NONCONVEX PROBLEMS FOR DIFFERENTIAL INCLUSIONS

António Ornelas

Ph D thesis 4 / Oct 88 / M

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I dedicate this thesis to Henrique Ornelas, who contributed very much to it
even if he died 18 years ago

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Nonconvex problems for differential inclusions

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Contents

Chapter 1 Preliminaries

1.1 Introduction and overview

We consider in this thesis Cauchy problems for differential inclusions:

(CP) $\quad x' \in F(t, x)$ a.e. on $[0, T]$, $x(0) = \xi$,

where $F: [0,T] \times \mathbb{R}^n \to \mathbb{R}^n$ is a multifunction with closed values. By multifunction we mean to say that F is a correspondence associating to each (t, x) in $[0,T] \times \mathbb{R}^n$ a nonempty subset $F(t, x)$ of $\mathbb{R}^n$. We define the solution set $S(\xi)$ of (CP) to be the set of all absolutely continuous functions $x: [0, T] \to \mathbb{R}^n$ verifying $x(0) = \xi$ and having the time derivative $x'(t)$ in the closed set $F(t, x(t))$ of allowed velocities for almost every t in the interval $[0, T]$. We always suppose $F(t, x)$ to depend on t in a measurable way, and to have a dependence on x either lsc (lower semicontinuous) or continuous or Lipschitz. In general we do not assume the values $F(t, x)$ to be convex. Convexity is assumed only in sections 2.1 and 3.1, and there this assumption is stressed by dealing with the closed convex hull $\text{co}\, F(t, x)$. In some cases the values $F(t, x)$ are compact, possibly integrably bounded; while in other cases they may be unbounded.

The Cauchy problem (CP) is usually studied using one of the following methods:

(a) continuous selections in $\mathbb{R}^n$;

(b) "polygonal" approximate solutions;

(c) successive projections;

(d) continuous selections in L^1;

(e) Baire category;

(f) directionally continuous selections in $\mathbb{R}^n$.

Here we do not deal with methods (b), (e) or (f). As to (b), see Filippov [28, 29], Kaczinsky-Olech [37], Olech [45], Lojasiewicz jr [40], Lojasiewicz jr [41], Himmelberg-Van Vleck [35]. For method (e) see Cellina, [14], DeBlasi-Pianigiani [23, 24, 25], Bressan-Colombo [10]. Method (f) can be seen in Bressan [6, 7], Bressan-Cortesi [11].

Consider now method (a). If F is measurable in (t, x) and lsc in x then we may apply Michael's theorem to obtain a selection f(t, x) from co F(t, x) which is measurable in t and continuous in x. Another possibility is to have F uniformly continuous relative to x and to construct a selection f(t, x) which is as uniformly continuous relative to x as F. For example, f(t, ·) Lipschitz in case F(t, ·) is Lipschitz. In section 2.1 we do more than this: we show that co F(t, x) can be represented as f(t, x, U), where f is uniformly continuous in (x, u), with a modulus of continuity equal to that of F(t, ·) multiplied by a constant, and U is a convex closed set in $\mathbb{R}^n$, which is compact provided the values F(t, x) are compact. Clearly this solves in particular the above selection problem, since each $u \in U$ gives automatically one such selection.

In section 3.1 this representation of co F(t, x) is applied to reduce the relaxed Cauchy problem in $\mathbb{R}^n$

(CPR) $\quad x' \in \text{co } F(t, x) \quad$ a.e. on $[0, T]$, $\quad x(0) = \xi$,

to a control differential equation in $\mathbb{R}^n$

(CDE) $\quad x' = f(t, x, u) \quad$ a.e. on $[0, T]$, $\quad x(0) = \xi$, $\quad u(t) \in U \subset \mathbb{R}^n$.

Such reductions were known, but the regularity conditions on f(t, ·) and U were not satisfactory. Namely, either f (t, ·) was non-Lipschitz for Lipschitz F(t, ·), or U was infinite dimensional. What we prove here is a new result: a representation with f(t, ·) Lipschitz whenever F(t, ·) is Lipschitz and $U \subset \mathbb{R}^n$. In particular it shows that differential inclusions with convex valued multifunctions co F(t, x) in $\mathbb{R}^n$ *do not* generalize differential equations with control in $\mathbb{R}^n$ (in case F(t, ·) is at least continuous).

We use method (d) in section 2.2, where we consider lsc multifunctions G defined on a compact metric space, with values which are closed bounded decomposable subsets of L^1. We construct "guided" continuous selections g from the multifunction G, using Liapunov's theorem on the range of vector measures.

In section 3.2 these continuous selections g are seen as Nemitskii operators, the multifunction G being there the Nemitskii multivalued operator induced in L^1 by the

multifunction F(t, x). In fact, the Nemitskii multivalued operator G turns out to be lsc provided the multifunction F is lsc relative to x. It follows that any solution of the operator equation $x' = g(x)$ is also a solution of the differential inclusion $x'(t) \in F(t, x(t))$; and this is used to prove the following approximation result. Let f(t, x) be any selection from the convexified multifunction co F(t, x) which is measurable in t and continuous in x. Then *at least one* solution of the differential equation $x' = f(t, x)$, $x(0) = \xi$ can be approximated by solutions of (CP).

Section 3.3 deals with multifunctions F(t, x) which are continuous in x and integrably bounded. In the first part we suppose that F has a modulus of continuity relative to x of Kamke type, namely implying uniqueness; and show that the solution set $S(\xi)$ of (CP) is dense in the solution set of the relaxed problem (CPR). This is not a new result but appears here under a new light: it is a straigthforward consequence of the results of sections 2.1 and 3.2, i.e. of methods (a) and (d).

In the second part of section 3.3 we consider multifunctions F(t, x) which are continuous relative to x, and prove a closure property: namely that the solution sets $S(\xi)$ of (CP) are closed *if and only if* the values F(t, x) are convex. The "if" part is well-known; what we prove here is that if some values F(t, x) are nonconvex then some solution sets $S(\xi)$ are nonclosed. The proof uses methods (a) and (d), through the results of sections 2.1 and 3.2. Notice that the closure property obtained here is the analogue for differential inclusions of a well known result in the calculus of variations, namely that integral functionals are weakly sequentially lsc if and only if integrands are convex. Notice also that this is a sharp result, in the sense that continuous means usc (upper semicontinuous) *and* lsc, while the above closure property turns up to be false if $F(t, \cdot)$ fails to be *either* usc *or* lsc.

Method (c) is used in section 2.3, dealing with multifunctions F which are Lipschitz in x with unbounded values. The first result is the construction of continuous selections $s(\xi)$ from the solution sets $S(\xi)$. By "cutting" the solution sets at the end time $t = T$ one then obtains continuous selections $a(\xi)$ from the attainable sets $A(\xi)$. Two things are specially interesting here: first, these continuous selections are obtained despite the fact that the solution sets and the attainable sets are nonclosed nonconvex in general; and second, the proof does not use neither Liapunov's theorem on the range of a vector measure nor any previous existence result. These selection results are used to parametrize the solution sets $S(\xi)$ and the attainable sets $A(\xi)$. Namely we show that $S(\xi)$ can be represented as $g(\xi, \mathcal{U})$, and $A(\xi)$ can be represented as $h(\xi, \mathcal{U})$, where $\mathcal{U}$ is a closed subset of a separable Banach space $\mathcal{X}$ and $g: \Xi \times \mathcal{X} \to C^0$, $h: \Xi \times \mathcal{X} \to \mathbb{R}^n$ are continuous maps. In particular each attainable set $A(\xi)$ is an analytic subset of $\mathbb{R}^n$. Another useful consequence of these continuous selections is the existence, under appropriate conditions, of periodic solutions.

In the second part of section 2.3 we construct a sharper selection result by using Liapunov's theorem, as follows. Fix some $y(\xi)$, depending continuously on ξ in AC, an approximate solution of (CP). We construct a true solution $x(\xi)$ of (CP), depending continuously on ξ in AC, at a distance from $y(\xi)$ which is almost minimal. The minimal distance, given by the Filippov-Gronwall inequality, is incompatible with continuous dependence, as simple examples show.

Finally in section 3.4 we consider again multifunctions $F(t, x)$ which are Lipschitz in x with unbounded values, and use method (c) to show that if some values $F(t, x)$ are nonconvex then some solutions of the relaxed problem (CPR) do not solve (CP).

The above description gives an overview of the original part of this thesis, chapters 2 and 3. All the results concerning continuous selections from multifunctions are grouped together in chapter 2 (even in case the multifunctions involved are solution sets), while chapter 3 contains results related to differential inclusions. An introduction is given at the begining of each section, comparing the results of this thesis with known results and giving the appropriate references.

Chapter 1 is completed below with a list of basic concepts, to fix notation and assumptions common to the whole thesis, together with some basic well-known results for easier reference.

1.2 General assumptions and basic results

Analysis

L^1 denotes, as usual, the Banach space of (equivalence classes of) functions $u : [0, T] \to \mathbb{R}^n$ which are Lebesgue integrable on the interval $[0, T]$. The norm in L^1 is $|u|_1 := \int_0^T |u(\tau)|\, d\tau$, where $|\cdot|$ denotes the euclidian norm of the m-dimensional euclidian space $\mathbb{R}^m$. The specific dimension m we have in mind in each case will be clear from the context. By integrating functions in L^1 one obtains the Banach space AC of absolutely continuous functions $x: [0, T] \to \mathbb{R}^m$ with norm $|x|_{AC} := |x(0)| + |x'|_1$. These AC functions are often considered with the uniform topology of the Banach space C^0 of continuous functions $x: [0, T] \to \mathbb{R}^m$ with norm $|x|_\infty := \sup\{ |x(t)|: t \in [0, T] \}$. Usually Ξ is a compact set in $\mathbb{R}^n$, where initial data ξ for the Cauchy problem (CP) is supposed to lie. The characteristic function χ_A of a set A is, as usual, $\chi_A(x) = 1$ for $x \in A$, $= 0$ for $x \notin A$. The set-theoretical difference $A \setminus B$ is $\{ a \in A : a \notin B \}$. A G_δ subset of a metric space is a countable intersection of open subsets, in particular it can be any open or closed subset. If the metric space is complete separable then any G_δ subset is metrizable complete (Cohn [18, 8.1.4]). Given two metric spaces X, Y with distances d, D then their cartesian product is a metric space with distance $d + D$.

Multifunctions

Fix $x \in \mathbb{R}^m$ and consider a closed nonempty set $A \subset \mathbb{R}^m$; the distance from x to A is $d(x, A) := \inf\{ |x - a| : a \in A \}$. Consider another closed nonempty set B in $\mathbb{R}^m$; the separation of A from B is $dl^+(A, B) := \sup\{ d(a, B): a \in A \}$. If $dl^+(A, B), dl^+(B, A)$ are finite then the Hausdorff distance between A and B is:

$$dl(A, B) := \max\{ dl^+(A, B),\ dl^+(B, A) \}.$$

The following properties are easily proved and widely used:

$$dl(A, B) = 0 \quad \text{iff} \quad A = B, \qquad dl(A, C) \le dl(A, B) + dl(B, C),$$

$$\forall a \in \mathbb{R}^m \quad \exists b \in B: \ |a-b| = d(a, B), \qquad d(a, A) = 0 \quad \text{iff} \quad a \in A,$$

$$\forall a \in A \quad \exists b \in B: \ |a-b| \leq dl(A, B),$$

$$d(y, B) \leq |y - x| + d(x, A) + dl(A, B).$$

Let A, B be any sets. We say that $F: A \to B$ is a multifunction if F is a correspondence associating to each point a in A a nonempty subset F(a) of B. The graph of F is the set $\{ (a, b) \in A \times B: b \in F(a) \}$. Given a set $C \subset B$, we may consider two types of inverse images of C by F:

$$F^-(C) := \{ a \in A: \ F(a) \cap C \neq \emptyset \}, \quad F^+(C) := \{ a \in A: \ F(a) \subset C \}.$$

Suppose now that F is single-valued, i.e. there exists a function $f: A \to B$ such that for each $a \in A$, F(a) is the set $\{ f(a) \}$; then clearly $F^-(C) = F^+(C) = f^{-1}(C)$.

Measurability concepts

Consider a Lebesgue measurable set I on [0, T], and let $\mathcal{L}$ be the σ-algebra of Lebesgue measurable subsets of I. More generally, we may suppose that I is a separable metrizable space together with a σ-algebra $\mathcal{A}$ which is the completion of the Borel σ-algebra of I relative to a σ- finite positive measure μ. Consider a metric space X, and let $\mathcal{B}$ be the σ-algebra of Borel subsets of X. Consider the product σ-algebra $\mathcal{A} \otimes \mathcal{B}$ on the cartesian product $I \times X$, generated by all sets of the form $A \times B$ with $A \in \mathcal{A}$, $B \in \mathcal{B}$. If $C \in \mathcal{A} \otimes \mathcal{B}$ and X is complete separable then the projection of C on I belongs to $\mathcal{A}$ (Cohn [18, 8.5.4]).

A multifunction $F: I \to X$ is said *measurable* provided $F^-(C) \in \mathcal{A}$ for each set C closed in X. If F is measurable with closed values then graph $F \in \mathcal{A} \otimes \mathcal{B}$ and the distance $t \longmapsto d(x, F(t))$ is measurable $\forall x \in X$ (Himmelberg [33,Theorem 3.5]). Suppose now that X is complete metric separable; then F is measurable *iff* there exists a countable family $\mathcal{F}$ of measurable selections f(t) from F(t) such that the closure of $\{ f(t): f \in \mathcal{F} \}$ is F(t) $\forall t$ *iff* graph $F \in \mathcal{A} \otimes \mathcal{B}$ *iff* the distance $t \longmapsto d(x, F(t))$ is measurable (Himmelberg [33, Theorem 3.5, Theorem 5.6]).

Consider now a multifunction $F: I \to \mathbb{R}^n$, measurable with closed values. Associate to F the multifunction co F, each value co F(t) being the closed convex hull of F(t); then co F is measurable (Himmelberg [33, Theorem 9.1]). Let $u: I \to \mathbb{R}^n$ be a measurable function; then, by an easy consequence of the Kuratowski-Ryll Nardzewski theorem [38], there exists a

measurable selection f from F at minimal distance, i.e. $f: I \to \mathbb{R}^n$ measurable such that $f(t) \in F(t)$ and $|u(t) - f(t)| = d(u(t, F(t))$ $\forall t$ (Himmelberg-Van Vleck [34, Proposition 1]).

