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FOUNDATIONS OF KINETIC THEORY FOR ASTROPHYSICAL PLASMAS WITH APPLICATIONS TO ACCRETION DISCS AND ELECTROMAGNETIC RADIATION-REACTION

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ultimum finis aestum decedere occasus longe
timori imminere memoriae praeteriti venti
temporis sidera nox silentium amittere fleto

Al mio nonno Andrea

Abstract

This thesis is devoted to laying down the theoretical foundations of kinetic theory for astrophysical plasmas that can be found in accretion discs around compact objects and in relativistic jets. The thesis aims at developing a self-consistent theoretical treatment of the subject which consists of two distinctive and complementary parts, sharing the description of plasmas both at the fundamental level of single-particle dynamics as well as in terms of the statistical description provided by kinetic theory.

In the first line of research, the formulation of a Vlasov-Maxwell kinetic theory for collisionless plasmas is addressed. An example of accretion disc plasmas of this type is provided by the radiatively inefficient flows (RIAFs). The case of non-relativistic axisymmetric magnetized and gravitationally-bound plasmas is considered in this part. Exact quasi-stationary kinetic solutions (equilibria) of the Vlasov equation are constructed and expressed in terms of generalized Maxwellian or bi-Maxwellian kinetic distribution functions (KDFs). These equilibria permit the treatment of quasi-stationary plasmas characterized by non-uniform fluid fields, with particular reference to temperature and pressure anisotropies, azimuthal and poloidal flows and differential rotation. The theory allows for the self-consistent analytical determination of the equilibrium KDFs and the perturbative treatment of the corresponding fluid fields, whose functional dependences are uniquely prescribed by means of suitably-imposed kinetic constraints. It is proved that the quasi-neutrality condition can be satisfied for these plasmas and an analytical solution for the electrostatic potential is reached under such an assumption. Then, analysis of the Ampere equation shows that collisionless plasmas can sustain both equilibrium azimuthal and poloidal charge currents, responsible for the self-generation of stationary poloidal and azimuthal magnetic fields. This intrinsically-kinetic mechanism is referred to here as kinetic dynamo. Regarding the azimuthal field, its origin can be entirely diamagnetic and due to phase-space anisotropies, finite-Larmor radius and energy-correction effects which are driven by temperature anisotropy. Finally, the possibility of describing quasi-stationary accretion flows is discussed, showing that these are admitted at the equilibrium for a non-vanishing toroidal magnetic field and/or plasma temperature anisotropy. A linear stability analysis of the Vlasov-Maxwell equilibria obtained is then performed by means of a fully-kinetic analytical treatment. It is proved that, for strongly-magnetized and gravitationally-bound plasmas in the presence

of a purely poloidal equilibrium magnetic field, these equilibria are asymptotically stable with respect to low-frequency and long-wavelength axisymmetric perturbations. Finally, as a side application, the kinetic theory is applied to laboratory plasmas for the kinetic description of rotating Tokamak plasmas in the collisionless regime.

The second line of research deals with the formulation of the relativistic kinetic and fluid theories appropriate for the treatment of relativistic collisionless plasmas. For this purpose, the classical electromagnetic (EM) radiation-reaction (RR) effect characterizing the dynamics of relativistic charged particles is first addressed. A variational formulation of the problem which is consistent with the basic principles of Classical Electrodynamics and Special Relativity is given. In contrast with previous treatments dealing with point charges and leading to the Lorentz-Abram-Dirac (LAD) and Landau-Lifschitz (LL) equations, here the case of classical finite-size charged particles is considered. The RR problem is given an exact analytical solution in which the occurrence of any possible divergence is naturally avoided thanks to the representation adopted for the particle 4-current density and the corresponding well-defined self 4-potential. The variational RR equation is obtained from the Hamilton variational principle and proved to be a delay-type ODE, with non-local contributions being due to the EM RR and arising from the finite-size charge distribution. Standard Lagrangian and conservative forms of the RR equation are obtained, expressed in terms of an effective Lagrangian function and a suitable stress-energy tensor respectively. The connection with the LAD equation is established, while the exact RR equation is proved to satisfy a fundamental existence and uniqueness theorem and to admit a classical dynamical system. On the basis of these results, a standard Hamiltonian formulation of the non-local RR equation is obtained in terms of an effective non-local Hamiltonian function. From the Hamiltonian representation of the non-local EM RR effect, a canonical formulation of the kinetic theory for relativistic collisionless plasmas subject to RR is presented. This permits to obtain the relativistic Liouville equation in Hamiltonian (conservative) form for the relativistic KDF in which the EM self-field due to the RR effect is included. Then, the corresponding relativistic fluid equations are obtained in both Eulerian and Lagrangian form, with the non-local EM RR contribution acting as a non-conservative collisional operator. Finally, the construction of a Hamiltonian structure characterizing N-body systems of finite-size particles subject to EM interactions is addressed and the corresponding variational, Lagrangian and Hamiltonian delay-type ODEs are obtained. On the basis of this result, the validity of the "no-interaction" theorem proposed by Currie is questioned. It is proved that the latter is violated by the non-local Hamiltonian structure determined here. Explicit counter-examples which overcome the limitations stated by the "no-interaction" theorem are provided.

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Contents

Collaborations

The research presented in this thesis was mainly conducted between November 2008 and October 2012 at SISSA-International School for Advanced Studies in Trieste with Prof. John C. Miller and at the Department of Mathematics-University of Trieste with Prof. Massimo Tessarotto. Concerning the organization of the thesis, each chapter is presented as a self-standing research investigation which contains the results of my work and is based on the following different published papers:

- CHAPTER 1
  - C. Cremaschini, J. C. Miller and M. Tessarotto, *Kinetic axisymmetric gravitational equilibria in collisionless accretion disk plasmas*, Phys. Plasmas **17**, 072902 (2010).
  - C. Cremaschini, J. C. Miller and M. Tessarotto, *Theory of quasi-stationary kinetic dynamos in magnetized accretion discs.* Proceedings of the International Astronomical Union, "Advances in Plasma Physics", Giardini Naxos, Sicily, Italy, Sept. 06-10, 2010. Cambridge University Press (Cambridge), vol. 6, p. 228-231.
  - C. Cremaschini, J. C. Miller and M. Tessarotto, *Kinetic closure conditions for quasi-stationary collisionless axisymmetric magnetoplasmas.* Proceedings of the International Astronomical Union "Advances in Plasma Physics", Giardini Naxos, Sicily, Italy, Sept. 06-10, 2010. Cambridge University Press (Cambridge), vol. 6, p. 236-238.
  - C. Cremaschini, M. Tessarotto and J. C. Miller, *Diamagnetic-driven kinetic dynamos in collisionless astrophysical plasmas*, Magnetohydrodynamics **48**, N. 1, pp.3-13 (2012)
- CHAPTER 2
  - C. Cremaschini, J. C. Miller and M. Tessarotto, *Kinetic description of quasi-stationary axisymmetric collisionless accretion disk plasmas with arbitrary magnetic field configurations*, Physics of Plasmas **18**, 062901 (2011).

- CHAPTER 3
  - C. Cremaschini, M. Tessarotto and J. C. Miller, *Absolute stability of axisymmetric perturbations in strongly-magnetized collisionless axisymmetric accretion disk plasmas*, Physical Review Letters **108**, 101101 (2012).
- CHAPTER 4
  - C. Cremaschini and M. Tessarotto, *Kinetic description of rotating Tokamak plasmas with anisotropic temperatures in the collisionless regime*, Physics of Plasmas **18**, 112502 (2011).
- CHAPTER 5
  - C. Cremaschini and M. Tessarotto, *Exact solution of the EM radiation-reaction problem for classical finite-size and Lorentzian charged particles*, The European Physical Journal Plus **126**, 42 (2011).
- CHAPTER 6
  - C. Cremaschini and M. Tessarotto, *Hamiltonian formulation for the classical EM radiation-reaction problem: Application to the kinetic theory for relativistic collisionless plasmas*, The European Physical Journal Plus **126**, 63 (2011).
- CHAPTER 7
  - C. Cremaschini and M. Tessarotto, *Hamiltonian structure of classical N-body systems of finite-size particles subject to EM interactions*, The European Physical Journal Plus **127**, 4 (2012).
  - C. Cremaschini and M. Tessarotto, *Addendum to: Hamiltonian structure of classical N-body systems of finite-size particles subject to EM interactions*, The European Physical Journal Plus **127**, 103 (2012).

The following collaborations are also warmly acknowledged:

- C. Cremaschini and M. Tessarotto, *Collisionless kinetic regimes for quasi-stationary axisymmetric accretion disc plasmas*, Physics of Plasmas **19**, 082905 (2012).
- Massimo Tessarotto, C. Asci, C. Cremaschini, A. Soranzo, Marco Tessarotto and G. Tironi, *Lagrangian dynamics of thermal tracer particles in Navier-Stokes fluids*, The European Physical Journal Plus **127**, 36 (2012).
- J. Kovar, P. Slany, Z. Stuchlik, V. Karas, C. Cremaschini and J. C. Miller, *Role of electric charge in shaping equilibrium configurations of fluid tori encircling black holes*, Physical Review D **84**, 084002 (2011).
- G. Marino, C. Arena, I. Bellia, G. Benintende, C. Cremaschini, S. Foglia et al., *CCD Minima of Eclipsing Binary Stars*, Information Bulletin on Variable Stars **1**, 5917 (2010).
- Marco Tessarotto, C. Cremaschini and Massimo Tessarotto, *Phase-space Lagrangian dynamics of incompressible thermofluids*, Physica A **388**, 3737 (2009).
- T. Widemann, B. Sicardy, R. Dusser, C. Martinez, W. Beisker, C. Cremaschini et al., *Titania's radius and an upper limit on its atmosphere from the September 8, 2001 stellar occultation*, Icarus **199**, 458-476 (2009).
- J.-E. Arlot, W. Thuillot, C. Ruatti, A. Ahmad, A. Amossè, P. Anbazhagan, C. Cremaschini et al., *The PHEMU03 catalogue of observations of the mutual phenomena of the Galilean satellites of Jupiter*, Astronomy & Astrophysics **493**, 1171 (2009).

Preface

The ambition to investigate the phenomenology of physical events occurring in nature has always involved the effort to provide a theoretical and rationalistic basis for physical sciences. I believe that the foundations of this thesis can be seen in that perspective.

In the present work, several problems, previously considered unsolved in the literature, have been investigated and corresponding solutions have been given. In addition, methods and techniques developed here, some of which are new at least for the context in which they are adopted, have led to a number of notable outcomes which are novel in several aspects.

These developments are relevant in principle both for astrophysical and laboratory plasma physics and mathematical physics. In all cases, the concept of "first-principle" approach has been adopted, i.e., based on classical electrodynamics, to warrant the validity of logical coherence and consistency with the principles and the laws of physics. The effort has been made to present each topic in closed analytical form and with a clear mathematical notation. In many cases, however, the apparent simplicity of the conclusions hides a notable underlying conceptual effort.

A preliminary remark should be made, concerning the structure of the thesis, in which each chapter is presented as a self-standing research investigation based on different published papers, as indicated above in the previous section.

From the point of view of the scientific relevance, two main research lines are developed in the thesis. The first one deals with the formulation of a kinetic theory for collisionless astrophysical and laboratory plasmas. The second one is about the solution of the so-called Electromagnetic (EM) Radiation-Reaction (RR) problem for relativistic finite-size charges, with applications to the kinetic and fluid theories of relativistic collisionless plasmas. The connection between the two issues is that they concern the description of plasmas both at the fundamental level of single-particle dynamics as well as in terms of the statistical description provided by kinetic theory. In connection with this, an introductory section containing some basic explanatory material for the understanding of the theoretical background underlying the developments presented in the thesis is included. Applications of the theories developed here are primarily intended for collisionless astrophysical plasmas in accretion discs around compact objects or relativistic jets. On the other hand, the generality of the formalism adopted makes it possible to apply the theory also to the case of laboratory plasmas in controlled fusion devices (with particular reference to Tokamak devices) or to laser plasmas.

The kinetic theory for astrophysical accretion-disc plasmas is developed here in the framework of the Vlasov-Maxwell description and is presented in Chapters 1 to 3. The case of non-relativistic collisionless axisymmetric magnetized and gravitationally-bound plasmas is considered. An example of astrophysical plasmas of this type is represented by the so-called radiatively inefficient accretion flows (RIAFs). For these plasmas, customary fluid or magnetofluid descriptions (like the ideal-MHD theory) can only provide at most a rough approximation for the comprehension of their phenomenology. A consistent treatment of collisionless plasmas of this kind requires necessarily the adoption of a kinetic formulation. The reason is that, at a microscopic level, phase-space anisotropies can develop in collisionless plasmas, with the result that the kinetic distribution function (KDF) can significantly differ from a simple isotropic Maxwellian KDF, both in stationary and non-stationary configurations. Kinetic theory provides a phase-space description of plasmas, allowing for the proper inclusion of information from single-particle dynamics and conservations laws together with their effects in the functional form of fluid fields at the macroscopic level. In this approach, fluid fields which identify the physical observables (e.g., particle number density, flow velocity, temperature and tensor pressure) are computed "a posteriori" from the kinetic solution and are expressed in terms of suitable integrals of the KDF over the velocity space. As discussed in Chapters 1 and 2, this provides at the same time the only consistent and physically-admissible solution to the closure problem which affects fluid theories, and which is usually expressed through the necessity of providing an equation of state for the plasma.

Specifically, the construction of quasi-stationary kinetic solutions (equilibria) of the Vlasov equation is first presented. For doing this, a first-principles approach is implemented, which consists of expressing the equilibrium KDF in terms of first integrals and adiabatic invariants of the system and imposing suitable kinetic constraints. Different realizations of the solution are obtained, depending on the configuration of the equilibrium magnetic field and the phenomenology one wants to study. These exact solutions are then expressed in terms of Chapman-Enskog representations obtained by implementing a Taylor-expansion of the KDF in a convenient asymptotic limit, here referred to as strongly-magnetized and gravitationally-bound plasmas. The physical properties of these equilibrium configurations as well as the implications for the physics of collisionless accretion-disc plasmas are discussed. Several notable results are obtained. First, it is proved that, contrary to previous literature, the issue of determining consistent equilibrium configurations for collisionless accretion-disc plasmas is not a trivial task, and cannot be exhausted by taking as equilibrium KDFs Maxwellian or bi-Maxwellian distributions with arbitrary prescriptions for their fluid fields. Instead, it is shown that the quasi-stationary KDF can only be expressed in terms of generalized Maxwellian or bi-Maxwellian KDFs, with the functional dependences of the corresponding fluid fields uniquely prescribed by the kinetic constraints imposed on the solution. These equilibria permit the treatment of quasi-stationary plasmas characterized by non-uniform fluid fields, temperature and pressure anisotropies, azimuthal and poloidal flows and differential rotation. Remarkably, the theory permits analytical treatment of the equilibrium KDFs and analytic calculation of the fluid fields, which are proved to be exact solutions of the fluid equations associated with the Vlasov equation.

Secondly, the consistency of the kinetic solution with the constraints imposed by the Maxwell equations is addressed. In this regard, it is proved that the quasi-neutrality condition can be satisfied and an analytical solution for the electrostatic potential is obtained by solving the Poisson equation. Then, analysis of the Ampere equation shows that collisionless plasmas can sustain both equilibrium azimuthal and poloidal charge currents, which allow for the self-generation of a stationary poloidal and azimuthal magnetic field. This represents an intrinsically-kinetic mechanism, which has been referred to here as kinetic dynamo. Regarding the azimuthal field, it is pointed out that its origin can be entirely diamagnetic and due to phase-space anisotropies, finite-Larmor radius and energy-correction effects which are driven by temperature anisotropy. In particular, the azimuthal field can be generated in stationary configurations in the absence of turbulence phenomena or instabilities and even when no net accretion of matter takes place in the disc. Finally, the possibility of describing quasi-stationary accretion flows is discussed, showing that these are admitted at the equilibrium provided that the toroidal magnetic field is non-vanishing and/or the plasma temperature is non-isotropic. In fact, phase-space anisotropies together with temperature anisotropy can give rise to a non-isotropic pressure tensor which can ultimately cause poloidal flows to take place, even at equilibrium. This in turn shows also that, for collisionless magnetized plasmas, phase-space anisotropies effectively play a role analogous to that of viscous stresses for the generation of accretion phenomena in discs.

The detailed analysis of the quasi-stationary solutions and their physical content is also a necessary prerequisite for the proper investigation of the stability of the same equilibria. In performing a linear stability analysis, it is the equilibrium solution which enters the resulting dispersion relations, with its information on the physical properties of the system. Therefore, a correct stability analysis can only be performed starting from correct equilibrium solutions. In Chapter 3 a linear stability analysis of Vlasov-Maxwell equilibria previously obtained is performed. It is proved that axisymmetric collisionless strongly-magnetized and gravitationally-bound plasmas are asymptotically stable with respect to low-frequency and long-wavelength axisymmetric perturbations. This result is of general validity for the range of frequencies and wavelengths considered and it is obtained by means of a fully-kinetic analytical treatment.

As a side application, in Chapter 4 this kinetic theory is applied to study laboratory fusion plasmas and a kinetic description of rotating Tokamak plasmas in the collisionless regime is given. This formulation shows the generality of the formalism developed, which equally applies for describing plasmas in fusion devices, even when the physical setting of the system is intrinsically different from the case of accretion-disc plasmas. New quasi-stationary kinetic solutions are obtained, which extend previous kinetic or fluid treatments and which permit the description of axisymmetric plasmas characterized by non-uniform fluid fields, differential azimuthal rotation, poloidal flows and temperature and pressure anisotropies.

The second part of the thesis is dedicated to the Electromagnetic Radiation-Reaction

(EM RR) phenomenon and its role for relativistic collisionless plasmas, such as those observed in relativistic jets. The results are presented in Chapters 5 to 7. In this work, the problem of giving a variational formulation of EM RR which is also consistent with the basic principles of Classical Electrodynamics and Special Relativity is addressed. This includes, in particular, the Hamilton variational principle, the validity of the classical Maxwell equations, the Newton principle of determinacy, the Einstein causality principle and the covariance property of the theory. In contrast with customary treatments dealing with point charges and leading to the well-known Lorentz-Abram-Dirac (LAD) or Landau-Lifschitz (LL) equations, here the case of classical finite-size charged particles is considered. In connection with this, it must be stressed that, at the classical level, any model for the description of classical particles is in principle admitted. By construction, the point-particle model is non-physical and is a purely mathematical representation. It does not require specification of the internal structure of the particle, but on the other hand it carries an intrinsic divergence which manifests itself in the divergence of the self EM 4-potential generated by the point charge. The choice of a point-particle model appears therefore not suitable for the treatment of EM self-forces. In contrast, finite-size classical particles require specification of the charge distribution but are not affected by any kind of intrinsic divergence. A physically acceptable solution of the EM RR problem which is also consistent with Classical Electrodynamics and Special Relativity can therefore be formulated on the basis of finite-size classical charged particles. To warrant the validity of the principle of energy-momentum conservation, the particle mass distribution must also be finite-sized and coincide with the charge distribution. For the theory developed here, spherical distributions are assumed and expressed in covariant form, with both mass and charge of the finite-size particle having the same support and belonging to a shell sphere. The RR problem is given an exact analytical solution in which any possible divergence is naturally excluded due to the non-divergent particle 4-current density and the corresponding well-defined self 4-potential.

For definiteness, open problems addressed and solved in Chapter 5 concern the variational formulation of the EM RR effect for the finite-size charge distribution introduced, in terms of the Hamilton variational principle. An analytical solution of the retarded self 4-potential is first obtained, and the symmetry properties of the variational functional together with the role of the Einstein causality principle are discussed. Then, the resulting variational dynamic RR equation is obtained and proved to be a delay-type ODE. From the physical point of view, the RR phenomenon is therefore explained as a non-local effect, where the non-locality feature is precisely due to the finite-size extension of the charge distribution. For consistency, standard Lagrangian and conservative forms of this equation are given, the former expressed in terms of an effective Lagrangian function and the latter in terms of a suitable stress-energy tensor associated with the particle's finite-sized charge and mass distributions. The connection with customary literature treatments, and in particular with the LAD equation, is addressed by implementing a suitable Taylor-expansion of the exact equation when the particle radius is assumed to be infinitesimal. The resulting asymptotic form recovers the LAD expression for the RR self-force together with a mass-renormalization term, which is divergent in the (unphysical) point-particle limit. Finally, a fundamental existence and uniqueness theorem is proved to hold for the exact RR equation, which therefore admits a classical dynamical system.

In Chapter 6 the possibility of deriving a Hamiltonian formulation for the description of particle dynamics in the presence of non-local interactions is first addressed. Customary formulations for point-particles do not admit a Hamiltonian representation of the equation of motion. Thus, for example, the LAD equation is neither variational, nor Lagrangian or Hamiltonian. However, the question arises of whether this is an intrinsic feature of the RR phenomenon or rather the consequence of the particle model adopted in its treatment. In particular, the search for a Hamiltonian theory of EM RR is not just a mathematical problem, but also a philosophical question which at the level of fundamental physics affects both classical and quantum formulations of charged particle dynamics. In Chapter 6 this issue is raised and solved for the first time, by means of the exact variational and non-asymptotic formulation of the EM RR effect displayed in Chapter 5 and holding for extended classical particles. Hence, it is proved that the covariant second-order delay-type Lagrangian ODEs can be equivalently expressed in Hamiltonian form as first-order delay-type ODEs in terms of an effective non-local Hamiltonian function. The same Hamiltonian structure is then proved to hold also for asymptotic expansions of the exact equations, after introduction of a suitable short delay-time approximation which differs from the customary expansion which recovers the LAD equation. Both the exact and asymptotic Hamiltonian formulations obtained here are new, and are uniquely due to the physically-significant finite-size charge model adopted. This result gives the classical EM RR problem a conceptual consistency which is missing in other treatments in the literature, while overall the theory developed is characterized by a notable formal elegance and simplicity.

The Hamiltonian formulation of the non-local EM RR effect is the starting point for the construction of a kinetic theory in canonical form for relativistic collisionless plasmas. A relativistic Liouville equation for the covariant KDF describing these plasmas with the inclusion of the EM self-field is derived and the connection with non-canonical representations is studied. Then, the corresponding relativistic fluid equations are obtained and expressed in both Eulerian and Lagrangian form, while asymptotic expansions of the EM RR contributions can always be performed "a posteriori" after velocity-integration, on the fluid equations themselves. In contrast with other literature works based on the LAD or the LL equations, it is proved that the fluid equations obtained here retain the customary closure conditions, so that no higher-order fluid moments enter in the relevant equations. A further notable aspect of this formulation is that the physical meaning of the EM RR term can be displayed. In particular, both explicit and implicit RR contributions enter in the fluid equations, the former through the definition of the Lorentz force due to the total EM field, and the latter through the definition of the canonical particle state. It is shown that the EM RR effect is analogous to a sort of scattering process, acting like a non-conservative collisional operator, in which single-particle energy and momentum are not conserved due to the EM radiation emission caused by the RR effect.

Finally, in the last Chapter the theory developed in Chapters 5 and 6 is implemented to address the construction of a Hamiltonian structure characterizing N-body systems of finite-size particles subject to EM interactions. Variational, Lagrangian and Hamiltonian delay-type ODEs are obtained, in which non-local interactions are due to both the EM RR self-force as well as the retarded binary EM interactions occurring between different finite-size charged particles of the system. Then, on the basis of this result, the validity of the so-called "no-interaction" theorem proposed by Currie is raised. This theorem prevents the possibility of defining a Hamiltonian system for isolated particles subject to mutual EM interactions. The Currie approach is based on the well-known Dirac generator formalism (DGF). The correctness of the "no-interaction" theorem has long been questioned and is of central importance for both classical and quantum formulations. In this work, interesting conclusions are first drawn concerning the validity of the DGF in the present context, with particular reference to the so-called instant-form representation of Poincarè generators for infinitesimal transformations of the inhomogeneous Lorentz group. It is proved that, in its original formulation, the DGF only applies to local Hamiltonian systems and therefore is inapplicable to the treatment of the EM-interacting particles considered here. To overcome this limitation, a modified formulation of the DGF, which we refer to as the non-local generator formalism, is developed starting from the non-local Hamiltonian structure previously determined. On the same grounds, the Currie "no-interaction" theorem is proved to be violated in any case by the non-local Hamiltonian structure determined here. Counter-examples which overcome the limitations stated by the "no-interaction" theorem are explicitly provided. In particular, the purpose of this work is to prove that indeed a standard Hamiltonian formulation for the N-body system of EM-mutually-interacting charged particles can be consistently obtained.

In conclusion, I hope that the results presented in this thesis and the methodology outlined here could serve as contributions for the description of physical phenomena connected with astrophysical and laboratory plasmas and the very foundations of the classical dynamics of finite-size charged particles in the context of Special Relativity. All these issues have been addressed in the spirit of philosophical deduction and have been inferred by invoking the respect of the fundamental principles of classical logic and physics. Obtaining these results required a systematic work, a deep and continuous effort as well as an untiring patience over the years. The final satisfaction in front of the regret for the past time may represent the right gratefulness for the conscience and any feeling aged through this attempt.

Trieste, October 19, 2012 — Claudio Cremaschini

Some basic background

This thesis concerns a systematic investigation of certain types of astrophysical plasma, based on a kinetic theory approach. We give here a brief introduction to the basic concepts involved.

In principle, discussion of fluids or plasmas should always be based on considerations of the microscopic dynamics of the N-body system formed by the constituent particles. These particles are generally subject to binary (or multiple) mutual interactions and, while each of them has a finite size, they can often be treated as point-like. Various descriptions can be formulated for studying the dynamics of such systems. Besides the basic deterministic description, these include microscopic and reduced statistical ones, such as the kinetic and fluid descriptions. For large N-body systems (with $N \gg 1$), it is generally essential to use statistical descriptions since the algorithmic complexity of the deterministic description (of order N^2 for binary interactions) would be prohibitive, whereas the reduced statistical descriptions have complexity independent of N.

In the present work, single-particle dynamics and statistical kinetic descriptions of systems of N particles will be regarded as fundamental descriptions of plasmas, whereas the corresponding fluid one is considered as a derived description which is obtained "a posteriori" from the kinetic solution.

The kinetic treatment provides a phase-space statistical description of the ensemble of N particles in terms of a single-particle kinetic distribution function (KDF), which satisfies a suitable kinetic equation. Examples of differential equations of this kind are represented by the Fokker-Planck-Landau kinetic equation, which applies to collisional plasmas, and the Vlasov equation, which describes collisionless plasmas. Once the KDF is prescribed, all of the continuum fluid moments can be represented in terms of well-defined constitutive equations, determined via appropriate velocity moments of the KDF. The kinetic description represents the fundamental background underlying any fluid description. In particular, the configuration-space information dealt with by fluid theories can always be obtained from the underlying phase-space kinetic treatment by means of suitable velocity-space integrals.

In the non-relativistic regime, the plasma is treated as being an ensemble of particle species, each being described by a KDF $f_s(\mathbf{y}, t)$, where $\mathbf{y}$ is the particle state vector and t is the time, with $\mathbf{y} \equiv (\mathbf{r}, \mathbf{v})$ where $\mathbf{r}$ is the position and $\mathbf{v}$ is the velocity. For collisional plasmas, the species KDFs are given by the Fokker-Planck-Landau equation

$$L_s f_s(\mathbf{y}, t) = C_s\left(f \mid f\right), \tag{1}$$

where L_s denotes the species Vlasov differential operator and $C_s\left(f \mid f\right)$ is the species Fokker-Planck-Landau collisional operator. The differential operator L_s is defined as

$$L_s \equiv \frac{\partial}{\partial t} + \mathbf{v} \cdot \frac{\partial}{\partial \mathbf{r}} + \frac{1}{M_s}\mathbf{F}_s \cdot \frac{\partial}{\partial \mathbf{v}}, \tag{2}$$

where $\mathbf{F}_s$ is total mean-force acting on a particle belonging to the species s with mass M_s, which in the case of non-relativistic astrophysical plasmas can generally be due to both EM and gravitational fields. In the case of collisionless plasmas the collisional operator $C_s\left(f \mid f\right)$ vanishes and the species KDFs satisfy the differential Vlasov kinetic equation

$$L_s f_s(\mathbf{y}, t) = 0. \tag{3}$$

The fluid fields can then be obtained as velocity moments of the KDF of the form

$$\int d^3v \left[G(\mathbf{y}) f_s(\mathbf{y}, t)\right], \tag{4}$$

where $G(\mathbf{y})$ represents a suitable phase-space weight function. Depending on the particular expression for $G(\mathbf{y})$, different fluid fields can be introduced. In particular, these include the following relevant ones:

a) species number density ($G(\mathbf{y}) = 1$)

$$n_s\left(\mathbf{r}, t\right) \equiv \int d^3v f_s(\mathbf{y}, t); \tag{5}$$

b) species flow velocity ($G(\mathbf{y}) = \frac{1}{n_s}\mathbf{v}$)

$$\mathbf{V}_s\left(\mathbf{r}, t\right) \equiv \frac{1}{n_s} \int d^3v \left[\mathbf{v} f_s(\mathbf{y}, t)\right]; \tag{6}$$

c) species tensor pressure ($G(\mathbf{y}) = M_s\left(\mathbf{v} - \mathbf{V}_s\right)\left(\mathbf{v} - \mathbf{V}_s\right)$)

$$\underline{\underline{\Pi}}_s\left(\mathbf{r}, t\right) \equiv M_s \int d^3v \left[\left(\mathbf{v} - \mathbf{V}_s\right)\left(\mathbf{v} - \mathbf{V}_s\right) f_s(\mathbf{y}, t)\right]; \tag{7}$$

d) species isotropic scalar temperature ($G(\mathbf{y}) = \frac{M_s}{3n_s}\left(\mathbf{v} - \mathbf{V}_s\right)^2$)

$$T_s\left(\mathbf{r}, t\right) \equiv \frac{M_s}{n_s} \int d^3v \left[\frac{\left(\mathbf{v} - \mathbf{V}_s\right)^2}{3} f_s(\mathbf{y}, t)\right]; \tag{8}$$

e) species heat flux ($G(\mathbf{y}) = \frac{M_s}{3}\left(\mathbf{v} - \mathbf{V}_s\right)\left(\mathbf{v} - \mathbf{V}_s\right)^2$)

$$\mathbf{Q}_s\left(\mathbf{r}, t\right) \equiv \frac{M_s}{3} \int d^3v \left[\left(\mathbf{v} - \mathbf{V}_s\right)\left(\mathbf{v} - \mathbf{V}_s\right)^2 f_s(\mathbf{y}, t)\right]. \tag{9}$$

In the same way, the velocity moments of the KDFs also determine the sources of the

EM self-field $\left\{\mathbf{E}^{self}, \mathbf{B}^{self}\right\}$. These are identified with the plasma charge density $\rho(\mathbf{r}, t)$ and the current density $\mathbf{J}(\mathbf{r}, t)$:

$$\rho(\mathbf{r}, t) = \sum_s Z_e e n_s(\mathbf{r}, t), \tag{10}$$

$$\mathbf{J}(\mathbf{r}, t) = \sum_s Z_e e n_s(\mathbf{r}, t) \mathbf{V}_s(\mathbf{r}, t). \tag{11}$$

Similarly, the relevant species fluid equations can be obtained by means of taking velocity integrals of the species kinetic equation of the form

$$\int d^3v \left[G(\mathbf{y}) L_s f_s(\mathbf{y}, t)\right] = \int d^3v \left[G(\mathbf{y}) C_s(f \mid f)\right], \tag{12}$$

with $G(\mathbf{y})$ still representing a suitable phase-space weight function. For example, consider the particular set of weight functions $G_1(\mathbf{y}) \equiv \left\{1, M_s\mathbf{v}, \frac{M_s}{2}(\mathbf{v} - \mathbf{V}_s)^2\right\}$ (these are the same as those given in a), b) and d) above, apart from the normalization constants which are chosen here so as to recover the standard form of fluid equations given below). When the set $G_1(\mathbf{y})$ is used in Eq.(12) and summation over species is taken, the form of the Fokker-Planck-Landau collisional operator implies that the equations

$$\sum_s \int d^3v \left[G_1(\mathbf{y}) C_s(f \mid f)\right] = 0 \tag{13}$$

are identically satisfied. The phase-functions of the set $G_1(\mathbf{y})$ are usually referred to as collisional invariants of the Fokker-Planck-Landau collisional operator. In the case of collisionless plasmas the collisional operator vanishes identically. For the set $G(\mathbf{y}) \equiv G_1(\mathbf{y})$ the velocity-moments of the Vlasov equation for each plasma species s are given by:

a) species continuity equation, obtained by setting $G(\mathbf{y}) = 1$ in Eq.(12):

$$\frac{\partial}{\partial t} n_s + \nabla \cdot (n_s \mathbf{V}_s) = 0; \tag{14}$$

b) species momentum equation (Euler equation), obtained for $G(\mathbf{y}) = M_s\mathbf{v}$:

$$M_s n_s \frac{\partial}{\partial t} \mathbf{V}_s + M_s n_s (\mathbf{V}_s \cdot \nabla) \mathbf{V}_s + \nabla \cdot \underline{\underline{\Pi}}_s - n_s \mathbf{F}_s = 0; \tag{15}$$

c) species energy equation, obtained for $G(\mathbf{y}) = \frac{M_s}{2}(\mathbf{v} - \mathbf{V}_s)^2$:

$$\frac{\partial}{\partial t} p_s + \nabla \cdot (p_s \mathbf{V}_s) + \nabla \cdot \mathbf{Q}_s + \frac{2}{3} \nabla \mathbf{V}_s : \underline{\underline{\Pi}}_s = 0, \tag{16}$$

where $p_s = n_s T_s$ denotes the scalar isotropic pressure.

We stress that the system of fluid moment equations obtained from Eq.(12) is intrinsically not closed. To clarify the issue, it is instructive to consider, for example, the two fluid equations (14) and (15) written above. Eq.(14) is a scalar equation which can be solved for the species number density n_s, while Eq.(15) is a vector equation for the vector field $\mathbf{V}_s$. However, in these equations the pressure tensor remains undetermined and needs to be suitably prescribed in order to close the system. Similarly, assuming a scalar pressure, so that $\nabla \cdot \underline{\underline{\Pi}}_s = \nabla p_s$, if Eq.(16) is used to solve for T_s, the system of equations still depends on the unknown fluid field $\mathbf{Q}_s$, which needs to be suitably prescribed in terms of the corresponding closure condition. This represents an intrinsic characteristic feature of fluid equations, which can only be consistently dealt with by obtaining the closure conditions from kinetic theory. While fluid or MHD equations require a separate prescription of closure conditions (i.e., equations of state), the kinetic equations already form a closed system and require no external closure condition. As explained above, knowledge of the species KDF provides complete information about the physical properties of plasmas and permits a unique determination "a posteriori" of the relevant fluid fields that characterize a system, including particle number density, flow velocity and pressure tensor. Referring directly to the KDF makes possible the consistent inclusion in the corresponding fluid fields and fluid equations of specifically kinetic effects which cannot appear in "stand-alone" fluid theories.

The statistical fluid description, on the other hand, can also in principle be introduced directly by itself, as a separate theory, i.e., independent of the underlying kinetic theory. Such an approach is typically adopted for the treatments of fluids or magnetofluids, i.e. continuous media prescribed in terms of a suitable set of fluid fields. Typical fluid models are based on the assumption that the set of fluid fields is complete and uniquely defined by a suitable ensemble of closed fluid equations. These models are widespread for the treatment of classical fluids and cold (i.e., low-temperature) and dense plasmas.

However, these treatments become unsatisfactory for the description of hot and/or diluted plasmas. In fact, the kinetic description becomes essential for plasmas in which binary Coulomb interactions are important, since these occur at a microscopic level and involve single particles (see for example S. I. Braginskii, *Transport processes in a plasma*, Review of Plasma Physics, Vol.1, 1965). In particular, in the case of hot dilute plasmas which are the main focus of interest in this thesis, these reduce to mean-field interactions which can be described within the framework of the Vlasov-Maxwell description. When this occurs the plasma is said to be collisionless. The proper definition of this regime requires comparing the characteristic time and length scales of the system (see the discussion below). For a collisionless plasma the single-particle KDF can become highly non-isotropic in the phase-space, and then "stand-alone" fluid or magneto-hydrodynamics (MHD) approaches formulated independently of the underlying kinetic theory can only provide, at best, a partial (i.e., qualitative) description of the plasma phenomenology. This is because of a number of possible inconsistencies which may arise due to the peculiar behavior of the KDF. Firstly, the fluid fields, which are identified with velocity moments of the KDF, no longer generally form a complete set, and so the corresponding fluid equations are not closed, but give an infinite set of moment equations. Secondly, the "stand-alone" fluid or MHD descriptions do not generally include the correct constitutive fluid equations for the fluid fields. In fact, the proper determination of these requires taking into account the actual form of the KDF. A special mention should be made of the issue of validity of a possible equation of state for the fluid pressure tensor. This is usually invoked as a closure condition for the linear momentum density equation. In ideal-MHD, for example, the fluid pressure is assumed to be isotropic, so that anisotropic effects typical of plasma phenomenology are ignored. These issues cannot be solved consistently without invoking a kinetic description where all fluid fields are, in principle, consistently determined from the underlying self-consistent KDF. This means that both the equations of state and the constitutive equations for the fluid fields then follow uniquely from the microscopic dynamics.

Further notable aspects to be mentioned concern deeper consequences of microscopic phase-space particle dynamics. For non-relativistic plasmas, these include single-particle conservation laws as well as phase-space plasma collective phenomena, both of which are usually referred to as kinetic effects. Instead, for relativistic plasmas, electromagnetic radiation-reaction effects also need to be taken into account. All of these, as explained in detail in the various chapters of the thesis, affect the very structure of the constitutive equations for the fluid fields as well as the form of the fluid equations. In particular, kinetic effects are associated - for example - with the occurrence of temperature anisotropy, finite Larmor-radius and diamagnetic effects in magnetized plasmas.

The proper treatment of the kinetic effects indicated above requires the identification of the appropriate plasma regimes in which different particle species may satisfy distinctive asymptotic orderings. The identification of these regimes, in contrast to single-species descriptions characteristic of typical MHD approaches, such as the ideal-MHD model, is a necessary prerequisite for multi-species kinetic treatments.

These issues are naturally addressed, in the present thesis, within the kinetic description of collisionless astrophysical and laboratory plasmas. For a system consisting of ion and electron species ($s = i, e$), the requirement of having a collisionless plasma involves the determination of the following characteristic parameters for each species (for the definitions see following chapters):

1) The Larmor radius $r_{Ls}$and the Larmor gyration time τ_{Ls}.

2) The Debye length λ_D and the Langmuir time τ_{ps}.

3) The mean-free-path $\lambda_{mfp,s}$ and the collision time τ_{Cs}.

For each species, these parameters depend on the plasma number density and temperature and on the magnitude of the magnetic field. For any phenomena occurring on timescales Δt and lengthscales ΔL, satisfying

$$\tau_{ps}, \tau_{Ls} \ll \Delta t \ll \tau_{Cs}, \tag{17}$$

$$\lambda_D, r_{Ls} \ll \Delta L \ll \lambda_{mfp,s}, \tag{18}$$

the plasma can be considered as:

#1 *Collisionless:* due to the inequalities between Δt and τ_{Cs} and between ΔL and

$\lambda_{mfp,s}$. *Any* plasma is effectively collisionless for processes whose timescales and lengthscales are short enough and, under those circumstances, one needs to use kinetic theory. For a plasma which is sufficiently diffuse, this can include almost *all* relevant processes.

#2 Characterized by a *mean-field EM interaction:* due to the collisionless assumption, charged particles can be taken to interact with the others only via a continuum mean EM field.

#3 *Quasi-neutral:* due to the inequality between Δt and τ_{ps} and between ΔL and λ_D, the plasma can be taken as being quasi-neutral on the lengthscale ΔL.

The inequalities involving the Larmor scales concern the degree of magnetization of the plasma and warrant the validity of a gyrokinetic description for the single-particle dynamics. When conditions #1-#3 are satisfied, the medium is referred to as a quasi-neutral *Vlasov-Maxwell plasma* and kinetic theory should be used. This is the setting that is considered in this thesis, in which a kinetic theory is formulated for classical ideal laboratory and astrophysical plasmas in both the non-relativistic and relativistic regimes.

In astrophysics, many different scenarios can be distinguished in which plasmas arise. These include, for example, astrophysical plasmas in the interstellar medium, molecular clouds, proto-stars, solar-type stars, red giants, planetary nebulae, pulsar magnetospheres, extra-galactic jets, accretion discs round ordinary stars, accretion discs round compact objects emitting X-rays, accretion discs associated with gamma-ray bursters, etc. Each of these contexts is characterized by different physical conditions as well as possibly different physical processes occurring in the plasma. Therefore, one needs to make a specific formulation of kinetic theory for each astrophysical scenario. In this thesis, we focus on collisionless astrophysical plasmas arising in some accretion discs around compact objects and in relativistic jets. In the first case, a non-relativistic kinetic theory is developed, for collisionless plasmas in which radiation-emission processes are negligible as far as the single-particle dynamics is concerned. Instead, a relativistic kinetic theory is formulated for the description of jet plasmas, with the consistent inclusion of electromagnetic radiation-reaction phenomena characterizing, at microscopic level, the dynamics of single charges in the plasma. The classification of these plasmas as collisionless requires the evaluation of the specific parameters listed above and the corresponding inequalities (17) and (18). This is particularly relevant in the case of accretion discs around compact objects, in which species densities and temperatures span a very wide range of values. It must be stressed, however, that, depending on the magnitude of the characteristic scales Δt and ΔL for the particular phenomena of interest, the same plasma may need to be considered as collisional for some purposes, but collisionless for processes which happen sufficiently fast. An important example of collisionless accretion-disc plasmas to which the theory developed here can apply is represented by the hydrodynamic model known in the astrophysics literature as radiatively inefficient accretion flows (RIAFs, see Narayan et al. in *Theory* *of Black Hole Accretion Discs,* 1998). The present thesis aims at overcoming in several ways the limitations of the fluid treatment on which this type of model is based.

Chapter 1

Kinetic axisymmetric gravitational equilibria in collisionless accretion disc plasmas

1.1 Introduction

The investigation described here is concerned with dynamical processes in the sub-set of astrophysical accretion disc (AD) plasmas which can be considered as collisionless (e.g. RIAFs) and their relationship with the accretion process. The aim of the research developed here is to provide a consistent theoretical formulation of kinetic theory for these AD plasmas, which can then be used for investigating their equilibrium properties and dynamical evolution. Note that what is meant here by the term "equilibrium" is in general a stationary-flow solution, which can also include a radial accretion velocity. Apart from the intrinsic interest of this study for the equilibrium properties of these accretion discs, the conclusions reached may have important consequences for other applications and for stability analyses of the discs. In this Chapter we study the particular case where the AD plasma contains domains of locally-closed magnetic surfaces where there is in fact no local net accretion. The corresponding treatment for arbitrary magnetic field configurations will be dealt with in the next Chapter.

Astrophysical background

Accretion discs are observed in a wide range of astrophysical contexts, from the small-scale regions around proto-stars or stars in binary systems to the much larger scales associated with the cores of galaxies and Active Galactic Nuclei (AGN). Observations tell us that these systems contain matter accreting onto a central object, losing angular momentum and releasing gravitational binding energy. This can give rise to an extremely powerful source of energy generation, causing the matter to be in the plasma state and allowing the discs to be detected through their radiation emission (1, 2, 3). A particularly interesting class of accretion discs consists of those occurring around black holes in binary systems, which give rise to compact X-ray sources. For these, one has both a strong gravitational field and also presence of significant magnetic fields which are mainly self-generated by the plasma current densities. Despite the information available about these systems, mainly provided by observations collected over the past forty years and concerning their macroscopic physical and geometrical properties (structure, emission spectrum, etc.), no complete theoretical description of the physical processes involved in the generation and evolution of the magnetic fields is yet available. While it is widely thought that the magneto-rotational instability (MRI) (3) plays a leading role in generating an effective viscosity in these discs, more remains to be done in order to obtain a full understanding of the dynamics of disc plasmas and the relation of this with the accretion process. This requires identifying the microphysical phenomena involved in the generation of instabilities and/or turbulence which may represent a plausible source for the effective viscosity which, in turn, is then related to the accretion rates (1, 3). There is a lot of observational evidence which cannot yet be explained or fully understood within the framework of existing theoretical descriptions and many fundamental questions still remain to be answered (4, 5).

Motivations for a kinetic theory

Historically, most theoretical and numerical investigations of accretion discs have been made in the context of hydrodynamics (HD) or magneto-hydro-dynamics (MHD) (*fluid approaches*) (1, 2, 3, 6, 7, 8, 9, 10, 11). Treating the medium as a fluid allows one to capture the basic large-scale properties of the disc structure and evolution. An interesting development within this context has been the work of Coppi (12) and Coppi and Rousseau (13) who showed that stationary magnetic configurations in AD plasmas, for both low and high magnetic energy densities, can exhibit complex magnetic structures characterized locally by plasma rings with closed nested magnetic surfaces. However, even the most sophisticated fluid models are still not able to give a good explanation for all of the complexity of the phenomena arising in these systems. While fluid descriptions are often useful, it is well known that, at a fundamental level, a correct description of microscopic and macroscopic plasma dynamics should be formulated on the basis of kinetic theory (*kinetic approach*) (14, 15), and for that there is still, remarkably, no satisfactory theoretical formulation in the existing literature. Going to a kinetic approach can overcome the problem characteristic of fluid theories of uniquely defining consistent closure conditions (14, 15), and a kinetic formulation is required, instead of MHD, for correctly describing regimes in which the plasma is either collisionless or weakly collisional (16, 17, 18). In these situations, the distribution function describing the AD plasma will be different from a Maxwellian, which is instead characteristic of highly collisional plasmas for which fluid theories properly apply. An interesting example of collisionless plasmas, arising in the context of astrophysical accretion discs around black holes, is that of radiatively inefficient accretion flows (19, 20). Theoretical investigations of such systems have suggested that the accreting matter consists of a two-temperature plasma, with the proton temperature being much higher than the electron temperature (4, 5). This in turn implies that the timescale for energy exchange by Coulomb collisions between electrons and ions must be longer than the other characteristic timescales of the system, in particular the inflow time. In this case, a correct physical description of the phenomenology governing these objects can only be provided by a kinetic formulation. The kinetic formulation is anyway more convenient for the inclusion of some particular physical effects, including ones due to temperature anisotropies, and is essential for making a complete study of the kinetic instabilities which can play a key role in causing the accretion process (16, 17, 18). We note here that, although there are in principle several possible physical processes which may explain the appearance of temperature anisotropies (see for example (17, 18)), the main reason for their maintenance in a collisionless and non-turbulent plasma may simply be the lack of any efficient mechanism for temperature isotropization (see also discussion below and in the next Chapter).

Previous work

Only a few studies have so far addressed the problem of deriving a kinetic formulation of steady-state solutions for AD plasmas.

The paper by Bhaskaran and Krishan (21), based on theoretical results obtained by Mahajan (22, 23) for laboratory plasmas, is a first example going in this direction. These authors assumed an equilibrium distribution function expressed as an infinite power series in the ratio of the drift velocity to the thermal speed (considered as the small expansion parameter) such that the zero-order term coincides with a homogeneous Maxwellian distribution. Assuming a prescribed profile for the external magnetic field and ignoring the self-generated field, they looked for analytic solutions of the Vlasov-Maxwell system for the coefficients of the series (typically truncated after the first few orders). However, the assumptions made strongly limited the applicability of their model.

Another approach proposed recently by Cremaschini *et al.* (15), gives an exact solution for the equilibrium Kinetic Distribution Function (KDF) of strongly magnetized non-relativistic collisionless plasmas with isotropic temperature and purely toroidal flow velocity. The strategy adopted was similar to that developed by Catto *et al.* (24) for toroidal plasmas, suitably adapted to the context of accretion discs. The stationary KDF was expressed in terms of the first integrals of motion of the system showing, for example, that the standard Maxwellian KDF is an asymptotic stationary solution only in the limit of a strongly magnetized plasma, and that the spatial profiles of the fluid fields are fixed by specific kinetic constraints (15).

In recent years, the kinetic formalism has also been used for investigating stability of AD plasmas, particularly in the collisionless regime and focused on studying the role and importance of MRI (3, 18, 25, 26, 27). The main goal of these studies (16, 17, 18, 26) was to provide and test suitable kinetic closure conditions for asymptotically-reduced fluid equations (referred to as "kinetic MHD"), so as to allow the fluid stability analysis to include some of the relevant kinetic effects for collisionless plasmas (16, 18, 26). However, none of them systematically treated the issue of kinetic equilibrium, and the underlying unperturbed plasma was usually taken to be described by either a

Maxwellian or a bi-Maxwellian KDF.

Finally, increasing attention has been paid to the role of temperature anisotropy and the related kinetic instabilities. Some recent numerical studies (18, 27) have tried to include the effects of temperature anisotropy but, although it is clear that this can give rise to an entirely new class of phenomena, all of these estimates rely on fluid models in which kinetic effects are included in only an approximate way. A kinetic approach is needed rather than a fluid one, in order to give a clear and self-consistent picture, and this needs to be based on equilibrium solutions suitable for accretion discs.

1.2 Open problems and goals of the research

Many problems remain to be addressed and solved regarding the kinetic formulation of AD plasma dynamics. Among them, in this Chapter we focus on the following:

1. The construction of a kinetic theory for collisionless AD plasmas within the framework of the Vlasov-Maxwell description, and the investigation of their kinetic equilibrium properties.

2. The inclusion of finite Larmor-radius (FLR) effects in the MHD equations. For magnetized plasmas, this can be achieved by making a kinetic treatment and representing the KDF in terms of gyrokinetic variables (Bernstein and Catto (28, 29, 30, 31)). The gyrokinetic formalism provides a simplified description of the dynamics of charged particles in the presence of magnetic fields, thanks to the symmetry of the Larmor gyratory motion of the particles around the magnetic field lines. Therefore *kinetic* and *gyrokinetic theory* are both fundamental tools for treating FLR effects in a consistent way.

