak{g}_p$, over $\mathbb{R}$ or $\mathbb{C}$, is called a pseudo-product if:

1. the algebra $\mathfrak{m} = \oplus_{p<0}\mathfrak{g}_p$ is fundamental;
2. there is a decomposition of the subspace $\mathfrak{g}_{-1} = \mathfrak{e} \oplus \mathfrak{f}$ into a direct sum of two irreducible $\mathfrak{g}_0$-modules (by the adjoint representation) such that $[\mathfrak{e}, \mathfrak{e}] = 0$ and $[\mathfrak{f}, \mathfrak{f}] = 0$.

When $\mathfrak{e}$ and $\mathfrak{f}$ are irreducible $\mathfrak{g}_0$-modules and $\mathfrak{g}_{-2} \neq 0$, then $\mathfrak{g}$ is called of irreducible type. We use the following weaker form of a result due to Yatsui (see [49]).

THEOREM 3.8.5 *If $\mathfrak{g}$ is a pseudo-product graded Lie algebra of irreducible type such that the subspaces*

$$\delta_{-1}(\mathfrak{g}) = \{X \in \mathfrak{g}_{-1} \mid [X, \oplus_{p<-1}\mathfrak{g}_p] = 0\}$$

and $\mathfrak{g}_1$ are not trivial, then we have:

1. *if $\mathfrak{g}_2 = 0$, $\mathfrak{g}$ is a semidirect product $\mathfrak{s} \oplus N$ of a semisimple graded Lie algebra $\mathfrak{s} = \mathfrak{s}_{-1} \oplus \mathfrak{s}_0 \oplus \mathfrak{s}_1$ and a nilpotent ideal $N = \oplus_{p<0}N_p$ (with $[N_{-1}, N_{-1}] = 0$);*
2. *if $\mathfrak{g}_2 \neq 0$, $\mathfrak{g}$ is simple.*

It is not difficult to show that a simple transitive graded Lie algebra with $\delta_{-1} \neq 0$ has kind $\mu \leq 3$.

Let $\mathfrak{g}$ be a complex pseudo-product of irreducible type, with $\mathfrak{e}$ and $\mathfrak{f}$ as above. Make it into a pseudocomplex Lie algebra defining $J : \mathfrak{g}_{-1} \to \mathfrak{g}_{-1}$ by

$$J(X) = \begin{cases} \sqrt{-1}X & \text{if } X \in \mathfrak{e}, \\ -\sqrt{-1}X & \text{if } X \in \mathfrak{f}. \end{cases}$$

Since $\mathfrak{e}$ and $\mathfrak{f}$ are $\mathfrak{g}_0$-modules the elements of $\rho_{-1}(\mathfrak{g}_0)$ commute with J on $\mathfrak{g}_{-1}$. Thus $\mathfrak{g}$ is a pseudocomplex prolongation of the pseudocomplex algebra $\mathfrak{m} = \oplus_{p<0}\mathfrak{g}_p$. Let $\tilde{\mathfrak{g}}$ be the canonical pseudocomplex prolongation of $\mathfrak{m}$. Then $\mathfrak{g} \subset \tilde{\mathfrak{g}}$ and $\tilde{\mathfrak{g}}$ is still a pseudo-product, with the same $\mathfrak{e}$ and $\mathfrak{f}$, because the elements of $\rho_{-1}(\tilde{\mathfrak{g}}_0)$, commuting with J, operate on the subspaces $\mathfrak{e}$ and $\mathfrak{f}$ which are the eigenspaces of J in $\mathfrak{g}_{-1}$. Assume now that $\mathfrak{g}$ also satisfies the assumptions in Theorem 3.8.5. If $\mathfrak{g}$ has kind $\mu \geq 4$, then $\tilde{\mathfrak{g}}$ is not simple and therefore $\mathfrak{g}_2 = 0$. The Levi-Tanaka algebra $\tilde{\mathfrak{g}}$ so obtained is weakly-solvable.

3.8.7 Finite dimensional Levi-Tanaka algebras $\mathfrak{g} = \oplus_{p\in\mathbb{Z}}\mathfrak{g}_p$ with $\dim_{\mathbb{R}}\mathfrak{g}_{-1} = 2$

Let $\mathfrak{m} = \oplus_{-\mu\leq p<0}\mathfrak{g}_p$ be a pseudocomplex fundamental graded Lie algebra with $n = \frac{1}{2}dim_{\mathbb{R}}\mathfrak{g}_{-1} = 1$ and let $\mathfrak{g} = \oplus_{p\geq-\mu}\mathfrak{g}_p$ be its canonical pseudocomplex prolongation. Suppose that $\mathfrak{m}$ is nondegenerate. This is equivalent to $\mu \geq 2$ and, by Theorem 3.1.1, to $\mathfrak{g}$ finite dimensional. Note that $\mathfrak{g}_0$ is abelian, $dim_{\mathbb{R}}\mathfrak{g}_0 \leq 2$ and the equality holds if and only if $\mathfrak{g}$ has the (J) property (see subsection 3.2.1).

We will prove that:

1. if $\mathfrak{g}_1 = 0$, then $\mathfrak{g}$ is solvable;
2. if $\mathfrak{g}_1 \neq 0$, then $\mathfrak{g}$ is simple and isomorphic to $\mathfrak{su}(2,1)$ with the graduation given in the example in 3.8.1.

As ρ_{-2} is irreducible, by Corollary 3.5.8, the algebra $\mathfrak{g}$ is simple or weakly-solvable. If $\mathfrak{g}$ is weakly-solvable, then $\mathfrak{g}$ decomposes as direct sum of a graded semisimple pseudocomplex Lie algebra $\mathfrak{s} = \mathfrak{s}_{-1} \oplus \mathfrak{s}_0 \oplus \mathfrak{s}_1$ and the radical $\mathfrak{r}$. If $\mathfrak{s}_{-1}$ were nontrivial, then it would coincide with $\mathfrak{g}_{-1}$ and so, by Corollary 3.5.6, $\mathfrak{g}$ would be semisimple. Then $\mathfrak{s}_{-1} = 0$ and $\mathfrak{g}$ is almost-solvable. Again by Corollary 3.5.8, $\mathfrak{g}_1 = 0$.

Let us assume now that $\mathfrak{g}$ is simple. The complexification $\mathfrak{g}^{\mathbb{C}} = \mathbb{C} \otimes_{\mathbb{R}} \mathfrak{g}$ of $\mathfrak{g}$ is a semisimple complex Lie algebra and a Levi-Tanaka algebra. By Lemma 3.2.5, $\mathfrak{g}^{\mathbb{C}}$ has a Cartan subalgebra $\mathfrak{h}^{\mathbb{C}}$ contained in $\mathfrak{g}_0^{\mathbb{C}} = \mathbb{C} \otimes_{\mathbb{R}} \mathfrak{g}_0$ and then its rank $\ell = dim_{\mathbb{C}}\mathfrak{h}^{\mathbb{C}}$ is less than or equal to $dim_{\mathbb{C}}\mathfrak{g}_0^{\mathbb{C}} \leq 2$. By the classification of simple complex Lie algebras we have that $\mathfrak{g}^{\mathbb{C}}$ is isomorphic to one of the following: $\mathfrak{so}(5,\mathbb{C})$, $\mathfrak{sl}(2,\mathbb{C})$, $\mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C})$, $\mathfrak{sl}(3,\mathbb{C})$ or the exceptional Lie algebra G_2. Since $dim_{\mathbb{C}}\mathfrak{g}^{\mathbb{C}} = dim_{\mathbb{R}}\mathfrak{g} \geq 7$, and using Proposition 3.6.9, we obtain that $\mathfrak{g}^{\mathbb{C}}$ is isomorphic to $\mathfrak{sl}(3,\mathbb{C})$. This implies $\mu = 2$ and $dim_{\mathbb{R}}\mathfrak{g}_{-2} = 1$. These two conditions characterize a fundamental graded Lie algebra $\mathfrak{m}$ whose prolongation is isomorphic to $\mathfrak{su}(2,1)$ (cf. example in 3.8.1).

Chapter 4

Classification of semisimple Levi-Tanaka algebras

In this chapter we classify finite dimensional semisimple Levi-Tanaka algebras up to isomorphisms. Every semisimple Levi-Tanaka algebra is a direct sum of simple ideals which are Levi-Tanaka algebras for the restriction of the partial complex structure. Hence, it suffices to classify simple Levi-Tanaka algebras.

In the classical cases we will give also matrix representations, while for the exceptional ones we will give in appendix a complete list, obtained by symbolic calculus.

4.1 Real semisimple graded Lie algebras

In this section we rehearse some general properties of finite dimensional real semisimple Lie algebras, for which we refer to [12], [37], [45], and consider their bearing to semisimple Levi-Tanaka algebras.

Let $\mathfrak{s}$ be a finite dimensional semisimple real Lie algebra. Every Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{s}$ decomposes into a direct sum $\mathfrak{h} = \mathfrak{h}^+ \oplus \mathfrak{h}^-$ where

$$\begin{aligned} \mathfrak{h}^+ &= \{X \in \mathfrak{h} \mid \mathrm{ad}_{\mathfrak{s}}(X) \text{ has purely imaginary eigenvalues}\} \\ \mathfrak{h}^- &= \{X \in \mathfrak{h} \mid \mathrm{ad}_{\mathfrak{s}}(X) \text{ has real eigenvalues}\} \end{aligned}$$

are called respectively the *toroidal* and *vectorial part* of $\mathfrak{h}$. A Cartan subalgebra $\mathfrak{h}$ whose toroidal part has minimal dimension is called *minimally compact.*

We have (cf. [37, Cor.2 to Th.3] or also [45, Cor.1.3.1.5]):

PROPOSITION 4.1.1 *All minimally compact Cartan subalgebras of a semisimple real Lie algebra are conjugated by the action of the adjoint group.*

The complexification $\mathfrak{h}^{\mathbb{C}}$ of a minimally compact Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{s}$ is a Cartan subalgebra of the complexification $\mathfrak{s}^{\mathbb{C}}$ of $\mathfrak{s}$. Denote by $\mathcal{R}$ the relative root system. Consider the real form $\mathfrak{h}_u = \mathfrak{h}^- \oplus \sqrt{-1}\mathfrak{h}^+$ of $\mathfrak{h}^{\mathbb{C}}$. We identify the complexification of the dual $\check{\mathfrak{h}}_u$ of $\mathfrak{h}_u$ with the dual $\check{\mathfrak{h}}^{\mathbb{C}}$ of $\mathfrak{h}^{\mathbb{C}}$. Then $\mathcal{R} \subset \check{\mathfrak{h}}_u \subset \check{\mathfrak{h}}^{\mathbb{C}}$. Let σ be the conjugation on $\mathfrak{s}^{\mathbb{C}}$ defined by the real form $\mathfrak{s}$ and denote by σ the corresponding involution of $\check{\mathfrak{h}}^{\mathbb{C}}$ given by $\check{\mathfrak{h}}^{\mathbb{C}} \ni \alpha \to \alpha^\sigma \in \check{\mathfrak{h}}^{\mathbb{C}}$, where

$$\alpha^\sigma(X) = \overline{\alpha(\sigma X)} \quad \forall X \in \mathfrak{h}^{\mathbb{C}}. \tag{4.1}$$

Then the root system $\mathcal{R}$ is σ-invariant.

In the following we will use $\mathfrak{s}$ to denote a finite dimensional semisimple real Lie algebra and $\mathfrak{g}$ for the graded Lie algebra obtained by fixing a graduation on $\mathfrak{s}$.

We recall that a finite dimensional semisimple graded Lie algebra $\mathfrak{g} = \oplus_{-\mu \le p \le \mu} \mathfrak{g}_p$ has the characteristic element E which is semisimple with integral eigenvalues. By Theorem 3.6.5, if $\mathfrak{g}$ is also pseudocomplex, then there exists an element $\tilde{J}$ in the center $\mathfrak{z}(\mathfrak{g}_0)$ of $\mathfrak{g}_0$ such that $J = \rho_{-1}(\tilde{J})$; this element is semisimple with purely imaginary eigenvalues. Then we have:

Remark 4.1.2 *A Cartan subalgebra $\mathfrak{h}$ of a semisimple graded Lie algebra $\mathfrak{g} = \oplus_{-\mu \le p \le \mu} \mathfrak{g}_p$ is contained in $\mathfrak{g}_0$ if and only if it contains the characteristic element E of $\mathfrak{g}$ (and the element $\tilde{J}$ if $\mathfrak{g}$ is also pseudocomplex); moreover $E \in \mathfrak{z}(\mathfrak{g}_0) \cap \mathfrak{h}^-$ (and $\tilde{J} \in \mathfrak{z}(\mathfrak{g}_0) \cap \mathfrak{h}^+$).*

We have (cf. [12]):

LEMMA 4.1.3 *The characteristic element E of a semisimple real graded Lie algebra $\mathfrak{g} = \oplus_{p \in \mathbb{Z}} \mathfrak{g}_p$ is contained in a minimally compact Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}_0$ of $\mathfrak{g}$.*

Proof. The lemma is a consequence of the fact (cf. [37], Theorem 2) that the vectorial part of any Cartan subalgebra of $\mathfrak{g}$ is conjugated by an element of the adjoint group to a subspace of a minimally compact Cartan subalgebra of $\mathfrak{g}$. Then we reduce to the observation that the semisimple element E belongs to a Cartan subalgebra of $\mathfrak{g}$. □

Let $\mathfrak{h}$ be a minimally compact Cartan subalgebra of a semisimple graded Lie algebra $\mathfrak{g}$, containing its characteristic element E, and let $\mathcal{R}$ be the root system of $\mathfrak{g}^{\mathbb{C}}$ with respect to $\mathfrak{h}^{\mathbb{C}}$. We set:

$$\begin{aligned}
\mathcal{R}_{\bullet} &= \{\alpha \in \mathcal{R} \mid \alpha(H) = 0 \quad \forall H \in \mathfrak{h}^{-}\}\,; \\
\mathcal{R}_{p} &= \{\alpha \in \mathcal{R} \mid \alpha(E) = p\} \quad \text{for} \quad p \in \mathbb{Z}\,; \\
\mathcal{R}_{-} &= \{\alpha \in \mathcal{R} \mid \alpha(E) \leq 0\} = \bigcup_{p \leq 0} \mathcal{R}_{p}\,.
\end{aligned}$$

We have the inclusions: $\mathcal{R}_{\bullet} \subset \mathcal{R}_0 \subset \mathcal{R}_{-} \subset \mathcal{R}$.

We call the integer $|\alpha| = \alpha(E)$ the *degree* of the root $\alpha \in \mathcal{R}$.

Remark 4.1.4 *As $\alpha^{\sigma}(H) = \overline{\alpha(\sigma H)} = \overline{\alpha(H)} = \alpha(H)$ for every $H \in \mathfrak{h}^{-}$, the root systems $\mathcal{R}_{\bullet}$ and $\mathcal{R}_0$ are invariant under the action of the conjugation σ defined by (4.1).*

For every $\alpha \in \mathcal{R}$ we denote by $\mathfrak{g}^{\alpha}$ the relative eigenspace:

$$\mathfrak{g}^{\alpha} = \{X \in \mathfrak{g}^{\mathbb{C}} \mid [H, X] = \alpha(H)X \quad \forall H \in \mathfrak{h}^{\mathbb{C}}\}.$$

We have:

$$\begin{aligned}
\mathfrak{g}_0^{\mathbb{C}} &= \mathfrak{h}^{\mathbb{C}} \oplus \bigoplus_{\alpha \in \mathcal{R}_0} \mathfrak{g}^{\alpha} \\
\mathfrak{g}_p^{\mathbb{C}} &= \bigoplus_{\alpha \in \mathcal{R}_p} \mathfrak{g}^{\alpha}\,.
\end{aligned}$$

Assume that $\mathfrak{g}$ is a semisimple pseudocomplex graded Lie algebra with partial complex structure $J \in \mathfrak{gl}_{\mathbb{R}}(\mathfrak{g}_{-1})$. We recall that:

$$\begin{aligned}
\mathfrak{g}_{-1}^{(1,0)} &= \{X - \sqrt{-1}JX \mid X \in \mathfrak{g}_{-1}\} \subset \mathfrak{g}_{-1}^{\mathbb{C}} \\
\mathfrak{g}_{-1}^{(0,1)} &= \{X + \sqrt{-1}JX \mid X \in \mathfrak{g}_{-1}\} = \sigma\left(\mathfrak{g}_{-1}^{(1,0)}\right) \subset \mathfrak{g}_{-1}^{\mathbb{C}}.
\end{aligned}$$

These are the eigenspaces corresponding respectively to the eigenvectors $\sqrt{-1}$ and $-\sqrt{-1}$ of J and are invariant under $\rho_{-1}(\mathfrak{g}_0)$. In particular each eigenspace $\mathfrak{g}^{\alpha}$ for $\alpha \in \mathcal{R}_{-1}$ is contained either in $\mathfrak{g}_{-1}^{(1,0)}$ or in $\mathfrak{g}_{-1}^{(0,1)}$ and is J-invariant. Moreover, $\mathfrak{g}^{\alpha} \subset \mathfrak{g}_{-1}^{(1,0)}$ if and only if $\mathfrak{g}^{\alpha^{\sigma}} = \sigma(\mathfrak{g}^{\alpha}) \subset \mathfrak{g}_{-1}^{(0,1)}$.

If X is an eigenvector corresponding to a root $\alpha \in \mathcal{R}$ with $|\alpha| \in \{-1, 0\}$, we define

$$\operatorname{sgn}(X) = \begin{cases} +1 & \text{if } |\alpha| = -1 \text{ and } \mathfrak{g}^\alpha \subset \mathfrak{g}_p^{(1,0)} \\ -1 & \text{if } |\alpha| = -1 \text{ and } \mathfrak{g}^\alpha \subset \mathfrak{g}_p^{(0,1)} \\ 0 & \text{if } |\alpha| = 0. \end{cases} \tag{4.2}$$

Note that $JX = \operatorname{sgn}(X)\sqrt{-1}X$ when $X \in \mathfrak{g}^\alpha$ with $|\alpha| = -1$.

We recall also that, by Theorem 3.6.5, if $\mathfrak{g}$ is a semisimple Levi-Tanaka algebra, there exists a unique element $\tilde{J}$ in the center $\mathfrak{z}(\mathfrak{g}_0)$ of $\mathfrak{g}_0$ whose representation $\rho_{-1}(\tilde{J})$ in $\mathfrak{g}_{-1}$ coincides with the partial complex structure J of $\mathfrak{g}$.

In the following we will use for simplicity the same symbol J to denote either the partial complex structure J of $\mathfrak{g}$ or the element $\tilde{J}$ of the center of $\mathfrak{g}_0$.

We list now some results, related to the classification of real semisimple graded Lie algebras, for which we refer to [12], [3], [19], [45].

Let $\mathfrak{g} = \oplus_{-\mu \le p \le \mu} \mathfrak{g}_p$ be a semisimple real graded Lie algebra of kind μ. Then there is a minimally compact Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ contained in $\mathfrak{g}_0$ and $\mathfrak{h}$ contains the characteristic element E of $\mathfrak{g}$. We denote by $\mathcal{R}$ the root system of $\mathfrak{g}^{\mathbb{C}}$ relative to $\mathfrak{h}^{\mathbb{C}}$. It admits a fundamental root system $\mathcal{B} = \{\alpha_1, ..., \alpha_\ell\}$ contained in $\mathcal{R}_-$, which is σ-fundamental according to [45, p.23] (cf. condition (2) below).

We denote by $\Delta_{\mathfrak{g}}$ the *weighted Dynkin diagram* of $\mathfrak{g}^{\mathbb{C}}$, which is obtained from the Dynkin diagram of $\mathfrak{g}^{\mathbb{C}}$, in which the vertices are identified to the corresponding roots in $\mathcal{B}$, by attaching to each vertex α_i its degree $|\alpha_i|$. The *weighted Satake diagram* $\Sigma_{\mathfrak{g}}$ of $\mathfrak{g}$ is obtained from the weighted Dynkin diagram $\Delta_{\mathfrak{g}}$ of $\mathfrak{g}^{\mathbb{C}}$ by the following procedure:

1. The vertices $\alpha \in \mathcal{B}_\bullet = \mathcal{B} \cap \mathcal{R}_\bullet$ are black and all other vertices are white. Note that black vertices have degree 0.

2. For every white vertex $\alpha \in \mathcal{B} - \mathcal{B}_\bullet$ there exists a unique white vertex $\alpha' \in \mathcal{B} - \mathcal{B}_\bullet$ such that $\alpha^\sigma - \alpha'$ is a linear combination of the black roots (cf. [45], Lemma 1.1.3.2). If $\alpha \neq \alpha'$, then we connect the pair $\{\alpha, \alpha'\}$ by a curved arrow. Note that roots connected by a curved arrow have the same degree.

Let $\mathfrak{s}$ be a semisimple real Lie algebra and $\mathfrak{g}$ a graded Lie algebra obtained by fixing a graduation of $\mathfrak{s}$. The *weighted Satake diagram* of $\mathfrak{g}$ is obtained by attaching to each vertex α of the Satake diagram of $\mathfrak{s}$ its degree $|\alpha|$ with respect to $\mathfrak{g}$. This construction provides a partition of $\mathcal{B}$ into the subsets $\mathcal{B}_p$, with $p \leq 0$, given by

$$\mathcal{B}_p = \mathcal{B} \cap \mathcal{R}_p = \{\alpha \in \mathcal{B} \,|\, |\alpha| = p\}\,. \tag{4.3}$$

Vice versa, every such partition of $\mathcal{B}$, indexed by the nonpositive integers, determines a unique graduation of $\mathfrak{s}$ and all graduations of $\mathfrak{s}$ are obtained in this way, up to isomorphisms of graded Lie algebras. We explicitly note that the fact that $\mathfrak{g}_{-1}$ generates $\mathfrak{m}(\mathfrak{g})$ is equivalent to have $\mathcal{B} = \mathcal{B}_0 \cup \mathcal{B}_{-1}$.

The classification of real semisimple graded Lie algebras is deduced from the following:

THEOREM 4.1.5 *A necessary and sufficient condition in order that two real graded Lie algebras be isomorphic is that they have isomorphic weighted Satake diagrams.*

We also note that (cf. [19], Theorem 2.4):

PROPOSITION 4.1.6 *1. The kind μ of $\mathfrak{g}$ is equal to the absolute value of the degree of the highest root of $\mathfrak{g}^{\mathbb{C}}$.*

2. Assume that the graduation of $\mathfrak{g}$ is associated to a partition $\mathcal{B} = \mathcal{B}_0 \cup \mathcal{B}_{-1}$ of the fundamental roots of $\mathfrak{g}$. Then $\mathfrak{g}_0$ is the direct sum of a semisimple Lie algebra, whose Satake diagram is obtained from $\Sigma_{\mathfrak{g}}$ by deleting all vertices in $\mathcal{B}_{-1}$ and all rods and arrows issuing from them, and its center $\mathfrak{z}(\mathfrak{g}_0)$, whose dimension equals the number of elements of $\mathcal{B}_{-1}$.

We say that a real simple Lie algebra is *of the complex type* if its Satake diagram is disconnected, i.e. if it is obtained from a simple complex Lie algebra by change of the base field; we say that a simple real Lie algebra is *of the real type* if it has a connected Satake diagram, i.e. if its complexification is also simple.

We note that for simple Lie algebras $\mathfrak{s}$ of the complex type the Satake diagram is obtained by taking two copies $D'_{\mathfrak{s}}$ and $D''_{\mathfrak{s}}$ of its Dynkin diagram $D_{\mathfrak{s}}$, painting white all vertices and connecting by a curved arrow each root in $D'_{\mathfrak{s}}$ to the same root in $D''_{\mathfrak{s}}$. Thus for simple real Lie algebras of the complex

type the Dynkin diagram already contains all information.

We collect some elementary facts on the Satake diagram of a Levi-Tanaka algebra in the following

THEOREM 4.1.7 *Let $\mathfrak{g}$ be a finite dimensional semisimple Levi-Tanaka algebra, and $\Sigma_{\mathfrak{g}}$ its weighted Satake diagram related to a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}_0$ of $\mathfrak{g}$. Then:*

(*i*) *the degrees of the vertices in $\mathcal{B}$ are either (-1) or 0, so that $\mathcal{B} = \mathcal{B}_{-1} \cup \mathcal{B}_0$; the set $\mathcal{B}_{-1}$ of fundamental roots of degree (-1) contains at least two elements;*

(*ii*) *each vertex α in $\mathcal{B}_{-1}$ is connected to another vertex α' in $\mathcal{B}_{-1}$ by a curved arrow.*

Proof. The first claim in (*i*) follows from the fact that $\mathfrak{g}_{-1}$ generates the subalgebra $\mathfrak{m}(\mathfrak{g}) = \oplus_{p<0}\mathfrak{g}_p$, the second from (*ii*) and the fact that the kind μ of a finite dimensional Levi-Tanaka algebra is ≥ 2.

We prove (*ii*) by contradiction. Assume that $\alpha \in \mathcal{B}_{-1}$ is not connected to any other root in $\mathcal{B}_{-1}$ by a curved arrow. Then $\alpha^\sigma = \alpha + \gamma$, where γ is a linear combination with integral coefficients of roots of $\mathcal{B}_0$. It follows that J acts as the multiplication by the same $\eta = \pm\sqrt{-1}$ both on $\mathfrak{g}^\alpha$ and on $\mathfrak{g}^{\alpha^\sigma}$. But we already noticed that this cannot be the case because $\mathfrak{g}^\alpha$ and $\mathfrak{g}^{\alpha^\sigma}$ are contained in eigenspaces of J corresponding to opposite eigenvalues. □

For the sake of conciseness we will call in the following *LT-admissible* the weighted Satake diagrams of semisimple Levi-Tanaka algebras.

We also call *admissible* a weighted Satake diagram satisfying conditions (*i*) and (*ii*) in Theorem 4.1.7.

4.2 The weighted Satake diagrams of simple Levi-Tanaka algebras of the complex type

First we investigate the Levi-Tanaka structures that can be defined on a simple real Lie algebra $\mathfrak{s}$ of the complex type.

We fix a minimally compact Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{s}$. Note that $\mathfrak{h}$ is simply a Cartan subalgebra of $\mathfrak{s}$ considered as a complex Lie algebra. Let $\mathcal{R}$ be the corresponding root system (attached to $\mathfrak{s}$ and $\mathfrak{h}$ considered as complex Lie algebras), $\mathcal{B} = \{\alpha_1, \dots, \alpha_\ell\}$ a fundamental root system for $\mathcal{R}$ and $D_{\mathfrak{s}}$ the associated Dynkin diagram. For simplicity we will call *connected* a subset Y of $\mathcal{B}$ if its points are the vertices of a connected subset of the graph $D_{\mathfrak{s}}$.

Up to equivalence, all admissible gradings of $\mathfrak{s}$ are obtained from a partition $\{\mathcal{B}_0, \mathcal{B}_{-1}\}$ of $\mathcal{B}$ into a set $\mathcal{B}_{-1}$ of fundamental roots of degree (-1) and a set $\mathcal{B}_0$ of fundamental roots of degree 0. Let $\mathcal{B}_{-1} = \{\alpha_{i_1}, \dots, \alpha_{i_\nu}\}$, with $1 \leq i_1 < \dots < i_\nu \leq \ell$, be the set of roots of degree (-1).

Every root $\alpha \in \mathcal{R}$ is a linear combination with integral coefficients

$$\alpha = \sum_{i=1}^{\ell} k_i(\alpha)\alpha_i\,, \quad k_i(\alpha) \in \mathbb{Z} \tag{4.4}$$

of the roots in $\mathcal{B}$. We associate to the root α its degree

$$|\alpha| = \sum_{i=1}^{\ell} k_i(\alpha)|\alpha_i| \tag{4.5}$$

where $|\alpha_i|$ equals (-1) if $\alpha_i \in \mathcal{B}_{-1}$ and 0 if $\alpha_i \in \mathcal{B}_0$. We recall that

$$Y(\alpha) = \{\alpha_i \,|\, k_i(\alpha) \neq 0\} \tag{4.6}$$

is a connected subset of $D_{\mathfrak{s}}$ and, for every connected $Y \subset D_{\mathfrak{s}}$,

$$\sum_{\alpha_i \in Y} \alpha_i \in \mathcal{R} \tag{4.7}$$

(cf. Corollary 3 to Proposition 19, Ch.VI §1 in [6]).

We denote by $\mathfrak{g}$ the graded Lie algebra obtained from $\mathfrak{s}$ by the partition $\{\mathcal{B}_{-1}, \mathcal{B}_0\}$ of $\mathcal{B}$: the subspace $\mathfrak{g}_p$ of homogeneous elements of degree p is defined by:

$$\mathfrak{g}_0 = \mathfrak{h} \oplus \bigoplus_{\alpha \in \mathcal{R}_0} \mathfrak{g}^\alpha$$

$$\mathfrak{g}_p = \bigoplus_{\alpha \in \mathcal{R}_p} \mathfrak{g}^\alpha \qquad \text{for} \quad p \neq 0.$$

To define on the so obtained graded Lie algebra $\mathfrak{g}$ a partial complex structure satisfying (2.2), by Theorem 3.6.5, we need to find an element $J \in \mathfrak{z}(\mathfrak{g}_0) \subset \mathfrak{h}$ such that

$$\begin{cases} \rho_{-1}(J)^2 = -\mathrm{Id}_{\mathfrak{g}_{-1}} \\ [J, \mathfrak{g}_{-2}] = 0. \end{cases} \tag{4.8}$$

We note that J must satisfy

$$\langle J, \alpha_i \rangle = \begin{cases} 0 & \text{if} \quad \alpha_i \in \mathcal{B}_0 \\ \pm\sqrt{-1} & \text{if} \quad \alpha_i \in \mathcal{B}_{-1} \end{cases} \tag{4.9}$$

because the eigenspaces of the roots α_i are contained in the eigenspaces of $\mathrm{ad}_{\mathfrak{g}}(J)$.

Then from (4.8) we obtain:

(a) *if Y is a connected subset of $D_{\mathfrak{s}}$ and $Y \cap \mathcal{B}_{-1} = \{\alpha_i, \alpha_j\}$, with $i < j$, then*

$$\langle J, \alpha_i \rangle = -\langle J, \alpha_j \rangle. \tag{4.10}$$

Let indeed Y be a connected subset of $D_{\mathfrak{s}}$ containing exactly two elements α_i, α_j of $\mathcal{B}_{-1}$. Decompose Y into two disjoint connected subsets Y_1, Y_2 such that $\{\alpha_i\} = Y_1 \cap \mathcal{B}_{-1}$ and $\{\alpha_j\} = Y_2 \cap \mathcal{B}_{-1}$. Then $\beta_1 = \sum_{\alpha_h \in Y_1} \alpha_h$ and $\beta_2 = \sum_{\alpha_h \in Y_2} \alpha_h$ are roots of degree (-1) and

$$\langle J, \beta_1 \rangle = \langle J, \alpha_i \rangle, \quad \langle J, \beta_2 \rangle = \langle J, \alpha_j \rangle.$$

Since $\beta_1 + \beta_2 \in \mathcal{R}$, we obtain for nonzero eigenvectors X_{β_1}, X_{β_2} of β_1, β_2 respectively:

$$0 \neq [X_{\beta_1}, X_{\beta_2}] = [JX_{\beta_1}, JX_{\beta_2}] = \langle J, \alpha_i \rangle \langle J, \alpha_j \rangle [X_{\beta_1}, X_{\beta_2}].$$

This shows that condition (a) is necessary.

From this observation, since any two roots in $\mathcal{B}$ can be joined by a segment in $D_{\mathfrak{s}}$, we deduce that J is uniquely determined by the value it assumes on one of the roots in $\mathcal{B}_{-1}$. In particular, for each admissibile structure of graded Lie algebra $\mathfrak{g}$ of $\mathfrak{s}$, either there is no partial complex structure J satisfying (2.2), or there are two such structures, one being conjugated to the other.

Moreover, condition (4.10) does not restrict the possibility of defining the partial complex structure J, unless D_s contains ramification points. In the last case, there are two possibilities:

(b) *either the ramification point α_i of D_s belongs to $\mathcal{B}_{-1}$, or at most two of the branches issuing from α_i contain elements of $\mathcal{B}_{-1}$.*

Assuming that the conditions (a) and (b) are fulfilled by a J defined on the elements of $\mathcal{B}$, then a necessary and sufficient condition in order that J extends to an $\mathbb{R}$-linear map definining a complex structure on $\mathfrak{g}_{-1}$ satisfying (2.2) is that:

(c) *there are no roots $\alpha \in \mathcal{R}$ with $|\alpha| = -2$ and $k_i(\alpha) = 2$ for some $\alpha_i \in \mathcal{B}_{-1}$.*

Indeed for $\alpha \in \mathcal{R}$ with $|\alpha| = -2$ there are the two possibilities:

$(*)$ $Y(\alpha) \cap \mathcal{B}_{-1} = \{\alpha_i, \alpha_j\}$ contains exactly two roots and $k_i(\alpha) = k_j(\alpha) = 1$;

$(**)$ $Y(\alpha) \cap \mathcal{B}_{-1} = \{\alpha_i\}$ contains only one root and $k_i(\alpha) = 2$.

We first extend J to a $\mathbb{C}$-linear map on $\mathfrak{g}_0 \oplus \mathfrak{g}_{-1}$ by setting $J = 0$ on $\mathfrak{g}_0$ and $JX_\beta = \langle J, \alpha_i \rangle X_\beta$ if $\{\alpha_i\} = Y(\beta) \cap \mathcal{B}_{-1}$.

A nonzero eigenvector X_α corresponding to a root $\alpha \in \mathcal{R}$ of degree (-2) is obtained as the Lie product $[X_\beta, X_\gamma]$ of two eigenvectors corresponding to roots $\beta, \gamma \in \mathcal{R}$ with $|\beta| = |\gamma| = -1$. If the root α satisfies $(**)$, then J acts on X_β and X_γ as the multiplication by the same $\eta = \pm\sqrt{-1}$ and hence, if (2.2) is satisfied,

$$X_\alpha = [X_\beta, X_\gamma] = [JX_\beta, JX_\gamma] = -[X_\beta, X_\gamma] \Rightarrow X_\alpha = 0,$$

gives a contradiction.

Vice versa, if all roots $\alpha \in \mathcal{R}$ with $|\alpha| = -2$ satisfy $(*)$, then each of these roots can be represented as $\alpha = \beta + \gamma$ for two roots $\beta, \gamma \in \mathcal{R}$ with $|\beta| = |\gamma| = -1$. Then $Y(\beta) \cap \mathcal{B}_{-1} = \{\alpha_i\}$ and $Y(\gamma) \cap \mathcal{B}_{-1} = \{\alpha_j\}$, where α_i, α_j are the edges of a segment in D_s which does not contain any other root in $\mathcal{B}_{-1}$. By condition (a), J acts on an eigenvector X_β of β as the multiplication by η and on an eigenvector X_γ of γ as the multiplication by $-\eta$, with $\eta = \pm\sqrt{-1}$. Then

$$X_\alpha = [X_\beta, X_\gamma] = [JX_\beta, JX_\gamma]$$

and by $\mathbb{C}$-linearity condition (2.2) is satisfied in this case.

We summarize the discussion above by:

THEOREM 4.2.1 *A necessary and sufficient condition in order that a weighted Dynkin diagram $\Delta_{\mathfrak{g}}$ with vertices $\mathcal{B} = \{\alpha_1, \dots, \alpha_\ell\}$, associated to a graduation $\mathfrak{g}$ of a simple Lie algebra $\mathfrak{s}$ of the complex type correspond to a LT-admissible Satake diagram is that the following three conditions be satisfied:*

1. *the graduation of $\mathfrak{g}$ is defined by a partition of $\mathcal{B}$ into a subset $\mathcal{B}_{-1}$ of roots of degree (-1) and a subset $\mathcal{B}_0$ of roots of degree 0 and $\mathcal{B}_{-1}$ contains at least two elements;*

2. *if $\alpha_i \in \mathcal{B}$ is a ramification point of $\Delta_{\mathfrak{g}}$, then either $\alpha_i \in \mathcal{B}_{-1}$ or at most two of the branches issuing from α_i contain vertices in $\mathcal{B}_{-1}$;*

3. *there are no roots $\alpha = \sum_{i=1}^{\ell} k_i(\alpha)\alpha_i \in \mathcal{R}_-$ with $|\alpha| = -2$ and $k_i(\alpha) = 2$ for some $\alpha_i \in \mathcal{B}_{-1}$.*

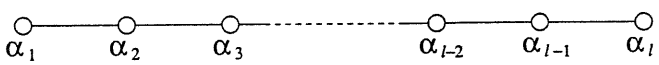
To every LT-admissible weighted Satake diagram of a simple Lie algebra of the complex type correspond exactly two partial complex structures on the corresponding graded Lie algebra $\mathfrak{g}$, one conjugated to the other.

Using this theorem, we can give the complete list of LT-admissible Satake diagrams for simple Lie algebras of the complex type. We refer to [6], Ch.VI for all relevant informations on the root systems. Note that the issue reduces to finding the partitions $\{\mathcal{B}_{-1}, \mathcal{B}_0\}$ of $\mathcal{B}$ into a subset $\mathcal{B}_{-1}$ of roots of degree (-1) and a subset $\mathcal{B}_0$ of roots of degree 0 leading to LT-admissible Satake diagrams. We will say in this case that $\mathcal{B}_{-1}$ is *LT-admissible.*

While considering the classification of a simple Levi-Tanaka algebra $\mathfrak{g}$ of the complex type, we must take into account the fact that an automorphism of its weighted Dynkin diagram $\Delta_{\mathfrak{g}}$ induces either an automorphism or an antiautomorphism (i.e. changing J into $-J$) of $\mathfrak{g}$.

4.2.1 Simple Levi-Tanaka algebras of the complex type A_ℓ $(\ell \geq 1)$

The basic roots $\alpha_1, \dots, \alpha_\ell$ are organized in the Dynkin diagram:



Since the Dynkin diagram has no ramification points and there are no roots in $\mathcal{R}_-$ with coefficients $k_i \geq 2$, every $\mathcal{B}_{-1}$ containing at least two elements is LT-admissible. Note that we need $\ell \geq 2$.