Continuity concepts

Let X,Y be metric spaces and $F: X \to Y$ a multifunction. To define some continuity concepts we consider sets O open in Y, points x_0 in X and sequences (x_i) converging in X to x_0. Then:

(a) F *lsc* (*lower semicontinuous*) *at* x_0 means:

$$F(x_0) \cap O \neq \emptyset \quad \Rightarrow \quad F(x_i) \cap O \neq \emptyset \qquad \text{for large } i;$$

(b) F *h-lsc* (*Hausdorff lsc)* *at* x_0 means:

$$dl^+ (F(x_0), F(x_i)) \to 0 \qquad \text{as} \qquad i \to \infty;$$

(c) F *usc* (*upper semicontinuous*) *at* x_0 means:

$$F(x_0) \subset O \quad \Rightarrow \quad F(x_i) \subset O \qquad \text{for large } i;$$

(d) F *h-usc* (*Hausdorff usc*) *at* x_0 means:

$$dl^+ (F(x_i), F(x_0)) \to 0 \qquad \text{as} \qquad i \to \infty;$$

(e) F *(h-) lsc* means F (h-) lsc at each $x_0 \in X$;

(f) F *(h-) usc* means F (h-) usc at each $x_0 \in X$;

(g) F *(h-) continuous* means F (h-) lsc and (h-) usc.

It is easy to see that "F h-lsc at x_0" *implies* "F lsc at x_0" ; and the two concepts are *equivalent* provided $F(x_0)$ is compact. Similarly "F usc at x_0" *implies* "F h-usc at x_0"; and the two concepts are equivalent provided $F(x_0)$ is compact. The following *equivalences* are also easy to prove:

F lsc $\Leftrightarrow$ $F^-(O)$ is open in X *provided* O is open in Y

$\Leftrightarrow$ $F^+(C)$ is closed in X *provided* C is closed in Y;

F usc $\Leftrightarrow$ $F^-(C)$ is closed in X *provided* C is closed in Y

$\Leftrightarrow$ $F^+(O)$ is open in X *provided* O is open in Y.

Moreover, "F usc with closed values" *implies* "graph F closed in $X \times Y$"; and "Y compact" with "graph F closed" *imply* "F usc" (Aubin-Cellina [2, Corollary 1.1.1]).

Let now Y be a Banach space. Consider the multifunction co F: $X \to Y$, each value co F(x) being the closed convex hull of F(x). One easily proves that "F lsc" *implies* "co F lsc"; and that $dl^+(\text{co } F(x), \text{co } F(\underline{x})) \leq dl^+(F(x), F(\underline{x}))$. In particular co F is h-usc or h-lsc provided F is, and

$$dl\,(\text{co } F(x), \text{co } F(\underline{x})) \leq dl\,(F(x), F(\underline{x})).$$

Therefore if F is lsc then by Michael's selection theorem (Michael [44]) there exists a continuous selection f from co F, i.e. a continuous function f: $X \to Y$ such that $f(x) \in \text{co } F(x)$ $\forall$ x.

Scorza-Dragoni property

Let I be as above (in "Measurability concepts") and let X be a separable space metrizable complete. Consider a multifunction $F: I \times X \to \mathbb{R}^n$ with closed values. Then:

(i) "F (·, x) is measurable $\forall x \in X$ and F(t, ·) is lsc with closed graph for a.e. $t \in I$"

if and only if

(ii) "$\forall\, \varepsilon > 0$ $\exists$ closed set $I_\varepsilon \subset I$ with $\mu\,(I \setminus I_\varepsilon) \leq \varepsilon$ such that $F_{|I_\varepsilon \times X}$ is lsc with closed graph" . (Lojasiewicz jr [41, Corollary 4 + Remark 19 + Remark 12]).

If moreover F(t, ·) is continuous and each value F(t, x) is compact then $F_{|I_\varepsilon \times X}$ is continuous (Himmelberg-Van Vleck [34, Theorem 1]), provided I is compact and μ is a Radon measure.

It is easy to see that condition (ii) implies the following :

(iii) $\exists\ I_0 \subset I$ such that $I \setminus I_0$ is a null set and $F_{|I_0 \times X}$ is $\mathcal{A} \otimes \mathcal{B}$– measurable.

In fact, let O be any open set in $\mathbb{R}^n$ and let (I_k) be a sequence of disjoint closed sets such that $I = \mathcal{N} \cup I_0$, $I_0 = \cup I_k$, $\mathcal{N}$ is a null set and $F_k := F_{|I_k \times X}$ is lsc . Then $F_k^{-}(O)$ is the intersection of an open set with the closed set $I_k \times X$, hence is a Borel set. Therefore each F_k, hence $F_{|I_0 \times X}$, is $\mathcal{A} \otimes \mathcal{B}$- measurable (Himmelberg [33, Theorem 3.5]).

Finally we show that condition (iii) implies :

(iv) $\forall\ x: I \to X$ measurable, the multifunction $\Phi: I \to X$, $\Phi(t) := F(t, x(t))$ is measurable.

In fact, for each closed set C in $\mathbb{R}^n$ we can write : $\Phi^{-}(C) = \{\ t \in I:\ F(t, x(t)) \cap C \neq \emptyset\ \} =$

$= \{\ t \in I:\ F(t, \xi) \cap C \neq \emptyset$ for some ξ with $(t, \xi) \in \text{graph}(x)\ \} =$

$=$ projection of $F^{-}(C) \cap \text{graph}(x)$ on I.

But, appart from a null set, this is the projection of an $\mathcal{A} \otimes \mathcal{B}$- measurable set, by (iii), hence is measurable.

Chapter 2 Selection theorems

2.1 Uniformly continuous selections in $\mathbb{R}^n$

Introduction

Let $F: X \to \mathbb{R}^n$ be a multifunction which is Lipschitz with constant ℓ and has values $F(x)$ bounded by m. We show that $\text{co}\, F(x)$ can be represented as $f(x, U)$, with U the unit closed ball in $\mathbb{R}^n$ and f Lipschitz with constant $6n(2\ell + m)$. Existing representations were: either with U the unit closed ball in $\mathbb{R}^n$ but f just continuous in (x, u) (Ekeland-Valadier [27]) ; or with f Lipschitz in (x, u) but U in some infinite dimensional space (LeDonne-Marchi [39]).

More generally, let $F: I \times X \to \mathbb{R}^n$ be a multifunction with $F(\cdot, x)$ measurable and $F(t, \cdot)$ uniformly continuous. We show that $\text{co}\, F(t, x)$ can be represented as $f(t, x, U)$, where U is either the unit closed ball in $\mathbb{R}^n$ (in case the values $F(t, x)$ are compact) or $U = \mathbb{R}^n$ (in case the values $F(t, x)$ are unbounded). As to f, we obtain $f(\cdot, x, u)$ measurable and $f(t, \cdot, \cdot)$ uniformly continuous (with modulus of continuity equal to that of $F(t, \cdot)$ multiplied by a constant).

Assumptions and main results

Let I be a Lebesgue measurable set in $\mathbb{R}^n$ (or, more generally, a separable metrizable space together with a σ-algebra $\mathcal{A}$ which is the completion of the Borel σ-algebra of I relative to a σ-finite positive measure μ). Let X be an open or closed set in $\mathbb{R}^n$ (or, more generally, a separable space metrizable complete, with a distance d and Borel σ-algebra $\mathcal{B}$). We consider multifunctions F with values $F(t, x)$ either bounded by a linear growth condition -- hypothesis (FLB) -- or unbounded -- hypothesis (FU).

Hypothesis (FLB)

$F : I \times X \to \mathbb{R}^n$ is a multifunction with:

(a) values $F(t, x)$ compact ;

(b) $F(\cdot, x)$ measurable ;

(c) $\exists\ \alpha, m : I \to \mathbb{R}^+$ measurable such that

$$y \in F(t, x) \Rightarrow |y| \leq \alpha(t)\ x + m(t) \quad \text{for a.e. } t ;$$

(d) X is compact, I is σ-compact, $F(t, \cdot)$ is continuous for a.e. t.

Hypothesis (FU)

$F : I \times X \to \mathbb{R}^n$ is a multifunction with:

(a') values $F(t, x)$ closed;

(b') $F(\cdot, x)$ measurable;

(d') $\exists\ w : I \times \mathbb{R}^+ \to \mathbb{R}^+$ such that: $dl\,(F(t, x), F(t, \underline{x})) \leq w\,(t, d(x, \underline{x}))$,

with $w(\cdot, r)$ measurable, $w(t, \cdot)$ continuous concave, $w(t, 0) = 0$ for a.e. t.

Proposition 1

Let F verify hypothesis **(FLB)**
Then F verifies hypothesis **(FU)** also, namely it verifies (d') with

$$w\,(t, r) \leq 2\ \alpha(t)\ r + 2\,m(t) .$$

Theorem 1

Let F verify hypothesis **(FU)**. Suppose moreover that each value $F(t, x)$ is compact, and set $M(t, x) := \max \{ |y| : y \in F(t, x) \}$.

Then there exists a function $f : I \times X \times U \to \mathbb{R}^n$, with U the unit closed ball in $\mathbb{R}^n$, such that:

(i) $\operatorname{co} F(t, x) = f(t, x, U) \quad \forall x$ for a.e. t;

(ii) $f(\cdot, x, u)$ is measurable;

(iii) $|f(t, x, u) - f(t, \underline{x}, \underline{u})| \leq 12n \; w(t, d(x, \underline{x})) + 6n \; M(t, x) \; |u - \underline{u}|$ for a.e. t.

If moreover F, w are jointly continuous then f is continuous.

Corollary 1

Let F verify hypothesis **(FU)**.
Let U be a convex closed set in $\mathbb{R}^n$ and let $h : I \times X \times U \to \mathbb{R}^n$ verify:

(α) $\operatorname{co} F(t, x) \subset h(t, x, U) \quad \forall x$ for a.e. t;

(β) $u \longmapsto h(t, x, u)$ has inverse $h^{-1}(t, x, \cdot) : h(t, x, u) \longmapsto u \quad \forall x, u$ for a.e. t;

(γ) $h(\cdot, x, u)$ and $h^{-1}(\cdot, x, u)$ are measurable;

(δ) $h(t, \cdot, \cdot)$ and $h^{-1}(t, \cdot, \cdot)$ are jointly continuous for a.e. t.

Then there exists a function $f : I \times X \times U \to \mathbb{R}^n$ such that (i), (ii) of Th. 1 hold and :

(iii') $|f(t, x, u) - f(t, \underline{x}, \underline{u})| \leq 6n \; w(t, d(x, \underline{x})) + 6n \; |h(t, x, u) - h(t, \underline{x}, \underline{u})|$ a.e..

Corollary 2

Let F verify hypothesis **(FU)**.

Then, setting $h(t, x, u) = u$ in Corollary 1, the conclusions of Theorem 1 hold with $U = \mathbb{R}^n$ and $M(t, x) \equiv 1$. (The final part provided F is jointly h - continuous.)

Theorem 2

Let F verify hypothesis **(FU)** and let I be σ-compact.

Then there exists a σ - compact set E in a Banach space, a function $\varphi : X \times E \to \mathbb{R}^n$ and a multifunction $\mathcal{U} : I \to E$ such that :

(i) $\operatorname{co} F(t, x) = \varphi(x, \mathcal{U}(t)) \ \forall x$ for a.e. t;

(ii) $\mathcal{U}(\cdot)$ is measurable with convex closed values;

(iii) $\varphi(x, \cdot)$ is linear nonexpansive;

(iv) $|\varphi(x, u) - \varphi(\underline{x}, u)| \leq 6n \ w(t, d(x, \underline{x})) \ \forall u \in \mathcal{U}(t)$ for a.e. t.

If moreover F is integrably bounded then the values $\mathcal{U}(t)$ are compact for a.e. t.

Intermediate results and proofs

Proof of Proposition 1

Apply the Scorza-Dragoni property in 1.2 (ii) to obtain a sequence (I_k) of compact disjoint sets such that $I = I_0 \cup \mathcal{N}$, $\mathcal{N}$ is a null set, $I_0 = \cup I_k$, and $F_k := \text{co}\, F_{|I_k \times X}$, $\alpha_{|I_k}$, $m_{|I_k}$ are continuous. Set $\alpha_k := \max \alpha_{|I_k}$, $m_k := \max m_{|I_k}$ and :

$$v_k(r) := \sup \{ \, dl\,(F_k(t, x), F_k(t, \underline{x})) : t \in I_k, |x - \underline{x}| \leq r \, \}.$$

It is clear that $v_k(\cdot)$ is nondecreasing and $v_k(r) \leq 2\alpha_k r + 2m_k$. Since I_k, X are compact and F_k is jointly h - continuous, we must have $v_k(r) \to 0$ as $r \to 0$, otherwise a contradiction would follow. By a lemma of McShane [42], there exists a continuous concave function $w_k : \mathbb{R}^+ \to \mathbb{R}^+$ such that $w_k(0) = 0$, $w_k(r) \geq v_k(r)$, hence

$$dl\,(F_k(t, x), F_k(t, \underline{x})) \leq w_k(|x - \underline{x}|) \quad \forall\, t \in I_k.$$

Set $w(t, r) := \min \{ w_k(r), 2\alpha(t) r + 2m(t) \}$ for $t \in I_k$,
$w(t, r) := 2m(t) + 2\alpha(t) r$ for $t \in \mathcal{N}$. ♦

Lemma 1

Let $\mathcal{K}$ be any family of nonempty closed convex sets in $\mathbb{R}^n$ such that $dl\,(K, \underline{K}) < \infty$ $\forall\, K, \underline{K}$ in $\mathcal{K}$. Let $B(y, K)$ be the closed ball around y with radius $r(y, K) := \sqrt{3}\; d(y, K)$.

Then the map $P : \mathbb{R}^n \times \mathcal{K} \to \mathcal{K}$, $P(y, K) := K \cap B(y, K)$ is well defined, verifies $P(y, K) = \{ y \}$ whenever $y \in K$, and :

$$dl\,(P(y, K), P(y, \underline{K}) \leq 3\; dl\,(K, \underline{K}) + [1 + \sqrt{3}]\; |y - \underline{y}|.$$

Remark

This lemma refines and simplifies the construction of LeDonne-Marchi. We have changed the expansion constant from 2 to $\sqrt{3}$ in the definition of the radius r because we believe this value to be the best possible. More precisely, we believe that the Lipschitz constant

3 for the above intersection cannot be improved, and that it is not obtainable unless one uses the expansion constant $\sqrt{3}$.

Moreover, in the definition of the radius r we do not use the Hausdorff distance between two sets, as LeDonne-Marchi, but rather the distance from a point to a set. This is not only conceptually simpler but also seems better fitted for applications (as in Theorem 1).

Proof

(a) First we fix y_* in $\mathbb{R}^n$ and prove that

$$dl\,(P(y_*, K), P(y_*, \underline{K})) \leq 3\ dl\,(K, \underline{K}) \quad \forall\, K, \underline{K} \in \mathcal{K}.$$

Choose any $K, \underline{K}$ in $\mathcal{K}$ and any $\underline{y} \in P(y_*, \underline{K})$. Set $\varepsilon_* := d(y_*, K)$, $\underline{\varepsilon} := dl\,(\underline{K}, K)$. We may suppose that $\varepsilon_*, \underline{\varepsilon} > 0$, otherwise just take $y := y_*, \underline{y}$ respectively. To prove the above inequality we need only find a point y in $P(y_*, K)$ such that $|y - \underline{y}| \leq 3\ \underline{\varepsilon}$.