3. The determination of suitable kinetic closure conditions to be used in the fluid description of the discs. These should include the kinetic effects of the plasma dynamics in a consistent way.

4. Extension of the known solutions to more general contexts, with the inclusion of important effects such as temperature anisotropy.

5. Development of a kinetic theory for stability analysis of AD plasmas. As already mentioned, this could cast further light on the physical mechanism giving rise to the effective viscosity and the related accretion processes. This is particularly interesting for collisionless plasmas with temperature anisotropy, since only kinetic theory could be able to explain how instabilities can originate and grow to restore the isotropic properties of the plasma.

The reference publications for the material presented in this Chapter are Refs.(15, 32, 33, 34, 35).

In particular, we pose here the problem of constructing analytic solutions for *exact kinetic and gyrokinetic axi-symmetric gravitational equilibria* (see definition below) in accretion discs around compact objects. The solution presented is applicable to collisionless magnetized plasmas with temperature anisotropy and mainly toroidal flow velocity. The kinetic treatment of the gravitational equilibria necessarily requires that the KDF is itself a stationary solution of the relevant kinetic equations. Ignoring possible weakly-dissipative effects, we shall assume - in particular - that the KDF and the electromagnetic (EM) fields associated with the plasma obey the system of Vlasov-Maxwell equations. The only restriction on the form of the KDF, besides assuming its strict positivity and it being suitably smooth in the relevant phase-space, is due to the requirement that it must be a function only of the independent first integrals of the motion or the adiabatic invariants for the system.

1.3 Kinetic theory for accretion disc plasmas: basic assumptions

The meaning of asymptotic kinetic equilibria in the present study and the physical conditions under which they can be realized are first discussed.

An asymptotic kinetic equilibrium must be one obtained within the context of kinetic theory and must be described by the stationary Vlasov-Maxwell equations. This means that the generic plasma KDF, f_s, must be a solution of the stationary Vlasov equation, as will be the case if f_s is expressed in terms of exact first integrals of the motion or adiabatic invariants of the system, which in turn implies that f_s for each species must be an exact first integral of the motion or an adiabatic invariant. The stationarity condition means that the equilibrium KDF cannot depend explicitly on time, although in principle it could contain an implicit time dependence via its fluid moments (in which case the kinetic equilibrium does not correspond to a fluid equilibrium and there are non-stationary fluid fields).

In the following, the AD plasma is taken to be: a) *non-relativistic*, in the sense that it has non–relativistic species flow velocities, that the gravitational field can be treated within the classical Newtonian theory, and that the non-relativistic Vlasov kinetic equation is used as the dynamical equation for the KDF; b) *collisionless*, so that the mean free path of the plasma particles is much longer than the largest characteristic scale length of the plasma; c) *axi-symmetric*, so that the relevant dynamical variables characterizing the plasma (e.g., the fluid fields) are independent of the toroidal angle φ, when referred to a set of cylindrical coordinates (R, φ, z); d) acted on by both gravitational and EM fields.

Also, the situation is considered where the equilibrium magnetic field $\mathbf{B}$ admits, at least locally, a family of nested axi-symmetric closed toroidal magnetic surfaces $\{\psi(\mathbf{r})\} \equiv \{\psi(\mathbf{r}) = const.\}$, where ψ denotes the poloidal magnetic flux of $\mathbf{B}$ (see (12, 13) for a proof of the possible existence of such configurations in the context of astrophysical accretion discs; see also (14, 15) for further discussions in this regard and Fig.1.1 for a schematic view of such a configuration). In this situation, a set of magnetic coordinates $(\psi, \varphi, \vartheta)$ can be defined locally, where ϑ is a curvilinear angle-like coordinate on the magnetic surfaces $\psi(\mathbf{r}) = const$. Each relevant physical quantity $A(\mathbf{r})$ can then be expressed as a function of these magnetic coordinates, i.e. $A(\mathbf{r}) = A(\psi, \vartheta)$, where the φ dependence has been suppressed due to the axi-symmetry. It follows that it is always possible to write the following decomposition: $A = A^{\sim} + \langle A \rangle$, where the *oscillatory part* $A^{\sim} \equiv A - \langle A \rangle$ contains the ϑ-dependencies and $\langle A \rangle$ is the $\psi-$surface average of the function $A(\mathbf{r})$ defined on a flux surface $\psi(\mathbf{r}) = const.$ as $\langle A \rangle = \xi^{-1} \oint d\vartheta A(\mathbf{r}) / |\mathbf{B} \cdot \nabla\vartheta|$, with ξ denoting $\xi \equiv \oint d\vartheta / |\mathbf{B} \cdot \nabla\vartheta|$.

For definiteness, we shall consider here a plasma consisting of at least two species of charged particles: one species of ions and one of electrons.

We also introduce some convenient dimensionless parameters which will be used in constructing asymptotic orderings for the relevant quantities of the theory. The first one, which enters into the construction of the gyrokinetic theory, is defined as $\varepsilon_{M,s} \equiv \frac{r_{Ls}}{L} \ll 1$, where $s = i, e$ denotes the species index. Here $r_{Ls} = v_{\perp ths}/\Omega_{cs}$ is the species average Larmor radius, with $v_{\perp ths} = \{T_{\perp s}/M_s\}^{1/2}$ denoting the species thermal velocity perpendicular to the magnetic field and $\Omega_{cs} = Z_s eB/M_s c$ being the species Larmor frequency. Moreover, L is the characteristic length-scale of the spatial inhomogeneities of the EM field, defined as $L \sim L_B \sim L_E$, where L_B and L_E are the characteristic magnitudes of the gradients of the absolute values of the magnetic field $\mathbf{B}(\mathbf{x}, t)$ and the electric field $\mathbf{E}(\mathbf{x}, t)$, defined as $\frac{1}{L_B} \equiv \max\left\{\left|\frac{\partial}{\partial r_i} \ln B\right|, i = 1, 3\right\}$ and $\frac{1}{L_E} \equiv \max\left\{\left|\frac{\partial}{\partial r_i} \ln E\right|, i = 1, 3\right\}$, where the vector $\mathbf{x}$ denotes $\mathbf{x} = (R, z)$. Then, a unique parameter $\varepsilon_M \equiv \max\{\varepsilon_{M,s}, s = i, e\}$ is defined. For temperatures and magnetic fields typical of AD plasmas, we have $0 < \varepsilon_M \ll 1$.

The second parameter is the *inverse aspect ratio* defined as $\delta \equiv \frac{r_{\max}}{R_0}$, where R_0 is the radial distance from the vertical axis to the center of the nested magnetic surfaces and $r_{\max}$ is the average cross-sectional poloidal radius of the largest closed toroidal magnetic surface; see Fig.1.1 for a schematic view of the configuration geometry and the meaning of the notation introduced here. Then, we impose the requirement $0 < \delta \ll 1$, which is referred to as "small inverse aspect ratio ordering". The main motivation for introducing this ordering is that we are discussing only local solutions where this asymptotic condition holds; this property also follows from the results presented in (12, 13), and has aready been used in other previous work on the subject (14, 15). The requirement $\delta \ll 1$ is also needed in order to satisfy the constraint condition imposed by Ampere's law, as discussed in Sec. VII. We stress that the $\delta-$ordering here introduced is consistent with the assumption of nested and closed magnetic surfaces that are assumed to be localized in space.

Finally we introduce a parameter δ_{Ts} which measures the magnitude of the species temperature anisotropy and is defined as $\delta_{Ts} \equiv \frac{T_{\parallel s} - T_{\perp s}}{T_{\parallel s}}$, where $T_{\parallel s}$ and $T_{\perp s}$ denote the parallel and perpendicular temperatures, as measured with respect to the magnetic field direction.

Note that, in the following, primed quantities will denote dynamical variables defined at the guiding-center position.

The treatment of EM and gravitational fields

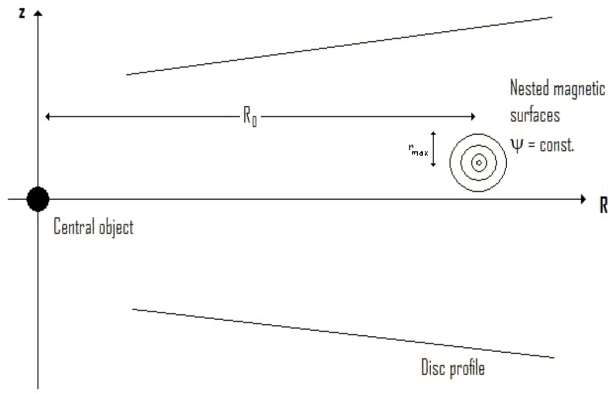

Figure 1.1: Schematic view of the configuration geometry.

We require the EM field to be slowly varying in time, i.e., to be of the form

$$\left[\mathbf{E}(\mathbf{x}, \varepsilon_M^k t), \mathbf{B}(\mathbf{x}, \varepsilon_M^k t)\right], \tag{1.1}$$

with $k \geq 1$ being a suitable integer. This time dependence is connected with either external sources or boundary conditions for the KDF. In particular, we shall assume that the magnetic field is of the form

$$\mathbf{B} \equiv \nabla \times \mathbf{A} = \mathbf{B}^{self}(\mathbf{x}, \varepsilon_M^k t) + \mathbf{B}^{ext}(\mathbf{x}, \varepsilon_M^k t), \tag{1.2}$$

where $\mathbf{B}^{self}$ and $\mathbf{B}^{ext}$ denote the self-generated magnetic field produced by the AD plasma and a finite external magnetic field produced by the central object (in the case of neutron stars or white dwarfs). We also impose the following relative ordering between the two components of the total magnetic field: $\frac{|\mathbf{B}^{ext}|}{|\mathbf{B}^{self}|} \sim O\left(\varepsilon_M^k\right)$, with $k \geq 1$. This means that the self-field is the dominant component: the magnetic field is primarily self-generated. However, for greater generality, we shall not prescribe any relative orderings between the various components of the total magnetic field, which are taken to be of the form

$$\mathbf{B}^{self} = I(\mathbf{x}, \varepsilon_M^k t)\nabla\varphi + \nabla\psi_p(\mathbf{x}, \varepsilon_M^k t) \times \nabla\varphi, \tag{1.3}$$
$$\mathbf{B}^{ext} = \nabla\psi_D(\mathbf{x}, \varepsilon_M^k t) \times \nabla\varphi. \tag{1.4}$$

In particular, here $\mathbf{B}_T \equiv I(\mathbf{x}, \varepsilon_M^k t)\nabla\varphi$ and $\mathbf{B}_P \equiv \nabla\psi_p(\mathbf{x}, \varepsilon_M^k t) \times \nabla\varphi$ are the toroidal and poloidal components of the self-field, while the external magnetic field $\mathbf{B}^{ext}$ is assumed to be purely poloidal and defined in terms of the vacuum potential $\psi_D(\mathbf{x}, \varepsilon_M^k t)$. As a consequence, the magnetic field can also be written in the equivalent form

$$\mathbf{B} = I(\mathbf{x}, \varepsilon_M^k t)\nabla\varphi + \nabla\psi(\mathbf{x}, \varepsilon_M^k t) \times \nabla\varphi, \tag{1.5}$$

where the function $\psi(\mathbf{x},\varepsilon_M^k t)$ is defined as $\psi(\mathbf{x},\varepsilon_M^k t) \equiv \psi_p(\mathbf{x},\varepsilon_M^k t) + \psi_D(\mathbf{x},\varepsilon_M^k t)$, with $k \geq 1$ and (ψ,φ,ϑ) defining a set of local magnetic coordinates (as implied by the equation $\mathbf{B}\cdot\nabla\psi = 0$ which is identically satisfied). Also, it is assumed that the charged particles of the plasma are subject to the action of *effective EM potentials* $\left\{\Phi_s^{eff}(\mathbf{x},\varepsilon_M^k t), \mathbf{A}(\mathbf{x},\varepsilon_M^k t)\right\}$, where $\mathbf{A}(\mathbf{x},\varepsilon_M^k t)$ is the vector potential corresponding to the magnetic field of Eq.(1.5), while $\Phi_s^{eff}(\mathbf{x},\varepsilon_M^k t)$ is given by

$$\Phi_s^{eff}(\mathbf{x},\varepsilon_M^k t) = \Phi(\mathbf{x},\varepsilon_M^k t) + \frac{M_s}{Z_s e}\Phi_G(\mathbf{x},\varepsilon_M^k t), \tag{1.6}$$

with $\Phi_s^{eff}(\mathbf{x},\varepsilon_M^k t)$, $\Phi(\mathbf{x},\varepsilon_M^k t)$ and $\Phi_G(\mathbf{x},\varepsilon_M^k t)$ denoting the *effective* electrostatic potential and the electrostatic and generalized gravitational potentials (the latter, in principle, being produced both by the central object and the accretion disc). The effective electric field $\mathbf{E}_s^{eff}$ can then be defined as

$$\mathbf{E}_s^{eff} \equiv -\nabla\Phi_s^{eff} - \frac{1}{c}\frac{\partial\mathbf{A}}{\partial t}. \tag{1.7}$$

1.4 First integrals of motion and guiding-center adiabatic invariants

In the present formulation, assuming axi-symmetry and stationary EM and gravitational fields, the exact first integrals of motion can be immediately recovered from the symmetry properties of the single charged particle Lagrangian function L. In particular, these are the total particle energy

$$E_s = \frac{M_s}{2}v^2 + Z_s e\Phi_s^{eff}(\mathbf{r}), \tag{1.8}$$

and the canonical momentum $p_{\varphi s}$ (conjugate to the ignorable toroidal angle φ)

$$p_{\varphi s} = M_s R\mathbf{v}\cdot\mathbf{e}_\varphi + \frac{Z_s e}{c}\psi \equiv \frac{Z_s e}{c}\psi_{*s}. \tag{1.9}$$

Gyrokinetic theory allows one to derive the adiabatic invariants of the system (28, 29); by construction, these are quantities conserved only in an asymptotic sense, i.e., only to a prescribed order of accuracy. As is well known, gyrokinetic theory is a basic prerequisite for the investigation both of kinetic instabilities (see for example (36, 37, 38)) and of equilibrium flows occurring in magnetized plasmas (24, 39, 40, 41, 42). For astrophysical plasmas close to compact objects, this generally involves the treatment of strong gravitational fields which needs to be based on a covariant formulation (see (43, 44, 45, 46)). However, for non-relativistic plasmas (in the sense already discussed), the appropriate formulation can also be directly recovered via a suitable reformulation of the standard (non-relativistic) theory for magnetically confined laboratory plasmas (see Refs.(29, 30, 31, 47, 48, 49, 50, 51, 52, 53) and also the next Chapter). In connection with this, consider again the Lagrangian function L of charged particle dynamics. By performing a gyrokinetic transformation of L, accurate to the prescribed order in ε_M, it follows that - by construction - the transformed Lagrangian L' becomes independent of the guiding-center gyrophase angle ϕ'. Therefore, by construction, the canonical momentum $p'_{\phi' s} = \partial L'/\partial \cdot \phi'$, as well as the related magnetic moment defined as $m'_s \equiv \frac{Z_s e}{M_s c} p'_{\phi' s}$, are adiabatic invariants. As shown by Kruskal (1962 (54)) it is always possible to determine L' so that m'_s is an adiabatic invariant of arbitrary order in ε_M, in the sense that $\frac{1}{\Omega'_{cs}} \frac{d}{dt} \ln m'_s = 0 + O(\varepsilon_M^{n+1})$, where $\Omega'_{cs} = Z_s e B'/M_s c$ denotes the Larmor frequency evaluated at the guiding-center and the integer n depends on the approximation used in the perturbation theory to evaluate m'_s. In addition, the guiding-center invariants corresponding to E_s and ψ_{*s} (denoted as E'_s and $p'_{\varphi s}$ respectively) can also be given in terms of L'. These are also, by definition, manifestly independent of ϕ'.,

This basic property of the magnetic moment m'_s is essential in the subsequent developments. Indeed, we shall prove that it allows the effects of temperature anisotropy to be included in the asymptotic stationary solution.

Let us now define the concept of *gyrokinetic* and *equilibrium* KDFs.

Def. - Gyrokinetic KDF (GK KDF)

A generic KDF $f_s(\mathbf{r}, \mathbf{v}, t)$ *will be referred to as gyrokinetic if it is independent of the gyrophase angle* ϕ' *(evaluated the guiding-center position) when its state* $\mathbf{x} = (\mathbf{r}, \mathbf{v})$ *is espressed as a function of an arbitrary gyrokinetic state* $\mathbf{z}' = (\mathbf{y}', \phi')$.

Def. - Equilibrium KDF

A generic KDF $f_s(\mathbf{r}, \mathbf{v}, t)$ *will be referred to as an equilibrium KDF if it identically satisfies the Vlavov equation* $\frac{d}{dt} f_s(\mathbf{r}, \mathbf{v}, t) = 0$ *and if* f_s *is also independent of time, namely* $f_s = f_s(\mathbf{r}, \mathbf{v})$. *More generally,* $f_s(\mathbf{r}, \mathbf{v}, t)$ *will be referred to as an asymptotic-equilibrium KDF if, neglecting corrections of order* $O\left(\varepsilon_M^{n+1}\right)$, $\frac{d}{dt} f_s(\mathbf{r}, \mathbf{v}, t) = 0$ *and to the same order* f_s *is independent of* t.

Let us first provide an example of a *GK equilibrium KDF*. This can be obtained by assuming that f_s depends only on the exact invariants, namely that it is of the form $f_s \equiv f_{*s}(E_s, \psi_{*s})$, with f_{*s} suitably prescribed and strictly positive. On the other hand, an *asymptotic GK equilibrium KDF* is manifestly of the form $f_s \equiv \widehat{f_{*s}}(E_s, \psi_{*s}, m'_s)$ [again to be assumed as strictly positive]. In fact, in this case, by construction, the KDF is an adiabatic invariant of prescribed order n, such that

$$\frac{1}{\Omega'_{cs}} \frac{d}{dt} \ln \widehat{f_{*s}} = 0 + O\left(\varepsilon_M^{n+1}\right), \tag{1.10}$$

(*asymptoptic Vlasov equation).* In particular, the order n (with $n \geq 0$) can in principle be selected at will. We stress, however, that since gyrokinetic theory is intrinsically asymptotic any GK equilibrium KDF depending on the magnetic moment m'_s is necessarily asymptotic in the sense of the previous two definitions.

Regarding the notations used in the following, we remark that, unless differently specified: 1) the symbol "$\wedge$" denotes physical quantities which refer to the treatment of anisotropic temperatures; 2) the symbol "$*$" is used to denote variables which depend on the canonical momentum ψ_{*s}.

1.5 Asymptotic equilibria with non-isotropic temperature

In this section an equilibrium solution for the KDF describing AD plasmas for which the temperature is anisotropic is derived.

Let us first assume that the AD plasma is characterized by a mainly toroidal flow velocity, where the toroidal component is expressed in terms of the angular frequency by $\mathbf{V}_s(R,z)\cdot\mathbf{e}_\varphi \equiv R\Omega_s(R,z)$. In the following, we will also require that the plasma is locally characterized by a family of nested magnetic surfaces which close inside the plasma in such a way that any $\psi = const.$ surface is a closed one. We also assume that the system can be satisfactorily described by a closed set of fluid equations in terms of four moments of the KDF, giving the number density, the flow velocity and the parallel and perpendicular temperatures.

The presence of a temperature anisotropy means that the plasma KDF cannot be a Maxwellian. As already mentioned, it remains in principle completely unspecified, with just the constraint that it must be a function only of the first integrals of motion or the adiabatic invariants of the system. Any non-negative KDF depending on the constants of motion and the adiabatic invariants is therefore an acceptable solution. This freedom in choosing a stationary solution is a well-known property of the Vlasov equation. Here it is shown that, in these circumstances, it is still possible to construct a satisfactory asymptotic GK equilibrium KDF which is an adiabatic invariant expressed in terms of the two first integrals of motion (1.8), (1.9) and the guiding-center magnetic moment m'_s. In particular, the form of the stationary KDF which we are going to introduce is characterized by the following properties: 1) it is analytically tractable; 2) it affords an explicit determination of the relevant kinetic constraints to be imposed on the fluid fields (see the discussion after Eq.(1.14)); 3) it represents a possible kinetic model which is consistent with fluid descriptions of collisionless plasmas characterized by temperature anisotropy; 4) it is suitable for comparisons with previous literature, in which astrophysical plasmas have been treated by means of a Maxwellian or a bi-Maxwellian KDF (see for example (16, 18, 26)). Then, following (15, 24), a convenient solution is given by

$$\widehat{f_{*s}} = \frac{\widehat{\beta_{*s}}}{(2\pi/M_s)^{3/2}\left(T_{\parallel *s}\right)^{1/2}} \exp\left\{-\frac{H_{*s}}{T_{\parallel *s}} - m'_s\widehat{\alpha_{*s}}\right\} \tag{1.11}$$

(*Generalized bi-Maxwellian KDF*), where

$$\widehat{\beta_{*s}} \equiv \frac{\eta_s}{\widehat{T}_{\perp s}}, \tag{1.12}$$

$$\widehat{\alpha_{*s}} \equiv \frac{B'}{\widehat{\Delta_{T_s}}}, \tag{1.13}$$

$$H_{*s} \equiv E_s - \frac{Z_s e}{c}\psi_{*s}\Omega_{*s}, \tag{1.14}$$

while E_s is given by Eq.(1.8), ψ_{*s} is given by Eq.(1.9) and $\frac{1}{\widehat{\Delta_{T_s}}} \equiv \frac{1}{\widehat{T}_{\perp s}} - \frac{1}{T_{\parallel *s}}$. In order for the solution (1.11) to be a function of the integrals of motion and the adiabatic invariants, the functions $\widehat{\beta_{*s}}$, $\widehat{\alpha_{*s}}$, $T_{\parallel *s}$ and Ω_{*s} must depend on the constants of motion by themselves. In general this would require a functional dependence on both the total particle energy and the canonical momentum. In this chapter, for simplicity, the case is considered in which only a dependence on ψ_{*s} is retained (15, 24). Namely, $\widehat{f_{*s}}$ depends, by assumption, on the flux functions $\left\{\widehat{\beta_{*s}}, T_{\parallel *s}, \widehat{\alpha_{*s}}, \Omega_{*s}\right\}$:

$$\widehat{\beta_{*s}} = \widehat{\beta_{*s}}\left(\psi_{*s}\right), \tag{1.15}$$
$$T_{\parallel *s} = T_{\parallel *s}\left(\psi_{*s}\right), \tag{1.16}$$
$$\widehat{\alpha_{*s}} = \widehat{\alpha_{*s}}\left(\psi_{*s}\right), \tag{1.17}$$
$$\Omega_{*s} = \Omega_{*s}(\psi_{*s}), \tag{1.18}$$

which in the following will be referred to as *kinetic constraints*. From these considerations it is clear that the KDF $\widehat{f_{*s}}$ is itself an adiabatic invariant, and is therefore an asymptotic solution of the stationary Vlasov equation, whose order of accuracy is uniquely determined by the magnetic moment, as already anticipated.

From definition (1.14), it follows immediately that an equivalent representation for $\widehat{f_{*s}}$ is given by:

$$\widehat{f_{*s}} = \frac{\widehat{\beta_{*s}} \exp\left[\frac{X_{*s}}{T_{\parallel *s}}\right]}{\left(2\pi/M_s\right)^{3/2}\left(T_{\parallel *s}\right)^{1/2}} \exp\left\{-\frac{M_s\left(\mathbf{v}-\mathbf{V}_{*s}\right)^2}{2T_{\parallel *s}} - m_s'\widehat{\alpha_{*s}}\right\}, \tag{1.19}$$

where $\mathbf{V}_{*s} = \mathbf{e}_\varphi R\Omega_{*s}(\psi_{*s})$ and

$$X_{*s} \equiv \left(M_s\frac{\left|\mathbf{V}_{*s}\right|^2}{2} + \frac{Z_s e}{c}\psi\Omega_{*s} - Z_s e\Phi_s^{eff}\right). \tag{1.20}$$

The same kinetic constraints (1.15)-(1.18) also apply to the solution (1.19). Note that the functions $\widehat{\beta_{*s}} \exp\left[\frac{X_{*s}}{T_{\parallel *s}}\right]$, $\mathbf{V}_{*s}$ and $T_{\parallel *s}$ cannot be regarded as *fluid fields*, since they have a dependence on the particle velocity via the canonical momentum ψ_{*s}. On the other hand, fluid fields must be computed as integral moments of the distribution function over the particle velocity $\mathbf{v}$.

Perturbative expansion

Next we show that a convenient asymptotic expansion for the adiabatic invariant $\widehat{f_{*s}}$ can be properly obtained in the following suitable limit. Consider, in fact, the quantity ε defined as $\varepsilon \equiv \max\{\varepsilon_s, s = i, e\}$, with $\varepsilon_s \equiv \left|\frac{L_{\varphi s}}{p_{\varphi s}-L_{\varphi s}}\right| = \left|\frac{M_s R v_\varphi}{\frac{Z_s e}{c}\psi}\right|$, where we have used the definition (1.9) with $v_\varphi \equiv \mathbf{v}\cdot\mathbf{e}_\varphi$, and where $L_{\varphi s}$ denotes the species particle angular momentum. We can give an average upper limit estimate for the magnitude of ε_s in terms of the species thermal velocity and the inverse aspect ratio previously defined. To do this, we first set $\psi \sim B_P r^2$, which is appropriate for the domain of closed nested magnetic surfaces. Recall that here r is the average poloidal radius of a generic nested magnetic surface. In this evaluation, the species thermal velocities v_{ths} and the toroidal flow velocities $R\Omega_s$ are considered to be of the same order with respect to the ε-expansion, i.e. $v_{ths}/R\Omega_s \sim O\left(\varepsilon^0\right)$ (referred to as *sonic flow*). Therefore, assuming $v_{\varphi s} \sim v_{ths}$ it follows immediately that $\varepsilon_s \sim \frac{r_{Ls}}{L_C}$, where r_{Ls} is the species Larmor radius and $L_C \equiv r\delta$, with δ the inverse aspect ratio. We shall say that the AD plasma is *strongly magnetized* whenever $0 < \varepsilon \ll 1$. This condition is realized if $r \geq r_{\min}$, where $r_{\min} = \max\left\{\frac{r_{Ls}}{\varepsilon_s \delta}, s = i, e\right\}$ is the minimum average poloidal radius of the toroidal nested magnetic surfaces for which $\varepsilon \ll 1$ is satisfied. In this case ε can be taken as a small parameter for making a Taylor expansion of the KDF and its related quantities, by setting $\psi_{*s} \simeq \psi + O\left(\varepsilon^k\right)$, $k \geq 1$. From the above discussion, it is clear that this asymptotic expansion is valid for r within an interval $r_{\min} \leq r \leq r_{\max}$, where the lower bound is fixed by the condition of having a strongly magnetized plasma, while the upper bound is given by the geometric properties of the system and the small inverse aspect ratio ordering. For the purpose of this study, in performing the asymptotic expansion we retain the leading-order expression for the guiding-center magnetic moment $m_s' \simeq \mu_s' = \frac{M_s w'^2}{2B'}$ (54). Then, it is straightforward to prove that for strongly magnetized plasmas, the following relation holds to first order in ε (i.e., retaining only linear terms in the expansion): $\widehat{f_{*s}} = \widehat{f_s}\left[1 + h_{Ds}\right] + O\left(\varepsilon^n\right)$, $n \geq 2$. Here, the zero order distribution $\widehat{f_s}$ is expressed as

$$\widehat{f_s} = \frac{n_s}{\left(2\pi/M_s\right)^{3/2}\left(T_{\parallel s}\right)^{1/2} T_{\perp s}} \exp\left\{-\frac{M_s\left(\mathbf{v}-\mathbf{V}_s\right)^2}{2T_{\parallel s}} - \frac{M_s w'^2}{2\Delta_{T_s}}\right\}, \tag{1.21}$$

which we will here call the *bi-Maxwellian KDF*, where $\frac{1}{\Delta_{T_s}} \equiv \frac{1}{T_{\perp s}} - \frac{1}{T_{\parallel s}}$, the number density $n_s = \eta_s \exp\left[\frac{X_s}{T_{\parallel s}}\right]$ and

$$X_s \equiv \left(M_s \frac{R^2\Omega_s^2}{2} + \frac{Z_s e}{c}\psi\Omega_s(\psi) - Z_s e \Phi_s^{eff}\right), \tag{1.22}$$

with η_s denoting the *pseudo-density*. Then, $\mathbf{V}_s = \mathbf{e}_\varphi R\Omega_s$ and the following kinetic constraints are implied from (1.15)-(1.18): $\beta_s = \beta_s(\psi) = \frac{\eta_s}{T_{\perp s}}$, $T_{\parallel s} = T_{\parallel s}(\psi)$, $\widehat{\alpha_s} = \widehat{\alpha_s}(\psi) = \frac{B'}{\Delta_{T_s}}$, $\Omega_s = \Omega_s(\psi)$. As can be seen, the functional form of the leading order number density, the flow velocity and the temperatures carried by the bi-Maxwellian KDF is naturally determined. In particular, note that the flow velocity is species-dependent, while the related angular frequency Ω_s must necessarily be constant on each nested toroidal magnetic surface $\{\psi(\mathbf{r}) = const.\}$. Finally, the quantity h_{Ds} represents the *diamagnetic part* of the KDF $\widehat{f_{*s}}$, given by

$$h_{Ds} = \left\{\frac{cM_sR}{Z_s e} Y_1 + \frac{M_s R}{T_{\parallel s}} Y_2\right\}\left(\mathbf{v}\cdot\mathbf{e}_\varphi\right), \tag{1.23}$$

with

$$Y_1 \equiv \left[A_{1s} + A_{2s}\left(\frac{H_s}{T_{\parallel s}} - \frac{1}{2}\right) - \mu_s' \widehat{A_{4s}}\right], \tag{1.24}$$

$$Y_2 \equiv \Omega_s(\psi)\left[1 + \psi A_{3s}\right], \tag{1.25}$$

and

$$H_s \equiv E_s - \frac{Z_s e}{c}\psi_s \Omega_s(\psi_s), \tag{1.26}$$

where we have introduced the following definitions: $A_{1s} \equiv \frac{\partial \ln \beta_s}{\partial \psi}$, $A_{2s} \equiv \frac{\partial \ln T_{\parallel s}}{\partial \psi}$, $A_{3s} \equiv \frac{\partial \ln \Omega_s(\psi)}{\partial \psi}$, $\widehat{A_{4s}} \equiv \frac{\partial \widehat{\alpha_s}}{\partial \psi}$. We remark here that: 1) in the ε-expansion of (1.11), performed around (1.21), no magnetic or electric field scale lengths enter, as can be seen from Eq.(1.23); 2) we also implicitly assume the validity of the ordering $\frac{\varepsilon_M}{\varepsilon} \ll 1$, which will be discussed below (see next section). For this reason, corrections of $O\left(\varepsilon_M^k\right)$, with $k \geq 1$, to (1.23) have been neglected; 3) in this $\varepsilon-$expansion we have also assumed that the scale-length L is of the same order (with respect to ε) as the characteristic scale-lengths associated with the species pseudo-densities η_s, the temperatures $T_{\parallel s}$ and $T_{\perp s}$, and the toroidal rotational frequencies Ω_s.

To conclude this section we point out that the very existence of the present asymptotic kinetic equilibrium solution and the realizability of the kinetic constraints implied by it, must be checked for consistency also with the constraints imposed by the Maxwell equations (see discussion below).

1.6 Moments of the KDF

It is well known that, given a distribution function, it is always possible to compute the fluid moments associated with it, which are defined through integrals of the distribution over the velocity space. Although an exact calculation of the fluid moments could be carried out (e.g., numerically) for prescribed kinetic closures, in this section we want to take advantage of the asymptotic expansion of the KDF in the limit of strongly magnetized plasmas to evaluate them analytically, thanks to the properties of the bi-Maxwellian KDF. In the following, we provide approximate expressions for the number density and the flow velocity, which allow one to write the Poisson and Ampere equations for the EM fields in a closed form, and for the non-isotropic species pressure tensor. Since these fluid fields are then known (in terms of suitable kinetic flux functions and with a prescribed accuracy), the closure problem characteristic of the fluid theories is then naturally solved as well.

The main feature of this calculation is that the number density and flow velocity are computed by performing a transformation of all of the guiding-center quantities appearing in the asymptotic equilibrium KDF to the actual particle position, to leading order in ε_M (according to the order of accuracy of the adiabatic invariant used), and they are then determined up to first order in ε, in agreement with the order of expansion previously set for the KDF. Terms of higher order, i.e. $O\left(\varepsilon_M^n\right)$, with $n \geq 1$, and $O\left(\varepsilon^k\right)$, with $k \geq 2$, as well as mixed terms of order $O\left([\varepsilon\varepsilon_M]^n\right)$ with $n \geq 1$, are therefore neglected in the present calculation. This approximation clearly holds if $\frac{\varepsilon_M}{\varepsilon} \ll 1$, which is consistent with the present assumptions. In fact, from the definitions given for these two small dimensionless parameters it follows that

$$\frac{\varepsilon_M}{\varepsilon} \sim O\left(\delta\right) \ll 1. \tag{1.27}$$

To first order in ε, the total number density n_s^{tot} is given by

$$n_s^{tot} \equiv \int d\mathbf{v}\widehat{f_{*s}} \simeq n_s \left[1 + \Delta_{n_s}\right]. \tag{1.28}$$

Note here that the resulting number density has two distinct contributions: n_s is the zero order term given in the previous section, while Δ_{n_s} represents the term of $O\left(\varepsilon\right)$ which carries all of the corrections due to the asymptotic expansion of the KDF for strongly magnetized plasmas. In particular, the term Δ_{n_s} is given by

$$\begin{aligned} \Delta_{n_s} \equiv \; & V_s\left[\gamma_1 + \gamma_3\left(\frac{T_{\parallel s}}{M_s} + \frac{4T_{\perp s}}{M_s} + V_s^2\right)\right] + \\ & + \frac{2\gamma_3 I^2}{B^2}\frac{\left(T_{\parallel s} - T_{\perp s}\right)V_s}{R^2 M_s} - \frac{\gamma_2}{B}V_s T_{\perp s}, \end{aligned} \tag{1.29}$$

where $V_s = R\Omega_s(\psi)$ and

$$\gamma_1 \equiv \left\{\frac{cM_s R}{Z_s e}K + \frac{M_s V_s}{T_{\parallel s}}\left[1 + \psi A_{3s}\right]\right\}, \tag{1.30}$$

$$\gamma_2 \equiv \left\{\frac{cM_s R}{Z_s e}A_{4s}\right\}, \tag{1.31}$$

$$\gamma_3 \equiv \left\{\frac{cM_s^2 R}{Z_s e}\frac{A_{2s}}{2T_{\parallel s}}\right\}, \tag{1.32}$$

in which

$$K \equiv \left[A_{1s} + A_{2s}\left(\frac{Z_s e\Phi_s^{eff} - \frac{Z_s e}{c}\psi\Omega_s(\psi)}{T_{\parallel s}} - \frac{1}{2}\right)\right], \tag{1.33}$$

and $A_{4s} \equiv \frac{\partial \alpha_s}{\partial \psi}$, with $\alpha_s\left(\psi\right) \equiv \frac{B}{\Delta_{T_s}}$. Note that here $\alpha_s\left(\psi\right)$ differs from $\widehat{\alpha_s}\left(\psi\right)$ because of the guiding-center transformation of the magnetic field B.

A similar integral can be performed also to compute the total flow velocity $\mathbf{V}_s^{tot}$. This has the form

$$n_s^{tot}\mathbf{V}_s^{tot} \equiv \int d\mathbf{v}\mathbf{v}\widehat{f_{*s}} \simeq n_s\left[\mathbf{V}_s + \Delta\mathbf{U}_s\right], \tag{1.34}$$

where by definition $\mathbf{V}_s = \Omega_s(\psi)R\mathbf{e}_\varphi$ and $\Delta\mathbf{U}_s$ represents the *self-consistent FLR ve-*

locity corrections given by:

$$\Delta \mathbf{U}_s \equiv \Delta_{\varphi s} \mathbf{e}_\varphi + \frac{\Delta_{3s}}{B} \nabla \psi \times \nabla \varphi, \tag{1.35}$$

where

$$\Delta_{\varphi s} \equiv \Delta_{n_s} \Omega_s R + \Delta_{2s} + \Delta_{3s} \frac{I}{RB}. \tag{1.36}$$

Note that in Eq.(1.35) the terms proportional to Δ_{n_s}, Δ_{2s} and Δ_{3s} come from the asymptotic expansion of the KDF for strongly magnetized plasmas and are of $O(\varepsilon)$ with respect to the toroidal velocity $\Omega_s R$. Here Δ_{n_s} is as given in Eq.(1.29), while Δ_{2s} and Δ_{3s} are given by

$$\Delta_{2s} \equiv \frac{T_{\perp s}}{M_s} \left(\gamma_1 + 3\gamma_3 V_s^2\right) - \frac{\gamma_2 2 T_{\perp s}^2}{B M_s} + \frac{\gamma_3 T_{\perp s}}{M_s^2} \left(T_{\parallel s} + 4 T_{\perp s}\right), \tag{1.37}$$

$$\begin{aligned} \Delta_{3s} \equiv\ & \frac{I \gamma_2 T_{\perp s}}{R B^2 M_s} \left(2 T_{\perp s} - T_{\parallel s}\right) + \frac{I \left(T_{\parallel s} - T_{\perp s}\right)}{R B M_s} \left(\gamma_1 + 3\gamma_3 V_s^2\right) + \\ & + \frac{I \gamma_3}{R B M_s^2} \left(3 T_{\parallel s}^2 - 4 T_{\perp s}^2 + T_{\parallel s} T_{\perp s}\right). \end{aligned} \tag{1.38}$$

The first-order term $\Delta \mathbf{U}_s$ provides corrections to the zero-order toroidal flow velocity with components in all of the three space directions and so we can conclude that, although the dominant fluid velocity is mainly toroidal, there is also a poloidal component of order ε, associated with the term $\frac{\Delta_{3s}}{B} \nabla \psi \times \nabla \varphi$. However, this is not necessarily an accretion velocity, especially under the hypothesis of closed nested magnetic surfaces which define a local domain in which the disc plasma is confined. Moreover, note that the ratio between the toroidal and poloidal velocities depends also on δ_{Ts}, in the sense that $\frac{\left|\frac{\Delta_{3s}}{B} \nabla \psi \times \nabla \varphi\right|}{|\mathbf{V}_s|} \sim O(\varepsilon) O(\delta_{Ts})$. The magnitude of the temperature anisotropy can therefore be relevant in further decreasing the poloidal velocity in comparison with the toroidal one, which on the contrary is not affected by δ_{Ts}. However, the real importance of this result in connection with the astrophysics of collisionless AD plasmas is, instead, the fact that this poloidal velocity is a primary source for a poloidal current density which in turn can generate a finite toroidal magnetic field (see the section on the Maxwell equations). This means that, even without any net accretion of disc material (which would require at least a redistribution of the angular momentum), the kinetic equilibrium solution provides a mechanism for the generation of a toroidal magnetic field, with serious implications for the stability analysis of these equilibria. The physical mechanism responsible for this poloidal drift is purely kinetic and is essentially due to the conservation of the canonical toroidal momentum and the FLR effects associated with the temperature anisotropy. These issues will be discussed in details in the following Sections.

1.7 The case of isotropic temperature

In this section, the case of isotropic temperature for the equilibrium distribution $\widehat{f_{*s}}$ is considered. When the condition $T_{\|s} = T_{\perp s} \equiv T_s$ is satisfied, the stationary KDF reduces to f_{*s}, where

$$f_{*s} = \frac{\eta_{*s}}{\pi^{3/2}\left(2T_{*s}/M_s\right)^{3/2}} \exp\left\{-\frac{H_{*s}}{T_{*s}}\right\} \tag{1.39}$$

is referred to as the *Generalized Maxwellian Distribution with isotropic temperature* (15). Here, H_{*s} retains its definition (1.14), while the kinetic constraints are expressed for the quantities $\widehat{n}_{*s}$ and T_{*s}, whose functional dependence is $\eta_{*s} = \eta_s(\psi_{*s})$ and $T_{*s} = T_s(\psi_{*s})$. By construction, this distribution function is expressed only in terms of the first integrals of motion of the system and is therefore an exact kinetic equilibrium solution. Performing an asymptotic expansion in the limit of strong magnetic field, as done before for $\widehat{f_{*s}}$, gives the following result: $f_{*s} = f_{Ms}\left[1 + h_{Ds}\right] + O(\varepsilon^n)$, $n \geq 2$, where

$$f_{Ms} = \frac{n_s}{\pi^{3/2}\left(2T_s/M_s\right)^{3/2}} \exp\left\{-\frac{M_s\left(\mathbf{v} - \mathbf{V}_s\right)^2}{2T_s}\right\} \tag{1.40}$$

is the zero-order term of the series, which coincides with a drifted Maxwellian KDF with $T_s = T_s(\psi)$, $\mathbf{V}_s = \Omega_s(\psi) R \mathbf{e}_\varphi$ and $n_s = \eta_s(\psi) \exp\left[\frac{X_s^{\sim}}{T_s}\right]$. In this case, the function h_{Ds} is given by

$$h_{Ds} = \left\{\frac{cM_sR}{Z_se} Y_1 + \frac{M_sR}{T_s} Y_2\right\}\left(\mathbf{v} \cdot \mathbf{e}_\varphi\right), \tag{1.41}$$

with $Y_1 \equiv \left[A_{1s} + A_{2s}\left(\frac{H_s}{T_s} - \frac{3}{2}\right)\right]$ and $Y_2 \equiv \Omega_s(\psi)\left[1 + \psi A_{3s}\right]$, where $A_{1s} \equiv \frac{\partial \ln \eta_s}{\partial \psi}$, $A_{2s} \equiv \frac{\partial \ln T_s}{\partial \psi}$, $A_{3s} \equiv \frac{\partial \ln \Omega_s(\psi)}{\partial \psi}$. Finally, as shown in (15), the angular frequency is given to leading order by $\Omega_s(\psi) = \frac{\partial \langle \chi \rangle}{\partial \psi}$, where $\chi \equiv c\Phi_s^{eff} + \frac{cT_s}{Z_se} \ln n_s$.

For reference, for isotropic temperature the number density becomes (with the same accuracy given above) $n_s^{tot} \simeq n_s\left[1 + \Delta_{n_s}\right]$, with n_s entering Eq.(1.40) and the first-order correction given by

$$\Delta_{n_s} \equiv V_s\left[\gamma_1 + \gamma_3\left(\frac{5T_s}{M_s} + V_s^2\right)\right]. \tag{1.42}$$

Here

$$\gamma_1 \equiv \left\{\frac{cM_sR}{Z_se} K + \frac{M_sV_s}{T_s}\left[1 + \psi A_{3s}\right]\right\}, \tag{1.43}$$

$$\gamma_3 \equiv \left\{\frac{cM_s^2R}{Z_se} \frac{A_{2s}}{2T_s}\right\}, \tag{1.44}$$

and

$$K \equiv \left[A_{1s} + A_{2s}\left(\frac{Z_se\Phi_s^{eff} - \frac{Z_se}{c}\psi\Omega_s(\psi)}{T_s} - \frac{1}{2}\right)\right]. \tag{1.45}$$

In the same limit, the calculation of the flow velocity gives an expression analogous expression as the one given in the previous Chapter. In particular, the leading-order term is still found to be $\mathbf{V}_s = \Omega_s(\psi)R\mathbf{e}_\varphi$, while, for the first-order term, Δ_{n_s} is as given by Eq.(1.42) and Δ_{2s} reduces to

$$\Delta_{2s} \equiv \frac{T_s}{M_s}\left(\gamma_1 + 3\gamma_3 V_s^2\right) + \frac{5\gamma_3 T_s^2}{M_s^2}. \tag{1.46}$$

Finally, $\Delta_{3s} \equiv 0$ since, in Eq.(1.38), the second and third terms on the right hand side necessarily vanish, while the first one is proportional to $A_{4s} \equiv \frac{\partial \alpha_s}{\partial \psi}$, where $\alpha_s(\psi) \equiv \frac{B}{\Delta_{T_s}} \equiv 0$, and hence vanishes too.

Before concluding this section, we stress again that the case with isotropic temperature represents a result whose accuracy is not limited by dependence on any gyrokinetic invariant and that it does not require any guiding-center variable transformation. For this reason, there are no restrictions of applicability of the solution (1.39) which, in principle, holds also in the limit of vanishing magnetic field.

1.8 Kinetic closure conditions

In this Section the issue concerned with the determination of kinetic closure conditions for collisionless magnetized axisymmetric accretion disc (AD) plasmas is discussed. The treatment of this problem is a prerequisite of primary importance for getting correct descriptions for the dynamics of collisionless plasmas in terms of suitable fluid equations and the corresponding fluid fields. In particular, here we focus on the calculation of the species pressure tensor, which represents the closure condition (i.e., the equation of state) for the fluid momentum equation (Euler equation). The publication reference is provided by Ref.(33).

The species pressure tensor is defined by the following moment of the KDF:

$$\underline{\underline{\Pi}}_s = \int d\mathbf{v} M_s(\mathbf{v} - \mathbf{V}_s^{tot})(\mathbf{v} - \mathbf{V}_s^{tot})\widehat{f_{*s}}. \tag{1.47}$$

Then, the overall pressure tensor of the system is obtained by summing the single species pressure tensors: $\underline{\underline{\Pi}} = \sum_{s=i,e} \underline{\underline{\Pi}}_s$. Since the AD plasma is collisionless, the KDF is not Maxwellian and we expect to recover some sort of anisotropy in the final form of the pressure tensor. For example, this can be due to the temperature anisotropy, whose origin is related to the conservation of the magnetic moment as an adiabatic invariant. Parallel and perpendicular temperatures are defined with respect to the local direction of the magnetic field. For this reason, it is convenient to introduce the set of orthogonal unitary vectors given by $(\mathbf{b}, \mathbf{e}_1, \mathbf{e}_2)$, where $\mathbf{b} \equiv \frac{\mathbf{B}}{B}$ is the tangent vector to the magnetic field while $\mathbf{e}_1$ and $\mathbf{e}_2$ are two orthogonal vectors in the plane perpendicular to the magnetic field line. Then, from this basis, we can construct the following unitary tensor: $\underline{\underline{\mathbf{I}}} \equiv \mathbf{bb} + \mathbf{e}_1\mathbf{e}_1 + \mathbf{e}_2\mathbf{e}_2$. The tensor pressure has its simplest representation when expressed in terms of the tensor $\underline{\underline{\mathbf{I}}}$ and the vectors $(\mathbf{b}, \mathbf{e}_1, \mathbf{e}_2)$.