Since the highest root in $\mathcal{R}_-$ is $\delta = \alpha_1 + \ldots + \alpha_\ell$, the kind μ equals the number ν of roots in $\mathcal{B}_{-1}$.

We note that the isomorphism of the Dynkin diagram $s : \mathcal{B} \ni \alpha_i \to \alpha_{\ell+1-i} \in \mathcal{B}$ yields isomorphisms of the Levi-Tanaka algebras corresponding to $\mathcal{B}_{-1}$ and $s(\mathcal{B}_{-1})$. Therefore, up to equivalence, the weighted Dynkin diagrams associated to simple Levi-Tanaka algebras of the complex type A_ℓ are parametrized by

$$\mathcal{B}_{-1} = \{\alpha_{i_1}, \ldots, \alpha_{i_\nu}\} \quad \text{with} \quad \nu \geq 2 \tag{4.11}$$

and, to take into account the isomorphism s, we impose that

(4.12)
$$(i_1, \ldots, i_\nu) \leq (\ell + 1 - i_\nu, \ldots, \ell + 1 - i_1) \quad \text{for the lexicographic order.}$$

To each $\mathcal{B}_{-1}$ in (4.11) there correspond two nonisomorphic Levi-Tanaka algebras when $(i_1, \ldots, i_\nu) < (\ell + 1 - i_\nu, \ldots, \ell + 1 - i_1)$ or when $(i_1, \ldots, i_\nu) = (\ell+1-i_\nu, \ldots, \ell+1-i_1)$ and ν is odd. We obtain, up to isomorphisms, only one structure of Levi-Tanaka algebra when $(i_1, \ldots, i_\nu) = (\ell+1-i_\nu, \ldots, \ell+1-i_1)$ and ν is even.

Let $\mathcal{B}_{-1} = \{\alpha_{i_1}, \ldots, \alpha_{i_\nu}\}$ with $\nu \geq 2$, $1 \leq i_1 < \ldots < i_\nu \leq \ell$ and let us define

$$d_0 = i_1,\ d_1 = i_2 - i_1,\ \ldots\ d_{\nu-1} = i_\nu - i_{\nu-1},\ d_\nu = \ell + 1 - i_\nu.$$

Let $\mathfrak{g} = \oplus_{-\nu \leq p \leq \nu} \mathfrak{g}_p$ be a Levi-Tanaka algebra associated to $\mathcal{B}_{-1}$. We know from Lemma 1.1 that $\mathfrak{g}_0$ is reductive. Its center $\mathfrak{z}(\mathfrak{g}_0)$ has dimension ν and the semisimple part is isomorphic to the direct sum:

$$\bigoplus_{d_i > 1} \mathfrak{sl}(d_i, \mathbb{C}).$$

Moreover we obtain:

$$\begin{cases} \dim_{\mathbb{C}} \mathfrak{g}_0 = (\sum_{i=0}^{\nu} d_i^2) - 1 \\ \dim_{\mathbb{C}} \mathfrak{g}_{\pm p} = \sum_{i=p}^{\nu} d_i d_{i-p} \qquad \text{for} \quad p = 1, \ldots, \nu. \end{cases}$$

To obtain a matrix representation of $\mathfrak{g}$, it is convenient to write every matrix X in $\mathfrak{sl}(\ell+1,\mathbb{C})$ as

$$X = (x_{ij})_{0\le i,j\le \nu} \quad \text{with} \quad x_{ij} \in \mathfrak{M}(d_i \times d_j, \mathbb{C}).$$

Then the characteristic element E of $\mathfrak{g}$ is given by the matrix

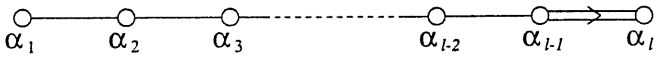
$$\begin{pmatrix} e_0 I_{d_0} & 0 & \dots & 0 \\ 0 & e_1 I_{d_1} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & e_\nu I_{d_\nu} \end{pmatrix} \quad \text{where } e_j = \frac{jd_0 + (j-1)d_1 + \ldots + (j-\nu)d_\nu}{\ell+1}$$

while the partial complex structure is represented by plus or minus the matrix

$$\begin{pmatrix} \eta_1 I_{d_0} & 0 & \dots & 0 \\ 0 & \eta_{-1} I_{d_1} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \eta_{(-1)^\nu} I_{d_\nu} \end{pmatrix} \quad \text{where} \quad \begin{cases} \eta_1 = \sqrt{-1}\frac{d_1+d_3+\ldots}{\ell+1} \\ \eta_{-1} = -\sqrt{-1}\frac{d_0+d_2+\ldots}{\ell+1} \end{cases}.$$

4.2.2 Simple Levi-Tanaka algebras of the complex type B_ℓ ($\ell \ge 2$)

The basic roots $\alpha_1, \ldots, \alpha_\ell$ are organized in the Dynkin diagram:



Also in this case there are no ramification points. The positive roots which have a coefficient larger than or equal to 2 are those of the form:

$$\sum_{i\le k<j} \alpha_i + 2\left(\sum_{j\le k\le \ell} \alpha_k\right) \quad \text{with} \quad 1 \le i < j \le \ell.$$

Hence, if $\mathcal{B}_{-1} = \{\alpha_{i_1}, \ldots, \alpha_{i_\nu}\}$ with $1 \le i_1 < \ldots < i_\nu \le \ell$, the necessary and sufficient condition for the existence of a Levi-Tanaka structure on $\mathfrak{g}$ is that

$$i_\nu = i_{\nu-1} + 1.$$

The highest root in $\mathcal{R}_-$ is $\delta = \alpha_1 + 2(\alpha_2 + \ldots + \alpha_\ell)$ and therefore for a LT-admissible $\mathcal{B}_{-1} = \{\alpha_{i_1}, \ldots, \alpha_{i_\nu}\}$ the kind μ of the corresponding Levi-Tanaka

algebras equals $2\nu-1$ or 2ν according to either α_1 belongs or does not belong to $\mathcal{B}_{-1}$.

There are no automorphisms of the Dynkin diagram of B_ℓ and therefore we distinguish the weighted Dynkin diagrams associated to simple Levi-Tanaka algebras of the complex type B_ℓ into two classes, that parametrize up to equivalence these algebras:

(4.13)
$$\begin{aligned}&\mathcal{B}_{-1}=\{\alpha_1,\alpha_{i_2},\dots,\alpha_{i_{\nu-1}},\alpha_{i_{\nu-1}+1}\}\\ &\quad\text{with}\quad \nu\geq 2,\ 1<i_2<\dots<i_{\nu-1}<\ell,\quad \text{of kind}\quad \mu=2\nu-1\end{aligned}$$

and

(4.14)
$$\begin{aligned}&\mathcal{B}_{-1}=\{\alpha_{i_1},\dots,\alpha_{i_{\nu-1}},\alpha_{i_{\nu-1}+1}\}\\ &\quad\text{with}\quad \nu\geq 2,\ 1<i_1<\dots<i_{\nu-1}<\ell,\quad \text{of kind}\quad \mu=2\nu.\end{aligned}$$

Let us set:

(4.15)
$$d_0=\ell-i_{\nu-1}-1,\ d_1=1,\ \dots,\ d_h=i_{\nu-h+1}-i_{\nu-h},\ \dots,\ d_\nu=i_1.$$

It is also convenient to set $d_h=0$ if $h\neq 0,1,\dots,\nu$.

The center $\mathfrak{z}(\mathfrak{g}_0)$ of $\mathfrak{g}_0$ has complex dimension ν. The subalgebra $\mathfrak{g}_0$ is the direct sum of its center and a semisimple part which is isomorphic to the direct sum

$$\begin{cases}\bigoplus_{d_i>1}\mathfrak{sl}(d_i,\mathbb{C})\oplus\mathfrak{so}(2d_0+1,\mathbb{C}) & \text{if}\quad d_0>0\\ \bigoplus_{d_i>1}\mathfrak{sl}(d_i,\mathbb{C}) & \text{if}\quad d_0=0.\end{cases}$$

Then we obtain

$$\begin{cases}\dim_{\mathbb{C}}\mathfrak{g}_0=1+\sum_{i=2}^{\nu}d_i^2+2d_0^2+d_0\\ \dim_{\mathbb{C}}\mathfrak{g}_{\pm(2p+1)}=\sum_{i=2}^{\nu-2p-1}d_id_{2p+1+i}+\sum_{i=2}^{p}d_id_{2p+1-i}+d_{2p}+(2d_0+1)d_{2p+1}+d_{2p+2}\\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\text{for}\quad p\geq 0\\ \dim_{\mathbb{C}}\mathfrak{g}_{\pm 2p}=\sum_{i=2}^{\nu-2p}d_id_{2p+i}+\sum_{i=2}^{p-1}d_id_{2p-i}+\frac{d_p(d_p-1)}{2}+d_{2p-1}+(2d_0+1)d_{2p}+d_{2p+1}\\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\text{for}\quad p>0.\end{cases}$$

To obtain a matrix representation of $\mathfrak{g}$ we introduce the $(2\ell+1)\times(2\ell+1)$ symmetric matrix

$$B = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & \dots & 0 & I_{d_\nu} \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & \dots & I_{d_{\nu-1}} & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & I_{d_1} & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & I_{2d_0+1} & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 1 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & I_{d_1} & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & I_{d_{\nu-1}} & \dots & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ I_{d_\nu} & 0 & \dots & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 \end{pmatrix}.$$

We can identify $\mathfrak{so}(2\ell+1,\mathbb{C})$ to the space of matrices $X \in \mathfrak{sl}(2\ell+1,\mathbb{C})$ such that

$${}^tXB + BX = 0.$$

We write these matrices in the form

$$X = \begin{pmatrix} x_{-i,-j} & x_{-i,0} & x_{-i,j} \\ x_{0,-j} & x_{0,0} & x_{0,j} \\ x_{i,-j} & x_{i,0} & x_{i,j} \end{pmatrix}_{1\le i,j\le\nu}.$$

They are characterized by:

$$\begin{cases} x_{i,j} \in \mathfrak{M}(d_{|i|}\times d_{|j|},\mathbb{C}) & \text{for} \quad |i|,|j|>0 \\ x_{0,j} \in \mathfrak{M}((2d_0+1)\times d_{|j|},\mathbb{C}) & \text{for} \quad |j|>0 \\ x_{i,0} \in \mathfrak{M}(d_{|i|}\times(2d_0+1),\mathbb{C}) & \text{for} \quad |i|>0 \\ x_{0,0} \in \mathfrak{M}((2d_0+1)\times(2d_0+1),\mathbb{C}) & \\ {}^tx_{i,j} = -x_{-j,-i} & \text{for} \quad |i|,|j| = 0,\dots,\nu. \end{cases}$$

We note that $x_{-1,1} = x_{1,-1} = 0$ because $d_1 = 1$. The characteristic element

E of $\mathfrak{g}$ is represented by the diagonal matrix:

$$\begin{pmatrix} -\nu I_{d_\nu} & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & (1-\nu)I_{d_{\nu-1}} & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & -1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0_{2d_0+1} & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & (\nu-1)I_{d_{\nu-1}} & 0 \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & \nu I_{d_\nu} \end{pmatrix}$$

i.e. $E = (e_{i,j})$ with $e_{i,i} = iI_{d_{|i|}}$ for $i = -\nu, \dots, 0, \dots, \nu$ and $e_{i,j} = 0$ for $i \neq j$; the element J which defines the partial complex structure of $\mathfrak{g}$ is defined by plus or minus the matrix

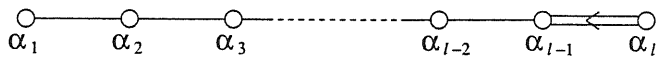
$$\begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \dots \\ \dots & \sqrt{-1}I_{d_3} & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ \dots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ \dots & 0 & 0 & \sqrt{-1} & 0 & 0 & 0 & 0 & \dots \\ \dots & 0 & 0 & 0 & 0_{2d_0+1} & 0 & 0 & 0 & \dots \\ \dots & 0 & 0 & 0 & 0 & -\sqrt{-1} & 0 & 0 & \dots \\ \dots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ \dots & 0 & 0 & 0 & 0 & 0 & 0 & -\sqrt{-1}I_{d_3} & \dots \\ \dots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

i.e. $J = (\eta_{i,j})$ with

$$\begin{cases} \eta_{2h+1,2h+1} = \sqrt{-1}I_{d_{2h+1}} & \text{for} \quad h = -1, \dots, -\left[\frac{\nu+1}{2}\right], \\ \eta_{2h+1,2h+1} = -\sqrt{-1}I_{d_{2h+1}} & \text{for} \quad h = 0, \dots, \left[\frac{\nu-1}{2}\right], \\ \eta_{i,j} = 0 & \text{if} \quad i \neq j \quad \text{or} \quad i = j \quad \text{even}. \end{cases}$$

4.2.3 Simple Levi-Tanaka algebras of the complex type C_ℓ ($\ell \geq 3$)

The basic roots $\alpha_1, \dots, \alpha_\ell$ are organized in the Dynkin diagram:



Since there are no ramification points, we only need to check that $\mathcal{B}_{-1} = \{\alpha_{i_1}, \dots, \alpha_{i_\nu}\}$ with $1 \leq i_1 < \dots < i_\nu \leq \ell$ satisfies condition (3) of Theorem 4.2.1. The positive roots which have a coefficient larger than or equal to two are:

$$\sum_{i \leq k < j} \alpha_k + 2\left(\sum_{j \leq k < \ell} \alpha_k\right) + \alpha_\ell \quad \text{for} \quad 1 \leq i < j < \ell$$

and

$$2\left(\sum_{i \leq k < \ell} \alpha_k\right) + \alpha_\ell \quad \text{for} \quad 1 \leq i < \ell.$$

Thus we must have

$$\alpha_\ell \in \mathcal{B}_{-1} \tag{4.16}$$

and this condition, together with $\nu \geq 2$, is also sufficient in order that there exist Levi-Tanaka structures corresponding to $\mathcal{B}_{-1}$.

The highest root in $\mathcal{R}_-$ is $\delta = 2(\alpha_1 + \dots + \alpha_{\ell-1}) + \alpha_\ell$ and therefore the Levi-Tanaka algebras corresponding to a LT-admissible $\mathcal{B}_{-1} = \{\alpha_1, \dots, \alpha_\nu\}$ have kind $\mu = 2\nu - 1$.

Since there are no automorphisms of the Dynkin diagram of C_ℓ, the weighted Dynkin diagrams associated to simple Levi-Tanaka algebras of the complex type C_ℓ are parametrized by:

$$\begin{aligned} &\mathcal{B}_{-1} = \{\alpha_{i_1}, \dots, \alpha_{i_{\nu-1}}, \alpha_\ell\} \quad \text{with} \quad \nu \geq 2, \\ &1 \leq i_1 < \dots < i_{\nu-1} < \ell \quad \text{and} \quad \mu = 2\nu - 1. \end{aligned} \tag{4.17}$$

Let us set:

(4.18)
$$d_1 = \ell - i_{\nu-1},\ d_2 = i_{\nu-1} - i_{\nu-2},\ \dots,\ d_h = i_{\nu-h+1} - i_{\nu-h},\ \dots,\ d_\nu = i_1.$$

We set also $d_h = 0$ if $h \neq 1, \dots, \nu$.

The center $\mathfrak{z}(\mathfrak{g}_0)$ of the subalgebra $\mathfrak{g}_0$ has complex dimension ν and $\mathfrak{g}_0$ is the direct sum of its center and a semisimple Lie algebra isomorphic to

$$\bigoplus_{d_i > 1} \mathfrak{sl}(d_i, \mathbb{C}).$$

Then we obtain:

$$\begin{cases} \dim_{\mathbb{C}} \mathfrak{g}_0 = \sum_{i=1}^{s} d_i^2 \\ \dim_{\mathbb{C}} \mathfrak{g}_{\pm(2p+1)} = \sum_{h=2p+2}^{\nu} d_h d_{h-2p-1} + \sum_{h=p+2}^{\nu} d_h d_{2p+2-h} + \frac{d_{p+1}(d_{p+1}+1)}{2} & \text{for} \quad p \geq 0 \\ \dim_{\mathbb{C}} \mathfrak{g}_{\pm 2p} = \sum_{h=2p+1}^{\nu} d_h d_{h-2p} + \sum_{h=p+1}^{\nu} d_h d_{2p+1-h} & \text{for} \quad p > 0. \end{cases}$$

To describe a matrix representation of a Levi-Tanaka algebra of the complex type C_ℓ, we consider the matrix:

$$A = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & I_{d_\nu} \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & I_{d_{\nu-1}} & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & 0 & I_{d_2} & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & I_{d_1} & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & -I_{d_1} & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & -I_{d_2} & 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & -I_{d_{\nu-1}} & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ -I_{d_\nu} & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \end{pmatrix}.$$

We identify $\mathfrak{sp}(\ell, \mathbb{C})$ to the space of complex $(2\ell) \times (2\ell)$ matrices X such that ${}^tXA + AX = 0$.

Denote by S_ν the set of indexes $\{\frac{1}{2}, \frac{3}{2}, \dots, \frac{2\nu-1}{2}\}$. Then we represent the matrices X in $\mathfrak{sp}(\ell, \mathbb{C})$ by

$$X = \begin{pmatrix} x_{-i,-j} & x_{-i,j} \\ x_{i,-j} & x_{i,j} \end{pmatrix}_{i,j \in S_\nu}$$

where

$$\begin{cases} x_{i,j} \in \mathfrak{M}(d_h \times d_k, \mathbb{C}) & \text{for} \quad i = \pm\frac{2h-1}{2},\ j = \pm\frac{2k-1}{2} \\ {}^t x_{i,j} = -\sigma(i)\sigma(j) x_{-j,-i} & \text{for} \quad i, j \in S_\nu \cup -S_\nu \end{cases}$$

and $\sigma(a)$ denotes the sign of the rational number a.

The characteristic element of $\mathfrak{g}$ is represented by the matrix E:

$$\begin{pmatrix} \frac{1-2\nu}{2}I_{d_\nu} & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & \frac{3-2\nu}{2}I_{d_{\nu-1}} & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & -\frac{3}{2}I_{d_2} & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & -\frac{1}{2}I_{d_1} & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & \frac{1}{2}I_{d_1} & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & 0 & \frac{3}{2}I_{d_2} & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & \frac{2\nu-3}{2}I_{d_{\nu-1}} & 0 \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & \frac{2\nu-1}{2}I_{d_\nu} \end{pmatrix}$$

i.e. $E = (e_{i,j})_{i,j \in S_\nu \cup -S_\nu}$ with

$$\begin{cases} e_{i,i} = iI_{d_h} & \text{for} \quad i = \pm\frac{2h-1}{2} \\ e_{i,j} = 0 & \text{for} \quad i \neq j. \end{cases}$$

The partial complex structure of $\mathfrak{g}$ is given by plus or minus the matrix J:

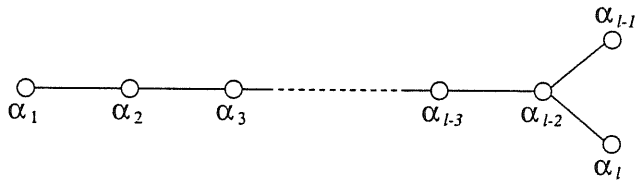
$$\begin{pmatrix} \frac{(-1)^\nu}{2\sqrt{-1}}I_{d_\nu} & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & \frac{(-1)^{\nu-1}}{2\sqrt{-1}}I_{d_{\nu-1}} & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & \dots & \frac{-\sqrt{-1}}{2}I_{d_2} & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & \frac{\sqrt{-1}}{2}I_{d_1} & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & \frac{-\sqrt{-1}}{2}I_{d_1} & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & 0 & \frac{\sqrt{-1}}{2}I_{d_2} & \dots & 0 & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & \frac{-(-1)^{\nu-1}}{2\sqrt{-1}}I_{d_{\nu-1}} & 0 \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & \frac{-(-1)^\nu}{2\sqrt{-1}}I_{d_\nu} \end{pmatrix}$$

i.e. $J = (\eta_{i,j})_{i,j \in S_\nu \cup -S_\nu}$ with

$$\begin{cases} \eta_{i,i} = \sigma(i)\frac{(-1)^h\sqrt{-1}}{2}I_{d_h} & \text{for} \quad i = \pm\frac{2h-1}{2} \\ \eta_{i,j} = 0 & \text{for} \quad i \neq j. \end{cases}$$

4.2.4 Simple Levi-Tanaka algebras of the complex type D_ℓ ($\ell \geq 4$)

The basic roots $\alpha_1, \ldots, \alpha_\ell$ are organized in the Dynkin diagram:



In this case $\alpha_{\ell-2}$ is a ramification point. If $\alpha_{\ell-2} \in \mathcal{B}_0$, then, according to condition (2) of Theorem 4.2.1, only two branches issued from $\alpha_{\ell-2}$ may contain elements of $\mathcal{B}_{-1}$.

Then we consider the five cases:

$(D_\ell I)$ $\quad \mathcal{B}_{-1} = \{\alpha_{\ell-1}, \alpha_\ell\}$;

$(D_\ell II)$ $\quad \alpha_{\ell-2}, \alpha_{\ell-1}, \alpha_\ell \in \mathcal{B}_{-1}$;

$(D_\ell III)$ $\quad \alpha_{\ell-1} \in \mathcal{B}_0,\ \alpha_\ell \in \mathcal{B}_{-1}$;

$(D_\ell III')$ $\quad \alpha_{\ell-1} \in \mathcal{B}_{-1},\ \alpha_\ell \in \mathcal{B}_0$;

$(D_\ell IV)$ $\quad \alpha_{\ell-1}, \alpha_\ell \in \mathcal{B}_0$.

We note that the cases $D_\ell III$ and $D_\ell III'$ are interchanged by the automorphism of the Dynkin diagram of D_ℓ which leaves α_i fixed for $i \leq \ell - 2$ and exchanges $\alpha_{\ell-1}$ with α_ℓ. Therefore, in order to give a classification of the LT-admissible weighted Dynkin diagrams of Levi-Tanaka algebras of the complex type D_ℓ it will suffice to consider the four cases $D_\ell I$, $D_\ell II$, $D_\ell III$ and $D_\ell IV$. Moreover, for $\ell = 4$, all permutations of the roots that leave α_2 fixed are automorphisms of the Dynkin diagram of D_4. Therefore we need to consider only the first three cases when $\ell = 4$.

The positive roots in $\mathcal{R}$ having a coefficient larger than or equal to 2 are given by:

$$\alpha = \sum_{i \leq k < j} \alpha_k + 2\left(\sum_{j \leq k \leq \ell-2} \alpha_k\right) + \alpha_{\ell-1} + \alpha_\ell \quad \text{for} \quad 1 \leq i < j \leq \ell - 2.$$

Thus condition (3) of Theorem 4.2.1 is always satisfied by $D_\ell I$, $D_\ell II$ and $D_\ell III$.

In case $D_\ell IV$, if $\mathcal{B}_{-1} = \{\alpha_{i_1}, \dots, \alpha_{i_\nu}\}$ with $1 \leq i_1 < \dots < i_\nu \leq \ell - 2$, the necessary and sufficient condition in order that $\mathcal{B}_{-1}$ be LT-admissible is that $\nu \geq 2$ and $i_\nu = i_{\nu-1} + 1$.

Since the highest root in $\mathcal{R}_-$ is $\delta = \alpha_1 + 2(\alpha_2 + \dots + \alpha_{\ell-2}) + \alpha_{\ell-1} + \alpha_\ell$, the kind μ of a Levi-Tanaka algebra associated to a LT-admissible $\mathcal{B}_{-1} = \{\alpha_{i_1}, \dots, \alpha_{i_\nu}\}$ is equal to $2\nu - \nu'$ where $0 \leq \nu' \leq 3$ is the number of elements of $\mathcal{B}_{-1} \cap \{\alpha_1, \alpha_{\ell-1}, \alpha_\ell\}$.

$\mathbf{D_\ell I}$: $\mathcal{B}_{-1} = \{\alpha_{\ell-1}, \alpha_\ell\}$.

In this case $\mathfrak{g}_0$ is the direct sum of its center $\mathfrak{z}(\mathfrak{g}_0)$, which has dimension 2, and of a simple Lie algebra isomorphic to $\mathfrak{sl}(\ell - 1, \mathbb{C})$. The corresponding Levi-Tanaka algebra has kind 2 and we have

$$\begin{cases} \dim_{\mathbb{C}} \mathfrak{g}_0 = \ell^2 - 2\ell + 2 \\ \dim_{\mathbb{C}} \mathfrak{g}_{\pm 1} = 2(\ell - 1) \\ \dim_{\mathbb{C}} \mathfrak{g}_{\pm 2} = \frac{(\ell-1)(\ell-2)}{2}. \end{cases}$$

To obtain a matrix representation of $\mathfrak{g}$ we introduce the matrix

$$B = \begin{pmatrix} 0 & 0 & I_{\ell-1} \\ 0 & I_2 & 0 \\ I_{\ell_1} & 0 & 0 \end{pmatrix}$$

and identify $\mathfrak{so}(2\ell, \mathbb{C})$ to the space of $(2\ell) \times (2\ell)$ matrices

$$X = \begin{pmatrix} x_{-1,-1} & x_{-1,0} & x_{-1,1} \\ x_{0,-1} & x_{0,0} & x_{0,1} \\ x_{1,-1} & x_{1,0} & x_{1,1} \end{pmatrix}$$

with

$$\begin{cases} x_{\pm 1, \pm 1} \in \mathfrak{M}((\ell - 1) \times (\ell - 1), \mathbb{C}), \\ x_{-1,0}, x_{1,0} \in \mathfrak{M}((\ell - 1) \times 2, \mathbb{C}), \\ x_{0,-1}, x_{0,1} \in \mathfrak{M}(2 \times (\ell - 1), \mathbb{C}), \\ {}^t x_{i,j} = -x_{-j,-i} \qquad \text{for} \quad i, j = -1, 0, 1. \end{cases}$$

The characteristic element E of $\mathfrak{g}$ is represented by the matrix:

$$\begin{pmatrix} -I_{\ell-1} & 0 & 0 \\ 0 & 0_2 & 0 \\ 0 & 0 & I_{\ell-1} \end{pmatrix}$$

and the partial complex structure J is defined by plus or minus the matrix:

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Note that for the case of D_4, all choices of $\mathcal{B}_{-1}$ equal to $\{\alpha_1, \alpha_3\}$, $\{\alpha_1, \alpha_4\}$, $\{\alpha_3, \alpha_4\}$ give equivalent LT-admissible weighted Dynkin diagrams of the type $D_4 I$, leading to Levi-Tanaka algebras of kind $\mu = 2$ with

$$\dim_{\mathbb{C}} \mathfrak{z}(\mathfrak{g}_0) = 2, \quad \mathfrak{g}_0 \simeq \mathfrak{z}(\mathfrak{g}_0) \oplus \mathfrak{sl}(3, \mathbb{C})$$

and with

$$\begin{cases} \dim_{\mathbb{C}} \mathfrak{g}_0 = 10 \\ \dim_{\mathbb{C}} \mathfrak{g}_{\pm 1} = 6 \\ \dim_{\mathbb{C}} \mathfrak{g}_{\pm 2} = 3. \end{cases}$$

$\mathbf{D}_\ell$II: $\mathcal{B}_{-1} = \{\alpha_{i_1}, \ldots, \alpha_{i_{\nu-3}}, \alpha_{\ell-2}, \alpha_{\ell-1}, \alpha_\ell\}$ with $1 \leq i_1 < \ldots < i_{\nu-3} < \ell - 2$.

We set $\nu - 1 = s$ and define

$$d_0 = 2, d_1 = 1, d_2 = \ell - 2 - i_{\nu-3}, \ \ldots, \ d_h = i_{\nu-h} - i_{\nu-h-1}, \ \ldots, \ d_s = i_1.$$

It is convenient to set $d_h = 0$ for $h \neq 0, 1, \ldots, s$.

The center $\mathfrak{z}(\mathfrak{g}_0)$ of $\mathfrak{g}_0$ has complex dimension ν and $\mathfrak{g}_0$ is isomorphic to the direct sum of $\mathfrak{z}(\mathfrak{g}_0)$ and the semisimple Lie algebra:

$$\bigoplus_{\substack{i>1 \\ d_i>1}} \mathfrak{sl}(d_i, \mathbb{C}).$$

Therefore we obtain:

$$\begin{cases} \dim_{\mathbb{C}} \mathfrak{g}_0 = \sum_{h=2}^{s} d_h^2 + 2 \\ \dim_{\mathbb{C}} \mathfrak{g}_{\pm(2p+1)} = 2d_{2p+1} + \sum_{h=1}^{s-2p-1} d_h d_{h+2p+1} + \sum_{h=1}^{p} d_h d_{2p+1-h} \quad \text{for} \quad p \geq 0 \\ \dim_{\mathbb{C}} \mathfrak{g}_{\pm 2p} = 2d_{2p} + \sum_{h=1}^{s-2p} d_h d_{h+2p} + \sum_{h=1}^{p-1} d_h d_{2p-h} + \frac{d_p(d_p-1)}{2} \quad \text{for} \quad p > 0. \end{cases}$$

To give a matrix representation of the corresponding Levi-Tanaka algebras $\mathfrak{g}$, we introduce the $(2\ell) \times (2\ell)$ matrix

$$B = \begin{pmatrix} 0 & \dots & 0 & 0 & 0 & 0 & 0 & \dots & I_{d_s} \\ \vdots & \dots & \vdots & \vdots & \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & \dots & 0 & 0 & 0 & 0 & I_{d_2} & \dots & 0 \\ 0 & \dots & 0 & 0 & 0 & 1 & 0 & \dots & 0 \\ 0 & \dots & 0 & 0 & I_2 & 0 & 0 & \dots & 0 \\ 0 & \dots & 0 & 1 & 0 & 0 & 0 & \dots & 0 \\ 0 & \dots & I_{d_2} & 0 & 0 & 0 & 0 & \dots & 0 \\ \vdots & \dots & \vdots & \vdots & \vdots & \vdots & \vdots & \dots & \vdots \\ I_{d_s} & \dots & 0 & 0 & 0 & 0 & 0 & \dots & 0 \end{pmatrix}$$

and identify $\mathfrak{so}(2\ell, \mathbb{C})$ with the space of $(2\ell) \times (2\ell)$ matrices X such that ${}^tXB + BX = 0$. It is convenient to write these matrices X in the form:

$$X = \begin{pmatrix} x_{-i,-j} & x_{-i,0} & x_{-i,j} \\ x_{0,-j} & x_{0,0} & x_{0,j} \\ x_{i,-j} & x_{i,0} & x_{i,j} \end{pmatrix}_{i,j=1,\dots,s}$$

with

$$\begin{cases} x_{i,j} \in \mathfrak{M}(d_{|i|} \times d_{|j|}, \mathbb{C}) \\ {}^tx_{i,j} = -x_{-j,-i} \\ \qquad \text{for} \quad i, j = -s, \dots, 0, \dots, s. \end{cases}$$

We note that the condition $d_2 = 1$ implies that $x_{-1,1} = x_{1,-1} = 0$.

Then the characteristic element E of $\mathfrak{g}$ is represented by the matrix:

$$\begin{pmatrix} -sI_{d_s} & \dots & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & \dots & -2I_{d_2} & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & \dots & 0 & -1 & 0 & 0 & 0 & \dots & 0 \\ 0 & \dots & 0 & 0 & 0_2 & 0 & 0 & \dots & 0 \\ 0 & \dots & 0 & 0 & 0 & 1 & 0 & \dots & 0 \\ 0 & \dots & 0 & 0 & 0 & 0 & 2I_{d_2} & \dots & 0 \\ \vdots & \dots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 0 & 0 & 0 & 0 & \dots & sI_{d_s} \end{pmatrix}$$

i.e. by the matrix $(e_{i,j})$ with $e_{i,i} = iI_{d_{|i|}}$ for $i = 0, \pm 1, \dots, \pm s$ and $e_{i,j} = 0$ for $i \neq j$; the partial complex structure is defined by the element J of $\mathfrak{z}(\mathfrak{g}_0)$ associated to plus or minus the matrix:

$$\begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \sqrt{-1}I_{d_3} & 0 & 0 & 0 & 0 & 0 & 0 & \vdots \\ \vdots & 0 & 0_{d_2} & 0 & 0 & 0 & 0 & 0 & \vdots \\ \vdots & 0 & 0 & \sqrt{-1} & 0 & 0 & 0 & 0 & \vdots \\ \vdots & 0 & 0 & 0 & 0_2 & 0 & 0 & 0 & \vdots \\ \vdots & 0 & 0 & 0 & 0 & -\sqrt{-1} & 0 & 0 & \vdots \\ \vdots & 0 & 0 & 0 & 0 & 0 & 0_{d_2} & 0 & \vdots \\ \vdots & 0 & 0 & 0 & 0 & 0 & 0 & -\sqrt{-1}I_{d_3} & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

i.e. by the matrix $(\eta_{i,j})$ with $\eta_{i,i} = \frac{\sigma(i)}{\sqrt{-1}} I_{d_i}$ if i is an odd integer, $\eta_{i,j} = 0$ otherwise.

When $\ell = 4$ and $\mathcal{B}_{-1} = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$, the algebra $\mathfrak{g}_0$ has complex dimension 4 and it is abelian, the kind of the corresponding Levi-Tanaka algebras is 5 and

$$\begin{cases} \dim_{\mathbb{C}} \mathfrak{g}_0 = 4 \\ \dim_{\mathbb{C}} \mathfrak{g}_{\pm 1} = 4 \\ \dim_{\mathbb{C}} \mathfrak{g}_{\pm 2} = 3 \\ \dim_{\mathbb{C}} \mathfrak{g}_{\pm 3} = 3 \\ \dim_{\mathbb{C}} \mathfrak{g}_{\pm 4} = 1 \\ \dim_{\mathbb{C}} \mathfrak{g}_{\pm 5} = 1. \end{cases}$$

We rewrite explicitly the matrices corresponding to the elements E and J in

this case. They are respectively:

$$\begin{pmatrix} -3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 \end{pmatrix} \text{ and } \begin{pmatrix} \sqrt{-1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{-1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\sqrt{-1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\sqrt{-1} \end{pmatrix}.$$

All the weighted Dynkin diagrams obtained from the choices of $\mathcal{B}_{-1}$ equal to $\{\alpha_1, \alpha_2, \alpha_3\}$, $\{\alpha_1, \alpha_2, \alpha_4\}$, $\{\alpha_2, \alpha_3, \alpha_4\}$ are isomorphic. In this case the subalgebra $\mathfrak{g}_0$ has center $\mathfrak{z}(\mathfrak{g}_0)$ of complex dimension 3 and it is the direct sum of its center and a simple Lie algebra isomorphic to $\mathfrak{sl}(2, \mathbb{C})$; the kind of $\mathfrak{g}$ is 4 and the complex dimensions of $\mathfrak{g}_0, \mathfrak{g}_{\pm 1}, \mathfrak{g}_{\pm 2}, \mathfrak{g}_{\pm 4}$ are respectively $6, 4, 4, 2, 1$.

The matrices corresponding to the elements E and J are in this case:

$$\begin{pmatrix} -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{-1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\sqrt{-1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

$\mathbf{D}_\ell$III: $\mathcal{B}_{-1} = \{\alpha_{i_1}, \ldots, \alpha_{i_{\nu-1}}, \alpha_\ell\}$ with $\nu \geq 2$, $1 \leq i_1 < \ldots < i_{\nu-1} < \ell - 1$. We set:

(4.19)
$$d_1 = \ell - i_{\nu-1}, d_2 = i_{\nu-1} - i_{\nu-2}, \ \ldots, \ d_h = i_{\nu-h+1} - i_{\nu-h}, \ \ldots, \ d_\nu = i_1$$

and $d_h = 0$ for $h \neq 1, \ldots, \nu$.

The center $\mathfrak{z}(\mathfrak{g}_0)$ has dimension ν and $\mathfrak{g}_0$ is the direct sum of its center and of a semisimple Lie algebra isomorphic to

$$\bigoplus_{d_i > 1} \mathfrak{sl}(d_i, \mathbb{C}).$$

We obtain

$$\begin{cases} \dim_{\mathbb{C}} \mathfrak{g}_0 = \sum_{i=1}^{k} d_i^2 \\ \dim_{\mathbb{C}} \mathfrak{g}_{\pm(2p+1)} = \sum_{h=1}^{\nu-2p-1} d_h d_{h+2p+1} + \sum_{h=1}^{p} d_h d_{2p+2-h} + \frac{d_{p+1}(d_{p+1}-1)}{2} \quad \text{for} \quad p \geq 0 \\ \dim_{\mathbb{C}} \mathfrak{g}_{\pm 2p} = \sum_{h=1}^{\nu-2p} d_h d_{h+2p} + \sum_{h=1}^{p} d_h d_{2p+1-h} \quad \text{for} \quad p > 0\,. \end{cases}$$

To give a matrix representation of $\mathfrak{g}$, we introduce the matrix

$$B = \begin{pmatrix} 0 & \dots & 0 & 0 & \dots & I_{d_\nu} \\ \vdots & \dots & \vdots & \vdots & \dots & \vdots \\ 0 & \dots & 0 & I_{d_1} & \dots & 0 \\ 0 & \dots & I_{d_1} & 0 & \dots & 0 \\ \vdots & \dots & \vdots & \vdots & \dots & \vdots \\ I_{d_\nu} & \dots & 0 & 0 & \dots & 0 \end{pmatrix}$$

and identify $\mathfrak{so}(2\ell, \mathbb{C})$ to the space of $(2\ell)\times(2\ell)$ complex matrices X such that ${}^tXB+BX = 0$. The matrices X are better written as block matrices indexed by the set of half odd numbers $S_\nu \cup -S_\nu$ where $S_\nu = \{\frac{1}{2}, \dots, \frac{2h-1}{2}, \dots, \frac{2\nu-1}{2}\}$. It is convenient to introduce the following notation: for every integer $r \neq 0$ we set

$$\hat{r} = \begin{cases} \frac{2r-1}{2} & \text{for} \quad r > 0 \\ \frac{2r+1}{2} & \text{for} \quad r < 0. \end{cases}$$

Then we write the matrices X in the form

$$X = \begin{pmatrix} x_{-\hat{i},-\hat{j}} & x_{-\hat{i},\hat{j}} \\ x_{\hat{i},-\hat{j}} & x_{\hat{i},\hat{j}} \end{pmatrix}_{i,j=1,\dots,\nu}$$

with

$$\begin{cases} x_{\hat{i},\hat{j}} \in \mathfrak{M}(d_{|i|} \times d_{|j|}, \mathbb{C}) \\ {}^t x_{\hat{i},\hat{j}} = x_{-\hat{j},-\hat{i}} \\ \qquad \text{for} \quad i, j = \pm 1, \dots, \pm\nu. \end{cases}$$

The characteristic element E of $\mathfrak{g}$ is represented by the matrix

$$\begin{pmatrix} -\hat{\nu} I_{d_\nu} & \dots & 0 & 0 & \dots & 0 \\ \vdots & \ddots & 0 & 0 & \dots & 0 \\ 0 & \dots & -\frac{1}{2} I_{d_1} & 0 & \dots & 0 \\ 0 & \dots & 0 & \frac{1}{2} I_{d_1} & \dots & 0 \\ \vdots & \dots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 0 & \dots & \hat{\nu} I_{d_\nu} \end{pmatrix}$$

i.e. by the matrix $(e_{\hat{i},\hat{j}})$ with $e_{\hat{i},\hat{i}} = \hat{i} I_{d_{|i|}}$ and $e_{\hat{i},\hat{j}} = 0$ when $i \neq j$.