To find y, choose points y_1, y_2 in K such that $|y_* - y_1| \leq \varepsilon_*$, $|y_2 - \underline{y}| \leq \underline{\varepsilon}$. If $|y_* - y_2| \geq \sqrt{3}\ \varepsilon_*$ then take $y := y_2$. Otherwise $y_2 \notin P(y_*, K)$; but in the segment $]\,y_1, y_2\,[$ certainly there exists some point y such that $|y_* - y| = \sqrt{3}\ \varepsilon_*$, hence $y \in P(y_*, K)$. If $|y - \underline{y}| \leq 3\ \underline{\varepsilon}$ then (a) is proved. Otherwise by the claim below we have $|y_* - \underline{y}| = |y_* - z| + |z - \underline{y}| > \sqrt{3}\ (\varepsilon_* + \underline{\varepsilon})$. But this is absurd because $\underline{y} \in P(y_*, \underline{K})$ hence $|y_* - \underline{y}| \leq \sqrt{3}\ d(y_*, \underline{K}) \leq \sqrt{3}\,(\varepsilon_* + \underline{\varepsilon})$. Therefore (a) is proved.

Trigonometrical **claim :** If $|y - \underline{y}| > 3\,\underline{\varepsilon}$ then $\exists\ z \in\,]\,y_*, \underline{y}\,[$ such that :

$$|y_* - z| > \sqrt{3}\ \varepsilon_* \qquad \text{and} \qquad |z - \underline{y}| > \sqrt{3}\ \underline{\varepsilon}\ .$$

In fact, as we prove below, in the triangle $\underline{y}, y, y_*$ the angle $\theta + \pi/2$ at y verifies $\operatorname{sen}\theta > 1/\sqrt{3}$, in particular $\theta > 0$. Therefore in the segment $]\,y_*, \underline{y}\,[$ certainly there exists a point z such that in the triangle y_*, y, z the angle at y is $\pi/2$. This implies that $|y_* - z| > |y_* - y| = \sqrt{3}\ \varepsilon_*$, and since $1/\sqrt{3} < \operatorname{sen}\theta \leq |z - \underline{y}|/|y - \underline{y}| < |z - \underline{y}|/(3\ \underline{\varepsilon})$, we have $|z - \underline{y}| > \sqrt{3}\ \underline{\varepsilon}$.

To prove $\operatorname{sen}\theta > \dfrac{1}{\sqrt{3}}$, set $0 < \beta_0 := \operatorname{arcsen}\dfrac{1}{3} < \dfrac{\pi}{6} < \alpha_0 := \operatorname{arcsen}\dfrac{1}{\sqrt{3}} < \dfrac{\pi}{4}$ and notice that we only need to show that $\theta > \alpha_0$. Since $\pi - \alpha_0 - \beta_0 = \alpha_0 + \pi/2$, it is enough to prove that $\theta + \pi/2 > \pi - \alpha_0 - \beta_0$. To prove this notice that in the triangle y_*, y, y_1 the

angle α at y verifies $\operatorname{sen} \alpha \le \varepsilon_*/(\sqrt{3}\,\varepsilon_*) = 1/\sqrt{3}$, hence $\alpha \le \alpha_0$. In fact we must have $0 \le \alpha \le \alpha_0$ and not $\pi - \alpha_0 \le \alpha \le \pi$ because the later is incompatible with the fact that the angle α has an adjacent side which is larger that the opposite side. Similarly, in the triangle $\underline{y}$, y, y_2 the angle β at y verifies $\operatorname{sen} \beta < \underline{\varepsilon}/3\underline{\varepsilon} = 1/3$, hence $\beta < \beta_0$. In fact we must have $0 \le \beta < \beta_0$ inside the claim and not $\pi - \beta_0 < \beta \le \pi$ because the later would imply $\beta \ge \pi/2$ hence $|\underline{y} - y| \le |\underline{y} - y_2| \le \underline{\varepsilon}$. Finally, to show that $\theta + \pi/2 > \pi - \alpha_0 - \beta_0$, we distinguish the following possibilities :

(i) let $\underline{y}$ be in the y_*, y_1, y_2- plane, in the same side of the y_1, y_2- line as y_* ; then the inequality $\theta + \pi/2 = \pi - \alpha - \beta > \pi - \alpha_0 - \beta_0$ is obvious;

(ii) let $\underline{y}$ be in the y_*, y_1, y_2- plane, in the side of the y_1, y_2- line opposite to y_*, and let $0 \le \beta \le \alpha$; then $\theta + \pi/2 = \pi - \alpha + \beta > \pi - \alpha - \beta > \pi - \alpha_0 - \beta_0$;

(iii) as in (ii) but with $\alpha \le \beta < \beta_0$; then $\theta + \pi/2 = \pi - \beta + \alpha > \pi - \alpha_0 - \beta_0$;

(iv) let $\underline{y}$ be outside the y_*, y_1, y_2- plane and let the projection $\underline{y}'$ of $\underline{y}$ into that plane fall in the side of the y_1, y_2- line opposite to y_* and let the angle β', projection of the angle β on that plane, verify $0 \le \beta' \le \alpha$; then $\theta + \pi/2 > \pi - \alpha_0 > \pi - \alpha_0 - \beta_0$;

(v) as in (iv) but $\alpha \le \beta' < \beta_0$; then $\theta + \pi/2 \ge \pi - \beta' - \alpha > \pi - \alpha_0 - \beta_0$;

(vi) as in (iv) but $\underline{y}'$ in the same side as y_* ; then it is clear that the situation is similar to that in (i), the difference being that $\theta + \pi/2 > \pi - \alpha - \beta$.

This proves the claim.

(b) Now consider points y, $\underline{y}$ in $\mathbb{R}^n$ and sets K, $\underline{K}$ in $\mathcal{K}$. Setting $\varepsilon := \sqrt{3}\ d(y, K)$, $\underline{\varepsilon} := \sqrt{3}\ d(\underline{y}, K)$, and using (a) one obtains:

$$\begin{aligned} dl(P(y,K), P(\underline{y},\underline{K})) &\le dl(P(y,K), P(\underline{y},K)) + dl(P(\underline{y},K), P(\underline{y},\underline{K})) \le \\ &\le dl(B(y,\varepsilon), B(\underline{y},\underline{\varepsilon})) + 3\ dl(K,\underline{K}) \le \\ &\le |y-\underline{y}| + |\varepsilon-\underline{\varepsilon}| + 3\ dl(K,\underline{K}) \le |y-\underline{y}| + \sqrt{3}\ |y-\underline{y}| + \\ &+ 3\ dl(K,\underline{K}). \end{aligned}$$

♦

To prove Theorem 1 we need the following result :

Proposition 2 (Bressan [3])

Denote by $\mathcal{K}^n$ the family of nonempty compact convex sets in $\mathbb{R}^n$. Then there exists a map $\sigma : \mathcal{K}^n \to \mathbb{R}^n$ that selects a point $\sigma(K) \in K$ for each K and verifies:

$$|\sigma(K) - \sigma(\underline{K})| \leq 2n \ \mathrm{dl}(K, \underline{K}).$$

Proof of Theorem 1

Fix any point $x_0 \in X$ and set : $M_0(t) := \sup \{ |y| : y \in F(t, x_0) \}$, $M(t, x) := M_0(t) + w(t, d(x, x_0))$. Clearly

$$y \in F(t, x) \Rightarrow |y| \leq M(t, x),$$

and using the subadditivity of w one obtains:

$$|M(t, x) - M(t, \underline{x})| \leq |w(t, d(x, x_0)) - w(t, d(\underline{x}, x_0)| \leq w(t, d(x, x_0)).$$

Consider the function $h : I \times X \times U \to \mathbb{R}^n$, $h(t, x, u) := M(t, x)\, u$.
Clearly $h(t, x, \cdot)$ is an homeomorphism between the ball U and the ball of radius $M(t, x)$; let $h^{-1}(t, x, y) := M(t, x)^{-1}\, y$ be the inverse homeomorphism.

Project now $h(t, x, u)$ into $\mathrm{co}\, F(t, x)$, i.e. set $f(t, x, u) := \sigma \circ P\, [h(t, x, u), \mathrm{co}\, F(t, x)]$, where σ is the selection in Proposition 2 and P is the multivalued projection in Lemma 2.

Claim : $f(\cdot, x, u)$ is measurable.

To prove this, notice first that $M_0(\cdot)$ is measurable by Himmelberg [32, Theorem 5.8]. Then $M(\cdot, x)$ and $h(\cdot, x, u)$ are measurable. Consider the closed ball $B(\cdot, x, u)$ of radius $r(\cdot, x, u) := \sqrt{3}\ d(h(\cdot, x, u), \mathrm{co}\, F(\cdot, x))$ around $h(\cdot, x, u)$. Then $r(\cdot, x, u)$ is measurable by Himmelberg [32, Theorem 3.5, Theorem 6.5], and since

$$d(y, B(\cdot, x, u)) = (|y - h(\cdot, x, u)| - r(\cdot, x, u))^+$$

by Himmelberg [32, Theorem 3.5, Theorem 4.1], $B(\cdot, x, u)$ and its intersection with $\mathrm{co}\, F(\cdot, x)$ are measurable. Therefore this intersection is a measurable map : $I \to \mathcal{K}^n$; and since $\sigma : \mathcal{K}^n \to \mathbb{R}^n$ is continuous, $f(\cdot, x, u)$ is measurable.

It is easy to prove (iii) using the Lipschitz properties of σ and P :

$$f(t, x, u) - f(t, \underline{x}, \underline{u})| \leq 6n \; |M(t, x)\, u - M(t, x)\, \underline{u}| +$$

$$+ 6n\, w(t, d(x, \underline{x})) \leq 6n \; M(t, x)\, |u - \underline{u}| + 6n \; |M(t, x) - M(t, \underline{x})| +$$

$$+ 6n\, w(t, d(x, \underline{x})) \leq 12n \; w(t, d(x, \underline{x})) + 6n \; M(t, x) \; |u - \underline{u}|.$$

It is clear that if F is jointly h–continuous then $M_0(\cdot)$ is continuous; and if also w is jointly continuous then M is jointly continuous hence h is jointly continuous. Then the ball B is continuous and its intersection with $\mathrm{co}\, F$ is continuous, by the h–continuity of $\mathrm{co}\, F$. This means that the intersection is a continuous map : $I \times X \times \mathcal{U} \to \mathcal{K}^n$, and since σ: $\mathcal{K}^n \to \mathbb{R}^n$ is continuous, f is jointly continuous.

To prove (i) fix some $\underline{t} \in I$, $\underline{x} \in X$; then for any $\underline{y} \in \mathrm{co}\, F(\underline{t}, \underline{x})$, set $\underline{u} := h^{-1}(\underline{t}, \underline{x}, \underline{y})$, obtaining $\underline{u} \in \mathcal{U}$, $h(\underline{t}, \underline{x}, \underline{u}) = \underline{y}$, hence $f(\underline{t}, \underline{x}, \underline{u}) = \sigma \circ P(\underline{y}, \mathrm{co}\, F(\underline{t}, \underline{x})) = \underline{y}$ because $\underline{y} \in \mathrm{co}\, F(\underline{t}, \underline{x})$ already . This means that $\mathrm{co}\, F(\underline{t}, \underline{x}) \subset f(\underline{t}, \underline{x}, U)$, and since the opposite inclusion is obvious, (i) is proved. ♦

Proof of Theorem 2

Since I is σ – compact, we can use the Scorza-Dragoni property in (1.2) (ii) to write $I = \mathcal{N} \cup I_0$, $\mathcal{N}$ a null set and $I_0 = \bigcup I_k$, where (I_k) is a sequence of compact disjoint sets such that $F_k := \text{co}\, F_{|I_k \times X}$ is lsc with closed graph, $w_k := w_{|I_k \times X}$ is continuous. If moreover there exists $m : I \to \mathbb{R}^+$ such that $y \in F(t, x) \Rightarrow |y| \leq m(t)$, and m is measurable then we may also suppose that $m_{|I_k}$ is continuous. Let $C^0(X, \mathbb{R}^n)$ be the Banach space of continuous bounded maps $u : X \to \mathbb{R}^n$ with the usual sup norm. Set, for $t \in I_0$,

$$E(t) := \{ u \in C^0(X, \mathbb{R}^n) : |u(x) - u(\underline{x})| \leq 6n\, w(t, d(x, \underline{x})), \text{ and, in case } F \text{ is integrably bounded, } |u(x)| \leq m(t) \}.$$

Set $E_k := \bigcup_{t \in I_K} E(t)$, and let E be the closed convex hull of $\bigcup_{K \in \mathbb{N}} E_k$. Clearly each bounded subset of $E(t)$ is totally bounded, in particular $E(t)$ is compact provided F is integrably bounded; in general $E(t)$ is σ-compact. Since I_k is compact and w_k is jointly continuous, each bounded subset of E_k is totally bounded; in particular E_k is σ-compact, hence E is σ-compact.

Define the function φ to be the evaluation map $\varphi(x, u) := u(x)$; then clearly (iii) holds. Define the multifunction $\mathcal{U}$ by:

$$\mathcal{U}(t) := \{ u \in E(t) : u(x) \in \text{co}\, F(t, x) \quad \forall x \in X \}.$$

Since $\mathcal{U}(t) \subset E(t)$, (iv) holds. Since $\text{co}\, F(t, x)$ and $E(t)$ are conevx closed, $\mathcal{U}(t)$ is convex closed. In particular $\mathcal{U}(t)$ is compact in case F is integrably bounded. Set now $\mathcal{U}_k := \mathcal{U}_{|I_k}$. Since F_k, w_k, m_k have closed graph, one easily shows that $\mathcal{U}_k$ has closed graph. In particular $\mathcal{U}_0 := \mathcal{U}_{|I_0}$ has measurable graph. By Himmelberg [32, Theorem 3.5], $\mathcal{U}_0$ is measurable hence $\mathcal{U}$ is measurable.

Finally, to prove (i), fix any $\underline{t} \in I_0$, $\underline{x} \in X$; then, for any $\underline{y} \in \text{co}\, F(\underline{t}, \underline{x})$, set $\underline{u}(x) := \sigma \circ P(\underline{y}, \text{co}\, F(\underline{t}, x))$. Clearly $\underline{u} \in E(\underline{t})$, and $\underline{u} \in \mathcal{U}(\underline{t})$; moreover $\varphi(\underline{x}, \underline{u}) = \underline{u}(\underline{x}) = \underline{y}$, so that $\text{co}\, F(\underline{t}, \underline{x}) \subset \varphi(\underline{x}, \mathcal{U}(\underline{t}))$. Since the opposite inclusion is obvious, (i) is proved. ♦

2.2 Continuous selections in L^1

Introduction

Let K be a compact metric space. We construct a "guided" continuous selection for multifunctions $G : K \to L^1$ which are lsc with closed bounded decomposable values. A set $D \subset L^1$ is said *decomposable* provided the following property holds:

" whenever u, v are in D and χ is the characteristic function of a measurable set $S \subset I$ then the function $w := \chi u + (1 - \chi) v$ is also in D ".