By Taylor expanding the equilibrium KDF, the species pressure tensor $\underline{\underline{\Pi}}_s^{tot}$ for magnetized plasmas can be written as follows: $\underline{\underline{\Pi}}_s^{tot} \simeq \underline{\underline{\Pi}}_s + \Delta\underline{\underline{\Pi}}_s$, where $\underline{\underline{\Pi}}_s$ is the leading-order term (with respect to all of the expansion parameters), while $\Delta\underline{\underline{\Pi}}_s$ represents the first-order (i.e., $O(\varepsilon)$) correction term. In particular, $\underline{\underline{\Pi}}_s$ is given by

$$\underline{\underline{\Pi}}_s = p_{\perp s}\underline{\underline{\mathbf{I}}} + \left(p_{\parallel s} - p_{\perp s}\right)\mathbf{bb}, \tag{1.48}$$

where $p_{\perp s} \equiv n_s T_{\perp s}$ and $p_{\parallel s} \equiv n_s T_{\parallel s}$ represent the leading-order perpendicular and parallel pressures. The divergence of the species pressure tensor is found to be

$$\nabla \cdot \underline{\underline{\Pi}}_s = \nabla p_{\perp s} + \mathbf{bB} \cdot \nabla \left(\frac{p_{\parallel s} - p_{\perp s}}{B}\right) - \Delta p_s \mathbf{Q}, \tag{1.49}$$

where $\mathbf{Q} \equiv \left[\mathbf{bb}\cdot\nabla \ln B + \frac{4\pi}{cB}\mathbf{b} \times \mathbf{J} - \nabla \ln B\right]$ and $\Delta p_s \equiv \left(p_{\parallel s} - p_{\perp s}\right)$. On the other hand, $\Delta\underline{\underline{\Pi}}_s$ can be written as

$$\Delta\underline{\underline{\Pi}}_s \equiv \Delta\Pi_s^1 \underline{\underline{\mathbf{I}}} + \Delta\Pi_s^2 \mathbf{bb} + \Delta\underline{\underline{\Pi}}_s^3, \tag{1.50}$$

in which $\Delta\Pi_s^1$ and $\Delta\Pi_s^2$ are the diagonal first-order anisotropic corrections to the pressure tensor, while $\Delta\underline{\underline{\Pi}}_s^3$ contains all of the non-diagonal contributions. More precisely, $\Delta\Pi_s^1$ is given by

$$\begin{aligned}\Delta\Pi_s^1 \equiv\ & \gamma_1 \Omega_s R T_{\perp s} n_s - \gamma_2 \frac{2\Omega_s R T_{\perp s}^2 n_s}{B} + \\ & + \gamma_3 \frac{\Omega_s R T_{\perp s}^2 n_s}{M_s}\left(5 + \frac{T_{\parallel s}}{\Delta_{T_s}}\right) + \gamma_3 \left(\Omega_s R\right)^3 T_{\perp s} n_s + \\ & + \gamma_3 \frac{2\Omega_s I^2 T_{\parallel s} T_{\perp s} n_s}{R B^2 M_s} + \gamma_3 \frac{8\Omega_s R T_{\perp s}^2 n_s}{M_s}\left[\frac{3}{4} - \frac{I^2}{R^2 B^2}\right],\end{aligned} \tag{1.51}$$

while $\Delta\Pi_s^2$ is defined as

$$\begin{aligned}\Delta\Pi_s^2 \equiv\ & \gamma_1 \Omega_s R n_s \left[T_{\parallel s} - T_{\perp s}\right] - \gamma_2 \frac{\Omega_s R T_{\parallel s} T_{\perp s} n_s}{B} + \\ & + \gamma_3 \frac{\Omega_s R n_s T_{\parallel s} T_{\perp s}}{M_s}\left(5 + 3\frac{T_{\parallel s}}{\Delta_{T_s}}\right) + + \gamma_3 \left(\Omega_s R\right)^3 n_s \left[T_{\parallel s} - T_{\perp s}\right] + \\ & + \gamma_3 \frac{2\Omega_s R n_s T_{\parallel s} T_{\perp s}}{M_s} - \gamma_3 \frac{8\Omega_s R T_{\perp s}^2 n_s}{M_s}\left[\frac{3}{4}\frac{I^2}{R^2 B^2} + 1\right] + \gamma_2 \frac{2\Omega_s R T_{\perp s}^2 n_s}{B} + \\ & - \gamma_3 \frac{\Omega_s R T_{\perp s}^2 n_s}{M_s}\left(5 + \frac{T_{\parallel s}}{\Delta_{T_s}}\right) + \gamma_3 \frac{6\Omega_s I^2 n_s T_{\parallel s}^2}{R B^2 M_s} - \gamma_3 \frac{4\Omega_s I^2 T_{\parallel s} T_{\perp s} n_s}{R B^2 M_s}.\end{aligned} \tag{1.52}$$

Finally, $\Delta\underline{\underline{\Pi}}_s^3$ is symmetric and is given by

$$\Delta\underline{\underline{\Pi}}_s^3 \equiv \gamma_3 \frac{16\Omega_s R T_{\perp s}^2 n_s}{M_s}\left(\mathbf{e}_1\mathbf{e}_2 \colon \mathbf{e}_\varphi \mathbf{e}_\varphi\right)\left[\mathbf{e}_1\mathbf{e}_2 + \mathbf{e}_2\mathbf{e}_1\right] +$$

$$+\gamma_3 \frac{4\Omega_s I T_{\|s} T_{\perp s} n_s}{B M_s} \left(\left(\mathbf{e}_2 \cdot \mathbf{e}_\varphi \right) \left[\mathbf{b}\mathbf{e}_2 + \mathbf{e}_2 \mathbf{b} \right] + \left(\mathbf{e}_1 \cdot \mathbf{e}_\varphi \right) \left[\mathbf{b}\mathbf{e}_1 + \mathbf{e}_1 \mathbf{b} \right] \right). \quad (1.53)$$

The following comments about the solution are in order:

- The total tensor pressure $\underline{\underline{\Pi}}_s^{tot}$ is symmetric in the system defined by the vectors $(\mathbf{b}, \mathbf{e}_1, \mathbf{e}_2)$.
- The leading-order pressure tensor $\underline{\underline{\Pi}}_s$ calculated in this approximation is diagonal but non-isotropic. We notice that the source of this anisotropy in Eq.(1.48) is just the temperature anisotropy. In the limit of isotropic temperature, the leading-order pressure tensor becomes diagonal and isotropic, as can be easily verified.
- The first-order correction $\Delta\underline{\underline{\Pi}}_s$ is non-diagonal and non-isotropic. In particular, non-diagonal terms are carried by the tensor $\Delta\underline{\underline{\Pi}}_s^3$ given in Eq.(1.53). Two main properties of the solution contribute to generating the non-isotropic feature of $\Delta\underline{\underline{\Pi}}_s$. The first is the temperature anisotropy, while the second is the existence of the *diamagnetic part* of the KDF obtained from the Taylor expansion of $\widehat{f_{*s}}$ and depending on the thermodynamic forces associated with the gradients of the fluid fields. Indeed, we notice that taking the limit of isotropic temperature is not enough to make the tensor $\Delta\underline{\underline{\Pi}}_s$ isotropic as well. In fact, even if the parallel and perpendicular temperatures are equal, since the plasma is magnetized and collisionless the KDF will not be perfectly Maxwellian and deviations carried by the diamagnetic part act as a source of anisotropy.

1.9 Quasi-neutrality

In this Section the validity of the quasi-neutrality condition is addressed for the kinetic equilibria determined here. Consider the Poisson equation for the electrostatic potential Φ, expressed as

$$\nabla^2 \Phi = -4\pi \sum_{s=i,e} q_s n_s \left[1 + \Delta_{n_s} \right], \quad (1.54)$$

where Δ_{n_s} is written out explicitly above. In the limit of a strongly magnetized plasma and considering the accuracy of the previous asymptotic analytical expansions, we shall say that the plasma is *quasi-neutral* if the ordering $\frac{-\nabla^2\Phi}{4\pi\sum_{s=i,e} q_s n_s [1+\Delta_{n_s}]} = 0 + O\left(\varepsilon^k\right)$, with $k \geq 2$ holds, whereas we call it *weakly non-neutral* if $\frac{-\nabla^2\Phi}{4\pi\sum_{s=i,e} q_s n_s [1+\Delta_{n_s}]} = 0 + O\left(\varepsilon\right)$. The kinetic equilibrium for a weakly non-neutral plasma is referred to as a *Hall kinetic equilibrium* (15), and the corresponding fluid configuration is referred to as a *Hall Gravitational MHD (Hall-GMHD) fluid equilibrium* (14).

Let us now show that quasi-neutrality (in the sense just defined) can be locally satisfied by imposing a suitable constraint on the electrostatic (ES) potential Φ. It can be shown that this constraint can always be satisfied since the leading-order contribution to the ES potential remains unaffected. The result follows by neglecting higher-order corrections to the number density (Δ_{n_s}) and setting $q_i = Ze$ and $q_e = -e$. Thanks to the arbitrariness in the choice of the flux functions introduced by the kinetic constraints, it is possible to show that quasi-neutrality implies the following constraint for the oscillatory part $\Phi^{\sim}$ of the ES potential, i.e., correct to both $O\left(\varepsilon^0\right)$ and $O\left(\varepsilon_M^0\right)$,

$$\Phi^{\sim}(\psi, \vartheta) \equiv \Phi - \langle\Phi\rangle \simeq \frac{S^{\sim}}{e\left(\frac{Z}{T_{\|i}} + \frac{1}{T_{\|e}}\right)}, \tag{1.55}$$

where

$$S^{\sim} \equiv \ln\left(\frac{\eta_e}{Z\eta_i}\right) + \left[\frac{\overline{X}_e}{T_{\|e}} - \frac{\overline{X}_i}{T_{\|i}}\right], \tag{1.56}$$

and

$$\overline{X}_s \equiv \left(M_s \frac{R^2\Omega_s^2}{2} + \frac{Z_s e}{c}\psi\Omega_s(\psi) - M_s\Phi_G\right). \tag{1.57}$$

In particular, the arbitrariness in the coefficient $\frac{\eta_e}{Z\eta_i}$ can be used to satisfy the constraint $\langle S^{\sim}\rangle = 0$. In fact, in view of Eq.(1.15), it follows that

$$\eta_s = \frac{\beta_s(\psi)\, T_{\|s}(\psi)}{1 + \frac{\alpha_s(\psi) T_{\|s}(\psi)}{B(\psi,\vartheta)}}, \tag{1.58}$$

where the flux functions still remain arbitrary. In conclusion, Eq.(1.55) determines only $\Phi^{\sim}$ and not the total ES potential. Note that this solution for the electrostatic potential $\Phi^{\sim}$ can be shown to be consistent with earlier treatments appropriate for Tokamak plasma equilibria (24, 40). This can be exactly recovered thanks to the arbitrariness in defining the pseudo-densities and by taking the limit of isotropic temperatures and zero gravitational potential, as in the case of laboratory plasmas. In this limit the species pseudo-densities become flux functions (24). Then, because of this arbitrariness, by taking

$$\frac{\eta_e}{Z\eta_i} = 1, \tag{1.59}$$

it follows that Eq.(1.55) reduces to the form presented in (24), which can only be used to determine the poloidal variation of the potential.

1.10 Diamagnetic-driven kinetic dynamo

In this Section the Ampere equation is considered and the existence of a diamagnetic-driven kinetic dynamo for collisionless magnetized plasmas is pointed out. The reference publications are Refs.(32, 34, 35).

Adopting the Taylor analytical expansion of the asymptotic equilibrium KDF and neglecting corrections of $O\left(\varepsilon^k\right)$, with $k \geq 2$, and $O\left(\varepsilon_M^{n+1}\right)$, with $n \geq 0$, the Ampere equation can be approximately written as follows:

$$\nabla \times \mathbf{B}^{self} = \frac{4\pi}{c} \sum_{s=i,e} q_s n_s \left[\mathbf{V}_s + \Delta \mathbf{U}_s \right], \tag{1.60}$$

where $\mathbf{B}^{self}$ is as defined in Eq.(1.3) and the expression for $\Delta \mathbf{U}_s$ is given by Eq.(1.35). The toroidal component of this equation gives the generalized Grad-Shafranov equation for the poloidal flux function ψ_p:

$$\Delta^* \psi_p = -\frac{4\pi}{c} R \sum_{s=e,i} q_s n_s \left[\Omega_s (\psi) R + \Delta_{\varphi s} \right], \tag{1.61}$$

where the elliptic operator Δ^* is defined as $\Delta^* \equiv R^2 \nabla \cdot \left(R^{-2} \nabla \right)$ (55). The remaining terms in Eq.(1.60) give the equation for the toroidal component of the magnetic field $\frac{I(\psi,\vartheta)}{R}$. In the same approximation, this is:

$$\nabla I(\psi,\vartheta) \times \nabla \varphi = \frac{4\pi}{c} \sum_{s=i,e} q_s n_s \frac{\Delta_{3s}}{B} \nabla \psi \times \nabla \varphi, \tag{1.62}$$

where Δ_{3s}, given above, contains the contributions of the species temperature anisotropies. For consistency with the approximation introduced, in the small inverse aspect ratio ordering, it follows that $\frac{\partial I(\psi,\vartheta)}{\partial \vartheta} = 0 + O\left(\delta^k\right)$, i.e., to leading order in δ: $I = I\left(\psi\right) + O\left(\delta^k\right)$, with $k \geq 1$. This in turn also requires that the corresponding current density in Eq.(1.62) is necessarily a flux function. Then, correct to $O(\varepsilon)$, $O(\varepsilon_M^0)$ and $O\left(\delta^0\right)$, the differential equation for $I\left(\psi\right)$ becomes:

$$\frac{\partial I(\psi)}{\partial \psi} = \frac{4\pi}{c} \sum_{s=e,i} q_s n_s \frac{\Delta_{3s}}{B}, \tag{1.63}$$

which uniquely determines an approximate solution for the toroidal magnetic field. This result is remarkable because it shows that there is a *stationary diamagnetic-driven kinetic dynamo effect* which generates an equilibrium toroidal magnetic field *without requiring any net accretion* and *in the absence of any possible instability/turbulence phenomena*. This new mechanism results from poloidal currents arising due to the FLR effects and temperature anisotropies which are characteristic of the equilibrium KDF. We remark that the self-generation of the stationary magnetic field is purely diamagnetic. In particular, the toroidal component is associated with the drifts of the plasma away from the flux surfaces. In the present formulation, possible dissipative phenomena leading to a non-stationary self field have been ignored. Such dissipative phenomena probably do arise in practice and could occur both in the local domain where the equilibrium magnetic surfaces are closed and nested and elsewhere. Temperature anisotropies are therefore an important physical property of collisionless AD plasmas, giving a possible mechanism for producing a stationary toroidal magnetic field. We stress that this effect disappears altogether in the case of isotropic temperatures. Finally, we consider the ratio between the toroidal and poloidal current densities ($\mathbf{J}_T$ and $\mathbf{J}_P$). In the small inverse aspect ratio ordering, neglecting corrections of order $O\left(\delta^k\right)$ with $k \geq 1$, this provides an estimate of the magnitude of the corresponding components of the magnetic field. In fact, in this limit we can write $\frac{|\nabla\times\mathbf{B}_T|}{|\nabla\times\mathbf{B}_P|} \sim \frac{|\mathbf{B}_T|}{|\mathbf{B}_P|} \sim \frac{|\mathbf{J}_P|}{|\mathbf{J}_T|}$ and so conclude that, although for the single species velocity, the ordering $\frac{\left|\frac{\Delta_{3s}}{B}\nabla\psi\times\nabla\varphi\right|}{|\mathbf{V}_s|} \sim O\left(\varepsilon\right) O\left(\delta_{Ts}\right)$ holds, this might no longer be the case for the magnetic field, which instead depends on the ratio between the total toroidal and poloidal current densities. In particular, the possibility of having finite stationary toroidal magnetic fields is, in principle, allowed by the present analysis, depending on the properties of the overall solution describing the system.

1.11 Conclusions

Getting a complete understanding of the dynamical properties of astrophysical accretion discs still represents a challenging task and there are many open problems remaining to be solved before one can get a full and consistent theoretical formulation for the physical processes involved.

The present investigation provides some important new results for understanding the equilibrium properties of accretion discs, obtained within the framework of a kinetic approach based on the Vlasov-Maxwell description. The derivation presented applies for collisionless non-relativistic and axi-symmetric AD plasmas under the influence of both gravitational and EM fields. A wide range of astrophysical scenarios can be investigated with the present theory, thanks to the possibility of properly setting the different parameters which characterize the physical and geometrical properties of the model. A possible astrophysical context is provided by radiatively inefficient accretion flows onto black holes, where the accreting material is thought to consist of a plasma of collisionless ions and electrons with different temperatures, in which the dominant magnetic field is generated by the plasma current density. We have considered here the specific case in which the structure of the magnetic field is locally characterized by a family of closed nested magnetic surfaces within which the plasma has mainly toroidal flow velocity. For this, we have proved that a kinetic equilibrium exists and can be described by a stationary KDF expressed in terms of the exact integrals of motion and the magnetic moment prescribed by the gyrokinetic theory, which is an adiabatic invariant. Many interesting new results have been pointed out. The most relevant ones for astrophysical applications are the following: 1) the possibility of including the effects of a non-isotropic temperature in the stationary KDF; 2) the proof that the Maxwellian and bi-Maxwellian KDFs are asymptotic stationary solutions, i.e. they can be regarded as approximate equilibrium solutions in the limit of strongly magnetized plasmas; 3) the possibility of computing the stationary fluid moments to the desired order of accuracy in terms of suitably prescribed flux functions; 4) the proof that a toroidal magnetic field can be generated in a stationary configuration even in the absence of any net accretion flow if and only if the plasma has a temperature anisotropy.

This last point, in particular, is of great interest because it gives a mechanism for generating a stationary toroidal field in the disc, independent of instabilities related to the accretion flow. The consistent kinetic formulation developed here permits the self-generation of such a field by the plasma itself, associated with localized poloidal drift-currents on the nested magnetic surfaces as a consequence of temperature anisotropies. This stationary poloidal motion is made possible in the framework of kinetic theory by the conservation of the canonical momentum as a result of FLR effects.

These results reveal and confirm the power of the kinetic treatment and the necessity for adopting such a formalism in order to correctly understand the physical phenomena occurring in accretion discs. This study may represent a significant step forward for understanding the physical properties of accretion discs in their kinetic equilibrium configurations, and it motivates making further investigations of the subject aimed at extending the present range of validity to more general physical configurations.

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Chapter 2

Kinetic description of quasi-stationary axisymmetric collisionless accretion disc plasmas with arbitrary magnetic field configurations

2.1 Introduction

In the previous Chapter, results were presented concerning formulation of kinetic theory for investigating stationary solutions for collisionless AD plasmas, focusing on configurations with locally-closed magnetic flux surfaces. In the present Chapter, we generalize the previous solution to arbitrary magnetic field configurations, which are no longer restricted to localized spatial domains in the disc. We refer to Fig.2.1 below for an explicit comparison of the two configurations.

The purpose of this investigation is to formulate a comprehensive kinetic treatment for collisionless axisymmetric AD plasmas including both accretion flows and collisionless dynamo effects. We include general relative orderings between the magnitudes of the external and self-generated magnetic fields and allow the magnetic field be non-uniform and slowly time-varying while possessing locally nested open magnetic surfaces.

Extending the investigation developed in Chapter 1, this is done by constructing particular quasi-stationary solutions of the Vlasov-Maxwell equations, characterized by generalized bi-Maxwellian phase-space distributions, which are referred to here as quasi-stationary asymptotic KDFs (QSA-KDFs). As discussed below, the functional form of these solutions is physically motivated. We will show that this makes possible the explicit inclusion of both *temperature anisotropies* and *parallel velocity perturba-*

tions in the QSA-KDFs (see the definition below in Section 3.4). This is done, first, by developing a formulation for GK theory in the presence of a gravitational field, making it possible to directly construct the relevant particle guiding-center adiabatic invariants. The QSA-KDFs are then expressed in terms of these. Remarkably, this allows also the consistent treatment of trapping phenomena due to spatial variations both of the magnetic field and of the total effective potential (gravitational EM trapping). Second, the QSA-KDFs are constructed by imposing appropriate *kinetic constraints* (see Section 3.4), requiring that suitable *structure functions* (see below) which enter the definition of the QSA-KDFs, depend only on the azimuthal canonical momentum and total particle energy. By invoking suitable perturbative expansions, it follows that the relevant moments and moment equations can be evaluated analytically. The solution thus obtained can be used for investigating the quasi-stationary dynamics of magnetized AD plasmas, including description of quasi-stationary accretion flows and "kinetic dynamo effects" allowing for the generation of finite poloidal and toroidal magnetic fields. In particular, the kinetic theory predicts the possibility of pure matter inflows as well as the independent coexistence of both inflows and outflows.

In this Chapter, we use the same basic assumptions and definitions as in Chapter 1 (see Section 1.3). Main difference here is that we focus on solutions for the equilibrium magnetic field $\mathbf{B}$ which admit, at least locally, a family of nested and *open* axisymmetric toroidal magnetic surfaces $\{\psi(\mathbf{x})\} \equiv \{\psi(\mathbf{x}) = const.\}$, where ψ denotes the poloidal magnetic flux of $\mathbf{B}$. See Fig.2.1 for a schematic comparison between the configuration of locally closed magnetic surfaces considered in the previous Chapter and the case of open magnetic surfaces analyzed in the present one. A set of magnetic coordinates $(\psi, \varphi, \vartheta)$ can be defined locally, where ϑ is a curvilinear angle-like coordinate on the magnetic surfaces $\psi(\mathbf{x}) = const$. Each relevant physical quantity $G(\mathbf{x}, t)$ can then be conveniently expressed either in terms of the cylindrical coordinates or as a function of the magnetic coordinates, i.e. $G(\mathbf{x}, t) = \overline{G}(\psi, \vartheta, t)$, where the φ dependence has been suppressed due to the axisymmetry. Contrary to the previous Chapter, here we do not longer introduce the inverse aspect ratio parameter δ, which is appropriate for treating locally closed nested magnetic surfaces.

The publication reference for the material presented in this Chapter is Ref.(1).

2.2 GK theory for magnetized accretion disc plasmas

In this section we recall the GK theory appropriate for the description of AD plasmas. Its formulation is in fact a prerequisite for the construction of the kinetic quasi-stationary equilibria to be developed later. The appropriate generalization of GK theory allowing for the presence of strong gravitational fields should in principle be based on a covariant formulation [see (2, 3, 4, 5)]. However, for non-relativistic plasmas within a gravitational field, the appropriate formulation can also be directly recovered via a suitable reformulation of the standard non-relativistic theory holding for magnetically confined plasmas (6, 7, 8, 9, 10, 11, 12, 13, 14).

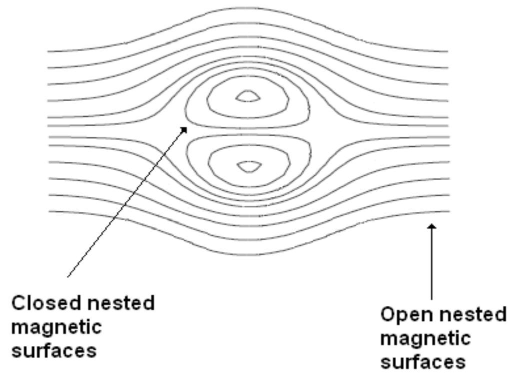

Figure 2.1: Schematic comparison between the configuration of locally closed magnetic surfaces considered in the previous Chapter and the case of open magnetic surfaces analysed in the present study.

In this case, the appropriate particle Lagrangian function can be represented in terms of the effective EM potentials $\left\{\Phi_s^{eff}(\mathbf{x},\varepsilon_M^k t),\mathbf{A}(\mathbf{x},\varepsilon_M^k t)\right\}$, with $k\geq 1$, where Φ_s^{eff} is defined in previous Chapter. In terms of the hybrid variables $\mathbf{z}\equiv(\mathbf{x},\mathbf{v})$ (with $\mathbf{x}$ and $\mathbf{v}$ denoting respectively the particle position and velocity vectors), this is expressed as

$$\mathcal{L}_s(\mathbf{z},\frac{d}{dt}\mathbf{z},\varepsilon_M^k t)\equiv\dot{\mathbf{r}}\cdot\mathbf{P}_s-\mathcal{H}_s(\mathbf{z},\varepsilon_M^k t),\tag{2.1}$$

where $\mathbf{P}_s\equiv\left[M_s\mathbf{v}+\frac{Z_s e}{c}\mathbf{A}(\mathbf{x},\varepsilon_M^k t)\right]$ and

$$\mathcal{H}_s(\mathbf{z},\varepsilon_M^k t)=\frac{M_s}{2}v^2+Z_s e\Phi_s^{eff}(\mathbf{x},\varepsilon_M^k t)\tag{2.2}$$

denotes the corresponding Hamiltonian function in hybrid variables. The GK treatment for the Lagrangian (2.1) involves the construction - in terms of a perturbative expansion determined by means of a power series in ε_M - of a diffeomorphism of the form

$$\mathbf{z}\equiv(\mathbf{r},\mathbf{v})\rightarrow\mathbf{z}'\equiv(\mathbf{r}',\mathbf{v}'),\tag{2.3}$$

referred to as the *GK transformation*. Again, in the following primed quantities will denote dynamical variables defined at the *guiding-center position* $\mathbf{r}'$ (or $\mathbf{x}'$ in axisymmetry). Here, by definition, the transformed variables $\mathbf{z}'$ (*GK state*) are constructed so that their time derivatives to the relevant order in ε_M have at least one ignorable coordinate (a suitably-defined gyrophase ϕ'). As an illustration, we show the formulation of the perturbative theory to leading-order in ε_M. In this case the GK transformation becomes simply

$$\begin{cases} \mathbf{r} = \mathbf{r}' - \frac{\mathbf{w}' \times \mathbf{b}'}{\Omega'_{cs}}, \\ \mathbf{v} = u'\mathbf{b}' + \mathbf{w}' + \mathbf{V}'_{eff}, \end{cases} \tag{2.4}$$

where $\mathbf{w}' = w' \cos\phi' \mathbf{e}'_1 + w' \sin\phi' \mathbf{e}'_2$, with ϕ' denoting the gyrophase angle. In the following, the GK transformation will be performed on all phase-space variables $\mathbf{z} \equiv (\mathbf{r}, \mathbf{v})$, *except* for the azimuthal angle φ which is left unchanged (15) and is therefore to be considered as one of the GK variables. Here $\mathbf{b}' = \mathbf{b}(\mathbf{x}', \varepsilon_M^k t)$, with $\mathbf{b} \equiv \mathbf{B}/B$, while $\Omega'_{cs} = \frac{Z_s e B'}{M_s c}$ and $\mathbf{V}'_{eff}$ are respectively the guiding-center Larmor frequency and the *effective drift velocity* produced by $\mathbf{E}_s^{\prime eff}$, namely

$$\mathbf{V}'_{eff}(\mathbf{x}, \varepsilon_M^k t) \equiv \frac{c}{B'} \mathbf{E}_s^{\prime eff} \times \mathbf{b}'. \tag{2.5}$$

The rest of the notation is standard, with u' and $\mathbf{w}'$ denoting respectively the parallel and perpendicular (guiding-center) velocities, both defined relative to the frame locally moving with velocity $\mathbf{V}'_{eff}$. It follows that, when expressed in terms of the GK variables $\mathbf{z}'$, the GK Lagrangian and Hamiltonian functions, $\mathcal{L}'_s$ and $\mathcal{H}'_s$, can be evaluated with the desired order of accuracy. In particular, to leading-order, i.e. neglecting corrections of $O(\varepsilon_M^n)$ with $n \geq 1$, $\mathcal{L}'_s = \mathcal{L}_s^{\prime(1)} + O(\varepsilon_M)$ and $\mathcal{H}'_s = \mathcal{H}_s^{\prime(1)} + O(\varepsilon_M)$, where $\mathcal{L}_s^{\prime(1)}$ and $\mathcal{H}_s^{\prime(1)}$ recover the customary expressions

$$\mathcal{L}_s^{\prime(1)} \equiv \dot{\mathbf{r}}' \cdot \frac{Z_s e}{c} \mathbf{A}_s^{\prime *} - \frac{\dot{\phi}'}{\Omega'_{cs}} m'_s B' - \mathcal{H}_s^{\prime(1)}, \tag{2.6}$$

$$\mathcal{H}_s^{\prime(1)} \equiv m'_s B' + \frac{M_s}{2} \left(u'\mathbf{b}' + \mathbf{V}'_{eff} \right)^2 + Z_s e \Phi_s^{\prime *}, \tag{2.7}$$

with the magnetic moment $m'_s \cong \mu'_s \equiv \frac{M_s w'^2}{2B'}$ to leading order, while the gyrophase-independent *modified EM potentials* $\left(\Phi_s^{\prime *}, \mathbf{A}_s^{\prime *} \right)$ are

$$\Phi_s^{\prime *} \cong \Phi_s^{\prime eff}, \tag{2.8}$$

$$\mathbf{A}_s^{\prime *} \cong \mathbf{A}' + \frac{M_s c}{Z_s e} \left(u'\mathbf{b}' + \mathbf{V}'_{eff} \right), \tag{2.9}$$

in the same approximation. It is important to stress that the GK theory can be performed in principle to arbitrary order in ε_M (6, 7, 8, 9, 10, 11, 12, 13, 14), thus permitting the explicit determination of m'_s and the modified EM potentials as well as the relevant guiding-center canonical momenta.

First integrals of motion and guiding-center adiabatic invariants for AD plasmas

The exact integrals of motion and the relevant adiabatic invariants corresponding respectively to Eqs.(2.1) and (2.6) can be immediately recovered. By definition, an adiabatic invariant P of order n with respect to ε_M is conserved only in an asymptotic sense, i.e., in the sense that $\frac{1}{\Omega'_{cs}} \frac{d}{dt} \ln P = 0 + O(\varepsilon_M^{n+1})$, where $n \geq 0$ is a suitable integer.

First we notice that, under the assumptions of axisymmetry, the only first integral of motion is the canonical momentum $p_{\varphi s} \equiv \frac{\partial \mathcal{L}_s}{\partial \cdot \varphi}$ conjugate to the ignorable azimuthal angle φ:

$$p_{\varphi s} = M_s R \mathbf{v} \cdot \mathbf{e}_\varphi + \frac{Z_s e}{c} \psi \equiv \frac{Z_s e}{c} \psi_{*s}. \tag{2.10}$$

Since the azimuthal angle φ is ignorable also for the GK Lagrangian $\mathcal{L}'_s$, it follows that the quantity $p'_{\varphi s} \equiv \frac{\partial \mathcal{L}'_s}{\partial \cdot \varphi}$ is an adiabatic invariant of the prescribed order, according to the accuracy of the GK transformation used to evaluate $\mathcal{L}'_s$. We shall refer to $p'_{\varphi s}$ as the *guiding-center canonical momentum.* In particular, correct to $O(\varepsilon_M^k)$, with $k \geq 1$, one obtains

$$p'_{\varphi s} \equiv \frac{M_s}{B'} \left(u' I' - \frac{c \nabla' \psi' \cdot \nabla' \Phi_s^{\prime eff}}{B'} \right) + \frac{Z_s e}{c} \psi', \tag{2.11}$$

which is an adiabatic invariant of $O(\varepsilon_M^{k+1})$, with $k \geq 1$. Furthermore, the total particle energy E_s

$$E_s = \frac{M_s}{2} v^2 + Z_s e \Phi_s^{eff}(\mathbf{x}, \varepsilon_M^n t), \tag{2.12}$$

with $n \geq 1$, and the GK Hamiltonian $\mathcal{H}'_s$

$$\mathcal{H}'_s \equiv m'_s B' + \frac{M_s}{2} \left(u' \mathbf{b}' + \mathbf{V}'_{eff} \right)^2 + Z_s e \Phi_s^{\prime eff} \tag{2.13}$$

are also adiabatic invariants of order n. Finally, in GK theory, by construction, the momentum $p'_{\phi' s} = \partial \mathcal{L}'_s / \partial \cdot \phi'$ conjugate to the gyrophase, as well as the related magnetic moment m'_s defined as $m'_s \equiv \frac{Z_s e}{M_s c} p'_{\phi' s}$, are adiabatic invariants. As shown by Kruskal (1962 (16)) it is always possible to determine $\mathcal{L}'_s$ so that m'_s is an adiabatic invariant of arbitrary order in ε_M. In particular, the leading-order approximation is $m'_s \cong \mu'_s \equiv \frac{M_s w'^2}{2B'}$.

2.3 Physical implications of the conservation laws

We now discuss the physical meaning of the conservation laws introduced here and their implications for particle dynamics in magnetized accretion discs.

Consider first the conservation of the toroidal canonical momentum. For a charged particle this is the sum of two terms: the particle angular momentum $L_{\varphi s} \equiv M_s R v_\varphi$ and a magnetic contribution $\frac{Z_s e}{c} \psi$. It follows that the angular momentum by itself is generally not conserved. As a consequence, the canonical momentum conservation law allows for the existence of radial particle motion inside a disc. In fact, since in AD plasmas the magnetic flux function ψ is necessarily spatially non-homogeneous, a moving particle must change its angular momentum $L_{\varphi s}$ while fulfilling the constraint $\psi_{*s} = const.$, namely staying on a $\psi_{*s}-$surface. Depending on the geometry of the magnetic surfaces, such particle motion may correspond to either a vertical or radial velocity towards regions of higher or lower magnetic flux. Since the magnetic contribution to ψ_{*s} depends on the sign of the charges, single ions and electrons exhibit motions in different directions while keeping ψ_{*s} constant. This feature is different from the situation for neutral particles, for which the angular momentum itself is conserved. Because of the presence of plasma boundaries, this can lead to the self-generation of quasi-stationary electric fields in the accretion disc as a result of charge separation.

We next focus on the conservation of the guiding-centre Hamiltonian (2.13) and the magnetic moment μ'_s (to leading-order approximation). These can be combined to represent the parallel velocity u' as

$$u' = \pm\sqrt{\frac{2}{M_s}\left[\mathcal{H}'_s - \mu'_s B' - Z_s e \Phi'^{eff}_s - \frac{M_s}{2}V'^2_{eff}\right]}. \tag{2.14}$$

Therefore u' is a local function of the guiding-centre position vector $\mathbf{x}'$ and, due to axisymmetry, of the corresponding flux coordinates (ψ', ϑ'). The above relationship is the basis of *particle trapping phenomena*, corresponding to the existence of allowed and forbidden regions of configuration space for the motion of charged particles. In fact, since u' is only defined in the subset of the configuration space spanned by (ψ', ϑ') where the argument of the square root is non-negative, it follows from Eq.(2.14) that particles must undergo spatial reflections when $u' = 0$. The points of the configuration space where this occurs are the so-called *mirror points* and the occurrence of such points may generate various kinetic phenomena. In particular, particles can in principle experience zero, one or two reflections corresponding respectively to *passing particles (PPs), bouncing particles (BPs)* and *trapped particles (TPs).* In the present case, since the right hand side of Eq.(2.14) depends on the magnitude of the magnetic field (B'), the effective potential energy $(Z_s e \Phi'^{eff}_s)$ and the centrifugal potential $(\frac{M_s}{2}V'^2_{eff})$, we will refer to the TP case as *gravitational EM trapping.*

Finally, an important qualitative property of collisionless magnetized plasmas follows from the conservation of the magnetic moment μ'_s. The expression for this relates the magnitude of the perpendicular velocity w' to that of the local magnetic field B'. Conservation of the adiabatic invariant μ'_s implies that when a charge is subject to a non-uniform or a non-stationary magnetic field, its kinetic energy of perpendicular motion, $M_s\frac{w'^2}{2}$, must change accordingly so as to keep μ'_s constant. On the other hand, particles moving on $\psi_{*s}-$surfaces generally necessarily experience a non-uniform magnetic field $B(\mathbf{x}, \varepsilon^k_M t)$. It can be shown that this property implies also the phenomenon of having a non-isotropic kinetic temperature (i.e. there being different effective temperatures parallel and perpendicular to the local direction of the magnetic field). From the statistical point of view of kinetic theory, this temperature anisotropy corresponds to an anisotropy in the kinetic energy of random motion of particles subject to the magnetic field. Such a feature is a characteristic kinetic phenomenon arising in magnetized collisionless plasmas. This physical mechanism operating at the level of single particle dynamics has important consequences also for the macroscopic properties of such plasmas. As we will see, conservation of μ'_s allows the effects of temperature anisotropy to be included consistently in the quasi-stationary solution for the KDF, and for its physical implications for the dynamics of the corresponding fluid system to be inferred. Another candidate source of temperature anisotropy is radiation emission (cyclotron radiation) due to Larmor rotation in the presence of a strong magnetic field. The signature of this is the simultaneous occurrence of radiation emission corresponding to the Larmor frequencies of the different plasma species.

2.4 Construction of the QSA-KDF: generalized solution

In this section we show that the equilibrium generalized bi-Maxwellian solution for the KDF obtained in Ref.(17) can be extended to QSA-KDFs describing axisymmetric AD plasmas with the following features:

1) The KDF is also axisymmetric;

2) Each species in the collisionless plasma is considered to be associated with a suitable set of *sub-species* (referring to the different populations mentioned above), each one having a different KDF;

3) Temperature anisotropy: for all of the species, it is assumed that different parallel and perpendicular temperatures are allowed (with respect to the local direction of the magnetic field);

4) Accretion flow velocity: a non-vanishing species dependent poloidal flow velocity is prescribed;

5) Open, locally nested magnetic flux surfaces: the magnetic field is taken to allow quasi-stationary solutions with magnetic flux lines belonging to open and locally nested magnetic surfaces;

6) Kinetic constraints: suitable functional dependencies are imposed so that the KDF is an adiabatic invariant;

7) Analytic form: the solution is required to be asymptotically "close" to a local bi-Maxwellian in order to permit comparisons with previous literature dealing with Maxwellian or a bi-Maxwellian KDFs (see for example (18, 19, 20)).

Requirement 2) is suggested by observations of collisionless plasmas. For example, in the solar wind plasma both ion and electron species are described by superpositions of shifted bi-Maxwellian distributions. Requirements 1) - 7) clearly imply that the solution cannot generally be a Maxwellian. However, it is possible to show that they can be fulfilled by a suitable modified bi-Maxwellian expressed solely in terms of first integrals of motion and adiabatic invariants (17, 21, 22). It follows that this is necessarily a QSA-KDF. A set of fluid equations can then readily be determined using this solution, expressed in terms of four moments of the KDF [corresponding to the species number density, flow velocity and the parallel and perpendicular temperatures]. These equations which, by construction, satisfy a kinetic closure condition, are also useful for comparing with previous fluid treatments.

For consistency with the notation of Ref.(17) and previous Chapter, we again use the symbol "$\wedge$ " to denote physical quantities which refer to the treatment of anisotropic temperatures, unless otherwise specified, but in the present work, for greater generality, the symbol "$*$" is used to denote variables which depend on both the canonical momentum ψ_{*s} and the total particle energy E_s.

In line with all of the previous requirements, it is possible to show that a particular solution for the QSA-KDF is given by:

$$\widehat{f_{*s}} = \frac{\widehat{\beta_{*s}}}{(2\pi/M_s)^{3/2}\left(T_{\parallel *s}\right)^{1/2}} \exp\left\{-\frac{K_{*s}}{T_{\parallel *s}} - m'_s\widehat{\alpha_{*s}}\right\}, \tag{2.15}$$

which we refer to as the *Generalized bi-Maxwellian KDF with parallel velocity perturbations*. Here $\widehat{f_{*s}}$ is defined in the phase-space $\Gamma = \Gamma_r \times \Gamma_u$, where Γ_r and Γ_u are both identified with suitable subsets of the Euclidean space $\mathbb{R}^3$. The notation is as follows:

$$\widehat{\beta_{*s}} \equiv \frac{\eta_s}{\widehat{T}_{\perp s}}, \tag{2.16}$$

$$\widehat{\alpha_{*s}} \equiv \frac{B'}{\widehat{\Delta_{T_s}}}, \tag{2.17}$$

$$K_{*s} \equiv E_s - \ell_{\varphi s}\varpi_{*s}, \tag{2.18}$$

with E_s and ψ_{*s} given by Eqs.(2.12) and (2.10) respectively, while $\frac{1}{\widehat{\Delta_{T_s}}} \equiv \frac{1}{\widehat{T}_{\perp s}} - \frac{1}{T_{\parallel *s}}$. By construction, $\ell_{\varphi s}$ has the dimensions of an angular momentum, while ϖ_{*s} has those of a frequency. The latter is not necessarily associated here with a purely azimuthal leading-order velocity. In general K_{*s} can, in fact, be represented as

$$K_{*s} = E_s - \frac{Z_s e}{c}\psi_{*s}\Omega_{*s} - p'_{\varphi s}\xi_{*s} = H_{*s} - p'_{\varphi s}\xi_{*s}. \tag{2.19}$$

Here $H_{*s} \equiv E_s - \frac{Z_s e}{c}\psi_{*s}\Omega_{*s}$ has the same meaning as the analogous quantity used in Chapter 2, with Ω_{*s} being related to the azimuthal rotational frequency. In Eq.(2.19) ξ_{*s} is a frequency associated with the leading-order guiding-center canonical momentum $p'_{\varphi s}$ defined in Eq.(2.11), which is an adiabatic invariant depending on u' and, by definition, is independent of the gyrophase angle. As we shall show at the end of this section, this feature can be used to require that the QSA-KDF carries a non-vanishing parallel flow velocity. This can be related to a net accretion flow arising in the AD plasma. Finally, by substituting Eq.(2.19) into Eq.(2.15) we reach the equivalent representation for the QSA-KDF:

$$\widehat{f_{*s}} = \frac{\widehat{\beta_{*s}}}{(2\pi/M_s)^{3/2}\left(T_{\parallel *s}\right)^{1/2}} \exp\left\{-\frac{H_{*s}}{T_{\parallel *s}} + \frac{p'_{\varphi s}\xi_{*s}}{T_{\parallel *s}} - m'_s\widehat{\alpha_{*s}}\right\}. \tag{2.20}$$

In order for the solution (2.20) [or equivalently (2.15)] to be a function of the integrals of motion and of the adiabatic invariants, the functions $\{\Lambda_{*s}\} \equiv \left\{\widehat{\beta_{*s}}, \widehat{\alpha_{*s}}, T_{\parallel *s}, \Omega_{*s}, \xi_{*s}\right\}$, which we will refer to as *structure functions*, must be adiabatic invariants by themselves. To further generalize the solution of Ref.(17), we shall here retain a functional dependence on both the total particle energy and the canonical momentum, thus imposing the functional dependencies

$$\Lambda_{*s} = \Lambda_{*s}\left(\psi_{*s}, E_s\right), \tag{2.21}$$

which will be referred to in the following as *kinetic constraints*. The kinetic constraints (2.21) provide the most general solution for $\widehat{f_{*s}}$. It can be shown that the physical motivation behind imposing these dependencies lies essentially in the fact that the asymptotic condition of small inverse aspect ratio is no longer valid. In the present context, the kinetic solution is no longer restricted to localized spatial domains in the disc but applies to the general configuration of open magnetic surfaces. This in turn implies that the structure functions are generally not simply flux-functions on the magnetic surfaces.

Some basic properties of $\widehat{f_{*s}}$ are:

Property 1: $\widehat{f_{*s}}$ is itself an adiabatic invariant, and is therefore an asymptotic solution of the stationary Vlasov equation, i.e., a QSA-KDF;

Property 2: $\widehat{f_{*s}}$ is only defined in the subset of phase-space where the adiabatic invariants $p'_{\varphi s}$, $\mathcal{H}_s^{\prime(1)}$ and m'_s are defined. It follows that $\widehat{f_{*s}}$ is suitable for describing both circulating and trapped particles;

Property 3: all of the velocity-moment equations obtained from the Vlasov equation (and in particular the continuity and linear momentum fluid equations) are identically satisfied in an asymptotic sense, i.e., neglecting corrections of $O\left(\varepsilon_M^{n+1}\right)$;

Property 4: its velocity moments, to be identified with the fluid fields, are unique once $\widehat{f_{*s}}$ is prescribed in terms of the structure functions;

Property 5: it generalizes the solution earlier presented: a) by using both $p'_{\varphi s}$ and m'_s as adiabatic invariants and b) because of the new kinetic constraints.

It follows immediately that the solution (2.20) does indeed carry finite parallel velocity perturbations. Invoking the definitions (2.11) and (2.19), Eq.(2.20) can be re-written as

$$\widehat{f_{*s}} = \frac{\widehat{\beta_{*s}} \exp\left[\frac{X_{*s}}{T_{\parallel *s}}\right]}{\left(2\pi/M_s\right)^{3/2}\left(T_{\parallel *s}\right)^{1/2}} \exp\left\{-\frac{M_s\left(\mathbf{v}-\mathbf{V}_{*s}-U'_{\parallel *s}\mathbf{b}'\right)^2}{2T_{\parallel *s}} - m'_s\widehat{\alpha_{*s}}\right\}, \tag{2.22}$$

where $\mathbf{V}_{*s} = \mathbf{e}_\varphi R\Omega_{*s}\left(\psi_{*s}, E_s\right)$ and

$$X_{*s} \equiv M_s\frac{\left|\mathbf{V}_{*s}\right|^2}{2} + \frac{Z_s e}{c}\psi\Omega_{*s} - Z_s e\Phi_s^{eff} + \Upsilon'_{*s}. \tag{2.23}$$

Here the function Υ'_{*s} is defined as

$$\Upsilon'_{*s} \equiv \frac{M_s U'^2_{\parallel *s}}{2}\left(1+\frac{2\Omega_{*s}}{\xi_{*s}}\right) - \left(\frac{M_s c\nabla'\psi'\cdot\nabla'\Phi_s^{\prime eff}}{B'^2} - \frac{Z_s e}{c}\psi'\right)\xi_{*s}, \tag{2.24}$$

with $U'_{\parallel *s} = \frac{I'}{B'}\xi_{*s}\left(\psi_{*s}, E_s\right)$. Note that $U'_{\parallel *s}$ is non-zero only if the toroidal magnetic field is non-vanishing. This quantity is independent of $\mathbf{V}_{*s}$ and is clearly associated with a parallel flow velocity (i.e., having both poloidal and toroidal components), referred to here as a *parallel velocity perturbation*. This perturbation enters the solution via the adiabatic invariant $p'_{\varphi s}$ and therefore its inclusion is consistent with the requirement that KDF is an adiabatic invariant.

Finally we note that the same kinetic constraints (2.21) also apply to the solution (2.22). However, the functions $\widehat{\beta_{*s}} \exp\left[\frac{X_{*s}}{T_{\|*s}}\right]$, $\mathbf{V}_{*s}$, $U'_{\|*s}$ and $T_{\|*s}$ cannot be directly regarded as *fluid fields*, since they still depend on the single particle velocity via the canonical momentum ψ_{*s} and the particle energy E_s.

2.5 Perturbative analytical expansion

Based on Properties 1-5, in this section we determine an approximate analytical expression for $\widehat{f_{*s}}$ obtained by means of suitable asymptotic expansions. These are carried out in terms of the following two dimensionless parameters:

1) ε_s: which is related to the canonical momentum ψ_{*s}. This is defined as $\varepsilon_s \equiv \left|\frac{L_{\varphi s}}{p_{\varphi s}-L_{\varphi s}}\right| = \left|\frac{M_s R v_\varphi}{\frac{Z_s e}{c}\psi}\right|$, where $v_\varphi \equiv \mathbf{v}\cdot\mathbf{e}_\varphi$ and $L_{\varphi s}$ denotes the species particle angular momentum. We refer to the AD plasma as being *strongly magnetized* if $0 < \varepsilon_s \ll 1$;

2) σ_s: which is related to the total particle energy E_s. This is defined as $\sigma_s \equiv \left|\frac{\frac{M_s}{2}v^2}{Z_s e \Phi_s^{eff}}\right|$, i.e., it is the ratio between the kinetic energy and potential energy of the particle. For bound orbits $E_s < 0$, and so $\sigma_s < 1$.

In the following, we treat ε_s and σ_s as infinitesimals of the same order, with $\varepsilon_s \sim \sigma_s \ll 1$ and then ε_s and σ_s can be used for performing a Taylor expansion of the implicit dependencies contained in the structure functions by setting $\psi_{*s} \cong \psi + O(\varepsilon_s)$ and $E_s \cong Z_s e \Phi_s^{eff} + O(\sigma_s)$ to leading order. This implies that the linear asymptotic expansion for the structure functions, obtained neglecting corrections of $O(\varepsilon_s \sigma_s)$, as well as of $O(\varepsilon_s^k)$ and $O(\sigma_s^k)$, with $k \geq 2$, is

$$\Lambda_{*s} \cong \Lambda_s + (\psi_{*s} - \psi)\left[\frac{\partial \Lambda_{*s}}{\partial \psi_{*s}}\right]_{\psi_{*s}=\psi,\ E_s=Z_s e\Phi_s^{eff}} + \tag{2.25}$$

$$+\left(E_s - Z_s e \Phi_s^{eff}\right)\left[\frac{\partial \Lambda_{*s}}{\partial E_s}\right]_{\psi_{*s}=\psi,\ E_s=Z_s e\Phi_s^{eff}}, \tag{2.26}$$

where

$$\Lambda_s \equiv \Lambda_{*s}|_{\psi_{*s}=\psi E_s=Z_s e \Phi_s^{eff}}. \tag{2.27}$$

To perform the corresponding expansion for $\widehat{f_{*s}}$, we leave unchanged the dependence in terms of the guiding-center canonical momentum $p'_{\varphi s}$, while retaining the leading-order approximation for the magnetic moment only in the linear perturbation terms of Eq.(2.26). Then, it is straightforward to prove that for strongly magnetized and bound plasmas, the following relation holds to leading-order:

$$\widehat{f_{*s}} \cong \widehat{f}_s\left(p'_{\varphi s}, m'_s\right)\left[1 + h^1_{Ds} + h^2_{Ds}\right], \tag{2.28}$$

where h^1_{Ds} and h^2_{Ds} represent the so-called *diamagnetic parts* of $\widehat{f_{*s}}$ (see the definition below). The definitions are then as follows:

First, the leading-order distribution $\widehat{f}_s\left(p'_{\varphi s}, m'_s\right)$ is expressed as

$$\widehat{f}_s\left(p'_{\varphi s}, m'_s\right) = \frac{n_s}{\left(2\pi/M_s\right)^{3/2}\left(T_{\parallel s}\right)^{1/2} T_{\perp s}} \exp\left\{-\frac{M_s\left(\mathbf{v} - \mathbf{V}_s - U'_{\parallel s}\mathbf{b}'\right)^2}{2T_{\parallel s}} - m'_s\frac{B'}{\Delta_{T_s}}\right\} \tag{2.29}$$

which we will here call the *bi-Maxwellian KDF with parallel velocity perturbations.* Here $\frac{1}{\Delta_{T_s}} \equiv \frac{1}{T_{\perp s}} - \frac{1}{T_{\parallel s}}$ is related to the temperature anisotropy, the quantity n_s is related to the number density and is defined as

$$n_s = \eta_s \exp\left[\frac{X_s}{T_{\parallel s}}\right] \tag{2.30}$$

and

$$X_s \equiv \left(M_s\frac{R^2\Omega_s^2}{2} + \frac{Z_s e}{c}\psi\Omega_s - Z_s e\Phi_s^{eff} + \Upsilon'_s\right), \tag{2.31}$$

with η_s denoting the *pseudo-density.* The function Υ'_s is defined as

$$\Upsilon'_s \equiv \frac{M_s U'^2_{\parallel s}}{2}\left(1 + \frac{2\Omega_s}{\xi'_s}\right) - \left(\frac{M_s c\nabla'\psi'\cdot\nabla'\Phi_s'^{eff}}{B'^2} - \frac{Z_s e}{c}\psi'\right)\xi_s. \tag{2.32}$$

Note that $\mathbf{V}_s = \mathbf{e}_\varphi R\Omega_s$ and $U'_{\parallel s} = \frac{I'}{B'}\xi_s$ define, respectively, the leading-order azimuthal flow velocity and the leading-order GK parallel velocity perturbation of the fluid. Then, the following kinetic constraints are implied from (2.21), to leading-order, for the structure functions:

$$\Lambda_s = \Lambda_s\left(\psi, Z_s e\Phi_s^{eff}\right). \tag{2.33}$$

Second, the diamagnetic parts h^1_{Ds} and h^2_{Ds} of $\widehat{f_{*s}}$, due respectively to the expansions of the canonical momentum and the total energy, are given by

$$h^1_{Ds} = \left\{\frac{cM_sR}{Z_se}\left[Y_1 + Y_3\right] + \frac{M_sR}{T_{\parallel s}}Y_2\right\}\left(\mathbf{v}\cdot\widehat{e}_\varphi\right), \tag{2.34}$$

$$h^2_{Ds} = \frac{M_s}{2Z_se}\left\{Y_4 - \frac{Z_se}{T_{\parallel s}}Y_5 + \frac{p'_{\varphi s}\xi_s}{T_{\parallel s}}C_{5s}\right\}v^2. \tag{2.35}$$

Here Y_i, $i=1,5$, is defined as

$$Y_1 \equiv \left[A_{1s} + A_{2s}\left(\frac{H_s}{T_{\parallel s}} - \frac{1}{2}\right) - \mu_s' \widehat{A_{4s}}\right], \tag{2.36}$$

$$Y_2 \equiv \Omega_s\left[1 + \psi A_{3s}\right], \tag{2.37}$$

$$Y_3 \equiv \left[\frac{p_{\varphi s}'\xi_s}{T_{\parallel s}} A_{5s} - A_{2s}\frac{p_{\varphi s}'\xi_s}{T_{\parallel s}}\right], \tag{2.38}$$

$$Y_4 \equiv \left[C_{1s} + C_{2s}\left(\frac{H_s}{T_{\parallel s}} - \frac{1}{2}\right) - \mu_s' \widehat{C_{4s}}\right], \tag{2.39}$$

$$Y_5 \equiv \left[1 + \frac{\Omega_s \psi}{c} C_{3s}\right], \tag{2.40}$$

where $H_s = E_s - \frac{Z_s e}{c}\psi_s\Omega_s$ and the following definitions have been introduced: $A_{1s} \equiv \frac{\partial \ln \beta_s}{\partial \psi}$, $A_{2s} \equiv \frac{\partial \ln T_{\parallel s}}{\partial \psi}$, $A_{3s} \equiv \frac{\partial \ln \Omega_s}{\partial \psi}$, $\widehat{A_{4s}} \equiv \frac{\partial \widehat{\alpha_s}}{\partial \psi}$, $A_{5s} \equiv \frac{\partial \ln \xi_s}{\partial \psi}$ and $C_{1s} \equiv \frac{\partial \ln \beta_s}{\partial \Phi_s^{eff}}$, $C_{2s} \equiv \frac{\partial \ln T_{\parallel s}}{\partial \Phi_s^{eff}}$, $C_{3s} \equiv \frac{\partial \ln \Omega_s}{\partial \Phi_s^{eff}}$, $\widehat{C_{4s}} \equiv \frac{\partial \widehat{\alpha_s}}{\partial \Phi_s^{eff}}$, $C_{5s} \equiv \frac{\partial \ln \xi_s}{\partial \Phi_s^{eff}}$.