The partial complex structure of $\mathfrak{g}$ is defined by the element J in $\mathfrak{z}(\mathfrak{g}_0)$ corresponding to plus or minus the matrix

$$\begin{pmatrix} \frac{(-1)^\nu}{2\sqrt{-1}} I_{d_\nu} & \dots & 0 & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \dots & \vdots \\ 0 & \dots & \frac{\sqrt{-1}}{2} & 0 & \dots & 0 \\ 0 & \dots & 0 & -\frac{\sqrt{-1}}{2} & \dots & 0 \\ \vdots & \dots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 0 & \dots & \frac{(-1)^\nu \sqrt{-1}}{2} I_{d_\nu} \end{pmatrix}$$

i.e. by the matrix $(\eta_{\hat{i},\hat{j}})$ where $\eta_{\hat{i},\hat{i}} = \frac{\sigma(i)(-1)^i\sqrt{-1}}{2} I_{d_{|i|}}$ and $\eta_{\hat{i},\hat{j}} = 0$ for $i \neq j$.

When $\ell = 4$, and $\mathcal{B}_{-1}$ contains three elements, the discussion reduces to the case $D_4 II$. The construction above yields an equivalent matrix representation for the choices of $\mathcal{B}_{-1}$ equal to $\{\alpha_1, \alpha_2, \alpha_3\}$, $\{\alpha_1, \alpha_2, \alpha_4\}$ and $\{\alpha_2, \alpha_3, \alpha_4\}$.

Consider now the equivalent weighted Dynkin diagrams corresponding to the choices of $\mathcal{B}_{-1}$ respectively equal to $\{\alpha_1, \alpha_2\}$, $\{\alpha_2, \alpha_3\}$ and $\{\alpha_2, \alpha_4\}$.

We obtain Levi-Tanaka algebras of kind 3 with $\mathfrak{z}(\mathfrak{g}_0)$ of complex dimension two and $\mathfrak{g}_0$ equal to the direct sum of its center and a semisimple algebra isomorphic to $\mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C})$. We have:

$$\begin{cases} \dim_{\mathbb{C}} \mathfrak{g}_0 = 8 \\ \dim_{\mathbb{C}} \mathfrak{g}_{\pm 1} = 5 \\ \dim_{\mathbb{C}} \mathfrak{g}_{\pm 2} = 4 \\ \dim_{\mathbb{C}} \mathfrak{g}_{\pm 3} = 1. \end{cases}$$

In this case $\mathfrak{g}$ is isomorphic to the Lie subalgebra of $\mathfrak{gl}(8,\mathbb{C})$ whose elements are the matrices:

$$\begin{pmatrix} x_{-\hat{2},-\hat{2}} & x_{-\hat{2},-\hat{1}} & x_{-\hat{2},\hat{1}} & x_{-\hat{2},\hat{2}} \\ x_{-\hat{1},-\hat{2}} & x_{-\hat{1},-\hat{1}} & x_{-\hat{1},\hat{1}} & x_{-\hat{1},\hat{2}} \\ x_{\hat{1},-\hat{2}} & x_{\hat{1},-\hat{1}} & x_{\hat{1},\hat{1}} & x_{\hat{1},\hat{2}} \\ x_{\hat{2},-\hat{2}} & x_{\hat{2},-\hat{1}} & x_{\hat{2},\hat{1}} & x_{\hat{2},\hat{2}} \end{pmatrix}$$

where all entries $x_{\hat{i},\hat{j}}$ are 2×2 complex matrices and ${}^t x_{\hat{i},\hat{j}} = -x_{-\hat{j},-\hat{i}}$ for $i, j = -2, -1, 1, 2$. The characteristic element E and the partial complex

structure J are associated respectively to the matrices:

$$\begin{pmatrix} -\frac{3}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{3}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{3}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{3}{2} \end{pmatrix}$$

and

$$\pm \begin{pmatrix} -\frac{\sqrt{-1}}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{\sqrt{-1}}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\sqrt{-1}}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\sqrt{-1}}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{\sqrt{-1}}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{\sqrt{-1}}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{\sqrt{-1}}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\sqrt{-1}}{2} \end{pmatrix}.$$

D_ℓIV: $\mathcal{B}_{-1} = \{\alpha_{i_1}, \dots, \alpha_{i_{\nu-1}}, \alpha_{i_{\nu-1}+1}\}$ with $\nu \geq 2$, $1 \leq i_1 < \dots < i_{\nu-1} < \ell - 2$.

We set:

$$d_0 = 2(\ell - 1 - i_{\nu-1}),\ d_1 = 1,\ d_2 = i_{\nu-1}1 - i_{\nu-2},\ \dots,\ d_{\nu-1} = i_2 - i_1,\ d_\nu = i_1.$$

We also set $d_h = 0$ for $h \neq 0, 1, \dots, \nu$.

The center $\mathfrak{z}(\mathfrak{g}_0)$ has complex dimension ν and the semisimple Lie algebra $\mathfrak{g}_0/\mathfrak{z}(\mathfrak{g}_0)$ is isomorphic to

$$\bigoplus_{\substack{i>1 \\ d_i>1}} \mathfrak{sl}(d_i, \mathbb{C}) \oplus \mathfrak{so}(d_0, \mathbb{C}).$$

We obtain:

$$\begin{cases} \dim_{\mathbb{C}} \mathfrak{g}_0 = \dfrac{d_0(d_0-1)}{2} + \sum\limits_{i=2}^{\nu} {d_i}^2 + 1 \\ \dim_{\mathbb{C}} \mathfrak{g}_{\pm(2p+1)} = d_{2p} + d_0 d_{2p+1} + d_{2p+2} + \sum\limits_{h=2}^{\nu-2p-1} d_h d_{h+2p+1} + \sum\limits_{h=2}^{p} d_h d_{2p+1-h} \\ \qquad\qquad \text{for} \quad p \geq 0 \\ \dim_{\mathbb{C}} \mathfrak{g}_{\pm 2p} = d_{2p-1} + d_0 d_{2p} + d_{2p+1} + \sum\limits_{h=2}^{\nu-2p} d_h d_{h+2p} + \sum\limits_{h=2}^{p-1} d_h d_{2p-h} + \dfrac{d_p(d_p-1)}{2} \\ \qquad\qquad \text{for} \quad p > 0. \end{cases}$$

To obtain a matrix representation of the Levi-Tanaka algebra $\mathfrak{g}$ we introduce the symmetric $(2\ell) \times (2\ell)$ matrix:

$$B = \begin{pmatrix} 0 & \dots & 0 & 0 & 0 & \dots & I_{d_\nu} \\ \vdots & \dots & \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & \dots & 0 & 0 & 1 & \dots & 0 \\ 0 & \dots & 0 & I_{d_0} & 0 & \dots & 0 \\ 0 & \dots & 1 & 0 & 0 & \dots & 0 \\ \vdots & \dots & \vdots & \vdots & \vdots & \dots & \vdots \\ I_{d_\nu} & \dots & 0 & 0 & 0 & \dots & 0 \end{pmatrix}$$

and identify $\mathfrak{so}(2\ell, \mathbb{C})$ to the Lie algebra of the complex $(2\ell) \times (2\ell)$ matrices X such that ${}^tXB + BX = 0$. We write the matrices X in the form:

$$X = \begin{pmatrix} x_{-i,-j} & x_{-i,0} & x_{-i,j} \\ x_{0,-j} & x_{0,0} & x_{0,j} \\ x_{i,-j} & x_{i,0} & x_{i,j} \end{pmatrix}_{i,j=1,\dots,\nu}$$

with

$$\begin{cases} x_{i,j} \in \mathfrak{M}(d_{|i|} \times d_{|j|}, \mathbb{C}), \\ {}^t x_{i,j} = -x_{-j,-i} \\ \qquad \text{for} \quad i,j = 0, \pm 1, \dots, \pm\nu. \end{cases}$$

Note that $x_{-1,1} = x_{1,-1} = 0$ because $d_1 = 1$. Then the characteristic element

of $\mathfrak{g}$ corresponds to the matrix:

$$\begin{pmatrix} -\nu I_{d_\nu} & \dots & 0 & 0 & 0 & \dots & 0 \\ \vdots & \ddots & 0 & 0 & 0 & \dots & 0 \\ 0 & \dots & -1 & 0 & 0 & \dots & 0 \\ 0 & \dots & 0 & 0_{d_0} & 0 & \dots & 0 \\ 0 & \dots & 0 & 0 & 1 & \dots & 0 \\ \vdots & \dots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 0 & 0 & \dots & \nu I_{d_\nu} \end{pmatrix}$$

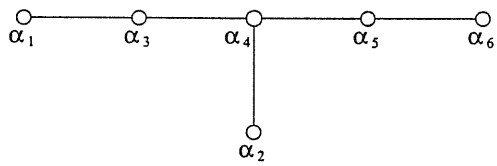
i.e. to the matrix $(e_{i,j})$ with $e_{i,i} = iI_{d_{|i|}}$ and $e_{i,j} = 0$ for $i \neq j$. The element $J \in \mathfrak{z}(\mathfrak{g}_0)$ that defines the partial complex structure of $\mathfrak{g}$ is associated to plus or minus the matrix:

$$\begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \dots \\ \dots & \sqrt{-1}I_{d_3} & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ \dots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ \dots & 0 & 0 & \sqrt{-1} & 0 & 0 & 0 & 0 & \dots \\ \dots & 0 & 0 & 0 & 0_{d_0} & 0 & 0 & 0 & \dots \\ \dots & 0 & 0 & 0 & 0 & -\sqrt{-1} & 0 & 0 & \dots \\ \dots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ \dots & 0 & 0 & 0 & 0 & 0 & 0 & -\sqrt{-1}I_{d_3} & \dots \\ \dots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

i.e. to the matrix $(\eta_{i,j})$ with $\eta_{i,j} = \frac{\sigma(i)}{\sqrt{-1}} I_{d_{|i|}}$ if $i = j$ is an odd integer, and $\eta_{i,j} = 0$ otherwise.

4.2.5 Simple Levi-Tanaka algebras of the complex type E_6

The basic roots $\alpha_1, \dots, \alpha_6$ of the complex Lie algebra E_6 are organized in the Dynkin diagram:



The root α_4 is a ramification point for the Dynkin diagram and therefore, according to condition (2) of Theorem 4.2.1, we must restrict to the following cases:

(i) $|\alpha_4| = -1$;

(ii) $\alpha_2 \in \mathcal{B}_{-1} \subset \{\alpha_1, \alpha_3, \alpha_2\}$;

(iii) $\alpha_2 \in \mathcal{B}_{-1} \subset \{\alpha_2, \alpha_5, \alpha_6\}$;

(iv) $\mathcal{B}_{-1} \subset \{\alpha_1, \alpha_3, \alpha_5, \alpha_6\}$.

To check condition (3) of Theorem 4.2.1, we only need to consider positive roots having at least one coefficient equal to 2 and at least one coefficient equal to 0. These roots are:

$$\begin{array}{ccccccccccc}
 & & \alpha_2 & + & \alpha_3 & + & 2\alpha_4 & + & \alpha_5 & & \\
\alpha_1 & + & \alpha_2 & + & \alpha_3 & + & 2\alpha_4 & + & \alpha_5 & & \\
 & & \alpha_2 & + & \alpha_3 & + & 2\alpha_4 & + & \alpha_5 & + & \alpha_6 \\
\alpha_1 & + & \alpha_2 & + & 2\alpha_3 & + & 2\alpha_4 & + & \alpha_5 & & \\
 & & \alpha_2 & + & \alpha_3 & + & 2\alpha_4 & + & 2\alpha_5 & + & \alpha_6.
\end{array}$$

In case (i) the LT-admissible $\mathcal{B}_{-1}$ are therefore given by:

all $\mathcal{B}_{-1}$ containing α_4 and at least one of the roots $\alpha_2, \alpha_3, \alpha_5$.

In case (ii) and (iii) all $\mathcal{B}_{-1}$, containing at least two elements are LT-admissible.

Finally, in case (iv), the necessary and sufficient condition for $\mathcal{B}_{-1}$ to be LT-admissible is that it equals one of the following:

$$\begin{array}{ll}
\{\alpha_1, \alpha_3\}, & \{\alpha_1, \alpha_3, \alpha_5\}, \\
\{\alpha_1, \alpha_6\}, & \{\alpha_1, \alpha_3, \alpha_6\}, \\
\{\alpha_3, \alpha_5\}, & \{\alpha_1, \alpha_5, \alpha_6\}, \\
\{\alpha_5, \alpha_6\}, & \{\alpha_3, \alpha_5, \alpha_6\}, \\
\{\alpha_1, \alpha_3, \alpha_5, \alpha_6\}. &
\end{array} \tag{4.20}$$

Since the highest root in $\mathcal{R}_-$ is $\delta = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6$, the kind μ of a Levi-Tanaka algebra associated to a LT-admissible $\mathcal{B}_{-1}$ is obtained by the formula:

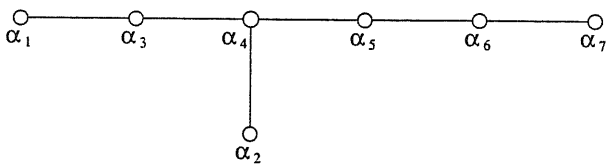
$$-\mu = |\alpha_1| + 2|\alpha_2| + 2|\alpha_3| + 3|\alpha_4| + 2|\alpha_5| + |\alpha_6|$$

and therefore we have $2 \leq \mu \leq 11$.

The Dynkin diagram of E_6 has the only non trivial automorphism that leaves α_2 and α_4 fixed and exchanges α_1 with α_6 and α_3 with α_5. There are therefore up to isomorphisms 26 nonisomorphic weighted Dynkin diagrams associated to simple Levi-Tanaka algebras of the complex type E_6, corresponding to 49 nonisomorphic Levi-Tanaka algebras. We will give the complete list in the appendix, also indicating their kind, the complex dimension of each subspace $\mathfrak{g}_p$ and the structure of the reductive subalgebra $\mathfrak{g}_0$.

4.2.6 Simple Levi-Tanaka algebras of the complex type E_7

The basic roots $\alpha_1, \ldots, \alpha_7$ of the complex Lie algebra E_7 are organized in the Dynkin diagram:



The root α_4 is of ramification for the Dynkin diagram. The roots α in $\mathcal{R}_-$ for which we have $k_i(\alpha) = 2$ and $k_j(\alpha) = 0$ for some $1 \leq i, j \leq 7$ are all roots from E_6 which have a coefficient $k_i(\alpha)$ equal to two (we have in this case $k_7(\alpha) = 0$) and the roots

$$\begin{array}{ccccccccccc} \alpha_2 & + & \alpha_3 & + & 2\alpha_4 & + & \alpha_5 & + & \alpha_6 & + & \alpha_7 \\ \alpha_2 & + & \alpha_3 & + & 2\alpha_4 & + & 2\alpha_5 & + & \alpha_6 & + & \alpha_7 \\ \alpha_2 & + & \alpha_3 & + & 2\alpha_4 & + & 2\alpha_5 & + & 2\alpha_6 & + & \alpha_7. \end{array}$$

In particular the LT-admissible $\mathcal{B}_{-1}$ which do not contain α_7 are obtained from the $\mathcal{B}_{-1}$ which are LT-admissible for E_6. When $\alpha_7 \in \mathcal{B}_{-1}$, we observe that:

1. the only admissible $\mathcal{B}_{-1}$ of the form $\{\alpha_i, \alpha_7\}$ are $\{\alpha_1, \alpha_7\}$ and $\{\alpha_6, \alpha_7\}$, because the root system of E_6 contains roots α with $k_i(\alpha) = 2$ for every $i \neq 1, 6$;

2. if $\mathcal{B}_{-1}$ contains α_7 and at least other two roots α_i with $1 \leq i \leq 6$, then the necessary and sufficient condition in order that $\mathcal{B}_{-1}$ be LT-admissibile is that $\mathcal{B}_{-1} \setminus \{\alpha_7\}$ be LT-admissible for E_6 and, when $\alpha_4 \in \mathcal{B}_0$, we need also that $\mathcal{B}_{-1}$ be contained in the union of only two of the three branches issued from α_4.

Since the highest root in $\mathcal{R}_-$ is $\delta = 2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7$, the kind μ of a Levi-Tanaka algebra associated to a LT-admissible $\mathcal{B}_{-1}$ is obtained by the formula:

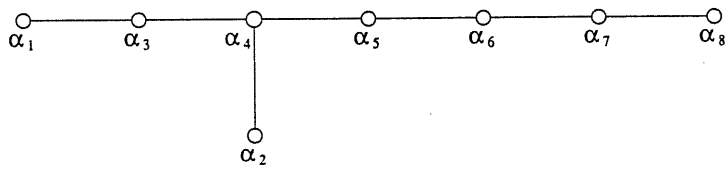
$$-\mu = 2|\alpha_1| + 2|\alpha_2| + 3|\alpha_3| + 4|\alpha_4| + 3|\alpha_5| + 2|\alpha_6| + |\alpha_7|$$

and therefore we have $3 \leq \mu \leq 17$.

Taking into account that there are no nontrivial automorphisms of the Dynkin diagram of E_7, we conclude that there are 84 nonequivalent weighted Dynkin diagrams, each one corresponding to two nonisomorphic simple Levi-Tanaka algebras $\mathfrak{g}$ of the complex type E_7. We list them in the appendix, also indicating their kind, the complex dimension of the subspaces $\mathfrak{g}_p$ and the structure of the reductive subalgebra $\mathfrak{g}_0$.

4.2.7 Simple Levi-Tanaka algebras of the complex type E_8

The basic roots $\alpha_1, \ldots, \alpha_8$ of the complex Lie algebra E_8 are organized in the Dynkin diagram:



Again the root α_4 is of ramification. The roots $\alpha \in \mathcal{R}_-$ for which we have $k_i(\alpha) = 2$ and $k_j(\alpha) = 0$ for some $1 \leq i, j \leq 8$ are the roots of E_7 (where we take $k_8(\alpha) = 0$) for which $k_i(\alpha) = 2$ for some $1 \leq i \leq 7$ and the roots

$$\begin{array}{ccccccccccccc}
\alpha_2 & + & \alpha_3 & + & 2\alpha_4 & + & \alpha_5 & + & \alpha_6 & + & \alpha_7 & + & \alpha_8 \\
\alpha_2 & + & \alpha_3 & + & 2\alpha_4 & + & 2\alpha_5 & + & \alpha_6 & + & \alpha_7 & + & \alpha_8 \\
\alpha_2 & + & \alpha_3 & + & 2\alpha_4 & + & 2\alpha_5 & + & 2\alpha_6 & + & \alpha_7 & + & \alpha_8 \\
\alpha_2 & + & \alpha_3 & + & 2\alpha_4 & + & 2\alpha_5 & + & 2\alpha_6 & + & 2\alpha_7 & + & \alpha_8.
\end{array}$$

We note that for all these four roots $k_8(\alpha) = 1$. We conclude that when $\mathcal{B}_{-1}$ does not contain α_8, a necessary and sufficient condition in order that it be LT-admissible is that it was admissible for E_7. When $\alpha_8 \in \mathcal{B}_{-1}$, we note that:

1. the only LT-admissible $\mathcal{B}_{-1}$ of the form $\mathcal{B}_{-1} = \{\alpha_i, \alpha_8\}$ with $1 \leq i \leq 7$ is $\{\alpha_7, \alpha_8\}$ because for every $1 \leq i \leq 6$ the root system of E_7 contains some root α with $k_i(\alpha) = 2$;

2. if $\alpha_8 \in \mathcal{B}_{-1}$ and $\mathcal{B}_{-1}$ contains at least three elements, then it is LT-admissible if and only if $\mathcal{B}_{-1} \setminus \{\alpha_8\}$ is LT-admissible for E_7 and is all contained in two of the three branches issued from α_4 in case $\alpha_4 \in \mathcal{B}_0$.

Since the highest root in $\mathcal{R}_-$ is $\delta = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 5\alpha_5 + 4\alpha_6 + 3\alpha_7 + 2\alpha_8$, the kind μ of a Levi-Tanaka algebra associated to a LT-admissible $\mathcal{B}_{-1}$ is obtained by the formula:

$$-\mu = 2|\alpha_1| + 3|\alpha_2| + 4|\alpha_3| + 6|\alpha_4| + 5|\alpha_5| + 4|\alpha_6| + 3|\alpha_7| + 2|\alpha_8|$$

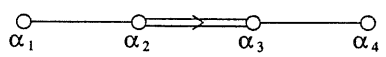
and therefore we have $5 \leq \mu \leq 29$ (with the one exception of 28).

Since there are no nontrivial automorphisms of the Dynkin diagram of E_8, we conclude that there are exactly 165 nonisomorphic weighted Dynkin diagrams corresponding to simple Levi-Tanaka algebras of the complex type E_8 (each corresponding to two nonisomorphic conjugated Levi-Tanaka algebras).

We list in the appendix all LT-admissible choices of $\mathcal{B}_{-1}$, giving for each one the kind of the corresponding Levi-Tanaka algebra $\mathfrak{g}$, the complex dimension of the $\mathfrak{g}_p$'s and describing the structure of the reductive subalgebra $\mathfrak{g}_0$.

4.2.8 Simple Levi-Tanaka algebras of the complex type F_4

The basic roots $\alpha_1, \ldots, \alpha_4$ of the complex Lie algebra F_4 are organized in the Dynkin diagram:



There are no ramification points, so that we have only to care for the positive roots having at least one coefficient equal to 2 and at least one coefficient equal to 0. The list of these roots is:

$$\begin{gathered}\alpha_2 + 2\alpha_3 \\ \alpha_1 + \alpha_2 + 2\alpha_3 \\ \alpha_2 + 2\alpha_3 + \alpha_4 \\ \alpha_1 + 2\alpha_2 + 2\alpha_3 \\ \alpha_2 + 2\alpha_3 + 2\alpha_4 .\end{gathered}$$

Again we can compute the kind μ of the Levi-Tanaka algebras associated to F_4, using the highest root $\delta = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4 \in \mathcal{R}_-$, by the formula

$$-\mu = 2|\alpha_1| + 3|\alpha_2| + 4|\alpha_3| + 2|\alpha_4| .$$

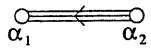
The possible values of μ are $5, 7, 9, 11$. There exists a Levi-Tanaka structure on $\mathfrak{g}$ if and only if $\mathcal{B}_{-1}$ is equal to one of the following sets:

$$\begin{array}{llll} \{\alpha_1, \alpha_2\}, & \mu = 5; & \{\alpha_1, \alpha_2, \alpha_4\}, & \mu = 7; \\ \{\alpha_2, \alpha_3\}, & \mu = 7; & \{\alpha_2, \alpha_3, \alpha_4\}, & \mu = 9; \\ \{\alpha_1, \alpha_2, \alpha_3\}, & \mu = 9; & \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}, & \mu = 11 . \end{array}$$

In the appendix we will list for each LT-admissible $\mathcal{B}_{-1}$ the kind of the corresponding Levi-Tanaka algebras $\mathfrak{g}$, together with the complex dimensions of the subspaces $\mathfrak{g}_p$ and the structure of the reductive subalgebra $\mathfrak{g}_0$.

4.2.9 Simple Levi-Tanaka algebras of the complex type G_2

The basic roots α_1, α_2 of the root system G_2 are organized in the Dynkin diagram:



There are no ramification points and the only possible choice $\mathcal{B}_{-1} = \{\alpha_1, \alpha_2\} = \mathcal{B}$ satisfies condition (3) of Theorem 4.2.1, so that there is a Levi-Tanaka structure on the corresponding graded Lie algebra $\mathfrak{g}$. Since there are no nontrivial automorphisms of the Dynkin diagram, there are exactly

two nonisomorphic Levi-Tanaka algebras of the complex type G_2. They have kind 5 and we have:

$$\begin{cases} \dim_{\mathbb{C}} \mathfrak{g}_0 = 2, \quad \mathfrak{g}_0 \simeq \mathfrak{d}_2(\mathbb{C}) \\ \dim_{\mathbb{C}} \mathfrak{g}_{\pm 1} = 2 \\ \dim_{\mathbb{C}} \mathfrak{g}_{\pm 2} = 1 \\ \dim_{\mathbb{C}} \mathfrak{g}_{\pm 3} = 1 \\ \dim_{\mathbb{C}} \mathfrak{g}_{\pm 4} = 1 \\ \dim_{\mathbb{C}} \mathfrak{g}_{\pm 5} = 1. \end{cases}$$

4.3 The weighted Satake diagrams of simple Levi-Tanaka algebras of the real type

We turn now to investigate the possibility of defining a Levi-Tanaka structure on a real simple Lie algebra of the real type. If $\mathfrak{g}$ is a Levi-Tanaka algebra, then the complexification $\mathfrak{g}^{\mathbb{C}}$ of $\mathfrak{g}$ is also a Levi-Tanaka algebra for the complexification of the partial complex structure. Therefore, if $\mathfrak{g}$ is a simple graded Lie algebra of the real type, admitting a structure of Levi-Tanaka algebra, it follows from Theorem 4.2.1 that there are only two possible partial complex structures on $\mathfrak{g}$, one being the opposite of the other. Moreover, the fact that $\mathfrak{g}^{\mathbb{C}}$ admits the structure of a Levi-Tanaka algebra is a necessary condition in order that $\mathfrak{g}$ could be made into a Levi-Tanaka algebra.

Let $\mathfrak{g}$ be a simple graded Lie algebra of the real type. By Theorem 4.1.7 in order that its weighted Satake diagram $\Sigma_{\mathfrak{g}}$ be LT-admissible, it must contain curved arrows and the graduation of $\mathfrak{g}$ will be determined by a partition $\{\mathcal{B}_0, \mathcal{B}_{-1}\}$ of its vertices, where the set $\mathcal{B}_{-1}$ of the roots having degree (-1) is a nonempty union of pairs of distinct white roots joined by a curved arrow.

If $\mathfrak{g}$ is a simple Levi-Tanaka algebra of the real type, the underlying weighted Dynkin diagram $\Delta_{\mathfrak{g}}$ corresponds to one which is associated to a Levi-Tanaka algebra of the complex type. Therefore we can use the results obtained in the case of simple graded Lie algebra of the complex type to obtain the classification of those of the real type.

Indeed we can use the following criterion:

THEOREM 4.3.1 *Let $\mathfrak{g}$ be a simple graded Lie algebra of the real type. Then a necessary and sufficient condition in order that $\mathfrak{g}$ admits the structure*

of a Levi-Tanaka algebra is that for its Satake diagram $\Sigma_{\mathfrak{g}}$ the following conditions (i) and (ii) hold true:

(*i*) $\mathcal{B} = \mathcal{B}_0 \cup \mathcal{B}_{-1}$ *and the set* $\mathcal{B}_{-1}$ *of vertices of weight* (-1) *is nonempty and consists of a disjoint union of pairs of white roots joined by a curved arrow;*

(*ii*) $\mathfrak{g}^{\mathbb{C}}$ *admits a structure of Levi-Tanaka algebra.*

Proof. A direct inspection of the Satake's diagrams of simple Lie algebras of the real type shows that, when (*i*) holds, we also have the following:

(*iii*) if $\alpha, \alpha' \in \mathcal{B}_{-1}$ are joined by a curved arrow, then the line joining α to α' in the Dynkin diagram $\Delta_{\mathfrak{g}}$ contains an even number of vertices in $\mathcal{B}_{-1}$.

Assume that $J \in \mathfrak{z}(\mathfrak{g}_0^{\mathbb{C}})$ defines a partial complex structure on $\mathfrak{g}^{\mathbb{C}}$ for which $\mathfrak{g}^{\mathbb{C}}$ is a simple Levi-Tanaka algebra of the complex type. To prove the theorem, it suffices to show that condition (*iii*) implies that $[J, \mathfrak{g}_{-1}] \subset \mathfrak{g}_{-1}$.

Let $\alpha \in \mathcal{B}_{-1}$. Write an eigenvector of α in the form $X + \sqrt{-1}Y$ with $X, Y \in \mathfrak{g}_{-1}$. We have:

$$[J, X + \sqrt{-1}Y] = \eta X + \eta\sqrt{-1}Y \quad \text{with} \quad \eta = \pm\sqrt{-1}.$$

If σ is the involution of $\mathfrak{g}^{\mathbb{C}}$ induced by the real form $\mathfrak{g}$, we obtain

$$\alpha^{\sigma} = \alpha' + \sum_{i=1}^{r} \beta_i, \quad \text{with} \quad \beta_1, \ldots, \beta_r \in \mathcal{R}_{\bullet}$$

for the root α' joined to α by a curved arrow. It follows that $\mathrm{ad}(J)$ acts on $\mathfrak{g}^{\alpha'}$ and on $\mathfrak{g}^{\alpha^{\sigma}}$ as the multiplication by the same factor $\pm\eta$. Since $X - \sqrt{-1}Y$ belongs to $\mathfrak{g}^{\alpha^{\sigma}}$, we obtain

$$[J, X] = \eta\sqrt{-1}Y, \quad [J, Y] = -\eta\sqrt{-1}X$$

because the line joining α to α' in the Dynkin diagram contains an even number of roots of $\mathcal{B}_{-1}$.

Because the real and imaginary parts of vectors in $\mathfrak{g}^{\alpha}$ for $\alpha \in \mathcal{B}_{-1}$, together with their images by $\mathrm{ad}(\mathfrak{g}_0)$, generate $\mathfrak{g}_{-1}$, we obtain that $[J, \mathfrak{g}_{-1}] \subset \mathfrak{g}_{-1}$. □

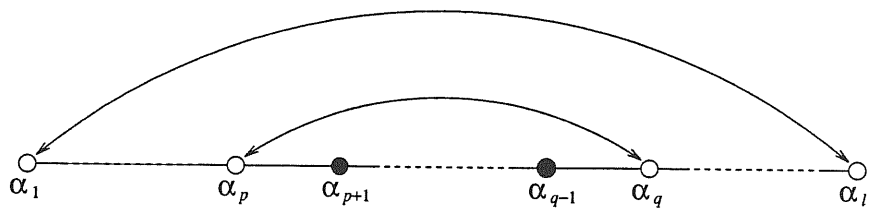
Hence the simple Levi-Tanaka algebras of the real type are, modulo isomorphisms, in a one to one correspondence with the simple Levi-Tanaka algebras of the complex type for which $\mathcal{B}_{-1}$ satisfies condition (i) above.

We can now proceed to classify, up to isomorphisms, all simple Levi-Tanaka algebras of the real type. We already know that their complexifications should be of the type A_ℓ, D_ℓ or E_6 (cf. Proposition 3.6.9).

4.3.1 Simple Levi-Tanaka algebras of the real type A_ℓ

There are only two types of Satake diagrams associated to A_ℓ that contain curved arrows. They correspond respectively to the real Lie algebras $\mathfrak{su}(p,q)$ with $p<q$ and $p+q=\ell+1$ and to the real Lie algebra $\mathfrak{su}(p,p)$ with $p\geq 2$ and $\ell=2p-1$. Accordingly, we divide the discussion of the case of Levi-Tanaka algebras of the real type A_ℓ into two subcases.

Subtype $\mathfrak{su}(p,q)$, $1\leq p<q$, $p+q=\ell+1$. The Satake diagram is:



According to Theorem 4.3.1, the weighted Satake's diagrams associated to Levi-Tanaka algebras isomorphic to $\mathfrak{su}(p,q)$ are those corresponding to the choices of

$$\mathcal{B}_{-1}=\{\alpha_{i_1},...,\alpha_{i_\nu},\alpha_{\ell-i_\nu+1},...,\alpha_{\ell-i_1+1}\}\quad\text{with}\quad 1\leq i_1<...<i_\nu\leq p.$$

The kind of these Levi-Tanaka algebras is 2ν, according to the discussion for the complex type A_ℓ.

Let us set

(4.21)

$$d_0=\ell+1-2i_\nu,\ d_1=i_\nu-i_{\nu-1},\ ...,\ d_h=i_{\nu-h+1}-i_{\nu-h},\ ...,\ d_\nu=i_1.$$

It is convenient to set also $d_h=0$ for $h\neq 0,1,...,\nu$. We obtain then

(4.22)

$$\mathfrak{g}_0\simeq\mathfrak{d}_{2\nu}(\mathbb{R})\oplus\mathfrak{su}(p-i_\nu,\ell+1-p-i_\nu)\oplus\bigoplus_{\substack{i>0\\ d_i>1}}\mathfrak{sl}(d_i,\mathbb{C})$$

and

$$\begin{cases} \dim_{\mathbb{R}} \mathfrak{g}_0 = d_0^2 + 2\sum_{i=1}^{\nu} d_i^2 - 1 \\ \dim_{\mathbb{R}} \mathfrak{g}_{\pm(2r+1)} = 2\sum_{i=0}^{\nu-2r-1} d_i d_{2r+1+i} + 2\sum_{i=1}^{r} d_i d_{2r+1-i} \qquad \text{for} \quad r \geq 0 \\ \dim_{\mathbb{R}} \mathfrak{g}_{\pm 2r} = 2\sum_{i=0}^{\nu-2r} d_i d_{2r+i} + 2\sum_{i=1}^{r-1} d_i d_{2r-i} + d_r^2 \qquad \text{for} \quad r > 0. \end{cases}$$

To obtain a matrix representation of the Levi-Tanaka algebra $\mathfrak{g}$ associated to this choice of $\mathcal{B}_{-1}$, we first introduce the $d_0 \times d_0$ matrix

$$\tilde{B} = \begin{pmatrix} -I_{p-i_\nu} & 0 \\ 0 & I_{\ell+1-p-i_\nu} \end{pmatrix}$$

and consider then the $(\ell+1) \times (\ell+1)$ matrix

$$B = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & I_{d_\nu} \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & I_{d_{\nu-1}} & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & I_{d_1} & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & \tilde{B} & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & I_{d_1} & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & I_{d_{\nu-1}} & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\ I_{d_\nu} & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \end{pmatrix}.$$

We identify $\mathfrak{g}$ to the Lie subalgebra of $\mathfrak{sl}(\ell+1, \mathbb{C})$ of the matrices X satisfying $X^*B + BX = 0$. It is convenient to write X as a block matrix:

$$X = \begin{pmatrix} x_{-i,-j} & x_{-i,0} & x_{-i,j} \\ x_{0,-j} & x_{0,0} & x_{0,j} \\ x_{i,-j} & x_{i,0} & x_{i,j} \end{pmatrix}_{i,j=1,\dots,\nu}$$

with

$$\begin{cases} x_{i,j} \in \mathfrak{M}(d_{|i|} \times d_{|j|}, \mathbb{C}) & \text{for} \quad i,j = 0, \pm 1, \dots, \pm\nu \\ x_{i,j}^* = -x_{-j,-i} & \text{for} \quad i,j = \pm 1, \dots, \pm\nu \\ x_{0,j}^* \tilde{B} + x_{-j,0} = 0 & \text{for} \quad j = \pm 1, \dots, \pm\nu \\ x_{i,0}^* + \tilde{B} x_{0,-i} = 0 & \text{for} \quad i = \pm 1, \dots, \pm\nu \\ x_{0,0}^* \tilde{B} + \tilde{B} x_{0,0} = 0 \end{cases}$$

The characteristic element E of $\mathfrak{g}$ is then associated to the matrix:

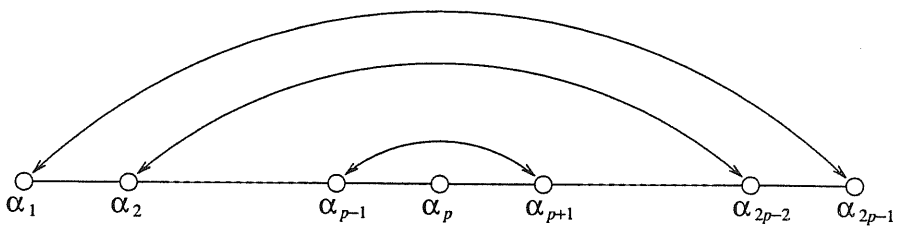
$$\begin{pmatrix} -\nu I_{d_\nu} & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & (1-\nu)I_{d_{\nu-1}} & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & \dots & -I_{d_1} & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0_{d_0} & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & I_{d_1} & \dots & 0 & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & (\nu-1)I_{d_{\nu-1}} & 0 \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & \nu I_{d_\nu} \end{pmatrix}$$

and the partial complex structure is defined by plus or minus the matrix:

$$\begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \dots \\ \dots & \eta_1 I_{d_3} & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ \dots & 0 & \eta_0 I_{d_2} & 0 & 0 & 0 & 0 & 0 & \dots \\ \dots & 0 & 0 & \eta_1 I_{d_1} & 0 & 0 & 0 & 0 & \dots \\ \dots & 0 & 0 & 0 & \eta_0 I_{d_0} & 0 & 0 & 0 & \dots \\ \dots & 0 & 0 & 0 & 0 & \eta_1 I_{d_1} & 0 & 0 & \dots \\ \dots & 0 & 0 & 0 & 0 & 0 & \eta_0 I_{d_2} & 0 & \dots \\ \dots & 0 & 0 & 0 & 0 & 0 & 0 & \eta_1 I_{d_3} & \dots \\ \dots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

where $\begin{cases} \eta_0 = \sqrt{-1}\frac{2(d_1+d_3+\dots)}{\ell+1} \\ \eta_1 = -\sqrt{-1}\frac{d_0+2(d_2+d_4+\dots)}{\ell+1} \end{cases}$.