For the history of decomposable sets see Hiai-Umegaki [32], Olech [46] and Colombo [19].

Fryszkowski [30] proved that the multifunction G, as above, has a continuous selection, thus showing that in some sense decomposability can substitute convexity. However this selection theorem resulted from abstracting a construction first developped by Antosiewicz-Cellina [1] and later applied by Pianigiani [51] and by Bressan [4]. Recently Bressan-Colombo [9] devised a method to avoid compactness assumptions on K. The result we describe here is a refinement of the construction of Fryszkowski. It is an abstract result that was developped to fit the needs in applications (as in sections 3.2 and 3.3).

Assumptions and main result

Let I be the interval $[0, T]$, set $L^1_+ := \{ \delta \in L^1(I, \mathbb{R}) : \delta(t) \geq 0 \text{ a.e.} \}$ and let K be a compact metric space.

Hypothesis (H)

$G : K \to L^1$ is a multifunction and $g_* : K \to L^1$ is a function, verifying :

(a) each value $G(u)$ is closed decomposable ;

(b) $\exists M : I \to \mathbb{R}^+$ integrable such that : $v \in G(u) \Rightarrow |v(t)| \leq M(t)$ a.e. ;

(c) $g_*(u)(t)$ is in the closed convex hull of $G(u)(t)$ $\forall u \in K$, for a.e. t ;

(d) G is lsc and g_* is continuous .

Theorem 1

Let G , g_* verify hypothesis **(H).**

Then there exists a sequence (g_i) of continuous selections from the multifunction G such that

$$\left| \int_0^t [g_*(u) - g_i(u)]\, ds \right| \leq 1/i \qquad \forall i \in \mathbb{N} \quad \forall t \in I \quad \forall u \in K.$$

Intermediate results and proofs

Proposition 1

Let Δ be a nonempty bounded decomposable subset of L^1_+.
Then there exists a uniquely determined element δ_0 in L^1_+ such that :

(i) $\delta \in \Delta \quad \Rightarrow \quad \delta_0 \leq \delta$ a.e.

(ii) if $\delta_1 \in L^1_+$ verifies " $\delta \in \Delta \quad \Rightarrow \quad \delta_1 \leq \delta$ a.e. "
then $\delta_1 \leq \delta_0$ a.e. .

Then we define Inf Δ as the unique element δ_0 in L^1_+ as above.

Proof Is obvious from Proposition 1 of Bressan-Colombo [9]. ♦

Proposition 2

Fix some element v and some closed bounded decomposable set V in L^1. Define the map

$$D : L^1 \times L^1 \to L^1_+ , \qquad D(u, v)(t) := |u(t) - v(t)| \quad \text{a.e.},$$

and set

$$D(u, V) := \text{Inf}\{ D(u, v) : v \in V \} ,$$

$$d_1(u, v) := \int D(u, v)(t)\, dt, \qquad d_1(u, V) := \int D(u, V)(t)\, dt.$$

Then there exists a measurable multifunction $\Gamma : I \to \mathbb{R}^n$ with closed values such that $\Gamma(t) = \{ v(t) : v \in V \}$. Moreover there exists a measurable selection γ from Γ such that

$$d(u(t), \Gamma(t)) = |u(t) - \gamma(t)|, \quad D(u, V) = D(u, \gamma) \quad \text{and} \quad d_1(u, V) = d_1(u, \gamma) .$$

Proof

The existence of Γ is obvious from Hiai-Umegaki [31] . For the existence of γ, see section 1.2. Finally, it is clear that if $v \in V$ then $v(t) \in \Gamma(t)$ a.e. hence $|u(t) - v(t)| \geq d(u(t), \Gamma(t)) = |u(t) - v(t)|$, i.e. $D(u, V) \geq D(u, \gamma)$ a. e.. Since the opposite inequality is obvious, the equality holds. ♦

Proposition 3

Let G verify hypothesis (H) and fix some $(u_0, v_0) \in$ graph G .
Then there exists a continuous map $\rho_{u_0 v_0} : K \to L^1_+$ such that

$$\rho_{u_0 v_0}(u_0) = 0 \ , \quad D(v_0, G(u)) \leq \rho_{u_0 v_0}(u) \qquad \forall\, u \in K .$$

Proof See Fryszkowski [29, Propositon 2.2, Lemma 3.1] or Bressan-Colombo [9, Proposition 4, Proposition 5] . ♦

To simplify the statement of the next proposition,we define a set $\wedge^m \subset L^1(I, \mathbb{R}^m)$ which represents a partition of I into m disjoint measurable subsets. Namely, we set

$$\wedge^m := \{ \lambda \in L^1(I, \mathbb{R}^m) : \lambda_i(t) \in \{0,1\} \text{ and } \sum_{i=1}^{m} \lambda_i(t) = 1 \quad \text{a.e.} \}.$$

Proposition 4

Let $p : K \to [0, 1]^m$ be a continuous partition of unity, let $\varphi : K \to L^1(I, \mathbb{R}^m)$ be a continuous map, and fix $\varepsilon > 0$.

Then there exists a continuous map $\lambda : K \to \wedge^m$ verifying :

(i) $\int \lambda(u)\, d\tau = p(u) . T$;

(ii) $\left| \int \lambda_i(u)(\tau)\, \varphi(u)(\tau)\, d\tau - p_i(u) \int \varphi(u)(\tau)\, d\tau \right| \leq \varepsilon / m$;

(iii) $p_i(u) = 1 \Rightarrow \lambda_i(u) \equiv 1$; $p_i(u) = 0 \Rightarrow \lambda_i(u) \equiv 0$, a.e. $\forall\, u \in K \ \forall\, i$.

Proof See Fryszkowski [29, Proposition 1.2] . ♦

Lemma 1

Let G verify hypothesis (H).
Then for each $\varepsilon > 0$ there exists a continuous map g such that $d_1(g(u), G(u)) \leq \varepsilon$,

$$\left| \int_0^t (g(u)(\tau) - g_*(u)(\tau))\, d\tau \right| \leq \varepsilon \qquad \forall\, t \in I \;\; \forall u \in K.$$

Proof

Using the integrable boundedness of G we can find a partition of I into subintervals $I_j = [\, t_{j-1}, t_j\,)$, $j = 1, \ldots, m_1$ such that

$$\forall\, u \in K \quad \forall\, v \in G(u), \quad \left| \int_{I_j} v\, ds \right| \leq \varepsilon/4, \quad j = 1, \ldots, m_1.$$

Since g_* is continuous on K, we can find ε' such that, denoting by d the distance in K,

$$u_1, u_2 \in K, \quad d(u_1, u_2) < \varepsilon' \;\Rightarrow\; d_1[\, g_*(u_1), g_*(u_2)] < \varepsilon / 4 .$$

Set $\varepsilon_1 := \frac{1}{4} \min\{\, \varepsilon, \varepsilon'\}$, and:

$$V_j(u) := \{\, v|_{I_j} : v \in G(u), \quad \int_{I_j} (\, g_*(u) - v\,)\, ds = 0 \,\}, \quad j = 1, \ldots, m_1$$

$$V(u) := \{\, v \in G(u) : \; v|_{I_j} \in V_j(u), \;\; \forall\, j = 1, \ldots, m_1 \,\}.$$

By Liapunov theorem on the range of vector measures (see[30]), $V_j(u)$ is nonempty $\forall j$, and since $G(u)$ is decomposable, we have $V(u) \neq \emptyset \ \forall u \in K$. If we fix some $u_o \in K$ and some $v_o \in V(u_o)$, by Proposition 3, there exists a continuous map $\rho_{u_o v_o}$ such that $\rho_{u_o v_o}(u_o) = 0$ and $D(v_0, G(u)) \leq \rho_{u_o v_o}(u) \ \forall u \in K$; therefore the set

$$U(u_o, v_o) := \{ u \in K: \ d(u, u_o) < \varepsilon_1, \ |\rho_{u_o v_o}(u)|_1 < \varepsilon_1 \}$$

is an open nbd of u_o. By compactness of K, the open cover $\{ U(u_o, v_o): u_o \in K, v_o \in V(u_o) \}$ has a finite subcover $\{ U_1, ..., U_m \}$, where $U_i = U(u_i, v_i)$, and:

$$u \in U_i \Rightarrow \ d(u, u_i) < \varepsilon_1 , \quad d_1[g_*(u), g_*(u_i)] < \varepsilon/4$$

$$v_i \in G(u_i) , \ D[v_i, G(u)] \leq \rho_i(u) := \rho_{u_i v_i}(u) , \ |\rho_i(u)|_1 < \varepsilon_1$$

$$\sum_{j=1}^{m_1} \left| \int_{I_j} [g_*(u) - v_i] \, ds \right| \leq d_1[g_*(u), g_*(u_i)] + \sum_{j=1}^{m_1} \left| \int_{I_j} [g_*(u_i) - v_i] \, ds \right| < \varepsilon/4$$

for $i = 1, ..., m$. Let $p: K \to [0, 1]^m$ be a subordinated continuous partition of unity, and apply Proposition 4 to $\varphi = (\varphi_1, \dots , \varphi_{m+m.m_1.n})$ defined by:

$$\varphi_i = \rho_i := \rho_{u_i v_i} \quad , \qquad \varphi_{m+k}(u)(t) = \chi_{I_j}(t) \, . \, [g_*(u)(t) - v_i(t)]_r \quad ,$$

for $i = 1, ..., m$, $j = 1, ..., m_1$, $r = 1, ..., n$, $k = 1, ..., m.m_1.n$, where $[\cdot]_r$ denotes the rth component of the vector $[.]$, with ε_1/m_1 in place of ε, obtaining a continuous map $\lambda: K \to \wedge^m$ verifying : $\int \lambda(u) \, d\tau = p(u) \, . \, T$;

$$\int \lambda_i(u) \ D[v_i, G(u)] \ (\tau) \ d\tau \leq p_i(u) \ |\rho_i(u)|_1 + \varepsilon_1/m.m_1 \leq (p_i(u) + 1/m) \ \varepsilon/4 ;$$

$$\int_0^{t_j} \lambda_i(u) \ [g^*(u) - v_i] \, ds \ | \leq \sum_{j=1}^{m_1} (p_i(u) \ | \int_{I_j} [g^*(u) - v_i] \, ds \ | + \varepsilon_1/m.m_1 \leq$$

$$\leq (p_i(u) + 1/m) \ \varepsilon/4 ;$$

$$p_i(u) = 1 \ \Rightarrow \ \lambda_i(u) \equiv 1 \ ; \quad p_i(u) = 0 \ \Rightarrow \ \lambda_i(u) \equiv 0 \ ,$$

a.e. $\forall\, u \in K$ for $i = 1, \dots, m$.

Define now $g: K \to L^1$, $g(u) := \sum_{i=1}^{m} \lambda_i(u)\, v_i$. To see that g is continuous, it is enough to see that

$$|g(u) - g(\underline{u})|_1 \le \sum_{i=1}^{m} \int |\lambda_i(u) - \lambda_i(\underline{u})|\ |v_i|\, ds \le \sum_{i=1}^{m} \int |\lambda_i(u) - \lambda_i(\underline{u})|\, M(s)\, ds,$$

and each term in this sum is the integral of M over a set of measure $\int |\lambda_i(u) - \lambda_i(\underline{u})|\, ds$, which clearly tends to 0 as $u \to \underline{u}$, since $\lambda_i: K \to L^1$ is continuous, $\forall i$. Moreover:

$$\left|\int_0^t [g_*(u) - g(u)]\, ds\right| \le \left|\int_{t_{j(t)}}^{t} g_*(u)\, ds\right| + \left|\int_{t_{j(t)}}^{t} g(u)\, ds\right| +$$

$$+\sum_{i=1}^{m} \left|\int_0^{t_j(t)} \lambda_i(u)\, [g_*(u) - v_i]\, ds\right| \le \tfrac{1}{4}\,\varepsilon + \tfrac{1}{4}\,\varepsilon + \tfrac{1}{4}\,\varepsilon \sum_{i=1}^{m} (p_i(u) + 1/m) = \varepsilon.$$

To see that g is an ε-approximate selection from G, recall that by Proposition 2,

$\forall\, u \in K$ $\exists\, v_i(u) \in G(u)$: $D(v_i, v_i(u)) = D(v_i, G(u))$, $i = 1, \dots, m$, so that, setting $v(u) := \sum_{i=1}^{m} \lambda_i(u)\, v_i(u) \in G(u)$ $\forall\, u \in K$, we have:

$$D[g(u), G(u)] \le D[g(u), v(u)] = \sum_{i=1}^{m} \lambda_i(u)\, D(v_i, v_i(u)) = \sum_{i=1}^{m} \lambda_i(u)\, D(v_i, G(u)).$$

Therefore

$$d_1[g(u), G(u)] \le \tfrac{1}{4}\,\varepsilon \sum_{i=1}^{m} [p_i(u) + 1/m] \le \varepsilon \quad \forall\, u \in K. \qquad \blacklozenge$$

Lemma 2

Let G verify hypothesis (H).

Let $g^{k-1} : K \to L^1$ be a continuous map verifying $d_1 (g^{k-1}(u), G(u)) \leq \varepsilon_{k-1}$ for some $\varepsilon_{k-1} > 0$.