We should make a number of comments here:

1) The functional forms of the leading-order number density, the parallel and azimuthal flow velocities and the temperatures carried by the bi-Maxwellian KDF, are naturally determined in terms of ψ and $Z_s e \Phi_s^{eff}$. The effective potential Φ_s^{eff} is generally a function of the form $\Phi_s^{eff} = \Phi_s^{eff}(\mathbf{x}, \varepsilon_M^k t)$, with $\mathbf{x} = (R, z)$, since generally neither the gravitational potential nor the electrostatic potential are expected to be flux functions in the present case. Hence, in magnetic coordinates, it follows that the structure functions are of the form $\Lambda_s \equiv \overline{\Lambda_s}\left(\psi, \vartheta, \varepsilon_M^k t\right)$;

2) The kinetic constraints imply a precise relationship between the magnitude of the temperature anisotropy and the guiding-centre magnetic field at two different spatial locations. In fact, the quantity $\frac{B'}{\Delta_{T_s}}$ in the KDF is necessarily an adiabatic invariant. To leading-order in the GK expansion, this implies that the asymptotic equation

$$\frac{[\Delta_{T_s}]_2}{[\Delta_{T_s}]_1} \cong \frac{[B]_2}{[B]_1} \tag{2.41}$$

must hold identically for any two arbitrary positions "1" and "2", with $([\Delta_{T_s}]_1, [B]_1)$ and $([\Delta_{T_s}]_2, [B]_2)$ denoting the temperature anisotropy and the magnitude of the magnetic field at these positions respectively.

3) The coefficients A_{is} and C_{is}, $i=1,5$, can be identified with effective *thermodynamic forces*: A_{5s} carries the contribution of the parallel velocity perturbation, while the C_{is}, $i=1,5$, are due to the energy dependence contained in the structure functions;

4) We stress that the energy dependence contained in the kinetic constraints is non trivial and cannot be included simply by redefining the structure functions (e.g., by transforming the magnetic coordinates). In fact, besides modifying the leading order structure functions (see point 1 above), it gives rise to the new diamagnetic contribution h_{Ds}^2. Eq.(2.28) is therefore a generalization of the analogous solution obtained in

Ref.(17), which also appears in standard tokamak transport theory (15), where the relevant structure functions were considered solely as flux functions. Including the effect of the parallel velocity perturbations gives rise to contributions to h_{Ds}^2 which are even with respect to u';

5) In the analytical expansion, we have assumed that the scale-length L is of the same order in ε_s as the characteristic scale-lengths associated with the structure functions;

6) We have performed the analysis distinguishing between the different plasma species. Since this is an asymptotic estimation, the analytical expansion can be different for ions and electrons, particularly for the terms appearing in the diamagnetic part, depending on the relative magnitudes of the parameters ε_s and σ_s. On the other hand, because of the double expansion and the energy dependence, the asymptotic solution for the two species can hold also in different spatial domains;

7) The KDF $\widehat{f}_s\left(p'_{\varphi s}, m'_s\right)$ also satisfies Property 2: namely, it is only defined in the subset of phase-space where the parallel velocity $|u'|$ is a real function. It is therefore suitable for properly describing particle trapping;

8) Finally, we stress that the QSA-KDF (2.20) obtained here, reduces asymptotically to the expression reported in Ref.(17) when the following conditions are satisfied: a) parallel velocity perturbations are ignored, namely the structure function ξ_{*s} is set to zero; b) closed nested magnetic surfaces are considered; c) large aspect ratio ordering, $1/\delta \gg 1$, is invoked. In this case, the effective potential is solely a flux-function to leading order, while the diamagnetic contribution h_{Ds}^2 can be shown to be of higher order than h_{Ds}^1.

2.6 Moment equations

In this section we discuss the connection between the kinetic treatment presented here and the corresponding fluid approach, obtained by describing the plasma in terms of a suitable set of fluid fields. The latter can in principle be specified as required by experimental observations and identified with the relevant physical observables. Important practical aspects of the present theory concern the explicit evaluation of the fluid fields associated with the QSA-KDF, and the conditions for validity of the relevant moment equations.

For definiteness, let us require that:

1. The KDF, the EM fields $\{\mathbf{E}, \mathbf{B}\}$ and the corresponding EM potentials $\{\Phi, \mathbf{A}\}$ are all exactly axisymmetric and, moreover, stationary in an asymptotic sense, i.e. neglecting corrections of $O(\varepsilon_M^{n+1})$;

2. The KDF is identified with the QSA-KDF $\widehat{f_{*s}}\left(E_s, \psi_{*s}, m'_s\right)$ which, by assumption, is required to be an adiabatic invariant of $O(\varepsilon_M^{n+1})$. By construction $\widehat{f_{*s}}\left(E_s, \psi_{*s}, m'_s\right)$

is a solution of the *asymptotic Vlasov equation*

$$\frac{1}{\Omega'_{cs}}\frac{d}{dt}\ln\widehat{f_{*s}} = 0 + O\left(\varepsilon_M^{n+1}\right). \tag{2.42}$$

This equation holds by definition up to infinitesimals of $O\left(\varepsilon_M^{n+1}\right)$, where n is an arbitrary positive integer.

As a basic consequence of these assumptions, the stationary fluid equations following from the Vlasov equation are necessarily all identically satisfied in an asymptotic sense, i.e., again neglecting corrections of $O\left(\varepsilon_M^{n+1}\right)$. In fact if $Z(\mathbf{x})$ is an arbitrary weight function, identified for example with $Z = \left(1, \mathbf{v}, v^2\right)$, then the generic moment of Eq.(2.42) is:

$$\int_{\Gamma_u} d^3v Z\frac{d}{dt}\widehat{f_{*s}} = 0 + O\left(\varepsilon_M^{n+1}\right), \tag{2.43}$$

where Γ_u denotes the appropriate velocity space of integration. Using the chain rule, this can be written as

$$\int_{\Gamma_u} d^3vZ\left\{\frac{d\psi_*}{dt}\frac{\partial\widehat{f_{*s}}}{\partial\psi_*} + \frac{dE_s}{dt}\frac{\partial\widehat{f_{*s}}}{\partial E_s} + \frac{dm'_s}{dt}\frac{\partial\widehat{f_{*s}}}{\partial m'_s}\right\} = 0 + O\left(\varepsilon_M^{n+1}\right). \tag{2.44}$$

On the other hand, Eq.(2.43) can also be represented as

$$\int_{\Gamma_u} d^3v\left\{\frac{d}{dt}\left[Z\widehat{f_{*s}}\right] - \widehat{f_{*s}}\frac{d}{dt}Z\right\} = 0 + O\left(\varepsilon_M^{n+1}\right), \tag{2.45}$$

which recovers the usual form of the velocity-moment equations in terms of suitable (and *uniquely defined*) fluid fields. For $Z = (1, \mathbf{v})$ one obtains, in particular, that the species continuity and linear momentum fluid equations are satisfied identically up to infinitesimals of $O\left(\varepsilon_M^{n+1}\right)$:

$$\nabla\cdot\left(n_s^{tot}\mathbf{V}_s^{tot}\right) = 0 + O\left(\varepsilon_M^{n+1}\right), \tag{2.46}$$

$$M_s\mathbf{V}_s^{tot}\cdot\nabla\mathbf{V}_s^{tot} + \nabla\cdot\underline{\underline{\Pi}}_s^{tot} + Z_sen_s^{tot}\nabla\Phi_s^{eff} - \frac{Z_se}{c}\mathbf{V}_s^{tot}\times\mathbf{B} = 0 + O\left(\varepsilon_M^{n+1}\right). \tag{2.47}$$

Here the notation is standard. In particular the following velocity moments of the QSA-KDF can be introduced:

a) *species number density*

$$n_s^{tot} \equiv \int_{\Gamma_u} d^3v\widehat{f_{*s}}; \tag{2.48}$$

b) *species flow velocity*

$$\mathbf{V}_s^{tot} \equiv \frac{1}{n_s^{tot}}\int_{\Gamma_u} d^3v\mathbf{v}\widehat{f_{*s}}; \tag{2.49}$$

c) *species tensor pressure*

$$\underline{\underline{\Pi}}_s^{tot} \equiv \int_{\Gamma_u} d^3v M_s \left(\mathbf{v} - \mathbf{V}_s^{tot}\right) \left(\mathbf{v} - \mathbf{V}_s^{tot}\right) \widehat{f_{*s}}. \tag{2.50}$$

It is worth remarking here that *the velocity moments are unique once the QSA-KDF* $\widehat{f_{*s}}$ [see Eq.(2.15)] *is prescribed in terms of the structure functions* Λ_{*s}. On the other hand, as a result of Eqs.(2.42) and (2.43), it follows that the stationary fluid moments calculated in terms of the QSA-KDF $\widehat{f_{*s}}$ are identically solutions of the corresponding stationary fluid moment equations.

Let us now illustrate explicitly how it is possible to carry out such a calculation within the present theory. The evaluation of the previous fluid fields can be made by using the asymptotic analytical solution of the QSA-KDF $\widehat{f_{*s}}$ derived in the previous section and given by Eq.(2.28). For example, adopting this expansion in the limit of strongly magnetized plasmas, from Eq.(2.48) the species number density becomes

$$n_s^{tot} \cong \int_{\Gamma_u} d^3v \left\{ \widehat{f}_s \left[1 + h_{Ds}^1 + h_{Ds}^2\right] \right\}, \tag{2.51}$$

in which the diamagnetic corrections to the bi-Maxwellian KDF $\widehat{f}_s$ are polynomial functions of the particle velocity. Analogous expressions can also be obtained in a straightforward way for the remaining fluid moments. The expansion procedure for $\widehat{f_{*s}}$ can in principle be performed to higher order, allowing for the analytical computation of the corresponding quasi-stationary fluid fields and the determination of the relevant kinetic closure conditions for the stationary moment equations. In the present context we stress that the theory allows the treatment of multiple-species plasmas including, in particular, particle trapping phenomena. This is taken into account by proper definition of the velocity sub-space Γ_u in which the integrations are performed. In fact, charged particles in both open and closed configurations can have mirror points (TPs and BPs) or be PPs, which are free to stream through the boundaries of the domain. These populations give different contributions to the relevant fluid fields and therefore require separate statistical treatments. The explicit calculation of fluid fields requires also a preliminary *inverse transformation* representing all quantities in terms of the actual particle positions (the FLR expansion, see Eq.(2.4)). This introduces further correction terms of order ε_M^k, $k \geq 1$, into the final analytical expressions. In contrast with the case discussed in the previous Chapter, here we expect these FLR corrections to be non-negligible due to the requirement $\varepsilon_{M,s} \lesssim \varepsilon_s$ holding for open-field configurations.

2.7 Slow time-evolution of the axisymmetric QSA-KDF

In this section we investigate the temporal evolution of the axisymmetric QSA-KDF. Two different issues must be addressed: giving an estimate of the maximum time interval over which the QSA-KDF can be regarded as an asymptotic stationary solution;

and determining the solution of the Vlasov equation for time intervals longer than the equilibrium one.

For our explicit determination of the time evolution of the QSA-KDF, we make the following assumptions:

1) That the plasma can be treated as a continuous medium in the kinetic description. This requires that the species kinetic equation holds on time and spatial scales which are much longer than the corresponding Langmuir characteristic times and Debye lengths;

2) That we are considering timescales much shorter than the species characteristic collisional time τ_C, so that it is appropriate to use the Vlasov equation;

3) That the species KDF and the EM fields vary slowly in time and space with respect to the corresponding Larmor times and radii, so that the GK description is valid;

4) That the EM and gravitational fields vary slowly in time, so that the total energy E_s is an adiabatic invariant. In particular, we require:

$$\frac{d}{dt}E_s = Z_s e \frac{\partial}{\partial t}\Phi_s^{eff} - \frac{Z_s e}{c}\mathbf{v}\cdot\frac{\partial}{\partial t}\mathbf{A}, \tag{2.52}$$

which implies that $\tau_{Ls}\frac{d}{dt}\ln E_s \sim O\left(\varepsilon_{M,s}^{n+1}\right)$, with $n \geq 0$. Consistently with the properties of solution (2.20), we take $n = 0$ as a specific case. Note that from here on, $\tau_{Ls} \equiv \frac{1}{\Omega'_{cs}}$ will denote the species characteristic time associated with the Larmor rotation (the *Larmor rotation time*). Since $\frac{d}{dt}\Omega_{*s} = \frac{dE_s}{dt}\frac{\partial}{\partial E_s}\Omega_{*s}$, it follows that

$$\frac{d}{dt}H_{*s} = \frac{d}{dt}E_s\left[1 - \frac{Z_s e}{c}\psi_{*s}\frac{\partial}{\partial E_s}\Omega_{*s}\right]; \tag{2.53}$$

5) That the magnetic moment m'_s and the guiding-center canonical momentum $p'_{\varphi s}$ can be taken as adiabatic invariants of $O\left(\varepsilon_{M,s}^{j}\right)$, with $j \geq n$. The ordering $\tau_{Ls}\frac{d}{dt}\ln p'_{\varphi s} \sim O\left(\varepsilon_{M,s}^{2}\right)$ holds for the leading-order expression for $p'_{\varphi s}$ adopted here as follows from Eq.(2.11) and the fact that, by definition, higher-order correction terms, $\Delta p'_{\varphi s}$, to $p'_{\varphi s}$ are independent of the gyrophase angle ϕ'. In fact, denoting by $\mathcal{L}_s'^{(2)}$ the second-order GK Lagrangian, $\Delta p'_{\varphi s}$ can be estimated as $\Delta p'_{\varphi s} = \frac{\partial}{\partial \cdot \varphi}\left[\mathcal{L}_s'^{(2)} - \mathcal{L}_s'^{(1)}\right]$ where, by construction, $\mathcal{L}_s'^{(1)}$ and $\mathcal{L}_s'^{(2)}$ are both gyrophase independent. Note that the assumption made here requires the construction of a higher-order GK theory in order to correctly determine m'_s to the required order in the Larmor-radius expansion.

The time evolution of the QSA-KDF is in principle determined by two different mechanisms: the explicit time variation of the EM and gravitational fields, and the time variation of the guiding-center adiabatic invariants. However, the choice of the orderings in 4) and 5) above, allows the time dependence produced only by the EM and gravitational fields to be singled out.

When assumptions 1) - 5) above hold, it follows that $\tau_{Ls}\frac{d}{dt}\ln\widehat{f_{*s}} = 0 + O\left(\varepsilon_{M,s}^{n+1}\right)$, with $n \geq 0$ being determined by Eq.(2.52). Then, ignoring higher-order corrections

$$\frac{d}{dt}\ln \widehat{f_{*s}} = \frac{dE_s}{dt} S_s, \tag{2.54}$$

where

$$\begin{aligned} S_s \quad &\equiv \quad \frac{\partial \ln \widehat{\beta_{*s}}}{\partial E_s} - m_s' \frac{\partial \widehat{\alpha_{*s}}}{\partial E_s} + \left(\frac{H_{*s}}{T_{\parallel *s}} - \frac{1}{2} + \frac{p'_{\varphi s}\xi_{*s}}{T_{\parallel *s}} \right) \frac{\partial \ln T_{\parallel *s}}{\partial E_s} + \\ &+ \frac{p'_{\varphi s}}{T_{\parallel *s}} \frac{\partial \xi_{*s}}{\partial E_s} - \frac{1}{T_{\parallel *s}} \left(1 - \frac{Z_s e}{c} \psi_{*s} \frac{\partial \Omega_{*s}}{\partial E_s} \right), \end{aligned} \tag{2.55}$$

and so the solution $\widehat{f_{*s}}$ can be regarded as an exact kinetic equilibrium for all times $t \geq 0$ such that

$$\tau_{Ls} \ll t \ll t_{\text{sup}} \ll \tau_C, \tag{2.56}$$

where $t_{\text{sup}} \equiv \frac{\tau_{Ls}}{\varepsilon_{M,s}^{n+1}}$. Within the scope of the above assumptions, we now determine the dynamical evolution equation which describes the slow time-evolution of the QSA-KDF $\widehat{f_{*s}}$, for time intervals such that t is within

$$t_{\text{sup}} \ll t \ll \tau_C. \tag{2.57}$$

In analogy with Ref.(15), we denote by

$$f_s \equiv \widehat{f_{*s}} + g_s' \tag{2.58}$$

the exact solution of the collisionless Vlasov equation, for which $\frac{d}{dt} f_s = 0$. Here g_s' is referred to as the *reduced KDF*. Following the discussion in Ref.(15), regarding the evaluation of $\frac{d}{dt} g_s'$: it is straightforward to prove that g_s' is gyrophase independent, to lowest order, in the sense that $\frac{\partial g_s'}{\partial \phi'} = 0$. Therefore, identifying the GK variables with the set $\mathbf{z} \equiv \left(\vartheta', \varphi, p'_{\varphi s}, \mathcal{H}_s'^{(1)}, m_s', \phi' \right)$, we shall assume that g_s' is axisymmetric and of the form $g_s' = g_s' \left(\vartheta', p'_{\varphi s}, \mathcal{H}_s'^{(1)}, m_s', t \right)$. The gyro-averaged dynamical equation for g_s' can then be obtained to next order by introducing the gyro-average operator $\langle ... \rangle_{\phi'}$ defined as

$$\langle ... \rangle_{\phi'} \equiv \frac{1}{2\pi} \int_0^{2\pi} (...) \, d\phi', \tag{2.59}$$

with the operation being performed while all of the other GK variables are held fixed (15). It follows that, to leading-order, $\frac{d}{dt} g_s' \cong \frac{\partial}{\partial t} g_s' + \dot{\vartheta}' \frac{\partial}{\partial \vartheta'} g_s'$, where the time variation of the guiding-center magnetic coordinate ϑ' is given by $\dot{\vartheta}' \cong \dot{\mathbf{r}}' \cdot \nabla' \vartheta' \cong \left[u' \mathbf{b}' + \mathbf{V}'_{eff} \right] \cdot \nabla' \vartheta'$ to leading-order, with the equation of motion for $\dot{\mathbf{r}}'$ following from the gyrokinetic Lagrangian [e.g. from the leading-order Eq.(2.6)]. Then, consistently with these assumptions and ignoring higher-order corrections, it is found that

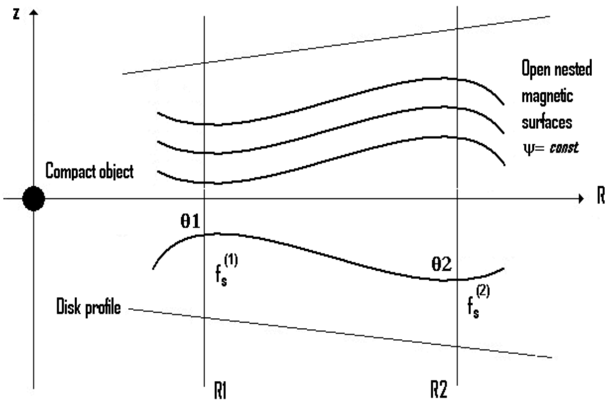

Figure 2.2: Schematic view of the configuration geometry (not to scale) and meaning of the notation.

the GK reduced KDF g_s' obeys the *reduced GK-Vlasov equation*

$$\frac{\partial}{\partial t} g_s' + \cdot\vartheta' \frac{\partial}{\partial \vartheta'} g_s' = -\left\langle \widehat{f_{*s}} S_s \frac{dE_s}{dt} \right\rangle_{\phi'}. \tag{2.60}$$

It follows that, to leading-order

$$\left\langle \widehat{f_{*s}} S_s \frac{dE_s}{dt} \right\rangle_{\phi'} \cong f_s' \left\langle S_s \frac{dE_s}{dt} \right\rangle_{\phi'}. \tag{2.61}$$

Denoting $\widehat{f_{*s}} \equiv F_s\left(\psi_{*s}, H_{*s}, p_{\varphi s}', m_s'\right)$, f_s' is then defined as $f_s' \equiv F_s\left(\frac{c}{Z_s e} p_{\varphi s}', \mathcal{H}_s'^{(1)}, p_{\varphi s}', m_s'\right)$. The remaining gyrophase average in the last equation can be performed in a straightforward way using Eqs.(2.52) and (2.55).

Eq.(2.60) clearly also holds in the time interval (2.56), and so it determines the slow time-evolution for all times $\tau_{Ls} \ll t \ll \tau_C$. For consistency, the non-stationary Maxwell equations must also be solved with the same accuracy. Eq.(2.60) must be supplemented by appropriate boundary conditions: for open magnetic surfaces with boundaries prescribed on a given magnetic surface $\psi = const.$, at $\vartheta = \vartheta_1$ and $\vartheta = \vartheta_2$, with $\vartheta_1 < \vartheta_2$ and ϑ_1, ϑ_2 representing the internal and external boundaries, these are defined respectively either by prescribing $f_s(\vartheta_1) = f_s^{(1)}$ or $f_s(\vartheta_2) = f_s^{(2)}$ (see Fig.2.2 for a schematic view of the configuration geometry and the meaning of the notation). Both $f_s^{(1)}$ and $f_s^{(2)}$ are necessarily of the form (2.58) but their moments remain arbitrary in principle.

2.8 Angular momentum

In this section we discuss the implications of the kinetic treatment for the law of conservation of fluid angular momentum. For doing this, we first define the *species fluid canonical toroidal momentum* as

$$L_{cs}^{tot} \equiv \frac{1}{n_s^{tot}} \int_{\Gamma_u} d^3v \frac{Z_s e}{c} \psi_{*s} \widehat{f_{*s}}. \tag{2.62}$$

Consider then the corresponding conservation law for the species total canonical momentum. This can be recovered by setting

$$\int_{\Gamma_u} d^3v \frac{d}{dt} \left[\psi_{*s} \widehat{f_{*s}} \right] = 0. \tag{2.63}$$

In the equilibrium case this implies the *species fluid angular momentum conservation law*

$$\nabla \cdot \left[R^2 \underline{\underline{\Pi}}_s^{tot} \cdot \nabla\varphi + n_s^{tot} \mathbf{V}_s^{tot} L_s^{tot} \right] + \frac{Z_s e}{c} \nabla\psi \cdot n_s^{tot} \mathbf{V}_s^{tot} = 0 \tag{2.64}$$

for the species angular momentum

$$L_s^{tot} \equiv M_s R^2 \mathbf{V}_s^{tot} \cdot \nabla\varphi. \tag{2.65}$$

In Eq.(2.64) a key role is played by the divergence of the species pressure tensor. For strongly-magnetized plasmas, using the leading-order expression (see previous Chapter), this is given by:

$$\nabla \cdot \underline{\underline{\Pi}}_s^{tot} \cong \nabla p_{\perp s} + \mathbf{bB} \cdot \nabla \left(\frac{p_{\parallel s} - p_{\perp s}}{B} \right) - \Delta p_s \mathbf{Q}, \tag{2.66}$$

where $\mathbf{Q} \equiv \left[\mathbf{bb} \cdot \nabla \ln B + \frac{4\pi}{cB} \mathbf{b} \times \mathbf{J} - \nabla \ln B \right]$ and $\Delta p_s \equiv \left(p_{\parallel s} - p_{\perp s} \right)$. It is clear that in this case $\nabla \cdot \underline{\underline{\Pi}}_s^{tot}$ has non-vanishing components in arbitrary spatial directions, including the azimuthal direction along $\nabla\varphi$.

For a single species, the total canonical momentum L_{cs}^{tot} and the total angular momentum L_s^{tot} in general differ because of the contribution of the magnetic part proportional to the flux function ψ. However, a different conclusion can be drawn for the corresponding canonical momentum density $n_s^{tot} L_{cs}^{tot}$ and angular momentum density $n_s^{tot} L_s^{tot}$. If one considers summation over species for both these quantities and imposes the *quasi-neutrality condition*

$$\sum_s Z_s e n_s^{tot} = 0, \tag{2.67}$$

then one obtains the identity

$$\sum_s n_s^{tot} L_s^{tot} \equiv \sum_s n_s^{tot} L_{cs}^{tot}. \tag{2.68}$$

We next investigate the consequences of Eq.(2.64) for the dynamical properties of collisionless plasmas. Note the following aspects:

1) In the usual interpretation the directional derivative of L_s^{tot} along the flow velocity $\mathbf{V}_s^{tot}$ vanishes. However, for strongly-magnetized plasmas Eq.(2.64) shows that equilibrium configurations are possible in which this is generally non-zero. This arises because of the non-isotropic pressure tensor and the poloidal components of the flow velocity which, in turn, are consequences of temperature anisotropy, the first-order energy-correction and FLR-diamagnetic effects which are not included in standard MHD treatments.

2) According to Eq.(2.64), spatial variation in the species angular momentum implies the possibility of having quasi-stationary radial matter flows in the disc without departing from the unperturbed equilibrium solution. These can correspond either to local outflows or inflows; both can be described consistently within the present kinetic solution for open magnetic field lines in strongly-magnetized plasmas. Local inflows and outflows can occur independently and are described consistently by their respective quasi-stationary KDFs. Radial flows arise due both to the parallel velocities $U_{\|s}$ and to the kinetic effects driven by the first-order energy-correction and FLR-diamagnetic effects. Therefore, species radial flows appear necessarily together with a non-isotropic pressure tensor and a non-vanishing toroidal magnetic field.

2.9 The Ampere equation and the kinetic dynamo

In this section we apply the kinetic solution for the QSA-KDF to discuss the properties of the Ampere equation and the implications for the self-generation of magnetic field by the quasi-stationary AD collisionless plasma. We refer here to this phenomenon as a *quasi-stationary kinetic dynamo effect*. Generalizing the treatment presented in the previous Chapter, the Ampere equation for the self magnetic field becomes:

$$\nabla \times \mathbf{B}^{self} = \frac{4\pi}{c}\left(\mathbf{J}^T + \mathbf{J}^B + \mathbf{J}^P\right), \tag{2.69}$$

where distinction is made between the contributions arising from PPs, BPs and TPs, denoting the corresponding total current densities as $\mathbf{J}^T, \mathbf{J}^B$ and $\mathbf{J}^P$. As described above, these fluid fields can be calculated in closed analytic form to the required order, by using the asymptotic expansion of the QSA-KDF. This gives:

$$\mathbf{J}^l \equiv \sum_{s=i,e} \mathbf{J}_s^l = \sum_{s=i,e} Z_s e \int_{\Gamma_u^l} d^3v \mathbf{v} \widehat{f_{*s}} \cong \sum_{s=i,e} Z_s e \int_{\Gamma_u^l} d^3v \mathbf{v} \left\{ \widehat{f}_s \left[1 + h_{Ds}^1 + h_{Ds}^2\right] \right\} \tag{2.70}$$

for $l = T, B, P$ and where Γ_u^l denotes the appropriate velocity space domain of integration for trapped, bouncing and passing particles respectively. For convenience of notation, in the following we shall denote as $\mathbf{J} \equiv \mathbf{J}^T + \mathbf{J}^B + \mathbf{J}^P$ the total current density entering Eq.(2.69). It is possible to prove that in the case of open magnetic surfaces the total current density $\mathbf{J}$ in general has non-vanishing components along all of the three directions identified by the set of magnetic coordinates $(\psi, \varphi, \vartheta)$. Hence, $\mathbf{J}$ can be represented as

$$\mathbf{J} = (J_\psi \nabla\vartheta \times \nabla\varphi, J_\varphi \nabla\varphi, J_\vartheta \nabla\psi \times \nabla\varphi) \,. \tag{2.71}$$

Let us now proceed with the study of the Ampere equation. The toroidal component of Eq.(2.69) gives, as usual, the generalized Grad-Shafranov equation for the poloidal flux function ψ_p:

$$\Delta^* \psi_p = -\frac{4\pi}{c} J_\varphi, \tag{2.72}$$

where the elliptic operator Δ^* is defined as $\Delta^* \equiv R^2 \nabla \cdot \left(R^{-2}\nabla\right)$. The remaining terms of Eq.(2.69) along the directions $\nabla\vartheta \times \nabla\varphi$ and $\nabla\psi \times \nabla\varphi$ give two equations for the toroidal component of the magnetic field I/R. These are respectively

$$\frac{\partial I}{\partial \psi} = \frac{4\pi}{c} J_\vartheta, \tag{2.73}$$

$$\frac{\partial I}{\partial \vartheta} = \frac{4\pi}{c} J_\psi, \tag{2.74}$$

yielding the constraint

$$\frac{\partial J_\psi}{\partial \psi} = \frac{\partial J_\vartheta}{\partial \vartheta} \tag{2.75}$$

which is a solubility condition for the structure functions. In this regard we notice that as a consequence of the kinetic constraints the function I in the previous equations is of the form $I(\psi, \vartheta, \varepsilon_M^k t)$, i.e., it is not a ψ flux-function. Therefore, the solubility condition (2.75) can always be satisfied. Eqs.(2.72)-(2.75) therefore provide consistent solutions for both poloidal and toroidal self magnetic fields in a collisionless AD plasma.

It is remarkable that in principle all of the populations of charged particles (PPs, BPs and TPs) can contribute to the generation of the toroidal magnetic field. More precisely, the following mechanisms can be involved:

#1) FLR and diamagnetic effects, driven by temperature anisotropy;

#2) Parallel velocity perturbations $U'_{\|*s}$, which generate a poloidal flow velocity, giving a related contribution to the electric current density through J_ψ and J_ϑ;

#3) FLR effects driven by the remaining thermodynamic forces. These contributions are produced by the diamagnetic KDF and arise because of the asymptotic ordering introduced here;

#4) Gyrophase-dependent contributions driven by the same thermodynamic forces. These are originated by the inverse GK transformation of the guiding-center quantities in the QSA-KDF.

As discussed above, contributions #2 and #4 were negligible under the circumstances discussed in the previous Chapter. Therefore they should be considered as characteristic features of open-field configurations.

We refer to the mechanism of self-generation of both poloidal and toroidal magnetic fields as a *quasi-stationary kinetic dynamo effect*. In contrast to customary MHD treatments, this type of dynamo effect occurs *in the absence of possible instabilities or turbulence phenomena*. In particular, in the case of TPs, the self generation of toroidal field could take place even *without any net accretion* in the domain of interest, in presence of open magnetic field lines. This phenomenon is analogous to that treated in the previous Chapter for closed-field configurations (i.e., the diamagnetic-driven kinetic dynamo). In particular, the toroidal field is associated with the existence of torques which cause redistribution of angular momentum, producing radial inflows and outflows of disc material. As a consequence, various scenarios can be envisaged in which stationary radial flows and kinetic dynamos are present in AD plasmas, both affected by processes of type #1-#4.

2.10 Quasi-stationary accretion flow

Let us now consider specifically the application of the kinetic solution developed here to the investigation of the accretion process in AD plasmas.

The inward accretion flow in ADs is usually "slow" in comparison with the characteristic Larmor time τ_{Ls}. For example, AD plasmas with $B \sim 10^1 - 10^8 G$ have Hydrogen-ion Larmor rotation times in the range $\tau_{Li} \sim 10^{-4} - 10^{-11} s$ which is shorter than the dynamical timescale at most relevant radii. For typical plasma densities and temperatures in the range $n_i \sim 10^9 - 10^{11} cm^{-3}$ and $T_i \sim 1 - 10 keV$, the (Spitzer) ion collision time (below which the plasma can be considered collisionless) is in the range $\tau_C \sim 10^2 - 10^5 s$ (the upper value corresponding to high temperature and low density). Independent of the physical origin of the accretion process, we can therefore expect that the present theory correctly describes phenomena occurring on all time-scales in the range $\tau_{Li} < t < \tau_C$.

We next determine the local poloidal and radial flow velocities for the various particle sub-species. By definition, these are given by

$$V_{ps} \equiv \mathbf{V}_s \cdot \mathbf{e}_p = \sum_{sub-species} \frac{1}{n_s^{tot}} \int_{\Gamma_u^l} d^3v \left[\mathbf{v} \cdot \mathbf{e}_p\right] \widehat{f_{*s}} \left[1 + g'_s\right], \tag{2.76}$$

$$V_{Rs} \equiv \mathbf{V}_s \cdot \mathbf{e}_R = \sum_{sub-species} \frac{1}{n_s^{tot}} J^l_{Rs}, \tag{2.77}$$

$$J^l_{Rs} \equiv \int_{\Gamma_u^l} d^3v \left[\mathbf{v} \cdot \mathbf{e}_R\right] \widehat{f_{*s}} \left[1 + g'_s\right], \tag{2.78}$$

where $\mathbf{e}_p \equiv \frac{\nabla\psi \times \nabla\varphi}{|\nabla\psi \times \nabla\varphi|}$ and $\mathbf{e}_R \equiv \frac{\nabla R}{|\nabla R|}$ and the summations are performed over the particle sub-species for $l = T, B, P$. We stress that the velocity-space integrals indicated above must contain the contributions from PPs, BPs and TPs and so $J_{Rs} = J^T_{Rs} + J^B_{Rs} + J^P_{Rs}$, where J^T_{Rs}, J^B_{Rs} and J^P_{Rs} are the corresponding mass currents.

We are interested in situations where there is a *net radial accretion flow* i.e. where the average radial mass current $\langle\langle J_{Rs}\rangle\rangle \equiv \frac{1}{z_2 - z_1} \int_{z_1}^{z_2} J_{Rs} dz$ (with z_1 and z_2 being suitably prescribed) is negative (inward flow). There are local contributions to $\langle\langle J_{Rs}\rangle\rangle$ from TPs, BPs and PPs, but the overall accretion flow is mainly associated with PPs.

Note also the following basic features involved in the accretion process. The ratio $\frac{\xi_s}{T_{\|s}}$ is approximately constant due to the kinetic constraints, and so

$$\frac{[\xi_s]_1}{[\xi_s]_2} \cong \frac{\left[T_{\|s}\right]_1}{\left[T_{\|s}\right]_2} \tag{2.79}$$

for any two arbitrary positions "1" and "2" prescribed in terms of the magnetic coordinates (ψ, ϑ). Then consider the case $\varepsilon_{M,s} \ll \varepsilon_s$, which allows one to approximate the guiding-centre quantities with the expression for them evaluated at the particle position. Assuming that $I = I(\psi)$ (see Ref.(1)) and considering the two positions (ψ, ϑ_1) and (ψ, ϑ_2) on the same flux surface, the *kinetic accretion law* follows

$$\frac{U_{\|s}(\psi, \vartheta_1)}{U_{\|s}(\psi, \vartheta_2)} \cong \frac{B(\psi, \vartheta_2)}{B(\psi, \vartheta_1)} \frac{T_{\|s}(\psi, \vartheta_1)}{T_{\|s}(\psi, \vartheta_2)}. \tag{2.80}$$

In this case, under the same assumptions, it follows from the continuity equation that the ratio of the corresponding species number densities must vary on a given ψ-surface according to the following relation:

$$\frac{n_s(\psi, \vartheta_1)}{n_s(\psi, \vartheta_2)} \cong \frac{B^2(\psi, \vartheta_1)}{B^2(\psi, \vartheta_2)} \frac{T_{\|s}(\psi, \vartheta_2)}{T_{\|s}(\psi, \vartheta_1)}. \tag{2.81}$$

Therefore, on a given ψ-surface:

1) the species parallel flow velocity increases with the parallel temperature while decreasing with respect to the magnitude of the magnetic field;

2) the species number density instead increases with the magnetic pressure and decreases with the parallel temperature.

The physical interpretation for both of these is clear: higher magnetic pressure slows down the matter accretion rate while increasing the number density, whereas higher parallel temperature corresponds to higher radial fluid mobility, thus decreasing the local species number density.

To summarize: the present theory provides a possible new collisionless physical mechanism giving an equilibrium accretion process in AD plasmas. In particular, we note that:

1) Only strongly-magnetized plasmas with open magnetic surfaces can sustain these equilibrium accretion flows.

2) The primary source of this equilibrium accretion flow mechanism is the appearance of equilibrium radial flows driven by temperature anisotropies and phase-space anisotropies. These are directly connected with the existence of non-isotropic species pressure tensors, which in turn play the role of an effective viscosity in driving quasi-stationary accretion flows.

3) Quasi-stationary accretion flows are consistent with the basic conservation laws

(for mass density and canonical momentum) and with the existence of a non-isotropic species pressure tensor.

4) First-order (as well as higher-order) perturbative corrections, can in principle be included consistently in the present theory.

2.11 Conclusions

In this Chapter, a consistent theoretical investigation of the slow kinetic dynamics of collisionless non-relativistic and axisymmetric AD plasmas has been presented. The formulation is based on a kinetic approach developed within the framework of the Vlasov-Maxwell description. We have considered here plasmas immersed in quasi-stationary magnetic fields characterized by open nested magnetic surfaces. This can be appropriate for radiatively inefficient accretion flows onto black holes, some of which are believed to be associated with a plasma of collisionless ions and electrons having different temperatures, and there can be other related applications to the inner regions of accretion flows onto magnetized neutron stars and white dwarfs. The discussion presented here provides a background for future investigations of instabilities and turbulence occurring in these plasmas.

We have shown that a new type of asymptotic kinetic equilibria exists, which can be described by QSA-KDFs expressed in terms of generalized bi-Maxwellian distributions. These solutions permit the consistent treatment of a number of physical properties characteristic of collisionless plasmas. The existence of these equilibrium solutions has been shown to be warranted by imposing suitable kinetic constraints for the structure functions entering the definition of the QSA-KDFs. In terms of these solutions, the slow dynamics of collisionless AD plasmas has been described by means of a suitable reduced GK-Vlasov equation. In addition, the theory permits the consistent treatment of gravitational EM particle trapping phenomena, allowing one to distinguish between different populations of charged particles.

We have shown that the kinetic approach is suitable for the description of quasi-stationary AD plasmas subject to accretion flows and kinetic dynamo effects responsible for the self-generation of both poloidal and toroidal magnetic fields. Four intrinsically-kinetic physical mechanisms have been included in the treatment of this, related to temperature anisotropy, parallel velocity perturbations and FLR-diamagnetic effects.

The novelty of the present approach, with respect to traditional fluid treatments, lies in the possibility of explicitly constructing asymptotic solutions for the fluid equations: the calculation of all of the relevant fluid fields involved (e.g. the plasma charge and mass current densities and the radial flow velocity) can be performed in a straightforward way using a species-dependent asymptotic expansion of the QSA-KDF.

This study makes a relevant contribution for the description of two-temperature collisionless AD plasmas and the improvement of the understanding of their physical properties. The kinetic treatment developed here can also provide a convenient starting point for making a kinetic stability analysis of these plasmas.

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Chapter 3

Absolute stability of axisymmetric perturbations in strongly-magnetized collisionless axisymmetric accretion disc plasmas

3.1 Introduction

A fundamental issue in the physics of accretion discs (ADs) concerns the stability of equilibrium or quasi-stationary configurations occurring in AD plasmas. The observed transport phenomena giving rise to the accretion flow are commonly ascribed to the existence of instabilities and the subsequent development of fluid or MHD turbulence (1, 2, 3, 4). In principle, these can include both MHD phenomena (such as drift instabilities driven by gradients of the fluid fields) and kinetic ones (due to velocity-space anisotropies, including, for example, trapped-particle modes, cyclotron and Alfven waves, etc.). Possible candidates for the angular momentum transport mechanism are usually identified either with the magneto-rotational instability (MRI) (5, 6) or the thermal instability (TMI) (7, 8, 9, 10), caused by unfavorable gradients of rotation/shear and temperature respectively. The validity of the above identifications needs to be checked in the case of collisionless AD plasmas, because they usually rely on incomplete physical descriptions, which ignore the microscopic (kinetic) plasma behavior. In fact, as discussed earlier, "stand-alone" fluid and MHD approaches which are not explicitly based on kinetic theory and/or do not start from consistent kinetic equilibria, may become inadequate or inapplicable for collisionless or weakly-collisional plasmas. Apart from possible gyrokinetic and finite Larmor-radius effects (which are typically not included for MRI and TMI), this concerns consistent treatment of the kinetic constraints which must be imposed on the fluid fields (see related discussion in Refs.(11, 12, 13)). This concerns, in particular, the correct determination of the constitutive equations for the relevant fluid fields. Because of this, the issue of stability of these systems is in need of further study.

In Chapters 1 and 2, a perturbative kinetic theory for collisionless plasmas has been developed and the existence of asymptotic kinetic equilibria has been demonstrated for axisymmetric magnetized plasmas. In AD plasmas, in particular, they are characterized by the presence of *stationary azimuthal and poloidal species-dependent flows* and can support *stationary kinetic dynamo effects*, responsible for the self-generation of azimuthal and poloidal magnetic fields (14), together with *stationary accretion flows*. This provides the basis for a systematic stability analysis of such systems. We stress that these features arise as part of the kinetic equilibrium solution, and are not dependent on perturbative instabilities. Furthermore, by assumption in the theory developed here, there is no background (i.e., externally-produced) radiation field. In principle, for a collisionless plasma in equilibrium, charged particles can still be subject to EM radiation produced by accelerating particles (EM radiation-reaction (15, 16, 17)). However, the effect of these physical mechanisms is negligible for the dynamics of non-relativistic plasmas, and therefore they can be safely ignored in the present treatment.

The goal of this Chapter is to address the stability of these equilibria with respect to infinitesimal axisymmetric perturbations. The investigation reported here has been published in Ref.(18).

3.2 Assumptions and equilibrium orderings

We restrict attention to the treatment of non-relativistic, strongly-magnetized and gravitationally-bound (see definition below) collisionless AD plasmas around compact objects for which the theory developed in the previous chapters applies (see also Refs.(11, 12)). The plasmas can be considered *quasi-neutral* and characterized by a *mean-field interaction.* Accretion discs fulfilling these requirements rely necessarily on kinetic theory in the so-called Vlasov-Maxwell statistical description, which represents the fundamental physical approach for these systems. In AD plasmas, electromagnetic (EM) fields can be present, which may either be externally produced or self-generated. At equilibrium, they are taken here to be axisymmetric and of the general form

$$\mathbf{B}^{(eq)} \equiv B^{(eq)}\mathbf{b} = \mathbf{B}_T^{(eq)} + \mathbf{B}_P^{(eq)} \tag{3.1}$$

and

$$\mathbf{E}^{(eq)} \equiv -\nabla\Phi^{(eq)}(\mathbf{x}) \,. \tag{3.2}$$

Here $\mathbf{B}_T^{(eq)} \equiv I(\mathbf{x})\nabla\varphi$ and $\mathbf{B}_P^{(eq)} \equiv \nabla\psi(\mathbf{x}) \times \nabla\varphi$ are the toroidal and poloidal components of the magnetic field respectively, with $I(\mathbf{x})$ and $\Phi^{(eq)}(\mathbf{x})$ being the toroidal current and the electrostatic potential. Furthermore, (R, φ, z) denote a set of cylindrical coordinates, with $\mathbf{x} = (R, z)$, while $(\psi, \varphi, \vartheta)$ is a set of local magnetic coordinates, with ψ being the so-called poloidal flux function. The validity of the previous representation for $\mathbf{B}^{(eq)}$ requires the existence of locally nested magnetic $\psi-$surfaces, represented by $\psi = const.$, while the expressions for $\psi(\mathbf{x})$, $I(\mathbf{x})$ and $\Phi^{(eq)}(\mathbf{x})$ follow from the stationary Maxwell equations. The gravitational field is treated here non-relativistically, by means of the gravitational potential $\Phi_G = \Phi_G(\mathbf{x})$. This means that the electrostatic and gravitational fields are formally replaced by the effective electric field $\mathbf{E}_s^{eff} = -\nabla\Phi_s^{eff}$, determined in terms of the effective electrostatic potential

$$\Phi_s^{eff} = \Phi^{(eq)}(\mathbf{x}) + \frac{M_s}{Z_s e}\Phi_G(\mathbf{x}), \tag{3.3}$$

with M_s and $Z_s e$ denoting the mass and charge, respectively, of the s-species particle (where s can indicate either ions or electrons). Based on astronomical observations, the magnetic field magnitudes are expected to range in the interval $B \sim 10^1 - 10^8 G$ (19, 20, 21). This implies that the proton Larmor radius r_{Li} is in the range $10^{-6} - 10^3 cm$ (the lower values corresponding to the lower temperature and the higher magnetic field). Additional important physical parameters are related to the species number density and temperature. Astrophysical AD plasmas can have a wide range of values for the particle number density n_s, depending on the circumstances considered. Here we focus on the case of collisionless and non-relativistic AD plasmas assuming values of the number density n_s in the range $n_s \sim 10^6 - 10^{15} cm^{-3}$. For reference, the highest value of this interval corresponds to ion mass density $\rho_i \sim 10^{-9} gcm^{-3}$. The choice of this parameter interval lies well in the range of values which can be estimated for the so-called radiatively inefficient accretion flows (RIAFs, (19, 22)). For these systems, estimates for species temperatures usually lie in the ranges $T_i \sim 1 - 10^5 keV$ and $T_e \sim 1 - 10 keV$ for ions and electrons respectively. Depending on the magnitude of the EM, gravitational and fluid fields, the AD plasmas can sustain a variety of notable physical phenomena. Their systematic treatment requires a classification in terms of suitable dimensionless parameters. These have been already defined in the previous chapters and are briefly recalled here for the sake of completeness. They are identified with $\varepsilon_{M,s}$, ε_s and σ_s, to be referred to as *Larmor-radius*, *canonical momentum* and *total-energy parameters*. Their definitions are respectively: $\varepsilon_{M,s} \equiv \frac{r_{Ls}}{(\Delta L)^{eq}}$, $\varepsilon_s \equiv \left|\frac{M_s R v_\varphi}{\frac{Z_s e}{c}\psi}\right|$ and $\sigma_s \equiv \left|\frac{\frac{M_s}{2}v^2}{Z_s e \Phi_s^{eff}}\right|$. Here, $r_{Ls} \equiv v_{ths}/\Omega_{cs}$ denotes the Larmor radius of the species s, v_{ths} and Ω_{cs} are the species thermal velocity and the Larmor frequency respectively, $(\Delta L)^{eq}$ is the characteristic scale-length of the equilibrium fluid fields, $\mathbf{v}$ is the single-particle velocity and $v_\varphi \equiv \mathbf{v} \cdot R\nabla\varphi$. Systems satisfying the asymptotic ordering $0 \leq \sigma_s, \varepsilon_s, \varepsilon, \varepsilon_{M,s} \ll 1$ are referred to as *strongly-magnetized and gravitationally-bound* plasmas (11, 12), with the parameters σ_s, ε_s and $\varepsilon_{M,s}$ to be considered as independent while $\varepsilon \equiv \max\{\varepsilon_s, s = 1, n\}$. In the following, we shall assume that the poloidal flux is of the form $\psi \equiv \frac{1}{\varepsilon}\overline{\psi}(\mathbf{x})$, with $\overline{\psi}(\mathbf{x}) \sim O(\varepsilon^0)$, while the equilibrium electric field satisfies the constraint $\frac{\mathbf{E}^{(eq)} \cdot \mathbf{B}^{(eq)}}{|\mathbf{E}^{(eq)}||\mathbf{B}^{(eq)}|} \sim O(\varepsilon)$. This implies that to leading-order $\Phi^{(eq)}$ is a function of ψ only, while Φ_s^{eff} remains generally a function of the type $\Phi_s^{eff} = \overline{\Phi}_s^{eff}(\overline{\psi}, \vartheta)$ (see Ref.(12)). At equilibrium, by construction, the particle toroidal canonical momentum $p_{\varphi s} \equiv \frac{Z_s e}{c}\psi_{*s} = M_s R v_\varphi + \frac{Z_s e}{c}\psi$, the total particle energy $E_s \equiv Z_s e \Phi_{*s} = \frac{M_s}{2}v^2 + Z_s e \Phi_s^{eff}$ and the magnetic moment m'_s predicted by gyrokinetic theory are either exact or adiabatic invariants (see also extended discussions presented in previous chapters). In particular, the above orderings imply the leading-order asymptotic perturbative expansions for the variables ψ_{*s} and Φ_{*s}:

$$\psi_{*s} = \psi \left[1 + O\left(\varepsilon_s\right)\right], \tag{3.4}$$

$$\Phi_{*s} = \Phi_s^{eff} \left[1 + O\left(\sigma_s\right)\right], \tag{3.5}$$

while similarly $m'_s = \frac{M_s w'^2}{2B'}\left[1 + O\left(\varepsilon_{M,s}\right)\right]$. In agreement with the notation used here, primed quantities are always evaluated at the guiding-center. In particular, $\mathbf{w}' = \mathbf{v} - u'\mathbf{b}' - \mathbf{V}'_{eff}$ denotes the perpendicular particle velocity in the local frame having the effective drift velocity $\mathbf{V}'_{eff} \equiv \frac{c}{B'}\mathbf{E}_s^{eff} \times \mathbf{b}'$, while $u' \equiv \mathbf{v} \cdot \mathbf{b}'$. In the following we shall also assume that the toroidal and poloidal magnetic fields and the species accretion and azimuthal flow velocities scale as $\frac{|\mathbf{B}_T|}{|\mathbf{B}_P|} \sim O(\varepsilon)$ and $\frac{|\mathbf{V}_{accr,s}|}{|\mathbf{V}_{\varphi,s}|} \sim O(\varepsilon)$ respectively.