Subtype $\mathfrak{su}(p,p)$, $2p = \ell+1$, $p \geq 2$. The Satake diagram is:



Therefore, according to Theorem 4.3.1, the choices of $\mathcal{B}_{-1}$ corresponding to Levi-Tanaka algebras are:

$$\begin{aligned}\mathcal{B}_{-1} &= \{\alpha_{i_1}, \ldots, \alpha_{i_\nu}, \alpha_{2p-i_\nu}, \ldots, \alpha_{2p-i_1}\} \\ &\qquad \text{with } \nu \geq 1 \quad \text{and } 1 \leq i_1 < \ldots < i_\nu \leq p-1.\end{aligned}$$

The corresponding Levi-Tanaka algebra $\mathfrak{g}$ has kind 2ν.

We set

(4.23)
$$d_0 = 2(p - i_\nu),\ d_1 = i_\nu - i_{\nu-1},\ \ldots,\ d_h = i_{\nu-h+1} - i_{\nu-h},\ \ldots,\ d_\nu = i_1$$

and also $d_h = 0$ for $h \neq 0, 1, \ldots, \nu$.

We obtain

$$\mathfrak{g}_0 \simeq \mathfrak{d}_{2\nu}(\mathbb{R}) \oplus \mathfrak{su}(p - i_\nu, p - i_\nu) \oplus \bigoplus_{\substack{i>0 \\ d_i>1}} \mathfrak{sl}(d_i, \mathbb{C})$$

and

$$\begin{cases} \dim_{\mathbb{R}} \mathfrak{g}_0 = d_0^2 + \sum_{i=1}^{\nu} d_i^2 - 1 \\ \dim_{\mathbb{R}} \mathfrak{g}_{\pm(2r+1)} = 2 \displaystyle\sum_{i=0}^{\nu-2r-1} d_i d_{2r+1+i} + 2 \sum_{i=1}^{r} d_i d_{2r+1-i} \quad \text{for} \quad r \geq 0 \\ \dim_{\mathbb{R}} \mathfrak{g}_{\pm 2r} = 2 \displaystyle\sum_{i=0}^{\nu-2r} d_i d_{2r+i} + 2 \sum_{i=1}^{r-1} d_i d_{2r-i} + d_r^2 \quad \text{for} \quad r > 0. \end{cases}$$

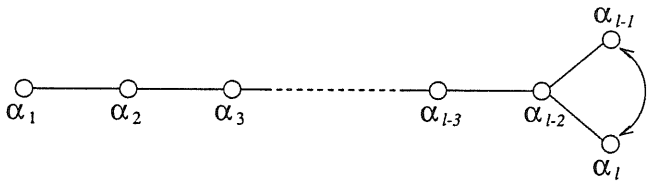
We obtain a matrix representation analogous to that of the case $\mathfrak{su}(p, q)$ taking instead of the matrix $\tilde{B}$ used before the new matrix

$$\tilde{B} = \begin{pmatrix} 0 & I_{p-i_\nu} \\ I_{p-i_\nu} & 0 \end{pmatrix}.$$

4.3.2 Simple Levi-Tanaka algebras of the real type D_ℓ

There are only two types of Satake diagrams associated to D_ℓ that contain curved arrows, which correspond to the real Lie algebras $\mathfrak{so}(\ell - 1, \ell + 1)$ and $\mathfrak{so}^*(2\ell)$ with $\ell = 2p + 1$, respectively. We discuss the two cases separately.

Subtype $\mathfrak{so}(\ell-1,\ell+1)$, $\ell \geq 4$. The Satake diagram is:



Therefore the only possible choice is:

$$\mathcal{B}_{-1} = \{\alpha_{\ell-1}, \alpha_\ell\}$$

and it is LT-admissible. A Levi-Tanaka algebra $\mathfrak{g}$ associated to the corresponding weighted Satake diagram has kind two and we obtain:

$$\mathfrak{g}_0 \simeq \mathfrak{d}_2(\mathbb{R}) \oplus \mathfrak{sl}(\ell-1,\mathbb{R})$$

and

$$\begin{cases} \dim_{\mathbb{R}} \mathfrak{g}_0 = \ell^2 - 2\ell + 2 \\ \dim_{\mathbb{R}} \mathfrak{g}_{\pm 1} = 2(\ell-1) \\ \dim_{\mathbb{R}} \mathfrak{g}_{\pm 2} = \frac{(\ell-1)(\ell-2)}{2}. \end{cases}$$

To obtain a matrix representation of $\mathfrak{g}$ we introduce the matrix

$$B = \begin{pmatrix} 0 & 0 & I_{\ell-1} \\ 0 & I_2 & 0 \\ I_{\ell-1} & 0 & 0 \end{pmatrix}$$

and identify $\mathfrak{g}$ to the subalgebra of $\mathfrak{gl}(\ell+1,\mathbb{R})$ of matrices X such that ${}^tXB + BX = 0$. We write these matrices in the form

$$X = \begin{pmatrix} x_{-1,-1} & x_{-1,0} & x_{-1,1} \\ x_{0,-1} & x_{0,0} & x_{0,1} \\ x_{1,-1} & x_{1,0} & x_{1,1} \end{pmatrix}$$

with

$$\begin{cases} x_{\pm 1,\pm 1} \in \mathfrak{M}((\ell-1)\times(\ell-1),\mathbb{R}) \\ x_{\pm 1,0} \in \mathfrak{M}((\ell-1)\times 2,\mathbb{R}) \\ x_{0,\pm 1} \in \mathfrak{M}(2\times(\ell-1),\mathbb{R}) \\ x_{0,0} \in \mathfrak{M}(2\times 2,\mathbb{R}) \\ {}^tx_{i,j} = -x_{-j,-i} \quad \text{for} \quad i,j = 0,\pm 1. \end{cases}$$

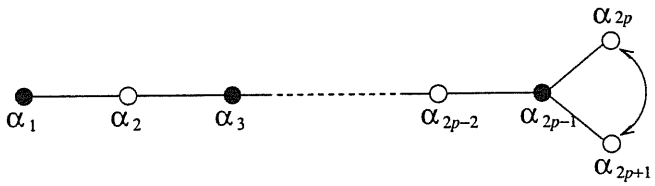
The characteristic element E is associated to the matrix

$$\begin{pmatrix} -I_{\ell-1} & 0 & 0 \\ 0 & 0_2 & 0 \\ 0 & 0 & I_{\ell-1} \end{pmatrix}$$

and the partial complex structure is defined by plus or minus the matrix

$$\begin{pmatrix} 0_{\ell-1} & 0 & 0 \\ 0 & \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) & 0 \\ 0 & 0 & 0_{\ell-1} \end{pmatrix}.$$

Subtype $\mathfrak{so}^*(2\ell)$, $\ell = 2p+1$, $p \geq 2$. The Satake diagram is:



Therefore the only possible choice of $\mathcal{B}_{-1}$ is:

$$\mathcal{B}_{-1} = \{\alpha_{2p}, \alpha_{2p+1}\}$$

and it is LT-admissible. If $\mathfrak{g}$ is a corresponding Levi-Tanaka algebra, then it has kind 2 and

$$\mathfrak{g}_0 \simeq \mathfrak{d}_2(\mathbb{R}) \oplus \mathfrak{su}^*(2\ell - 4).$$

Moreover we obtain:

$$\begin{cases} \dim_{\mathbb{R}} \mathfrak{g}_0 = 4p^2 + 1 \\ \dim_{\mathbb{R}} \mathfrak{g}_{\pm 1} = 4p \\ \dim_{\mathbb{R}} \mathfrak{g}_{\pm 2} = p(2p-1). \end{cases}$$

Let us describe a matrix representation of $\mathfrak{g}$. For every positive integer h we denote by $\check{I}_{2h}$ the $(2h) \times (2h)$ matrix

$$\begin{pmatrix} 0 & -I_h \\ I_h & 0 \end{pmatrix}.$$

Then we introduce the two $(4p+2)\times(4p+2)$ matrices:

$$A=\begin{pmatrix}\check{I}_{2p} & 0 & 0\\ 0 & \check{I}_2 & 0\\ 0 & 0 & \check{I}_{2p}\end{pmatrix} \quad \text{and} \quad B=\begin{pmatrix}0 & 0 & I_{2p}\\ 0 & I_2 & 0\\ I_{2p} & 0 & 0\end{pmatrix}.$$

Then we identify $\mathfrak{g}$ to the space of complex $(4p+2)\times(4p+2)$ matrices X such that

$$\overline{X}A=AX \quad \text{and} \quad {}^tXB+BX=0.$$

Using the block notation

$$X=\begin{pmatrix}x_{-1,-1} & x_{-1,0} & x_{-1,1}\\ x_{0,-1} & x_{0,0} & x_{0,1}\\ x_{1,-1} & x_{1,0} & x_{1,1}\end{pmatrix}$$

with

$$\begin{cases}x_{\pm1,\pm1}\in\mathfrak{M}(2p\times 2p,\mathbb{C})\\ x_{\pm1,0}\in\mathfrak{M}(2p\times 2,\mathbb{C})\\ x_{0,\pm1}\in\mathfrak{M}(2\times 2p,\mathbb{C})\\ x_{0,0}\in\mathfrak{M}(2\times 2,\mathbb{C})\end{cases}$$

we obtain the relations

$$\begin{cases}\overline{x}_{i,j}\check{I}=\check{I}x_{i,j} & \text{for} \quad i,j=0,\pm1\\ {}^tx_{i,j}=-x_{-j,-i} & \text{for} \quad i,j=0,\pm1\end{cases}$$

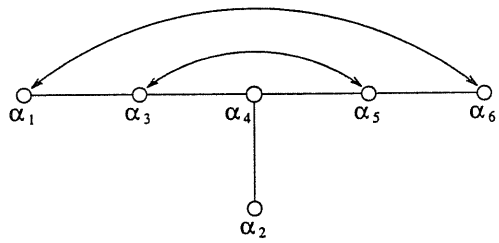
where $\check{I}$ is either $\check{I}_{2p}$ or $\check{I}_2$ according to the sizes of the matrices involved. The characteristic element and the partial complex structure correspond respectively to the matrices

$$\begin{pmatrix}-I_{2p} & 0 & 0\\ 0 & 0_2 & 0\\ 0 & 0 & I_{2p}\end{pmatrix} \quad \text{and} \quad \pm\begin{pmatrix}0_{2p} & 0 & 0\\ 0 & \left(\begin{smallmatrix}0 & -1\\ 1 & 0\end{smallmatrix}\right) & 0\\ 0 & 0 & 0_{2p}\end{pmatrix}.$$

4.3.3 Simple Levi-Tanaka algebras of the real type E_6

There are only two types of Satake diagrams associated to E_6 that contain curved arrows: they are usually referred to as E_6II and E_6III. Accordingly, we divide the description into two parts.

Subtype E_6II. The Satake diagram of E_6II is:



The choices of $\mathcal{B}_{-1}$ corresponding to Levi-Tanaka algebras are:

$$\{\alpha_3, \alpha_5\}, \quad \{\alpha_1, \alpha_6\} \quad \text{and} \quad \{\alpha_1, \alpha_3, \alpha_5, \alpha_6\}.$$

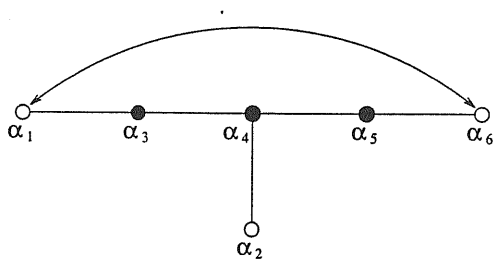
Due to the automorphisms of E_6II, there is, modulo isomorphisms, a unique Levi-Tanaka algebra corresponding to each LT-admissible choice of $\mathcal{B}_{-1}$. We list below the main features of each of these algebras:

$$\begin{cases} \mathcal{B}_{-1} = \{\alpha_1, \alpha_6\}\,, \qquad \mu = 2 \\ \mathfrak{g}_0 \simeq \mathfrak{d}_2(\mathbb{R}) \oplus \mathfrak{so}(3,5) \\ \dim_{\mathbb{R}} \mathfrak{g}_0 = 30 \\ \dim_{\mathbb{R}} \mathfrak{g}_{\pm 1} = 16 \\ \dim_{\mathbb{R}} \mathfrak{g}_{\pm 2} = 8\,, \end{cases} \tag{4.24}$$

$$\begin{cases} \mathcal{B}_{-1} = \{\alpha_3, \alpha_5\}\,, \qquad \mu = 4 \\ \mathfrak{g}_0 \simeq \mathfrak{d}_2(\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{sl}(3,\mathbb{R}) \\ \dim_{\mathbb{R}} \mathfrak{g}_0 = 16 \\ \dim_{\mathbb{R}} \mathfrak{g}_{\pm 1} = 12 \\ \dim_{\mathbb{R}} \mathfrak{g}_{\pm 2} = 12 \\ \dim_{\mathbb{R}} \mathfrak{g}_{\pm 3} = 4 \\ \dim_{\mathbb{R}} \mathfrak{g}_{\pm 4} = 3\,, \end{cases} \tag{4.25}$$

$$
(4.26) \qquad \begin{cases} \mathcal{B}_{-1} = \{\alpha_1, \alpha_3, \alpha_5, \alpha_6\}\,, \qquad \mu = 6 \\ \mathfrak{g}_0 \simeq \mathfrak{d}_4(\mathbb{R}) \oplus \mathfrak{sl}(3, \mathbb{R}) \\ \dim_{\mathbb{R}} \mathfrak{g}_0 = 12 \\ \dim_{\mathbb{R}} \mathfrak{g}_{\pm 1} = 8 \\ \dim_{\mathbb{R}} \mathfrak{g}_{\pm 2} = 9 \\ \dim_{\mathbb{R}} \mathfrak{g}_{\pm 3} = 6 \\ \dim_{\mathbb{R}} \mathfrak{g}_{\pm 4} = 5 \\ \dim_{\mathbb{R}} \mathfrak{g}_{\pm 5} = 2 \\ \dim_{\mathbb{R}} \mathfrak{g}_{\pm 6} = 3\,. \end{cases}
$$

Subtype $E_6 III$. The Satake diagram of the root system $E_6 III$ is:



Therefore the only possible choice of $\mathcal{B}_{-1}$ is:

$$\mathcal{B}_{-1} = \{\alpha_1, \alpha_6\}.$$

It is LT-admissible and, due to the automorphisms of $E_6 III$, there is, modulo isomorphisms, a unique Levi-Tanaka algebra $\mathfrak{g}$ associated to it. Its main features are:

$$
(4.27) \qquad \begin{cases} \mathcal{B}_{-1} = \{\alpha_1, \alpha_6\}, \qquad \mu = 2 \\ \mathfrak{g}_0 \simeq \mathfrak{d}_2(\mathbb{R}) \oplus \mathfrak{so}(1, 7) \\ \dim_{\mathbb{R}} \mathfrak{g}_0 = 30 \\ \dim_{\mathbb{R}} \mathfrak{g}_{\pm 1} = 16 \\ \dim_{\mathbb{R}} \mathfrak{g}_{\pm 2} = 8. \end{cases}
$$

Chapter 5

Finite dimensional Levi-Tanaka algebras of codimension 2

In this chapter we shall consider finite dimensional Levi-Tanaka algebras of the second kind. These are particularly interesting because they correspond to those CR manifolds for which general results on the behaviour of the CR complexes are known (cf. [30]).

A finite dimensional Levi-Tanaka algebra of CR-dimension n, CR-codimension k and kind $\mu = 2$, is completely determined by the datum of a k-dimensional linear subspace of the n^2-dimensional linear space of Hermitian symmetric forms. The case $k = 1$ is the one discussed in [39], [11], [41]. Here we shall be interested to the case of higher codimension $k \geq 2$. The results that are known about the canonical form of a pair of Hermitian symmetric matrices allow us to give a complete description of the Levi-Tanaka algebras arising in the case $k = 2$.

The theory developed by Tanaka in [39], [40] and [42] makes it possible to obtain results on the group of CR automorphisms of CR manifolds of codimension 2, complementing and completing those obtained in [28] and in [23].

5.1 Canonical forms

Let $\mathfrak{m} = \oplus_{-\mu \leq p < 0} \mathfrak{m}_p$ be a pseudocomplex fundamental graded Lie algebra. As shown in Proposition 1.3.1, the alternating map

$$\mathfrak{m}_{-1} \times \mathfrak{m}_{-1} \ni (X, Y) \to [X, Y] \in \mathfrak{m}_{-2}$$

uniquely defines a Hermitian symmetric (vector valued) form

$$\mathfrak{f} : \mathfrak{m}_{-1} \times \mathfrak{m}_{-1} \ni (X, Y) \to \mathfrak{f}(X, Y) \in \mathbb{C} \otimes_{\mathbb{R}} \mathfrak{m}_{-2}$$

such that

$$[X, Y] = \Im \mathfrak{f}(X, Y) \quad \forall X, Y \in \mathfrak{m}_{-1}.$$

The form $\mathfrak{f}$ is defined by

$$\mathfrak{f}(X, Y) = [JX, Y] + \sqrt{-1}[X, Y] \quad \forall X, Y \in \mathfrak{m}_{-1}.$$

We consider the natural map

$$\mathcal{L} : \mathfrak{m}_{-2}^* \ni \xi \to \mathfrak{f}_\xi \in \mathfrak{H}_s(\mathfrak{m}_{-1})$$

from the dual space $\mathfrak{m}_{-2}^*$ of $\mathfrak{m}_{-2}$ to the real linear space $\mathfrak{H}_s(\mathfrak{m}_{-1})$ of Hermitian symmetric forms on $\mathfrak{m}_{-1}$, which is given by

$$\mathfrak{f}_\xi(X, Y) = \langle \mathfrak{f}(X, Y), \xi \rangle \quad \forall \xi \in \mathfrak{m}_{-2}^*, \forall X, Y \in \mathfrak{m}_{-1}.$$

Vice versa, given a finite dimensional $\mathbb{C}$-linear space V and a linear subspace F of the space $\mathfrak{H}_s(V)$ of Hermitian symmetric forms on V, there is a pseudocomplex fundamental graded Lie algebra $\mathfrak{m} = \mathfrak{m}_{-2} \oplus \mathfrak{m}_{-1}$ of kind 2 such that

$$\mathfrak{m}_{-1} = V \quad \text{and} \quad \mathcal{L}(\mathfrak{m}_{-2}^*) = F.$$

This algebra is unique up to isomorphisms and can be described by setting

$$\mathfrak{m}_{-2} = F^*, \quad [\mathfrak{m}_{-2}, \mathfrak{m}_{-2}] = [\mathfrak{m}_{-2}, \mathfrak{m}_{-1}] = 0$$

while the Lie product $[X, Y]$ of two elements $X, Y \in \mathfrak{m}_{-1} = V$ is the $\mathbb{R}$-linear functional on F:

$$[X, Y] : F \ni h \to \Im h(X, Y) \in \mathbb{R}.$$

The complex structure J of $\mathfrak{m}_{-1}$ is the complex structure of V and $\mathfrak{m}$ is a graded pseudocomplex Lie algebra $\mathfrak{m}$ of kind 2 and type (n, k) with $n = dim_{\mathbb{C}} V$ and $k = dim_{\mathbb{R}} F$.

Remark 5.1.1 *The group* $\mathbf{GL}_{\mathbb{C}}(V)$ *of* $\mathbb{C}$*-linear automorphisms of* V *acts on the space* $\mathfrak{H}(V)$ *of the Hermitian forms on* V *by*

$$\mathbf{GL}_{\mathbb{C}}(V) \times \mathfrak{H}(V) \ni (a, h) \to a \cdot h \in \mathfrak{H}(V),$$

where

$$a \cdot h(X, Y) = h(aX, aY) \quad \forall a \in \mathbf{GL}_{\mathbb{C}}(V), \forall h \in \mathfrak{H}(V), \forall X, Y \in V.$$

The subspace $\mathfrak{H}_{\mathfrak{s}}(V)$ *is stable under this action of* $\mathbf{GL}_{\mathbb{C}}(V)$*. Moreover,* $\mathbf{GL}_{\mathbb{C}}(V)$ *transforms* k*-dimensional subspaces of* $\mathfrak{H}_{\mathfrak{s}}(V)$ *into* k*-dimensional subspaces of* $\mathfrak{H}_{\mathfrak{s}}(V)$*.*

We denote by $\mathbf{Gr}_k(\mathfrak{H}_{\mathfrak{s}}(V))$ the Grassmannian of k-dimensional subspaces of $\mathfrak{H}_{\mathfrak{s}}(V)$ and by $\mathfrak{O}_k(\mathfrak{H}_{\mathfrak{s}}(V))$ the space of orbits of $\mathbf{Gr}_k(\mathfrak{H}_{\mathfrak{s}}(V))$ for the action of the linear group $\mathbf{GL}_{\mathbb{C}}(V)$.

Given a k-uple $(\mathfrak{f}_1, \ldots, \mathfrak{f}_k)$ of independent Hermitian symmetric forms, we denote by $F =< \mathfrak{f}_1, \ldots, \mathfrak{f}_k >$ the subspace of $\mathfrak{H}_{\mathfrak{s}}(V)$ that they generate, and by $[F]$ the orbit of F in $\mathfrak{O}_k(\mathfrak{H}_{\mathfrak{s}}(V))$.

PROPOSITION 5.1.2 *Let* n, k *be positive integers, with* $1 \leq k \leq n^2$ *and let* V *be a complex vector space of dimension* n*. There is a* 1*-to-*1 *correspondence between pseudocomplex fundamental graded Lie algebras of kind* 2 *and type* (n, k) *-modulo isomorphisms- and the orbits of* $\mathbf{GL}_{\mathbb{C}}(V)$ *in* $\mathfrak{O}_k(\mathfrak{H}_{\mathfrak{s}}(V))$*.*

Proof. Let $\mathfrak{m} = \mathfrak{m}_{-2} \oplus \mathfrak{m}_{-1}$ be a pseudocomplex fundamental graded Lie algebra. Let $a \in \mathbf{GL}_{\mathbb{C}}(\mathfrak{m}_{-1})$ and $b \in \mathbf{GL}_{\mathbb{R}}(\mathfrak{m}_{-2})$. Then we obtain another isomorphic fundamental graded Lie algebra $\tilde{\mathfrak{m}} = \tilde{\mathfrak{m}}_{-2} \oplus \tilde{\mathfrak{m}}_{-1}$ by setting $\tilde{\mathfrak{m}}_{-1} = \mathfrak{m}_{-1}$ as $\mathbb{C}$-linear spaces and $\tilde{\mathfrak{m}}_{-2} = \mathfrak{m}_{-2}$ as $\mathbb{R}$-linear spaces and defining the Lie product by:

$$[\tilde{\mathfrak{m}}_{-1}, \tilde{\mathfrak{m}}_{-2}]' = [\tilde{\mathfrak{m}}_{-2}, \tilde{\mathfrak{m}}_{-2}]' = 0$$

and

$$[X, Y]' = b([a(X), a(Y)]) \qquad \forall X, Y \in \mathfrak{m}_{-1} = \tilde{\mathfrak{m}}_{-1}.$$

The isomorphism $\phi : \tilde{\mathfrak{m}} \to \mathfrak{m}$ is given by

$$\tilde{\mathfrak{m}}_{-1} \ni X \to a(X) \in \mathfrak{m}_{-1} \quad \text{and} \quad \tilde{\mathfrak{m}}_{-2} \ni T \to b^{-1}(T) \in \mathfrak{m}_{-2}.$$

Indeed the equation $\phi([X, Y]') = [\phi(X), \phi(Y)]$ reduces to the definition of the Lie product in $\tilde{\mathfrak{m}}$.

By this remark, the statement of the proposition becomes clear. □

Using this proposition, we can parametrize pseudocomplex fundamental graded Lie algebras of kind 2 and type (n, k) -modulo isomorphisms- by fixing a complex n-dimensional vector space V and a point F in one of the orbits of $\mathfrak{O}_k(\mathfrak{H}_s(V))$. We will denote by $\mathfrak{m}(F)$ the corresponding pseudocomplex fundamental graded Lie algebra and by $\mathfrak{g}(F)$ its canonical pseudocomplex prolongation.

The subspace F and the corresponding algebra $\mathfrak{m}(F)$ are called *nonsingular* if the space $F \in \mathfrak{H}_s(V)$ contains a nondegenerate form.

Note that a nonsingular algebra is nondegenerate. The converse is in general false (see the example in 3.8.3).

Remark 5.1.3 *Let $\mathbb{P}\mathfrak{H}_s(V)$ denote the projective (n^2-1)-dimensional space corresponding to the linear space $\mathfrak{H}_s(V)$. The action of $\mathbf{GL}_{\mathbb{C}}(V)$ defines, by passing to the quotient, an action on $\mathbb{P}\mathfrak{H}_s(V)$. Let us denote by $\mathcal{C}$ the image in $\mathbb{P}\mathfrak{H}_s(V)$ of the cone of positive definite Hermitian symmetric forms on V. This is a convex body in $\mathbb{P}\mathfrak{H}_s(V)$. The corresponding Hilbert distance in $\mathcal{C}$ is given by*

$$d([h_1],[h_2]) = \sup_{1\leq i,j\leq n} \log \frac{\lambda_i}{\lambda j}$$

where $[h_1]$ and $[h_2]$ are the points of $\mathcal{C}$ corresponding to two positive definite Hermitian symmetric forms h_1, h_2 on V and $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of h_2 with respect to h_1 (i.e., denoting still by h_1 and h_2 the anti-$\mathbb{C}$-linear maps $V \to V^*$ corresponding to the forms h_1 and h_2, the eigenvalues of the $\mathbb{C}$-linear endomorphism $h_1^{-1} \circ h_2$ of V). The group $\mathbf{GL}_{\mathbb{C}}(V)$ operates on $\mathcal{C}$. Its image in its representation in the group of permutations of $\mathcal{C}$ is the connected component of the identity in the Lie group of isometries of the Hilbert metric.

For the study of the Levi-Tanaka algebras of the second kind with CR-codimension equal to 2, we will employ the following lemma (taken from [43]).

LEMMA 5.1.4 *Let $F \subset \mathfrak{H}_s(V)$ be a two-dimensional linear subspace, such that $\mathfrak{m}(F)$ is nondegenerate. Then there is a direct sum decomposition of V:*

$$V = V_1 \oplus \ldots \oplus V_m \tag{5.1}$$

with the following properties:

(i) V_i is orthogonal to V_j with respect to every form $\mathfrak{f} \in F$ for $0 \leq i \neq j \leq m$ (biorthogonality);

(ii) for each $i = 1, \dots, m$, the subspace V_i is not a direct sum of two non-trivial subspaces that are orthogonal with respect to all $\mathfrak{f} \in F$ (indecomposability).

Let $\mathfrak{f}_1, \mathfrak{f}_2$ be a basis of F. Then, for each $i = 1, \dots, m$, we can choose a basis of V_i in such a way that, for all $\lambda, \mu \in \mathbb{C}$, the matrix corresponding to the restriction of $\mu\mathfrak{f}_2 - \lambda\mathfrak{f}_1$ to V_i has one of the following forms, involving parameters $\gamma \in \mathbb{R}, \Gamma \in \mathbb{C}$, $e, E, \mathcal{E}$ natural numbers, and $\epsilon = \pm 1$:

(I) an e-square matrix:

$$\epsilon D_e(\gamma, \mu, \lambda) = \epsilon \begin{pmatrix} & & & & \mu\gamma - \lambda \\ & & & \cdot & \mu \\ & & \cdot & \cdot & \\ & \cdot & \cdot & & \\ \mu\gamma - \lambda & \mu & & & \end{pmatrix} \quad \text{with } \gamma \in \mathbb{R};$$

(II) a $2E$-square matrix:

$$D_E^{\mathbb{C}}(\Gamma, \mu, \lambda) = \begin{pmatrix} 0 & D_E(\Gamma, \mu, \lambda) \\ D_E(\bar{\Gamma}, \mu, \lambda) & 0 \end{pmatrix} \quad \text{with } \Gamma \in \mathbb{C} \setminus \mathbb{R};$$

(III) an e-square matrix:

$$\epsilon D_e(\infty, \mu, \lambda) = \epsilon \begin{pmatrix} & & & & -\mu \\ & & & \cdot & \lambda \\ & & \cdot & \cdot & \\ & \cdot & \cdot & & \\ -\mu & \lambda & & & \end{pmatrix};$$

(IV) a $(2E-1)$-square matrix:

$$D_{\mathcal{E}}(\mu, \lambda) = \begin{pmatrix} & & & & -\lambda & & \\ & & & & \mu & \ddots & \\ & 0_{\mathcal{E}} & & & & \ddots & -\lambda \\ & & & & & & \mu \\ -\lambda & \mu & & & & & \\ & \ddots & \ddots & & & 0_{\mathcal{E}-1} & \\ & & -\lambda & \mu & & & \end{pmatrix}.$$

The decomposition (5.1) is essentially unique: this means that the number and the dimension of the subspaces are uniquely determined and, after fixing a basis $\mathfrak{f}_1, \mathfrak{f}_2$ *of* F*, the forms of the blocks and the parameters involved are uniquely determined.*

The parameter $\epsilon = \pm 1$ above is called the *sign* or *inertial signature* associated to the correspondent block.

The numbers γ's and Γ's are called *roots* or *eigenvalues* of the pair $(\mathfrak{f}_1, \mathfrak{f}_2)$. The set $\Sigma = \Sigma(\mathfrak{f}_1, \mathfrak{f}_2)$ of the roots, including $\infty \in \mathbb{CP}^1$ if the form given in Lemma 5.1.4 contains blocks of type (*III*), is called the *spectrum* of $(\mathfrak{f}_1, \mathfrak{f}_2)$.

Remark 5.1.5 *When* F *is nonsingular and* $\mathfrak{f}_1, \mathfrak{f}_2$ *are chosen with a nondegenerate* $\mathfrak{f}_1$*, the basis of* V *given in the lemma is one for which the matrix corresponding to* $\mathfrak{f}_1^{-1}\mathfrak{f}_2$ *is in the Jordan canonical form (here we used the same letters both for the Hermitian symmetric forms and for the anti-*$\mathbb{C}$*-linear maps into the dual that they define); we call it a* canonical basis.

In particular, the subspaces in the decomposition (5.1) are either indecomposable subspaces for $\mathfrak{f}_1^{-1}\mathfrak{f}_2$ *corresponding to a real eigenvalue or the direct sum of two indecomposable subspaces for* $\mathfrak{f}_1^{-1}\mathfrak{f}_2$ *corresponding to a non real eigenvalue and its complex conjugate. In this case the spectrum* Σ *coincides with the spectrum of the eigenvalues of* $\mathfrak{f}_1^{-1}\mathfrak{f}_2$.

We consider more closely the different blocks in the matrix representation of $\mathfrak{f}_1, \mathfrak{f}_2$ given in the previous lemma. The blocks of type (*I*) are characterized by the data of distinct real numbers $\gamma_1, \gamma_2, \ldots, \gamma_l$, with γ_i occuring in blocks of distinct sizes e_{i1} (m_{i1} times), e_{i2} (m_{i2} times), ... , e_{is_i} (m_{is_i} times), $e_{i1} > e_{i2} > \cdots > e_{is_i}$, and corresponding inertial signatures ϵ_{ijp}, for $1 \leq p \leq m_{ij}$, $1 \leq j \leq s_i$, $1 \leq i \leq l$.

The blocks of type (*II*) are described by distinct conjugate pairs $(\Gamma_1, \bar{\Gamma}_1)$, ... , $(\Gamma_L, \bar{\Gamma}_L)$ of nonreal numbers, with Γ_I, $\bar{\Gamma}_I$ occuring in blocks of distinct sizes $2E_{I1}$ (M_{I1} times), ... , $2E_{IS_I}$ (M_{IS_I} times), $E_{I1} > \cdots > E_{IS_I}$, $I = 1, \ldots, L$.

The blocks of type (*III*) are described by their distinct sizes $e_{\infty 1}$ ($m_{\infty 1}$ times), ... ,$e_{\infty s_\infty}$ ($m_{\infty s_\infty}$ times), with $e_{\infty 1} > \cdots > e_{\infty s_\infty}$, and associated inertial signatures $\epsilon_{\infty jp}$, for $p = 1, \ldots, m_{\infty j}$, $j = 1, \ldots, s_\infty$.

The blocks of type (*IV*) are given by the datum of distinct parameters $2 \leq \mathcal{E}_1 \leq \cdots \leq \mathcal{E}_s$.

The matrix associated to $\mu\mathfrak{f}_2 - \lambda\mathfrak{f}_1$ in a suitable basis of V takes then the form:

$$\begin{aligned}
[\mu\mathfrak{f}_2 - \lambda\mathfrak{f}_1] &= \bigoplus_{i=1}^{l}\bigoplus_{j=1}^{s_i}\bigoplus_{p=1}^{m_{ij}} \epsilon_{ijp} D_{e_{ij}}(\gamma_i, \mu, \lambda) \\
&\oplus \bigoplus_{j=1}^{s_\infty}\bigoplus_{p=1}^{m_{\infty j}} \epsilon_{\infty pj} D_{e_{\infty j}}(\infty, \mu, \lambda) \\
&\oplus \bigoplus_{I=1}^{L}\bigoplus_{J=1}^{S_I}\bigoplus_{P=1}^{M_{IJ}} \begin{pmatrix} 0 & D_{E_{IJ}}(\Gamma_I, \mu, \lambda) \\ D_{E_{IJ}}(\bar{\Gamma}_I, \mu, \lambda) & 0 \end{pmatrix} \\
&\oplus \bigoplus_{j=1}^{s} D_{\mathcal{E}_j}(\mu, \lambda)\,,
\end{aligned}$$

where we used the symbol of direct sum to indicate diagonal block decomposition. This is called *the canonical reduced form for the ordered pair of Hermitian forms* $(\mathfrak{f}_1, \mathfrak{f}_2)$.

We consider now a 2-dimensional subspace F of $\mathfrak{H}_s(V)$ generated by a pair of Hermitian symmetric forms $(\mathfrak{f}_1, \mathfrak{f}_2)$. If $\mathfrak{f}_1', \mathfrak{f}_2'$ is another basis of F, then there is $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{GL}(2, \mathbb{R})$ such that

$$(\mathfrak{f}_1', \mathfrak{f}_2') = (a\mathfrak{f}_1 + b\mathfrak{f}_2, c\mathfrak{f}_1 + d\mathfrak{f}_2).$$

The canonical form of the pair $(\mathfrak{f}_1', \mathfrak{f}_2')$ has blocks of the same type corresponding to the same biorthogonal decomposition (5.1) of V. The spectrum Σ' of $(\mathfrak{f}_1', \mathfrak{f}_2')$ is obtained from the spectrum Σ of $(\mathfrak{f}_1, \mathfrak{f}_2)$ by the action of ${}^tB$ as a real projectivity on $\mathbb{CP}^1$. Indeed the spectrum Σ' of $(\mathfrak{f}_1', \mathfrak{f}_2')$ is the image of Σ by the transformation

$$\phi : \mathbb{CP}^1 \ni [z, w] \longrightarrow [az + cw, bz + dw] \in \mathbb{CP}^1\,. \tag{5.2}$$

In particular, we can always choose a basis $\mathfrak{f}_1, \mathfrak{f}_2$ of F such that in the canonical form of the pair $(\mathfrak{f}_1, \mathfrak{f}_2)$ given by Lemma 5.1.4 do not appear blocks of type *(III)*.

The question of the equivalence of Levi-Tanaka algebras corresponding to different 2-dimensional subspaces of $\mathfrak{H}_s(V)$ can be solved using the action of $\mathbf{GL}(2, \mathbb{R})$ on the set of data

$$(\gamma_i, e_{ij}, m_{ij}, \epsilon_{ijp}, \infty, e_{\infty j}, m_{\infty j}, \epsilon_{\infty jp}, \Gamma_I, E_{IJ}, M_{IJ}, \mathcal{E}_h).$$

5.2 Computation of $\mathfrak{g}_0$

Let V be an n-dimensional complex linear space and $\mathfrak{m} = \mathfrak{m}(F) = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$ be the graded pseudocomplex fundamental Lie algebra associated to a given two dimensional subspace F of $\mathfrak{H}_5(V)$. We fix a basis $\mathfrak{f}_1, \mathfrak{f}_2$ of F and denote by (F_1, F_2) the matrices corresponding to $\mathfrak{f}_1, \mathfrak{f}_2$ in a canonical basis of V as described in Lemma 5.1.4. The elements of $\mathfrak{g}_0$ correspond to matrices $A \in \mathfrak{gl}(n, \mathbb{C})$ such that

$$\begin{cases} A^* F_1 + F_1 A = aF_1 + bF_2 \\ A^* F_2 + F_2 A = cF_1 + dF_2 \end{cases} \tag{5.3}$$

for some $a, b, c, d \in \mathbb{R}$.

We assume, as we can, that $\mathfrak{f}_1$ has maximal rank in F. Consider first the case of a nonsingular F. Then F_1 is invertible and we can define the matrix $L = F_1^{-1} F_2$.