Then for any $0 < \varepsilon_k < \varepsilon_{k-1}$ there exists a continuous map $g^k : K \to L^1$ such that

$$d_1 (g^k(u), G(u)) \leq \varepsilon_k , \qquad d_1 (g^k(u), g^{k-1}(u),) \leq \varepsilon_k + \varepsilon_{k-1} .$$

Proof

Since g^{k-1} is continuous on K, we can find ε' such that

$$u, \underline{u} \in K , \ d(u, \underline{u}) < \varepsilon' \Rightarrow d_1[g^{k-1}(u), g^{k-1}(\underline{u})] \leq \varepsilon_k / 2.$$

Set $\varepsilon = ½ \min \{ \varepsilon_k, \varepsilon' \}$ and $V(u) := \{ v \in G(u): d_1[g^{k-1}(u), v] = d_1[g^{k-1}(u), G(u)] \}$; then, by Proposition 2, $V(u) \neq \emptyset$, $\forall u \in K$. As in Lemma 1, for each $u_o \in K$ and each $v_o \in V(u_o)$, the set

$$U(u_o, v_o) = \{ u \in K: d(u, u_o) < \varepsilon , \ |\rho_{u_o v_o}(u)|_1 < \varepsilon \}$$

is an open nbd of u_o, and the rest of the proof follows the steps of the proof of Lemma 1. ♦

Proof of Theorem 1

Choose a positive decreasing sequence (ε_k) such that $\sum \varepsilon_k = 1/(2i)$, and apply Lemma 1 with ε_o in place of ε, obtaining a continuous map:

$$g^o: K \to L^1 \text{ such that } d_1[g^o(u), G(u)] \le \varepsilon_o, \quad \left| \int_0^t [g_*(u) - g^o(u)]\, ds \right| \le \varepsilon_o$$

, $\forall t \in I \quad \forall u \in K$. For $k = 1, 2, \ldots,$ apply Lemma 2, obtaining a continuous $g^k: K \to L^1$ such that

$$d_1(g^k(u), G(u)) \le \varepsilon_k \ , \quad d_1(g^k(u), g^{k-1}(u)) \le \varepsilon_k + \varepsilon_{k-1}.$$

In particular the sequence $(g^k(u))$ is Cauchy, uniformly in $u \in K$, hence the sequence (g^k) is a Cauchy sequence of continuous maps, and $(g^k) \to g_1$ uniformly, $g_1: K \to L^1$ is continuous, and:

$$d_1[g_1(u), G(u)] \le d_1[g_1(u), g^k(u)] + d_1[g^k(u), G(u)] \le$$

$$\le d_1[g_1(u), g^k(u)] + \varepsilon_k \to 0 \text{ as } k \to \infty, \text{ hence } g_1(u) \in G(u) \ \forall\, u \in K.$$

This means that g_1 is a continuous selection from G, and:

$$\left| \int_0^t [g_*(u) - g_1(u)]\, ds \right| \le \left| \int_0^t [g_*(u) - g^o(u)]\, ds \right| + d_1[g^o(u), g^1(u)] + \cdots +$$

$$+ d_1[g^{k-1}(u), g^k(u)] + d_1[g^k(u), g_1(u)] \le \varepsilon_o + (\varepsilon_o + \varepsilon_1) + \cdots +$$

$$+ (\varepsilon_{k-1} + \varepsilon_k) + d[g^k(u), g_1(u)] \to 2 \sum \varepsilon_k = 1/i \ , \forall t \in I \ \forall u \in K. \ \blacklozenge$$

2.3 Continuous selections from solution sets

Introduction

Consider a multifunction $F(t, x)$ Lipschitz in x with unbounded values. Let $S(\xi)$, $A(\xi)$ be the solution set and the attainable set at time $t = T$ for the associated Cauchy problem (CP). If we let initial data $x(0) = \xi$ vary over a compact set Ξ in $\mathbb{R}^n$ then we can associate to F two new multifunctions $S : \Xi \to C^0$ and $A : \Xi \to \mathbb{R}^n$, with values $S(\xi)$, $A(\xi)$ which are in general nonconvex nonclosed (Hermes-LaSalle [31]). However these multifunctions have continuous selections through each point of the graph, as we show in the first part of this section. Notice that we do not use neither Liapunov's theorem on the range of vector measures nor any previous existence result.

In the second part we consider approximate solutions of (CP), i.e. functions y in AC such that the distance $\rho(t) := d(y'(t), F(t, y(t))$ is integrable. Filippov [28] showed that (CP) has solutions *iff* it has approximate solutions. He obtained also an estimate for the distance between a given approximate solution y and a true solution x, as follows :

$$|x(t) - y(t)| \le \zeta(t) := \int_0^t \exp\left(\int_\tau^t \ell\right) \rho(\tau)\, d\tau, \quad |x'(t) - y'(t)| \le \ell(t)\, \zeta(t) + \rho(t) \quad \text{a.e.,}$$

where $\ell(t)$ is the Lipschitz constant of $F(t, \cdot)$. This is somehow a multivalued version of the well known Gronwall inequality. What we prove here is a "continuous" version of this estimate. Namely, we let y and y' vary continuously with ξ and show that there exists a true solution x of (CP), also depending continuously on ξ, at a distance from y which equals the Filippov estimate plus a small error δ. This error is unavoidable because Filippov's estimate is incompatible with continuous dependence, as simple examples show.

The first continuous selection from solution sets was obtained by Cellina [15], supposing F to have values contained in a bounded set, and using Liapunov's theorem. Previously, approximate selections had been obtained also by Cellina [13], for usc multifunctions with convex values (also in Aubin-Cellina [2, Corollary 2.2.2]). Notice that by using these continuous selections (or approximate continuous selections), one can deduce the existence, under appropriate conditions, of periodic solutions (see [13] or Aubin-Cellina [2, Corollary 2.2.3], and [15]). As to the Filippov estimate, it was originally proved for F jointly continuous and subsequently extended by Himmelberg-Van Vleck [34] to allow measurability in t.

Assumptions and results

Let I be the interval $[0, T]$ and let Ξ be a compact set in $\mathbb{R}^n$, with diameter D. Consider the following :

Hypothesis (H)

$F : I \times \mathbb{R}^n \to \mathbb{R}^n$ is a multifunction with:

(a) values $F(t, x)$ closed ;

(b) $F(\cdot, x)$ measurable;

(c) $\exists\ \ell : I \to \mathbb{R}^+$ integrable such that $dl(F(t, x), F(t, \underline{x})) \leq \ell(t)\ |x - \underline{x}|$ for a.e. t ;

(d) $\exists\ y$ in AC such that $t \longmapsto d(y'(t), F(t, y(t)))$ is integrable .

We associate to F the multifunctions

$$S : \Xi \to C^0 , \quad S(\xi) := \{ x \in C^0 : x(0) = \xi,\ x' \in F(t, x) \text{ a.e. } \} ,$$

$$A : \Xi \to \mathbb{R}^n , \quad A(\xi) := \{ \eta \in \mathbb{R}^n : \eta = x(T) \text{ for some } x \in S(\xi) \} .$$

$S(\xi)$ is the solution set, and $A(\xi)$ is the attainable set, for (CP) .

Theorem 1

Let F satisfy hypothesis **(H)**.

Then for each solution x_0 of $x' \in F(t, x)$ a.e. there exists a continuous selection s from the solution multifunction S verifying $s(x_0(0)) = x_0$.

Corollary 1

Let F satisfy hypothesis **(H)**.

Then there exists a closed set $\mathcal{U}$ in a separable Banach space $\mathcal{X}$, and continuous functions $g : \Xi \times \mathcal{U} \to C^0$, $h : \Xi \times \mathcal{U} \to \mathbb{R}^n$ such that :

$$S(\xi) = g(\xi, \mathcal{U}) \ , \qquad A(\xi) = h(\xi, \mathcal{U})$$

for any ξ in Ξ.

Theorem 2

Let F satisfy hypothesis **(H)**.

Then for each $\delta > 0$ and each continuous map $y : \Xi \to AC$, there exists a continuous map $x : \Xi \to AC$ such that $x(\xi)'(t) \in F(t, x(\xi)(t))$ a.e. on I and

$$\int_0^t | x(\xi)'(\tau) - y(\xi)'(\tau) | \, d\tau \leq \delta + \int_0^t \exp\left(\int_\tau^t \ell \right) \rho(\xi)(\tau) \, d\tau \ ,$$

where $\rho(\xi)(t) := d(y(\xi)'(t), F(t, y(\xi)(t)))$.

Proof of Theorem 1

We construct a continuous $y : \Xi \to AC$ such that $y(\xi)'(t) \in F(t, y(\xi)(t))$ a.e. on I and $y(\xi_0) = x_0$, where $\xi_0 = x_0(0)$, and set $s : \Xi \to C^0$, $s(\xi)(t) = y(\xi)(t)$ $\forall \xi$ $\forall t$. To construct the map y we use the following

Proposition 1

Let $v_0, \ldots, v_m$ be in L^1, let $(I_j(\xi))$ be a partition of I *into* a finite number of subintervals with endpoints depending continuously on ξ. Consider the map

$$\varphi : \xi \to \xi + \int_0^t \sum_{j=0}^{m} \chi_{I_j(\xi)}(\tau)\, v_j(\tau)\, d\tau \ .$$

Then there exists α in $L^1(I)$ such that: for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|\xi' - \xi| < \delta \quad \text{implies} \quad |\varphi(\xi')'(t) - \varphi(\xi)'(t)| \le \alpha(t)\, \chi_E(t) \ ,$$

for some set E with measure(E) $\le \varepsilon$.

The construction of y is based on Filippov's successive approximations. As a function of the initial data, each approximation would not be continuous. We modify it in order to obtain continuity, by interpolating through continuous partitions of the interval I, as in Antosiewicz-Cellina [1].

(a) Set $\quad y : \Xi \to AC$ to be $y(\xi)(t) := \xi + \int_0^t x_0'(\tau)\, d\tau$,

and notice that y is continuous and verifies:

$$d[y(\xi)'(t), F(t, y(\xi)(t))] = d[x_0'(t), F(t, y(\xi)(t))] \le$$

$$\leq \mathrm{dl}\,[F(t,y(\xi_0)(t)),\, F(t,y(\xi)(t))] \leq \ell(t)\ |\xi_0 - \xi| \ .$$

Choose $v^0(\xi)(t)$ to be a measurable selection from $F[t,y(\xi)(t)]$ such that

$$|y(\xi)'(t) - v^0(\xi)(t)| = d[y(\xi)'(t)\,,\, F(t,y(\xi)(t))] \leq \ell(t)\ |\xi_0 - \xi| \ .$$

Hence $v^0(\xi)$ belongs to L^1. Fix some $\eta > 0$ and define

$$\delta(\xi) := \min\{\ 2^{-3}\eta\ ,\ |\xi - \xi_0|/2\ \}\ \text{ for } \xi \neq \xi_0\ ,\ \delta(\xi_0) := 2^{-3}\eta\ .$$

Cover Ξ with balls $B(\xi,\delta(\xi))$, and let $(B(\xi_j, \delta(\xi_j)))_{j=0,\dots,m}$, be a finite subcovering; in particular ξ_0 belongs only to $B(\xi_0, \delta(\xi_0))$. Let $(p_j)_{j=0,\dots,m}$, be a continuous partition of unity subordinate to this covering, and define $I_0(\xi) := [0,T\ p_0(\xi)]$ and, for $j > 0$,

$$I_j(\xi) := [T(p_0(\xi) + \dots + p_{j-1}(\xi))\ ,\ T(p_0(\xi) + \dots + p_j(\xi))]\ .$$

Set

$$y^1(\xi)(t) := \xi + \int_0^t \sum_{j=0}^m \chi_{I_j(\xi)}(\tau)\ v^0(\xi_j)(\tau)d\tau\ .$$

From Proposition 1 it follows that y^1 is continuous from Ξ to AC. Moreover, $y^1(\xi_0) = s_0$, since $I_0(\xi) = [0,T]$. We have:

$$(1) \qquad \int_0^t |y^1(\xi)' - y(\xi)'|\, d\tau \leq \int_0^t \sum_j |v^0(\xi_j) - y(\xi)'|\, d\tau \leq$$

$$\leq \int_0^t \sum_j \chi_{I_j(\xi)}\, k(\tau)\ |\xi_0 - \xi_j|\, d\tau \leq D\ m(t)\ ,$$

where $m(t) := \int_0^t \ell(\tau)\, d\tau$.

Fix t and let j be such that $t \in I_j(\xi)$. Then:

(2) $$d[\, y^1(\xi)'(t)\, ,\, F(t,y(\xi)(t))\,] \;=\; d[\, v^0(\xi_j)(t)\, ,\, F(t,y(\xi)(t))\,] \;\le$$

$$\le\; dl\, [\, F(t,y(\xi_j)(t))\, ,\, F(t,y(\xi)(t))\,] \;\le\; \ell(t)\, |\xi_j - \xi| \;\le$$

$$\le\; 2^{-3}\, \eta\, \ell(t)\ .$$

This estimate is independent of j, hence it holds on I. By the same reasoning,

(3) $$d[\, y^1(\xi)'(t)\, ,\, F(t,y^1(\xi)(t))\,] \;=\; d[\, y^1(\xi)'(t)\, ,\, F(t,y(\xi)(t))\,] \;+$$

$$+\; dl\, [\, F(t,y(\xi)(t))\, ,\, F(t,y^1(\xi)(t))\,] \;\le\; \ell(t)\, [\, 2^{-3}\, \eta \;+ D\, m(t)\,]\ .$$

(b) In general we claim that for $n = 1, 2, \ldots$, we can define a continuous map $y^n : \Xi \to AC$ verifying $y^n(\xi_0) = x_0$ and:

(i) $$\int_0^t |y^n(\xi)' - y^{n-1}(\xi)'|\, d\tau \;\le\; D\, \frac{m^n(t)}{n!} \;+$$

$$+\; \eta\, 2^{-n-1}\, [\, 2^{-2} \;+ \sum_{i=1}^{n} \frac{(2m(t))^i}{i!}\,]\ ;$$

(ii) $$d[\, y^n(\xi)'(t)\, ,\, F(t,y^{n-1}(\xi)(t))\,] \;\le\; \eta\, 2^{-n-2}\, \ell(t)\ ;$$

(iii) $$d[\, y^n(\xi)'(t)\, ,\, F(t,y^n(\xi)(t))\,] \;\le\; D\, \ell(t)\, \frac{m^n(t)}{n!} \;+$$

$$+\; \eta\, 2^{-n-1}\, \ell(t) \sum_{i=0}^{n} \frac{(2m(t))^i}{i!}\ ;$$

(iv) there exists α^n in L^1 such that : for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|\xi' - \xi| < \delta \text{ implies } |y^n(\xi)'(t) - y^n(\xi')'(t)| \;\le\; \alpha^n(t)\, \chi_E(t)\ ,$$

for some $E \subset I$ with measure$(E) \le \varepsilon$.

From the definition of y^1 and the Proposition, this claim holds for $n = 1$. Assume that it holds for $n - 1$.

Choose $v^{n-1}(\xi)(t) \in F[t, y^{n-1}(\xi)(t)]$ such that

$$|y^{n-1}(\xi)'(t) - v^{n-1}(\xi)(t)| = d[\, y^{n-1}(\xi)'(t) \,,\, F(t, y^{n-1}(\xi)(t))\,] \leq$$

$$\leq D\, \ell(t) \frac{m^{n-1}(t)}{(n-1)!} + \eta\, 2^{-n}\, \ell(t) \sum_{i=0}^{n-1} \frac{(2m(t))^i}{i!} \,.$$

By (iv) of the recursive hypothesis, there exists $\delta_n > 0$ such that: $|\xi' - \xi| < \delta_n$ implies

$$|y^{n-1}(\xi')'(t) - y^{n-1}(\xi)'(t)| \leq \alpha^{n-1}(t)\, dt \leq \alpha^{n-1}\, \chi_E(t) \,,$$

for some E such that $\int_E \alpha^{n-1}(t)\, dt \leq \eta\, 2^{-n-3}$.