3.3 Equilibrium distribution function

In validity of the previous assumptions, an explicit asymptotic solution of the Vlasov equation can be obtained for the kinetic distribution function f_{*s}^{eq} (KDF). As pointed out in the previous chapters (see also Ref.(12)), ignoring slow-time dependencies, this is of the generic form

$$f_{*s}^{eq} = f_{*s}^{eq}\left(X_{*s}, (\psi_{*s}, \Phi_{*s})\right). \tag{3.6}$$

Here X_{*s} are the invariants $X_{*s} \equiv \left(E_s, \psi_{*s}, p'_{\varphi s}, m'_s\right)$, while the brackets (ψ_{*s}, Φ_{*s}) denote implicit dependencies for which the perturbative expansions (3.4) and (3.5) are performed (see Section V in Ref.(12) for the details on the perturbative expansion). Therefore, f_{*s}^{eq} is by construction an adiabatic invariant, defined on a subset of the phase-space $\Gamma = \Omega \times U$, with $\Omega \subset \mathbb{R}^3$ and $U \equiv \mathbb{R}^3$ being, respectively, a bounded subset of the Euclidean configuration space and the velocity space. Hence, f_{*s} varies slowly in time on the slow-time-scale $(\Delta t)^{eq}$, i.e.

$$\frac{d}{dt}\ln f_{*s}^{eq} \sim \frac{1}{(\Delta t)^{eq}}. \tag{3.7}$$

In view of the previous orderings holding for AD plasmas, this implies also $\frac{(\Delta t)^{eq}}{\tau_{col,s}} \ll 1$, where $\tau_{col,s}$ denotes the Spitzer collision time for the species s. Therefore, this requirement is consistent with the assumption of a collisionless plasma. A possible realization of f_{*s}^{eq} is provided by a non-isotropic generalized bi-Maxwellian KDF. As shown in Ref.(12), f_{*s}^{eq} determined in this way describes Vlasov-Maxwell equilibria characterized by quasi-neutral plasmas which exhibit species-dependent azimuthal and poloidal flows as well as temperature and pressure anisotropies. The existence of these equilibria is warranted by the validity of suitable kinetic constraints (see the discussion in Ref.(12, 14)). As a consequence, the same equilibria are characterized by the presence of fluid fields (number density, flow velocity, pressure tensor, etc.) which are generally non-uniform on the $\psi-$surfaces.

3.4 Stability analysis

Let us now pose the problem of linear stability for Vlasov-Maxwell equilibria of the type considered here. This can generally be set for perturbations of both the EM field and the equilibrium KDF, which exhibit appropriate time and space scales $\{(\Delta t)^{osc}, (\Delta L)^{osc}\}$. Here both are prescribed to have *fast time* and *fast space* dependencies with respect to those of the equilibrium quantities, in the sense that

$$\frac{(\Delta t)^{osc}}{(\Delta t)^{eq}} \sim \frac{(\Delta L)^{osc}}{(\Delta L)^{eq}} \sim O(\lambda), \tag{3.8}$$

with λ being a suitable infinitesimal parameter. In the case of strongly-magnetized AD plasmas, to permit a direct comparison with the literature, we also assume that these perturbations are *non-gyrokinetic*. In other words, they are characterized by typical wave-frequencies and wave-lengths which are much larger than the Larmor gyration frequency Ω_{cs} and radius r_{Ls}. This implies that the following inequalities must hold:

$$\frac{\tau_{Ls}}{(\Delta t)^{osc}} \sim \frac{r_{Ls}}{(\Delta L)^{osc}} \ll 1, \tag{3.9}$$

with $\tau_{Ls} = 1/\Omega_{cs}$, while λ must satisfy $\lambda \gg \sigma_s, \varepsilon_s, \varepsilon, \varepsilon_{M,s}$. These will be referred to as *low-frequency* and *long-wavelength perturbations* with respect to the corresponding Larmor scales. Notice that Eqs.(3.8) and (3.9) are independent and complementary, establishing the upper and lower limits for the range of magnitudes of both $(\Delta t)^{osc}$ and $(\Delta L)^{osc}$. We now determine the generic form of the perturbations as implied by the above assumptions. For this purpose, we shall require in the following that the EM field is subject to *axisymmetric EM perturbations* of the form

$$\delta\mathbf{B} = \nabla \times \delta\mathbf{A}, \tag{3.10}$$

$$\delta\mathbf{E} = -\nabla\delta\phi - \frac{1}{c}\frac{\partial\delta\mathbf{A}}{\partial t}, \tag{3.11}$$

with $\left\{\delta\phi\left(\frac{\overline{\psi}}{\lambda}, \frac{\vartheta}{\lambda}, \frac{t}{\lambda}\right), \delta\mathbf{A}\left(\frac{\overline{\psi}}{\lambda}, \frac{\vartheta}{\lambda}, \frac{t}{\lambda}\right)\right\}$ both assumed to be *analytic* (with respect to $\overline{\psi}$ and ϑ) and *infinitesimal*, i.e., such that $\frac{\delta\mathbf{E}}{\left|\mathbf{E}^{(eq)}\right|}, \frac{\delta\mathbf{B}}{\left|\mathbf{B}^{(eq)}\right|} \sim O(\varepsilon)$. This implies that the corresponding perturbations for the EM potentials must scale as

$$\frac{\delta\phi}{\left|\Phi^{(eq)}\right|}, \frac{\delta\mathbf{A}}{\left|\mathbf{A}^{(eq)}\right|} \sim O(\varepsilon)O(\lambda), \tag{3.12}$$

with $\mathbf{A}^{(eq)}$ denoting the equilibrium vector potential. As a consequence

$$\frac{d}{dt}E_s = q_s \left[\frac{\partial \delta \phi}{\partial t} - \frac{1}{c}\mathbf{v} \cdot \frac{\partial \delta \mathbf{A}}{\partial t} \right]. \tag{3.13}$$

Similarly, the perturbation of the equilibrium KDF is taken of the general form

$$\delta f_s \equiv \delta f_s \left(X_{*s}, (\psi_{*s}, \Phi_{*s}), \frac{\overline{\psi}}{\lambda}, \frac{\vartheta}{\lambda}, \frac{t}{\lambda} \right), \tag{3.14}$$

with

$$\frac{\delta f_s}{f_{*s}^{eq}} \sim O(\varepsilon) O(\lambda). \tag{3.15}$$

It follows that the corresponding KDF (the solution of the Vlasov kinetic equation) must now be of the general form

$$f_s = f_s \left(X_{*s}, (\psi_{*s}, \Phi_{*s}), \frac{\overline{\psi}}{\lambda}, \frac{\vartheta}{\lambda}, \frac{t}{\lambda} \right), \tag{3.16}$$

while, from the Maxwell equations, the perturbations $\{\delta\phi, \delta\mathbf{A}\}$ are necessarily linear functionals of δf_s. However, for analytic perturbations of the form (3.16), f_s must itself be regarded as an analytic function of $\overline{\psi}$ and ϑ. Therefore, invoking Eqs.(3.4) and (3.5), the same KDF can always be considered as an asymptotic approximation obtained by Taylor expansion of a suitable *generalized KDF* of the form

$$f_s^{(gen)} \equiv f_s^{(gen)} \left(X_{*s}, (\psi_{*s}, \Phi_{*s}, Y_{*s}), \frac{t}{\lambda} \right), \tag{3.17}$$

with $Y_{*s} \equiv \left[\frac{\varepsilon_s \psi_{*s}}{\lambda}, \frac{\sigma_s \Phi_{*s}}{\lambda} \right]$. In particular, denoting $\delta f_s^{(gen)} \equiv f_s^{(gen)} - f_{*s}^{eq}$, it follows that also $\delta f_s^{(gen)}$ is such that

$$\delta f_s^{(gen)} \equiv \delta f_s^{(gen)} \left(X_{*s}, (\psi_{*s}, \Phi_{*s}, Y_{*s}), \frac{t}{\lambda} \right). \tag{3.18}$$

Then, by Taylor expansion with respect to the variables Y_{*s}, the perturbation $\delta f_s^{(gen)}$ can be shown to be related to δf_s (defined by Eq.(3.14)) by

$$\delta f_s^{(gen)} \cong \delta \widehat{f}_s \left(X_{*s}, (\psi_{*s}, \Phi_{*s}), \frac{\overline{\psi}}{\lambda}, \frac{\vartheta}{\lambda} \right) e^{i\omega t}, \tag{3.19}$$

where corrections of $\frac{O(\varepsilon_s)}{O(\lambda)}$ and $\frac{O(\sigma_s)}{O(\lambda)}$ have been neglected and ω is the complex time-frequency which, according to Eq.(3.8), is ordered as $\omega (\Delta t)^{eq} \sim 1/O(\lambda)$. Similarly, invoking again Eqs.(3.4) and (3.5), for the analytic perturbations $\{\delta\phi, \delta\mathbf{A}\}$ we can introduce the corresponding *generalized perturbations* $\left\{ \delta\phi^{(gen)}, \delta\mathbf{A}^{(gen)} \right\}$. Neglecting in the similar way corrections of $\frac{O(\varepsilon_s)}{O(\lambda)}$ and $\frac{O(\sigma_s)}{O(\lambda)}$, these are given as follows:

$$\delta\phi^{(gen)}\left(Y_{*s},\frac{t}{\lambda}\right) \cong \delta\widehat{\phi}\left(\frac{\overline{\psi}}{\lambda},\frac{\vartheta}{\lambda}\right)e^{i\omega t}, \tag{3.20}$$

$$\delta\mathbf{A}^{(gen)}\left(Y_{*s},\frac{t}{\lambda}\right) \cong \delta\widehat{\mathbf{A}}\left(\frac{\overline{\psi}}{\lambda},\frac{\vartheta}{\lambda}\right)e^{i\omega t}. \tag{3.21}$$

Analogous expressions for the corresponding generalized perturbations can be readily obtained. In particular, using Eq.(3.19), we get the following representation for $\delta f_s^{(gen)}$:

$$\delta f_s^{(gen)} = \delta\widehat{f}_s^{(gen)}\left(X_{*s},(\psi_{*s},\Phi_{*s},Y_{*s})\right)e^{i\omega t}, \tag{3.22}$$

where, expanding the Fourier coefficient and neglecting again corrections of $\frac{O(\varepsilon_s)}{O(\lambda)}$ and $\frac{O(\sigma_s)}{O(\lambda)}$, $\delta\widehat{f}_s^{(gen)} \cong \delta\widehat{f}_s\left(X_{*s},(\psi_{*s},\Phi_{*s}),\frac{\varepsilon\psi}{\lambda},\frac{\vartheta}{\lambda}\right)$. Therefore, in view of Eq.(3.13), for infinitesimal axisymmetric analytical EM perturbations $\{\delta\phi,\delta\mathbf{A}\}$, to leading order in λ the Vlasov equation implies the dispersion equation

$$-i\omega\delta\widehat{f}_s\left(X_{*s},(\psi_{*s},\Phi_{*s}),\frac{\overline{\psi}}{\lambda},\frac{\vartheta}{\lambda}\right) = i\omega q_s\left[\delta\widehat{\phi}\left(\frac{\overline{\psi}}{\lambda},\frac{\vartheta}{\lambda}\right)-\frac{1}{c}\mathbf{v}\cdot\delta\widehat{\mathbf{A}}\left(\frac{\overline{\psi}}{\lambda},\frac{\vartheta}{\lambda}\right)\right]\frac{\partial f_{*s}^{eq}}{\partial E_s}. \tag{3.23}$$

Apart from the trivial solution $\omega = 0$ (i.e., a stationary perturbation of the equilibrium), this requires that, for $\omega \neq 0$, one must have

$$\delta\widehat{f}_s = -q_s\left[\delta\widehat{\phi}-\frac{1}{c}\mathbf{v}\cdot\delta\widehat{\mathbf{A}}\right]\frac{\partial f_s^{(eq)}}{\partial E_s}, \tag{3.24}$$

where, by construction, $\delta\widehat{f}_s$, $\delta\widehat{\phi}$ and $\delta\widehat{\mathbf{A}}$ are manifestly independent of ω. Hence, Eq.(3.24) necessarily holds also when $|\omega|$ is arbitrarily small. In this limit $\left\{\delta\widehat{\phi},\delta\widehat{\mathbf{A}},\delta\widehat{f}_s\right\}$ tend necessarily to infinitesimal stationary perturbations of the equilibrium solutions. On the other hand, Eqs.(3.20), (3.21) and (3.22) show that $\left\{\delta\widehat{\phi},\delta\widehat{\mathbf{A}},\delta\widehat{f}_s\right\}$ are always asymptotically close to the generalized quantities $\left\{\delta\widehat{\phi}^{(gen)},\delta\widehat{\mathbf{A}}^{(gen)},\delta\widehat{f}_s^{(gen)}\right\}$, which are by definition equilibrium perturbations [i.e., functions of $\left(\frac{\varepsilon_s\psi_{*s}}{\lambda},\frac{\sigma_s\Phi_{*s}}{\lambda}\right)$]. Since the latter again represent an equilibrium and are independent of ω, it follows that the only admissible solution of the dispersion equation (3.24) is clearly independent of ω as well and coincides with the null solution, i.e.

$$\delta\widehat{\phi}\left(\frac{\overline{\psi}}{\lambda},\frac{\vartheta}{\lambda}\right) \equiv 0, \tag{3.25}$$

$$\delta\widehat{\mathbf{A}}\left(\frac{\overline{\psi}}{\lambda},\frac{\vartheta}{\lambda}\right) \equiv 0, \tag{3.26}$$

$$\delta \widehat{f}_s \left(X_{*s}, (\psi_{*s}, \Phi_{*s}), \frac{\overline{\psi}}{\lambda}, \frac{\vartheta}{\lambda} \right) \equiv 0. \tag{3.27}$$

In summary: *no analytic, low-frequency and long-wavelength axisymmetric unstable perturbations can exist in non-relativistic strongly-magnetized and gravitationally-bound axisymmetric collisionless AD plasmas.*

We stress that this result follows from two basic assumptions. The first one is the requirement that the equilibrium magnetic field admits locally nested $\psi-$surfaces. The second one is due to the assumed property of AD plasmas to be gravitationally-bound. This implies (as pointed out above) that the effective ES potential Φ_s^{eff} is necessarily a function of both ψ and ϑ, and therefore the perturbation of the KDF is actually close to a function of the exact and adiabatic invariants X_{*s}.

A notable aspect of the conclusion is that it applies to collisionless Vlasov-Maxwell equilibria having, in principle, arbitrary topology of the magnetic field lines which can belong to either closed or open magnetic $\psi-$surfaces. Also, as pointed out in Refs.(11, 12), for strongly-magnetized plasmas these equilibria can give rise to kinetic dynamo effects simultaneously with having accretion flows. These results are important for understanding the phenomenology of collisionless AD plasmas of this type. In particular, they completely rule out the possibility that axisymmetric perturbations, which are long-wavelength and low-frequency in the sense of the inequalities (3.9), could give rise to kinetic instabilities in such systems. This conclusion applies for collisionless AD plasmas (having in particular particle densities within the range mentioned earlier) which are strongly-magnetized and simultaneously gravitationally-bound . Since fluid descriptions of these plasmas can only be arrived at on the basis of the present Vlasov-Maxwell statistical description, also MHD instabilities, such as the axisymmetric MRI (2, 23), the axisymmetric TMI (see for example (8, 9, 10)), and axisymmetric instabilities driven by temperature anisotropy (e.g., the firehose instability (24)) remain definitely forbidden for collisionless plasmas under these conditions.

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Chapter 4

A side application: kinetic description of rotating Tokamak plasmas with anisotropic temperatures in the collisionless regime

4.1 Introduction

As discussed in previous chapters, plasma dynamics is most frequently treated in the framework of stand-alone MHD approaches, i.e., formulated independent of an underlying kinetic theory. However, these treatments can provide at most a partial description of plasma phenomenology, because of two basic inconsistencies of customary fluid approaches. First, the set of fluid equations may not be closed, requiring in principle the prescription of arbitrary higher-order fluid fields. Secondly, in these approaches typically no account is given of microscopic phase-space particle dynamics as well as of phase-space plasma collective phenomena. It is well known that only in the context of kinetic theory can these difficulties be consistently met. Such a treatment in fact permits one to obtain well-defined constitutive equations for the relevant fluid fields describing the plasma state, overcoming at the same time the closure problem. Kinetic theory is appropriate, for example, in the case of collisionless or weakly-collisional plasmas where phase-space particle dynamics is expected to play a dominant role.

Unfortunately, for a wide range of physical effects arising in magnetically-confined plasmas and relevant for controlled fusion research, a fully consistent approach of this type is still missing. Surprisingly, these include even the description of equilibrium or slowly-time varying phenomena occurring in realistic laboratory Tokamak plasmas. The issue concerns specifically the description of finite pressure anisotropies, strong toroidal differential rotation as well as concurrent poloidal flows observed in Tokamak devices. This deficiency may represent a serious obstacle for meaningful developments in plasma physics (both theoretical and computational) and controlled fusion research. In particular, it is well-known that both toroidal and poloidal plasma equilibrium rotation flows may exist in Tokamak plasmas (1, 2). The observation of intrinsic rotation, occurring without any external momentum source (3), remains essentially unexplained to date, being mostly ascribed to turbulence or boundary-layer phenomena occurring in the outer regions of the plasma (4, 5). Such an effect, potentially combining both toroidal and poloidal flow velocities with temperature anisotropy, may be of critical importance both for stability and suppression of turbulence (6, 7, 8).

The goal of the present investigation is the construction of slowly-time varying particular solutions of the Vlasov-Maxwell system for collisionless axisymmetric plasmas immersed in strong magnetic and electric fields. In principle, two approaches are possible for the investigation of the problem. One is based on the Chapman-Enskog solution of the drift-kinetic Vlasov equation, achieved by seeking a perturbative solution of the form $f_s = f_{Ms} + \varepsilon f_{1s} + \ldots$, where $0 < \varepsilon \ll 1$ is an appropriate dimensionless parameter to be defined below (see Section 4.3) and f_{Ms} a suitable equilibrium kinetic distribution function (KDF). In customary formulations this is typically identified with a drifted Maxwellian KDF. An example is provided by Hinton *et al.* (9) where an approximate equilibrium KDF carrying both toroidal and poloidal flows was introduced to describe ion poloidal flows in Tokamaks near the plasma edge. An alternative approach is represented by the construction of exact or asymptotic solutions of the Vlasov equations of the form $f_s = f_{*s}$, with f_{*s} to be considered only as a function of particle exact and adiabatic invariants, via the introduction of suitable *kinetic constraints*. This technique is exemplified by Ref.(10), where f_{*s} was assumed to be a function of only two invariants, namely the particle energy $E_s \equiv Z_s e\Phi_{*s}$ and the toroidal canonical momentum $p_{\varphi s} \equiv \frac{Z_s e}{c}\psi_{*s}$ [see their definitions given below], and identified with a generalized Maxwellian distribution of the form

$$f_{*s} = \frac{n_{*s}}{\pi^{3/2}\left(2T_{*s}/M_s\right)^{3/2}} \exp\left\{-\frac{H_{*s}}{T_{*s}}\right\}. \tag{4.1}$$

Here H_{*s} is the invariant $H_{*s} \equiv E_s - \frac{Z_s e}{c}\int_0^{\psi_{*s}} d\psi\Omega_0(\psi)$, while $\Lambda_{*s} \equiv \{n_{*s}, T_{*s}\}$ denotes suitable "structure functions", i.e., properly defined functions of the particle invariants. In Refs.(10, 11, 12) these were prescribed by imposing the kinetic constraint $\Lambda_{*s} = \Lambda_{*s}(\psi_{*s})$. By performing a perturbative expansion in the canonical momentum (see also the related discussion in Section 4.6), it was shown that f_{*s} recovers the Chapman-Enskog form, with the leading-order Maxwellian KDF carrying isotropic temperature $T_s(\psi)$, species-independent toroidal angular rotation velocity $\Omega_0(\psi)$ (see the definition given by Eq.(4.39)) and finite toroidal differential rotation, i.e., $\frac{\partial}{\partial\psi}\Omega_0(\psi) \neq 0$. A basic aspect of Tokamak plasmas is the property of allowing toroidal rotation velocities $R\Omega_0$ comparable to the ion thermal velocity $v_{thi} = \{2T_i/M_i\}^{1/2}$. As shown in Ref.(10) this implies the fundamental consequence that, for kinetic equilibria characterized by purely toroidal differential rotation as described by the KDF (4.1), necessarily the self-generated electrostatic (ES) potential Φ in the plasma must satisfy the ordering $\left(M_i v_{thi}^2\right)/\left(Z_i e\Phi\right) \sim O\left(\varepsilon\right)$. If the ion and electron temperatures are comparable, in the sense that $T_i/T_e \sim O\left(\varepsilon^0\right)$, it follows that an analogous ordering must hold also for the electron species. *Therefore, the same asymptotic condition must be adopted for all thermal particles of the plasma, namely those for which* $|\mathbf{v}| \sim v_{ths}$*, independent of species.*

In the following, utilizing this type of ordering, the second route is adopted. Hence, the theory developed here applies to a two-species ion-electron plasma characterized by a toroidal rotation velocity of the same order as the ion thermal speed. It relies on the perturbative kinetic theory developed in Refs.(13, 14). The aim is to provide a systematic generalization of the theory presented in Ref.(10), allowing f_{*s} to depend on the complete set of independent adiabatic invariants, and therefore to vary slowly in time ("equilibrium" KDF). In particular, here we intend to show that, besides the properties indicated above, also temperature anisotropy, finite poloidal flow velocities and first-order perturbative corrections, including finite Larmor-radius (FLR) corrections, can be consistently dealt with at the equilibrium level. A remarkable feature of the approach is that, by construction, all of the moment equations stemming from the Vlasov equation are identically satisfied, together with their related solubility conditions (i.e., those following from the condition of periodicity of the KDF and its moments in the poloidal angle). An interesting development consists of the inclusion of both diamagnetic (i.e., FLR) and energy corrections arising from the Taylor-expansions of the relevant structure functions. In this case the structure functions are identified with smooth functions of both the particle energy and toroidal canonical momentum, of the general form

$$\Lambda_{*s} = \Lambda_s(\psi_{*s}, \Phi_{*s}), \tag{4.2}$$

with the functions $\Lambda_s(\psi, \Phi)$ being identified with suitable fluid fields, s denoting the species index. This permits the construction of a systematic perturbative expansion also for the KDF itself, allowing retention of perturbative corrections (of arbitrary order) expressed as polynomial functions in terms of the particle velocity. In particular, under suitable assumptions, the leading-order KDF is shown to be determined by a bi-Maxwellian distribution carrying anisotropic temperature and non-uniform (toroidal and poloidal) flow velocities. Thanks to the kinetic constraints, constitutive equations are determined for the related equilibrium fluid fields. First-order corrections with respect to ε are shown to be linear functions of suitably-generalized thermodynamic forces. These now include, besides the customary ones (10), additional thermodynamic forces associated with energy derivatives of the relevant structure functions.

The constraints imposed by the Maxwell equations are then investigated. First, the Poisson equation is analyzed within the quasi-neutrality approximation. As a development with respect to Ref.(10), it is proved that the perturbative scheme determines uniquely, correct through $O\left(\varepsilon^0\right)$, the equilibrium ES potential, including the $1/O\left(\varepsilon\right)$ contribution. Secondly, the solubility conditions for Ampere's law are shown to prescribe constraints on the species poloidal and toroidal flow velocities and the corresponding current densities. The theory applies for magnetic configurations with nested and closed toroidal magnetic surfaces characterized by finite aspect ratio.

The reference publication for this investigation is given by Ref.(15).

4.2 Vlasov-Maxwell asymptotic orderings

In the following, for particles belonging to the s-species, we introduce the characteristic time and length scales $\Delta t_s \equiv \frac{2\pi r}{v_{\perp ths}}$ and $\Delta L_s = \Delta L \equiv 2\pi r$, with $2\pi r$ and $v_{\perp ths} = \{T_{\perp s}/M_s\}^{1/2}$ denoting respectively the *connection length* and the thermal velocity associated with the species perpendicular temperature $T_{\perp s}$ (defined with respect to the local magnetic field direction). We shall consider phenomena occurring in time intervals Δt_s within the ranges $\tau_{ps} \ll \Delta t_s \ll \tau_{Cs}$, where $\tau_{ps} \equiv \left(\frac{M_s}{4\pi n_s (Z_s e)^2}\right)^{1/2}$, and for isotropic species temperatures $\tau_{Cs} \equiv \frac{3\sqrt{M_s} T_s^{3/2}}{4\sqrt{2\pi} n_s \ln \Lambda (Z_s e)^4}$ denote respectively the Langmuir time and the Spitzer ion self-collision time. A similar ordering follows for the corresponding scale-length ΔL_s letting $\Delta L_s = \Delta t_s v_{ths}$, with v_{ths} being the species isotropic-temperature thermal velocity. For definiteness, we shall consider here a plasma consisting of n species of charged particles, with $n \geq 2$. Such a plasma can be regarded, respectively, as:

(#1) *Collisionless:* when the inequality between Δt_s and τ_{Cs} holds, contributions proportional to the ratios $\varepsilon_{Cs} \equiv \frac{\Delta t_s}{\tau_{Cs}} \ll 1$, here referred to as the *collision-time parameter*, can be ignored. Thus, Coulomb binary interactions are negligible, so that all particle species in the plasma can be regarded as collisionless.

(#2) *Continuous:* due to the left-side inequality between Δt_s and τ_{ps}, plasma particles interact with each other only via a continuum mean EM field. In particular, the inequality $\varepsilon_{Lg,s} \equiv \frac{\tau_{ps}}{\Delta t_s} \ll 1$ is assumed to hold, with $\varepsilon_{Lg,s}$ denoting the *Langmuir-time parameter*.

(#3) *Quasi-neutral:* due again to the same inequality, the plasma is quasi-neutral on the spatial scale ΔL_s corresponding to Δt_s.

Systems fulfilling requirements #1-#2 - the so-called *Vlasov-Maxwell plasmas* - are described by kinetic theory, since fluid MHD approaches are inapplicable in that case. Such plasmas are described in the framework of the so-called Vlasov-Maxwell kinetic theory. In this case the plasma is treated as an ensemble of particle $s-$species (subsets of like particles) each one described by a KDF $f_s(\mathbf{x}, t)$ defined in the phase-space $\Gamma = \Gamma_r \times \Gamma_u$ (with $\Gamma_r \subset \mathbb{R}^3$ and $\Gamma_u \equiv \mathbb{R}^3$ denoting respectively the configuration and velocity spaces) and satisfying the Vlasov kinetic equation. Velocity moments of $f_s(\mathbf{x}, t)$ are then defined as integrals of the form $\int_{\Gamma_u} d^3v Q(\mathbf{x}, t) f_s(\mathbf{x}, t)$, with $Q(\mathbf{x}, t)$ being a suitable phase-space weight function. In particular, for $Q(\mathbf{x}, t) = \{1, \mathbf{v}\}$ the velocity moments determine the source of the EM self-field $\left\{\mathbf{E}^{self}, \mathbf{B}^{self}\right\}$, identified with the plasma charge and current density $\{\rho(\mathbf{r}, t), \mathbf{J}(\mathbf{r}, t)\}$.

In addition, we require the plasma to be *axisymmetric*, so that, when referred to a set of cylindrical coordinates (R, φ, z), all relevant dynamical variables characterizing the plasma (e.g., the fluid fields and the EM field) are independent of the azimuthal angle φ. Here, by assumption, the configuration space is identified with the bounded internal domain of an axisymmetric torus, which can be parametrized in terms of the scale-lengths (a, R_0), with a denoting $a \equiv \sup \{r, r \in \Gamma_r\}$ and R_0 being the radius of the plasma magnetic axis.

4.3 Basic assumptions

In this Section the basic hypotheses of the model, which include the EM field and the magnetized-plasma orderings, are pointed out.

The EM field

Here we restrict our analysis to EM fields which are slowly-time varying in the sense $\left[\mathbf{E}(\mathbf{x}, \varepsilon^k t), \mathbf{B}(\mathbf{x}, \varepsilon^k t)\right]$, with $k \geq 1$ being a suitable integer (*quasi-stationarity condition*). This type of time dependence is thought to arise either due to external sources or boundary conditions. In particular, the magnetic field $\mathbf{B}$ is assumed of the form

$$\mathbf{B} \equiv \nabla \times \mathbf{A} = \mathbf{B}^{self}(\mathbf{x}, \varepsilon^k t) + \mathbf{B}^{ext}(\mathbf{x}, \varepsilon^k t), \tag{4.3}$$

where $\mathbf{B}^{self}$ and $\mathbf{B}^{ext}$ denote the self-generated magnetic field produced by the plasma and a finite external magnetic field produced by external coils. In particular the magnetic field $\mathbf{B}$ is assumed to admit a family of nested and closed axisymmetric toroidal magnetic surfaces $\{\psi(\mathbf{x})\} \equiv \{\psi(\mathbf{x}) = const.\}$, where ψ denotes the poloidal magnetic flux of $\mathbf{B}$ and, because of axisymmetry, $\mathbf{x}$ can be identified with the coordinates $\mathbf{x} = (R, z)$. In such a setting a set of magnetic coordinates $(\psi, \varphi, \vartheta)$ can be defined, where ϑ is a curvilinear angle-like coordinate on the magnetic surfaces $\psi(\mathbf{x}) = const.$ It is assumed that the vectors $(\nabla\psi, \nabla\varphi, \nabla\vartheta)$ define a right-handed system. Each relevant physical quantity $G(\mathbf{x}, t)$ can then be conveniently expressed either in terms of the cylindrical coordinates or as a function of the magnetic coordinates, i.e. $G(\mathbf{x}, t) = \overline{G}(\psi, \vartheta, t)$. The total magnetic field is then decomposed as

$$\mathbf{B} = I(\mathbf{x}, \varepsilon^k t)\nabla\varphi + \nabla\psi(\mathbf{x}, \varepsilon^k t) \times \nabla\varphi, \tag{4.4}$$

where $\mathbf{B}_T \equiv I(\mathbf{x}, \varepsilon^k t)\nabla\varphi$ and $\mathbf{B}_P \equiv \nabla\psi(\mathbf{x}, \varepsilon^k t) \times \nabla\varphi$ are the toroidal and poloidal components of the field. In particular, the following ordering is assumed to hold: $\frac{|\mathbf{B}_P|}{|\mathbf{B}_T|} \sim O\left(\varepsilon^0\right)$. Finally, the corresponding electric field expressed in terms of the EM potentials $\left\{\Phi(\mathbf{x}, \varepsilon^k t), \mathbf{A}(\mathbf{x}, \varepsilon^k t)\right\}$ is considered to be primarily electrostatic, namely

$$\mathbf{E}(\mathbf{x}, \varepsilon^k t) \equiv -\nabla\Phi - \varepsilon^k \frac{1}{c}\frac{\partial \mathbf{A}}{\partial \tau} \cong -\nabla\Phi, \tag{4.5}$$

with τ denoting the slow-time variable $\tau \equiv \varepsilon^k t$, and quasi-orthogonal to the magnetic field, in the sense that $\frac{\mathbf{E}\cdot\mathbf{B}}{|\mathbf{E}||\mathbf{B}|} \sim O(\varepsilon)$, while $\frac{c|\mathbf{E}|}{|\mathbf{B}|}\frac{1}{v_{ths}} \sim O\left(\varepsilon^0\right)$. Together with the quasi-stationarity condition, this implies that, to leading order in ε, $\Phi = \Phi(\psi, \varepsilon^k t)$. In particular, assuming that both Φ and $\mathbf{A}$ are analytic with respect to ε, it can be shown that, consistent with gyrokinetic (GK) theory and the asymptotic orderings indicated below, they must be considered as being of the general form

$$\Phi = \frac{1}{\varepsilon}\Phi_{-1}\left(\psi,\varepsilon^k t\right)+\varepsilon^0\Phi_0\left(\psi,\vartheta,\varepsilon^k t\right)+.., \tag{4.6}$$

$$\mathbf{A} = \frac{1}{\varepsilon}\mathbf{A}_{-1}\left(\mathbf{r},\varepsilon^k t\right)+\varepsilon^0\mathbf{A}_0\left(\mathbf{r},\varepsilon^k t\right)+.., \tag{4.7}$$

where Φ is expressed in terms of the magnetic coordinates and $\mathbf{A}_{-1}$ is $\mathbf{A}_{-1}\equiv\psi\nabla\varphi+g\left(\psi,\vartheta,\varepsilon^k t\right)\nabla\vartheta$, with g being a suitable function.

The magnetized-plasma orderings

Next, let us introduce the *magnetized plasma ordering* appropriate for the treatment of single-particle dynamics in magnetized plasmas, i.e. for which in particular $B^2 \gg E^2$. For $s=i,e$, this requires the definition of the following additional dimensionless parameters:

1) *Larmor-radius parameter* $\varepsilon_{M,s}\equiv\frac{r_{Ls}}{\Delta L_s}$ and *Larmor-time parameter* $\varepsilon_{Lr,s}\equiv\frac{\tau_{Ls}}{\Delta t_s}$: here τ_{Ls} and r_{Ls} are respectively the Larmor time and the Larmor radius of the species s, with $s=1,n$, defined as $r_{Ls}\equiv v_{\perp ths}/\Omega_{cs}$, with $\Omega_{cs}=Z_seB/M_sc\equiv 1/\tau_{Ls}$ being the species Larmor frequency. Imposing the requirement that $\tau_{Ls}\ll\Delta t_s$ and $r_{Ls}\ll\Delta L_s$, it follows that $\varepsilon_{M,s}$ and $\varepsilon_{Lr,s}$ are infinitesimals of the same order, i.e., $0\leq\varepsilon_{Lr,s}\sim\varepsilon_{M,s}\ll 1$. Requiring again that $T_i\sim T_e$, and furthermore $Z_i\sim O\left(1\right)$, it follows that $\varepsilon_{M,i}\sim\left(\frac{M_i}{M_e}\right)^{1/2}\varepsilon_{M,e}$.

2) *Canonical-momentum parameter:* $\varepsilon_s\equiv\left|\frac{L_{\varphi s}}{p_{\varphi s}-L_{\varphi s}}\right|=\left|\frac{M_sRv_\varphi}{\frac{Z_se}{c}\psi}\right|$, where $v_\varphi\equiv\mathbf{v}\cdot\mathbf{e}_\varphi$ and $L_{\varphi s}$ denotes the species particle angular momentum.

3) *Total-energy parameter:* $\sigma_s\equiv\left|\frac{\frac{M_s}{2}v^2}{Z_se\Phi}\right|$, where $\frac{M_s}{2}v^2\sim T_s$ and $Z_se\Phi$ are respectively the particle kinetic and ES energy.

In principle, the parameters ε_s and σ_s are independent (in particular, they might differ from $\varepsilon_{M,s}$). More precisely, here we shall consider the subset of phase-space for which the following ordering holds:

$$0\leq\sigma_s\sim\varepsilon_s\sim\varepsilon_{Lr,s}\sim\varepsilon_{M,s}\ll 1, \tag{4.8}$$

which applies in the subset of thermal particles. Notice that the assumption concerning ε_s is consistent with the requirement of finite inverse aspect-ratio (see below), while, as recalled above, the ordering of σ_s is required for the treatment of Tokamak equilibria in the presence of strong toroidal differential rotation (10, 11, 12, 16). The same orderings are of course invoked also for the validity of the GK theory (see Ref.(17) and also Eq.(4.13) in the next section and the related discussion). The assumption concerning the σ_s-ordering can be shown to be consistent with the quasi-neutrality condition (see Corollary to THM.1 in Section 6.9). These requirements imply, for all species, the asymptotic perturbative expansions in the variables ψ_{*s} and Φ_{*s}:

$$\psi_{*s} = \psi\left[1+O\left(\varepsilon_{M,s}\right)\right], \tag{4.9}$$

$$\Phi_{*s} \quad = \quad \Phi\left[1+O\left(\varepsilon_{M,s}\right)\right]. \tag{4.10}$$

Finally, to warrant the validity of the Vlasov equation on the Larmor-radius scale, we shall impose also that $0 \ll \varepsilon_{mfp,s} \sim \varepsilon_{Cs} \leq \varepsilon_{M,s}$, with $\varepsilon_{mfp,s} \equiv \frac{\Delta L}{\lambda_{Cs}}$ and $\varepsilon_{Cs} \equiv \frac{\Delta t_s}{\tau_{Cs}}$ denoting respectively the *mean-free-path parameter* and *the collision-time parameter*. Then, consistent with quasi-neutrality, we demand also that $\varepsilon_{Lg,s} \sim \varepsilon_D \leq \varepsilon_{M,s} \leq \varepsilon \ll 1$, with $\varepsilon = \sup\left\{\varepsilon_{M,s}, s = e, i\right\}$. Finally, the *inverse aspect-ratio parameter* $\delta \equiv \frac{a}{R_0}$ will be considered finite, i.e. such that $\delta \sim O\left(\varepsilon^0\right)$. We remark that the parameters $\{\sigma_s, \varepsilon_s, \varepsilon_{Lr,s}, \varepsilon_{M,s}\}$ deal with the single-particle dynamics, $\{\varepsilon_{Lg,s}, \varepsilon_{D,s}, \varepsilon_{mfp,s}, \varepsilon_{Cs}\}$ concern collective properties of the plasma, while δ is a purely geometrical quantity.

4.4 The particle adiabatic invariants

For single-particle dynamics, the exact first integrals of motion and the relevant adiabatic invariants are well-known. In particular, the adiabatic invariants can be defined either in the context of Hamiltonian dynamics or GK theory (18, 19, 20). In both cases, for a magnetized plasma, they can be referred to the Larmor frequency. Hence, by definition, a phase-function P_s depending on the s-species particle state is denoted as an adiabatic invariant of order n with respect to $\varepsilon_{M,s}$ if it is conserved asymptotically, namely in the sense $\frac{1}{\Omega'_{cs}}\frac{d}{dt}\ln P_s = 0+O(\varepsilon_{M,s}^{n+1})$, where $n \geq 0$ is a suitable integer and Ω'_{cs} is the Larmor frequency evaluated at the guiding-center position $\mathbf{x}'$. Note that, in the following, primed quantities denote dynamical variables defined at the *guiding-center position* $\mathbf{r}'$ (or $\mathbf{x}'$ in axisymmetry). If there is axisymmetry, the only first integral of motion is the canonical momentum $p_{\varphi s}$ conjugate to the azimuthal angle φ:

$$p_{\varphi s} = M_s R\mathbf{v}\cdot\mathbf{e}_\varphi + \frac{Z_s e}{c}\psi \equiv \frac{Z_s e}{c}\psi_{*s}. \tag{4.11}$$

Furthermore, the total particle energy

$$E_s = \frac{M_s}{2}v^2 + Z_s e\Phi(\mathbf{x}, \varepsilon^n t) \equiv Z_s e\Phi_{*s}, \tag{4.12}$$

with $n \geq 1$, is assumed to be an adiabatic invariant of order n.

Let us now analyze the adiabatic invariants predicted by GK theory. As usual, the GK treatment involves the construction - in terms of an asymptotic perturbative expansion determined by means of a power series in $\varepsilon_{M,s}$ - of a diffeomorphism of the form $\mathbf{z} \equiv (\mathbf{r}, \mathbf{v}) \rightarrow \mathbf{z}' \equiv (\mathbf{r}', \mathbf{v}')$, referred to as the *GK transformation*. The GK transformation is performed on all phase-space variables $\mathbf{z} \equiv (\mathbf{r}, \mathbf{v})$, *except* for the azimuthal angle φ which is left unchanged and is therefore to be considered as one of the GK variables. Here, by definition, the transformed variables $\mathbf{z}'$ (*GK state*) are constructed so that their time derivatives to the relevant order in $\varepsilon_{M,s}$ have at least one ignorable coordinate, to be identified with a suitably-defined gyrophase ϕ'. The starting point is then the representation of the particle Lagrangian in terms of the hybrid variables $\mathbf{z}$. This is expressed as $\mathcal{L}_s(\mathbf{z}, \frac{d}{dt}\mathbf{z}, \varepsilon^k t) \equiv \dot{\mathbf{r}}\cdot\mathbf{P}_s - \mathcal{H}_s(\mathbf{z}, \varepsilon^k t)$, where $\mathbf{P}_s \equiv \left[M_s\mathbf{v} + \frac{Z_s e}{c}\mathbf{A}(\mathbf{x}, \varepsilon^k t)\right]$

and $\mathcal{H}_s(\mathbf{z},\varepsilon^k t) = \frac{M_s}{2}v^2 + Z_s e\Phi(\mathbf{x},\varepsilon^k t)$ denotes the corresponding Hamiltonian function in hybrid variables. The development of GK theory is well known. It involves a phase-space transformation to a local reference frame in which the particle guiding-center is instantaneously at rest with respect to the ψ-surface to which it belongs. In this case, the leading-order GK transformation can be proved to be necessarily of the form

$$\begin{cases} \mathbf{r} = \mathbf{r}' - \frac{\mathbf{w}'\times\mathbf{b}'}{\Omega'_{cs}}, \\ \mathbf{v} = u'\mathbf{b}' + \mathbf{w}' + \mathbf{U}', \end{cases} \tag{4.13}$$

Here, in particular, $\mathbf{U}' \equiv \mathbf{U}(\mathbf{x}',\varepsilon^k t)$, with $\mathbf{U}(\mathbf{x},\varepsilon^k t)$ being the fluid-field identified with the $\mathbf{E}\times\mathbf{B}-$drift velocity:

$$\mathbf{U}(\mathbf{x},\varepsilon^k t) \equiv -\frac{c}{B}\nabla\Phi\times\mathbf{b}. \tag{4.14}$$

This coincides with the so-called frozen-in velocity, namely the fluid velocity with respect to which each line of force is carried into itself. The rest of the notation is standard. Thus, u' and $\mathbf{w}'$ denote respectively the parallel and perpendicular (guiding-center) velocities, with $\mathbf{w}' = w'\cos\phi'\mathbf{e}'_1 + w'\sin\phi'\mathbf{e}'_2$ and ϕ' denoting the gyrophase angle, $\Omega'_{cs} = \frac{Z_s eB'}{M_s c}$ and $\mathbf{b}' = \mathbf{b}(\mathbf{x}',\varepsilon^k t)$, with $\mathbf{b}(\mathbf{x},\varepsilon^k t) \equiv \mathbf{B}(\mathbf{x},\varepsilon^k t)/B(\mathbf{x},\varepsilon^k t)$. Notice that, here, by construction, $\left|\frac{\mathbf{w}'\times\mathbf{b}'}{\Omega'_{cs}}\right|$ must be considered to be of $O(\varepsilon_{M,s})$ with respect to $|\mathbf{r}'|$, while for thermal particles $|u'|$ and $|\mathbf{w}'|$ are all of the same order as v_{ths}. In particular, due to the previous orderings, for the validity of GK theory the EM potentials $(\Phi,\mathbf{A})$ entering the Lagrangian must be considered to be of the form indicated above (see Eqs.(4.6) and (4.7)), namely both being of $1/O(\varepsilon)$ with respect to the remaining terms. As a consequence, the ordering (4.8) for σ_s necessarily applies, under the assumption $T_i/T_e \sim O(\varepsilon^0)$ considered here. On the other hand, as in Ref.(10), $|\mathbf{U}'|$ is to be taken as being of the order of the ion thermal velocity v_{thi}, while $|\mathbf{U}'| \sim \Omega_0 R$, with Ω_0 being the toroidal angular rotation frequency, defined below by Eq.(4.39). *It is important to stress here that these two conditions imply that Φ must satisfy the asymptotic ordering given above (4.6). Therefore, the previous orderings for σ_s and Φ must be regarded as basic prerequisites for the description of Tokamak plasmas characterized by toroidal rotation speeds comparable to the ion thermal velocity.*

By construction, in the GK description the gyrophase angle is ignorable, so that the magnetic moment m'_s is an adiabatic invariant of prescribed accuracy. In particular, the leading-order approximation is $m'_s \cong \mu'_s \equiv \frac{M_s w'^2}{2B'}$. Two further adiabatic invariants can immediately be obtained from the previous considerations. In fact, since the azimuthal angle φ is ignorable also in GK theory, the conjugate GK canonical momentum $p'_{\varphi s}$, referred to as the *guiding-center canonical momentum*, is necessarily an adiabatic invariant. Neglecting corrections of $O(\varepsilon_{M,s})$ this is given by

$$p'_{\varphi s} \equiv \frac{M_s}{B'}\left(u'I' + \frac{c\nabla'\psi'\cdot\nabla'\Phi'}{B'}\right) + \frac{Z_s e}{c}\psi', \tag{4.15}$$

which provides a third-order adiabatic invariant. We remark that both m'_s and $p'_{\varphi s}$ can in principle be identified with adiabatic invariants of $O(\varepsilon_{M,s}^{k+1})$, with $k \geq 1$ arbitrarily prescribed (21). In the following we shall make use of the local invariants (ψ_{*s}, E_s, m_s') to represent the particle state, while adopting $p_{\varphi s}'$ to deal with the dependences in terms of u'.

4.5 Vlasov kinetic theory: equilibrium KDF

Let us now proceed with constructing asymptotic solutions of the Vlasov equation holding for collisionless Tokamak plasmas when the above assumptions are valid. The treatment is based on Refs.(13, 14), where equilibrium generalized bi-Maxwellian solutions for the KDF were proved to hold for accretion disc plasmas. In particular, the following features are required for the equilibrium KDF:

1) For all of the species, different parallel and perpendicular temperatures are allowed (temperature anisotropy).

2) Non-vanishing species dependent differential toroidal and poloidal rotation velocities are included.

3) The KDF is required to be an adiabatic invariant asymptotically "close" to a local bi-Maxwellian. Hence, in particular, in the case of a locally non-rotating plasma (i.e., one for which both toroidal and poloidal rotation velocities vanish identically on a given ψ-surface) the KDF must be close to a locally non-rotating bi-Maxwellian.

It is possible to show that Requirements 1) - 3) can be fulfilled by a suitable modified bi-Maxwellian expressed solely in terms of first integrals and adiabatic invariants, including also a suitable set of structure functions $\{\Lambda_{*s}\}$ of the form (4.2) (see the precise definition below). Hence, the desired KDF is identified with an adiabatic invariant of the form

$$f_{*s} = f_{*s}\left(E_s, \psi_{*s}, p_{\varphi s}', m_s', (\psi_{*s}, \Phi_{*s}), \varepsilon^n t\right), \tag{4.16}$$

with $n \geq 1$, and where the brackets (ψ_{*s}, Φ_{*s}) denote the dependence in terms of the structure functions $\{\Lambda_{*s}(\Phi_{*s}, \psi_{*s})\}$. In particular, in agreement with assumptions 1) - 3), f_{*s} is identified with KDF of the form:

$$f_{*s} = \frac{\beta_{*s}}{(2\pi/M_s)^{3/2}\left(T_{\|*s}\right)^{1/2}} \exp\left\{-\frac{E_{*s}}{T_{\|*s}} - m_s'\alpha_{*s}\right\}, \tag{4.17}$$

which we refer to here to as the *generalized bi-Maxwellian KDF with parallel velocity perturbations*. The notation is as follows. First, $\{\Lambda_{*s}\} \equiv \left\{\beta_{*s}, \alpha_{*s}, T_{\|*s}, \Omega_{*s}, \xi_{*s}\right\}$ are structure functions subject to kinetic constraints of the type (4.2), assumed to be analytic functions of both ψ_{*s} and Φ_{*s}. These are, by definition, suitably close to appropriate fluid fields $\Lambda_s = \Lambda_s(\psi, \Phi)$. In particular, the functions Λ_s are defined as $\{\Lambda_s\} \equiv \left\{\beta_s \equiv \frac{\eta_s}{T_{\perp s}}, \alpha_s \equiv \frac{B'}{\Delta_{T_s}}, T_{\|s}, \Omega_s, \xi_s\right\}$, where η_s denotes the *pseudo-density*, $T_{\|s}$ and $T_{\perp s}$ the parallel and perpendicular temperatures, with $\frac{1}{\Delta_{T_s}} \equiv \frac{1}{T_{\perp s}} - \frac{1}{T_{\|s}}$, while Ω_s and ξ_s are the toroidal and parallel rotation frequencies. Secondly, the phase-function E_{*s}

is defined as $E_{*s} \equiv H_{*s} - p'_{\varphi s}\xi_{*s}$, while H_{*s} is identified with

$$H_{*s} \equiv E_s - \frac{Z_s e}{c}\psi_{*s}\Omega_{*s}. \tag{4.18}$$

Note that the form of f_{*s} [see Eq.(4.17)] is obtained consistent with assumption 3), namely such that when the constraint $\Omega_{*s} = \xi_{*s} = 0$ *locally holds*, f_{*s} reduces to the *non-rotating generalized bi-Maxwellian KDF* $f_{*s} = \frac{\beta_{*s}}{(2\pi/M_s)^{3/2}(T_{\|*s})^{1/2}} \exp\left\{-\frac{E_s}{T_{\|*s}} - m'_s\alpha_{*s}\right\}$. In particular, unlike Ref.(10), the definition given above for H_{*s} follows by requiring that E_{*s}, and hence also H_{*s}, is a *local* linear function of the frequencies Ω_{*s} and ξ_{*s} and of the canonical momenta $p_{\varphi s}$ and $p'_{\varphi s}$.