From the first equation in (5.3) we obtain

$$F_1^{-1} A^* F_1 = -A + aI + bL \tag{5.4}$$

and hence, substituting this expression into the second, we get:

$$[A, L] = bL^2 + (a - d)L - cI\,. \tag{5.5}$$

The following easy lemma will be useful for computing $\mathfrak{g}_0$:

LEMMA 5.2.1 *Let A and L be two endomorphisms of a finite dimensional complex vector space V. Assume that $[L, [L, A]] = 0$. Then for every $p \in \mathbb{C}[x]$ we have:*

$$[p(L), A] = [L, A]p'(L).$$

In particular the generalized eigenspaces V^λ of the spectral decomposition of L are A-invariant.

It follows that all generalized eigenspaces V^λ of $L = F_1^{-1} F_2$ are invariant under all endomorphisms $A \in \mathbf{GL}(n, \mathbb{C})$ which are solutions of (5.3). We will denote by A^λ the restriction of A to V^λ.

5.2.1 Computation of $\rho_{-2}(\mathfrak{g}_0)$

We begin the study of $\mathfrak{g}_0 = \mathfrak{g}_0(F)$ by computing its ρ_{-2} representation. To describe $\rho_{-2}(\mathfrak{g}_0)$, it is convenient to introduce four ℓ-square matrices that will be used in the discussion below:

$$J_\ell = \begin{pmatrix} 0 & 1 & & & \\ & \cdot & \cdot & & \\ & & \cdot & \cdot & \\ & & & \cdot & 1 \\ & & & & 0 \end{pmatrix}, \quad D_\ell = \begin{pmatrix} & & & & 1 \\ & & & \cdot & \\ & & \cdot & & \\ & \cdot & & & \\ 1 & & & & \end{pmatrix},$$

$$M_\ell = \begin{pmatrix} \ell - \frac{1}{2} & & & & \\ & \ell - \frac{5}{2} & & & \\ & & \ddots & & \\ & & & \frac{7}{2} - \ell & \\ & & & & \frac{3}{2} - \ell \end{pmatrix}$$

and

$$N_\ell = \begin{pmatrix} 0 & \frac{\ell-1}{2} & & & & & \\ & 0 & \frac{\ell-3}{2} & & & & \\ & & 0 & \cdot & & & \\ & & & \cdot & \cdot & & \\ & & & & \cdot & \cdot & \\ & & & & 0 & \frac{5-\ell}{2} & \\ & & & & & 0 & \frac{3-\ell}{2} \\ & & & & & & 0 \end{pmatrix}.$$

We distinguish different cases, according to the nature of the spectrum Σ of the pair $(\mathfrak{f}_1, \mathfrak{f}_2)$.

(0): $\Sigma = \emptyset$, i.e. there are only blocks of type (IV).
For

$$[\mu\mathfrak{f}_2 - \lambda\mathfrak{f}_1] = \bigoplus_{j=1}^{s} D_{\varepsilon_j}(\mu, \lambda),$$

we set

$$A(h) = \bigoplus_{j=1}^{s} A_{\varepsilon_j}(h) \qquad \text{for } h = 1, 2, 3, 4,$$

where, for each integer $\mathcal{E} \geq 2$,

$$\begin{aligned}
A_{\mathcal{E}}(1) &= \frac{1}{2} I_{2\mathcal{E}-1} \\
A_{\mathcal{E}}(2) &= \begin{pmatrix} M_{\mathcal{E}} - \frac{1}{2} I_{\mathcal{E}} & \\ & D_{\mathcal{E}-1} M_{\mathcal{E}-1} D_{\mathcal{E}-1} - \frac{1}{2} I_{\mathcal{E}-1} \end{pmatrix} \\
A_{\mathcal{E}}(3) &= \begin{pmatrix} N_{\mathcal{E}} + \frac{\mathcal{E}-1}{2} J_{\mathcal{E}} & \\ & D_{\mathcal{E}-1}(N_{\mathcal{E}-1} - \frac{\mathcal{E}}{2} J_{\mathcal{E}-1}) D_{\mathcal{E}-1} \end{pmatrix} \\
A_{\mathcal{E}}(4) &= \begin{pmatrix} D_{\mathcal{E}}(N_{\mathcal{E}} + \frac{\mathcal{E}-1}{2} J_{\mathcal{E}}) D_{\mathcal{E}} & \\ & N_{\mathcal{E}-1} - \frac{\mathcal{E}}{2} J_{\mathcal{E}-1} \end{pmatrix} .
\end{aligned}$$

Then:

$$\begin{aligned}
\rho_{-2}(A(1)) &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
\rho_{-2}(A(2)) &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\
\rho_{-2}(A(3)) &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\
\rho_{-2}(A(4)) &= \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} .
\end{aligned}$$

Therefore $\rho_{-2}(\mathfrak{g}_0) = \mathfrak{gl}_{\mathbb{R}}(\mathfrak{g}_{-2})$ in this case.

Remark 5.2.2 *From the discussion of case* (0) *it follows that, when* $\Sigma \neq \emptyset$, *the existence of blocks of type* (*IV*) *imposes no restriction on* $\rho_{-2}(\mathfrak{g}_0) \subset \mathfrak{gl}_{\mathbb{R}}(\mathfrak{g}_{-2})$; *by this we mean that the form of* $\rho_{-2}(\mathfrak{g}_0)$ *only depends on* Σ.

Using Remark 5.2.2, we can assume for simplicity, while discussing the cases where $\Sigma \neq \emptyset$, that the canonical form of (F_1, F_2) does not contain blocks of type (*IV*).

(1): $\Sigma = \{\gamma\}$, $\gamma \in \mathbb{R}$.
By computing the trace of the two sides of (5.5), we have $0 = b\gamma^2 + (a-d)\gamma - c$. Assuming, as we can, that $\gamma = 0$, we obtain $c = 0$.

For

$$[\mu \mathfrak{f}_2 - \lambda \mathfrak{f}_1] = \bigoplus_{j=1}^{s_1} \bigoplus_{p=1}^{m_{1j}} \epsilon_{1jp} D_{e_{1j}}(0, \mu, \lambda) ,$$

we set:

$$\begin{aligned} A(1) &= I_n \\ A(2) &= \bigoplus_{j=1}^{s_1} (\underbrace{M_{e_{1j}} \oplus \ldots \oplus M_{e_{1j}}}_{m_{1j}-times}) \\ A(3) &= \bigoplus_{j=1}^{s_1} (\underbrace{N_{e_{1j}} \oplus \ldots \oplus N_{e_{1j}}}_{m_{1j}-times}) \end{aligned}$$

to obtain:

$$\begin{aligned} \rho_{-2}(A(1)) &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ \rho_{-2}(A(2)) &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ \rho_{-2}(A(3)) &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

In this case $\rho_{-2}(\mathfrak{g}_0)$ turns out to be isomorphic to the Lie algebra of upper triangular 2×2 real matrices.

(2): $\Sigma = \{\gamma_1, \gamma_2\}$, $\gamma_1, \gamma_2 \in \mathbb{R}$, $\gamma_1 \neq \gamma_2$.
In this case, by taking the restriction of the two sides of (5.5) to the subspaces V^{γ_1}, V^{γ_2}, we obtain:

$$\begin{cases} b(\gamma_1)^2 + (a-d)\gamma_1 - c = 0 \\ b(\gamma_2)^2 + (a-d)\gamma_2 - c = 0\,. \end{cases}$$

Taking, as we can, $\gamma_1 = -1$, $\gamma_2 = 1$, we obtain

$$\begin{cases} a = d \\ b = c\,. \end{cases} \tag{5.6}$$

For

$$[\mu \mathfrak{f}_2 - \lambda \mathfrak{f}_1] = \bigoplus_{j=1}^{s_1} \bigoplus_{p=1}^{m_{1j}} \epsilon_{1jp} D_{e_{1j}}(-1, \mu, \lambda) \oplus \bigoplus_{j=1}^{s_2} \bigoplus_{p=1}^{m_{2j}} \epsilon_{2jp} D_{e_{2j}}(1, \mu, \lambda)\,,$$

we set:

$$\begin{aligned} A(1) &= I_n \\ A(2) &= \bigoplus_{j=1}^{s_1} \underbrace{(N_{e_{1j}} - M_{e_{1j}})}_{m_{1j}-times} \oplus \bigoplus_{j=1}^{s_2} \underbrace{(N_{e_{2j}} + M_{e_{2j}})}_{m_{2j}-times} . \end{aligned}$$

Then:

$$\begin{aligned} \rho_{-2}(A(1)) &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ \rho_{-2}(A(2)) &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} . \end{aligned}$$

Hence in this case $\rho_{-2}(\mathfrak{g}_0)$ is isomorphic to the Lie algebra of the group of conformal linear transformations of the real hyperbolic plane.

(2′): $\Sigma = \{\Gamma, \bar{\Gamma}\}$, $\Gamma \in \mathbb{C} \setminus \mathbb{R}$.
In this case we have the system:

$$\begin{cases} b\,\Gamma^2 + (a-d)\Gamma - c = 0 \\ b\,\bar{\Gamma}^2 + (a-d)\bar{\Gamma} - c = 0 . \end{cases}$$

Taking $\Gamma = \sqrt{-1}$, we have

$$\begin{cases} a = d \\ b = -c . \end{cases} \tag{5.7}$$

For

$$[\mu \mathfrak{f}_2 - \lambda \mathfrak{f}_1] = \bigoplus_{J=1}^{S_1} \bigoplus_{P=1}^{M_{1J}} \begin{pmatrix} 0 & D_{E_{1J}}(\sqrt{-1}, \mu, \lambda) \\ D_{E_{1J}}(-\sqrt{-1}, \mu, \lambda) & 0 \end{pmatrix},$$

we set:

$$\begin{aligned} A(1) &= I_n \\ A(2) &= \bigoplus_{J=1}^{S_1} \underbrace{(N_{E_{1J}} - \sqrt{-1} M_{E_{1J}}) \oplus (N_{E_{1J}} + \sqrt{-1} M_{E_{1J}})}_{M_{1J}-times} . \end{aligned}$$

Then:

$$\begin{aligned}\rho_{-2}(A(1)) &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ \rho_{-2}(A(2)) &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.\end{aligned}$$

Thus $\rho_{-2}(\mathfrak{g}_0)$ is isomorphic to the Lie algebra of the group of linear conformal transformations of the Euclidean plane.

(3): Σ contains at least three distinct elements $\gamma_1, \gamma_2, \gamma_3 \in \mathbb{C}$.
We obtain the system

$$\begin{cases} b(\gamma_1)^2 + (a-d)\gamma_1 - c = 0 \\ b(\gamma_2)^2 + (a-d)\gamma_2 - c = 0 \\ b(\gamma_3)^2 + (a-d)\gamma_3 - c = 0 \end{cases}$$

which has solutions

$$\begin{cases} a = d \\ b = c = 0\,. \end{cases} \tag{5.8}$$

Then $\rho_{-2}(\mathfrak{g}_0)$ is the 1-dimensional Lie algebra of the real multiples of the identity on $\mathfrak{g}_{-2}$.

5.2.2 The "homogeneous" system

To complete the computation of $\mathfrak{g}_0 = \mathfrak{g}_0(F)$, we observe that all solutions of (5.3) are obtained by adding to those found in the previous subsection the elements of $\ker \rho_{-2}$, i.e. the solutions of the "homogeneous" system:

$$\begin{cases} A^* F_1 + F_1 A = 0 \\ A^* F_2 + F_2 A = 0\,. \end{cases} \tag{5.9}$$

When F_1 is nonsingular, the homogeneous system is equivalent to

$$\begin{cases} A^* = -F_1 A F_1^{-1} \\ [A, L] = 0\,, \end{cases} \tag{5.10}$$

where $L = F_1^{-1}F_2$.

Note that, when (F_1, F_2) is in the canonical form, $F_1 = F_1^{-1} = F_1^*$ and L is a Jordan matrix. Moreover L is self-adjoint with respect to F_1.

To compute $\ker \rho_{-2}$ we use the classical Frobenius theorem from linear algebra. The matrix corresponding to an element of $\mathfrak{g}_0$ is block-diagonal, of the form:

$$\operatorname{diag}(A^{\gamma_1}, \dots, A^{\gamma_l}, A^{\Gamma_1}, A^{\bar{\Gamma}_1}, \dots, A^{\Gamma_L}, A^{\bar{\Gamma}_L})$$

where $\gamma_1, \dots, \gamma_l, \Gamma_1, \dots, \Gamma_L$ are respectively the eigenvalues of L which are real and which have a positive imaginary part.

Let $n_i = \dim_{\mathbb{C}} V^{\gamma_i}$ and $n_{i1} \leq n_{i2} \leq \dots \leq n_{ir_i}$ be the dimension of the subspaces of a Jordan decomposition of V^{γ_i} for the restriction of L. In a basis adapted to the Jordan decomposition, $A^{\gamma_i} = (A^{\gamma_i}_{hk})$ and we have:

(5.11)

$$\begin{cases} A^{\gamma_i}_{hh} \in \sqrt{-1}\mathbb{R}[J_{n_{ih}}] & \text{for } h = 1, \dots, r_i \\ A^{\gamma_i}_{hk} = (0, B)\,, \quad A^{\gamma_i}_{kh} = -\begin{pmatrix} D_h B^* D_h \\ 0 \end{pmatrix} & \\ & \text{with } B \in \mathbb{C}[J_{n_{ih}}] \quad \text{for } 1 \leq h < k \leq r_i\,. \end{cases}$$

In particular, the dimension of the Lie subalgebra of $\mathfrak{gl}_{\mathbb{C}}(V^{\gamma_i})$ obtained by restricting to V^{γ_i} the ρ_{-1} representation of $\ker \rho_{-2}$ has real dimension

$$(2r_i - 1)n_{i1} + (2r_i - 3)n_{i2} + \dots + 3n_{i,r_i-1} + n_{i,r_i}\,.$$

Let $n_I = \dim_{\mathbb{C}} V^{\Gamma_I} = \dim_{\mathbb{C}} V^{\bar{\Gamma}_I}$ and $n_{I1} \leq n_{I2} \leq \dots \leq n_{Ir_I}$ be the dimension of the subspaces of a Jordan decomposition of V^{Γ_I} (resp. $V^{\bar{\Gamma}_I}$) for the restriction of L. In this case we obtain

$$A^{\bar{\Gamma}_I} = -(A^{\Gamma_I})^*$$

with $A^{\Gamma_I} = (A^{\Gamma_I}_{hk})_{1 \leq h,k \leq r_I}$ satisfying

(5.12)

$$\begin{cases} A^{\Gamma_I}_{hh} \in \mathbb{C}[J_{n_{Ih}}] & \text{for } h = 1, \dots, r_I \\ A^{\Gamma_I}_{hk} = (0, B) & \text{with } B \in \mathbb{C}[J_{n_{Ih}}] \quad \text{for } 1 \leq h < k \leq r_I \\ A^{\Gamma_I}_{kh} = \begin{pmatrix} B \\ 0 \end{pmatrix} & \text{with } B \in \mathbb{C}[J_{n_{Ik}}] \quad \text{for } 1 \leq k < h \leq r_I\,. \end{cases}$$

The dimension of the real Lie subalgebra of $\mathfrak{gl}_{\mathbb{C}}(V^{\Gamma_I} \oplus V^{\bar{\Gamma}_I})$ obtained by restricting to $V^{\Gamma_I} \oplus V^{\bar{\Gamma}_I}$ the ρ_{-1} representation of $\ker \rho_{-2}$ is

$$2\left((2r_I - 1)n_{I1} + (2r_I - 3)n_{I2} + \cdots + 3n_{I,r_I-1} + n_{I,r_I}\right).$$

LEMMA 5.2.3 *Let (F_1, F_2) be a pair of independent Hermitian symmetric matrices with F_1 nonsingular. Let $\Sigma = \{\gamma_1, \dots, \gamma_l, \Gamma_1, \bar{\Gamma}_1, \dots, \Gamma_L, \bar{\Gamma}_L\}$ be the relative spectrum. Then the solutions $A \in \mathbf{GL}(n, \mathbb{C})$ of the homogeneous system (5.3) are block diagonal matrices $(A^{\gamma_1}, \dots, A^{\gamma_l}, A^{\Gamma_1}, \dots, A^{\Gamma_L})$.*

Assume that (F_1, F_2) is given in canonical form. Then the A^{γ_i}'s are block matrices $(A^{\gamma_i}_{hk})$ solutions of the system

$$\begin{cases} (A^{\gamma_i}_{hk})^* = -D^{\gamma_i}_k A^{\gamma_i}_{kh} D^{\gamma_i}_h \\ J^{\gamma_i}_h A^{\gamma_i}_{hk} - A^{\gamma_k}_{hk} J^{\gamma_i}_k = 0\,, \end{cases} \tag{5.13}$$

where $J^{\gamma_i}_h$ and $D^{\gamma_i}_h$ are J_ℓ and D_ℓ matrices of suitable size.

In particular: $A^{\gamma_i}_{hh} \in \sqrt{-1}\mathbb{R}[J^{\gamma_i}_h]$ (see, for instance, [18, vol.II, p.107] or [21, § 64]).

The A^{Γ_I}'s are block matrices $A^{\Gamma_I}_{hk}$ such that

$$A^{\Gamma_I}_{hk} = \begin{pmatrix} A'^{\Gamma_I}_{hk} & \\ & A''^{\Gamma_I}_{hk} \end{pmatrix}$$

with $A^{\Gamma_I}_{hk}$ solutions of the system

$$\begin{cases} (A'^{\Gamma_I}_{hk})^* = -D^{\Gamma_I}_k A''^{\Gamma_I}_{kh} D^{\Gamma_I}_h \\ J^{\Gamma_I}_h A'^{\Gamma_I}_{hk} - A'^{\Gamma_I}_{hk} J^{\Gamma_I}_k = 0 \\ J^{\Gamma_I}_h A''^{\Gamma_I}_{hk} - A''^{\Gamma_I}_{hk} J^{\Gamma_I}_k = 0\,, \end{cases} \tag{5.14}$$

where $J^{\Gamma_I}_h$ and $D^{\Gamma_I}_h$ are J_ℓ and D_ℓ matrices of suitable size.

In particular: $A'^{\Gamma_I}_{hh} = -\overline{A''^{\Gamma_I}_{hh}} \in \mathbb{C}[J^{\Gamma_I}_h]$.

Assume now that F is singular, i.e. all forms in F are degenerate. We consider a biorthogonal decomposition

$$V = U \oplus W \tag{5.15}$$

where U is the direct sum of the subspaces of the decomposition (5.1) corresponding to blocks of type (IV) and W is the direct sum of the subspaces of

the decomposition (5.1) corresponding to blocks of types (I), (II) and (III). Accordingly, we write:

$$F_1 = \begin{pmatrix} F_1^U & \\ & F_1^W \end{pmatrix} \quad \text{and} \quad F_2 = \begin{pmatrix} F_2^U & \\ & F_2^W \end{pmatrix}.$$

and decompose the matrices in $\mathfrak{gl}(n, \mathbb{C})$ as:

$$A = \begin{pmatrix} A^{UU} & A^{UW} \\ A^{WU} & A^{WW} \end{pmatrix}.$$

Note that (5.9) yields

$$\begin{cases} (A^{WW})^* F_1^W + F_1^W A^{WW} = 0 \\ (A^{WW})^* F_2^W + F_2^W A^{WW} = 0\,. \end{cases} \tag{5.16}$$

In particular, Lemma 5.2.3 applies to describe the structure of A^{WW}. For the other pieces, we have the equations:

$$\begin{cases} (A^{UU})^* F_1^U + F_1^U A^{UU} = 0 \\ (A^{UU})^* F_2^U + F_2^U A^{UU} = 0 \end{cases} \tag{5.17}$$

and

$$\begin{cases} (A^{WU})^* F_1^W + F_1^U A^{UW} = 0 \\ (A^{WU})^* F_2^W + F_2^U A^{UW} = 0\,. \end{cases} \tag{5.18}$$

We solve (5.17) in the case where there is a unique $(2\mathcal{E}-1) \times (2\mathcal{E}-1)$ block of type (IV). Let

$$Q_1 = Q_1(\mathcal{E}) = \begin{pmatrix} I_{\mathcal{E}-1} \\ 0 \end{pmatrix}, \quad Q_2 = Q_2(\mathcal{E}) = \begin{pmatrix} 0 \\ I_{\mathcal{E}-1} \end{pmatrix}.$$

Note that

$$\begin{aligned} &Q_i^* Q_i = I_{\mathcal{E}-1}, \\ &Q_1 Q_1^* = \begin{pmatrix} I_{\mathcal{E}-1} & 0 \\ 0 & 0 \end{pmatrix}, \quad Q_2 Q_2^* = \begin{pmatrix} 0 & 0 \\ 0 & I_{\mathcal{E}-1} \end{pmatrix} \\ &Q_1 Q_2^* = J_{\mathcal{E}}, \qquad\qquad Q_1^* Q_2 = J_{\mathcal{E}-1}\,. \end{aligned}$$

If $A^{UU} = \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix}$, we obtain

$$\begin{cases} \alpha_{11}^* Q_i + Q_i \alpha_{22} = 0 \\ \alpha_{21}^* Q_i^* + Q_i \alpha_{21} = 0 \\ \alpha_{12}^* Q_i + Q_i^* \alpha_{12} = 0 \,. \end{cases}$$

We have, from the first equation equation:

$$Q_i^* \alpha_{11}^* Q_i + \alpha_{22} = 0 \qquad \text{for } i = 1, 2.$$

This implies that α_{11}^* is a matrix having terms which are constant on the diagonals. On the other hand $\alpha_{11}^* Q_1 + Q_1 \alpha_{22} = 0$ implies that all terms in the last row of α_{11}^* which are not on the principal diagonal are zero and $\alpha_{11}^* Q_2 + Q_2 \alpha_{22} = 0$ implies that all terms in the first row of α_{11}^* which are not on the principal diagonal are zero. This shows that $\alpha_{11} = \lambda I_\mathcal{E}$ for a complex λ and therefore $\alpha_{22} = -\bar{\lambda} I_{\mathcal{E}-1}$. The equation $\alpha_{21}^* Q_i^* + Q_i \alpha_{21} = 0$ implies that

$$\alpha_{21}^* + Q_i \alpha_{21} Q_i = 0.$$

After deducing from this equation that all columns of α_{21} are equal, we obtain $\alpha_{21} = 0$.

Finally, the third equation shows that both $Q_1^* \alpha_{12}$ and $Q_2^* \alpha_{12}$ are Hermitian antisymmetric. This gives α_{12} in the form

(5.19)

$$\begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \cdots & \alpha_{\mathcal{E}-1} \\ \alpha_2 & \alpha_3 & & \alpha_{\mathcal{E}-1} & \alpha_\mathcal{E} \\ \alpha_3 & & \iddots & \alpha_\mathcal{E} & \\ \vdots & \alpha_{\mathcal{E}-1} & \iddots & & \vdots \\ \alpha_{\mathcal{E}-1} & \alpha_\mathcal{E} & & & \alpha_{2\mathcal{E}-3} \\ \alpha_\mathcal{E} & & \cdots & \alpha_{2\mathcal{E}-3} & \alpha_{2\mathcal{E}-2} \end{pmatrix} \quad \text{with } \alpha_1, \ldots, \alpha_{2\mathcal{E}-2} \in \sqrt{-1}\mathbb{R}.$$

Therefore the Lie algebra of matrices satisfying (5.17) is in this case the algebra of matrices of the form

$$\begin{pmatrix} \lambda I_\mathcal{E} & \alpha \\ 0 & -\bar{\lambda} I_{\mathcal{E}-1} \end{pmatrix} \tag{5.20}$$

with α as in (5.19), and is a solvable real Lie algebra of dimension $2\mathcal{E}$.

In case there are several blocks of type (IV), organized in such a way that $\mathcal{E}_1 \leq \mathcal{E}_2 \leq \ldots \leq \mathcal{E}_s$, the analogue of Frobenius formula yields an algebra of real dimension

$$2\left((2s-1)\mathcal{E}_1 + (2s-3)\mathcal{E}_2 + \cdots + 3\mathcal{E}_{s-1} + \mathcal{E}_s\right).$$

5.3 Computation of $\mathfrak{g}_1$

Before computing the $\mathfrak{g}_1 = \mathfrak{g}_1(F)$ term of the canonical pseudocomplex prolongation of an $\mathfrak{m}(F)$ of type $(n, 2)$, we give some remarks valid for general Levi-Tanaka algebras of the second kind.

An element of $\mathfrak{g}_1$ is described by the datum of two $\mathbb{R}$-linear maps (B, D), $B : \mathfrak{g}_{-1} \to \mathfrak{g}_0$ and $D : \mathfrak{g}_{-2} \to \mathfrak{g}_{-1}$ such that

$$B(X)Y - B(Y)X = D[X, Y] \qquad \forall X, Y \in \mathfrak{g}_{-1} \tag{5.21}$$

$$[DT, X] = B(X)T \qquad \forall X \in \mathfrak{g}_{-1}, \forall T \in \mathfrak{g}_{-2}. \tag{5.22}$$

When the Levi form is nondegenerate, the homomorphism D completely determines the corresponding element of $\mathfrak{g}_1$. To make this observation more precise, we prove:

LEMMA 5.3.1 *Let (B, D) be the pair of $\mathbb{R}$-linear homomorphisms associated to an element of $\mathfrak{g}_1$. Then*

$$\begin{aligned} 2[B(X)Y, Z] &= [D[Y, Z] + JD[JY, Z], X] + \\ &+ [D[X, Z] + JD[JX, Z], Y] + [D[X, Y] + JD[JX, Y], Z] \end{aligned} \tag{5.23}$$

for every $X, Y, Z \in \mathfrak{g}_{-1}$.

Proof. For $X, Y, Z \in \mathfrak{g}_{-1}$ we obtain, since $B(X) \in \mathfrak{g}_0$ and (5.22) holds true

$$[B(X)Y, Z] + [Y, B(X)Z] = B(X)[Y, Z] = [D[Y, Z], X].$$

Thus we have:

$$
\begin{aligned}
&[B(X)Y, JZ] = [B(Y)X, JZ] + [D[X,Y], JZ] \\
&= -[X, B(Y)JZ] + [D[X, JZ], Y] + [D[X,Y], JZ] \\
&= [JX, B(Z)Y] + [JX, D[Y,Z]] + [D[X, JZ], Y] + [D[X,Y], JZ] \\
&= [B(Z)X, JY] + [D[JX,Y], Z] + [JD[Y,Z], X] + \\
&\quad +[D[X, JZ], Y] + [D[X,Y], JZ] \\
&= [B(X)Z, JY] + [D[Z,X] + JY] + [D[JX,Y], Z] + \\
&\quad +[JD[Y,Z], X] + [D[X, JZ], Y] + [D[X,Y], JZ] \\
&= -[Z, B(X)JY] + [D[Z, JY], X] + +[D[Z,X] + JY] + [D[JX,Y], Z] + \\
&\quad +[JD[Y,Z], X] + [D[X, JZ], Y] + [D[X,Y], JZ]\,.
\end{aligned}
$$

We note that

$$
-[Z, B(X)JY] + [JZ, B(X)Y] = -[B(X)Y, JZ]
$$

and hence

$$
\begin{aligned}
2\,[B(X)Y, JZ] &= [D[Y, JZ] + JD[JY, JZ], X] + \\
&\quad +[D[X, JZ] + JD[JX, JZ], Y] + \\
&\quad +[D[X,Y] + JD[JX,Y], JZ]\,,
\end{aligned}
$$

from which (5.23) follows. □

We still denote by $D : \mathbb{C} \otimes_{\mathbb{R}} \mathfrak{g}_{-2} \to \mathfrak{g}_{-1}$ the $\mathbb{C}$-linear extension of D. Then we have

$$
\begin{aligned}
D[Y,Z] + JD[JY,Z] &= J(D[JY,Z] + JD[Z,Y]) \\
&= JD([JZ,Y] + \sqrt{-1}[Z,Y]) = JD\mathfrak{f}(Z,Y),
\end{aligned}
$$

where $\mathfrak{f} : \mathfrak{g}_{-1} \times \mathfrak{g}_{-1} \to \mathbb{C} \otimes_{\mathbb{R}} \mathfrak{g}_{-2}$ is the Hermitian symmetric form associated to the vector valued Levi form.

Formula (5.23) for the pair (B, D) of $\mathbb{R}$-linear homomorphisms associated to an element of $\mathfrak{g}_1$ can be written in the form

$$
\begin{aligned}
(5.24) \qquad & 2\,[B(X)Y, Z] = \\
& [JD\mathfrak{f}(Z,Y), X] + [JD\mathfrak{f}(Z,X), Y] + [JD\mathfrak{f}(Y,X), Z]
\end{aligned}
$$

for every $X, Y, Z \in \mathfrak{g}_{-1}$. We have the following:

PROPOSITION 5.3.2 *Let* $B \in \mathrm{Hom}_{\mathbb{R}}(\mathfrak{g}_{-1}, \mathrm{Hom}_{\mathbb{R}}(\mathfrak{g}_{-1}, \mathfrak{g}_{-1}))$ *and* $D \in \mathrm{Hom}_{\mathbb{R}}(\mathfrak{g}_{-2}, \mathfrak{g}_{-1})$ *be two* $\mathbb{R}$*-linear maps such that (5.24) is valid for every* $X, Y, Z \in \mathfrak{g}_{-1}$. *Then* $B(X) \in \rho_{-1}(\mathfrak{g}_0)$ *for every* $X \in \mathfrak{g}_{-1}$ *and there is a unique* $\xi \in \mathfrak{g}_1$ *such that:*

$$\begin{cases} [\xi, X] = B(X) & \forall X \in \mathfrak{g}_{-1} \\ [\xi, T] = D(T) & \forall T \in \mathfrak{g}_{-2}. \end{cases} \tag{5.25}$$

Proof. From (5.24) we deduce

$$\begin{aligned} &2\mathfrak{f}(B(X)Y, Z) = \\ &\mathfrak{f}(JX, D\mathfrak{f}(Z, Y)) + \mathfrak{f}(JY, D\mathfrak{f}(Z, X)) + \mathfrak{f}(JD\mathfrak{f}(Y, X), Z). \end{aligned} \tag{5.26}$$

We note that the right-hand side is $\mathbb{C}$-linear in Y and anti-$\mathbb{C}$-linear in Z and this shows that $B(X)$ is $\mathbb{C}$-linear on $\mathfrak{g}_{-1}$ because $\mathfrak{f}$ is nondegenerate. Next we show that $B(X) \in \mathfrak{g}_0$ for every $X \in \mathfrak{g}_{-1}$. Indeed we note that the last two summands in the right-hand side of (5.23) are interchanged when we interchange Y and Z and that also $[JY, Z] = [JZ, Y]$. Therefore we obtain

$$\begin{aligned} [B(X)Y, Z] + [Y, B(X)Z] &= [B(X)Y, Z] - [B(X)Z, Y] \\ &= \frac{1}{2}\left([D[Y, Z], X] - [D[Z, Y], X]\right) \\ &= [D[Y, Z], X] \qquad \forall Y, Z \in \mathfrak{g}_{-1}. \end{aligned}$$

This shows that $B(X) \in \mathfrak{g}_0$ and that (5.22) holds true. To prove (5.21), we note that the first two summands in the right-hand side of (5.23) are interchanged when we interchange X and Y and that also $[JX, Y]$ is symmetric with respect to X and Y. Therefore

$$\begin{aligned} [B(X)Y - B(Y)X, Z] &= \frac{1}{2}\left([D[X, Y], Z] - [D[Y, X], Z]\right) \\ &= [D[X, Y], Z] \qquad \forall X, Y, Z \in \mathfrak{g}_{-1}. \end{aligned}$$

The proof is complete. □

Let $k = \dim_{\mathbb{R}} \mathfrak{g}_{-2}$ and let us fix a basis $T_1, \ldots, T_k$ of $\mathfrak{g}_{-2}$ in such a way that the scalar components $\mathfrak{f}_1, \ldots, \mathfrak{f}_k$ of the vector valued Levi form $\mathfrak{f}$ have

been selected. Set $JDT_j = \xi_j \in \mathfrak{g}_{-1}$ for $j = 1, \dots, k$. Then $JD\mathfrak{f}(X,Y) = \sum \xi_j \mathfrak{f}_j(X,Y)$ and we deduce from (5.26) that

$$(5.27) \qquad 2\mathfrak{f}_j(B(X)Y,Z) = \\ -\sum_h \mathfrak{f}_j(X,\xi_h)\mathfrak{f}_h(Y,Z) - \sum_h \mathfrak{f}_j(Y,\xi_h)\mathfrak{f}_h(X,Z) + \sum_h \mathfrak{f}_h(Y,X)\mathfrak{f}_j(\xi_h,Z).$$

If we assume that $\mathfrak{f}_1$ is nondegenerate, then there are $\mathfrak{f}_1$-self-adjoint $\mathbb{C}$-linear maps $L_2, \dots, L_k : \mathfrak{g}_{-1} \to \mathfrak{g}_{-1}$ with:

$$\mathfrak{f}_j(X,Y) = \mathfrak{f}_1(L_jX,Y) \qquad \forall X,Y \in \mathfrak{g}_{-1},\ j = 2, \dots, k.$$

Then the equality (5.27) yields, for $j = 1, \dots, k$ and $L_1 = id$:

$$2L_jB(X)Y = -\sum_h \mathfrak{f}_j(X,\xi_h)L_hY - \sum_h \mathfrak{f}_j(Y,\xi_h)L_hX + \sum_h \mathfrak{f}_h(Y,X)L_j\xi_h\,.$$

In this case we obtain the following criterion:

PROPOSITION 5.3.3 *Assume that F is nonsingular. A necessary and sufficient condition in order that D be the homomorphism $D : \mathfrak{g}_{-2} \to \mathfrak{g}_{-1}$ associated to an element of $\mathfrak{g}_1$ is that, for $\xi_j = JDT_j$, $j = 1, \dots, k$, we have:*

$$(5.28) \qquad \sum_h \mathfrak{f}_1(X,\xi_h)L_jL_hY + \sum_h \mathfrak{f}_1(X,\xi_h)L_jL_hX = \\ \sum_h \mathfrak{f}_j(X,\xi_h)L_hY + \sum_h \mathfrak{f}_j(X,\xi_h)L_hX \qquad \forall X,Y \in \mathfrak{g}_{-1}\,.$$

After the preliminaries, we turn to the discussion of the special case where $\dim_{\mathbb{R}} F = 2$. The situation of a nonsingular F and that of a singular F are completely different: in case F is singular, the component $\mathfrak{g}_1(F)$ is completely determined by the singular part in the decomposition (5.1).

We begin by describing $\mathfrak{g}_1 = \mathfrak{g}_1(F)$ in the nonsingular case. We assume that $\mathfrak{f}_1$ is nondegenerate and we write $L = L_2$. Using Proposition 5.3.3, a necessary and sufficient condition for a pair $\xi_1, \xi_2 \in \mathfrak{g}_{-1}$ to define an element of $\mathfrak{g}_1$ is that the equation

$$(5.29) \qquad \mathfrak{f}_1(X,\xi_1)LY + \mathfrak{f}_1(X,\xi_2)L^2Y + \mathfrak{f}_1(Y,\xi_1)LX + \mathfrak{f}_1(Y,\xi_2)L^2X = \\ \mathfrak{f}_1(LX,\xi_1)Y + \mathfrak{f}_1(LX,\xi_2)LY + \mathfrak{f}_1(LY,\xi_1)X + \mathfrak{f}_1(LY,\xi_2)LX$$

be satisfied for every $X, Y \in \mathfrak{g}_{-1}$.

We consider several cases, depending on the spectrum Σ of $(\mathfrak{f}_1, \mathfrak{f}_2)$.

(1): $\Sigma = \{\gamma\}$, $\gamma \in \mathbb{R}$.
We can assume $\gamma = 0$. Note that, as $\mathfrak{f}_1$ and $\mathfrak{f}_2$ are linearly independent, at least one of the subspaces V_i of the biorthogonal decomposition (5.1) has dimension greater than or equal to 2. We distinguish two subcases:
(a). $\dim V_i \leq 2$ for all subspaces V_i in the biorthogonal decomposition (5.1).

In these cases all pairs (ξ_1, ξ_2) with $\xi_1 = L\xi_2$ for an arbitrary $\xi_2 \in V$ define solutions of (5.29). Thus $\mathfrak{g}_1$ is not zero and $\dim_{\mathbb{C}} \mathfrak{g}_1 = \dim_{\mathbb{C}} \mathfrak{g}_{-1}$. When $n = \dim_{\mathbb{C}} \mathfrak{g}_{-1} = 2$, the prolongation $\mathfrak{g}$ is described in the example in 3.8.5.
(b). At least one of the subspaces V_i in the biorthogonal decomposition (5.1) has dimension greater than or equal to 3.

In these cases (5.29) has only the trivial solution $\xi_1 = \xi_2 = 0$, so that $\mathfrak{g}_1 = 0$.

Let $V = V' \oplus V''$ where V' is one of the subspaces V_i, having dimension greater than or equal to 3 and V'' is the direct sum of the remaining subspaces in the decomposition (5.1). Fixing $Y \in V'$ with $L^2Y \neq 0$, we obtain for every $X \in V''$:

$$\begin{cases} \mathfrak{f}_1(LX, \xi_1) = 0 \\ \mathfrak{f}_1(X, \xi_1) = \mathfrak{f}_1(LX, \xi_2) \\ \mathfrak{f}_1(X, \xi_2) = 0\,. \end{cases}$$

The first and the third equations imply that $\xi_1, \xi_2 \in V'$.

Considering $X = Y \in V'$ in (5.29), we obtain:

$$\mathfrak{f}_1(X, \xi_1)LX + \mathfrak{f}_1(X, \xi_2)L^2X = \mathfrak{f}_1(LX, \xi_1)X + \mathfrak{f}_1(LX, \xi_2)LX\,.$$

For X in an open dense subset of V', the vectors X, LX, L^2X are linearly independent. Therefore we obtain:

$$\begin{cases} \mathfrak{f}_1(LX, \xi_1) = 0 \\ \mathfrak{f}_1(X, \xi_1) = \mathfrak{f}_1(LX, \xi_2) \\ \mathfrak{f}_1(X, \xi_2) = 0\,. \end{cases}$$

Since $\xi_1, \xi_2, L\xi_1, L\xi_2 \in V'$, we obtain that $\xi_1 = \xi_2 = 0$. Thus $\mathfrak{g}_1 = 0$ and $\mathfrak{g}(F)$ is solvable in this case.