Define

$$\delta_n(\xi) := \min\{\, \delta_n \,,\, 2^{-n-3}\eta \,,\, |\xi - \xi_0|/2 \,\} \text{ for } \xi \neq \xi_0 \,,$$

$$\delta_n(\xi_0) := \min\{\, \delta_n \,,\, 2^{-n-3}\eta \,\} \,.$$

Cover Ξ with balls $B(\xi, \delta_n(\xi))$ and let

$$B(\xi_j^n \,,\, \delta_n(\xi_j^n)) \,,\, j = 0 \,,\, \ldots,\, m_n \,,\, \xi_0^n = \xi_0 \,,$$

be a finite subcover; in particular ξ_0 belongs only to $B(\xi_0, \delta_n(\xi_0))$. Let $(p_j^n)_{j=0,\ldots,m_n}$, be a continuous partition of unity subordinate to this covering, and define $I_0^n(\xi) := [0, Tp_0^n(\xi)]$ and, for $j > 0$,

$$I_j^n(\xi) := [\, T(p_0^n(\xi) + \ldots + p_{j-1}^n(\xi)) \,,\, T(p_0^n(\xi) + \ldots + p_j^n(\xi))\,] \,.$$

Set

$$y^n(\xi)(t) := \xi + \int_0^t \sum_{j=0}^{m_n} \chi_{I_j^n(\xi)}(\tau)\, v^{n-1}(\xi_j^n)(\tau)d\tau \ .$$

From the Proposition it follows that y^n is continuous from Ξ into AC. Moreover, $y^n(\xi_0) = s_0$ since $I_0^n(\xi_0) = [0,T]$. We have:

$$\int_0^y |y^n(\xi)' - y^{n-1}(\xi)'|d\tau \le \int_0^t \sum_j \chi_{I_j^n(\xi)} |v^{n-1}(\xi_j^n) - y^{n-1}(\xi)'|\, d\tau \le$$

$$\le \int_0^t \sum_j \chi_{I_j^n(\xi)} |v^{n-1}(\xi_j^n) - y^{n-1}(\xi_j^n)'|\, d\tau +$$

$$+ \int_0^t \sum_j \chi_{I_j^n(\xi)} |y^{n-1}(\xi_j^n)' - y^{n-1}(\xi)'|\, d\tau \le$$

$$\le \int_0^t \Big(\sum_j \chi_{I_j^n(\xi)}\Big) [\, D\, k(t) \frac{m^{n-1}(t)}{(n-1)!} + \eta\, 2^{-n} \ell(t) \cdot \sum_{i=0}^{n-1} \frac{(2m(t))^i}{i!}]\, d\tau +$$

$$+ \int_0^t \Big(\sum_j \chi_{I_j^n(\xi)}\Big) \alpha^{n-1}(\tau)\, \chi_E(\tau)\, d\tau \le D\, \frac{m^n(t)}{n!} + \eta\, 2^{-n-1} \sum_{i=1}^{n} \frac{(2m(t))^i}{i!}] + \eta\, 2^{-n-3}.$$

Hence point (i) of the recursive hypothesis holds. Fix t and let j be such that $t \in I_j^n(\xi)$. Then

$$d[\, y^n(\xi)'(t)\, ,\, F(t, y^{n-1}(\xi)(t))\,] = d[\, v^{n-1}(\xi_j^n)(t)\, ,\, F(t, y^{n-1}(\xi)(t))\,] \le$$

$$\le dl\, [\, F(t, y^{n-1}(\xi_j^n)(t))\, ,\, F(t, y^{n-1}(\xi)(t))\,] \le$$

$$\leq \ell(t)\,[\,|\xi_j^n - \xi| + \int_0^t |y^{n-1}(\xi_j^n)' - y^{n-1}(\xi)'|d\tau \leq$$

$$\leq \ell(t)\,[\,\eta\, 2^{-n-3} + \eta\, 2^{-n-3}\,] = \eta\, 2^{-n-2}\,\ell(t)\ .$$

This estimate is independent of j, so it holds on I. Thus (i) is proved.
By the same reasoning,

$$d[\,y^n(\xi)'(t)\,,\,F(t,y^n(\xi)(t))\,] \leq d[\,y^n(\xi)'(t)\,,\,F(t,y^{n-1}(\xi)(t))\,] +$$

$$+\ dl\,[\,F(t,y^{n-1}(\xi)(t))\,,\,F(t,y^n(\xi)(t))\,] \leq$$

$$\leq \ell(t)\,[\,\eta\, 2^{-n-2} + D\,\ell(t)\,\frac{m^n(t)}{(n)!} + \eta\, 2^{-n-1}\sum_{i=1}^{n}\frac{(2m(t))^i}{i!}\,] + \eta\, 2^{-n-3}\,] \leq$$

$$\leq D\,\ell(t)\,\frac{m^n(t)}{n!} + \eta\, 2^{-n-1}\,\ell(t)\sum_{i=0}^{n}\frac{(2m(t))^i}{i!}\,]\ .$$

Applying the Proposition to y^n the recurrence is completed.

(c) From (i) we have that

$$\|y^n(\xi) - y^{n-1}(\xi)\|_{AC} \leq D\,\frac{m^n(t)}{n!} + \eta\, 2^{-n-1}\, e^{2m(t)},$$

so that the sequence of continuous functions $y^n : I \to AC$ converges uniformly to a continuous function y such that $y(\xi_0) = x_0$. By (iii), $y(\xi)$ belongs to $S(\xi)$. ♦

Proof of Corollary 1

Set $\mathcal{X}$ to be the separable Banach space of continuous maps φ from the compact Ξ into the separable Banach space C^0, with the usual sup norm, and let $\mathcal{U} \subset \mathcal{X}$ be the set of continuous selections from the map $\xi \to S(\xi)$. Define g to be the evaluation map $g(\xi, u) := u(\xi)$. Then the continuity of g is obvious, and the above theorem gives $g(\xi, \mathcal{U}) = S(\xi)$. Similarly for h. ♦

Proof of Theorem 2

The proof is essentially Filippov's construction of successive approximations. However, unlike in Theorem 1, we need Fryszkowski's [29] version of the Liapunov theorem on the range of a vector measure because $y(\xi)'(t)$ now depends on ξ and we aim at a sharper result.

(a) Set $\varepsilon := \delta\, e^{-2m(t)} / 6$, where $m(t) := \int_0^t \ell(\tau)\, d\tau$. Consider the map $\rho : \Xi \to L^1(I)$, with $\rho(\xi)(t)$ as in the statement; using (d) of hypothesis (H) one easily sees that ρ is well-defined. Moreover ρ is continuous because

$$\int |\rho(\xi) - \rho(\underline{\xi})|\, d\tau \le \int |y(\xi)' - y(\underline{\xi})'|\, d\tau + \int \ell(\tau)\, |y(\xi) - y(\underline{\xi})|\, d\tau \le$$

$$\le |y(\xi) - y(\underline{\xi})|_{AC}\, [1 + m(T)] .$$

For each $\xi_j \in \Xi$, the maps $\beta_j, \sigma_j : \Xi \to L^1(I)$, $\beta_j(\xi)(t) := |y(\xi_j)'(t) - y(\xi)'(t)|$, $\sigma_j(\xi)(t) := |\rho(\xi_j)(t) - \rho(\xi)(t)|$, are continuous and verify $\beta_j(\xi_j) = \sigma_j(\xi_j) = 0$; hence the set

$$\Xi_j := \left\{ \xi \in \Xi : |y(\xi) - y(\xi_j)|_{AC} < \frac{\varepsilon}{2} ,\ \int \sigma_j(\xi) d\tau < \frac{\varepsilon}{2} \right\}$$

is an open nbd of ξ_j. Therefore there exists an open cover $\Xi_1, \ldots, \Xi_m$ of Ξ and a subordinate continuous partition of unity $\pi_1, \ldots, \pi_m : \Xi \to [0,1]$. Choose $\Delta > 0$ such that

$$\text{measure}(E) \le \Delta \Rightarrow \int_E \rho_1(\tau)d\tau < \frac{\varepsilon}{2} ,$$

where $\rho_1 \in L^1(I)$ is such that $\rho(\xi_j)(t) \le \rho_1(t)$ a.e. on I for $j = 1, ..., m$. For each $\xi \in \Xi$ find a map $v^0(\xi) \in L^1(I)$ such that $v^0(\xi)(t) \in F[t, y(\xi)(t)]$, $|v^0(\xi)(t) - y(\xi)'(t)| = \rho(\xi)(t)$ a.e. on I, using the measurable selection theorem of Kuratowski-Ryll Nardzewski [36]. Apply the technique of [1] or [29] or section 2.2 to find sets $E_1(\xi), ..., E_m(\xi)$ such that:

$$\text{measure}[E_j(\xi)] = T \cdot \pi_j(\xi)$$

$$\int_0^{\ell\Delta} \chi_{E_j(\xi)}\, \rho(\xi_j)d\tau \le \pi_j(\xi) \int_0^{\ell\Delta} \rho(\xi_j)d\tau + \frac{\varepsilon}{2m} ,$$

$$\int \chi_{E_j}(\xi)\, \beta_j(\xi)d\tau \le \pi_j(\xi) \int \beta_j(\xi)d\tau + \frac{\varepsilon}{2m} \le \left[\pi_j(\xi) + \frac{1}{m}\right]\frac{\varepsilon}{2} .$$

Define

$$x^1(\xi)(t) := y(\xi)(0) + \int_0^t \sum_{j=1}^m \chi_{E_j(\xi)}\, v^0(\xi_j)d\tau ,$$

obtaining a continuous map $x^1 : \Xi \to AC$ such that

$$\int_0^t |x^1(\xi)' - y(\xi)'|d\tau = \int_0^t \sum_{j=1}^m \chi_{E_j(\xi)}\, |v^0(\xi_j) - y(\xi)'|d\tau \le$$

$$\le \int_0^{\ell\Delta} \sum_j \chi_{E_j(\xi)}\, \rho(\xi_j)d\tau + \int \sum_j \chi_{E_j(\xi)}\, \beta_j(\xi)d\tau +$$

$$+ \int_{\ell\Delta}^t \rho_1(\tau)d\tau \le \sum_{j=1}^m [\pi_j(\xi) \int_0^{\ell\Delta} \rho(\xi_j)d\tau + \frac{\varepsilon}{2m}] +$$

$$+ \sum_{j=1}^{m} \left[\pi_j(\xi) + \frac{1}{m}\right]\frac{\varepsilon}{2} + \frac{\varepsilon}{2} \le \int_0^t \rho(\xi)d\tau +$$

$$+ \sum_j \pi_j(\xi) \int \sigma_j(\xi)d\tau + 4\,\varepsilon/2 \le \int_0^t \rho(\xi)d\tau + 5\,\varepsilon/2 \ .$$

If we fix $t \in I$ and let j be such that $t \in E_j(\xi)$ then

$$d[x^1(\xi)'(t)\ ,\ F(t,y(\xi)(t))] = d[v^0(\xi_j)(t)\ ,\ F(t,y(\xi)(t))] \le$$

$$\le dl\ [F(t,y(\xi_j)(t))\ ,\ F(t,y(\xi)(t))] \le \ell(t)\ |y(\xi_j)(t) - y(\xi)(t)| \le$$

$$\le \ell(t)\ |y(\xi_j) - y(\xi)|_{AC} \le \ell(t)\ \varepsilon/2\ ,$$

hence:

$$d[x^1(\xi)'(t)\ ,\ F(t,x^1(\xi)(t))] \le d[x^1(\xi)'(t)\ ,\ F(t,y(\xi)(t))] +$$

$$+ dl\ [F(t,y(\xi)(t))\ ,\ F(t,x^1(\xi)(t))] \le \ell(t)\ [\int_0^t \rho(\xi)d\tau + 6\,\varepsilon/2] \ .$$

(b) **Claim.** For $n = 1, 2, \ldots$, it is possible to define a continuous map $x^n : \Xi \to AC$ verifying $x^n(\xi)(0) = y(\xi)(0)$ and:

(i)
$$\int_0^t |x^n(\xi)' - x^{n-1}(\xi)'|d\tau \le \int_0^t \frac{[m(t) - m(\tau)]^{n-1}}{(n-1)!}\rho(\xi)(\tau)d\tau +$$

$$+ 5 \cdot 2^{-n}\varepsilon + 6 \cdot 2^{-n}\varepsilon \sum_{i=1}^{n-1} \frac{[2m(t)]^i}{i!}\ ;$$

(ii)
$$d[x^n(\xi)'(t)\ ,\ F(t,x^n(\xi)(t))] \le \ell(t) \int_0^t \frac{[m(t) - m(\tau)]^{n-1}}{(n-1)!}\rho(\xi)(\tau)d\tau +$$

$$+ \; 6 \cdot 2^{-n} \varepsilon \, \ell(t) \sum_{i=0}^{n-1} \frac{[2m(t)]^i}{i!} .$$

To prove this claim notice first that for $n = 1$ it has been proved above. Supposing that it holds true for $n - 1$, we prove it now for n.

Consider the map $\rho^n : \Xi \to L^1(I)$,

$$\rho^n(\xi)(t) := d[x^{n-1}(\xi)'(t) \, , \, F(t, x^{n-1}(\xi)(t))] \; ;$$

since $x^{n-1} : \Xi \to AC$ is continuous by hypothesis, reasoning as for ρ at the beginning of the proof one easily sees that ρ^n is well-defined and continuous.

Moreover the estimate (ii) of the claim for $n - 1$ gives

$$\int_0^t \rho^n(\xi) d\tau \le \int_0^t \ell(\tau) \int_0^\tau \frac{[m(\tau) - m(r)]^{n-2}}{(n-2)!} \rho(\xi)(r) dr d\tau \; +$$

$$+ \; 6 \cdot 2^{-n+1} \varepsilon \int_0^t \ell(\tau) \sum_{i=0}^{n-2} \frac{[2m(\tau)]^i}{i!} d\tau \le$$

$$\le \int_0^t \frac{[m(t) - m(\tau)]^{n-1}}{(n-1)!} \rho(\xi)(\tau) d\tau \; + \; 6 \cdot 2^{-n} \varepsilon \sum_{i=1}^{n-1} \frac{[2m(t)]^i}{i!} .$$

As for the case $n = 1$ above, it is possible to find an open cover $\Xi_1^n, \ldots, \Xi_{m_n}^n$ of Ξ and a subordinate continuous partition of unity $\pi_1^n, \ldots, \pi_{m_n}^n : \Xi \to [0,1]$ such that each set Ξ_j^n is an open nbd of ξ_j^n given by:

$$\Xi_j^n := \{\xi \in \Xi : |x^{n-1}(\xi) - x^{n-1}(\xi_j^n)|_{AC} < 2^{-n} \varepsilon \, , \int \sigma_j^n(\xi) d\tau < 2^{-n} \varepsilon\} \, ,$$

where

$$\sigma_j^n(\xi)(t) := |\rho^n(\xi_j^n)(t) - \rho^n(\xi)(t)| \; .$$

Set

$$\beta_j^n(\xi)(t) := |x^{n-1}(\xi_j^n)'(t) - x^{n-1}(\xi)'(t)| ,$$

and choose $\Delta^n > 0$ such that: measure(E) $\leq \Delta^n \Rightarrow \int_E \rho_n(\tau)d\tau < 2^{-n}\varepsilon$, where ρ_n is some map in $L^1(I)$ such that $\rho^n(\xi_j^n)(t) \leq \rho_n(t)$ a.e. on I , $\forall j = 1, ..., m_n$.