An equivalent representation for (4.17) can be obtained by invoking the previous definitions. This gives:

$$f_{*s} = \frac{\beta_{*s} \exp\left[\frac{X_{*s}}{T_{\|*s}}\right]}{(2\pi/M_s)^{3/2}\left(T_{\|*s}\right)^{1/2}} \exp\left\{-\frac{M_s\left(\mathbf{v} - \mathbf{W}_{*s} - U'_{\|*s}\mathbf{b}'\right)^2}{2T_{\|*s}} - m'_s\alpha_{*s}\right\}, \tag{4.19}$$

where $\mathbf{W}_{*s} = \mathbf{e}_\varphi R\Omega_{*s}$, $U'_{\|*s} = \frac{I'}{B'}\xi_{*s}$ and

$$X_{*s} \equiv M_s\frac{|\mathbf{W}_{*s}|^2}{2} + \frac{Z_s e}{c}\psi\Omega_{*s} - Z_s e\Phi + \Upsilon'_{*s}, \tag{4.20}$$

$$\Upsilon'_{*s} \equiv \frac{M_s U'^2_{\|*s}}{2}\left(1 + \frac{2\Omega_{*s}}{\xi_{*s}}\right) + \left(\frac{M_s c\nabla'\psi'\cdot\nabla'\Phi'}{B'^2} + \frac{Z_s e}{c}\psi'\right)\xi_{*s}. \tag{4.21}$$

Note that $U'_{\|*s}$ is non-zero only if the toroidal magnetic field is non-vanishing.

The following comments are in order:

1) f_{*s} is by construction a solution of the *asymptotic Vlasov equation*

$$\frac{1}{\Omega'_{cs}}\frac{d}{dt}\ln f_{*s} = 0 + O\left(\varepsilon^{n+1}\right). \tag{4.22}$$

2) f_{*s} is defined in the phase-space $\Gamma = \Gamma_r \times \Gamma_u$, where Γ_r and Γ_u are both identified with suitable subsets of the Euclidean space $\mathbb{R}^3$. In particular, f_{*s} is non-zero in the subset of phase-space where the adiabatic invariants $p'_{\varphi s}$, $\mathcal{H}'_s$ and m'_s are defined. It follows that f_{*s} is suitable for describing both circulating and trapped particles.

3) The velocity moments of f_{*s}, to be identified with the corresponding fluid fields, are unique once f_{*s} is prescribed in terms of the structure functions.

4.6 Perturbative theory

In this Section a perturbative kinetic theory for the KDF f_{*s} is developed. This is obtained by performing on f_{*s} a double-Taylor expansion for the implicit functional dependences in the variables ψ_{*s} and Φ_{*s} *carried only by the structure functions* $\{\Lambda_{*s}\}$, while leaving unchanged all of the remaining phase-space dependences. As indicated above, such asymptotic expansions can be expressed in terms of the dimensionless parameters σ_s and ε_s when the ordering (4.8) is valid. Then the double Taylor expansion gives:

$$\Lambda_{*s} \cong \Lambda_s + (\psi_{*s} - \psi)\left[\frac{\partial \Lambda_{*s}}{\partial \psi_{*s}}\right]_{\psi_{*s}=\psi,\Phi_{*s}=\Phi} + (\Phi_{*s} - \Phi)\left[\frac{\partial \Lambda_{*s}}{\partial \Phi_{*s}}\right]_{\psi_{*s}=\psi,\Phi_{*s}=\Phi} + .., \quad (4.23)$$

where both Λ_s and the partial derivatives in (4.23) are by construction functions depending only on (ψ, Φ). This implies also their general dependence in terms of the magnetic coordinates (ψ, ϑ) (see Section 6.9). We notice that the asymptotic order of the "gradients" of the structure functions $\frac{\partial \Lambda_{*s}}{\partial \psi_{*s}}$ and $\frac{\partial \Lambda_{*s}}{\partial \Phi_{*s}}$ depends whether in Λ_{*s}, ψ_{*s} and/or Φ_{*s} are considered "fast" or "slow" variables with respect to ε, in the sense that the same gradients can be considered respectively $O(\varepsilon^0)$ or $O(\varepsilon)$. In principle, different possible orderings are allowed for the perturbative expansion of f_{*s}. Here we shall assume in particular that the structure functions $\beta_{*s}, \alpha_{*s}, T_{\parallel *s}$ have fast dependences, while Ω_{*s}, ξ_{*s} have only slow ones. As a consequence, the set of derivatives $\left\{\frac{\partial \Omega_{*s}}{\partial \psi_{*s}}, \frac{\partial \Omega_{*s}}{\partial \Phi_{*s}}\right\}$ and $\left\{\frac{\partial \xi_{*s}}{\partial \psi_{*s}}, \frac{\partial \xi_{*s}}{\partial \Phi_{*s}}\right\}$ are both taken here as being $O(\varepsilon)$. It follows that to first order in ε the KDF f_{*s} can be approximated as:

$$f_{*s} \cong \widehat{f}_s \left[1 + h_{Ds}^{(1)} + h_{Ds}^{(2)}\right], \quad (4.24)$$

where the leading-order KDF $\widehat{f}_s$ does not depend on the gradients of Λ_s. Hence, all of the information about the gradients of the structure functions appears only through the first-order (in ε) perturbations $h_{Ds}^{(1)}$ and $h_{Ds}^{(2)}$. These are denoted respectively as the *diamagnetic-correction* (see Ref.(10)) and the *energy-correction* (see Ref.(14)), which result from the leading-order Taylor expansions with respect to ψ_{*s} and Φ_{*s}. In particular, the following results apply. First, $\widehat{f}_s$ is expressed as

$$\widehat{f}_s = \frac{n_s}{(2\pi/M_s)^{3/2}\left(T_{\parallel s}\right)^{1/2} T_{\perp s}} \exp\left\{-\frac{M_s\left(\mathbf{v} - \mathbf{W}_s - U'_{\parallel s}\mathbf{b}'\right)^2}{2T_{\parallel s}} - m'_s \frac{B'}{\Delta_{T_s}}\right\} \quad (4.25)$$

and hence is identified with a *bi-Maxwellian KDF with parallel velocity perturbations.* In Eq.(4.25) $\mathbf{W}_s = \Omega_s R^2 \nabla\varphi$ and $U'_{\parallel s} = \frac{I'}{B'}\xi_s$ are related to the leading-order toroidal and parallel flow velocities and depend on angular frequencies of the general form $\Omega_s = \Omega_s(\psi, \Phi)$ and $\xi_s = \xi_s(\psi, \Phi)$. In addition, the function n_s is defined in terms of the pseudo-density η_s as

$$n_s(\psi, \Phi) \equiv \eta_s(\psi, \vartheta, \Phi) \exp\left[\frac{X_s}{T_{\parallel s}}\right] \quad (4.26)$$

and

$$X_s \equiv \left(M_s \frac{|\mathbf{W}_s|^2}{2} + \frac{Z_s e}{c}\psi\Omega_s - Z_s e\Phi + \Upsilon_s' \right), \tag{4.27}$$

$$\Upsilon_s' \equiv \frac{M_s U_{\parallel s}'^2}{2}\left(1+\frac{2\Omega_s}{\xi_s}\right) + \left(\frac{M_s c\nabla'\psi'\cdot\nabla'\Phi'}{B'^2} + \frac{Z_s e}{c}\psi'\right)\xi_s. \tag{4.28}$$

Secondly, the diamagnetic and energy-correction contributions $h_{Ds}^{(1)}$ and $h_{Ds}^{(2)}$ are given by

$$h_{Ds}^{(1)} = \left\{ \frac{cM_s R}{Z_s e}\left[Y_1 + Y_3\right] + \frac{M_s R}{T_{\parallel s}}\psi\Omega_s A_3 \right\}(\mathbf{v}\cdot\widehat{e}_\varphi), \tag{4.29}$$

$$h_{Ds}^{(2)} = \frac{M_s}{2Z_s e}\left\{ Y_4 - \frac{Z_s e}{T_{\parallel s}}\frac{\psi\Omega_s}{c}C_{3s} + \frac{p_{\varphi s}'\xi_s}{T_{\parallel s}}C_{5s} \right\} v^2. \tag{4.30}$$

Here Y_i, $i = 1, 5$, is defined as

$$Y_1 \equiv \left[A_{1s} + A_{2s}\left(\frac{H_s}{T_{\parallel s}} - \frac{1}{2}\right) - \mu_s' A_{4s} \right], \tag{4.31}$$

$$Y_3 \equiv \left[\frac{p_{\varphi s}'\xi_s}{T_{\parallel s}}A_{5s} - A_{2s}\frac{p_{\varphi s}'\xi_s}{T_{\parallel s}} \right], \tag{4.32}$$

$$Y_4 \equiv \left[C_{1s} + C_{2s}\left(\frac{H_s}{T_{\parallel s}} - \frac{1}{2}\right) - \mu_s' C_{4s} \right], \tag{4.33}$$

where $H_s = E_s - \frac{Z_s e}{c}\psi_{*s}\Omega_s$ and the following definitions have been introduced: $A_{1s} \equiv \frac{\partial \ln \beta_s}{\partial \psi}$, $A_{2s} \equiv \frac{\partial \ln T_{\parallel s}}{\partial \psi}$, $A_{3s} \equiv \frac{\partial \ln \Omega_s}{\partial \psi}$, $A_{4s} \equiv \frac{\partial \alpha_s}{\partial \psi}$, $A_{5s} \equiv \frac{\partial \ln \xi_s}{\partial \psi}$ and $C_{1s} \equiv \frac{\partial \ln \beta_s}{\partial \Phi}$, $C_{2s} \equiv \frac{\partial \ln T_{\parallel s}}{\partial \Phi}$, $C_{3s} \equiv \frac{\partial \ln \Omega_s}{\partial \Phi}$, $C_{4s} \equiv \frac{\partial \alpha_s}{\partial \Phi}$, $C_{5s} \equiv \frac{\partial \ln \xi_s}{\partial \Phi}$.

The outcome of the perturbative theory is as follows:

1) The asymptotic expansion in terms of ψ_{*s} and leading to the diamagnetic-correction $h_{Ds}^{(1)}$ is formally analogous to that presented in Ref.(10). The Taylor expansion in terms of Φ_{*s} (energy expansion) is instead a novel feature of the present approach and leads to the energy-correction $h_{Ds}^{(2)}$.

2) The kinetic equilibrium f_{*s} is compatible with the species-dependent rotational frequencies Ω_s and ξ_s. No restriction follows from the KDF for their relative magnitudes, so that the general ordering $\frac{\xi_s}{\Omega_s} \sim O\left(\varepsilon^0\right)$ is permitted.

3) A fundamental feature is related to the functional dependences imposed by the kinetic constraints on the structure functions. As a basic consequence, the latter depend both on the poloidal flux ψ and the ES potential Φ. As proved below, the ES potential Φ is generally a function of the form $\Phi = \Phi(\mathbf{x}, \varepsilon^k t)$, with $\mathbf{x} = (R, z)$, i.e. it is not simply a ψ-flux function. Hence, when expressed in magnetic coordinates, the structure functions become generally of the form $\Lambda_s \equiv \overline{\Lambda_s}\left(\psi, \vartheta, \varepsilon^k t\right)$. This type of functional

dependence is expected to apply for arbitrary nested magnetic surfaces having finite inverse aspect ratio. On the other hand, in the case of large aspect ratio ($1/\delta \gg 1$), the poloidal dependences in Λ_s are expected to become negligible. Nevertheless, $h_{Ds}^{(2)}$ remains finite even in this case. The reason is that also in this limit the double Taylor expansion (4.23) still applies.

4) The coefficients A_{is} and C_{is}, $i = 1, 5$, can be identified with *effective thermodynamic forces*, containing the spatial variations of Λ_s across the $\psi = const.$ and $\Phi = const.$ surfaces respectively.

4.7 The Vlasov fluid approach

An elementary consequence concerns the fluid approach defined in terms of the Vlasov description, i.e., based on the moment equations following from the asymptotic Vlasov kinetic equation (see Eq.(4.22)). In fact, assuming that the KDF is identified with the adiabatic invariant given by Eq.(4.16), these equations are *necessarily all identically satisfied* in an asymptotic sense, namely neglecting corrections of $O\left(\varepsilon^{n+1}\right)$. Furthermore, because f_{*s} is by construction periodic, also the corresponding solubility conditions, related to the requirement of periodicity in terms of the ϑ-coordinate, are necessarily fulfilled. To prove these statements we notice that if $Q(\mathbf{x})$ is an arbitrary weight function, identified for example with $Q = \left(1, \mathbf{v}, v^2\right)$, then the generic moment of Eq.(4.22) is:

$$\int_{\Gamma_u} d^3vQ\frac{d}{dt}f_{*s} = 0 + O\left(\varepsilon^{n+1}\right), \tag{4.34}$$

where Γ_u denotes the appropriate velocity space of integration. Using the chain rule, and taking into account explicitly also the dependence in terms of $p'_{\varphi s}$, this can be written as

$$\int_{\Gamma_u} d^3vQ\left\{\frac{d\psi_{*s}}{dt}\frac{\partial f_{*s}}{\partial \psi_{*s}} + \frac{dE_s}{dt}\frac{\partial f_{*s}}{\partial E_s} + \frac{dm'_s}{dt}\frac{\partial f_{*s}}{\partial m'_s} + \frac{dp'_{\varphi s}}{dt}\frac{\partial f_{*s}}{\partial p'_{\varphi s}}\right\} = 0 + O\left(\varepsilon^{n+1}\right). \tag{4.35}$$

On the other hand, Eq.(4.34) can also be represented as

$$\int_{\Gamma_u} d^3v\left\{\frac{d}{dt}\left[Qf_{*s}\right] - f_{*s}\frac{d}{dt}Q\right\} = 0 + O\left(\varepsilon^{n+1}\right), \tag{4.36}$$

which recovers the usual form of the velocity-moment equations in terms of suitable (and *uniquely defined*) fluid fields. For $Q = (1, \mathbf{v})$ one obtains, in particular, that the species continuity and linear momentum fluid equations are satisfied identically up to infinitesimals of $O\left(\varepsilon^{n+1}\right)$. Similarly, the law of conservation of the species total canonical momentum can be recovered by setting $Q = \psi_{*s}$, namely

$$\int_{\Gamma_u} d^3v\frac{d}{dt}\left[\psi_{*s}f_{*s}\right] = 0 + O\left(\varepsilon^{n+1}\right). \tag{4.37}$$

In the stationary case this implies the customary species angular momentum conservation law for the species angular momentum $L_s^{tot} \equiv M_s R^2 n_s^{tot} \mathbf{V}_s^{tot} \cdot \nabla\varphi$. Here the notation is standard. In particular the velocity moments of the KDF $\left\{ n_s^{tot}, \mathbf{V}_s^{tot}, \underline{\underline{\Pi}}_s^{tot}, L_{cs}^{tot} \right\}$ can be introduced, to be referred to as *species number density, flow velocity, tensor pressure* and *canonical toroidal momentum.* They are defined by the integrals $\int_{\Gamma_u} d^3v Q f_{*s}$, where Q is now identified respectively with $Q = \left\{ 1, \frac{\mathbf{v}}{n_s^{tot}}, M_s \left(\mathbf{v} - \mathbf{V}_s^{tot}\right)\left(\mathbf{v} - \mathbf{V}_s^{tot}\right), \frac{Z_s e}{c}\psi_{*s} \right\}$. It is worth remarking here that *the velocity moments are unique once the KDF* f_{*s} [see Eq.(4.17)] *is prescribed in terms of the structure functions* $\{\Lambda_{*s}\}$. On the other hand, as a result of Eqs.(4.22) and (4.34), it follows that the stationary fluid moments calculated in terms of the KDF f_{*s} are identically solutions of the corresponding stationary fluid moment equations.

We conclude this section by pointing out that no restrictions can possibly be required on the KDF and the EM potentials as a consequence of the validity of these moment equations. Therefore, *the only possible constraints on the KDF are necessarily only those arising from the solubility conditions of the Maxwell equations.*

4.8 Constitutive equations for species number density and flow velocity

In this Section we present the leading-order expressions of the species number density and flow velocities predicted by the kinetic equilibrium. The calculation of these fluid moments is required for the subsequent analysis of the Maxwell equations. An explicit calculation of the moment integrals can be carried out by adopting the perturbative asymptotic expansion of f_{*s} described in Section 6.6. This also requires performing an *inverse GK transformation*, by expressing all of the guiding-center quantities appearing in the equilibrium KDF in terms of the actual particle position, according to Eq.(4.13).

Consider first the evaluation of the species flow velocity $\mathbf{V}_s^{tot}$. Adopting the GK representation for the particle velocity, the leading-order contribution to the flow velocity is found to be

$$\mathbf{V}_s \cong \mathbf{U} + \frac{I}{B}\left(\Omega_s + \xi_s\right)\mathbf{b} + \frac{T_{\perp s}}{T_{\parallel s}}\left[R^2\left(\Omega_s + \xi_s\right)\nabla\varphi - \mathbf{U}\right] \cdot \left(\underline{\underline{\mathbf{1}}} - \mathbf{bb}\right), \tag{4.38}$$

where $\mathbf{U}$ is the frozen-in velocity defined by Eq.(4.14). Then, ignoring correction of $O(\varepsilon)$, $\mathbf{U}$ can be approximated as $\mathbf{U} \cong R^2 \Omega_o \nabla\varphi \cdot \left(\underline{\underline{\mathbf{1}}} - \mathbf{bb}\right)$. Here Ω_o is the *species-independent* and *ψ-flux function* (10)

$$\Omega_0\left(\psi, \varepsilon^k t\right) \equiv c\frac{\partial\langle\Phi\rangle}{\partial\psi} \tag{4.39}$$

and $\langle\Phi\rangle = \varkappa^{-1}\frac{d\vartheta}{\mathbf{B}\cdot\nabla\vartheta}\Phi$ denotes the ψ-surface average, with $\varkappa^{-1} \equiv \frac{d\vartheta}{\mathbf{B}\cdot\nabla\vartheta}$. Then, in terms of the relative toroidal frequency $\Delta\Omega_s \equiv \Omega_s - \Omega_o$, the leading-order flow velocity becomes

$$\mathbf{V}_s \cong \left[\Omega_o + \frac{T_{\perp s}}{T_{\parallel s}}\left[\Delta\Omega_s + \xi_s\right]\right] R^2\nabla\varphi + \left[\Delta\Omega_s + \xi_s\right]\frac{I}{B}\left(1 - \frac{T_{\perp s}}{T_{\parallel s}}\right)\mathbf{b}. \tag{4.40}$$

This implies that $\mathbf{V}_s$ can be decomposed in terms of the *total toroidal and poloidal rotation velocities*

$$V_{Ts}(\psi,\vartheta,\Phi) \equiv R\Omega_{Ts} = \mathbf{V}_s\cdot\mathbf{e}_\varphi, \tag{4.41}$$

$$V_{Ps}(\psi,\vartheta,\Phi) \equiv \frac{\Omega_{Ps}}{|\nabla\vartheta|} = \mathbf{V}_s\cdot\mathbf{e}_P, \tag{4.42}$$

where $\mathbf{e}_P \equiv \frac{\nabla\vartheta}{|\nabla\vartheta|}$, and the corresponding rotation frequencies Ω_{Ts} and Ω_{Ps} are respectively:

$$\Omega_{Ts} = \Omega_o + \frac{T_{\perp s}}{T_{\parallel s}}\left[\Delta\Omega_s + \xi_s\right] + \left[\Delta\Omega_s + \xi_s\right]\frac{I^2}{B^2R^2}\left(1 - \frac{T_{\perp s}}{T_{\parallel s}}\right), \tag{4.43}$$

$$\Omega_{Ps} = \left[\Delta\Omega_s + \xi_s\right]\frac{I}{B^2J}\left(1 - \frac{T_{\perp s}}{T_{\parallel s}}\right), \tag{4.44}$$

with $\frac{1}{J} \equiv \nabla\psi\times\nabla\varphi\cdot\nabla\vartheta$.

We remark that:

1) To leading-order in ε, the poloidal flow velocity (4.42) is non-zero *only in the presence of temperature anisotropy.* More precisely, provided that $\frac{T_{\perp s}}{T_{\parallel s}} \neq 1$, a non-vanishing V_{Ps} may arise only if $\Delta\Omega_s + \xi_s \neq 0$. Therefore, even if Ω_s coincides with the frozen-in frequency Ω_0, Ω_{Ps} is different from zero if $\xi_s \neq 0$.

2) The effect of the contributions $\Delta\Omega_s$ and ξ_s is analogous, although their physical origins are different. In particular $\Delta\Omega_s$ represents the departure from the frozen-in rotation velocity Ω_o, while ξ_s determines the parallel velocity perturbation in the KDF.

3) If the frozen-in condition is invoked, namely $\Omega_s \equiv \Omega_o$, Eq.(4.40) becomes

$$\mathbf{V}_s \cong \Omega_o R^2\nabla\varphi + \xi_s\left[\frac{T_{\perp s}}{T_{\parallel s}}R^2\nabla\varphi + \frac{I}{B}\left(1 - \frac{T_{\perp s}}{T_{\parallel s}}\right)\mathbf{b}\right], \tag{4.45}$$

which takes into account both finite poloidal rotation and temperature anisotropy. In the case of isotropic temperatures, i.e., $\frac{T_{\perp s}}{T_{\parallel s}} = 1$, this equation provides a purely toroidal flow given by $\mathbf{V}_s \cong (\Omega_o + \xi_s) R^2\nabla\varphi$. When $\xi_s \equiv 0$ this reduces to the customary result (10), namely $\mathbf{V}_s \cong \Omega_o R^2\nabla\varphi$.

Finally, the calculation for the number density n_s^{tot} is as follows. Neglecting again first-order diamagnetic and energy-correction contributions, the leading-order species number density is found to be

$$n_s = \eta_s(\psi,\Phi)\exp\left[\frac{\widehat{X}_s - Z_s e\Phi}{T_{\parallel s}}\right], \tag{4.46}$$

where

$$\begin{aligned}\widehat{X}_s \equiv &\frac{M_s}{2}\frac{I^2}{B^2}(\Omega_s+\xi_s)^2-\frac{M_s}{2}U^2+\left[M_sR^2\mathbf{U}\cdot\nabla\varphi+\frac{Z_se}{c}\psi\right]\Omega_s+\\ &+\left[\frac{M_s}{B}\frac{c\nabla\psi\cdot\nabla\Phi}{B}+\frac{Z_se}{c}\psi\right]\xi_s+\frac{M_s}{2}\frac{T_{\perp s}}{T_{\parallel s}}(\Delta\Omega_s+\xi_s)^2\left[R^2-\frac{I^2}{B^2}\right]\end{aligned} \tag{4.47}$$

contains the combined contribution of the kinetic energies carried by the rotation frequencies Ω_s, ξ_s and the frozen-in velocity $\mathbf{U}$.

4.9 Quasi-neutrality

In this Section the implications of the quasi-neutrality condition following from the Poisson equation are investigated. Here by quasi-neutrality we mean that the equation

$$\sum_s Z_s e n_s^{tot} = 0 \tag{4.48}$$

is satisfied asymptotically in the sense that

$$\frac{|\nabla\cdot\mathbf{E}|}{\left|\sum_s Z_s e n_s^{tot}\right|}\sim\frac{O\left(\varepsilon_D^2\right)}{O\left(\varepsilon\right)}, \tag{4.49}$$

with $\varepsilon_D \equiv \frac{\lambda_D}{\Delta L} \ll 1$ denoting the Debye-length dimensionless parameter, with $\lambda_D \sim \lambda_{Ds} = \tau_{ps} v_{ths}$, $\Delta L \sim \Delta L_s = \Delta t_s v_{ths}$, and n_s^{tot} being the total species-number density. We intend to show that the first two terms in the Laurent expansion (4.6) of Φ can be determined from Eq.(4.48) by prescribing n_s^{tot} to leading-order in ε, namely in terms of Eq.(4.46). In particular the following result holds.

THM.1 - Explicit form of the ES potential Φ.

Let us assume that the species KDF is defined by Eq.(4.17) and the finite aspect-ratio ordering applies. Then, imposing the quasi-neutrality condition (4.48) in the case of a two-species ion-electron plasma, the following propositions hold:

T1$_1$) *Correct through $O\left(\varepsilon^0\right)$, the ES potential satisfies the asymptotic implicit equation*

$$\Phi \simeq \frac{S(\psi,\vartheta,\Phi)}{e\left(\frac{Z_i}{T_{\parallel i}}+\frac{1}{T_{\parallel e}}\right)}, \tag{4.50}$$

where $S(\psi,\vartheta,\Phi)$ is the source term given by

$$S(\psi,\vartheta,\Phi)\equiv\ln\left(\frac{\eta_e}{Z_i\eta_i}\right)+\left[\frac{\widehat{X}_e}{T_{\parallel e}}-\frac{\widehat{X}_i}{T_{\parallel i}}\right], \tag{4.51}$$

with η_s being the species pseudo-density and the quantity $\widehat{X}_s = \widehat{X}_s(\psi,\vartheta,\Phi)$ defined by

Eq.(4.47).

T1$_2$) *If the temperatures are non-isotropic, then the species pseudo-density is generally of the form* $\eta_s = \eta_s(\psi, \vartheta, \Phi)$. *Instead, in the case of isotropic temperatures,* $\eta_s = \eta_s(\psi, \Phi)$.

T1$_3$) *A particular solution consistent with the kinetic constraints is obtained letting* $Z_i \eta_i = \eta_e$.

T1$_4$) *In particular, in validity of T1$_3$, correct through* $O(\varepsilon^0)$ *the ES potential* Φ *is uniquely determined by Eq.(4.50) and is necessarily of the form (4.6), where* Φ_{-1} *obeys the equation*

$$\Phi_{-1}(\psi) \cong \frac{\psi\left[\frac{Z_i(\Omega_i+\xi_i)}{T_{\parallel i}} + \frac{\Omega_e+\xi_e}{T_{\parallel e}}\right]}{c\left(\frac{Z_i}{T_{\parallel i}} + \frac{1}{T_{\parallel e}}\right)}, \tag{4.52}$$

while Φ_0 *is obtained subtracting* Φ_{-1} *from Eq.(4.50).*

PROOF - T1$_1$ - The proof of the first statement can be obtained from Eq.(4.48) by substituting for the species number density the leading-order solution given by Eq.(4.46). T1$_2$ - The proof follows by noting that, in validity of the kinetic constraint on β_{*s}, the species pseudo-density is such that $\frac{\eta_s}{T_{\perp s}} = \frac{\eta_s}{T_{\perp s}}(\psi, \Phi)$. On the other hand, from the kinetic constraint imposed on α_{*s} and the prescriptions that $B = B(\psi, \vartheta)$ and $T_{\parallel s} = T_{\parallel s}(\psi, \Phi)$, it must be that $T_{\perp s}$ is necessarily of the type $T_{\perp s} = T_{\perp s}(\psi, \vartheta, \Phi)$. Therefore, the general dependence of the pseudo-density is also necessarily of the form $\eta_s = \eta_s(\psi, \vartheta, \Phi)$. On the other hand, in the limit of isotropic temperatures $T_{\perp s} = T_{\parallel s} = T_s(\psi, \Phi)$ and $\alpha_{*s} = 0$. The functional dependence of η_s becomes therefore of the type $\eta_s = \eta_s(\psi, \Phi)$. T1$_3$ - Due to the arbitrariness of the structure function β_{*s}, it follows that β_e and β_i can always be defined in such a way that $\frac{\beta_e}{\beta_i}\frac{T_{\perp e}}{T_{\perp i}} = 1$ even when $T_{\perp i} \neq T_{\perp e}$. In particular, this constraint is consistent with the requirement that the ES potential vanishes identically in the absence of toroidal and poloidal rotations. T1$_4$ - By definition the Poisson equation, subject to suitable boundary conditions, must determine completely (i.e., uniquely) the ES potential Φ. Therefore, Eq.(4.50) necessarily gives the complete solution, correct through $O(\varepsilon^0)$. In particular, by inspecting the order of magnitude of the different contributions in the source term $\widehat{X}_s(\psi, \vartheta, \Phi)$, the Laurent expansion (4.6) can be introduced. In particular, by retaining in $\widehat{X}_s(\psi, \vartheta, \Phi)$ only contributions of $1/O(\varepsilon)$, $\Phi_{-1}(\psi)$ is found to obey Eq.(4.52). **Q.E.D.**

A fundamental implication of THM.1, and in particular of the validity of Eq.(4.6), is to assure the consistency of the perturbative σ_s-expansion as well as the orderings introduced in Sections 6.3 and 6.4. In fact, let us inspect the order of magnitude (with respect to the parameter ε) of the r.h.s. of Eq.(4.52). For definiteness, let us assume that $(\Omega_i + \xi_i) \sim (\Omega_e + \xi_e) \sim \Omega_0$, requiring $T_{\parallel i}/T_{\parallel e} \sim O(\varepsilon^0)$ and $Z_i \sim O(\varepsilon^0)$. Due to Eq.(4.39) it follows that the order of magnitude of Φ_{-1} is $\Phi_{-1} \sim \psi\frac{\Omega_0}{c}$. On the basis of this conclusion, the following statement holds.

Corollary to THM.1 - Consistency with the σ_s-expansion.

Given validity of THM.1 and the quasi-neutrality condition, invoking the previous assumptions it follows that $\sigma_i \sim \sigma_e \sim O(\varepsilon)$.

PROOF - First, by assumption $\sigma_i \sim \left|\frac{\frac{M_i}{2}v_{thi}^2}{Z_i e\Phi}\right|$ and $\sigma_e \sim \left|\frac{\frac{M_e}{2}v_{the}^2}{e\Phi}\right|$. As a consequence, due to the previous hypotheses $\sigma_i \sim \sigma_e$. Furthermore, due to quasi-neutrality, it follows that

$$\sigma_i \sim \left|\frac{M_s v_{thi}\frac{1}{2}v_{thi}}{Z_i e\psi\frac{\Omega_0}{c}}\right| \sim \left|\frac{M_s v_{thi}R}{\frac{Z_i e\psi}{c}}\right|\left|\frac{\frac{1}{2}v_{thi}}{\Omega_0 R}\right|. \tag{4.53}$$

The order of magnitude of the two factors on the r.h.s follows from the asymptotic ordering for the canonical-momentum parameter and the requirement indicated above that $\Omega_0 R \sim v_{thi}$ (see also Ref.(10)). It is concluded that, since by construction $O(\varepsilon_i) \sim O(\varepsilon)$, $\left|\frac{M_s v_{thi}R}{\frac{Z_i e\psi}{c}}\right| \sim O(\varepsilon)$, while $\left|\frac{\frac{1}{2}v_{thi}}{\Omega_0 R}\right| \sim O(\varepsilon^0)$, which manifestly implies the thesis. **Q.E.D.**

The following further remarks are useful in order to gain insight in the previous results.

1) Eq.(4.50) represents the general solution holding in the case of a two-species plasma characterized by temperature anisotropy, poloidal and toroidal flow velocities.

2) In Eq.(4.52) all quantities $\Lambda_s^{(1)} \equiv \left\{\Omega_s, \xi_s, T_{\|s}\right\}$ can be considered (to leading-order in ε) as being only ψ-functions, namely of the form $\Lambda_s^{(1)} = \Lambda_s^{(1)}(\psi)$. Therefore, Eq.(4.52) provides an ODE for $\Phi_{-1}(\psi)$.

4.10 The Ampere equation

Let us now investigate the constraints imposed by the Ampere law on the leading-order current densities and equilibrium flows. Let us consider the case of a two-species plasma. The following results apply.

THM.2 - Constraints on poloidal and toroidal flows.

Given validity of the asymptotic Vlasov kinetic equation (4.22) for the species KDF defined by Eq.(4.17), the quasi-neutrality condition (4.48) and the magnetized-plasma asymptotic orderings (see Section 6.3), for a two-species plasma the following propositions hold:

T2$_1$) *The poloidal flow velocity $V_{Ps}(\psi, \vartheta, \Phi)$ may be either species-dependent or independent. In the first case necessarily the constraint condition*

$$\frac{\partial}{\partial\vartheta}\left[\sum_s Z_s e n_s V_{Ps}(\psi, \vartheta, \Phi)\right] = 0 \tag{4.54}$$

must be fulfilled. In the second case, if Eq.(4.54) is not satisfied, the corresponding total equilibrium current density must vanish identically.

T2$_2$) *In both cases, the toroidal flow velocity remains species-dependent, so that the corresponding current density is generally non-vanishing.*

T2$_3$) *Both poloidal and toroidal magnetic fields can be self-generated by the plasma.*

PROOF - T2$_1$ - Let us consider first the component of Ampere's equation along the directions orthogonal to $\nabla\varphi$. This gives the following set of two scalar equations for the toroidal magnetic field $I\nabla\varphi$:

$$\frac{\partial I}{\partial \psi} = \frac{4\pi}{c}\sum_s Z_s e n_s V_{Ps}(\psi,\vartheta,\Phi)\left[1+O(\varepsilon)\right], \tag{4.55}$$

$$\frac{\partial I}{\partial \vartheta} = 0+O(\varepsilon), \tag{4.56}$$

implying manifestly the solubility condition (4.54). Therefore, either the total poloidal current density is a ψ-function, or the poloidal flow velocity $V_{Ps}(\psi,\vartheta,\Phi)$ must be species-independent. The first condition can always be satisfied by suitably selecting the species pseudo-density. In fact, even in validity of $T1_3$, the species pseudo-density can be defined in such a way as to satisfy the constraint (4.54). Therefore, excluding the null solution, a non-vanishing current density must appear when V_{Ps} is species-dependent.

$T2_2$ - The proof of the second statement follows by noting that, when proposition $T2_1$ is valid, the quantity $\frac{T_{\perp s}}{T_{\parallel s}}\left[\Delta\Omega_s+\xi_s\right]$ may still remain species-dependent. As a consequence, by direct inspection of Eq.(4.41), it follows that the toroidal current density is generally non-null.

$T2_2$ - Thanks to the previous propositions, it follows that both the toroidal and poloidal magnetic fields can be self generated. In particular, the self toroidal field necessarily requires the presence of temperature anisotropy, while the poloidal self field may arise even in the case of isotropic temperature, due to deviations from the frozen-in condition $\Omega_s=\Omega_o$ and/or parallel velocity perturbations associated to ξ_s. **Q.E.D.**

We briefly mention the case of a multi-species plasma. In fact, in collisionless systems plasma sub-species can be introduced, simply based on the topology of their phase-space trajectories. For example, different species can be identified distinguishing between circulating and magnetically-trapped particles. These components can in principle be characterized by KDFs carrying different structure functions, and in particular different poloidal flow velocities. In this case both the poloidal and toroidal flow velocities remain generally species-dependent. Therefore, the corresponding current densities may be expected to be non-vanishing.

4.11 Comparisons with literature

An interesting issue is related to comparisons with the literature. For what concerns the kinetic formulation, the relevant benchmark is represented by Ref.(10), where the theory of collisional transport in toroidally rotating plasmas was investigated. Although the conceptual foundations of the perturbative kinetic approach adopted here have already been exhaustively detailed in Sections 6.2 to 6.10, it is worth analyzing some differences arising between the two approaches. In detail, besides the inclusion of temperature anisotropy, parallel and toroidal velocity perturbations as well as the prescription of the kinetic constraints, the main differences from Ref.(10) are as follows:

1) The first one lies in the choice of equilibrium KDF. This is related, in particular, to the different definition adopted here for the dynamical variable H_{*s} (see Eq.(4.18)). The motivation for this definition have been detailed in Section 5. This choice permits one to obtain an explicit analytical solution for the leading-order ES potential $\Phi_{-1}(\psi)$ (see THM.1), based uniquely on the quasi-neutrality condition rather than imposing fluid constraints (see Ref.(10)).

2) In the present approach *no constraints arising from the moment (i.e., fluid) equations* are placed on the structure functions $\{\Lambda_{*s}\}$ [see Eq.(4.2)] and consequently on the velocity moments of the KDF f_{*s}. In particular, in our case, unlike the case of collisional plasmas treated in Ref.(10), the general form of the equilibrium species-fluid velocity $\mathbf{V}_s$ is merely a consequence of the form prescribed for the equilibrium KDF. Therefore, it cannot follow from imposing the validity of fluid equations, but only from the solubility conditions of the Maxwell equations.

3) The analysis of the Ampere equation has been carried out to investigate its consequences for the toroidal and poloidal species-flow velocities in the presence of temperature anisotropy (see THM.2). The discussion extends the treatment given in Ref.(10), where only differential toroidal flows were retained in the kinetic treatment.

Let us now consider, for the sake of reference, also the case of statistical fluid approaches. Such treatments (including those adopting multi-fluid formulations) typically do not rely on kinetic closure conditions and/or include FLR as well as perturbative kinetic effects, such as diamagnetic and energy-correction contributions. Further issues include:

1) The treatment of kinetic constraints. As shown here, kinetic constraints are critical for the construction of the KDF. They allow the structure functions to retain, in principle, both ψ (leading-order) and ϑ (first-order) dependences. The correct functional form of the fluid fields, arising as a consequence of the kinetic constraints, may not be correctly retained in customary fluid treatments (see for example Refs.(22, 23)).

2) The proper inclusion of slowly time-dependent temperature anisotropies and pressure anisotropies. As pointed out here, the functional form of the parallel and perpendicular temperature is related to microscopic conservation laws, in particular particle magnetic moment conservation. On the other hand, fluid approaches normally ignore such constraints. Even when kinetic closure conditions are invoked for the pressure tensor (see for example Ref.(23)), their validity may become questionable if they are not based on consistent equilibrium solutions for the KDF.

3) Another example-case is provided by the kinetic prescription for the expression of the number density, here shown to exhibit a complex dependence in terms of the ES potential, centrifugal potential and toroidal and parallel frequencies (for comparison see Ref.(23)).

4) Finally, the functional form of the poloidal flow velocity may differ from what can be obtained by adopting a two-fluid approach (22). In particular, in our treatment the toroidal and parallel rotation frequencies are considered independent of each other, so that kinetic constraints need to be imposed separately on Ω_s and ξ_s. Furthermore, according to the kinetic treatment, a non-vanishing equilibrium poloidal flow velocity

can only appear in the presence of temperature anisotropy.

4.12 Concluding remarks

In this Chapter, a theoretical formulation of quasi-stationary configurations for collisionless and axisymmetric Tokamak plasmas has been presented. This is based on a kinetic approach developed within the framework of the Vlasov-Maxwell description. It has been shown that a new type of asymptotic kinetic equilibrium exists, which can be described in terms of generalized bi-Maxwellian distributions. By construction, these are expressed in terms of the relevant particle first integrals and adiabatic invariants. Such solutions permit the consistent treatment of a number of physical properties characteristic of collisionless plasmas. These include, in particular, differential toroidal rotation and finite temperature anisotropy and poloidal flows in non-uniform multi-species Tokamak plasmas subject to intense quasi-stationary magnetic and electric fields. The existence of these solutions has been shown to be warranted by imposing appropriate kinetic constraints for the structure functions which appear in the species distribution functions. By construction, the theory assures the validity of the fluid moment equations associated with the Vlasov equation. In particular, the novelty of the approach lies in the explicit construction of asymptotic solutions for the fluid equations in terms of constitutive equations for the fluid fields. The approach is based on a perturbative asymptotic expansion of the equilibrium distribution function, which allows also the determination of diamagnetic and energy-correction contributions. The latter are found to be linearly proportional to suitable effective thermodynamic forces. Finally, the constraints placed by the Maxwell equations have been investigated. As a result, the electrostatic potential has been determined by imposing the quasi-neutrality condition. Furthermore, it has been shown that non-trivial solutions for the toroidal and poloidal species rotation frequencies are allowed consistent with the solubility conditions arising from the Ampere law. The discussion presented here can provide a useful background for future investigations of Tokamak plasmas.

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Chapter 5

Exact solution of the EM radiation-reaction problem for classical finite-size and Lorentzian charged particles

5.1 Introduction

An unsolved theoretical problem is related to the description of the dynamics of classical charges with the inclusion of their electromagnetic (EM) self-fields, the so-called *radiation-reaction* (RR) *problem* (Dirac (1), Pauli (2), Feynman (3)). Despite efforts spent by the scientific community in more than a century of intensive theoretical research, an exact solution is still missing (see related discussion in Ref.(4); for a review see Refs.(5, 6, 7, 8, 9)). In this regard, of fundamental importance is the construction of the *exact (i.e., non-asymptotic) relativistic equation of motion for a classical charged particle in the presence of its EM self-field,* also known as *RR equation.* This concerns, in particular, its treatment in the context of *special relativity* (SR) and *classical electrodynamics* (CE), namely imposing the following basic physical requirements, hereafter referred to as *SR-CE Axioms*:

1 Axiom #1: the Maxwell equations are fulfilled everywhere in the flat space-time $M^4 \subseteq \mathbb{R}^4$, with metric tensor $g_{\mu\nu}$. The Minkowski metric tensor is denoted as $\eta_{\mu\nu} \equiv diag(1, -1, -1, -1)$. In particular the EM 4-potential A^μ is assumed to be of class $C^k(M^4)$, with $k \geq 2$;

2 Axiom #2: the Hamilton variational principle holds for a suitable functional class of variations $\{f\}$. In particular, the Hamilton principle must uniquely prescribe the particle world-line as a real function $r^\mu(s) \in C^k(\mathbb{R})$, with $k \geq 2$ for all $s \in \mathbb{R}$. The RR equation is then determined by the corresponding Euler-Lagrange (E-

L) equations. Hence, $\{f\} \equiv \{f_i(s), i = 1, n\}$ is identified with the set of real functions of class $C^k(\mathbb{R})$, with $k \geq 2$:

$$\{f\} \equiv \left\{ \begin{array}{c} f_i(s) : f_i(s) \in C^k(\mathbb{R}); \\ i = 1, n; \;\; and k \geq 2 \end{array} \right\}, \tag{5.1}$$

with functions $f_i(s)$ (for $i = 1, n$) to be properly defined. In particular, we shall require that the action functional is allowed to be of the general form

$$S_1(f, [f]) \equiv \int_{-\infty}^{+\infty} ds L_1 \left(f(s), \frac{df(s)}{ds}, [f(s)], \left[\frac{df(s)}{ds} \right] \right). \tag{5.2}$$

Here L_1 denotes a non-local *variational particle Lagrangian*, by assumption defined on a *finite-dimensional* phase-space, which depends at most on first-order derivatives $\frac{df(s)}{ds}$, with $f(s)$ belonging to the functional class $\{f\}$, while $\left\{ f(s), \frac{df(s)}{ds} \right\}$ and $\left\{ [f(s)], \left[\frac{df(s)}{ds} \right] \right\}$ indicate respectively local and non-local dependencies in terms of $f(s)$ and $\frac{df(s)}{ds}$;

3 Axiom #3: the Newton determinacy principle (NDP) holds. This implies the validity of an existence and uniqueness theorem for the corresponding E-L equations. As a consequence, there exists necessarily a *classical dynamical system*, namely a diffeomorphism

$$\mathbf{x}_0 \equiv \mathbf{x}(s_0) \rightarrow \mathbf{x}(s), \tag{5.3}$$

with $\mathbf{x} \in I$ and s representing respectively the state of a classical particle and a suitable proper time, where $I \subseteq \mathbb{R}$ is an appropriate finite interval of the real axis;

4 Axiom #4: the Einstein causality principle (ECP) and the Galilei inertia principle (GIP) both apply;

5 Axiom #5: the general covariance property of the theory, and in particular the so-called manifest Lorentz covariance (MLC), i.e., the covariance with respect to the group of special Lorentz transformations, are satisfied.

Manifestly, these axioms are understood as *identically* fulfilled, i.e., they must apply for arbitrary choices of both the initial conditions for the dynamics of the charged particles and the applied external EM field.

The RR problem was first posed by Lorentz in his historical work (Lorentz, 1892 (10); see also Abraham, 1905 (11)). Traditional approaches are based either on the RR equation due to Lorentz, Abraham and Dirac (first presented by Dirac in 1938 (1)), nowadays popularly known as the *LAD equation*, or the equation derived from it by Landau and Lifschitz (12) via a suitable "reduction process" , the so-called *LL equation*. As recalled elsewhere (4) several aspects of the RR problem - and of the LAD and LL equations - are yet to find a satisfactory formulation/solution. Common feature of all previous approaches is the adoption of an asymptotic expansion for the EM self-field (or for the corresponding EM 4-potential), rather than of its exact representation. This, in turn, implies that such methods allow one to determine - at most - only an asymptotic approximation for the correct RR equation.

For contemporary science the solution of the RR problem represents a fundamental prerequisite for the proper formulation of all relativistic theories, both classical and quantum ones, which are based on the description of relativistic dynamics for classical charged particles.

Since Lorentz famous paper (10) several textbooks and research articles have appeared on the subject of RR. Many of them have criticized aspects of the RR theory, and in particular the LAD and LL equations (for a review see (5, 6, 7, 8, 9), where one can find the discussion of the related problems). However, despite contrary claims (13, 14, 15, 16, 17, 18), rigorous results are scarce (4). In particular, most of previous investigations concern the treatment of point charges. These are usually based either on suitable asymptotic approximations or regularization schemes to deal with intrinsic divergences of the point-charge model. On the other hand, there is no obvious classical physical mechanism, consistent with the SR-CE axioms, which can explain the appearance of a *finite* EM self-force acting on a point charge. This should arise as a consequence of a *finite delay time* occurring between the particle position at the time of the generation of its EM self-field and the instantaneous particle position. It is well-known, as discovered by Lorentz himself (Lorentz, 1892 (10); see also for example Landau and Lifschitz, 1951 (12)) that such a force can act on a charged particle only if the particle itself is actually *finite-size*. Therefore, although "ad hoc" models based on the adoption of a finite delay time have been known for a long time (see for example the heuristic approach to the RR problem by Caldirola, 1956 (18) leading to a delay-type differential equation), the treatment of extended charge distributions emerges as the only possible alternative, in analogy with the case of the Debye screening problem in electrostatics (19). In this regard, a first approach in this direction is provided by the paper by Nodvik (Nodvik,1964 (20)), where a variational treatment for point mass particles having finite-size charge distributions was developed. However, charge and mass are expected to have the same support, as required, for example, by the energy-momentum conservation law in both special and general relativity. Therefore a fully consistent relativistic theory should actually be formulated for *finite-size particles.* From the analysis of previous literature two important related problems arise:

- Issue #1 - *Existence of an exact variational RR equation:* this refers to the lack of an exact RR equation, based on Hamilton variational principle, even for classical point-particles (or point-masses). In fact, previous approaches have all been based on approximate (i.e., asymptotic) estimates. Example of this type leading to the well-known LAD equation (Lorentz, Abraham and Dirac (1, 10, 11, 21)) are those due to Nodvik (20) and Medina (16). A critical aspect of the LAD equation, as well as of the related LL (Landau and Lifschitz, 1951 (12)) equation, is that it does not satisfy a variational principle in the customary sense, i.e., according to Axiom #2 (4). In particular, the resulting LAD equation is only asymptotic and *non-variational* in the sense of Axiom #2. Instead, the LL is non-variational, i.e., it *does not* admit a variational action at all. However, the problem arises whether, in the context of special relativity, an *exact RR equation* actually exists which holds for suitable classical finite-size charged particles, and for Lorentzian particles as a limiting case, namely finite-size charges having point-mass distributions. Important related issues follow, such as the possibility for the resulting equation to admit a *standard Lagrangian form* in terms of a *non-local effective Lagrangian function*, and to be cast in an equivalent *conservative form,* as the divergence of an *effective stress-energy tensor.* Finally, the recovery of the customary LAD equation in a suitable approximation must be verified.

- Issue #2 - *Existence and uniqueness problem:* the second issue is related to the consistency of the variational RR equation with the SR-CE axioms and in particular with NDP. Therefore, the question arises whether an existence and uniqueness theorem for the corresponding initial value problem can be reached or not. Clearly the problem is relevant only for the exact RR equation.

The possible solution of these problems has potential wide-ranging implications which are related to the description of relativistic dynamics of systems of classical finite-size particles both in special and general relativity.

5.2 Goals of the investigation

The aim of the research program presented in this Chapter is to provide a consistent and exact theoretical formulation of the RR problem for classical charged particles with *finite-size charge and mass distributions*, addressing precisely issues #1 and #2 (22). In this investigation the case is considered of extended particles having mass and charge distributions localized on the same support, identified with a surface shell (see below for a complete rigorous definition). The result is obtained without introducing any perturbative or asymptotic expansion for the evaluation of EM self 4-potential and/or "ad hoc" regularization schemes for its point-particle limit. In particular, finite-size charge distributions are introduced in order to avoid intrinsic divergences (characteristic of the point-charge treatment) and achieve an analytical description of the RR phenomena which is consistent with the SR-CE axioms. A covariant representation for the EM self 4-potential is obtained, uniquely determined by the prescribed charge current density. This allows us to point out the characteristic non-local feature of the EM self-field, which is due to a causal retarded effect, produced by the finite spatial extension of the charge. Here we shall restrict the analysis to the treatment of charge and mass translational motion, leaving the inclusion of rotational dynamics to a subsequent study. Therefore, a suitable mathematical formulation of the problem is given, in which spatial rotational degrees of freedom are effectively excluded from the present investigation. As a further result, it is proved that the exact RR equation obtained here also holds for *classical non-rotating Lorentzian particles* (Lorentz, 1892(10)), i.e., in the case in which the mass is regarded as point-wise localized and only the charge has a finite spatial extension. The approach adopted here is based on the variational formulation for finite-size charged particles earlier pointed out by Tessarotto *et al.* (23), in turn relying on the hybrid form of the synchronous variational principle (24, 25). A key feature of this variational principle is the adoption of superabundant dynamic variables (26)(see also related discussion in Sections 5.5 and 5.7). Due to the arbitrariness of their definition, they can always be identified with the components of the particle position and velocity 4-vectors r^μ and u^μ. This also implies that, by construction, the variational functional necessarily satisfies the property of covariance and MLC. Then, the corresponding E-L equations yield both the RR equation and also the required physical realizability constraints for r^μ and u^μ, which allow one to identify them with physical observables.