(2): $\Sigma = \{\gamma_1, \gamma_2\}$, $\gamma_1, \gamma_2 \in \mathbb{R}$, $\gamma_1 \neq \gamma_2$.
We have two cases:
(a). If all subspaces V_i of the biorthogonal decomposition (5.1) have dimension one, then the algebra $\mathfrak{m} = \mathfrak{m}_{-2} \oplus \mathfrak{m}_{-1}$ decomposes into a direct sum of two pseudocomplex graded ideals of codimension one and its canonical pseudocomplex prolongation is isomorphic to $\mathfrak{su}(1+p,1) \oplus \mathfrak{su}(1+q,1)$, where p and q are the dimensions of the eigenspaces relative respectively to γ_1 and γ_2.
(b). Assume now that at least one of the subspaces V_i has dimension greater than or equal to 2. We can as well assume that this subspace is contained in V^{γ_1} and moreover that $\gamma_1 = 0$, $\gamma_2 = 1$. We use (5.29) to obtain

$$\begin{cases} L\xi_1 = 0 \\ (\xi_1 - L\xi_2) + \xi_2 = 0\,. \end{cases} \tag{5.30}$$

From

$$\mathfrak{f}_1(X, \xi_1 - L\xi_2) = 0$$

for every X such that $LX \neq 0$, we obtain $\xi_1 = L\xi_2$, because the set of the X with $LX \neq 0$ is a set of generators of V and $\mathfrak{f}_1$ is assumed to be nondegenerate. The second equation of (5.30) allows us to conclude that $\xi_1 = \xi_2 = 0$. Hence, in this case $\mathfrak{g}(F)$ is solvable.

(2′): $\Sigma = \{\Gamma, \bar{\Gamma}\}$, $\Gamma \in \mathbb{C} \setminus \mathbb{R}$.
Note that the CR-dimension n is even. We have two cases:
(a). If L is semisimple, i.e. all subspaces V_i in (5.1) have dimension 2, then $\mathfrak{g}(F)$ is isomorphic to $\mathfrak{sl}(\frac{n}{2}+2, \mathbb{C})$ with the structure of a Levi-Tanaka algebra described in the example in 3.8.2 (see formula (3.18)).
(b). Assume now that there is a subspace V_i in (5.1) of dimension $2d$, with $d \geq 2$. From (5.29) we obtain

$$\begin{cases} L\xi_1 - \xi_2 + \sqrt{-1}(\xi_1 + L\xi_2) = 0 \\ \xi_1 + L\xi_2 + 2\sqrt{-1}\xi_2 = 0\,. \end{cases}$$

Then

$$\begin{cases} L\xi_1 + \xi_2 = 0 \\ L\xi_2 + \xi_1 = 0 \end{cases}$$

and therefore $L^2\xi_1 = \xi_1$. Since L has no real eigenvalues, we conclude that $\xi_1 = \xi_2 = 0$. Hence $\mathfrak{g}_1 = 0$ and $\mathfrak{g}(F)$ is solvable.

(3): Σ contains at least three distinct elements $\gamma_1, \gamma_2, \gamma_3 \in \mathbb{C}$.
In this case (5.29) gives us

$$\begin{cases} -L\xi_1 + \gamma_1(\xi_1 - L\xi_2) + \gamma_1^2\xi_2 = 0 \\ -L\xi_1 + \gamma_2(\xi_1 - L\xi_2) + \gamma_2^2\xi_2 = 0 \\ -L\xi_1 + \gamma_3(\xi_1 - L\xi_2) + \gamma_3^2\xi_2 = 0\,. \end{cases}$$

Then $\xi_1 = \xi_2 = 0$ and so $\mathfrak{g}_1 = 0$. We note that we would obtain the same conclusion using Theorem 3.4.1 because $\rho_{-2}(\mathfrak{g}_0)$ operates on $\mathfrak{g}_{-2}$ as multiple of the identity (see 5.2.1). Therefore also in this case $\mathfrak{g}(F)$ is solvable.

Next we take up the case in which all Hermitian symmetric forms in F are degenerate. We fix a basis $\mathfrak{f}_1, \mathfrak{f}_2$ of F with $\mathfrak{f}_1$ and $\mathfrak{f}_2$ both of maximal rank, and denote by N_i, for $i = 1, 2$, the subspaces of V:

$$N_i = \{X \in V \mid \mathfrak{f}_i(X, Y) = 0, \quad \forall Y \in V\}\,.$$

We decompose V into a direct sum

$$V = U \oplus W$$

where U and W are orthogonal with respect to all elements of F, while the canonical form of the restriction of the pair $(\mathfrak{f}_1, \mathfrak{f}_2)$ to U is a direct sum of blocks of type (IV), whereas the restriction of $(\mathfrak{f}_1, \mathfrak{f}_2)$ to W leads only to types (I), (II), (III).

In this case, the necessary and sufficient condition in order that the datum of $\xi_1, \xi_2 \in V$ defines an element of $\mathfrak{g}_1$ is that, for some $B \in \mathrm{Hom}(V, \mathfrak{gl}_{\mathbb{C}}(V))$,

$$\begin{aligned} (5.31) \quad 2\mathfrak{f}_j(B(X)Y, Z) \;=\; & -\sum_{h=1}^{2} \mathfrak{f}_j(X, \xi_h)\mathfrak{f}_h(Y, Z) - \sum_{h=1}^{2} \mathfrak{f}_j(Y, \xi_h)\mathfrak{f}_h(X, Z) \\ & + \sum_{h=1}^{2} \mathfrak{f}_h(Y, X)\mathfrak{f}_j(\xi_h, Z) \qquad \text{for } j = 1, 2\,. \end{aligned}$$

Choose $j = 1$ and $Z \in N_1 \setminus \{0\}$. Then (5.31) yields:

$$\mathfrak{f}_1(X, \xi_2)\mathfrak{f}_2(Y, Z) + \mathfrak{f}_1(Y, \xi_2)\mathfrak{f}_2(X, Z) = 0 \quad \forall X, Y \in V\,.$$

Since $\mathfrak{m}(F)$ is nondegenerate, we can fix $X = Y$ in such a way that $\mathfrak{f}_2(X, Z) = \mathfrak{f}_2(Y, Z) \neq 0$. Then we obtain $\mathfrak{f}_1(X, \xi_2) = 0$ for every $X \in V$, i.e. $\xi_2 \in N_1$.

In the same way, we prove that $\xi_1 \in N_2$.

Using these information, we can rewrite (5.31) in the form:

$$\begin{aligned} 2\mathfrak{f}_j(B(X)Y, Z) &= -\mathfrak{f}_j(X, \xi_j)\mathfrak{f}_j(Y, Z) - \mathfrak{f}_j(Y, \xi_j)\mathfrak{f}_j(X, Z) \\ &\quad +\mathfrak{f}_j(Y, X)\mathfrak{f}_j(\xi_j, Z) \qquad \forall X, Y, Z \in V\,, \quad j = 1, 2. \end{aligned}$$

This can be written also as:

$$\begin{cases} 2\mathfrak{f}_1(B(X)Y, Z) = \mathfrak{f}_1(-\mathfrak{f}_1(X, \xi_1)Y - \mathfrak{f}_1(Y, \xi_1)X + \mathfrak{f}_1(Y, X)\xi_1, Z) \\ 2\mathfrak{f}_2(B(X)Y, Z) = \mathfrak{f}_2(-\mathfrak{f}_2(X, \xi_2)Y - \mathfrak{f}_2(Y, \xi_2)X + \mathfrak{f}_2(Y, X)\xi_2, Z) \end{cases}$$

From these relations we obtain that, for suitable bilinear forms $\phi_i : \mathfrak{g}_{-1} \times \mathfrak{g}_{-1} \to N_i$ (i=1,2) which are $\mathbb{C}$-linear in the first and $\mathbb{R}$-linear in the second variable:

(5.32)
$$\begin{cases} 2B(X)Y = -\mathfrak{f}_1(X, \xi_1)Y - \mathfrak{f}_1(Y, \xi_1)X + \mathfrak{f}_1(Y, X)\xi_1 + \phi_1(Y, X) \\ 2B(X)Y = -\mathfrak{f}_2(X, \xi_2)Y - \mathfrak{f}_2(Y, \xi_2)X + \mathfrak{f}_2(Y, X)\xi_2 + \phi_2(Y, X)\,. \end{cases}$$

First we exploit the two equations taking $X = Y$ not belonging to $N_1 \oplus N_2$. This yields the equation:

$$\mathfrak{f}_1(X, \xi_1) = \mathfrak{f}_2(X, \xi_2)$$

which first is valid for all $X \notin N_1 \oplus N_2$ and then, since $V \setminus N_1 \oplus N_2$ is dense in V, we obtain:

$$\mathfrak{f}_1(X, \xi_1) = \mathfrak{f}_2(X, \xi_2) \quad \forall X \in \mathfrak{g}_{-1} = V\,. \tag{5.33}$$

Taking on each subspace V_i of the biorthogonal decomposition (5.1) such that $F|_{V_i}$ is singular a canonical basis

$$Z_1^i, \dots, Z_{2\mathcal{E}_i - 1}^i,$$

we obtain that

$$\begin{cases} \xi_1 = \sum \lambda_i Z_1^i \\ \xi_2 = \sum \mu_i Z_{\mathcal{E}_i}^i \end{cases}$$

for suitable $\lambda_i, \mu_i \in \mathbb{C}$.

If $\mathcal{E}_i > 2$, taking $X = Z^i_{\mathcal{E}_i+1}$ in (5.33) we obtain

$$\bar{\lambda}_i = \mathfrak{f}_2(Z^i_{\mathcal{E}_i+1}, \mu_i Z^i_{\mathcal{E}_i}) = 0$$

and taking $X = Z^i_{2\mathcal{E}_i-1}$ we obtain

$$0 = \mathfrak{f}_1(Z^i_{2\mathcal{E}_i-1}, \lambda_i Z^i_1) = \bar{\mu}_i\,.$$

In the case $\mathcal{E}_i = 2$, by substituting $X = Z_3$ in (5.33) we obtain that $\lambda_i = \mu_i$. Hence

$$\begin{cases} \xi_1 = \sum_{\mathcal{E}_i=2} \lambda_i Z^i_1 \\ \xi_2 = \sum_{\mathcal{E}_i=2} \lambda_i Z^i_2\,. \end{cases}$$

The equations (5.32) give then

$$\mathfrak{f}_1(Y,X)\xi_1 + \phi_1(Y,X) = \mathfrak{f}_2(Y,X)\xi_2 + \phi_2(Y,X) \qquad \forall X, Y \in V$$

and therefore

$$\phi_1(Y,X) = \mathfrak{f}_2(Y,X)\xi_2\,, \qquad \phi_2(Y,X) = \mathfrak{f}_1(Y,X)\xi_1\,.$$

Hence we obtain the expression

$$\begin{aligned} (5.34)\quad B(X) &= \frac{1}{2}\{-\mathfrak{f}_1(X,\xi_1)Y - \mathfrak{f}_1(Y,\xi_1)X + \mathfrak{f}_1(Y,X)\xi_1 + \mathfrak{f}_2(Y,X)\xi_2\} \\ &= \frac{1}{2}\{-\mathfrak{f}_2(X,\xi_2)Y - \mathfrak{f}_2(Y,\xi_2)X + \mathfrak{f}_1(Y,X)\xi_1 + \mathfrak{f}_2(Y,X)\xi_2\} \end{aligned}$$

for every $X, Y \in \mathfrak{g}_{-1} = V$, with

$$\xi_1 = \sum_{\mathcal{E}_i=2} \lambda_i Z^i_1\,, \qquad \xi_2 = \sum_{\mathcal{E}_i=2} \lambda_i Z^i_2\,.$$

It follows that

PROPOSITION 5.3.4 *If F is singular, then $\mathfrak{g}_1(F)$ is a $\mathbb{C}$-vector space whose dimension equals the number of three-dimensional singular subspaces in the decomposition (5.1).*

5.4 Computation of $\mathfrak{g}_2$ in the singular case

Assume that F is singular. Let U be the direct sum of the subspaces V_i in the decomposition (5.1) such that $\dim V_i = 3$ and $F|_{V_i}$ is singular. We consider the restriction $F|_U$ of F to U and

$$\mathfrak{m}(F|_U) \cong \mathfrak{g}_{-2} \oplus U.$$

We have a natural inclusion

$$\mathfrak{g}_0(F|_U) \hookrightarrow \mathfrak{g}_0(F)$$

and an isomorphism

$$\mathfrak{g}_1(F|_U) \cong \mathfrak{g}_1(F)$$

in which

$$\begin{array}{ccc} \mathfrak{g}_{-2} & \xrightarrow{D} & \mathfrak{g}_{-1} = V \\ \| & & \uparrow{\scriptstyle 1-1} \\ \mathfrak{g}_{-2} & \xrightarrow{D|_U} & U \end{array} \qquad \begin{array}{ccc} \mathfrak{g}_{-1} & \xrightarrow{B} & \mathfrak{g}_0 \\ {\scriptstyle 1-1}\uparrow & & \downarrow{\scriptstyle \pi_U^* \circ \pi_{U*}} \\ U & \xrightarrow{B|_U} & \mathfrak{g}_0(F|_U) \end{array}$$

where B and D denote the linear applications given in Proposition 5.3.2.

An element $\eta \in \mathfrak{g}_2(F)$ defines by "restriction" an element of $\mathfrak{g}_2(F|_U)$. Since an element of $\mathfrak{g}_2(F)$ which vanishes on $\mathfrak{g}_{-2}(F) = \mathfrak{g}_{-2}(F|_U)$ is zero by Theorem 2.5.11, the map $\mathfrak{g}_2(F) \longrightarrow \mathfrak{g}_2(F|_U)$ is injective.

We note that $\rho_{-2}(\mathfrak{g}_0(F|_U))$ is irreducible. Then the Levi-Tanaka algebra $\mathfrak{g}(F|_U)$ is either simple or weakly solvable with $\mathfrak{g}_2 = 0$. We already know that all simple Levi-Tanaka algebras of type $(n, 2)$ have a nonsingular F. It follows that $\mathfrak{g}_2(F|_U) = 0$ and therefore $\mathfrak{g}_2(F) = 0$ and $\mathfrak{g}(F)$ is weakly solvable.

We conclude this chapter with the following:

EXAMPLE 5.4.1 *Let $F \subset \mathfrak{H}_5(V)$ be a two dimensional linear space of Hermitian symmetric forms on V such that the fundamental pseudocomplex graded Lie algebra $\mathfrak{m} = \mathfrak{m}(F) = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$ associated to F is nondegenerate. Assume that $\dim_{\mathbb{C}} V = 3$ and F singular, i.e. $[\mu \mathfrak{f}_2 - \lambda \mathfrak{f}_1] = D_2(\mu, \lambda)$. Then we obtain for $\ker \rho_{-2} \subset \mathfrak{g}_0$, as subset of $\mathfrak{gl}(3, \mathbb{C})$, the subalgebra generated by:*

$$\begin{pmatrix} \frac{1}{2} & & \\ & \frac{1}{2} & \\ & & \frac{1}{2} \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & \\ 0 & -1 & \\ & & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & \\ 0 & 0 & \\ & & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & \\ 1 & 0 & \\ & & 0 \end{pmatrix}.$$

Note that the last three matrices generate a subalgebra isomorphic to $\mathfrak{sl}(2,\mathbb{R})$.
The "homogeneous" system provides a subalgebra of $\mathfrak{gl}(3,\mathbb{C})$ *generated by:*

$$\begin{pmatrix} \sqrt{-1} & & \\ & \sqrt{-1} & \\ & & \sqrt{-1} \end{pmatrix}, \begin{pmatrix} 1 & & \\ & 1 & \\ & & -1 \end{pmatrix}, \begin{pmatrix} 0 & & \\ & 0 & \sqrt{-1} \\ & & 0 \end{pmatrix}, \begin{pmatrix} 0 & & \sqrt{-1} \\ & 0 & \\ & & 0 \end{pmatrix}.$$

In particular, $\dim_{\mathbb{R}} \mathfrak{g}_0 = 8$. *Moreover,* $\dim_{\mathbb{C}} \mathfrak{g}_1 = 1$ *and* $\mathfrak{g}_2 = 0$.
Note that in this case $\mathfrak{g}$ *has a pseudocomplex Levi factor not contained in* $\mathfrak{g}_0$.

Chapter 6

Homogeneous CR manifolds

6.1 Standard homogeneous CR manifolds

Let $\mathfrak{g} = \oplus_{p\in\mathbb{Z}}\mathfrak{g}_p$ be a finite dimensional Levi-Tanaka algebra. In this section we construct homogeneous CR manifolds $\mathbf{M} = (M, HM, J)$ having at each point $x \in M$ a Levi-Tanaka algebra $\mathfrak{g}(x)$ isomorphic to $\mathfrak{g}$ and such that the group of CR automorphisms of $\mathbf{M}$ is a Lie group with Lie algebra isomorphic to $\mathfrak{g}$.

Let us set

$$\mathfrak{m} = \oplus_{p<0}\mathfrak{g}_p \quad \text{and} \quad \mathfrak{g}_+ = \oplus_{p\geq 0}\mathfrak{g}_p.$$

We denote by $\mathbf{G}$ a connected and simply connected Lie group with Lie algebra $\mathfrak{g}$. We note that $\mathfrak{g}_+$ is a Lie subalgebra of $\mathfrak{g}$ and therefore generates a connected Lie subgroup $\mathbf{G}_+$ of $\mathbf{G}$.

LEMMA 6.1.1 *The subgroup $\mathbf{G}_+$ is a closed in $\mathbf{G}$.*

Proof. Let

$$\mathrm{Ad} : \mathbf{G} \to \mathbf{GL}(\mathfrak{g})$$

denote the adjoint representation of $\mathbf{G}$. Then

$$\mathbf{H} = \{g \in \mathbf{G} \mid \mathrm{Ad}(g)(\mathfrak{g}_+) = \mathfrak{g}_+\}$$

is a closed subgroup of $\mathbf{G}$ and hence a Lie subgroup of $\mathbf{G}$. Clearly the Lie algebra of $\mathbf{H}$ is $\mathfrak{g}_+$ and then $\mathbf{G}_+$, being the connected component of the identity in $\mathbf{H}$, is closed in $\mathbf{G}$. □

We identify $\mathfrak{g}$ to the Lie algebra of left invariant vector fields on $\mathbf{G}$. For $-\mu \leq p \leq 0$ we set $\mathfrak{g}_{(p)} = \oplus_{q \geq p} \mathfrak{g}_q$ and denote by $\tilde{\mathfrak{g}}_{(p)}$ the vector distribution generated by $\mathfrak{g}_{(p)}$. For $g \in \mathbf{G}$, we denote by L_g and R_g respectively the left and right translations with respect to g.

LEMMA 6.1.2 *For every $-\mu \leq p \leq 0$ the vector distribution $\tilde{\mathfrak{g}}_{(p)}$ is invariant with respect to left translations by elements of $\mathbf{G}$ and right translations by elements of $\mathbf{G}_+$.*

Proof. The invariance under $(L_g)_*$ for $g \in \mathbf{G}$ is obvious. For $X \in \mathfrak{g}$ and $g \in \mathbf{G}$, we have:

$$(R_{g^{-1}})_*(X) = \mathrm{Ad}(g)(X).$$

Since

$$\mathrm{ad}_{\mathfrak{g}}(Y)(X) = [Y, X] \in \mathfrak{g}_{(p)} \quad \forall X \in \mathfrak{g}_{(p)},\ Y \in \mathfrak{g}_+,$$

the Lie algebra of the Lie subgroup $\mathbf{A}$ of the elements $g \in \mathbf{G}$ such that $(R_g)_*(\mathfrak{g}_{(p)}) \subset \mathfrak{g}_{(p)}$ contains $\mathfrak{g}_+$. Hence $\mathbf{G}_+ \subset \mathbf{A}$ because $\mathbf{G}_+$ is connected. □

Using these lemmas we obtain:

THEOREM 6.1.3 *The homogeneous space $M = \mathbf{G}/\mathbf{G}_+$ is a simply connected real-analytic manifold. We can endow M with a natural CR structure, in such a way that $\mathbf{G}$ acts on M as a group of CR automorphisms and the Levi-Tanaka algebra $\mathfrak{g}(x)$ of M at every point x of M is isomorphic to $\mathfrak{g}$.*

Proof. Since $\mathbf{G}_+$ is a closed subgroup of $\mathbf{G}$, the homogeneous space $M = \mathbf{G}/\mathbf{G}_+$ is a real-analytic manifold, on which the elements of $\mathbf{G}$ define real-analytic diffeomorphisms. Moreover, M is simply connected because $\mathbf{G}$ is simply connected and $\mathbf{G}_+$ is connected.

Let us describe the CR structure of M. We denote by $\pi : \mathbf{G} \to M$ the natural projection, and by $\mathbf{G} \times M \ni (g, x) \to g \cdot x \in M$ the left action of $\mathbf{G}$ on M. Let $\tilde{\mathfrak{g}}_{-1}$, $\tilde{\mathfrak{g}}_+ = \tilde{\mathfrak{g}}_{(0)}$ and $\tilde{\mathfrak{g}}_{(-1)}$ denote the vector distribution generated respectively by $\mathfrak{g}_{-1}$, $\mathfrak{g}_+$ and $\mathfrak{g}_{(-1)} = \oplus_{p \geq -1} \mathfrak{g}_p$. They are all invariant by left translations and $\tilde{\mathfrak{g}}_+$ is the vertical distribution of the $\mathbf{G}_+$-principal bundle $\mathbf{G} \xrightarrow{\pi} M$.

Let $o = \pi(e)$ be the image of the identity of $\mathbf{G}$ in M and $H_oM = \pi_*((\tilde{\mathfrak{g}}_{-1})_e)$. If $h \in \mathbf{G}_+$, we have

$$\pi_*((\tilde{\mathfrak{g}}_{-1})_h) = H_oM.$$

Indeed, since $\pi \circ R_{h^{-1}} = \pi$ for $h \in \mathbf{G}_+$, we obtain:

$$\begin{aligned}
\pi_*\left((\tilde{\mathfrak{g}}_{-1})_h\right) &= \pi_*\left(\left(\tilde{\mathfrak{g}}_{(-1)}\right)_h\right) \\
&= \pi_* \circ (R_{h^{-1}})_*\left((\tilde{\mathfrak{g}}_{(-1)})_h\right) \\
&= \pi_*((\tilde{\mathfrak{g}}_{(-1)})_e) = \pi_*((\tilde{\mathfrak{g}}_{-1})_e) = H_o M.
\end{aligned}$$

This implies that

$$H_{\pi(g)}M = g_* H_o M$$

is well defined at all points of M and is invariant by the action of $\mathbf{G}$ on M.

If X_x^* is in $H_x M$ and $g \in \mathbf{G}$ is such that $x = g \cdot o$, then we can find a unique $X \in \mathfrak{g}_{-1}$ such that $X_x^* = g_* \pi_*(X_e)$. We want to define the partial complex structure J_M of M in such a way that

$$J_M X_x^* = g_* \pi_*(J X_e).$$

This would imply also that $M \ni x \to g \cdot x \in M$ is a CR diffeomorphism for every $g \in \mathbf{G}$.

To this aim, we only need to show that the definition is consistent, i.e. that, if γ is another element of $\mathbf{G}$ such that $\gamma \cdot o = x$ and $Y \in \mathfrak{g}_{-1}$ is such that $\gamma_* \pi_*(Y_e) = X_x^*$, then

$$\gamma_* \pi_*(J Y_e) = g_* \pi_*(J X_e).$$

We note that $\gamma^{-1} g \in \mathbf{G}_+$ and thus we are reduced to show that

$$\pi_*(\mathrm{Ad}(h)(J X_e)) = \pi_*(J Y_e) \tag{6.1}$$

if $h \in \mathbf{G}_+$, $X, Y \in \tilde{\mathfrak{g}}_{-1}$, and $Y - \mathrm{Ad}(h)X \in \tilde{\mathfrak{g}}_+$. Let $H = \sum_{p \geq 0} H_p \in \mathfrak{g}_+$, expressed as a sum of its homogeneous components. Then we have $\mathrm{Ad}(\exp(tH))X - \mathrm{Ad}(\exp(tH_0))X \in \mathfrak{g}_+$ for $X \in \mathfrak{g}_{-1}$ and $t \in \mathbb{R}$. This shows that (6.1) holds for the elements of $\mathbf{G}_+$ which are of the form $\exp(H)$ for $H \in \mathfrak{g}_+$ and therefore for all $h \in \mathbf{G}_+$ because $\mathbf{G}_+$ is connected.

To show that the Levi-Tanaka algebra $\mathfrak{g}(x)$ of M at every point $x \in M$ is isomorphic to $\mathfrak{g}$, it suffices to note that by construction $\mathfrak{m}(o)$ is isomorphic to $\mathfrak{m}$ and hence $\mathfrak{g}(o) \simeq \mathfrak{g}$: the general statement follows because $\mathbf{G}$ operates on M as a group of CR diffeomorphisms. $\square$

The $\mathbf{G}$-homogeneous CR manifold obtained in Theorem 6.1.3 will be denoted by $S_{\mathfrak{g}}$ and called the *standard (homogeneous) CR manifold associated to the Levi-Tanaka algebra* $\mathfrak{g}$.

We have

THEOREM 6.1.4 *Let Γ be the kernel of the representation of $\mathbf{G}$ as a group of CR automorphisms of the standard CR manifold $S_{\mathfrak{g}}$. Then Γ is the discrete subgroup $\mathbf{Z}(\mathbf{G}) \cap \mathbf{G}_+$, where $\mathbf{Z}(\mathbf{G})$ denotes the center of $\mathbf{G}$, and $\mathbf{G}/\Gamma$ is the connected component of the identity in the group of CR automorphisms of $S_{\mathfrak{g}}$.*
Every local one-parameter group of CR automorphisms of $S_{\mathfrak{g}}$ extends to a one-parameter subgroup of $\mathbf{G}$.

If M is another connected $\mathbf{G}$-homogeneous CR manifold with the same Levi-Tanaka algebra $\mathfrak{g}$, then there is a CR covering map $S_{\mathfrak{g}} \to M$ commuting to the action of $\mathbf{G}$.

Proof. We note that $\Gamma = \bigcap_{g \in \mathbf{G}} (g\mathbf{G}_+ g^{-1})$ is a closed normal subgroup of $\mathbf{G}$ contained in $\mathbf{G}_+$. Its Lie algebra is an ideal contained in $\mathfrak{g}_+$ and then is null because $\mathfrak{g}$ is transitive. This shows tht Γ is a normal discrete subgroup of the connected Lie group $\mathbf{G}$ and hence is contained in its center. So we have $\Gamma = \mathbf{Z}(\mathbf{G}) \cap \mathbf{G}_+$. Vive versa every element of $\mathbf{Z}(\mathbf{G}) \cap \mathbf{G}_+$ is obviously in the kernel Γ.

To show that $\mathbf{G}$ is the component of the identity in the group of CR automorphisms of M we essentially follow [40]; the proof in the case of homogeneous manifolds is actually simpler.

(a) Let us denote by $\mathbf{A}$ the connected subgroup of $\mathbf{G}$ with Lie algebra $\mathfrak{m}$. If θ is the Maurer-Cartan form of $\mathbf{G}$, then the Maurer-Cartan form ξ of $\mathbf{A}$ is the pullback of θ to $\mathbf{A}$. The natural projection $\pi : \mathbf{G} \to \mathbf{G}/\mathbf{G}_+ = M$ induces a diffeomorphism of an open neighborhood $\mathbf{U}_e$ of e in $\mathbf{A}$ onto an open neighborhood $\mathbf{U}_o$ of $o = \pi(e)$ in M. Let $\tilde{\xi} = (\pi|_{U_e})_* \xi$ and set $\tilde{\xi}^p = (\pi|_{U_e})_* \xi^p$, where $\xi = \sum_{p<0} \xi^p$ is the decomposition of ξ according to the graduation of the fundamental algebra $\mathfrak{m}$. We note that we obtain the equations

$$d\tilde{\xi}^p = -\frac{1}{2} \sum_{r+s=p} [\tilde{\xi}^r, \tilde{\xi}^s] \qquad \text{for} \quad p < 0.$$

(b) Let X be a vector field defined on an open neighborhood of o in $S_{\mathfrak{g}}$. We can as well assume that X is defined on U_o. We want to take X as the

infinitesimal generator of a 1-parameter family of local CR diffeomorphisms on M. If $\phi_X(t)$ is the local 1-parameter group defined by X, this condition means that $d\phi_X(t) : T_xM \to T_{\phi_X(t)(x)}M$ induces, by passing to the quotient, an isomorphism of pseudocomplex fundamental graded Lie algebras

$$\widehat{d\phi_X(t)} : \mathfrak{m}(x) \to \mathfrak{m}(\phi_X(t)x)$$

for x in a small neighborhood of o and t in a small neighborhood of 0. In particular, using the identification of $\mathfrak{m}(x)$ to $\mathfrak{m}$ for all $x \in U_o$, the differential at o of the map $\mathfrak{m} \to \mathfrak{m}$ induced by the diagram

$$\begin{array}{ccc} \mathfrak{m} & \longrightarrow & \mathfrak{m} \\ \downarrow & & \downarrow \\ \mathfrak{m}(x) & \xrightarrow{\widehat{d\phi_X(t)}} & \mathfrak{m}(\phi_X(t)x) \end{array}$$

gives a map $f^0 : U_o \to \mathfrak{g}_0$.

Let us set, for $p < 0$, $f^p(x) = \tilde{\xi}^p(X_x) \in \mathfrak{g}_p$. Then the definition of f^0 can be rewritten by

$$df^p(x) = \sum_{r=p}^{-1} [f^{p-r}(x), \tilde{\xi}^r] \qquad (\text{mod } \tilde{\xi}^{p-1}, \dots, \tilde{\xi}^{-\mu}) \qquad \text{for} \quad p < 0.$$

Indeed, we have for every $Y \in \mathfrak{X}(M)$

$$\begin{aligned} (L_X\tilde{\xi}^p)(Y) &= X(\tilde{\xi}^p(Y)) - \tilde{\xi}^p[X,Y] = \frac{d}{dt}(\phi_X(t)^*\tilde{\xi}^p)(Y)|_{t=0} \\ &= [f^0(x), \tilde{\xi}^p](Y) \qquad (\text{mod } \tilde{\xi}^{p-1}, \dots, \tilde{\xi}^{-\mu}) \qquad \text{for} \quad p < 0. \end{aligned}$$

Hence we deduce that

$$\begin{aligned} df^p(x) &= d(\tilde{\xi}^p(X)) = d(X \rfloor \tilde{\xi}^p) = L_X\tilde{\xi}^p - X \rfloor d\tilde{\xi}^p \\ &= L_X\tilde{\xi}^p + X \rfloor \left(\frac{1}{2} \sum_{r+s=p} [\tilde{\xi}^r, \tilde{\xi}^s] \right) \\ &= [f^0(x), \tilde{\xi}^p] + \sum_{r+s=p} [f^r(x), \tilde{\xi}^s] \qquad (\text{mod } \tilde{\xi}^{p-1}, \dots, \tilde{\xi}^{-\mu}). \end{aligned}$$

Then we can define f^p also for $p > 0$ in such a way that

$$df^p = \sum_{r<0} [f^{p-r}(x), \tilde{\xi}^r] \quad \forall p \in \mathbb{Z}.$$

We have already constructed f^p for $p \leq 0$. Now we note that these equations yield:

$$\begin{aligned} df^0(x) &= \sum_{r<0}[f^{-r}(x), \tilde{\xi}^r] \\ df^1(x) &= \sum_{r<0}[f^{1-r}(x), \tilde{\xi}^r] \\ &\cdots \end{aligned}$$

which is a completely integrable system (see [40]).

(c) Let us denote by $\tilde{\mathfrak{X}}_o$ the Lie algebra of germs at o of infinitesimal generators of 1-parameter groups of local CR diffeomorphisms. By Lemma 6.4 in [40], we have

$$f^p_{[X,Y]} = -\sum_{r+s=p}[f^r_X, f^s_Y] \quad \forall p \in \mathbb{Z} \quad \forall X, Y \in \tilde{\mathfrak{X}}_o$$

where for $Z \in \tilde{\mathfrak{X}}_o$ we used f^p_Z for the set of functions associated to Z as in (b).

The map $\tilde{\mathfrak{X}}_o \ni X \to \sum f^p_X(0) \in \mathfrak{g}$ is therefore an anti-homomorphism of Lie algebras and is injective by Lemma 6.3 in [40]. But this map is trivially surjective and therefore is an anti-isomorphism. This proves the first statement.

The second statement is a consequence of the proof above and the fact that left-invariant vector fields on a Lie group are complete, i.e. generate a one-parameter subgroup.

To prove the last statement of the theorem, it suffices to note that $M \cong \mathbf{G}/\mathbf{Q}$ for a closed subgroup $\mathbf{Q}$ of $\mathbf{G}$ whose Lie algebra is isomorphic to $\mathfrak{g}_+$. Indeed from (a), (b), (c) above we deduce that M and $S_{\mathfrak{g}}$ are locally CR diffeomorphic and therefore the Lie algebras of the stabilizer of a point in the group of local CR automorphism of M and $S_{\mathfrak{g}}$ (respectively) are isomorphic. □

In the example contained in 3.8.1 we showed that the Lie algebra $\mathfrak{su}(p+1, q+1)$ (for $p+q>0$) admits a structure of Levi-Tanaka algebra of codimension one.

Remark 6.1.5 *The standard CR manifold $S_{\mathfrak{g}}$ associated to $\mathfrak{su}(1, n+1)$ is CR diffeomorphic to the sphere S^{2n+1} contained in $\mathbb{C}^{n+1}$.*

Indeed, every biholomorphism of the open unit ball B in $\mathbb{C}^{n+1}$ extends to a neighborhood of the closure of B and then defines a CR automorphisms of the sphere. Conversely, every automorphism of the sphere extends to a biholomorphism of the open ball (cf. [29]). Then there is an isomorphism between the group of CR-diffeomorphisms of the sphere and the group of biholomorphisms of the ball, which is isomorphic to $\mathbf{SU}(1,n+1)/\{\pm I\}$ (see, for instance, [35]). In particular, S^{2n+1} is a $\mathbf{SU}(1,n+1)$-homogeneous CR manifolds. Then, by Theorem 6.1.4, there exists a CR covering $S_{\mathfrak{g}} \to S^{2n+1}$, which is a CR-diffeomorphism as S^{2n+1} is simply connected.

The following theorem is a slight extension of a result in [40]:

THEOREM 6.1.6 *If a (finite dimensional) Levi-Tanaka algebra $\mathfrak{g}$ is semisimple, then the standard homogeneous CR manifold $S_{\mathfrak{g}}$ is compact.*

The proof of this theorem relies on the following

LEMMA 6.1.7 *Let $\mathfrak{g} = \oplus_{p\in\mathbb{Z}}\mathfrak{g}_p$ be a finite dimensional semisimple Levi-Tanaka algebra and let*

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$$

be a Cartan decomposition of $\mathfrak{g}$, where $\mathfrak{k}$ is a maximal Lie subalgebra of $\mathfrak{g}$ on which the Killing form $\kappa_{\mathfrak{g}}$ is negative defined. Then, for $\mathfrak{g}_+ = \oplus_{p\geq 0}\mathfrak{g}_p$, we have

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{g}_+.$$

Proof. Let $d = \dim_{\mathbb{R}} \mathfrak{g}_0$ and $m = \dim_{\mathbb{R}} \mathfrak{m}$ where $\mathfrak{m} = \oplus_{p<0}\mathfrak{g}_p$. The Killing form $\kappa_{\mathfrak{g}}$ is nondegenerate on $\mathfrak{g}_0$ and therefore its restriction to $\mathfrak{g}_0$ has a signature (σ^+, σ^-) with $\sigma^+ + \sigma^- = d$. Since $\mathfrak{m}$ is totally isotropic, the Killing form $\kappa_{\mathfrak{g}}$ has signature $(\sigma^+ + m, \sigma^- + m)$. Given a Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$, we claim that $\mathfrak{k} \cap \mathfrak{g}_+$ is a Lie subalgebra of dimension σ^- of $\mathfrak{g}$. Indeed, if $X = \sum_{p\geq 0} X_p$ is a nonzero vector in $\mathfrak{k} \cap \mathfrak{g}_+$ decomposed into its homogeneous components, then

$$0 > \kappa_{\mathfrak{g}}(X, X) = \kappa_{\mathfrak{g}}(X_0, X_0)$$

shows that the natural projection $\mathfrak{k} \cap \mathfrak{g}_+ \to \mathfrak{g}_0$ is injective and its image is a subspace of $\mathfrak{g}_0$ on which $\kappa_{\mathfrak{g}}$ is negative definite. This shows that $\dim_{\mathbb{R}} \mathfrak{k} \cap \mathfrak{g}_+ \leq \sigma^-$. On the other hand, the projection $\mathfrak{k} \to \mathfrak{g}/\mathfrak{g}_+$, having kernel $\mathfrak{k} \cap \mathfrak{g}_+$, is necessarily surjective and therefore has rank m and σ^--dimensional kernel. In particular we obtain that $\mathfrak{g} = \mathfrak{k} + \mathfrak{g}_+$. $\square$

Proof (of Theorem 6.1.6). Let $\mathbf{G}$ be a connected and simply connected Lie group with Lie algebra $\mathfrak{g}$ and let $\mathbf{G}_+$ and κ be the connected Lie subgroups of $\mathbf{G}$ having Lie algebras $\mathfrak{g}_+$ and $\mathfrak{k}$ respectively, with $\mathfrak{k}$ the direct summand in a Cartan decomposition of $\mathfrak{g}$. Then κ is a compact subgroup of $\mathbf{G}$. We consider the map $\kappa \to S_{\mathfrak{g}} = \mathbf{G}/\mathbf{G}_+$ induced by the restriction of the natural projection. Its image is compact and hence closed. On the other hand, the decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{g}_+$ shows that this map is a submersion and then open. Therefore, since $S_{\mathfrak{g}}$ is connected, this map is onto and $S_{\mathfrak{g}}$ is compact. □

Denote by κ_0 the connected Lie subgroup of $\mathbf{G}$ having Lie algebra $\mathfrak{k} \cap \mathfrak{g}_+$. (Note that $\mathfrak{k} \cap \mathfrak{g}_+ \subset \mathfrak{g}_0$ if we use a Cartan decomposition with the properties of Proposition 3.6.10.) Then the natural map $\kappa/\kappa_0 \to S_{\mathfrak{g}}$ is a diffeomorphism because is a connected covering of a simply connected manifold.