Find a map $v^{n-1}(\xi)$ in L^1 such that

$$v^{n-1}(\xi)(t) \in F[t, x^{n-1}(\xi)(t)] , |v^{n-1}(\xi)(t) - x^{n-1}(\xi)'(t)| = \rho^n(\xi)(t)$$

a.e. on I , and apply the technique of [1] or [29] or section 2.2 to find sets $E_1^n(\xi), ..., E_{m_n}^n(\xi)$ such that

$$\text{measure}[E_j^n(\xi)] = T \cdot \pi_j^n(\xi) ,$$

$$\int_0^{\ell\Delta^n} \chi_{E_j^n(\xi)} \rho^n(\xi_j^n)d\tau \leq \pi_j^n(\xi) \int_0^{\ell\Delta^n} \rho^n(\xi_j^n)d\tau + 2^{-n}\frac{\varepsilon}{m_n} ,$$

$$\int \chi_{E_j^n(\xi)} \beta_j^n d\tau \leq \pi_j^n(\xi) \int \beta_j^n(\xi)d\tau + 2^{-n}\frac{\varepsilon}{m_n} \leq$$

$$\leq \left[\pi_j^n(\xi) + \frac{1}{m_n}\right] 2^{-n}\varepsilon .$$

Define

$$x^n(\xi)(t) := y(\xi)(0) + \int_0^t \sum_{j=1}^{m_n} \chi_{E_j^n(\xi)} v^{n-1}(\xi_j^n)d\tau ,$$

obtaining a continuous map $x^n : \Xi \to AC$ such that, reasoning as for the case $n = 1$ above,

$$\int_0^t |x^n(\xi)' - x^{n-1}(\xi)'| d\tau \leq \int_0^t \rho^n(\xi) d\tau + 5 \cdot 2^{-n} \varepsilon ,$$

$$d[x^n(\xi)'(t) , F(t,x^n(\xi)(t))] \leq k(t)[\int_0^t \rho^n(\xi) d\tau + 6 \cdot 2^{-n} \varepsilon] ,$$

and, using the estimate obtained above for $\int_0^t \rho^n(\xi) d\tau$, obtain the estimates (i) and (ii), thus completing the proof of the claim.

(c) From the claim it is clear that

$$\int_0^t |x^n(\xi)' - x^{n-1}(\xi)'| d\tau \leq \int_0^t \frac{[m(t) - m(\tau)]^{n-1}}{(n-1)!} \rho(\xi)(\tau) d\tau + 6 \cdot 2^{-n} e^{2m(t)} \varepsilon ,$$

so that the sequence of continuous maps $x^n : \Xi \to AC$ converges uniformly to a continuous map $x : \Xi \to AC$ verifying, by (ii), $x(\xi)'(t) \in F[t,x(\xi)(t)]$ a.e. on I. Moreover we have:

$$\int_0^t |x(\xi)' - y(\xi)'| \, d\tau \leq \int_0^t \exp[\int_\tau^t \ell(r) dr] \, \rho(\xi)(\tau) d\tau + \delta .$$

◆

Chapter 3 Differential inclusions

3.1 The convex continuous case

Introduction

We show that differential inclusions in $\mathbb{R}^n$ with convex valued multifunctions, continuous in x, do not generalize differential equations with control in $\mathbb{R}^n$. In fact, consider the relaxed problem (CPR) with $F(t, x)$ measurable in t and continuous in x. The results of section 2.1 show how to construct a function $f(t, x, u)$ and a convex closed set U in $\mathbb{R}^n$ such that $\text{co}\, F(t, x) = f(t, x, U)$. Moreover U is compact provided the values $F(t, x)$ are compact, and $f(t, \cdot, u)$ is Lipschitz provided $F(t, \cdot)$ is Lipschitz. We show here that any solution of $x' \in \text{co}\, F(t, x)$ also solves $x' = f(t, x, u)$, $u(t) \in U$.

Equivalence between differential inclusions with convex valued continuous multifunctions in $\mathbb{R}^n$ and control differential equations was known; but the regularity conditions were not satisfactory. Namely, either f was non-Lipschitz for Lipschitz F (Ekeland-Valadier [27]) or U was infinite dimensional (LeDonne-Marchi [39] or Lojasiewicz-Plis-Suarez [42] added to Ioffe [36]).

Assumptions and results

Let I be an interval, bounded or unbounded, and let Ω be an open or closed set in $\mathbb{R}^n$. Let $F : I \times \Omega \to \mathbb{R}^n$ be a multifunction with values either bounded by a linear growth condition -- hypothesis (FLB) in section 2.1 -- or unbounded -- hypothesis (FU). Notice that hypothesis (FLB) (d) now simply asks the boundedness of I and the continuity of $F(t, \cdot)$; in fact I is already σ- compact, and for X we can take an adequate compact subset of Ω, using an exponential a priori estimate for solutions of (CP) based on Gronwall's inequality (and supposing Ω large enough or I small enough).

Theorem 1

Let F verify hypothesis **(FU)** of section 2.1.

Then the relaxed Cauchy problem (CPR) has the same absolutely continuous solutions as the control differential equation

(CDE) $$x' = f(t, x, u) \quad \text{a.e. on } I, \quad x(0) = \xi, \quad u(t) \in U,$$

where f, U are as in Theorem 2.1.1 or Corollary 2.1.1 .

If moreover F, w are jointly h-continuous then for each continuous differentiable solution $\underline{x}$ of (CPR) there exists a continuous control $\underline{u} : I \to U$ such that

$$\underline{x}'(t) = f(t, \underline{x}(t), \underline{u}(t)) \quad \forall\, t.$$

Proof For each solution $\underline{x}$ of (CPR) set $\underline{y}(t) := \underline{x}'(t)$ and apply Proposition 1. ♦

A special case which appears more commonly in applications is covered by the simpler :

Corollary 1

Let $F : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ be a multifunction with $F(\cdot, x)$ measurable, $F(t, \cdot)$ Lipschitz with constant $\ell(t)$ integrable, and $y \in F(t, x) \Rightarrow |y| \leq 2\, m(t)$, $m(\cdot)$ integrable.

Then the Cauchy problem

$$x' \in \text{co}\, F(t, x) \quad \text{a.e. on } I, \quad x(0) = \xi,$$

has the same absolutely continuous solutions as the control differential equation

$$x' = f(t, x, u) \quad \text{a.e. on } I, \quad x(0) = \xi, \quad |u(t)| \leq 1,$$

where $f : \mathbb{R} \times \mathbb{R}^n \times B \to \mathbb{R}^n$ is measurable in t and Lipschitz in (x, u) with constant $6\,n\,[2\ell(t) + m(t)]$, and B is the unit closed ball in $\mathbb{R}^n$.

Proposition 1

Let F verify hypothesis **(FU)** of section 2.1.

Let f, U be as in Theorem 2.1.1 or Corollary 2.1.2.

Then for each $\underline{x} : I \to X$, $\underline{y} : I \to \mathbb{R}^n$ measurable verifying $\underline{y}(t) \in \text{co } F(t, \underline{x}(t))$ a.e. there exists $\underline{u} : I \to U$ measurable such that $\underline{y}(t) = f(t, \underline{x}(t), \underline{u}(t))$ a.e. .

If moreover F, w are jointly h-continuous and $\underline{x}, \underline{y}$ are continuous then $\underline{u}$ is continuous.

Proof Consider the homeomorphism h as in Corollary 2.1.1 or Theorem 2.1.1, and set $\underline{u}(t) := h^{-1}(t, \underline{x}(t), \underline{y}(t))$. ♦

3.2 The lsc case

Introduction

Consider a multifunction F with integrably bounded values F(t, x), and let S (ξ) be the solution set for the corresponding Cauchy problem (CP). If F(t, x) were Lipschitz in x then the solution set S (ξ) would be dense in the solution set of the relaxed problem (CPR), relative to the uniform topology (see section 3.3). However Plis [52] constructed a counter example showing that this density does not hold if we just assume continuity of F(t, ·) .

In this section we prove an approximation result for multifunctions F(t, x) lsc in x, which is weaker than the above density result and contains it as a special case (see section 3.3). This approximation result was originally obtained by Pianigiani [51] under stronger hypothesis, namely supposing F jointly continuous with values contained in a ball of $\mathbb{R}^n$. His proof is based on the method of Antosiewicz-Cellina, as explained in section 2.2. Other treatments of differential inclusions by this method are: Bressan [4], for multifunctions F(t, x) jointly lsc ; Colombo-Fonda-Ornelas [21] and Tolstonogov-Finogenko [53] for measurable multifunctions F(t, x) lsc in x.

Assumptions and result

Let I be the interval $[0, T]$, let Ξ be a compact convex set in $\mathbb{R}^n$ and Ω an open or closed set in $\mathbb{R}^n$.

Hypothesis (H')

$F : I \times \Omega \to \mathbb{R}^n$ is a multifunction with:

(a') values $F(t, x)$ compact ;

(b') $\exists\ I_0 \subset I$ such that $I \setminus I_0$ is a null set and $F_{|\, I_0 \times \Omega}$ is $\mathcal{L} \otimes \mathcal{B}$ - measurable ;

(c') $\exists\ M : I \to \mathbb{R}^+$ integrable such that $y \in F(t, x) \Rightarrow |y| \leq M(t)$ a.e.

and $d(y, \Xi) \leq |M|_1 \Rightarrow y \in \Omega$;

(d') $F(t, \cdot)$ is lsc .

Theorem 1

Let F verify hypothesis (H').

Let (ξ_i) be a sequence converging to some ξ_* in Ξ. Let $f(t, x)$ be a selection from co $F(t, x)$, measurable in t and continuous in x.

Then there exists a solution x_* of $x' = f(t, x)$, $x(0) = \xi_*$, and a sequence (x_i) of solutions of (CP) with $x_i(0) = \xi_i$, such that (x_i) converges uniformly to x_*.

Consider the compact convex subset of C^0 defined by :

$$K_\infty := \{ x \in C^0 : \ x \in AC, \ x(0) \in \Xi, \ |x'(t)| \leq M(t) \ \text{ a.e. } \}$$

Lemma 1

Let F verify hypothesis **(H')**.
Let $f(t, x)$ be a selection from $\text{co}\, F(t, x)$, measurable in t and continuous in x.
Then the function $g_*: K_\infty \to L^1$ and the multifunction $G : K_\infty \to L^1$ defined by

$$g_*(x)(t) := f(t, x(t)), \qquad G(u) := \{ v \in L^1 : \ v(t) \in F(t, x(t)) \ \text{ a.e. } \}$$

verify hypothesis **(H)** of section 2.2.

Proof

Clearly we need only prove that G is lsc. Notice first that for $u \in K_\infty$ the map $t \mapsto F(t, u(t))$ is measurable (see 1.2 (iii), (iv)). Let C be a closed set in L^1, and consider a sequence $(u_k) \to u_o$ such that $G(u_k) \subset C \ \forall\, k \in \mathbb{N}$. Fix any $v_o \in G(u_o)$; since $G(u_k)$ is closed decomposable, by Proposition 2.2.2 there exists $v_k \in G(u_k)$ such that $D(v_o, v_k) = D(v_o, G(u_k))$, hence for a.e. t we have:

$$|v_o(t) - v_k(t)| = D(v_o, v_k)(t) = D(v_o, G(u_k))(t) = d(v_o(t), F(t, u_k(t)))$$

; but $F(t, \cdot)$ is lsc, $(u_k(t)) \to u_o(t)$, and $v_o(t) \in F(t, u_o(t))$, hence $|v_o(t) - v_k(t)| \to 0$ as $k \to \infty$. This means that $d_1(v_o, v_k) \to 0$, and since $(v_k) \subset C$, we have $v_o \in C$. ♦

Proof of the theorem

Define g_* and G as in Lemma 1. Then by Theorem 2.2.1 there exists a sequence (g_i) of continuous selections from the multivalued Nemitskii operator G associated to F such that, setting

$$h_i, h_*: K_\infty \to K_\infty, \quad h_i(x)(t) = \xi_i + \int_0^t g_i(x)(\tau)\, d\tau, \qquad h_*(x)(t) = \xi_* + \int_0^t g_*(x)(\tau)\, d\tau,$$

then $(h_i) \to h_*$ uniformly.

It is clear that $h_i(K_\infty) \subset K_\infty$, and that h_i is continuous. By Schauder theorem, for each $i \in \mathbb{N}$ there exists a fixpoint $x_i = h_i(x_i)$, i.e. $x_i' = g_i(x_i) \in G(x_i)$, $x_i(0) = \xi_i$. This means that $x_i'(t) \in F(t, x(t))$ a.e.. Since (x_i) is a sequence in the compact K_∞, a subsequence, which we denote again by (x_i), converges to some x_*. It is clear that $x_* = h_*(x_*)$, so that $x_*'(t) = f(t, x_*(t))$ a.e..

♦

3.3 The continuous case

Introduction

Let $F(t, x)$ be a multifunction continuous in x and integrably bounded. Consider the solution set $S(\xi)$ for the associated Cauchy problem (CP). In the first part of this section we suppose that $F(t, \cdot)$ has a modulus of continuity of the Kamke type, namely implying uniqueness; and prove that the solution set $S(\xi)$ is dense in the relaxed solution set associated to the convexifed problem (CPR). This is not a new result: for $F(t, x)$ Lipschitz in x it was found partially by Wazewski [54] and then completed by Filippov [28]; and for Kamke type conditions it is due to Pianigiani [51], in case $F(t, x)$ is jointly continuous, and to Tolstonogov-Finogenko [53], under measurability conditions (see also Bressan [5], for the locally Lipschitz case). However in those papers the relationship between density and uniqueness is somewhat hidden, while here it appears very clearly. In fact the density result follows directly from the results in sections 2.1 and 3.2 .

We now describe the contents of the second part of this section. Supposing that F is continuous in x, we prove that the solution sets $S(\xi)$ are closed if and only if the values $F(t, x)$ are convex. The "if" part is well known (see Aubin-Cellina [2, Th. 2.2.1] and also [17]). What we prove here is that some solution sets $S(\xi)$ are nonclosed provided some values $F(t, x)$ are nonconvex. Also this proof is based on the results of sections 2.1 and 3.2. Notice that the result cannot be improved, in the sense that if $F(t, \cdot)$ fails to be continuous then the equivalence between convexity and closure does not hold anymore. More precisely, when $F(t, x)$ is not usc then some solution sets $S(\xi)$ may be nonclosed even if all values $F(t, x)$ are convex, as simple examples show; and when $F(t, \cdot)$ is not lsc then some solution sets $S(\xi)$ may be closed even if some values $F(t, x)$ are nonconvex (see [17]).

Assumptions and results

Let I be the interval $[0, T]$.