The reference publications for the results presented in this Chapter are Refs.(22, 27).

5.3 Charge and mass current densities

In this section the model of finite-size classical particle is defined, prescribing its mass and charge distributions, and determining the corresponding covariant expressions for the charge and mass current densities, both needed for the subsequent developments. Here we consider the treatment in the special relativity setting.

By definition, the particle is characterized by a positive constant rest mass m_o and a non-vanishing constant charge q, with surface mass and charge densities ρ_m and ρ_c respectively. We shall assume that the mass and charge distributions have supports $\partial\Omega_m$ and $\partial\Omega_\sigma$. To define the particle mass and charge distributions on $\partial\Omega_m$ and $\partial\Omega_\sigma$, let us assume initially that in a time interval $[-\infty, t_o]$ the particle is at rest with respect to an inertial frame (i.e., that external forces acting on the particle vanish identically). As a consequence, by assumption in the subset of the space-time $\mathcal{M}^4 \subseteq \mathbb{R}^4$ in which $t \in [-\infty, t_o]$, there is an inertial frame in which both the particle mass and charge distributions are at rest (particle rest-frame $\mathcal{R}_o$). In this frame, it is assumed that there exists a point, hereafter referred to as *center of symmetry (COS)*, whose position 4-vector $r^\mu_{COS} \equiv (ct, \mathbf{r}_o)$ spans the Minkowski space-time $\mathcal{M}^4 \subseteq \mathbb{R}^4$ and with respect to which:

1) $\partial\Omega_\sigma$ and $\partial\Omega_m$ are stationary spherical surfaces of radii $\sigma > 0$ and $\sigma_m > 0$ of equations $(\mathbf{r} - \mathbf{r}_o)^2 = \sigma^2$ and $(\mathbf{r} - \mathbf{r}_o)^2 = \sigma_m^2$;

2) the particle is *quasi-rigid*, i.e., the mass and charge distributions are stationary and spherically-symmetric respectively on $\partial\Omega_m$ and $\partial\Omega_\sigma$[1];

3) in addition, consistent with the principle of energy-momentum conservation (see further discussion below), the distributions of mass and charge densities are assumed

[1]In order to warrant the condition of rigidity in a manner consistent with the SR-CE Axioms, following the literature a possibility is to assume that the extended particle is acted upon by a local non-EM force "whose precise nature is left unspecified" (see Nodvik (20) and further references indicated there).

to have the same support $\partial\Omega_\sigma \equiv \partial\Omega_m$, hence letting

$$\sigma_m = \sigma. \tag{5.4}$$

Finally, the case in which the mass is considered localized point-wise (*Lorentzian particle*) is recovered letting $\sigma_m \neq \sigma$, with $\sigma > 0$ and $\sigma_m = 0$. In both cases the particle mass and charge distributions remain uniquely defined in any reference frame for arbitrary particle motion.

In this research, we are concerned only with the investigation of the EM RR phenomenon on the translational dynamical motion of the charged particle. Hence, it is required that the mass density (and, as a consequence, also the charge density) does not possess pure spatial rotation, nevertheless still allowing for space-time rotations (i.e., Thomas precession, see below). For definiteness, let us introduce here the Euler angles $\alpha(s) \equiv \{\varphi(s), \vartheta(s), \psi(s)\}$ which define the orientation of the body-axis system K' with respect to the rest system K (according to the notations used by Nodvik (20)). Introducing the generalized velocities $\frac{d\alpha(s)}{ds} \equiv \left\{\frac{d\varphi}{ds}, \frac{d\vartheta}{ds}, \frac{d\psi}{ds}\right\}$, the condition of *vanishing mass and charge spatial rotation* in a time interval $I \subseteq \mathbb{R}$ is thus prescribed imposing that the particular solution

$$\begin{aligned} \alpha(s) &= \alpha_o, \\ \frac{d\alpha(s)}{ds} &\equiv 0, \end{aligned} \tag{5.5}$$

holds for all $s \in I$. For a physical motivation for this assumption we refer to the discussion reported by Yaghjian (21).

Having specified the physical properties of the particle by means of the mass and charge distributions, we can now move on to obtaining the covariant expression for the corresponding charge and mass current densities. Since the charge and the mass have the same support, the mathematical derivation is formally the same for both of them. For convenience, the charge current $j^\mu(r)$ is first considered, introducing for it the representation used by Nodvik. For definiteness, let us denote (20)

$$\begin{aligned} s &\equiv \textit{proper time of the COS}, \\ r^\mu(s) &\equiv \textit{COS 4-position}, \\ \zeta^\mu &\equiv \textit{charge element 4-position}. \end{aligned}$$

Then, we define the displacement vector ξ^μ as follows:

$$\xi^\mu \equiv \varsigma^\mu - r^\mu(s), \tag{5.6}$$

from which we also have that $\varsigma^\mu = r^\mu(s) + \xi^\mu$. The physical meaning of the 4-vector ξ^μ is that of a displacement between the particle COS and its boundary, where the charge is located. According to this representation, ξ^μ is subject to the following two constraints (20):

$$\begin{aligned} \xi^\mu \xi_\mu &= -\sigma^2, && (5.7) \\ \xi_\mu u^\mu(s) &= 0, && (5.8) \end{aligned}$$

where

$$u^\mu(s) \equiv \frac{d}{ds} r^\mu(s) \tag{5.9}$$

is the 4-velocity of the COS. The first equality (5.7) defines the boundary $\partial\Omega_\sigma = \partial\Omega_m$. The second constraint (5.8) represents instead the constraint of rigidity for the particle. This implies that in the particle rest frame the 4-vector ξ^μ has only spatial components. We can use the information from Eq.(5.7) to define the internal and the external domains with respect to the mass and charge distributions. In particular, if we define a generic displacement 4-vector $X^\mu \in M^4$ as

$$X^\mu = r^\mu - r^\mu(s), \tag{5.10}$$

which is subject to the constraint

$$X^\mu u_\mu(s) = 0, \tag{5.11}$$

then the following relations hold:

$$\begin{aligned} X^\mu X_\mu &\leq -\sigma^2 : \textit{external domain}, && (5.12) \\ X^\mu X_\mu &> -\sigma^2 : \textit{internal domain}, \\ X^\mu X_\mu &= \xi^\mu \xi_\mu = -\sigma^2 : \textit{boundary location}. \end{aligned}$$

To derive the current density 4-vector corresponding to the spherical charged shell we follow the presentation by Nodvik (20). Consider first the charge-current density $\Delta j^\mu(r)$ corresponding to a charge element Δq on the shell. This is expressed as follows:

$$\Delta j^\mu(r) = c\Delta q \int_1^2 d\zeta^\mu \delta^4\left(r^\mu - \zeta^\mu\right) = c\Delta q \int_{-\infty}^{+\infty} ds \left[u^\mu + \frac{d\xi^\mu}{ds}\right] \delta^4\left(x^\mu - \xi^\mu\right), \tag{5.13}$$

where

$$x^\mu = r^\mu - r^\mu(s). \tag{5.14}$$

Note that, for the simplicity of the notation, here and in the rest of the Chapter the symbol r stands for the generic 4-vector r^α when used as an argument of a function. Since the charge does not possess any pure spatial rotation, the relation

$$\frac{d\xi^\mu}{ds} = \Gamma u^\mu \tag{5.15}$$

holds, where $\Gamma \equiv -\left(\frac{du_\alpha}{ds}\xi^\alpha\right)$ carries the effect associated with the Thomas precession(20).

The expression for $\Delta j^{\mu}(r)$ then becomes

$$\Delta j^{\mu}(r) = c\Delta q \int_{-\infty}^{+\infty} ds u^{\mu} \left[1+\Gamma\right] \delta^{4}\left(x^{\mu} - \xi^{\mu}\right). \tag{5.16}$$

To compute the total current of the charged shell we express the charge element Δq according to the constraint (5.8) as follows: $\Delta q = qf(|\xi|)\delta(\xi^{\alpha}u_{\alpha}(s))d^{4}\xi$, where $d^{4}\xi$ is the 4-volume element in the ξ-space. Moreover, $f(|\xi|)$ is referred to as the form factor, which describes the charge distribution of the moving body. In particular, for a spherically symmetric distribution this has the following representation:

$$f(|\xi|) = \frac{1}{4\pi\sigma^{2}}\delta(|\xi| - \sigma), \tag{5.17}$$

where $|\xi| \equiv \left|\sqrt{\xi^{\mu}\xi_{\mu}}\right|$. The total current density $j^{\mu}(r)$ can therefore be obtained by integrating $\Delta j^{\mu}(r)$ over $d^{4}\xi$. We get

$$\begin{aligned} j^{\mu}(r) &\equiv qc\int_{-\infty}^{+\infty} ds u^{\mu} \int_{1}^{2} d^{4}\xi f(|\xi|)\delta(\xi^{\alpha}u_{\alpha})\left[1+\Gamma\right]\delta^{4}\left(x^{\mu}-\xi^{\mu}\right) = \\ &= qc\int_{-\infty}^{+\infty} ds u^{\mu} f(|x|)\delta(x^{\alpha}u_{\alpha})\left[1+\Gamma\right], \end{aligned} \tag{5.18}$$

where

$$f(|x|) = \frac{1}{4\pi\sigma^{2}}\delta(|x| - \sigma) \tag{5.19}$$

with $|x| \equiv |\sqrt{x^{\mu}x_{\mu}}|$. Then we notice that

$$\delta(x^{\alpha}u_{\alpha}(s)) = \frac{1}{\left|\frac{d[x^{\alpha}u_{\alpha}]}{ds}\right|}\delta(s-s_{1}) = \frac{1}{|1+\Gamma|}\delta(s-s_{1}), \tag{5.20}$$

where by definition s_1 is the root of the algebraic equation

$$u_{\mu}(s_{1})\left[r^{\mu} - r^{\mu}\left(s_{1}\right)\right] = 0. \tag{5.21}$$

Combining these relations, it follows that the integral covariant expression for the charge current density is given by

$$j^{\mu}(r) = \frac{qc}{4\pi\sigma^{2}}\int_{-\infty}^{+\infty} ds u^{\mu}(s)\delta(|x|-\sigma)\delta(s-s_{1}). \tag{5.22}$$

Finally, an analogous expression for the mass current density $j^{\mu}_{mass}(r)$ can be easily obtained from $j^{\mu}(r)$ by replacing the total charge q with the total mass m_o, thus giving

$$j^{\mu}_{mass}(r) = \frac{m_{o}c}{4\pi\sigma^{2}}\int_{-\infty}^{+\infty} ds u^{\mu}(s)\delta(|x|-\sigma)\delta(s-s_{1}). \tag{5.23}$$

It is worth remarking that in both equations (5.22) and (5.23):

1) the dependence in terms of the 4-position r enters explicitly through $|x| = |r^\mu - r^\mu(s)|$ in the form factor and implicitly through the root s_1;

2) consistent with assumption (5.5), possible charge and mass spatial rotations have been set to be identically zero.

5.4 EM self 4-potential

A prerequisite for the subsequent developments is the determination of the EM self-potential ($A_\mu^{(self)}$) produced by the spherical charged particle shell introduced here. In principle the problem could be formally treated by solving the Maxwell equations with the 4-potential written in terms of a suitable Green function according to standard methods. Remarkably, the solution can also be achieved in a more straightforward way based on the relativity principle and the covariance of Maxwell's equations. This implies the possibility of obtaining a covariant representation of the EM 4-vector in a generic reference system once its definition is known in a particular reference frame. The approach is analogous to the derivation presented by Landau and Lifschitz (12) for the treatment of a point charge. The solution is provided by the following Lemmas.

Lemma 1 - Covariant representation for $A_\mu^{(self)}(r)$

Given validity of the assumptions on the particle structure introduced in the previous section and the results obtained for the current density, the following statements hold:

$L1_1$: *Particle at rest in an inertial frame.*

Let us assume that the particle is at rest in an inertial frame S_0 and, according to (5.5), is non-rotating in this frame. By definition, in S_0 the 4-vector potential of the self-field is written as $A_\mu^{(self)}(r) = A_{S_0\mu}^{(self)}(r) \equiv \left\{\Phi^{(self)}, \mathbf{0}\right\}$, where

$$\Phi^{(self)}(\mathbf{r}, t) = \begin{cases} \frac{q}{R} & (R \geq \sigma), \\ \frac{q}{\sigma} & (R < \sigma), \end{cases} \tag{5.24}$$

(rest-frame representation) denote respectively the external and internal solutions with respect to the boundary of the shell. Here

$$R \equiv |\mathbf{R}|, \tag{5.25}$$

$$\mathbf{R} = \mathbf{r} - \mathbf{r}(t'), \tag{5.26}$$

with $r^\mu = (ct, \mathbf{r})$, $r'^\mu = (ct', \mathbf{r}' \equiv \mathbf{r}(t'))$, and $\mathbf{r}, \mathbf{r}' \equiv \mathbf{r}(t')$ being respectively a generic position 3-vector of $\mathbb{R}^3$ and the (stationary) position 3-vector of the particle COS. It follows that $\Phi^{(self)}(\mathbf{r}, t)$ can be equivalently represented as

$$\Phi^{(self)}(\mathbf{r}, t) = \begin{cases} \frac{q}{c(t-t')} \equiv \frac{q}{R} & (R \geq \sigma), \\ \frac{q}{c(t-t')} \equiv \frac{q}{\sigma} & (R < \sigma), \end{cases} \tag{5.27}$$

where $t_{ret} \equiv t - t'$ *is the following positive root*

$$t_{ret} \equiv t - t' = \begin{cases} t_{ret}^{(ext)} \equiv \pm \frac{R}{c} & (R \geq \sigma), \\ t_{ret}^{(int)} \equiv \pm \frac{\sigma}{c} & (R < \sigma). \end{cases} \tag{5.28}$$

L1_2 : *Particle with inertial motion in an arbitrary inertial frame.*

Let us assume that when the particle is referred to an arbitrary inertial frame S_I *it has a constant* $4-$*velocity* $u^\alpha \equiv \frac{dr^\mu(s')}{ds'}$. *Then, let us require that* $t_{ret} \equiv t - t'$ *is the positive root of the delay-time equation*

$$\widehat{R}^\alpha\ \widehat{R}_\alpha = \rho^2, \tag{5.29}$$

with $\widehat{R}^\alpha$ *being the bi-vector*

$$\widehat{R}^\alpha = r^\alpha - r^\alpha(t') \tag{5.30}$$

and

$$\rho^2 = \begin{cases} 0 & (X^\alpha X_\alpha \leq -\sigma^2), \\ \rho^2 \equiv \sigma^2 \left[1 + \frac{X^\alpha X_\alpha}{\sigma^2}\right] & (X^\alpha X_\alpha > -\sigma^2), \end{cases} \tag{5.31}$$

where the displacement vector X^α *is defined by Eqs.(5.10) and (5.11). For consistency, Eq.(5.31) provides the solution Eq.(5.28) when evaluated in the COS comoving frame.*

It follows that in the reference frame S_I *the EM self 4-potential have the internal and external solutions*

$$A_\mu^{(self)}(r) = \begin{cases} q\frac{u_\mu}{\widehat{R}^\alpha u_\alpha}\Big|_{t_{ret}=t_{ret}^{(ext)}} & (X^\alpha X_\alpha \leq -\sigma^2), \\ q\frac{u_\mu}{\widehat{R}^\alpha u_\alpha}\Big|_{t_{ret}=t_{ret}^{(int)}} & (X^\alpha X_\alpha > -\sigma^2), \end{cases} \tag{5.32}$$

where $\widehat{R}^\alpha$ *is given by Eq.(5.30).*

L1_3 : *Particle with a non-inertial motion in an arbitrary frame.*

Let us assume that the same particle is now referred to an arbitrary frame in which it has a time-dependent velocity $u_\mu(t')$. *In this frame the EM self 4-potential* $A_\mu^{(self)}(r)$ *takes the form:*

$$A_\mu^{(self)}(r) = \begin{cases} q\frac{u_\mu(t')}{\widehat{R}^\alpha u_\mu(t')}\Big|_{t_{ret}=t_{ret}^{(ext)}} & (X^\alpha X_\alpha \leq -\sigma^2), \\ q\frac{u_\mu(t')}{\widehat{R}^\alpha u_\mu(t')}\Big|_{t_{ret}=t_{ret}^{(int)}} & (X^\alpha X_\alpha > -\sigma^2), \end{cases} \tag{5.33}$$

where $u_\mu(t')$ *is the 4-velociy of the COS with 4-position* $r^\alpha(t')$, *i.e.,*

$$u_\mu(t') \equiv \frac{dr^\beta(t')}{ds'} = \gamma(t')\frac{dr^\beta(t')}{cdt'}, \tag{5.34}$$

and $t_{ret}^{(ext)}$, $t_{ret}^{(int)}$ *are the positive roots of the delay-time equation (5.29).*

Proof - L1_1) If the particle is at rest in an inertial frame S_0, from the form of the charge density (5.22) and the condition of non-rotation (5.5), the EM self 4-potential is stationary in S_0. Hence it takes necessarily the form $A_\mu^{(self)}(r) = A_{S_0\mu}^{(self)}(r) \equiv \left\{\Phi^{(self)}, \mathbf{0}\right\}$. Thus, denoting

$$R \equiv |\mathbf{R}|, \tag{5.35}$$

$$\mathbf{R} = \mathbf{r} - \mathbf{r}\left(t - \frac{\left|\mathbf{r} - \mathbf{r}(t - \frac{R}{c})\right|}{c}\right), \tag{5.36}$$

with $\mathbf{r}$ a generic position 3-vector of $\mathbb{R}^3$ and $\mathbf{r}(t') \equiv \mathbf{r}(t - \frac{R}{c})$ the retarded-time position 3-vector, $\Phi^{(self)}$ is written as

$$\Phi^{(self)}(\mathbf{r}, t) = \begin{cases} \frac{q}{R} & (R \geq \sigma), \\ \frac{q}{\sigma} & (R < \sigma). \end{cases} \tag{5.37}$$

In other words, in the external/internal sub-domains (respectively defined by the inequalities $R \geq \sigma$ and $R < \sigma$) the ES potential $\Phi^{(self)}$ coincides with the ES potential of a point charge and a constant potential. In terms of the delay time $t_{ret} = t = t'$ determined by Eq.(5.28) it is immediate to prove Eq.(5.27).

L1$_2$) Next, let us consider the same particle referred to an arbitrary inertial frame S_I in which the COS position vector $r^\alpha(s')$ has a constant velocity

$$u_\alpha \equiv u^\alpha(s') = \frac{d}{ds'} r^\alpha(s') = const. \tag{5.38}$$

Since by definition $A_\mu^{(self)}(r)$ is a covariant 4-vector, its form in S_I is simply obtained by applying a Lorentz transformation (12) according to Eq.(5.38). This requires

$$A_\mu^{(self)}(r) = q\frac{u_\mu}{\widehat{R}^\alpha u_\alpha}, \tag{5.39}$$

where $\widehat{R}^\alpha = r^\alpha - r^\alpha(s')$. Denoting $s' \equiv s'(t')$ and $r^\alpha(s') \equiv (ct', \mathbf{r}(t'))$, let us now impose that $t - t'$ is the positive root of the delay-time equation (5.29). The external and internal solutions in this case are given respectively by Eq.(5.32), as can be seen by noting that when $u_\mu = (1, \mathbf{0})$ the correct external and internal solutions (5.24) are recovered.

L1$_3$) The proof of the third statement is a basic consequence of the principle of relativity and of the covariance of the Maxwell equations. In fact we notice that both the solution (5.32) for the 4-vector potential and Eq.(5.29) for the delay time, which have been obtained for the specific case of an inertial frame, are already written in covariant form by means of the 4-vector notation. Hence, according to the principle of relativity, this solution is valid in any reference system related by a Lorentz transformation, and for a generic form of the 4-velocity u_μ (cf Landau and Lifshitz (12)).

Q.E.D.

We remark that Eq.(5.33) provides an exact representation (defined up to a gauge transformation) for the EM self 4-potential generated by the non-rotating finite-size charge considered here.

On the base of the conclusions of Lemma 1 it follows that $A_{\mu}^{(self)}(r)$ can also be represented by means of an equivalent integral representation as proved by the following Lemma.

Lemma 2 - Integral representation for $A_{\mu}^{(self)}(r)$

Given validity of Lemma 1, the EM self 4-potential Eq.(5.33) admits the equivalent integral representation

$$A_{\mu}^{(self)}(r) = 2q \int_{1}^{2} dr'_{\mu} \delta(\widehat{R}^{\alpha} \widehat{R}_{\alpha} - \rho^2), \tag{5.40}$$

with ρ^2 defined by Eq.(5.31) and $r'_{\mu} \equiv r_{\mu}(s')$.

Proof - In fact in the external and internal domains

$$\delta(\widehat{R}^{\alpha} \widehat{R}_{\alpha} - \rho^2) = \begin{cases} \frac{\delta(s-s')}{2\left|\widehat{R}_{\alpha} \frac{dr'^{\alpha}}{ds'}\right|} & (X^{\alpha} X_{\alpha} \leq -\sigma^2), \\ \frac{\delta(s-s')}{\left|2\widehat{R}_{\alpha} \frac{dr'^{\alpha}}{ds'} + \frac{d\rho^2}{ds}\right|} & (X^{\alpha} X_{\alpha} > -\sigma^2), \end{cases} \tag{5.41}$$

where $\frac{d\rho^2}{ds} = \frac{dX^{\alpha} X_{\alpha}}{ds'} = 2X_{\alpha} u_{\alpha}(s') \equiv 0$ because of Eq.(5.11), while s' is determined by the delay-time equation (5.29). Hence, Eq.(5.40) manifestly implies Eq.(5.33).

Q.E.D.

5.5 The action integral

In this section we derive the Hamilton action functional suitable for the variational treatment of finite-size charged particles introduced here and the investigation of their dynamics. As indicated above, the contributions due to pure spatial charge and mass rotations will be ignored. In this case, the action integral is conveniently expressed in hybrid superabundant variables (see Tessarotto *et al.* (26)) as follows:

$$S_1(r, u, \chi, [r]) = S_M(r, u) + S_C^{(self)}(r, [r]) + S_C^{(ext)}(r) + S_{\chi}(u, \chi), \tag{5.42}$$

where S_M, $S_C^{(self)}$, $S_C^{(ext)}$ and S_{χ} are respectively the inertial mass, the EM-coupling with the self and external fields, and the kinematic constraint contributions. For what concerns the notation, here r and u represent *local* depepndencies with respect to the 4-vector position r^{μ} and the 4-velocity u^{μ}, $[r]$ stands for *non-local* dependencies on the 4-vector position r^{μ}, while $\chi \equiv \chi(s)$ is a Lagrange multiplier (see also below and the related discussion in THM.1 of Section 5.7).

Before addressing the explicit evaluation of $S_1(r, u, \chi, [r])$ we prove the following preliminary Lemma concerning the transformation properties of 4-volume elements under Lorentz transformations.

Lemma 3 - Lorentz transformations and 4-volume elements

Let us consider a Lorentz transformation (Lorentz boost) from an inertial reference frame S_I to a reference frame S_{NI} whose origin has 4-velocity $u_\mu(s_2)$ with respect to S_I, with s_2 being considered here an arbitrary proper time independent of $r^\mu \in S_I$. By assumption $u_\mu(s_2)$ is constant both with respect to the 4-positions $r^\mu \in S_I$ and $r'^\mu \in S_{NI}$ in the two reference frames. The relationship between the two 4-vectors $r^\mu \in S_I$ and $r'^\mu \in S_{NI}$ is expressed by the transformation law(28)

$$r'^\mu = \Lambda_\nu^\mu \left(u_\mu(s_2)\right) r^\nu, \tag{5.43}$$

where $\Lambda_\nu^\mu \left(u_\mu(s_2)\right)$ is the matrix of the Lorentz boost, which by definition depends only on the relative 4-velocity $u_\mu(s_2)$ between S_I and S_{NI}. Then it follows that the 4-volume element $d\Omega \in S_I$ is invariant with respect to the Lorentz boost (5.43), in the sense:

$$d\Omega = d\Omega', \tag{5.44}$$

with $d\Omega' \in S_{NI}$ denoting the corresponding volume element in the transformed frame S_{NI}.

Proof - The proof of this statement follows by considering the general transformation property of volume elements under arbitrary change of coordinates. Consider the invariant 4-volume element $d\Omega \in S_I$ and assume a Minkowski metric tensor. By definition (12), for a generic change of reference frame the volume element transforms according to the law

$$d\Omega = \frac{1}{J} d\Omega', \tag{5.45}$$

where

$$J \equiv \left| \frac{\partial r^\mu}{\partial r'^\mu} \right| \tag{5.46}$$

is the Jacobian of the corresponding coordinate transformation. In the case considered here, the Lorentz boost (5.43) is described by the matrix $\Lambda_\nu^\mu \left(u_\mu(s_2)\right)$ which depends only on the 4-velocity $u_\mu(s_2)$, by assumption independent of the coordinates r^ν and r'^ν. It follows that $J \equiv 1$, implying in turn Eq.(5.44).

Q.E.D.

We can now proceed to evaluate the various contributions to the action integral $S_1(r, u, \chi, [r])$ defined in Eq.(5.42).

$S_C^{(self)}(r, [r])$: EM coupling with the self-field

The action integral $S_C^{(self)}(r, [r])$ containing the coupling between the EM self-field and the electric 4-current is of critical importance. According to the standard approach (12), $S_C^{(self)}$ is defined as the 4-scalar

$$S_C^{(self)}(r, [r]) = \int_1^2 d\Omega \frac{1}{c^2} A^{(self)\mu}(r) j_\mu(r), \tag{5.47}$$

where $A^{(self)\mu}(r)$ is given by Eq.(5.40), $j_{\mu}(r)$ by Eq.(5.22) and $d\Omega$ is the invariant 4-volume element. In particular, in an inertial frame S_I with Minkowski metric tensor $\eta_{\mu\nu}$, this can be represented as

$$d\Omega = cdtdxdydz, \tag{5.48}$$

where (x, y, z) are orthogonal Cartesian coordinates. The functional can be equivalently represented as

$$\begin{aligned} S_C^{(self)}(r,[r]) &= \frac{q}{4\pi\sigma^2 c}\int_1^2 d\Omega A^{(self)\mu}(r)\int_{-\infty}^{+\infty} ds_2\delta(s_2-s_1)\times \\ &\times\int_{-\infty}^{+\infty} dsu^{\mu}(s)\delta(|x(s)|-\sigma)\delta(s-s_2), \end{aligned}\tag{5.49}$$

where s_1 is the root of the equation

$$u_{\mu}(s_1)\left[r^{\mu}-r^{\mu}(s_1)\right]=0. \tag{5.50}$$

Because of the principle of relativity, the integral (5.47) can be evaluated in an arbitrary reference frame. The explicit calculation of the integral (5.47) is then achieved, thanks to Lemma 3, by invoking the Lorentz boost (5.43) to the reference frame S_{NI} moving with 4-velocity $u_{\mu}(s_2)$. In this frame, by construction $d\Omega' = cdt'dx'dy'dz' \equiv d\Omega$. In particular, introducing the spherical spatial coordinates $(ct', \rho', \vartheta', \varphi')$ it follows that the transformed spatial volume element can also be written as $cdt'dx'dy'dz' \equiv cdt'd\rho'd\vartheta'd\varphi'\rho'^2\sin\vartheta'$. In this frame the previous scalar equation becomes

$$u'_{\mu}(s_1)\left[r'^{\mu}-r'^{\mu}(s_1)\right]=0. \tag{5.51}$$

On the other hand, performing the integration with respect to s_2 in Eq.(5.49), it follows that necessarily $s_2 = s_1$, so that from Eq.(5.51) s_1 is actually given by

$$s_1 = ct' = s_2. \tag{5.52}$$

As a result, the integral $S_C^{(self)}$ reduces to

$$S_C^{(self)}(r',[r']) = \frac{q}{4\pi\sigma^2 c}\int_1^2 dx'dy'dz'A'^{(self)\mu}(r')\int_{-\infty}^{+\infty} dsu'^{\mu}(s)\delta(\left|x'(s)\right|-\sigma), \tag{5.53}$$

with $x'^{\mu}(s) = r'^{\mu} - r'^{\mu}(s)$. Moreover

$$A_{\mu}^{\prime(self)}(r') = 2q\int_{-\infty}^{+\infty} ds''u'_{\mu}(s'')\,\delta(\widehat{R}'^{\alpha}\widehat{R}'_{\alpha}-\rho'^2), \tag{5.54}$$

with $\widehat{R}'^{\alpha} = r'^{\alpha} - r'^{\alpha}(s'')$ and, thanks to Lemma 1,

$$\rho'^2 = \left\{ \begin{array}{cc} 0 & (X'^{\alpha}X'_{\alpha} \leq -\sigma^2), \\ \rho'^2 \equiv \sigma^2 \left[1 + \frac{X'^{\alpha}X'_{\alpha}}{\sigma^2}\right] & (X'^{\alpha}X'_{\alpha} > -\sigma^2). \end{array} \right. \tag{5.55}$$

Notice here that in $S_C^{(self)}(r', [r'])$ the contributions of the external and internal domains for the self-field can be explicitly taken into account letting

$$\begin{aligned} \delta(\widehat{R}'^{\alpha}\widehat{R}'_{\alpha} - \rho'^2) &= \Theta(\sigma^2 + \xi^{\alpha}\xi_{\alpha})\delta(\widehat{R}'^{\alpha}\widehat{R}'_{\alpha} - \sigma^2 - X'^{\alpha}X'_{\alpha}) + \\ &+ \widehat{\Theta}(-\xi^{\alpha}\xi_{\alpha} - \sigma^2)\delta(\widehat{R}'^{\alpha}\widehat{R}'_{\alpha}), \end{aligned} \tag{5.56}$$

with $\Theta(x)$ and $\widehat{\Theta}(x)$ denoting respectively the strong and weak Heaviside step functions

$$\widehat{\Theta}(x) = \left\{ \begin{array}{ll} 1 & x \geq 0 \\ 0 & x < 0 \end{array} \right. \tag{5.57}$$

$$\Theta(x) = \left\{ \begin{array}{ll} 1 & x > 0 \\ 0 & x \leq 0. \end{array} \right. \tag{5.58}$$

On the other hand, the only contribution to the integral (5.53) arises (because of the Dirac-delta in the current density) from the subdomain for which $-\xi^{\alpha}\xi_{\alpha} - \sigma^2 = 0$. Hence, $S_C^{(self)}$ simply reduces to the functional form:

$$\begin{aligned} S_C^{(self)}(r', [r']) &= \frac{2q^2}{4\pi\sigma^2 c} \int_0^{\pi} d\vartheta' \sin\vartheta' \int_0^{2\pi} d\varphi' \int_0^{+\infty} d\rho' \rho'^2 \times \\ &\times \int_{-\infty}^{+\infty} ds'' u'_{\mu}(s'')\, \delta(\widehat{R}'^{\alpha}\widehat{R}'_{\alpha}) \int_{-\infty}^{+\infty} ds u'^{\mu}(s)\delta(\left|x'(s)\right| - \sigma) \end{aligned} \tag{5.59}$$

The remaining spatial integration can now be performed letting

$$\rho' \equiv \left|x'(s)\right| \tag{5.60}$$

and making use of the spherical symmetry of the charge distribution. The constraints placed by the two Dirac-delta functions $\delta(\widehat{R}'^{\alpha}\widehat{R}'_{\alpha})$ and $\delta(|x'(s)| - \sigma)$ in the previous equation imply that both $\widehat{R}'^{\alpha}\widehat{R}'_{\alpha}$ and $|x'(s)|$ are 4-scalars. Then, introducing the representation

$$\widehat{R}'^{\alpha} \equiv r'^{\alpha} - r'^{\alpha}(s'') = \widetilde{R}'^{\alpha} + x'^{\alpha}(s), \tag{5.61}$$

with

$$\widetilde{R}'^{\alpha} \equiv r'^{\alpha}(s) - r'^{\alpha}(s''), \tag{5.62}$$

$$x'^{\alpha}(s) \equiv r'^{\alpha} - r'^{\alpha}(s), \tag{5.63}$$

it follows that

$$\widehat{R}'^{\alpha}\widehat{R}'_{\alpha} = \widetilde{R}'^{\alpha}\widetilde{R}'_{\alpha} + x'^{\alpha}(s)\, x'_{\alpha}(s) + 2\widetilde{R}'^{\alpha}x'_{\alpha}(s) \tag{5.64}$$

is necessarily a 4-scalar independent of the integration angles (φ', ϑ') when evaluated on the hypersurface $\Sigma : \widehat{R}'^{\alpha}\widehat{R}'_{\alpha} = 0$. Similarly, the Dirac-delta $\delta(|x'(s)| - \sigma)$ warrants that $x'^{\alpha}(s)\, x'_{\alpha}(s) = -\sigma^2$, which is manifestly a 4-scalar too. It then follows that necessarily

$$\widetilde{R}'^{\alpha} x'_{\alpha}(s) \equiv 0 \tag{5.65}$$

(see Ref.(22) for the details of the calculations leading to this result).

Hence, as a result of the integration, the action integral $S_C^{(self)}$ is expressed as

$$S_C^{(self)}(r', [r']) = \frac{2q^2}{c} \int_1^2 dr'_{\mu}(s'') \int_1^2 dr'^{\mu}(s')\delta(\widetilde{R}'^{\alpha}\widetilde{R}'_{\alpha} - \sigma^2). \tag{5.66}$$

Finally, since by construction $S_C^{(self)}$ is a 4-scalar, it follows that the primes can be dropped thus yelding the following representation holding in a general reference frame:

$$S_C^{(self)}(r, [r]) = \frac{2q^2}{c} \int_1^2 dr_{\mu}(s) \int_1^2 dr^{\mu}(s')\delta(\widetilde{R}^{\alpha}\widetilde{R}_{\alpha} - \sigma^2), \tag{5.67}$$

where for simplicity of notation s'' has been replaced with s' and $\widetilde{R}^{\alpha}$ now denotes

$$\widetilde{R}^{\alpha} \equiv r^{\alpha}(s) - r^{\alpha}(s'). \tag{5.68}$$

It is worth pointing out the following basic properties of the functional $S_C^{(self)}$:

1) it is a non-local functional in the sense that it contains a coupling between the past and the future of the dynamical system (see Eq.(5.3)). In fact it can be equivalently represented as

$$S_C^{(self)}(r, [r]) = \frac{2q^2}{c} \int_{-\infty}^{+\infty} ds \frac{dr_{\mu}(s)}{ds} \int_{-\infty}^{+\infty} ds' \frac{dr^{\mu}(s')}{ds'} \delta(\widetilde{R}^{\alpha}\widetilde{R}_{\alpha} - \sigma^2); \tag{5.69}$$

2) furthermore, it is symmetric, namely it fulfills the property

$$S_C^{(self)}(r_A, [r_B]) = S_C^{(self)}(r_B, [r_A]), \tag{5.70}$$

where r_A and r_B are two arbitrary curves of the functional class $\{f\}$ (see Eq.(5.1)).

$S_C^{(ext)}(r)$: EM coupling with the external field

The action integral $S_C^{(ext)}(r)$ of the EM coupling with the external field is a 4-scalar defined as

$$S_C^{(ext)}(r) = \int_1^2 d\Omega \frac{1}{c^2} A^{(ext)\mu}(r) j_{\mu}(r), \tag{5.71}$$

where $A^{(ext)\mu}(r)$ is the 4-vector potential of the external field, assumed to be assigned, and $j_{\mu}(r)$ is the current density given by Eq.(5.22). The evaluation of the action integral $S_C^{(ext)}$ proceeds exactly in the same way as outlined for $S_C^{(self)}$, with the introduction of the Lorentz boost (5.43), the spherical spatial coordinates and the use of the result from Lemma 3. The only difference now is that the vector potential $A^{(ext)\mu}(r)$ does not possess spherical symmetry when evaluated in S_{NI}. As a result, spatial integration over the angle variables ϑ' and φ' cannot be computed explicitly. This leads to the introduction of the *surface average EM external 4-potential* $\overline{A}^{(ext)\mu}$, which is defined in S_{NI} as

$$\overline{A}'^{(ext)\mu}\left(r'(s), |x'|\right) \equiv \frac{1}{4\pi}\int_0^{2\pi} d\varphi' \int_0^{\pi} d\vartheta' \sin\vartheta' \left[A'^{(ext)\mu}(r'^{\mu}(s) + x'^{\mu})\right], \tag{5.72}$$

where we have used the relation (5.14). With this definition, the time and radial integrals can then be calculated using the Dirac-delta functions as outlined for the self-coupling action integral. After performing a final transformation to an arbitrary reference frame, this gives the following expression for $S_C^{(ext)}$:

$$S_C^{(ext)}(r) = \frac{q}{c}\int_1^2 \overline{A}^{(ext)\mu}\left(r^{\mu}(s), \sigma\right) dr_{\mu}(s). \tag{5.73}$$

$S_\chi(u,\chi)$: kinematic constraint

The kinematic constraint concerns the normalization of the extremal 4-velocity of the COS. This is defined as

$$S_\chi(u,\chi) \equiv \int_{-\infty}^{+\infty} ds\chi(s)\left[u_\mu(s)u^\mu(s) - 1\right], \tag{5.74}$$

where $\chi(s)$ is a Lagrange multiplier.

$S_M(r,u)$: inertial mass functional

The action integral S_M of the inertial mass for the extended particle is here defined as the following 4-scalar:

$$S_M(r,u) \equiv \int_1^2 d\Omega \frac{1}{c} g_{\mu\nu} T_M^{\mu\nu}(r), \tag{5.75}$$

where $d\Omega$ denotes the invariant 4-volume element and $T_M^{\mu\nu}$ the stress-energy tensor corresponding to the mass distribution of the finite-size charged particle. Notice that the choice of S_M is consistent with the customary definition of the stress-energy tensor $T^{\mu\nu}$ (for a fluid or a field) in terms of $T^{\mu\nu} \equiv \frac{\delta L}{\delta g_{\mu\nu}}$, with L being a suitable Lagrangian function and δ representing the variational derivative (12). Therefore, it is natural to identify S_M with the trace of the mass stress-energy tensor for the extended particle. In particular, the explicit representation of $T_M^{\mu\nu}$ follows by projecting the mass current density $j^{\mu}_{mass}(r)$ given in Eq.(5.23) along the velocity of a generic shell mass-element parameterized in terms of the proper time s of the COS. We notice that the stress-energy tensor thus defined is symmetric. By performing the volume integrals as outlined before, one obtains for $S_M(r,u)$ the final expression:

$$S_M(r,u) \equiv \int_1^2 m_o c u_\mu dr^\mu \tag{5.76}$$

holding in an arbitrary reference frame. Concerning this solution, it is worth noting that, for the treatment of the translational motion of the extended particle, the choice of S_M given by Eq.(5.75) leads to Eq.(5.76) which is formally the same action integral of a point particle, with the difference that here u_μ represents the 4-velocity of the COS rather than the one of a point mass.

5.6 The variational Lagrangian

From the results of the previous section we can write the action integral S_1 as a line integral in terms of a variational Lagrangian $L_1(r,[r],u,\chi)$ as follows [see Eq.(5.2)]:

$$S_1 = \int_{-\infty}^{+\infty} ds L_1(r,[r],u,\chi). \tag{5.77}$$

More precisely, $L_1(r,[r],u,\chi)$ is defined as:

$$L_1(r,[r],u,\chi) = L_M(r,u) + L_\chi(u,\chi) + L_C^{(ext)}(r) + L_C^{(self)}(r,[r]), \tag{5.78}$$

where

$$\begin{aligned} L_M(r,u) &= m_o c u_\mu \frac{dr^\mu}{ds}, & (5.79)\\ L_\chi(u,\chi) &= \chi(s)\left[u_\mu(s)u^\mu(s)-1\right], & (5.80)\\ L_C^{(ext)}(r) &= \frac{dr^\mu}{ds}\frac{q}{c}\overline{A}_\mu^{(ext)}(r(s),\sigma), & (5.81)\end{aligned}$$

denote the local contributions respectively from the inertial, the constraint and the external EM field coupling terms, while

$$L_C^{(self)}(r,[r]) = \frac{2q^2}{c}\frac{dr^\mu}{ds}\int_1^2 dr'_\mu \delta(\widetilde{R}^\mu \widetilde{R}_\mu - \sigma^2) \tag{5.82}$$

represents the non-local contribution arising from the EM self-field coupling.

The conclusion is remarkable. Indeed, although the extended particle can be regarded as a continuous system carrying mass and charge current densities, the variational functional determined here is similar to that of a point particle subject to appropriate interactions. In fact, because of the rigidity constraint and the spherical symmetry imposed on the charge and mass distributions, the variational action S_1 is actually reduced from a volume integral to a line integral over the proper time of the COS. This is realized by means of the volume integration performed in the reference frame S_{NI} and thanks to Lemma 3.

The procedure introduces the surface-average operator acting both on the external and the self EM coupling terms. As a result, the Lagrangian (5.78) must be interpreted as prescribing the dynamics for the COS of the charged particle in terms of averaged EM fields, integrating all the force contributions to the translational motion on the shell. Furthermore, we recall once again the formal analogy between the Lagrangian $L_M(r,u)$ and the one of a point particle, when u_μ is interpreted as the 4-velocity of the point mass rather than that of the COS of the shell. This means that the dynamics of the finite-size particle is effectively described in terms of a point particle with a finite-size charge distribution. Hence, *the mathematical problem is formally the same of that for a Lorentzian particle.* Therefore, this proves that the particular case of a Lorentzian particle is formally included in the present description, in the limit in which the radius of the mass distribution σ_m is sent to zero while keeping the charge spatial extension fixed ($\sigma > 0$). The conclusion manifestly follows within the framework of special relativity, in which any possible curvature effects due to the EM field and the mass of the particle itself are neglected.

5.7 The variational principle and the RR equation

In this section the explicit form of the relativistic RR equation for a non-rotating charged particle is determined. As pointed out earlier (23), this goal can be uniquely attained by means of a *synchronous variational principle*, in analogy with the approach originally developed for point particles by Nodvik in terms of an asynchronous principle (Nodvik, 1964 (20)). In particular, we intend to prove that, in the present case, the exact RR equation can be *uniquely* and *explicitly* obtained by using the hybrid synchronous Hamilton variational principle defined in the previous section and given by Eq.(5.42). In this case the action functional is expressed by means of superabundant hybrid (i.e., non-Lagrangian) variables and the variations are considered as synchronous, i.e., they are performed by keeping constant the particle COS proper time. Taking into account the results presented in the previous sections, the appropriate form of the Hamilton variational principle is given by the following theorem:

THM.1 - Hybrid synchronous Hamilton variational principle
In validity of the SR-CE axioms, let us assume that:

1. *the Hamilton action $S_1(r,u,\chi,[r])$ is defined by Eq.(5.42), with $A_\mu^{(self)}$ given by Eq.(5.40) and $\chi(s)$ being a suitable Lagrange multiplier;*
2. *the real functions $f(s)$ in the functional class $\{f\}$ [see Eq.(5.1)] are identified with*
$$f(s) \equiv [r^\mu(s), u_\mu(s), \chi(s)]\,, \tag{5.83}$$
with synchronous variations $\delta f(s) \equiv f(s) -\ f_1(s)$ belonging to
$$\{\delta f\} \equiv \delta f_i(s) : \delta f_i(s) = f_i(s) - f_{1i}(s);$$

$$i = 1, n \ and \forall f(s), f_1(s) \in \{f\}, \tag{5.84}$$

here referred to as the functional class of synchronous variations;

3. *the extremal curve $f \in \{f\}$ of S_1, which is the solution of the equation*

$$\delta S_1(r, u, \chi, [r]) = 0, \tag{5.85}$$

exists for arbitrary variations $\delta f(s)$ (hybrid synchronous Hamilton variational principle);

4. *if $r^\mu(s)$ is extremal, the line element ds satisfies the constraint $ds^2 = \eta_{\mu\nu} dr^\mu(s) dr^\nu(s)$;*

5. *the 4-vector field $A_\mu^{(ext)}(r)$ is suitably smooth in the whole Minkowski space-time M^4;*

6. *the E-L equation for the extremal curve $r^\mu(s)$ is determined subject to the constraint that the delay-time s_{ret} (namely the root of the delay-time equation (5.93) below) must be chosen consistently with ECP.*

It then follows that:

T1$_1$) If all the synchronous variations $\delta f_i(s)$ (i=1,n) are considered as being independent, the E-L equations for $\chi(s)$ and u_μ following from the synchronous hybrid Hamilton variational principle (5.85) give respectively

$$\frac{\delta S_1}{\delta \chi(s)} = u_\mu u^\mu - 1 = 0, \tag{5.86}$$

$$\frac{\delta S_1}{\delta u_\mu} = m_o c dr^\mu + 2\chi u^\mu ds = 0. \tag{5.87}$$

Instead, the E-L equation for r_μ

$$\frac{\delta S_1}{\delta r^\mu(s)} = 0 \tag{5.88}$$

yields the following covariant (and hence also MLC) 4-vector, second-order delay-type ODE:

$$m_o c \frac{du_\mu(s)}{ds} = \frac{q}{c} \overline{F}_{\mu\nu}^{(ext)}(r(s)) \frac{dr^\nu(s)}{ds} + \frac{q}{c} \overline{F}_{\mu k}^{(self)}\left(r(s), r(s')\right) \frac{dr^k(s)}{ds}, \tag{5.89}$$

which is identified with the RR equation of motion for the COS of a spherical shell non-rotating charge particle. Here

$$\overline{F}_{\mu\nu}^{(ext)} \equiv \partial_\mu \overline{A}_\nu^{(ext)} - \partial_\nu \overline{A}_\mu^{(ext)} \tag{5.90}$$

denotes the surface-average [defined according to Eq.(5.72)] of the Faraday tensor carried by the externally-generated EM 4-vector and evaluated at the particle 4-position

$r^{\mu}(s)$. In addition, $\overline{F}_{\mu k}^{(self)}$ - in MLC 4-vector representation - is the surface-averaged Faraday tensor of the corresponding EM self-field, given by

$$\overline{F}_{\mu k}^{(self)} = -\frac{2q}{\left|\widetilde{R}^{\alpha} u_{\alpha}(s')\right|} \frac{d}{ds'} \left\{ \frac{u_{\mu}(s')\widetilde{R}_{k} - u_{k}(s')\widetilde{R}_{\mu}}{\widetilde{R}^{\alpha} u_{\alpha}(s')} \right\}_{s'=s-s_{ret}}. \tag{5.91}$$

Imposing the constraint $ds' = \gamma(t')\, cdt'$, this implies also

$$\begin{aligned} \overline{F}_{\mu k}^{(self)} &= -\frac{2q}{c\left|(t-t') - \frac{1}{c^2}\frac{d\mathbf{r}(t')}{dt'} \cdot (\mathbf{r} - \mathbf{r}(t'))\right|} \\ &\quad \frac{d}{dt'} \left\{ \frac{v_{\mu}(t')\widetilde{R}_{k} - v_{k}(t')\widetilde{R}_{\mu}}{c^2 \left[(t-t') - \frac{1}{c^2}\frac{d\mathbf{r}(t')}{dt'} \cdot (\mathbf{r} - \mathbf{r}(t'))\right]} \right\}_{t'=t-t_{ret}}. \end{aligned} \tag{5.92}$$

Here $u^{\mu} = \frac{dr^{\mu}}{ds}$ denotes the COS 4-velocity and $v^{\mu}(t) = \frac{dr^{\mu}}{dt}$, while $s_{ret} = s - s'$ is the positive root of the delay-time equation

$$\widetilde{R}^{\alpha}\widetilde{R}_{\alpha} - \sigma^2 = 0. \tag{5.93}$$

$T1_2$) The E-L equations (5.86),(5.87) and (5.88) imply that the extremal functional takes the form

$$S(r, [r], u,) = S_1(r, [r], u, \chi(s) = -\frac{m_o c}{2}). \tag{5.94}$$

$T1_3$) If $F_{\mu}^{(ext)\nu}(r) \equiv 0$ for all $s \leq s_1 \in \mathbb{R}$, a particular solution of Eq.(5.89), holding for all $s \leq s_1$ is provided by the inertial motion, i.e.,

$$\frac{dr^{\mu}(s)}{ds} = u_o^{\mu} = const., \tag{5.95}$$

$$\frac{du^{\mu}}{ds} = 0, \tag{5.96}$$

in agreement with the Galilei principle of inertia.

$T1_4$) The RR equation Eq.(5.89) also holds for a Lorentzian particle having the same charge distribution of the finite-size particle ($\sigma > 0$) and carrying a point-mass with position and velocity 4-vectors $r^{\mu}(s)$, $u^{\mu}(s)$.