6.2 Canonical immersions of standard CR manifolds

In [2], Andreotti and Fredricks proved that for every real-analytic CR manifold M there exists a (global) embedding into a complex manifold X such that the CR structure of M is induced by the complex structure of X. In this section, for every standard CR manifold $S_{\mathfrak{g}}$ we give an immersion of $S_{\mathfrak{g}}$ into a complex manifold $X_{\mathfrak{g}}$, homogeneous with respect to a group $\mathbf{G}^{\mathbb{C}}$ of complex transformations, in such a way that $S_{\mathfrak{g}}$ is an orbit with respect to the action of a real subgroup of $\mathbf{G}^{\mathbb{C}}$.

Let $\mathfrak{g} = \oplus_{p\in\mathbb{Z}}\mathfrak{g}_p$ be a finite dimensional Levi-Tanaka algebra and $\mathfrak{g}^{\mathbb{C}} = \mathbb{C} \otimes_{\mathbb{R}} \mathfrak{g}$ be its complexification. We denote by $\mathbf{G}^{\mathbb{C}}$ a connected and simply connected Lie group having Lie algebra $\mathfrak{g}^{\mathbb{C}}$ and by $\mathbf{G}^{\mathbb{R}}$ the connected Lie subgroup of $\mathbf{G}^{\mathbb{C}}$ having Lie algebra $\mathfrak{g}$. This is a closed Lie subgroup of $\mathbf{G}^{\mathbb{C}}$, as $\mathbf{G}^{\mathbb{R}}$ is the connected component of the identity of the closed subgroup of $\mathbf{G}^{\mathbb{C}}$

$$\{g \in \mathbf{G}^{\mathbb{C}} \mid \mathrm{Ad}_{\mathbf{G}^{\mathbb{C}}}(g)(\mathfrak{g}) = \mathfrak{g}\},$$

where $\mathrm{Ad}_{\mathbf{G}^{\mathbb{C}}} : \mathbf{G}^{\mathbb{C}} \to \mathbf{GL}_{\mathbb{C}}(\mathfrak{g}^{\mathbb{C}})$ is the adjoint representation. We also use the notation $\mathfrak{g}_+^{\mathbb{C}} = \mathbb{C} \otimes_{\mathbb{R}} \mathfrak{g}_+$ for the complexification of the Lie subalgebra

$\mathfrak{g}_+ = \oplus_{p\geq 0}\mathfrak{g}_p$ and $\mathbf{G}^{\mathbb{R}}_+$ for the connected Lie subgroup of $\mathbf{G}^{\mathbb{R}}$ having Lie algebra $\mathfrak{g}_+$.

LEMMA 6.2.1 *Let* $\mathfrak{g}_{-1}^{(0,1)} = \{X + \sqrt{-1}JX \mid X \in \mathfrak{g}_{-1}\}$. *Then* $\mathfrak{q} = \mathfrak{g}_{-1}^{(0,1)} \oplus \mathfrak{g}_+^{\mathbb{C}}$ *is a complex Lie subalgebra of* $\mathfrak{g}^{\mathbb{C}}$.

Proof. First we remark that $\mathfrak{g}_{-1}^{(0,1)}$ is a complex subspace of $\mathfrak{g}^{\mathbb{C}}$. Indeed, for $X \in \mathfrak{g}_{-1}$ we have

$$\sqrt{-1}(X + \sqrt{-1}JX) = (-JX) + \sqrt{-1}J(-JX) \quad \text{and} \quad JX \in \mathfrak{g}_{-1}.$$

Moreover

$$[X + \sqrt{-1}JX, Y + \sqrt{-1}JY] = 0 \quad \forall X, Y \in \mathfrak{g}_{-1}$$

and $[\mathbb{C} \otimes_{\mathbb{R}} \mathfrak{g}_0, \mathfrak{g}_{-1}^{(0,1)}] \subset \mathfrak{g}_{-1}^{(0,1)}$ because $\mathfrak{g}_{-1}^{(0,1)}$ is a complex subspace of $\mathfrak{g}^{\mathbb{C}}$ and the elements of $\rho_{-1}(\mathfrak{g}_0)$ commute with J on $\mathfrak{g}_{-1}$. Finally, it is obvious that $[\mathbb{C} \otimes \mathfrak{g}_p, \mathfrak{g}_{-1}^{(0,1)}] \subset \mathbb{C} \otimes \mathfrak{g}_{p-1} \subset \mathfrak{q}$ for $p > 0$. □

Let $\mathbf{Q}$ be the connected complex Lie subgroup of $\mathbf{G}^{\mathbb{C}}$ corresponding to the Lie subalgebra $\mathfrak{q}$.

LEMMA 6.2.2 $\mathbf{Q}$ *is a closed Lie subgroup of* $\mathbf{G}^{\mathbb{C}}$.

Proof. We consider the adjoint representation $\mathrm{Ad}_{\mathbf{G}^{\mathbb{C}}} : \mathbf{G}^{\mathbb{C}} \to \mathbf{GL}_{\mathbb{C}}(\mathfrak{g}^{\mathbb{C}})$. Then

$$\mathbf{H} = \{g \in \mathbf{G}^{\mathbb{C}} \mid \mathrm{Ad}_{\mathbf{G}^{\mathbb{C}}}(g)(\mathfrak{q}) = \mathfrak{q}\}$$

is a closed subgroup of $\mathbf{G}^{\mathbb{C}}$ and $\mathbf{Q}$ is the connected component of the identity of $\mathbf{H}$. □

THEOREM 6.2.3 *The* $\mathbf{G}^{\mathbb{C}}$*-homogeneous space* $X_{\mathfrak{g}} = \mathbf{G}^{\mathbb{C}}/\mathbf{Q}$ *is a complex manifold.*

The $\mathbf{G}^{\mathbb{R}}$*-homogeneous space* $S_{\mathfrak{g}}^{\mathbb{R}} = \mathbf{G}^{\mathbb{R}}/\mathbf{G}^{\mathbb{R}}_+$ *is a differentiable manifold with a unique CR structure which makes the covering map*

$$S_{\mathfrak{g}} \to S_{\mathfrak{g}}^{\mathbb{R}}$$

defined by the commutative diagram

$$\begin{array}{ccc} \mathbf{G} & \longrightarrow & \mathbf{G}^{\mathbb{R}} \\ \downarrow & & \downarrow \\ S_{\mathfrak{g}} & \longrightarrow & S_{\mathfrak{g}}^{\mathbb{R}} \end{array}$$

a local CR diffeomorphism.

The composition $\mathbf{G} \to \mathbf{G}^{\mathbb{R}} \to \mathbf{G}^{\mathbb{C}}$ *induces a CR immersion*

$$S_{\mathfrak{g}} \to X_{\mathfrak{g}}$$

whose image $S_{\mathfrak{g}}^{\mathbb{C}}$ *is a locally closed CR submanifold of* $X_{\mathfrak{g}}$.

Proof. $X_{\mathfrak{g}}$ is a connected smooth complex manifold because $\mathbf{Q}$ is a closed subgroup of $\mathbf{G}^{\mathbb{C}}$. Analogously $S_{\mathfrak{g}}^{\mathbb{R}}$ is a connected real-analytic CR manifold because $\mathbf{G}_{+}^{\mathbb{R}}$ is a closed subgroup of $\mathbf{G}^{\mathbb{R}}$.

The group $\mathbf{G}$ is a covering of $\mathbf{G}^{\mathbb{R}}$ and $S_{\mathfrak{g}}^{\mathbb{R}}$ is $\mathbf{G}$-homogeneous by the action

$$\mathbf{G} \times S_{\mathfrak{g}}^{\mathbb{R}} \ni (g, x) \to p(g) \cdot x \in S_{\mathfrak{g}}^{\mathbb{R}}$$

where $p : \mathbf{G} \to \mathbf{G}^{\mathbb{R}}$ is the covering map.

We consider the orbit $S_{\mathfrak{g}}^{\mathbb{C}}$ in $X_{\mathfrak{g}}$ of the image o of the identity of $\mathbf{G}^{\mathbb{C}}$ in $X_{\mathfrak{g}}$ with respect to the closed subgroup $\mathbf{G}^{\mathbb{R}}$. Since $\mathfrak{g}_{+}$ is the Lie algebra of the stabilizer in $\mathbf{G}^{\mathbb{R}}$ of o, we obtain an immersion $S_{\mathfrak{g}}^{\mathbb{R}} \to X_{\mathfrak{g}}$ which is a surjective local diffeomorphism onto the orbit $S_{\mathfrak{g}}^{\mathbb{C}}$. Let $\alpha : \mathbf{G}^{\mathbb{R}} \to X_{\mathfrak{g}}$ denote the map

$$g \to g \cdot o.$$

We note that for the elements X of $\mathfrak{g}_{-1}$ we obtain, by the definition of $\mathfrak{q}$, $\alpha_*(JX) = \sqrt{-1}\alpha_*(X)$ and therefore the map $S_{\mathfrak{g}}^{\mathbb{R}} \to X_{\mathfrak{g}}$ is a CR immersion.

Let $\mathbf{A}$ and $\tilde{\mathbf{A}}$ be the connected Lie subgroups of $\mathbf{G}^{\mathbb{C}}$ having Lie algebras $\mathfrak{m}$ and $\mathfrak{l} = \mathfrak{g}_{-1} \oplus \left(\oplus_{p<-1}\mathfrak{g}_p^{\mathbb{C}}\right)$ respectively. We fix convex open neighborhoods U_0 of 0 in $\mathfrak{g}^{\mathbb{C}}$ and V_0 of 0 in $\mathfrak{l}$ such that the exponential maps

$$\exp : U_0 \to U_e \subset \mathbf{G}^{\mathbb{C}}, \quad \exp : V_0 \to V_e \subset \tilde{\mathbf{A}}, \quad \exp : U_0 \cap \mathfrak{g} \to U_e \cap \mathbf{G}^{\mathbb{R}}$$

be diffeomorphisms. We can assume that $V_0 = U_0 \cap \mathfrak{l}$, so that $V_e = U_e \cap \tilde{\mathbf{A}}$. If $a \in \mathbf{G}^{\mathbb{R}} \cap \tilde{\mathbf{A}} \cap U_e$, we have $a = \exp(Z) = \exp(X + \sqrt{-1}Y)$ with $Z \in \mathfrak{g} \cap U_0$,

$X \in \mathfrak{m}$, $Y \in \oplus_{p<-1}\mathfrak{g}_p$ and $Z, X + \sqrt{-1}Y \in U_0$. By the injectivity of the exponential on U_0, we obtain $Z = X + \sqrt{-1}Y$, hence $Y = 0$ and $Z \in \mathfrak{m}$. This shows that $\mathbf{G}^{\mathbb{R}} \cap \tilde{\mathbf{A}} \cap U_e = \mathbf{A} \cap U_e$. Moreover, $\mathbf{A} \cap U_e$ is closed and connected in $\tilde{\mathbf{A}} \cap U_e$. We note now that the projection $\pi : \mathbf{G}^{\mathbb{C}} \to S_{\mathfrak{g}}^{\mathbb{C}}$ induces a diffeomorphism of a neighborhood $W \subset V_e$ of e in $\tilde{\mathbf{A}}$ onto a neighborhood W_o of $o = \pi(e)$ in $X_{\mathfrak{g}}$ and, since $\mathbf{Q} \cap \tilde{\mathbf{A}} \cap V_e = \{e\}$, we have $(\pi_{|W_o})^{-1}(S_{\mathfrak{g}}^{\mathbb{R}}) = V_e \cap \mathbf{G}^{\mathbb{R}}$. This shows that $W_o \cap S_{\mathfrak{g}}^{\mathbb{R}}$ is closed in W_o. Since $S_{\mathfrak{g}}^{\mathbb{R}}$ is homogeneous, it is locally closed in $X_{\mathfrak{g}}$. □

We call $X_{\mathfrak{g}}$ the *standard (homogeneous) complex manifold* associated to $\mathfrak{g}$ and the map $S_{\mathfrak{g}} \to X_{\mathfrak{g}}$ the *canonical immersion* of $S_{\mathfrak{g}}$.

THEOREM 6.2.4 *If a Levi-Tanaka algebra $\mathfrak{g}$ is semisimple, then the standard homogeneous complex manifold $X_{\mathfrak{g}}$ associated to $\mathfrak{g}$ is compact.*

Proof. Let $\theta : \mathfrak{g} \to \mathfrak{g}$ be the Cartan involution found in Proposition 3.6.10 and let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the corresponding Cartan decomposition. Then $\mathfrak{u} = \mathfrak{k} \oplus \sqrt{-1}\mathfrak{p}$ is a compact form of the complexification $\mathfrak{g}^{\mathbb{C}}$ of $\mathfrak{g}$. We set

$$\begin{aligned}
\mathfrak{u}^{(1,0)} &= \{X - \sqrt{-1}JX + \theta(X) + \sqrt{-1}J\theta(X) \mid X \in \mathfrak{g}_{-1}\} \\
\mathfrak{u}^{(0,1)} &= \{X + \sqrt{-1}JX + \theta(X) - \sqrt{-1}J\theta(X) \mid X \in \mathfrak{g}_{-1}\} \\
\mathfrak{u}_p &= \mathfrak{u} \cap (\mathfrak{g}^{\mathbb{C}}{}_{-p} + \mathfrak{g}^{\mathbb{C}}{}_p) \quad \text{for} \quad p \geq 0.
\end{aligned}$$

Next we define the real Lie subalgebra $\mathfrak{h}$ of $\mathfrak{g}^{\mathbb{C}}$ by

$$\mathfrak{h} = \mathfrak{u} \cap \mathfrak{q} = \mathfrak{u}_0 \oplus \mathfrak{u}^{(0,1)}.$$

Let $\mathbf{U}$ denote the connected Lie subgroup of $\mathbf{G}^{\mathbb{C}}$ having Lie algebra $\mathfrak{u}$ and $\mathbf{H}$ the connected Lie subgroup of $\mathbf{G}^{\mathbb{C}}$ having Lie algebra $\mathfrak{h}$. The group $\mathbf{U}$ is compact and hence closed in $\mathbf{G}^{\mathbb{C}}$, and also $\mathbf{H}$ is compact, being the connected component of the identity in the intersection $\mathbf{U} \cap \mathbf{Q}$.

Consider the commutative diagram:

$$\begin{array}{ccc}
\mathbf{U} & \longrightarrow & \mathbf{G}^{\mathbb{C}} \\
\downarrow & & \downarrow \\
\mathbf{U}/_{\mathbf{H}} & \longrightarrow & \mathbf{G}^{\mathbb{C}}/_{\mathbf{Q}}.
\end{array}$$

Since $\mathfrak{u} + \mathfrak{q} = \mathfrak{g}^{\mathbb{C}}$, the map $\mathbf{U} \to \mathbf{G}^{\mathbb{C}}/_{\mathbf{Q}}$ is a submersion and therefore is open. It is also closed, being a continuous map from a compact space into a Hausdorff space. Since $X_{\mathfrak{g}} = \mathbf{G}^{\mathbb{C}}/_{\mathbf{Q}}$ is connected, this map is surjective and therefore $\mathbf{U}/_{\mathbf{H}} \to \mathbf{G}^{\mathbb{C}}/_{\mathbf{Q}}$ is a covering map. Since $\mathbf{G}^{\mathbb{C}}/_{\mathbf{Q}}$ is simply connected, this map is a diffeomorphism. This proves the theorem. □

PROPOSITION 6.2.5 *If the component $\mathfrak{g}_1$ of $\mathfrak{g}$ is zero, then the manifolds $S_{\mathfrak{g}}^{\mathbb{C}}$ and $X_{\mathfrak{g}}$ are both Euclidean and $S_{\mathfrak{g}}^{\mathbb{C}}$ is embedded in $X_{\mathfrak{g}}$ as a closed submanifold.*

Proof. By Lemma 3.18.4 of [44], $S_{\mathfrak{g}}^{\mathbb{C}}$ is closed in $X_{\mathfrak{g}}$ and simply connected and, by Lemma 3.18.11 of [44], it is also Euclidean. Indeed, $\mathfrak{m}$ is an ideal in $\mathfrak{g}$ and therefore the map $\mathfrak{m} \oplus \mathfrak{g}_0 \ni (X, Y) \to \exp(X)\exp(Y) \in \mathbf{G}$ is a diffeomorphism. □

6.3 Canonical projective immersions of standard CR manifolds

The problem of finding an immersion of the standard homogeneous CR manifold $S_{\mathfrak{g}}$ into a complex projective space is equivalent, by Theorem 6.1.4, to the one of finding, given a Levi-Tanaka algebra $\mathfrak{g}$, $\mathbf{G}$-homogeneous CR submanifolds of complex projective spaces having at each point a Levi-Tanaka algebra isomorphic to $\mathfrak{g}$.

Our construction is akin to the one used in [4]. We use the complexification of the adjoint representation $\mathrm{Ad}_{\mathbf{G}^{\mathbb{C}}} : \mathbf{G}^{\mathbb{C}} \to \mathbf{GL}_{\mathbb{C}}(\mathfrak{g}^{\mathbb{C}})$ and denote by $\mathbf{G}_{\mathbb{C}}^{\mathbb{P}}$ and $\mathbf{G}^{\mathbb{P}}$ respectively the image $\mathrm{Ad}_{\mathbf{G}^{\mathbb{C}}}(\mathbf{G}^{\mathbb{C}})$ and $\mathrm{Ad}_{\mathbf{G}^{\mathbb{C}}}(\mathbf{G}^{\mathbb{R}})$. They are Lie subgroups of $\mathbf{GL}_{\mathbb{C}}(\mathfrak{g}_{\mathbb{C}})$. We also set $\mathbf{Q}^{\mathbb{P}}$ and $\mathbf{G}_{+}^{\mathbb{P}}$ for the connected Lie subgroups of $\mathbf{G}_{\mathbb{C}}^{\mathbb{P}}$ having Lie algebra equal respectively to the Lie subalgebra $\mathfrak{q}$ defined in Lemma 6.2.1 and to $\mathfrak{g}_{+}$.

We consider the Grassmannian $\mathbf{Gr}_{\ell}(\mathfrak{g}^{\mathbb{C}})$ of complex subspaces of $\mathfrak{g}^{\mathbb{C}}$ having dimension ℓ equal to the complex dimension of $\mathfrak{q}$. The orbits $X_{\mathfrak{g}}^{\mathbb{P}}$ and $S_{\mathfrak{g}}^{\mathbb{P}}$ of $\mathfrak{q}$ by the action of $\mathbf{G}_{\mathbb{C}}^{\mathbb{P}}$ and $\mathbf{G}^{\mathbb{P}}$ are respectively a $\mathbf{G}_{\mathbb{C}}^{\mathbb{P}}$-homogeneous complex manifold and a $\mathbf{G}^{\mathbb{R}}$-homogeneous CR submanifold (and therefore $\mathbf{G}^{\mathbb{C}}$ and $\mathbf{G}$-homogeneous). In this way we obtain a CR submanifold of a projective

manifold having the prescribed Levi-Tanaka algebra $\mathfrak{g}$ at each point.
We take up now the question of the existence of a closed embedding into a projective space in the case where the Levi-Tanaka algebra is semisimple.

We recall that a *Borel subalgebra* $\mathfrak{b}$ of a Lie algebra $\mathfrak{g}$ is a maximal solvable Lie subalgebra of $\mathfrak{g}$ and a Lie subalgebra $\mathfrak{q}$ of $\mathfrak{g}$ is said to be *parabolic* if it contains a Borel subalgebra. Accordingly, a connected Lie subgroup $\mathbf{B}$ (resp. $\mathbf{Q}$) of a Lie group $\mathbf{G}$ is a Borel (resp. parabolic) subgroup if its Lie algebra $\mathfrak{b}$ (resp. $\mathfrak{q}$) is Borel (resp. parabolic). In particular a Borel subgroup of $\mathbf{G}$ is a maximal connected solvable subgroup of $\mathbf{G}$.

LEMMA 6.3.1 *Let $\mathfrak{g} = \oplus_{p\in\mathbb{Z}}\mathfrak{g}_p$ be a finite dimensional Levi-Tanaka algebra. Then the following facts are equivalent:*

(i) $\mathfrak{g}$ *is semisimple;*

(ii) $\mathfrak{g}_+ = \oplus_{p\geq 0}\mathfrak{g}_p$ *is parabolic;*

(iii) $\mathfrak{q} = \mathfrak{g}_{-1}^{(0,1)} \oplus \mathfrak{g}_+^{\mathbb{C}}$ *is a parabolic Lie subalgebra of* $\mathfrak{g}^{\mathbb{C}}$.

Proof.

(i) $\Leftrightarrow$ *(ii)*. Let E be the element of $\mathfrak{g}_0$ described in Lemma 3.2.1. Then $\oplus_{p>0}\mathfrak{g}_p \oplus \mathbb{R}\cdot E$ is a solvable Lie subalgebra of $\mathfrak{g}$ and hencefore is contained in a Borel subalgebra $\mathfrak{b}$. If $\mathfrak{r}$ is the radical of $\mathfrak{g}$, then $\mathfrak{r} \subset \mathfrak{b}$. By Corollary 3.5.6, $\mathfrak{r}$ is contained in $\mathfrak{g}_+$ if and only if $\mathfrak{g}$ is semisimple and $\mathfrak{r} = 0$. The condition is therefore necessary.

To prove sufficiency, we first note that the representation $\rho : \mathfrak{b} \to \mathfrak{l}(\mathfrak{g})$ obtained by restriction from the adjoint representation is faithful. Then, by the criterion of Cartan, $\rho(\mathfrak{b})$, and thus $\mathfrak{b}$, is solvable if and only if $[\mathfrak{b}, \mathfrak{b}]$ is orthogonal to $\mathfrak{b}$ with respect to the Killing form $\kappa_{\mathfrak{g}}$ of $\mathfrak{g}$. Assume by contradiction that $\mathfrak{b}$ contains an element $X = \sum_p X_p$ with homogeneous component $X_q \neq 0$ for some $q < 0$. Since $\mathfrak{g}$ was assumed to be semisimple, we can find $Y_{-q} \in \mathfrak{g}_{-q} \subset \mathfrak{b}$ such that $\kappa_{\mathfrak{g}}(X_q, Y_{-q}) \neq 0$. Then we obtain

$$\kappa_{\mathfrak{g}}([E, X], Y_{-q}) = q\,\kappa_{\mathfrak{g}}(X_q, Y_{-q}) \neq 0$$

which contradicts the Cartan criterion.

(i) $\Leftrightarrow$ *(iii)* If $\mathfrak{r}$ is the radical of $\mathfrak{g}$, then $\mathbb{C}\otimes_{\mathbb{R}}\mathfrak{r}$ is the radical of $\mathfrak{g}^{\mathbb{C}}$. Clearly, if $X \in \mathfrak{r}_{-1}$, then $X - \sqrt{-1}JX$ belongs to the radical of $\mathfrak{g}^{\mathbb{C}}$ and therefore, if $\mathfrak{q}$ is parabolic, the radical of $\mathfrak{g}$ is contained in $\mathfrak{g}_+$. $\square$

THEOREM 6.3.2 *A necessary and sufficient condition in order that $X_{\mathfrak{g}}^{\mathbb{P}}$ be compact is that $\mathfrak{g}$ be semisimple.*

If $\mathfrak{g}$ is semisimple, then $S_{\mathfrak{g}}^{\mathbb{P}} \to X_{\mathfrak{g}}^{\mathbb{P}}$ is a closed embedding of $S_{\mathfrak{g}}^{\mathbb{P}}$ into a compact projective complex manifold.

Proof. The first part of the statement is a consequence of Lemma 6.3.1 and of [5] (Theorem 11.1 and Corollary 11.2) because $\mathbf{G}_{\mathbb{C}}^{\mathbb{P}}$ is an algebraic group.

The second part follows because $S_{\mathfrak{g}}^{\mathbb{P}}$ is compact when $\mathfrak{g}$ is semisimple because it is the quotient of $S_{\mathfrak{g}}$ with respect to the action of a discrete subgroup of $\mathbf{G}$. □

We call $X_{\mathfrak{g}}^{\mathbb{P}}$ the *standard (homogeneous) projective manifold* associated to the Levi-Tanaka algebra $\mathfrak{g}$ and the map $S_{\mathfrak{g}} \to S_{\mathfrak{g}}^{\mathbb{P}} \to X_{\mathfrak{g}}^{\mathbb{P}}$ the *canonical projective immersion* of $S_{\mathfrak{g}}$.

THEOREM 6.3.3 *Let $\mathfrak{g}$ be a finite dimensional Levi-Tanaka algebra. Then a necessary and sufficient condition in order that $X_{\mathfrak{g}}$ be compact is that $\mathfrak{g}$ be semisimple.*

Proof. We already proved that $X_{\mathfrak{g}}$ is compact when $\mathfrak{g}$ is semisimple. When $\mathfrak{g}$ is not semisimple, then $X_{\mathfrak{g}}^{\mathbb{P}}$ is not compact and hence also $X_{\mathfrak{g}}$ is not compact, because it is a covering space of $X_{\mathfrak{g}}^{\mathbb{P}}$. □

Remark 6.3.4 *It follows from [14] that, when $\mathfrak{g}$ is semisimple, the standard homogeneous projective manifold $S_{\mathfrak{g}}^{\mathbb{P}}$ associated to a semisimple Levi-Tanaka algebra $\mathfrak{g} = \oplus_{-\mu \le p \le \mu} \mathfrak{g}_p$ with $\dim_{\mathbb{C}} \mathfrak{g}_{-1} = n$ and $\dim_{\mathbb{R}} \mathfrak{n} = \dim_{\mathbb{R}} \oplus_{p<-1} \mathfrak{g}_p = k$, has a CR embedding into the space $\mathbb{CP}^{[2n+(3/2)k]}$.*

Remark 6.3.5 *Every Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$ is splittable. Indeed the splittable envelope of a solvable subalgebra of $\mathfrak{g}$ is still solvable and therefore the splittable envelope of $\mathfrak{b}$ is equal to $\mathfrak{b}$ by maximality.*

Appendix

Tables of exceptional Levi-Tanaka algebras

SIMPLE LEVI-TANAKA ALGEBRAS OF THE COMPLEX TYPE E_6

	$-\lvert\alpha_i\rvert$						μ	$\dim_{\mathbb{C}} \mathfrak{g}$ 0 ±1 ±2 ±3 ±4 ±5 ±6 ±7 ±8 ±9 ±10 ±11	$\mathfrak{g}_0$
01.	1	0	0	0	0	1	02	30 16 08 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_2(\mathbb{C})\oplus\mathfrak{so}(8,\mathbb{C})$
02.	1	1	0	0	0	0	03	26 15 10 01 00 00 00 00 00 00 00 00	$\mathfrak{d}_2(\mathbb{C})\oplus\mathfrak{sl}(5,\mathbb{C})$
03.	1	0	1	0	0	0	03	26 11 10 05 00 00 00 00 00 00 00 00	$\mathfrak{d}_2(\mathbb{C})\oplus\mathfrak{sl}(5,\mathbb{C})$
04.	0	1	1	0	0	0	04	20 12 12 04 01 00 00 00 00 00 00 00	$\mathfrak{d}_2(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(4,\mathbb{C})$
05.	0	0	1	0	1	0	04	16 12 12 04 03 00 00 00 00 00 00 00	$\mathfrak{d}_2(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
06.	1	0	1	0	0	1	04	18 11 10 05 04 00 00 00 00 00 00 00	$\mathfrak{d}_3(\mathbb{C})\oplus\mathfrak{sl}(4,\mathbb{C})$
07.	0	0	1	1	0	0	05	16 08 12 06 03 02 00 00 00 00 00 00	$\mathfrak{d}_2(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})$
08.	0	1	0	1	0	0	05	18 10 09 09 01 01 00 00 00 00 00 00	$\mathfrak{d}_2(\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})$
09.	1	1	1	0	0	0	05	18 09 10 06 04 01 00 00 00 00 00 00	$\mathfrak{d}_3(\mathbb{C})\oplus\mathfrak{sl}(4,\mathbb{C})$
10.	1	0	1	0	1	0	05	14 10 09 07 03 03 00 00 00 00 00 00	$\mathfrak{d}_3(\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
11.	1	1	0	1	0	0	06	14 09 09 06 06 01 01 00 00 00 00 00	$\mathfrak{d}_3(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})$
12.	1	0	1	1	0	0	06	14 08 07 09 03 03 02 00 00 00 00 00	$\mathfrak{d}_3(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})$
13.	1	0	0	1	1	0	06	12 08 10 06 05 02 02 00 00 00 00 00	$\mathfrak{d}_3(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
14.	1	0	1	0	1	1	06	12 08 09 06 05 02 03 00 00 00 00 00	$\mathfrak{d}_4(\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})$
15.	0	1	1	1	0	0	07	14 06 09 06 06 03 01 01 00 00 00 00	$\mathfrak{d}_3(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})$
16.	0	0	1	1	1	0	07	12 06 08 08 04 04 01 02 00 00 00 00	$\mathfrak{d}_3(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
17.	1	1	0	1	0	1	07	10 09 08 06 05 04 01 01 00 00 00 00	$\mathfrak{d}_4(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
18.	1	0	1	1	0	1	07	10 08 07 07 05 03 02 02 00 00 00 00	$\mathfrak{d}_4(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
19.	1	1	1	1	0	0	08	12 06 07 06 06 03 03 01 01 00 00 00	$\mathfrak{d}_4(\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})$
20.	1	1	0	1	1	0	08	10 07 07 07 04 05 02 01 01 00 00 00	$\mathfrak{d}_4(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
21.	1	0	1	1	1	0	08	10 06 07 06 06 03 03 01 02 00 00 00	$\mathfrak{d}_4(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
22.	0	1	1	1	1	0	09	10 06 05 08 04 04 04 01 01 01 00 00	$\mathfrak{d}_4(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
23.	1	1	1	1	0	1	09	08 07 06 06 05 04 03 02 01 01 00 00	$\mathfrak{d}_5(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
24.	1	0	1	1	1	1	09	08 06 06 06 05 04 03 02 01 02 00 00	$\mathfrak{d}_5(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
25.	1	1	1	1	1	0	10	08 06 05 06 05 04 03 03 01 01 01 00	$\mathfrak{d}_5(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
26.	1	1	1	1	1	1	11	06 06 05 05 05 04 03 03 02 01 01 01	$\mathfrak{d}_6(\mathbb{C})$

In cases 1), 5), 14) the two conjugated Levi-Tanaka algebras corresponding to the same weighted Satake diagram are isomorphic, in the other cases they are not isomorphic.

SIMPLE LEVI-TANAKA ALGEBRAS OF THE COMPLEX TYPE E_7

	$-\lvert\alpha_i\rvert$	μ	$\dim_{\mathbb{C}} \mathfrak{g}$ 0 ±1 ±2 ±3 ±4 ±5 ±6 ±7 ±8 ±9 ±10 ±11 ±12 ±13 ±14 ±15 ±16 ±17	$\mathfrak{g}_0$
01.	1000001	03	47 26 16 01 00 00 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_2(\mathbb{C})\oplus\mathfrak{so}(10,\mathbb{C})$
02.	0000011	03	47 17 16 10 00 00 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_2(\mathbb{C})\oplus\mathfrak{so}(10,\mathbb{C})$
03.	1100000	04	37 21 20 06 01 00 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_2(\mathbb{C})\oplus\mathfrak{sl}(6,\mathbb{C})$
04.	0100010	04	29 20 20 07 05 00 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_2(\mathbb{C})\oplus\mathfrak{sl}(5,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
05.	1000011	05	31 17 16 09 08 01 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_3(\mathbb{C})\oplus\mathfrak{so}(8,\mathbb{C})$
06.	0100011	05	27 16 15 11 06 05 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_3(\mathbb{C})\oplus\mathfrak{sl}(5,\mathbb{C})$
07.	1010000	05	37 16 15 15 01 01 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_2(\mathbb{C})\oplus\mathfrak{sl}(6,\mathbb{C})$
08.	0110000	05	29 15 20 10 05 02 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_2(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(5,\mathbb{C})$
09.	0100100	05	25 16 18 12 04 04 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_2(\mathbb{C})\oplus\mathfrak{sl}(4,\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})$
10.	0000110	05	29 12 20 10 05 05 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_2(\mathbb{C})\oplus\mathfrak{sl}(5,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
11.	1010001	06	27 16 15 10 10 01 01 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_3(\mathbb{C})\oplus\mathfrak{sl}(5,\mathbb{C})$
12.	0100101	06	21 14 16 10 09 03 04 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_3(\mathbb{C})\oplus\mathfrak{sl}(4,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
13.	0000111	06	27 12 11 15 05 05 05 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_3(\mathbb{C})\oplus\mathfrak{sl}(5,\mathbb{C})$
14.	0010100	06	21 15 18 09 09 03 02 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_2(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})$
15.	0101000	06	25 13 12 18 04 04 03 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_2(\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})\oplus\mathfrak{sl}(4,\mathbb{C})$
16.	1110000	07	27 11 15 10 10 05 01 01 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_3(\mathbb{C})\oplus\mathfrak{sl}(5,\mathbb{C})$
17.	1010010	07	21 15 14 10 09 06 01 01 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_3(\mathbb{C})\oplus\mathfrak{sl}(4,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$

18.	1000110	07	21 12 16 10 09 04 04 01 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_3(\mathbb{C})\oplus\mathfrak{sl}(4,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
19.	0100110	07	21 10 14 12 08 06 02 04 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_3(\mathbb{C})\oplus\mathfrak{sl}(4,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
20.	0010101	07	17 14 15 10 08 06 03 02 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_3(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
21.	0101001	07	19 13 12 12 10 04 03 03 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_3(\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})$
22.	0011000	07	23 10 16 12 06 08 01 02 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_2(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(4,\mathbb{C})$
23.	0001100	07	21 09 18 09 09 06 02 03 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_2(\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})$
24.	1010011	08	19 12 14 09 09 05 06 01 01 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_4(\mathbb{C})\oplus\mathfrak{sl}(4,\mathbb{C})$
25.	1000111	08	19 12 11 11 09 05 04 04 01 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_4(\mathbb{C})\oplus\mathfrak{sl}(4,\mathbb{C})$
26.	0100111	08	19 10 11 10 10 05 05 02 04 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_4(\mathbb{C})\oplus\mathfrak{sl}(4,\mathbb{C})$
27.	1010100	08	19 13 12 12 06 09 03 01 01 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_3(\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})$
28.	0010110	08	17 11 12 13 06 08 03 03 02 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_3(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
29.	1101000	08	21 11 12 10 12 04 04 02 01 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_3(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(4,\mathbb{C})$
30.	0101010	08	17 11 12 09 12 05 04 02 03 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_3(\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
31.	0011001	08	17 11 14 10 09 05 06 01 02 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_3(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})$
32.	0001101	08	17 10 13 12 06 08 04 02 03 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_3(\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
33.	1010101	09	15 12 12.10 08 06 06 03 01 01 00 00 00 00 00 00 00 00	$\mathfrak{d}_4(\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
34.	0010111	09	15 11 10 10 09 06 05 03 03 02 00 00 00 00 00 00 00 00	$\mathfrak{d}_4(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})$
35.	1101001	09	15 12 11 09 10 07 04 03 02 01 00 00 00 00 00 00 00 00	$\mathfrak{d}_4(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})$
36.	0101011	09	15 10 11 09 09 07 05 03 02 03 00 00 00 00 00 00 00 00	$\mathfrak{d}_4(\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$

37.	1011000	09	21 10 09 14 06 06 08 01 01 01 00 00 00 00 00 00 00 00	$\mathfrak{d}_3(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(4,\mathbb{C})$
38.	0111000	09	21 07 12 08 12 06 04 04 01 02 00 00 00 00 00 00 00 00	$\mathfrak{d}_3(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(4,\mathbb{C})$
39.	0011010	09	15 10 12 10 09 06 05 04 01 02 00 00 00 00 00 00 00 00	$\mathfrak{d}_3(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
40.	1001100	09	17 09 14 09 09 06 06 02 02 01 00 00 00 00 00 00 00 00	$\mathfrak{d}_3(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})$
41.	0101100	09	19 07 12 09 09 09 03 04 01 03 00 00 00 00 00 00 00 00	$\mathfrak{d}_3(\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})$
42.	0001110	09	17 09 08 15 06 06 07 02 02 03 00 00 00 00 00 00 00 00	$\mathfrak{d}_3(\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
43.	1010110	10	15 09 12 09 09 05 07 03 03 01 01 00 00 00 00 00 00 00	$\mathfrak{d}_4(\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
44.	1101010	10	13 11 10 09 08 09 04 04 02 02 01 00 00 00 00 00 00 00	$\mathfrak{d}_4(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
45.	1011001	10	15 11 09 11 08 06 05 06 01 01 01 00 00 00 00 00 00 00	$\mathfrak{d}_4(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})$
46.	0111001	10	15 09 10 09 08 09 04 04 03 01 02 00 00 00 00 00 00 00	$\mathfrak{d}_4(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})$
47.	0011011	10	13 09 12 08 09 06 06 03 04 01 02 00 00 00 00 00 00 00	$\mathfrak{d}_4(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
48.	1001101	10	13 10 11 10 08 06 06 04 02 02 01 00 00 00 00 00 00 00	$\mathfrak{d}_4(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
49.	0101101	10	15 08 10 09 09 06 07 03 03 01 03 00 00 00 00 00 00 00	$\mathfrak{d}_4(\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
50.	0001111	10	15 09 08 10 09 06 05 05 02 02 03 00 00 00 00 00 00 00	$\mathfrak{d}_4(\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
51.	0011100	10	17 07 10 12 06 09 03 06 02 01 02 00 00 00 00 00 00 00	$\mathfrak{d}_3(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})$
52.	1010111	11	13 09 10 09 08 06 06 04 03 03 01 01 00 00 00 00 00 00	$\mathfrak{d}_5(\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})$
53.	1101011	11	11 10 10 08 08 07 06 04 03 02 02 01 00 00 00 00 00 00	$\mathfrak{d}_5(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
54.	1111000	11	19 07 09 08 10 06 06 04 04 01 01 01 00 00 00 00 00 00	$\mathfrak{d}_4(\mathbb{C})\oplus\mathfrak{sl}(4,\mathbb{C})$
55.	1011010	11	13 10 09 09 09 06 05 05 04 01 01 01 00 00 00 00 00 00	$\mathfrak{d}_4(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$