Hypothesis (H)

$F : I \times \mathbb{R}^n \to \mathbb{R}^n$ is a multifunction with:

(a) values $F(t, x)$ compact ;

(b) $F(\cdot, x)$ measurable ;

(c) $\exists\ m : I \to \mathbb{R}^+$ integrable such that $y \in F(t, x) \Rightarrow |y| \leq m(t)$ for a.e. t ;

(d) $\exists\ w : I \times \mathbb{R}^+ \to \mathbb{R}^+$ such that $dl(F(t, x), F(t, \underline{x})) \leq w(t, |x - \underline{x}|)$,

with $w(\cdot, r)$ measurable, $w(t, \cdot)$ continuous concave, $w(t, 0) = 0$ and $w(t, r) \leq 2\, m(t)$, for a.e. $t \in I$.

Condition (C)

$f_* : I \times \mathbb{R}^n \to \mathbb{R}^n$ is a function with:

(b') $f_*(\cdot, x)$ measurable ;

(c') $\exists\ m$ as in (c) such that $|f_*(t, x)| \leq m(t)$ for a.e. t ;

(d') $\exists\ w$ as in (d) such that $|f_*(t, x) - f_*(t, \underline{x})| \leq 12\, n\ w(t, |x - \underline{x}|)$ for a.e. t ;

(e) the differential equation $r'(t) = 12\, n\ w(t, r)$, $r(0) = 0$

has a unique AC solution on $[0, \underline{t}]$, for each $\underline{t}$ in $[0, T]$.

Theorem 1

Let F verify hypothesis (H) and suppose that w verifies (e) of condition (C).
Then for each solution x_* of the relaxed Cauchy problem (CPR) there exists a selection f_* from co F(t, x) verifying condition (C) such that x_* is the unique solution of the differential equation

$$x' = f_*(t, x), \quad x(0) = \xi .$$

In particular the solution set $S(\xi)$ of (CP) is dense in the solution set of the relaxed Cauchy problem (CPR).

Theorem 2

Let F verify hypothesis (H).
Then the solution sets $S(\xi)$ of (CP) are closed $\forall \xi \in \mathbb{R}^n$ if and only if the values F(t, x) are convex $\forall x \in \mathbb{R}^n$ for a.e. $t \in I$.

Corollary 1

Let F verify hypothesis (H) and suppose that $F(\cdot, x)$ is continuous $\forall x \in \mathbb{R}^n$.
Then the solution set $S(\xi_0)$ of (CP) is closed if and only if F(t, x) is convex $\forall (t, x) \in \text{graph}\,[t \longmapsto S(\xi_0)(t)]$ for a.e. $t \in I$.

Proof of Theorem 1

As in Theorem 3.1.1, find f such that $f(t, x, B) = \text{co}\, F(t, x)$, B the unit ball in $\mathbb{R}^n$; and $u_* : I \to B$ such that $f(t, x_*(t), u_*(t)) = x_*'(t)$ a.e., in such a way that the function f_* defined by $f_*(t, x) := f(t, x, u_*(t))$ verifies condition (C). Now apply Theorem 3.2.1 .

Proof of Theorem 2

The proof goes as follows. Supposing $F(\cdot, x_0)$ nonconvex on a set of positive measure, we construct an integrable selection $v(t) \in \text{co}\, F(t, x_0)$ such that $d(v(t), F(t, x_0)) \geq \eta_0 > 0$ on a set of positive measure. This v gives an approximate relaxed solution and near it there exists a true relaxed solution $\underline{x}$ verifying $\underline{x}' \notin F(t, \underline{x})$ on a set of positive measure. On the other hand the results of sections 2.1 and 3.2 yield a sequence (x_i) of solutions of (CP) converging uniformly to $\underline{x}$.

(i) Let S_1 be a subset of I with positive measure and x_0 a point in $\mathbb{R}^n$ such that $F(t, x_0)$ is nonconvex $\forall t \in S_1$. By the Scorza-Dragoni property in 1.2, there exists a compact set $S \subset S_1$ with positive measure such that $F|_{S\times\mathbb{R}^n}$ is lsc with closed graph. By Lusin's theorem we may suppose that $w|_{S\times\mathbb{R}^+}$ is continuous.

Fix any $t_0 \in S$ and choose a point $v_0 \in \text{co}\, F(t_0, x_0)$ such that $v_0 \notin F(t_0, x_0)$. Then $\eta_0 := \frac{1}{6} d[v_0, F(t_0, x_0)]$ is positive. By Michael's theorem [44] there exists a continuous selection $v(\cdot)$ from $\text{co}\, F(\cdot, x_0)|_S$ such that $v(t_0) = v_0$. By the theorem of Kuratowski-Ryll Nardzewski [38] we can extend v as a measurable map to the whole interval I so as to verify $v(t) \in F(t, x_0)$ $\forall t \in I \setminus S$. Set $\eta(t) := \frac{1}{6} d[v(t), F(t, x_0)]$. Since $v|_S$ is continuous and $F(\cdot, x_0)|_S$ is lsc with closed graph, the map $\eta|_S$ is continuous. Set $I(t_0) := I \cap (t_0 - \delta_0, t_0 + \delta_0)$, $S(t_0) := S \cap I(t_0)$, where $\delta_0 = \delta(t_0) > 0$ is chosen to be so small that

(1) $$\eta(t) \geq \frac{1}{2}\eta_0 \qquad \forall t \in S(t_0),$$

(2) $$w(t, M) \leq \eta_0 / (12n), \text{ where } M := \int_{I(t_0)} m(\tau)\, d\tau, \qquad \forall t \in S(t_0).$$

We can repeat this procedure for each $t_0 \in S$, obtaining an open cover $\{ S(t_0) : t_0 \in S \}$ of S. If $S(t_0), \ldots, S(t_N)$ is a finite subcover then at least one of the sets, let it be $S(t_0)$, has positive measure. Let S_0 be the closure of $S(t_0)$ and $I_0 = [\underline{t}, \underline{t} + \Delta]$ be the closure of $I(t_0)$. It is clear from the construction that $S_0 \subset I_0$ has positive measure and

(3) $$v(t) \in \text{co}\, F(t, x_0), \quad |v(t)| \leq m(t) \qquad \forall t \in I_0,$$

(4) $$d(v(t), F(t, x_0)) \geq 3\eta_0 \qquad \forall t \in S_0.$$

Consider the Cauchy problem

(CP_0) $\qquad x' \in F(t, x)$ a.e. on I_0, $x(\underline{t}) = x_0$,

and its relaxed version

(CPR_0) $\qquad x' \in \text{co}\, F(t, x)$ a.e. on I_0, $x(\underline{t}) = x_0$.

We construct a solution $\underline{x}$ of (CPR_0) such that $d(\underline{x}'(t), F(t, \underline{x}(t))) \geq \eta_0$ for a.e. $t \in S_0$, showing that $\underline{x}$ does not solve (CP_0). Clearly $\underline{x}$ can be extended to the whole interval I so as to be a solution of the relaxed problem (CPR), but it is certainly not a solution of (CP).

(ii) Using Theorem 2.1.1, find $f(t, x, u)$ with modulus of continuity $12\, n\, w(t, \cdot)$ relative to x such that $f(t, x, B) = \text{co}\, F(t, x)$, with B the unit closed ball of $\mathbb{R}^n$. By (3) and Proposition 3.1.1 there exists $\underline{u} : I \to B$ measurable such that, setting $\underline{f}(t, x) := f(t, x, \underline{u}(t))$, we have $\underline{f}(t, x_0) = v(t)$ $\forall\, t \in I_0$, and, by (2),

(5) $\quad |\underline{f}(t, x) - v(t)| \leq 12\, n\, w(t, |x - x_0|) \leq \eta_0$ provided $t \in S_0$, $|x - x_0| \leq M$.

Then, by Theorem 3.2.1 there exists a sequence (x_i) of solutions of (CP_0) which converges uniformly to a solution $\underline{x}$ of $x' = \underline{f}(t, x)$, $x(\underline{t}) = x_0$. Clearly $\underline{x}$ solves (CPR_0); however since

$$|\underline{x}(t) - x_0| \leq \int_{\underline{t}}^{t} m(\tau)\, d\tau \leq M$$

we have $\qquad dl(F(t, x_0), F(t, \underline{x}(t))) \leq w(t, M) \leq \eta_0 \quad \forall\, t \in S_0 \qquad$ by (2),

and $\qquad |\underline{x}'(t) - v(t)| \leq \eta_0 \qquad$ for a.e. $t \in S_0 \qquad$ by (5).

Therefore (4) yields

(6) $$3\eta_0 \leq d(v(t), F(t, x_0)) \leq |v(t) - \underline{x}'(t)| + d(\underline{x}'(t), F(t, \underline{x}(t))) +$$

$$+ dl(F(t, \underline{x}(t)), F(t, x_0)) \leq \eta_0 + d(\underline{x}'(t), F(t, \underline{x}(t))) + \eta_0,$$

hence $\qquad d(\underline{x}'(t), F(t, \underline{x}(t))) \geq 3\eta_0 - \eta_0 - \eta_0 = \eta_0$ for a.e. $t \in S_0$. ♦

3.4 The Lipschitz case

Introduction

Consider a multifunction $F(t, x)$ Lipschitz in x with unbounded values, and let $S(\xi)$ be the solution set of the associated Cauchy problem (CP). In section 2.3 we presented already some results for the multifunction $S: \Xi \to C^0$. Here we prove that there exists a solution of the relaxed Cauchy problem (CPR) which does not solve (CP). A more precise result can be obtained in case the multifunction $F(t, x)$ is integrably bounded, as was explained in section 3.3.

Assumptions and result

Let I be the interval $[0, T]$.

Hypothesis (H)

$F : I \times \mathbb{R}^n \to \mathbb{R}^n$ is a multifunction with:

(a) values $F(t, x)$ closed;

(b) $F(\cdot, x)$ measurable;

(c) $\exists\ m$ in $L^1(I)$ and x_1, y_1 in $\mathbb{R}^n$ such that

$$d(y_1, F(t, x_1)) \leq m(t) \quad \text{for a.e. } t \in I;$$

(d) $\exists\ \ell$ in $L^1(I)$ such that $\quad dl(F(t, x), F(t, \underline{x})) \leq \ell(t)\ |x - \underline{x}|$ for a.e. $t \in I$.

Notice that this hypothesis is equivalent to hypothesis (H) in section 2.3.

Theorem 1

Let F satisfy hypothesis (H) and exclude the trivial case in which all values $F(t, x)$ are convex $\forall\, x \in \mathbb{R}^n$ for a.e. $t \in I$.

Then for some $\xi \in \mathbb{R}^n$ there exists a solution of the relaxed problem (CPR) which does not solve (CP).

Proof

(i) Let S_1 be a subset of I with positive measure and x_0 a point in $\mathbb{R}^n$ such that $F(t, x_0)$ is nonconvex $\forall\, t \in S_1$. Let $S \subset S_1$ be a compact set with positive measure such that $F|_{S\times\mathbb{R}^n}$ is lsc with closed graph, and let ℓ_0 be a constant such that $\ell(t) \le \ell_0$ $\forall\, t \in S$. Possibly changing m we may suppose, using (c) and (d), that $d\,(0, F(t, x_0)) \le m\,(t)$ for a.e. $t \in I$.

Fix any $t_0 \in S$, choose a vector $v_0 \in \text{co}\, F(t_0, x_0)$ such that $v_0 \notin F(t_0, x_0)$, and set $\eta_0 := \frac{1}{6} d(v_0, F(t_0, x_0))$. As in the proof of Theorem 3.3.2 find a measurable selection $v(\cdot)$ from $\text{co}\, F(\cdot, x_0)$, an interval I_0 containing t_0 and a set $S_0 \subset I_0$ with positive measure such that

(1) $$d\,(v(t),\ F\,(t, x_0)) \ge 3\,\eta_0 \qquad \forall\, t \in S_0\ ,$$

(2) $$\ell_0\, e^L M \le \eta_0, \quad \text{where} \quad M := \int_{I(t_0)} m_0(\tau)\, d\tau, \qquad L := \int_{I(t_0)} \ell\,(\tau)\, d\tau\ .$$

(ii) Setting

$$x(t)\ :=\ x_0 + \int_{\underline{t}}^{t} v(\tau) d\tau\ ,$$

we have

$$d\,(x'(t), \text{co}\, F(t, x(t))) \le dl\,(\text{co}\, F(t, x_0),\ \text{co}\, F(t, x(t))) \le$$

$$\le \ell\,(t)\ |x_0 - x(t)| \le \ell\,(t) \int_{\underline{t}}^{t} m_0(\tau)\, d\tau \le \ell\,(t)\ M \qquad \forall\, t \in I_0\ .$$

Consider the Cauchy problem (CP_0) as in the proof of Theorem 3.3.2, and let (CPR_0) be its relaxed version. The above inequality shows that x is an approximate solution of (CPR_0),

hence by the Filippov-Gronwall inequality (as in section 2.3) there exists a solution $\underline{x}$ of (CPR_0) such that:

$$|\underline{x}(t) - x_0| \le |\underline{x}(t) - x(t)| + |x(t) - x_0| \le$$

$$\le \int_{\underline{t}}^{t} \exp\left(\int_s^t \ell\right) \ell(s) \int_{\underline{t}}^{s} m(\tau)\, d\tau \; ds + \int_{\underline{t}}^{t} m(\tau)\, d\tau =$$

$$= \int_{\underline{t}}^{t} \exp\left(\int_s^t \ell\right) m(s)\, ds \le e^L M \quad \forall\, t \in I_0, \qquad \text{so that}$$

$$dl\,(F(t, \underline{x}(t)), F(t, x_0)) \le \ell(t)\; |\underline{x}(t) - x_0| \le \ell(t)\; e^L M\;;$$

and $$|\underline{x}'(t) - v(t)| \le \ell(t) \int_{\underline{t}}^{t} \exp\left(\int_s^t \ell\right) m(\tau)\, d\tau \le \ell(t)\, e^L M\;,\; \forall\, t \in I\;.$$

For $t \in S_0$ we have $\ell(t) \cdot e^L M \le \ell_0 e^L M \le \eta_0$, by (2), hence (1) yields

(3) $$3\,\eta_0 \le d(v(t), F(t, x_0)) \le |v(t) - \underline{x}'(t)| + d(\underline{x}'(t), F(t, \underline{x}(t))) +$$

$$+ dl\,(F(t, \underline{x}(t)), F(t, x_0)) \le \eta_0 + d(\underline{x}'(t), F(t, \underline{x}(t))) + \eta_0\;, \text{ hence}$$

$$d(\underline{x}'(t), F(t, \underline{x}(t))) \ge 3\,\eta_0 - \eta_0 - \eta_0 = \eta_0 \quad \text{for a.e. } t \in S_0\,.$$

This shows that $\underline{x}$ does not solve (CP_0). However $\underline{x}$ can be extended so as to be a solution of (CPR) for some adequate $\xi = \underline{x}(0)$, while it certainly does not solve (CP) with that same ξ. ♦

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