56.	0111010	11	13 09 08 10 06 09 06 03 04 02 01 02 00 00 00 00 00 00	$\mathfrak{d}_4(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
57.	1101100	11	15 08 09 10 06 09 06 03 04 01 02 01 00 00 00 00 00 00	$\mathfrak{d}_4(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})$
58.	1001110	11	13 09 08 11 08 06 05 06 02 02 02 01 00 00 00 00 00 00	$\mathfrak{d}_4(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
59.	0101110	11	15 07 08 09 09 06 06 05 03 02 01 03 00 00 00 00 00 00	$\mathfrak{d}_4(\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
60.	0011101	11	13 08 09 10 08 06 06 04 04 02 01 02 00 00 00 00 00 00	$\mathfrak{d}_4(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
61.	1111001	12	13 09 08 08 08 07 06 04 04 03 01 01 01 00 00 00 00 00	$\mathfrak{d}_5(\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})$
62.	1011011	12	11 09 09 09 07 07 05 05 03 04 01 01 01 00 00 00 00 00	$\mathfrak{d}_5(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
63.	0111011	12	11 08 09 08 07 07 06 05 03 03 02 01 02 00 00 00 00 00	$\mathfrak{d}_5(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
64.	1101101	12	11 09 08 09 07 07 06 05 03 03 01 02 01 00 00 00 00 00	$\mathfrak{d}_5(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
65.	1001111	12	11 09 08 08 09 06 05 05 04 02 02 02 01 00 00 00 00 00	$\mathfrak{d}_5(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
66.	0101111	12	13 07 08 07 09 06 06 04 05 02 02 01 03 00 00 00 00 00	$\mathfrak{d}_5(\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})$
67.	1011100	12	15 07 09 08 09 06 06 03 06 02 01 01 01 00 00 00 00 00	$\mathfrak{d}_4(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})$
68.	0111100	12	15 07 06 11 06 06 09 03 03 04 01 01 02 00 00 00 00 00	$\mathfrak{d}_4(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})$
69.	0011110	12	13 07 08 08 10 05 06 04 05 02 02 01 02 00 00 00 00 00	$\mathfrak{d}_4(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
70.	1111010	13	11 09 07 08 07 07 06 05 03 04 02 01 01 01 00 00 00 00	$\mathfrak{d}_5(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
71.	1101110	13	11 08 07 08 08 06 06 05 04 03 02 01 02 01 00 00 00 00	$\mathfrak{d}_5(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
72.	1011101	13	11 08 08 08 08 06 06 04 04 04 02 01 01 01 00 00 00 00	$\mathfrak{d}_5(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
73.	0111101	13	11 08 06 09 07 06 06 06 03 03 03 01 01 02 00 00 00 00	$\mathfrak{d}_5(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
74.	0011111	13	11 07 08 07 08 07 05 04 05 03 02 02 01 02 00 00 00 00	$\mathfrak{d}_5(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$

75.	1111011	14	09 08 08 07 07 06 06 05 04 03 03 02 01 01 01 00 00 00	$\mathfrak{d}_6(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
76.	1101111	14	09 08 07 07 07 07 05 05 04 04 02 02 01 02 01 00 00 00	$\mathfrak{d}_6(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
77.	1111100	14	13 07 06 08 07 06 06 06 03 03 04 01 01 01 01 00 00 00	$\mathfrak{d}_5(\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})$
78.	1011110	14	11 07 07 08 07 07 05 05 03 05 02 02 01 01 01 00 00 00	$\mathfrak{d}_5(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
79.	0111110	14	11 07 06 07 08 06 05 06 04 03 03 02 01 01 02 00 00 00	$\mathfrak{d}_5(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
80.	1111101	15	09 08 06 07 07 06 05 06 04 03 03 03 01 01 01 01 00 00	$\mathfrak{d}_6(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
81.	1011111	15	09 07 07 07 07 06 06 04 04 04 03 02 02 01 01 01 00 00	$\mathfrak{d}_6(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
82.	0111111	15	09 07 06 07 06 07 05 05 04 04 03 02 02 01 01 02 00 00	$\mathfrak{d}_6(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
83.	1111110	16	09 07 06 06 07 06 05 05 05 03 03 03 02 01 01 01 01 00	$\mathfrak{d}_6(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
84.	1111111	17	07 07 06 06 06 06 05 05 04 04 03 03 02 02 01 01 01 01	$\mathfrak{d}_7(\mathbb{C})$

	$-\lvert\alpha_i\rvert$	μ	$\dim_{\mathbb{C}} \mathfrak{g}$	$\mathfrak{g}_0$
			0 ±1 ±2 ±3 ±4 ±5 ±6 ±7 ±8 ±9 ±10 ±11 ±12 ±13 ±14 ±15 ±16 ±17 ±18 ±19 ±20 ±21 ±22 ±23 ±24 ±25 ±26 ±27 ±28 ±29	
001.	1 1 0 0 0 0 0 0	05	50 28 35 21 08 07 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_2(\mathbb{C})\oplus\mathfrak{sl}(7,\mathbb{C})$
002.	0 0 0 0 0 0 1 1	05	80 28 27 27 01 01 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_2(\mathbb{C})\oplus\mathfrak{e}_6$
003.	1 0 1 0 0 0 0 0	06	50 22 20 36 07 07 07 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_2(\mathbb{C})\oplus\mathfrak{sl}(7,\mathbb{C})$
004.	1 0 0 0 0 0 1 1	07	48 27 26 18 17 10 01 01 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_3(\mathbb{C})\oplus\mathfrak{so}(10,\mathbb{C})$
005.	0 1 1 0 0 0 0 0	07	40 18 29 21 15 12 03 06 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_2(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(6,\mathbb{C})$
006.	0 1 0 0 0 1 0 0	07	34 25 30 18 16 10 05 03 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_2(\mathbb{C})\oplus\mathfrak{sl}(5,\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})$
007.	0 0 0 0 0 1 1 0	07	50 18 32 20 10 16 01 02 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_2(\mathbb{C})\oplus\mathfrak{so}(10,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
008.	1 0 1 0 0 0 0 1	08	38 22 21 20 22 07 06 06 01 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_3(\mathbb{C})\oplus\mathfrak{sl}(6,\mathbb{C})$
009.	0 1 0 0 1 0 0 0	08	32 20 24 24 10 16 06 04 04 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_2(\mathbb{C})\oplus\mathfrak{sl}(4,\mathbb{C})\oplus\mathfrak{sl}(4,\mathbb{C})$
010.	1 1 1 0 0 0 0 0	09	38 13 21 14 21 15 06 07 02 06 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_3(\mathbb{C})\oplus\mathfrak{sl}(6,\mathbb{C})$

011.	1 0 1 0 0 0 1 0	09	30 21 20 15 22 12 07 05 05 02 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_3(\mathbb{C})\oplus\mathfrak{sl}(5,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
012.	0 1 0 0 0 1 0 1	09	30 22 25 17 16 11 10 05 02 01 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_3(\mathbb{C})\oplus\mathfrak{sl}(5,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
013.	1 0 0 0 0 1 1 0	09	34 18 24 18 17 10 09 08 01 02 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_3(\mathbb{C})\oplus\mathfrak{so}(8,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
014.	0 0 0 0 0 1 1 1	09	48 18 17 26 10 10 16 01 01 01 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_3(\mathbb{C})\oplus\mathfrak{so}(10,\mathbb{C})$
015.	0 0 1 0 1 0 0 0	09	28 18 24 15 19 12 09 06 03 04 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_2(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})\oplus\mathfrak{sl}(4,\mathbb{C})$
016.	0 0 0 0 1 1 0 0	09	34 13 30 15 15 15 05 10 01 03 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_2(\mathbb{C})\oplus\mathfrak{sl}(5,\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})$
017.	0 1 0 1 0 0 0 0	09	34 16 15 30 10 10 15 03 03 05 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_2(\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})\oplus\mathfrak{sl}(5,\mathbb{C})$
018.	0 1 0 0 0 1 1 0	10	30 17 20 21 12 15 06 10 05 01 02 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_3(\mathbb{C})\oplus\mathfrak{sl}(5,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
019.	1 0 1 0 0 1 0 0	10	26 19 18 15 16 18 07 07 04 04 03 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_3(\mathbb{C})\oplus\mathfrak{sl}(4,\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})$
020.	0 1 0 0 1 0 0 1	10	26 19 22 18 16 10 12 06 04 03 01 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_3(\mathbb{C})\oplus\mathfrak{sl}(4,\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})$
021.	0 0 1 1 0 0 0 0	10	32 12 20 19 11 20 05 10 04 02 05 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_2(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(5,\mathbb{C})$
022.	1 0 1 0 0 0 1 1	11	28 17 20 15 15 13 12 06 05 05 01 01 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_4(\mathbb{C})\oplus\mathfrak{sl}(5,\mathbb{C})$

023.	1 0 0 0 0 1 1 1	11	32 18 17 17 17 10 09 09 08 01 01 01 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_4(\mathbb{C})\oplus\mathfrak{so}(8,\mathbb{C})$
024.	0 1 0 0 1 0 1 0	11	24 16 20 16 17 10 11 08 06 04 02 02 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_3(\mathbb{C})\oplus\mathfrak{sl}(4,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
025.	1 0 1 0 1 0 0 0	11	26 16 15 18 09 19 12 05 07 03 03 04 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_3(\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})\oplus\mathfrak{sl}(4,\mathbb{C})$
026.	0 0 1 0 1 0 0 1	11	22 18 21 15 15 13 11 07 06 03 03 01 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_3(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})$
027.	1 0 0 0 1 1 0 0	11	26 13 22 15 15 12 12 07 05 06 01 03 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_3(\mathbb{C})\oplus\mathfrak{sl}(4,\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})$
028.	0 0 0 0 1 1 0 1	11	30 14 21 20 10 15 10 05 10 01 02 01 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_3(\mathbb{C})\oplus\mathfrak{sl}(5,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
029.	1 1 0 1 0 0 0 0	11	30 13 15 15 20 10 10 10 06 03 02 05 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_3(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(5,\mathbb{C})$
030.	0 1 0 1 0 0 0 1	11	26 17 15 21 16 10 09 12 03 03 04 01 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_3(\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})\oplus\mathfrak{sl}(4,\mathbb{C})$
031.	0 0 0 1 1 0 0 0	11	28 10 24 12 18 12 08 12 03 06 01 04 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_2(\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(4,\mathbb{C})$
032.	0 1 0 0 0 1 1 1	12	28 17 16 16 16 11 10 06 10 05 01 01 01 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_4(\mathbb{C})\oplus\mathfrak{sl}(5,\mathbb{C})$
033.	1 0 1 0 0 1 0 1	12	22 17 18 14 14 13 13 07 06 04 04 02 01 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_4(\mathbb{C})\oplus\mathfrak{sl}(4,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
034.	0 0 1 0 1 0 1 0	12	20 16 18 16 12 15 09 10 05 06 03 02 02 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_3(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$

035.	0 1 0 0 1 1 0 0	12	26 11 18 18 12 15 06 13 04 06 04 01 03 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_3(\mathbb{C})\oplus\mathfrak{sl}(4,\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})$
036.	0 0 0 0 1 1 1 0	12	30 13 12 25 10 10 15 05 05 10 01 01 02 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_3(\mathbb{C})\oplus\mathfrak{sl}(5,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
037.	0 1 0 1 0 0 1 0	12	22 16 15 15 19 10 09 09 09 03 03 03 02 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_3(\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
038.	1 0 1 1 0 0 0 0	12	30 12 11 20 09 11 20 05 05 07 02 02 05 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_3(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(5,\mathbb{C})$
039.	0 0 1 1 0 0 0 1	12	24 14 18 16 13 13 13 06 08 04 02 04 01 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_3(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(4,\mathbb{C})$
040.	1 0 1 0 0 1 1 0	13	22 13 18 13 13 12 13 08 07 05 04 04 01 02 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_4(\mathbb{C})\oplus\mathfrak{sl}(4,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
041.	0 1 0 0 1 0 1 1	13	22 15 18 14 15 11 11 07 08 06 04 02 01 01 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_4(\mathbb{C})\oplus\mathfrak{sl}(4,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
042.	1 0 1 0 1 0 0 1	13	20 16 15 15 12 12 13 10 05 06 03 03 03 01 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_4(\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})$
043.	1 0 0 0 1 1 0 1	13	22 14 17 16 14 10 12 09 06 05 06 01 02 01 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_4(\mathbb{C})\oplus\mathfrak{sl}(4,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
044.	0 0 1 0 1 1 0 0	13	22 12 15 19 08 16 09 09 09 03 06 03 01 03 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_3(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})$
045.	1 1 0 1 0 0 0 1	13	22 15 14 13 17 12 10 08 09 05 03 02 04 01 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_4(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(4,\mathbb{C})$
046.	0 1 1 1 0 0 0 0	13	30 08 15 10 19 11 10 10 05 10 02 03 01 05 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_3(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(5,\mathbb{C})$

047.	0 0 1 1 0 0 1 0	13	20 14 16 14 14 11 13 08 07 06 04 02 03 02 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_3(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
048.	0 1 0 1 0 1 0 0	13	22 13 15 12 18 12 09 07 10 06 03 03 02 03 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_3(\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})$
049.	1 0 0 1 1 0 0 0	13	24 10 18 12 14 12 12 08 08 06 03 04 01 04 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_3(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(4,\mathbb{C})$
050.	0 0 0 1 1 0 0 1	13	22 12 19 15 12 15 08 09 09 03 06 01 03 01 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_3(\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})$
051.	1 0 1 0 1 0 1 0	14	18 14 15 13 13 09 14 09 08 05 05 03 03 02 02 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_4(\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
052.	0 0 1 0 1 0 1 1	14	18 15 17 13 14 10 12 08 08 05 06 03 02 01 01 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_4(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
053.	0 1 0 0 1 1 0 1	14	22 12 15 16 14 10 11 08 09 04 06 04 01 02 01 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_4(\mathbb{C})\oplus\mathfrak{sl}(4,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
054.	1 0 0 0 1 1 1 0	14	22 13 12 17 14 10 09 12 06 05 05 06 01 01 02 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_4(\mathbb{C})\oplus\mathfrak{sl}(4,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
055.	0 0 0 0 1 1 1 1	14	28 13 12 16 15 10 10 10 05 05 10 01 01 01 01 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_4(\mathbb{C})\oplus\mathfrak{sl}(5,\mathbb{C})$
056.	1 1 0 1 0 0 1 0	14	18 15 13 12 14 14 09 09 07 08 04 03 02 03 02 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_4(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
057.	0 1 0 1 0 0 1 1	14	20 14 15 15 13 13 09 09 06 09 03 03 03 01 01 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_4(\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})$
058.	1 0 1 1 0 0 0 1	14	22 14 11 16 12 09 13 13 05 05 06 02 02 04 01 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_4(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(4,\mathbb{C})$

059.	0 0 1 1 0 1 0 0	14	20 12 14 14 12 13 09 12 05 08 04 04 02 02 03 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_3(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})$
060.	0 1 0 1 1 0 0 0	14	26 08 15 12 12 18 06 10 04 12 03 03 03 01 04 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_3(\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})\oplus\mathfrak{sl}(4,\mathbb{C})$
061.	0 0 0 1 1 0 1 0	14	20 12 14 18 09 14 11 06 10 06 03 06 01 02 02 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_3(\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
062.	1 0 1 0 0 1 1 1	15	20 13 15 13 13 09 13 08 08 06 05 04 04 01 01 01 00 00 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_5(\mathbb{C})\oplus\mathfrak{sl}(4,\mathbb{C})$
063.	0 1 0 0 1 1 1 0	15	22 11 12 14 16 09 10 08 10 05 04 06 04 01 01 02 00 00 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_4(\mathbb{C})\oplus\mathfrak{sl}(4,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
064.	1 0 1 0 1 1 0 0	15	20 10 15 12 13 08 13 09 09 06 05 04 03 03 01 03 00 00 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_4(\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})$
065.	0 0 1 0 1 1 0 1	15	18 13 13 16 12 10 11 09 08 07 03 06 03 01 02 01 00 00 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_4(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
066.	1 1 1 1 0 0 0 0	15	28 08 11 10 15 09 11 10 10 05 05 06 02 02 01 05 00 00 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_4(\mathbb{C})\oplus\mathfrak{sl}(5,\mathbb{C})$
067.	0 1 1 1 0 0 0 1	15	22 11 13 11 14 13 09 10 07 06 08 02 03 01 04 01 00 00 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_4(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(4,\mathbb{C})$
068.	1 0 1 1 0 0 1 0	15	18 14 11 13 13 09 10 13 08 05 05 05 02 02 03 02 00 00 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_4(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
069.	0 0 1 1 0 0 1 1	15	18 12 17 12 13 10 12 08 09 05 06 04 02 03 01 01 00 00 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_4(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})$
070.	1 1 0 1 0 1 0 0	15	18 13 12 12 11 15 09 09 07 07 07 03 03 02 02 03 00 00 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_4(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})$

071.	0 1 0 1 0 1 0 1	15	18 13 14 12 15 11 11 07 08 07 06 03 03 02 02 01 00 00 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_4(\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
072.	1 0 0 1 1 0 0 1	15	18 12 15 13 12 11 12 08 08 07 05 03 04 01 03 01 00 00 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_4(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})$
073.	0 0 1 1 1 0 0 0	15	24 08 12 16 08 15 07 12 08 04 09 02 04 02 01 04 00 00 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_3(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(4,\mathbb{C})$
074.	0 0 0 1 1 1 0 0	15	22 10 09 21 09 09 15 06 06 11 03 03 06 01 01 03 00 00 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_3(\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})$
075.	1 0 1 0 1 0 1 1	16	16 13 14 13 11 11 09 11 07 08 04 05 03 03 02 01 01 00 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_5(\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
076.	1 0 0 0 1 1 1 1	16	20 13 12 12 15 10 09 09 09 05 05 05 06 01 01 01 01 00 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_5(\mathbb{C})\oplus\mathfrak{sl}(4,\mathbb{C})$
077.	0 0 1 0 1 1 1 0	16	18 12 11 13 15 08 11 08 09 07 05 03 06 03 01 01 02 00 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_4(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
078.	1 1 0 1 0 0 1 1	16	16 13 14 11 13 11 11 08 08 06 07 04 03 02 03 01 01 00 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_5(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})$
079.	0 1 1 1 0 0 1 0	16	18 12 11 12 10 14 10 08 09 05 07 06 02 03 01 03 02 00 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_4(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
080.	0 1 0 1 0 1 1 0	16	18 11 13 12 12 13 09 09 06 09 04 06 03 03 02 01 02 00 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_4(\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
081.	1 0 1 1 0 1 0 0	16	18 12 11 11 13 09 10 09 12 05 05 05 04 02 02 02 03 00 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_4(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})$
082.	0 0 1 1 0 1 0 1	16	16 12 14 12 13 09 12 08 09 06 06 04 04 02 02 02 01 00 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_4(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$

083.	1 1 0 1 1 0 0 0	16	22 09 11 13 08 14 12 06 10 04 08 06 02 03 02 01 04 00 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_4(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(4,\mathbb{C})$
084.	0 1 0 1 1 0 0 1	16	20 10 13 12 12 12 12 07 07 06 09 03 03 03 01 03 01 00 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_4(\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})$
085.	1 0 0 1 1 0 1 0	16	16 12 12 14 11 10 11 10 06 08 06 04 03 04 01 02 02 00 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_4(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
086.	0 0 0 1 1 0 1 1	16	18 11 15 13 12 11 10 09 07 07 06 03 06 01 02 01 01 00 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_4(\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
087.	0 1 0 0 1 1 1 1	17	20 11 12 11 14 11 09 07 10 06 05 04 06 04 01 01 01 01 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_5(\mathbb{C})\oplus\mathfrak{sl}(4,\mathbb{C})$
088.	1 0 1 0 1 1 0 1	17	16 11 13 12 12 09 10 09 09 07 06 04 04 03 03 01 02 01 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_5(\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
089.	1 1 1 1 0 0 0 1	17	20 11 10 10 12 11 09 09 10 07 05 05 05 02 02 01 04 01 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_5(\mathbb{C})\oplus\mathfrak{sl}(4,\mathbb{C})$
090.	1 0 1 1 0 0 1 1	17	16 12 12 13 10 11 08 11 08 08 05 04 05 02 02 03 01 01 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_5(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})$
091.	1 1 0 1 0 1 0 1	17	14 13 12 11 11 12 10 09 07 07 06 06 03 03 02 02 02 01 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_5(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
092.	0 1 1 1 0 1 0 0	17	18 11 09 13 08 12 13 06 09 07 04 08 04 02 03 01 02 03 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_4(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})$
093.	0 0 1 1 0 1 1 0	17	16 10 14 10 13 09 11 08 09 06 07 04 04 04 02 02 01 02 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_4(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
094.	0 1 0 1 1 0 1 0	17	18 10 11 12 12 09 13 08 07 05 08 06 03 03 03 01 02 02 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_4(\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$

095.	1 0 1 1 1 0 0 0	17	22 08 11 10 12 10 09 07 12 08 04 05 05 03 02 02 01 04 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_4(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(4,\mathbb{C})$
096.	0 0 1 1 1 0 0 1	17	18 10 11 14 10 11 09 10 08 07 05 07 02 04 02 01 03 01 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_4(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})$
097.	1 0 0 1 1 1 0 0	17	18 10 09 15 11 09 09 12 06 06 08 05 03 03 04 01 01 03 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_4(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})$
098.	0 0 0 1 1 1 0 1	17	18 11 09 16 12 09 10 10 06 07 08 03 03 06 01 01 02 01 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_4(\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
099.	1 0 1 0 1 1 1 0	18	16 10 11 12 11 10 08 10 07 09 05 06 03 04 03 03 01 01 02 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_5(\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
100.	0 0 1 0 1 1 1 1	18	16 12 11 11 12 12 07 09 08 08 05 05 03 06 03 01 01 01 01 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_5(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})$
101.	1 1 1 1 0 0 1 0	18	16 12 09 10 10 11 09 09 08 09 05 05 05 04 02 02 01 03 02 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_5(\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
102.	0 1 1 1 0 0 1 1	18	16 10 13 10 11 11 09 10 07 07 06 05 06 02 03 01 03 01 01 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_5(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})$
103.	1 1 0 1 0 1 1 0	18	14 11 12 10 11 10 11 08 08 06 07 05 05 03 03 02 02 01 02 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_5(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
104.	0 1 0 1 0 1 1 1	18	16 11 12 11 12 10 11 07 08 07 06 04 06 03 03 02 01 01 01 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_5(\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
105.	1 0 1 1 0 1 0 1	18	14 12 11 11 11 10 08 10 08 09 05 05 04 04 02 02 02 02 01 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_5(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
106.	1 1 0 1 1 0 0 1	18	16 11 10 12 09 11 11 09 07 07 05 07 05 02 03 02 01 03 01 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_5(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})$

107.	1 0 0 1 1 0 1 1	18	14 11 13 11 12 09 10 09 08 06 07 05 04 03 04 01 02 01 01 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_5(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
108.	0 1 1 1 1 0 0 0	18	22 08 07 14 08 08 15 07 06 10 04 04 09 02 02 03 01 01 04 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_4(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(4,\mathbb{C})$
109.	0 0 1 1 1 0 1 0	18	16 10 10 12 12 08 10 09 09 06 06 06 05 02 04 02 01 02 02 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_4(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
110.	0 1 0 1 1 1 0 0	18	20 08 09 12 12 09 09 12 06 06 04 10 03 03 03 03 01 01 03 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_4(\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})$
111.	0 0 0 1 1 1 1 0	18	18 10 09 11 15 09 08 10 07 06 08 05 03 03 06 01 01 01 02 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_4(\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
112.	1 1 1 1 0 1 0 0	19	16 11 08 10 09 10 09 10 06 09 07 04 05 05 03 02 02 01 02 03 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_5(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})$
113.	0 1 1 1 0 1 0 1	19	14 11 10 11 09 11 09 10 07 07 06 05 06 04 02 03 01 02 02 01 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_5(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
114.	1 0 1 1 0 1 1 0	19	14 10 11 11 09 11 07 10 07 09 06 05 05 03 04 02 02 02 01 02 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_5(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
115.	0 0 1 1 0 1 1 1	19	14 10 13 10 11 10 09 08 09 06 07 05 04 04 04 02 02 01 01 01 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_5(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
116.	1 1 0 1 1 0 1 0	19	14 11 09 11 10 09 10 10 07 07 05 06 06 04 02 03 02 01 02 02 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_5(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
117.	0 1 0 1 1 0 1 1	19	16 09 12 10 12 09 10 09 08 05 07 05 06 03 03 03 01 02 01 01 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_5(\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
118.	1 0 1 1 1 0 0 1	19	16 10 10 10 11 09 10 06 10 08 07 04 05 04 03 02 02 01 03 01 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_5(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})$

119.	1 0 0 1 1 1 0 1	19	14 11 09 12 12 09 08 10 08 06 06 07 04 03 03 04 01 01 02 01 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_5(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
120.	0 0 1 1 1 1 0 0	19	18 08 09 10 14 07 08 10 09 06 06 05 07 03 02 04 02 01 01 03 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_4(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})$
121.	1 0 1 0 1 1 1 1	20	14 10 11 10 11 09 10 06 09 07 07 05 05 03 04 03 03 01 01 01 01 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_6(\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})$
122.	1 1 1 1 0 0 1 1	20	14 10 11 09 10 09 10 07 09 07 07 05 05 04 04 02 02 01 03 01 01 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_6(\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})$
123.	1 1 0 1 0 1 1 1	20	12 11 11 10 10 10 09 09 07 07 06 06 04 05 03 03 02 02 01 01 01 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_6(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
124.	0 1 1 1 0 1 1 0	20	14 09 11 09 10 09 09 10 07 07 06 05 06 04 04 02 03 01 02 01 02 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_5(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
125.	1 1 1 1 1 0 0 0	20	20 08 07 10 09 08 10 09 07 06 10 04 04 05 05 02 02 02 01 01 04 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_5(\mathbb{C})\oplus\mathfrak{sl}(4,\mathbb{C})$
126.	0 1 1 1 1 0 0 1	20	16 10 07 12 09 08 11 09 07 07 07 04 05 07 02 02 03 01 01 03 01 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_5(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})$
127.	1 0 1 1 1 0 1 0	20	14 10 09 10 10 09 09 07 08 09 06 06 04 05 03 03 02 02 01 02 02 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_5(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
128.	0 0 1 1 1 0 1 1	20	14 09 11 11 10 10 08 08 09 07 05 07 04 05 02 04 02 01 02 01 01 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_5(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
129.	1 1 0 1 1 1 0 0	20	16 09 08 10 11 08 09 09 09 06 06 04 07 05 03 02 03 02 01 01 03 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_5(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})$
130.	0 1 0 1 1 1 0 1	20	16 09 09 10 12 09 09 08 10 05 05 06 07 03 03 03 03 01 01 02 01 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_5(\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$

131.	1 0 0 1 1 1 1 0	20	14 10 09 09 13 09 08 08 09 06 06 06 06 03 03 03 04 01 01 01 02 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_5(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
132.	0 0 0 1 1 1 1 1	20	16 10 09 11 10 12 08 08 07 07 07 05 05 03 03 06 01 01 01 01 01 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_5(\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
133.	1 1 1 1 0 1 0 1	21	12 11 09 09 09 09 09 08 08 07 07 06 04 05 04 03 02 02 01 02 02 01 00 00 00 00 00 00 00 00	$\mathfrak{d}_6(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
134.	1 0 1 1 0 1 1 1	21	12 10 10 11 09 09 09 07 08 08 06 06 05 04 03 04 02 02 02 01 01 01 00 00 00 00 00 00 00 00	$\mathfrak{d}_6(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
135.	1 1 0 1 1 0 1 1	21	12 10 10 10 09 10 08 09 08 07 05 06 05 05 04 02 03 02 01 02 01 01 00 00 00 00 00 00 00 00	$\mathfrak{d}_6(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
136.	0 1 1 1 1 0 1 0	21	14 10 07 10 10 08 08 10 08 06 07 05 04 06 05 02 02 03 01 01 02 02 00 00 00 00 00 00 00 00	$\mathfrak{d}_5(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
137.	0 1 0 1 1 1 1 0	21	16 08 09 08 12 09 09 07 09 07 05 04 08 04 03 03 03 03 01 01 01 02 00 00 00 00 00 00 00 00	$\mathfrak{d}_5(\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
138.	1 0 1 1 1 1 0 0	21	16 08 08 10 09 10 07 08 07 09 06 06 05 04 05 02 03 02 02 01 01 03 00 00 00 00 00 00 00 00	$\mathfrak{d}_5(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})$
139.	0 0 1 1 1 1 0 1	21	14 09 09 09 12 09 08 06 11 06 06 05 06 05 03 02 04 02 01 01 02 01 00 00 00 00 00 00 00 00	$\mathfrak{d}_5(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
140.	1 1 1 1 0 1 1 0	22	12 09 10 08 09 08 09 07 09 06 07 06 05 04 05 03 03 02 02 01 02 01 02 00 00 00 00 00 00 00	$\mathfrak{d}_6(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
141.	0 1 1 1 0 1 1 1	22	12 09 10 10 08 10 08 08 08 07 06 05 06 04 04 04 02 03 01 02 01 01 01 00 00 00 00 00 00 00	$\mathfrak{d}_6(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
142.	1 1 1 1 1 0 0 1	22	14 10 07 09 09 08 08 10 06 07 07 07 04 04 05 04 02 02 02 01 01 03 01 00 00 00 00 00 00 00	$\mathfrak{d}_6(\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})$

143.	1 0 1 1 1 0 1 1	22	12 09 10 09 10 08 09 07 07 08 07 05 06 04 04 03 03 02 02 01 02 01 01 00 00 00 00 00 00 00	$\mathfrak{d}_6(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
144.	1 1 0 1 1 1 0 1	22	12 10 08 09 10 09 08 08 08 08 05 05 05 06 04 03 02 03 02 01 01 02 01 00 00 00 00 00 00 00	$\mathfrak{d}_6(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
145.	1 0 0 1 1 1 1 1	22	12 10 09 09 10 10 08 08 07 07 06 06 05 05 03 03 03 04 01 01 01 01 01 00 00 00 00 00 00 00	$\mathfrak{d}_6(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
146.	0 1 1 1 1 1 0 0	22	16 08 07 08 11 08 07 08 10 06 06 06 04 04 07 03 02 02 03 01 01 01 03 00 00 00 00 00 00 00	$\mathfrak{d}_5(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})$
147.	0 0 1 1 1 1 1 0	22	14 08 09 08 10 11 07 06 09 08 05 06 04 07 03 03 02 04 02 01 01 01 02 00 00 00 00 00 00 00	$\mathfrak{d}_5(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
148.	1 1 1 1 1 0 1 0	23	12 10 07 08 09 08 07 09 07 07 06 07 05 04 04 05 03 02 02 02 01 01 02 02 00 00 00 00 00 00	$\mathfrak{d}_6(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
149.	0 1 1 1 1 0 1 1	23	12 09 08 10 08 09 08 08 07 08 06 05 05 05 04 05 02 02 03 01 01 02 01 01 00 00 00 00 00 00	$\mathfrak{d}_6(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
150.	1 1 0 1 1 1 1 0	23	12 09 08 08 09 10 07 08 07 08 06 05 04 06 05 03 03 02 03 02 01 01 01 02 00 00 00 00 00 00	$\mathfrak{d}_6(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
151.	0 1 0 1 1 1 1 1	23	14 08 09 08 10 09 09 07 08 06 07 04 06 05 04 03 03 03 03 01 01 01 01 01 00 00 00 00 00 00	$\mathfrak{d}_6(\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})$
152.	1 0 1 1 1 1 0 1	23	12 09 08 09 09 09 08 07 06 09 06 06 05 05 04 04 02 03 02 02 01 01 02 01 00 00 00 00 00 00	$\mathfrak{d}_6(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
153.	1 1 1 1 0 1 1 1	24	10 09 09 09 08 08 08 08 06 08 06 06 05 05 04 04 03 03 02 02 01 02 01 01 01 01 00 00 00 00	$\mathfrak{d}_7(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
154.	1 1 1 1 1 1 0 0	24	14 08 07 07 09 08 07 07 08 07 06 06 06 04 04 04 05 02 02 02 02 01 01 01 03 03 00 00 00 00	$\mathfrak{d}_6(\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})$

155.	0 1 1 1 1 1 0 1	24	12 09 07 08 09 09 07 08 06 09 06 05 05 04 05 05 03 02 02 03 01 01 01 02 01 01 00 00 00 00	$\mathfrak{d}_6(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
156.	1 0 1 1 1 1 1 0	24	12 08 08 08 09 08 09 06 06 08 07 05 06 04 05 04 03 02 03 02 02 01 01 01 02 02 00 00 00 00	$\mathfrak{d}_6(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
157.	0 0 1 1 1 1 1 1	24	12 08 09 08 09 09 09 06 07 07 07 05 05 05 05 03 03 02 04 02 01 01 01 01 01 01 00 00 00 00	$\mathfrak{d}_6(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
158.	1 1 1 1 1 0 1 1	25	10 09 08 08 08 08 07 08 07 06 07 06 05 05 04 04 04 03 02 02 02 01 01 02 01 01 00 00 00 00	$\mathfrak{d}_7(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
159.	1 1 0 1 1 1 1 1	25	10 09 08 08 08 09 08 07 07 07 06 06 04 05 05 04 03 03 02 03 02 01 01 01 01 01 00 00 00 00	$\mathfrak{d}_7(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
160.	0 1 1 1 1 1 1 0	25	12 08 07 08 07 10 07 07 06 08 07 05 05 04 04 06 03 03 02 02 03 01 01 01 01 01 00 00 00 00	$\mathfrak{d}_6(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
161.	1 1 1 1 1 1 0 1	26	10 09 07 07 08 08 07 07 07 06 07 06 05 05 04 04 04 04 02 02 02 02 01 01 01 01 01 00 00 00	$\mathfrak{d}_7(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
162.	1 0 1 1 1 1 1 1	26	10 08 08 08 08 08 08 07 06 06 07 06 05 05 04 05 03 03 02 03 02 02 01 01 01 01 01 00 00 00	$\mathfrak{d}_7(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
163.	1 1 1 1 1 1 1 0	27	10 08 07 07 07 08 07 07 06 06 07 06 05 05 04 04 04 04 03 02 02 02 02 01 01 01 01 02 00 00	$\mathfrak{d}_7(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
164.	0 1 1 1 1 1 1 1	27	10 08 07 08 07 08 08 07 06 06 07 06 05 04 04 05 04 03 03 02 02 03 01 01 01 01 01 01 00 00	$\mathfrak{d}_7(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
165.	1 1 1 1 1 1 1 1	29	08 08 07 07 07 07 07 07 06 06 05 07 05 05 04 04 04 04 03 03 02 02 02 02 01 01 01 01 01 01	$\mathfrak{d}_8(\mathbb{C})$

SIMPLE LEVI-TANAKA ALGEBRAS OF THE COMPLEX TYPE F_4

	$-\lvert\alpha_i\rvert$				μ	$\dim_{\mathbb{C}} \mathfrak{g}$ 0 ±1 ±2 ±3 ±4 ±5 ±6 ±7 ±8 ±9 ±10 ±11	$\mathfrak{g}_0$
1.	1	1	0	0	05	10 08 06 06 01 01 00 00 00 00 00 00	$\mathfrak{d}_2(\mathbb{C}) \oplus \mathfrak{sl}(3,\mathbb{C})$
2.	1	1	0	1	07	06 06 05 05 03 03 01 01 00 00 00 00	$\mathfrak{d}_3(\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C})$
3.	0	1	1	0	07	08 04 04 07 03 02 01 02 00 00 00 00	$\mathfrak{d}_2(\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C})$
4.	1	1	1	0	09	06 04 03 06 03 03 02 01 01 01 00 00	$\mathfrak{d}_3(\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C})$
5.	0	1	1	1	09	06 04 03 04 03 04 02 01 01 02 00 00	$\mathfrak{d}_3(\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C})$
6.	1	1	1	1	11	04 04 03 03 03 04 02 02 01 01 01 01	$\mathfrak{d}_4(\mathbb{C})$

SIMPLE LEVI-TANAKA ALGEBRAS OF THE COMPLEX TYPE G_2

$$\mathcal{B}_{-1} = \{\alpha_1, \alpha_2\}, \quad \mu = 5, \quad \begin{cases} \dim_{\mathbb{C}} \mathfrak{g}_0 = 2 \\ \dim_{\mathbb{C}} \mathfrak{g}_{\pm 1} = 2 \\ \dim_{\mathbb{C}} \mathfrak{g}_{\pm 2} = 1 \\ \dim_{\mathbb{C}} \mathfrak{g}_{\pm 3} = 1 \\ \dim_{\mathbb{C}} \mathfrak{g}_{\pm 4} = 1 \\ \dim_{\mathbb{C}} \mathfrak{g}_{\pm 5} = 1 \end{cases}, \qquad \mathfrak{g}_0 \simeq \mathfrak{d}_2(\mathbb{C}).$$

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