Proof - *$T1_1$*) and *$T1_2$*) The proof proceeds as follows. Since $\frac{\partial}{\partial u^{\mu}}\delta(\widetilde{R}^{\alpha}\widetilde{R}_{\alpha} - \sigma^2) = \frac{\partial}{\partial u'^{\mu}}\delta(\widetilde{R}^{\alpha}\widetilde{R}_{\alpha} - \sigma^2) \equiv 0$, the variations with respect to $\chi(s)$ and u_{μ} deliver respectively the two E-L equations (5.86) and (5.87). Hence, the Lagrange multiplier χ must be for consistency

$$2\chi = -m_o c, \tag{5.97}$$

so that, ignoring gauge contributions with respect to χ, the extremal functional $S_1(r, u, \chi, [r])$ takes the form (5.94) [statement *$T1_2$*]. To prove also Eq.(5.88), we notice that the synchronous variation of $S_C^{(self)}$ has the form

$$\delta S_C^{(self)} = \delta A + \delta B, \tag{5.98}$$

where

$$\begin{array}{c} \delta A \equiv -\frac{4q^2}{c}\eta_{\mu\nu}\int_1^2 \delta r^\mu d\left[\int_1^2 dr'^\nu \delta(\widetilde{R}^\alpha \widetilde{R}_\alpha - \sigma^2)\right], \\ \delta B \equiv \frac{4q^2}{c}\eta_{\alpha\beta}\int_1^2 dr'^\beta \int_1^2 dr^\alpha \delta r^\mu \frac{\partial}{\partial r^\mu}\delta(\widetilde{R}^k \widetilde{R}_k - \sigma^2), \end{array} \tag{5.99}$$

and $r'^\nu \equiv r^\nu(s')$ and $r^\nu \equiv r^\nu(s)$. By noting that

$$d\left[\int_1^2 dr'^\nu \delta(\widetilde{R}^\alpha \widetilde{R}_\alpha - \sigma^2)\right] = dr^k \int_{-\infty}^{\infty} ds' u^\nu(s') \frac{\partial}{\partial r^k}\left[\delta(\widetilde{R}^\alpha \widetilde{R}_\alpha - \sigma^2)\right], \tag{5.100}$$

the variations δA and δB become respectively

$$\delta A = -\frac{4q^2}{c}\eta_{\mu\nu}\int_1^2 \delta r^\mu dr^k \int_{-\infty}^{\infty} ds' u^\nu(s') \frac{\partial}{\partial r^k}\left[\delta(\widetilde{R}^\alpha \widetilde{R}_\alpha - \sigma^2)\right], \tag{5.101}$$

and

$$\delta B = \frac{4q^2}{c}\eta_{\alpha\beta}\int_1^2 dr^\alpha \delta r^\mu \int_{-\infty}^{\infty} ds' u^\beta(s') \frac{\partial}{\partial r^\mu}\delta(\widetilde{R}^k \widetilde{R}_k - \sigma^2). \tag{5.102}$$

Let us now evaluate the partial derivative $\frac{\partial}{\partial r^k}\delta(\widetilde{R}^\alpha \widetilde{R}_\alpha - \sigma^2)$. Invoking the chain rule, this becomes

$$\frac{\partial}{\partial r^k}\delta(\widetilde{R}^\alpha \widetilde{R}_\alpha - \sigma^2) = \frac{\partial(\widetilde{R}^\alpha \widetilde{R}_\alpha)}{\partial r^k}\frac{d\delta(\widetilde{R}^\alpha \widetilde{R}_\alpha - \sigma^2)}{d(\widetilde{R}^\alpha \widetilde{R}_\alpha)} = \frac{d\delta(\widetilde{R}^\alpha \widetilde{R}_\alpha - \sigma^2)}{ds'}\frac{2\widetilde{R}_k}{\frac{d(\widetilde{R}^\alpha \widetilde{R}_\alpha)}{ds'}}, \tag{5.103}$$

and so

$$\frac{\partial}{\partial r^k}\delta(\widetilde{R}^\alpha \widetilde{R}_\alpha - \sigma^2) = -\frac{\widetilde{R}_k}{\widetilde{R}^\alpha u_\alpha(s')}\frac{d}{ds'}\left\{\frac{\delta(s - s' - s_{ret})}{2\left|\widetilde{R}^\alpha u_\alpha(s')\right|}\right\}. \tag{5.104}$$

It follows that

$$\begin{aligned} \frac{\partial}{\partial r^k}\delta(\widetilde{R}^\alpha \widetilde{R}_\alpha - \sigma^2) &= -\frac{\widetilde{R}_k}{c^2\left[(t-t') - \frac{1}{c^2}\frac{d\mathbf{r}'}{dt'}\cdot(\mathbf{r}-\mathbf{r}')\right]}\times \\ &\quad \times\frac{d}{dt'}\left\{\frac{\delta(t-t'-t_{ret})}{2c^2\gamma(t')\left|(t-t') - \frac{1}{c^2}\frac{d\mathbf{r}(t')}{dt'}\cdot(\mathbf{r}-\mathbf{r}')\right|}\right\}, \end{aligned} \tag{5.105}$$

where $\mathbf{r}' \equiv \mathbf{r}(t')$, $t \equiv t(s)$ and $t' \equiv t(s')$. Substituting Eq.(5.105) into Eqs.(5.101) and (5.102), and then directly integrating, it follows that δA and δB have the form

$$\delta A \equiv \frac{2q}{c}\eta_{\mu\nu}\int_1^2 \delta r^\mu dr^k \left[B_k^\nu\right]_{t'=t-t_{ret}}, \tag{5.106}$$

$$\delta B \equiv -\frac{2q}{c}\eta_{\alpha\beta}\int_1^2 \delta r^\mu dr^\alpha \left[B_\mu^\beta\right]_{t'=t-t_{ret}},$$

where B_k^ν is

$$B_k^\nu \equiv -\frac{q}{c\left|(t'-t)-\frac{1}{c^2}\frac{d\mathbf{r}(t')}{dt'}\cdot(\mathbf{r}'-\mathbf{r})\right|}\frac{d}{dt'}\left\{\frac{v^\nu(t')\widetilde{R}_k}{c^2\left[(t'-t)-\frac{1}{c^2}\frac{d\mathbf{r}(t')}{dt'}\cdot(\mathbf{r}-\mathbf{r}')\right]}\right\}. \quad (5.107)$$

Finally, he variation with respect to r^μ yields

$$\frac{\delta S_1}{\delta r^\mu} = -m_o c du_\mu(s) + \frac{q}{c}dr^k\overline{F}_{\mu k}^{(self)} + \frac{q}{c}\left[\partial_\mu \overline{A}_\nu^{(ext)}(r(s)) - \partial_\nu \overline{A}_\mu^{(ext)}(r(s))\right]dr^\nu, \quad (5.108)$$

where

$$\overline{F}_{\mu k}^{(self)} = 2(B_{k\mu} - B_{\mu k}), \quad (5.109)$$

from which Eqs.(5.89)-(5.93) follow. This yields the *RR equation* being sought, i.e., *the exact relativistic equation of motion for the translational dynamics of the COS of a finite-size spherical shell charge particle subject to the simultaneous action of a prescribed external EM field and of its EM self-field.*

T1$_3$) The proof of Eqs.(5.95)-(5.96) is straightforward. In fact, let us assume that in the interval $[-\infty, s_1]$ the motion is inertial, namely that $\frac{d}{ds}u_\mu \equiv 0, \forall s \in [-\infty, s_1]$. This implies that in $[-\infty, s_1]$ it must be $u_\mu \equiv u_{0\mu}$, with $u_{0\mu}$ denoting a constant 4-vector velocity. It follows that $\forall s, s' \in [-\infty, s_1]$, $r_\mu(s) = r_\mu(s') + u_{0\mu}(s')(s-s')$ and $R_\mu = u_{0\mu}(s)(s-s')$. Hence, by direct substitution in Eq.(5.92) we get that $v_\mu(t')\widetilde{R}_k - v_k(t')\widetilde{R}_\mu = 0$, which by consequence implies also that $dr^k H_{\mu k} \equiv 0$ identically in this case.

T1$_4$) The proof follows immediately from the definition of Lorentzian particle given above by noting that in the context of SR the variational particle Lagrangian $L_1(r, [r], u, \chi)$ [see Eq.(5.78)] formally coincides with that of a Lorentzian particle characterized by a finite charge distribution [i.e., with $\sigma > 0$], subject to the simultaneous action of the averaged external and EM self-fields $\overline{F}_{\mu\nu}^{(ext)}$ and $\overline{F}_{\mu k}^{(self)}$.

Q.E.D.

We notice that, by assumption, the varied functions $f(s) \equiv [r^\mu(s), u_\mu(s), \chi(s)]$ are *unconstrained*, namely they are solely subject to the requirement that end points and boundary values are kept fixed. This implies that all of the 9 components of the variations $\delta f(s)$, namely $\delta r^\mu(s), \delta u_\mu(s), \delta\chi(s)$, must be considered independent. On the other hand, the extremal curves $f(s)$ of $S_1(r, u, \chi, [r])$, the solution of the hybrid Hamilton variational principle, satisfy all of the required physical constraints, so that only 6 of them are actually independent. In fact, the resulting E-L equations determine, besides the RR equation (5.89), also the relationship between $r^\mu(s)$ and $u_\mu(s)$, namely

$$u^\mu(s) = \frac{dr^\mu(s)}{ds}, \quad (5.110)$$

as well as the physical constraint

$$u^{\mu}(s)u_{\mu}(s) = 1. \tag{5.111}$$

As a consequence, $r^{\mu}(s)$ and $u^{\mu}(s)$ coincide respectively with the physical 4-position and 4-velocity of the COS mass particle. Therefore only 3 components of the 4-velocity are actually independent, while the first component of the 4-position ct can always be represented in terms of the proper length s (so that only the spatial part of the position 4-vector actually defines a set of independent Lagrangian coordinates).

A further basic feature of the RR equation concerns the validity of GIP and its meaning in this context. In fact, let us assume that the external EM field is non-vanishing in the time interval $I_{12} \equiv [s_1, s_2]$, while it vanishes identically in $I_2 \equiv [s_2, +\infty]$. Then, the inertial solution (5.95) and (5.96) does not hold, by definition, in I_{12} and is only achieved in an asymptotic sense in I_2, i.e., in the limit $s \to +\infty$. In fact, the non-local feature of the RR effect prevents the particle from reaching the inertial state in a finite time interval. It is concluded, therefore, that GIP must be intended as holding *in the past,* namely in the time interval $s \leq s_1 \in \mathbb{R}$, where by assumption no external EM field is acting on the particle.

5.8 Standard Lagrangian and conservative forms of the RR equation

The variational principle presented in THM.1 implies that the E-L equations (5.86)-(5.89) can be cast in an equivalent way either:

1) in a standard Lagrangian form, namely expressed in the form of Lagrange equations defined in terms of a suitable non-local effective Lagrangian L_{eff};

2) in a conservative form, as the divergence of a suitable effective stress-energy tensor.

The result is provided by the following theorem.

THM.2 - RR equation in standard Lagrangian and conservative forms
Given validity of THM.1, it follows that:
$T2_1$) Introducing the non-local real function

$$L_{eff} \equiv L_M(r,u) + L_{\chi}(u,\chi) + L_C^{(ext)}(r) + 2L_C^{(self)}(r,[r]), \tag{5.112}$$

here referred to as non-local effective Lagrangian, the E-L equations (5.86),(5.87) and (5.108) take respectively the form

$$\frac{\partial L_{eff}}{\partial \chi(s)} = 0, \tag{5.113}$$

$$\frac{\partial L_{eff}}{\partial u_{\mu}(s)} = 0, \tag{5.114}$$

$$\frac{d}{ds}\frac{\partial L_{eff}}{\partial \frac{dr^{\mu}(s)}{ds}} - \frac{\partial L_{eff}}{\partial r^{\mu}(s)} = 0. \tag{5.115}$$

These will be referred to as E-L equations in standard Lagrangian form.

T2$_2$) The stress-energy tensor of the system $T_{\mu\nu}$ is uniquely determined in terms of L_{eff}. As a consequence, the RR equation (5.89) can also be written in conservative form as

$$\overline{T}_{\mu\nu,\nu} = 0, \tag{5.116}$$

where $\overline{T}_{\mu\nu} \equiv \overline{T}_{\mu\nu}^{(M)} + \overline{T}_{\mu\nu}^{(EM)}$ is the surface-averaged total stress energy tensor, obtained as the sum of the corresponding tensors for the mass distribution and the EM field which characterize the system.

Proof - T2$_1$) The proof follows immediately by noting that the Hamiltonian action (5.42) defines a symmetric functional with respect to local and non-local dependencies, i.e., such that

$$S_1(r_A, [r_B], u, \chi) = S_1(r_B, [r_A], u, \chi). \tag{5.117}$$

Because the E-L equations (5.113)-(5.115) are written in terms of local partial derivative differential operators, the effective Lagrangian L_{eff} must be therefore distinguished from the corresponding variational Lagrangian function L_1 which enters the Hamilton action and which contains non-local contributions. These features imply the definition (5.112), which manifestly satisfies the E-L equations in standard form (5.113)-(5.115).

T2$_2$) The proof of this statement is straightforward, by first recalling that the Lagrangian of the distributed mass is analogous to that of a point mass particle. Moreover, the stress-energy tensor of the total EM field $T_{\mu\nu}^{(EM)}$, to be defined in terms of L_{eff} according to the standard definition (see for example Landau and Lifshitz (12)) becomes

$$T_{\mu\nu}^{(EM)} = T_{\mu\nu}^{(EM-ext)} + T_{\mu\nu}^{(EM-self)}. \tag{5.118}$$

Then, given validity to the Maxwell equations, it follows that

$$T_{\mu\nu,\nu}^{(EM)} = F_{\mu\nu} j^{\nu} = \left[F_{\mu\nu}^{(ext)} + F_{\mu\nu}^{(self)}\right] j^{\nu}. \tag{5.119}$$

Gathering the mass and the field contributions, substituting the expressions for $F_{\mu\nu}^{(ext)}$ and $F_{\mu\nu}^{(self)}$ obtained in THM.1, and performing the integration over the 4-volume element finally proves that the equation (5.116) actually coincides with the extremal RR equation (5.89).

Q.E.D.

The expression (5.116) represents the conservative form of Eq.(5.89), and hence - consistent with the surface integration procedure here adopted - it holds for the *surface-averaged* EM external and self-fields $\overline{F}_{\mu\nu}^{(ext)}$ and $\overline{F}_{\mu\nu}^{(self)}$, defined respectively by Eqs.(5.90) and (5.91). It is important to remark that the result holds both for finite-size and Lorentzian particles. On the other hand, a local form of the conservative equation - analogous to Eq.(5.116) - and holding for the local EM fields is in principle achievable too. However, this last conclusion generally applies only to finite-size particles with the same support for the mass and charge distributions, i.e., for which Eq.(5.4) holds.

5.9 Short delay-time asymptotic approximation

In this section the asymptotic properties of the RR equation are addressed, considering the customary approximation in the treatment of the problem, which leads to the LAD equation (Dirac, 1938 (1)). This is the power-series expansion of the retarded EM self-potential in terms of the dimensionless parameter $\epsilon \equiv \frac{(s-s')}{s}$, to be assumed as infinitesimal (*short delay-time ordering*), $s - s'$ denoting the proper-time difference between observation (s) and emission (s'). The same approach was also adopted by Nodvik (20) in the case of flat space-time and by DeWitt and Brehme (29) and Crowley and Nodvik (30) in their covariant generalizations of the LAD equation valid in curved space-time. It is immediate to show that the following result holds:

THM.3 - First-order, short delay-time asymptotic approximation
Let us introduce the 4-vector G_μ defined as

$$dsG_\mu = \frac{q}{c}\overline{F}_{\mu k}^{(self)} dr^k, \tag{5.120}$$

and invoke the asymptotic ordering

$$0 < \epsilon \ll 1. \tag{5.121}$$

Then:

$T3_1$) Neglecting corrections of order ϵ^N, with $N \geq 1$ (first-order approximation), the following asymptotic approximation holds for G_μ

$$G_\mu \cong \left\{ -m_{oEM} c \frac{d}{ds} u_\mu + g_\mu \right\} [1 + O(\epsilon)] \,, \tag{5.122}$$

where g_μ denotes the 4-vector

$$g_\mu = \frac{2}{3}\frac{q^2}{c} \left[\frac{d^2}{ds^2} u_\mu - u_\mu(s) u^k(s) \frac{d^2}{ds^2} u_k \right] , \tag{5.123}$$

with

$$m_{oEM} \equiv \frac{q^2}{2c^2\sigma} \tag{5.124}$$

being the leading-order EM mass.

$T3_2$) The point-charge limit of the RR equation (5.89) does not exist.

Proof - T3$_1$) The proof is straightforward and follows by performing explicitly the perturbative expansion with respect to ϵ. By dropping the terms which vanish in the limit $\epsilon \to 0$, this yields Eq.(5.122). The proof of *T3$_2$*), instead, follows by noting that the limit obtained by letting

$$\sigma \to 0^+ \tag{5.125}$$

(*point-charge limit*) is *not defined,* since

$$\lim_{\sigma \to 0^+} m_{oEM} = \infty. \tag{5.126}$$

Q.E.D.

As basic consequences, in the first-order approximation the RR equation (5.89) recovers the LAD equation. Moreover, in a similar way, by introducing a suitable approximate reduction scheme, also the LL equation (Landau and Lifschitz, 1951 (12)) can be immediately obtained.

5.10 The fundamental existence and uniqueness theorem

THMs.1 and 2 of this Chapter show that in the presence of RR the non-local Lagrangian system $\{\mathbf{x}, L\}$ admits E-L equations [Eq.(5.89)] which are of *delay differential type.* This feature is not completely unexpected, since model equations of this type have been proposed before for the RR problem (see for example (18)). In general, for a delay-type differential equation there is nothing similar to the existence and uniqueness theorem holding for an initial condition of the type

$$\mathbf{x}(s_o) = \mathbf{x}_o. \tag{5.127}$$

In fact, no finite set of initial data is generally enough to determine a unique solution. The possibility of having, under suitable physical assumptions, an existence and uniqueness theorem therefore plays a crucial role in the proper formulation of the RR problem. In fact, for consistency with the SR-CE axioms, and in particular with NPD, the existence of a classical dynamical system (5.3) must be warranted. The result can be obtained by requiring that there exists an initial time s_o before which for all $s < s_o$ the particle motion is inertial (see also the related discussion in Ref.(18)). The assumption has also been invoked to define the particle mass and charge distributions (see Section 5.3). In view of THM.1 this happens if the external EM force vanishes identically for all $s < s_o$ and is (smoothly) "turned on" at $s = s_o$. In this regard, we here point out the following theorem:

THM.4 - The fundamental theorem for the RR equation
Given validity of THM.1, let us assume that:

1. *REQUIREMENT #1: at time t_o the initial condition (5.127) holds;*

2. *REQUIREMENT #2: the external force* $\overline{F}_{\mu\nu}^{(ext)}(r,s)$ *is of the form* $\overline{F}_{\mu\nu}^{(ext)}(r,s) = \Theta(s-s_o)\overline{F}_{1\mu\nu}^{(ext)}(r))$, *i.e.,* $\overline{F}_{\mu\nu}^{(ext)}$ *is "turned on" at the proper time* $s = s_o$. *In particular we shall take* $\overline{F}_{\mu\nu}^{(ext)}(r,s)$ *to be a smooth function of* s, *of class* $C^k\left(M^4 \times I\right)$, *with* $k \geq 1$;

3. *REQUIREMENT #3: more generally, let us require that for an arbitrary initial state* $\mathbf{x}(s_1) = \mathbf{x}_1 \in \Gamma$ *there always exists* $\{\mathbf{x}(s_o) = \mathbf{x}_o, s_o\} \in \Gamma \times I$, *with* $s_o = s_1 - s_{ret}$, *such that at time* s_o, $\mathbf{x}(s_o)$ *is inertial, i.e., before* s_o *the external force* $\overline{F}_{\mu\nu}^{(ext)}$ *vanishes identically, so that the dynamics is of the form provided by Eqs.(5.95)-(5.96).*

It then follows that the solution of the initial-value problem (5.89)-(5.127), subject to REQUIREMENTS #1-#3, exists at least locally in a subset $I \equiv [-\infty, s_0] \cup [s_0, s_n] \subseteq \mathbb{R}$ *with* $[s_0, s_n]$ *a bounded interval, and is unique (fundamental theorem).*

Proof - Eq.(5.89) can be cast in the form of a delay-differential equation, i.e.,

$$\frac{d\mathbf{x}(s)}{ds} = \mathbf{X}(\mathbf{x}(s), \mathbf{x}(s - s_{ret}), s), \tag{5.128}$$

subject to the initial condition

$$\mathbf{x}(s_o) = \mathbf{x}_o. \tag{5.129}$$

Here $\mathbf{x}(s)$ and $\mathbf{x}(s-s_{ret})$ denote respectively the "instantaneous" and "retarded" states $\mathbf{x}(s)$ and $\mathbf{x}(s - s_{ret})$, while $\mathbf{X}(\mathbf{x}(s), \mathbf{x}(s - s_{ret}), s)$ is a suitable C^2 real vector field depending smoothly on both of them. The proof of local existence and uniqueness for Eq.(5.128), with the initial conditions (5.129) and the Requirements #1-#3, requires a generalization of *the fundamental theorem* holding for ordinary differential equations (in which the vector field $\mathbf{X}$ depends only on the local state $\mathbf{x}(s)$).

Let us first consider the case in which the solution $\mathbf{x}(s)$ of the initial-value problem (5.128) and (5.129) is defined in the half-axis $[-\infty, s_o]$: by assumption this solution exists, is unique and is that of inertial motion [see Eqs.(5.95)-(5.96)].

Next, let us consider the proper time interval $I_{o,1} \equiv [s_o, s_1 \equiv s_o + s_{ret}]$. Thanks to the Requirement #3, by assumption in $I_{o,1}$ the particle is subject only to the action of the external force (produced by $A_\mu^{(ext)}$), since $\overline{F}_{\mu\nu}^{(self)}$ vanishes by definition if $s < s_o + s_{ret}$. Hence, in the same time interval the solution exists and is unique because the differential equation (5.128) is of the form

$$\frac{d\mathbf{x}(s)}{ds} = \mathbf{X}^{ext}(\mathbf{x}(s), s), \tag{5.130}$$

with $\mathbf{X}^{ext}(\mathbf{x}(s), s)$ being, by assumption, a smooth vector field (see THM.1). Eq.(5.130) is manifestly a local ODE for which the fundamental theorem (for local ODEs) holds. Hence, existence and uniqueness is warranted also in $I_{o,1}$.

Finally, let us consider the sequence of proper time intervals $I_{k,k+1} \equiv [s_k, s_{k+1} = s_k + s_{ret}]$, for the integer $k = 1, 2, 3...n$, where $n \geq 2$. In this case, for any proper time $s \in I_{k,k+1}$,

the advanced-time solution $\mathbf{x}(s - s_{ret})$ appearing in the vector field $\mathbf{X} \equiv \mathbf{X}(\mathbf{x}(s), \mathbf{x}(s - s_{ret}), s)$ can be considered as a *prescribed function of* s, determined in the previous time interval $I_{k,k-1}$. Therefore, $\mathbf{X}$ is necessarily of the form $\mathbf{X} \equiv \widehat{\mathbf{X}}(\mathbf{x}(s), s)$, so that for $s > s_1$, Eq.(5.128) can be viewed again as a local ODE. We conclude that, thanks to the fundamental theorem holding for local ODEs, the local existence (in a suitable bounded proper time interval $I \equiv [s_1, s_n]$) and uniqueness of solutions of the problem (5.128)-(5.129) is assured under the Requirements #1-#3. This proves the statement.
Q.E.D.

5.11 Conclusions

In this Chapter we have shown that the RR problem originally posed by Lorentz for classical non-rotating finite-size and Lorentzian particles can exactly be solved analytically within the SR setting.

For these particles, the resulting relativistic dynamics in the presence of the RR force, i.e., the *classical RR equation*, has been found analytically by taking into account the exact covariant form of the EM self 4-potential. In particular, this has been uniquely determined consistently with the basic principles of classical electrodynamics and special relativity. In addition, the RR equation has been proved to be *variational* in the functional class of synchronous variations (5.1) with respect to the Hamilton variational principle, defined in terms of a non-local variational Lagrangian function. The same equation has been shown: 1) to admit the standard Lagrangian form in terms of the non-local effective Lagrangian L_{eff}; 2) to admit a conservative form; 3) to recover the usual *asymptotic* LAD and LL equations in the first-order short delay-time approximation; 4) not to admit the point-charge limit. From the mathematical point of view, the RR equation is a delay-type second order ODE, which fulfills *GIP* in the sense of THM.1*, relativistic covariance and MLC*. As a consequence, provided suitable physical requirements are imposed, *the initial-value problem for the RR equation is well-posed*, defining the classical dynamical system required by NDP.

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Chapter 6

Hamiltonian formulation for the classical EM radiation-reaction problem: Application to the kinetic theory for relativistic collisionless plasmas

6.1 Introduction

An open problem in relativistic theories is related to the Hamiltonian description of particle dynamics for which non-local interactions typically occur. In this regard, a basic difficulty which is usually met is the lack of a Hamiltonian formalism for non-local Lagrangian systems. In fact, for arbitrary non-local Lagrangians it is generally impossible to define the notion of Legendre transformation (1). As a consequence even the phase-space itself may not be well-defined.

Most approaches to the construction of a Hamiltonian formalism for non-local first-order Lagrangians have tried to change the functional part of the Euler-Lagrange equations (2, 3, 4, 5). In principle this delivers infinite-order Euler-Lagrange equations and a corresponding infinite-dimensional phase-space. As an alternative, a finite dimensional phase-space can be recovered by introducing appropriate asymptotic approximations, i.e., truncating the expansion of the Lagrangian in terms of finite-order derivatives (4, 6).

A typical situation of this kind occurs for the relativistic equation of motion for single isolated charged particles, subject both to external and self EM forces, namely the radiation-reaction (RR) equation. There is an extensive literature devoted to this subject, most of which dealing with point charges. As remarked by Dorigo *et al.* (7), customary formulations based either on the LAD (8, 9, 10) or LL (11) equations are *asymptotic,* i.e., obtained by means of asymptotic expansions of different sort. In particular, as a consequence it follows that the LAD equation is represented by a third-order ODE, so that it does not admit a Hamiltonian formulation in the customary sense (12, 13). The LL, instead, is *intrinsically* non-variational, although it is a second-order differential equation, being obtained by means of a one-step "reduction process" from the LAD equation (7). As a consequence, the LAD equation does not define a dynamical system in the customary sense, since it requires, for non-rotating particles, a 12-dimensional phase-space involving also the particle acceleration. Therefore, for different reasons, both the LAD and LL equations are *manifestly non-Hamiltonian.* In particular, for the LL equation, this implies that the corresponding phase-space volume is not conserved. Moreover, within these treatments particles are treated as point-like, so that non-local EM effects produced by the RR self-interaction may remain undetermined.

Fundamental problems arise when attempting to formulate classical statistical mechanics (CSM) for systems of relativistic charged particles based on the LAD or LL equations. In fact even the proper axiomatic formulation of the relativistic CSM for radiating particles is missing. This requires the precise identification of the corresponding phase-space and the definition of an invariant probability measure on this set. For a system of charged particles subject solely to an external EM field and the RR self-force this involves the construction of a Vlasov kinetic treatment. In this regard, important issues concern:

1) The lack of a standard Hamiltonian formulation of relativistic CSM based on such asymptotic equations, which implies the lack of a flow-preserving measure. This feature is shared by both the LAD and LL equations.

2) The proper definition of a phase-space. The problem is relevant for the LAD equation. In fact, although the construction of kinetic theory is still formally possible (14, 15), the corresponding fluid statistical description seems inhibited.

3) The explicit dependence of the kinetic distribution function (KDF) in terms of the retarded EM self 4-potential are excluded. In fact, in the LAD and LL approximations the self-potential does not appear explicitly (see for example Refs.(16, 17)). Indeed, within the point-particle model, underlying both treatments, the retarded self-potential is divergent.

On the other hand, for the fluid treatment:

1) The precise form of the fluid closure conditions may depend on the approximations adopted in the kinetic description for the representation of the EM RR self-force. An example-case is provided by Ref.(17) where relativistic fluid equations are obtained based on the LL equation. As a result, it was found that, with the exception of the continuity equation, all moment equations involve higher-order fluid moments associated to the RR self-force. It is unclear whether this is an intrinsic physical feature of the RR interaction or simply a result of the approximations involved.

2) The fluid fields may in principle depend implicitly on the EM self 4-potential. In the framework of the LL equation it is unclear how such an effect can be dealt with. However, the treatment of such effects seems to present objective difficulties.

In fact, in principle non-local effects might arise in this way in which *retarded velocity contributions* appear in the kinetic equation. In such a case the explicit construction of fluid equations would be ambiguous (and might involve an infinite set of higher-order moments).

The interesting question is whether *these difficulties can be overcome in physically realizable situations, namely for exactly solvable classical systems* (of particles) for which the relativistic equations of motion are both *variational* and *non-asymptotic*. The prerequisite is provided by the possibility of constructing an exact representation for the RR equation for a suitable type of classical charged particles. In the past, their precise identification with physically-realizable systems has remained elusive because of the difficulty of the problem. However, as pointed out in the previous Chapter (see also Tessarotto *et al.* (18) and Cremaschini *et al.* (19)) in the framework of special relativity an exact variational RR equation can be obtained for classical finite-size charged particles. This refers to particles having a finite-size mass and charge distributions which are quasi-rigid, non-rotating, spherically symmetric and radially localized on a spherical surface $\partial\Omega$ having a finite radius $\sigma > 0$ (see (20) and the related discussion in Ref.(19)). In this formulation, contrary to the point-particle case, the retarded EM self 4-potential is well-defined, namely, it does not diverge, and can be determined analytically. As shown in Ref.(19), it follows that the RR equation is variational and the corresponding Hamilton variational functional is symmetric with respect to the non-local contributions. The latter are due to the retarded EM self interaction arising from the finite spatial extension of the charge distribution. As a consequence, the resulting exact RR equation is a second-order delay-type ODE which admits a Lagrangian formulation in standard form (see discussion below). Furthermore, under suitable conditions, the same equation defines a classical dynamical system (*RR dynamical system*).

In this Chapter it is proved that, based on the results of Ref.(19) exposed in the previous Chapter, the RR dynamical system admits also a Hamiltonian representation in terms of an effective non-local Hamiltonian function H_{eff}. This implies that the exact RR equation can also be cast in the equivalent *standard Hamiltonian form* represented by first-order delay-type ODEs

$$\frac{dr^\mu}{ds} = \frac{\partial H_{eff}}{\partial P_\mu}, \tag{6.1}$$

$$\frac{dP_\mu}{ds} = -\frac{\partial H_{eff}}{\partial r_\mu}, \tag{6.2}$$

with $\mathbf{y} = (r^\mu, P_\mu)$ denoting, in superabundant variables, the particle canonical state which spans the eight-dimensional phase-space $\Gamma \equiv \Gamma_r \times \Gamma_u$, where Γ_r and Γ_u are respectively the Minkowski M^4-configuration space and the 4-dimensional velocity-space, both with metric $\eta_{\mu\nu} \equiv diag\,(1, -1, -1, -1)$. Remarkably, here it is found that the Hamiltonian structure can be retained also after the introduction of a suitable short delay-time approximation of the RR force. The result is an intrinsic feature of the extended particle model adopted in the present treatment.

As a consequence, the statistical description of the RR dynamical system follows in a standard way. In particular, here both the exact and asymptotic kinetic and fluid formulations are reported. These are developed for collisionless relativistic plasmas in the Vlasov-Maxwell description, including consistently the contribution carried by the RR self-field. Applications of the theory here developed concern:

1) The kinetic and fluid treatments of relativistic astrophysical plasmas observed, for example, in accretion disks, relativistic jets, active galactic nuclei and mass inflows around compact objects.

2) The kinetic and fluid treatments of laboratory plasmas subject to ultra-intense and pulsed-laser sources.

The reference publications for the results presented in this Chapter are Refs.(21, 22).

6.2 Non-local Lagrangian formulation

The natural mathematical apparatus for an abstract description of Lagrangian and Hamiltonian mechanics is that of variational principles, whose methods have been studied for a long time by mathematicians and can be found in the textbooks. Nevertheless, actual problems of interest in classical relativistic dynamics involving the treatment of non-local interactions have escaped a solution. In particular, in the literature the prevailing view is that, while a non-local variational formulation is possible, a corresponding Hamiltonian representation is generally excluded. In the following we intend to point out that for a particular class of non-local Lagrangian systems the problem can be given a complete solution. The latter correspond to variational problems in which the variational functional is symmetric. To this end, in this section we briefly recall basic notions holding for local and non-local Lagrangian systems. This task represents a necessary prerequisite for the establishment of a corresponding Hamiltonian formulation and for the subsequent investigation of the Hamiltonian dynamics of finite-size charged particles with the inclusion of the RR self-force.

Definition #1 - Local and non-local Lagrangian systems.

A local (respectively, non-local) Lagrangian system is defined by the set $\{\mathbf{x}, L\}$ such that the following conditions are satisfied.

1. $\mathbf{x} \equiv \left(r^{\mu}(s), \frac{dr^{\mu}(s)}{ds}\right)$ is the Lagrangian state spanning the Lagrangian phase space $\Gamma_L \subseteq \mathbb{R}^{2N}$.

2. The Lagrangian action functional S is a 4-scalar of the form

$$S = \int_{s_1}^{s_2} ds L, \tag{6.3}$$

   with L to be referred to as *variational Lagrangian function.* In particular, the functional dependencies of S and L are respectively of the form:

   - $S \equiv S_0(r)$ and $L \equiv L_0\left(r, \frac{dr}{ds}\right)$ for local systems;

- $S \equiv S_1(r,[r])$ and $L \equiv L_1\left(r, \frac{dr}{ds}, [r]\right)$ for non-local systems, with $[r]$ denoting non-local dependencies.

3. In the functional class

$$\{r^\mu\} \equiv \left\{r^\mu(s) : r^\mu(s_i) = r_i^\mu, s_i \in I,\ i=1,2, s_1 < s_2, r^\mu(s) \in C^2(I)\right\}, \tag{6.4}$$

the synchronous variations $\delta r^\mu(s)$ are considered independent and vanish at the endpoints $r^\mu(s_i) = r_i^\mu$. Hereafter δ denotes, as usual, the Frechet functional derivative. For a synchronous variational principle the interval ds is such that $\delta ds = 0$ and is subject to the constraint

$$ds^2 = g_{\mu\nu} dr^\mu(s)\, dr^\nu(s), \tag{6.5}$$

where $r^\mu(s)$ are the extremal curves.

4. The Lagrangian action (6.3) admits a unique extremal curve $r^\mu(s)$ such that, for all synchronous variations $\delta r^\mu(s)$ in the functional class (6.4) the Hamilton variational principle

$$\delta S = 0 \tag{6.6}$$

holds identically. For non-local systems the non-local Lagrangian must be suitably constructed in such a way that the extrema curves $r^\mu(s)$ satisfy the constraint (6.5).

In particular, for local systems the extremal curves of S_0 are provided by the Euler-Lagrange (E-L) equations

$$\frac{\delta S_0}{\delta r^\mu} \equiv F_\mu(r) L_0 = 0, \tag{6.7}$$

where, for an arbitrary set of Lagrange coordinates q^μ, $F_\mu(q)$ denotes the *E-L differential operator*

$$F_\mu(q) \equiv \frac{d}{ds} \frac{\partial}{\partial\left(\frac{d}{ds} q^\mu(s)\right)} - \frac{\partial}{\partial q^\mu}. \tag{6.8}$$

On the other hand, for non-local systems the extremal curves of the functional S_1 are provided by the Euler-Lagrange equations

$$\frac{\delta S_1}{\delta r^\mu} \equiv \left.\frac{\delta S_1}{\delta r^\mu}\right|_{[r]} + \left.\frac{\delta S_1}{\delta [r^\mu]}\right|_r = 0, \tag{6.9}$$

where $\left.\frac{\delta S_1}{\delta r^\mu}\right|_{[r]}$ and $\left.\frac{\delta S_1}{\delta [r^\mu]}\right|_r$ carry respectively the contributions due to the local and non-local dependencies.

Definition #2 - Non-local Lagrangian systems in standard form.

A non-local Lagrangian system $\{\mathbf{x}, L_1\}$ will be said to admit a *standard form* if the variational derivative (6.9) yields the E-L equations in *the standard form:*

$$\frac{\delta S_1}{\delta r^\mu} + \frac{\delta S_1}{\delta [r^\mu]} \equiv F_\mu(r) L_{eff} = 0, \tag{6.10}$$

with

$$L_{eff} \equiv L_{eff}\left(r, \frac{dr}{ds}, [r]\right) \tag{6.11}$$

denoting a suitable *effective non-local Lagrangian.*

On the base of these definitions, the following theorem holds.

THM.1 - Non-local and Effective Lagrangian functions
Given validity of the definitions #1 and #2, it follows that:
T1$_1$) The non-local Lagrangian L_1 and the effective Lagrangian L_{eff} are generally different, namely

$$L_1 \neq L_{eff}. \tag{6.12}$$

T1$_2$) If $S_1(r, [r])$ admits the general decomposition

$$S_1(r, [r]) = S_a(r) + S_b(r, [r]), \tag{6.13}$$

with $S_a(r) \equiv \int_{s_1}^{s_2} ds L_a\left(r, \frac{dr}{ds}\right)$ and $S_b(r, [r]) \equiv \int_{s_1}^{s_2} ds L_b\left(r, \frac{dr}{ds}, [r]\right)$, and moreover $S_b(r, [r])$ defines a symmetric functional such that

$$S_b(r, [r]) = S_b([r], r), \tag{6.14}$$

then the effective Lagrangian L_{eff} is related to the variational non-local Lagrangian $L_1 \equiv L_a + L_b$ as

$$L_{eff} = L_a + 2L_b = L_1 + L_b. \tag{6.15}$$

Proof - T1$_1$) The proof is an immediate consequence of Eqs.(6.9) and (6.10). In fact, by definition the E-L differential operator $F_\mu(r)$ is a local differential operator that is required to preserve its form also for non-local systems. On the other hand, the variational derivative (6.9) is different from (6.7). Hence, in order to write the E-L equations associated to the non-local function L_1 in standard form, a suitable effective Lagrangian L_{eff} must be introduced, which must differ from L_1 and be expressed in such a way that the non-local dependencies contained in L_1 can be equivalently treated by means of $F_\mu(r)$.

T1$_2$) The proof follows by inspecting the general definition (6.9). In this case, in view of the symmetry property (6.14), it follows manifestly that

$$\frac{\delta S_1}{\delta r^\mu} \equiv \left.\frac{\delta S_1}{\delta r^\mu}\right|_{[r]} + \left.\frac{\delta S_1}{\delta [r^\mu]}\right|_r = \left.\frac{\delta S_a}{\delta r^\mu}\right|_{[r]} + 2\left.\frac{\delta S_b}{\delta r^\mu}\right|_{[r]} = 0. \tag{6.16}$$

Then, by comparing this relation with the definitions both of the E-L differential operator (6.8) and the standard form representation of the E-L equations (6.10), it follows that the effective Lagrangian L_{eff} takes necessarily the form given in Eq.(6.15). This completes the proof of the statement.

Q.E.D.

A basic consequence of Definition #2 and THM.1 concerns the covariance property of the E-L equations (6.10). The result is stated in the following Corollary.

Corollary 1 to THM.1 - Covariance of the E-L equations for arbitrary point transformations.

The Euler-Lagrange equations (6.10) are covariant with respect to arbitrary point transformations

$$r^{\mu} \rightarrow q^{\mu}(r) \tag{6.17}$$

represented by a diffeomeorphism of class C^k, with $k \geq 2$, which requires they are of the form

$$F_{\mu}(q)\widetilde{L}_{eff} = \frac{\partial r^{\nu}}{\partial q^{\mu}} F_{\nu}(r) L_{eff} = 0, \tag{6.18}$$

with $\widetilde{L}_{eff}$ denoting

$$\widetilde{L}_{eff}\left(q, \frac{dq}{ds}, [q]\right) \equiv L_{eff}\left(r, \frac{dr}{ds}, [r]\right). \tag{6.19}$$

As a consequence, Eq.(6.10) satisfies also the covariance property with respect to arbitrary infinitesimal Lorentz transformations (Manifest Lorentz Covariance).

Proof - The Euler-Lagrange equations (6.10) are by definition covariant provided the variational Lagrangian $L_1\left(r, \frac{dr}{ds}, [r]\right)$ is a 4-scalar (as it is by construction). Then, it is sufficient to represent the Lagrangian action in terms of the Lagrangian coordinates q^{μ}, yielding

$$\widetilde{S_1}(q, [q]) \equiv S_1(r, [r]), \tag{6.20}$$

with $\widetilde{S_1}(q, [q])$ denoting the transformed action

$$\widetilde{S_1}(q, [q]) = \int_{s_1}^{s_2} ds \widetilde{L_1}\left(q, \frac{dq}{ds}, [q]\right) \tag{6.21}$$

and $\widetilde{L_1}$ denoting the transformed variational non-local Lagrangian. Hence, the Hamilton variational principle $\delta\widetilde{S_1}(q, [q]) = 0$ yields precisely the E-L equations (6.18). This proves the statement. The covariance property of Eqs.(6.10) with respect to point transformations (6.17) includes, as particular case, Lorentz transformations. Therefore, Eqs.(6.10) are also *Manifestly Lorentz Covariant* (MLC).

Q.E.D.

We notice the following notable features of this treatment:

1) In general, in absence of any kind of symmetry, a non-local Lagrangian system does not admit a standard form representation in terms of the effective Lagrangian L_{eff} (1).

2) As shown in T1$_2$, the possibility of getting an explicit relationship between L_1 and L_{eff} is a consequence solely of the symmetry property (6.14) of the functional S_b. This also proves the existence of L_{eff} and, as a consequence, of the standard form representation for non-local systems satisfying Eq.(6.14).

3) The symmetry assumption (6.14) can be effectively realized in physical systems. As it will be shown below, this condition is satisfied by the variational functional which describes the dynamics of finite-size classical charged particles with the inclusion of the RR effects associated to the interaction with the EM self-field.

6.3 Non-local Hamiltonian formulation

In this section we deal with the basic features concerning the Hamiltonian formulation for non-local systems which admit a variational treatment in terms of non-local Lagrangian functions. This requires the introduction of the following preliminary definitions.

Definition #3 - Local and non-local Hamiltonian systems.

A local (respectively, non-local) Lagrangian system $\{\mathbf{x}, L\}$ is said to admit a local (non-local) Hamiltonian system $\{\mathbf{y} \equiv (r^\mu, p_\mu), H\}$ provide the following conditions are satisfied.

1. The *variational Hamiltonian* H is defined as the Legendre transformation of the local (non-local) variational Lagrangian L
$$H = p_\mu \frac{dr^\mu}{ds} - L, \tag{6.22}$$
with
$$p_\mu = \frac{\partial L}{\partial \frac{dr^\mu}{ds}} \tag{6.23}$$
being the corresponding canonical momentum, with corresponding action functional $S_H \equiv \int_{s_1}^{s_2} ds \left[p_\mu \frac{dr^\mu}{ds} - H\right]$.

2. It is assumed that H is respectively of the form:
   - $H \equiv H_0\,(r, p)$ for local systems;
   - $H \equiv H_1\,(r, p, [r])$ for non-local systems, namely it is a local function of (r, p) and a functional of $[r]$.

The corresponding *Hamilton action functionals* are denoted respectively as
$$S_{H_0}(r, p) = \int_{s_1}^{s_2} ds \left[p_\mu \frac{dr^\mu}{ds} - H_0\right] \tag{6.24}$$

for local systems, and as

$$S_{H_1}(r,p,[r]) = \int_{s_1}^{s_2} ds \left[p_\mu \frac{dr^\mu}{ds} - H_1 \right] \tag{6.25}$$

for non-local systems.

3. In the functional class

$$\{\mathbf{y} \equiv (r^\mu, p_\mu)\} \equiv \left\{ \mathbf{y}(s) : \mathbf{y}(s_i) = \mathbf{y}_i, s_i \in I, i = 1,2, s_1 < s_2, \mathbf{y}(s) \in C^2(I) \right\} \tag{6.26}$$

the synchronous variations $(\delta r^\mu(s), \delta p_\mu(s))$ are all considered independent and vanish at the endpoints $\mathbf{y}(s_i) = \mathbf{y}_i$. By assumption, synchronous variations imply that $\delta ds = 0$, with the interval ds satisfying the constraint

$$ds^2 = g_{\mu\nu} dr^\mu (s) \, dr^\nu (s) \, , \tag{6.27}$$

where $r^\mu (s)$ are the extremal curves.

4. The *modified Hamilton variational principle*

$$\delta S_H = 0 \tag{6.28}$$

with variations $(\delta r^\mu(s), \delta p_\mu(s))$ is equivalent to the Hamilton principle (6.6), i.e., it yields the same extremal curves in the functional class $\{\mathbf{y}\}$.

In particular, for local systems the extremal curves of S_{H_0} can be cast in the *standard Hamiltonian form as first-order ODEs*

$$\frac{\delta S_{H_0}}{\delta p_\mu} = \frac{dr^\mu}{ds} = \frac{\partial H_0}{\partial p_\mu} = [r^\mu, H_0] \, , \tag{6.29}$$

$$\frac{\delta S_{H_0}}{\delta r^\mu} = -\frac{dp_\mu}{ds} = \frac{\partial H_0}{\partial r^\mu} = [p_\mu, H_0] \, , \tag{6.30}$$

where the customary Poisson bracket formalism has been used.

On the other hand, for non-local systems the extremal curves of the functional S_{H_1} are provided by the set of first-order ODEs

$$\frac{\delta S_{H_1}}{\delta p_\mu} = 0, \tag{6.31}$$

$$\frac{\delta S_{H_1}}{\delta r^\mu} \equiv \left. \frac{\delta S_{H_1}}{\delta r^\mu} \right|_{[r]} + \left. \frac{\delta S_{H_1}}{\delta \left[r^\mu \right]} \right|_r = 0, \tag{6.32}$$

where $\left. \frac{\delta S_{H_1}}{\delta r^\mu} \right|_{[r]}$ and $\left. \frac{\delta S_{H_1}}{\delta [r^\mu]} \right|_r$ carry respectively the contributions due to the local and non-local dependencies.

Definition #4 - Non-local Hamiltonian systems in standard form.

A non-local Hamiltonian system $\{\mathbf{y}, H_1\}$ will be said to admit a *standard form* if the extremal first-order ODEs (6.31) and (6.32) can be cast in the *standard Hamiltonian form* in terms of the *effective canonical momentum* P_μ *and Hamiltonian function* H_{eff} as

$$\frac{\delta S_{H_1}}{\delta p_\mu} = \frac{dr^\mu}{ds} = \frac{\partial H_{eff}}{\partial P_\mu} = \left[r^\mu, H_{eff}\right], \tag{6.33}$$

$$\frac{\delta S_{H_1}}{\delta r^\mu} \equiv \left.\frac{\delta S_{H_1}}{\delta r^\mu}\right|_{[r]} + \left.\frac{\delta S_{H_1}}{\delta\left[r^\mu\right]}\right|_r = -\frac{dP_\mu}{ds} = \frac{\partial H_{eff}}{\partial r^\mu} = \left[P_\mu, H_{eff}\right]. \tag{6.34}$$

Here both $H_{eff} = H_{eff}\left(r, P, [r]\right)$ and P_μ must be defined in terms of the effective Lagrangian function introduced in Eq.(6.10) respectively as

$$H_{eff} \equiv P_\mu \frac{dr^\mu}{ds} - L_{eff} \tag{6.35}$$

and

$$P_\mu \equiv \frac{\partial L_{eff}}{\partial \frac{dr^\mu}{ds}}. \tag{6.36}$$

From this definition it follows that, if the non-local Hamiltonian system $\{\mathbf{y}, H_1\}$ admits a standard form, then the Poisson bracket representation holds for H_{eff} and P_μ.

The following theorem can be stated concerning the relationship between H_1 and H_{eff}.

THM.2 - Non-local and Effective Hamiltonian functions

Given validity of the definitions #3 and #4 and the results of THM.1, if $S_{H_1}\left(r, p, [r]\right)$ *admits the general decomposition*

$$S_{H_1}\left(r, p, [r]\right) = S_{H_a}(r, p) + S_{H_b}(r, p, [r]), \tag{6.37}$$

with

$$S_{H_a}(r, p) \equiv \int_{s_1}^{s_2} ds \left[p_{a\mu}\frac{dr^\mu}{ds} - H_a\left(r, p\right)\right], \tag{6.38}$$

$$S_{H_b}(r, p, [r]) \equiv \int_{s_1}^{s_2} ds \left[p_{b\mu}\frac{dr^\mu}{ds} - H_b\left(r, p, [r]\right)\right], \tag{6.39}$$

where the canonical momenta $p_{a\mu}$ *and* $p_{b\mu}$ *are defined respectively as*

$$p_{a\mu} \equiv \frac{\partial L_a}{\partial \frac{dr^\mu}{ds}}, \tag{6.40}$$

$$p_{b\mu} \equiv \frac{\partial L_b}{\partial \frac{dr^\mu}{ds}}, \tag{6.41}$$

and moreover $S_{H_b}(r,p,[r])$ defines a symmetric functional such that

$$S_{H_b}(r,p,[r]) = S_{H_b}([r]\,,p,r), \tag{6.42}$$

then the effective Hamiltonian H_{eff} is related to the variational non-local Hamiltonian $H_1 \equiv H_a + H_b$ as

$$H_{eff} = H_a + 2H_b = H_1 + H_b, \tag{6.43}$$

where, by definition

$$H_1 \equiv p_\mu \frac{dr^\mu}{ds} - L_1, \tag{6.44}$$

$$H_a \equiv p_{a\mu} \frac{dr^\mu}{ds} - L_a, \tag{6.45}$$

$$H_b \equiv p_{b\mu} \frac{dr^\mu}{ds} - L_b. \tag{6.46}$$

Proof - The proof follows from THM.1 and by invoking the general definitions (6.33) and (6.34). In fact, in view of the symmetry property (6.42), it follows manifestly that

$$\frac{\delta S_{H_1}}{\delta r^\mu} \equiv \left.\frac{\delta S_{H_1}}{\delta r^\mu}\right|_{[r]} + \left.\frac{\delta S_{H_1}}{\delta\,[r^\mu]}\right|_{r} = \left.\frac{\delta S_{H_a}}{\delta r^\mu}\right|_{[r]} + 2\left.\frac{\delta S_{H_b}}{\delta r^\mu}\right|_{[r]}. \tag{6.47}$$

Then, by comparing this relation with the definitions (6.35)-(6.36) for the standard Hamiltonian form and using Eqs.(6.44)-(6.46), from the analogous results in THM.1 which concerns the relationship between L_1 and L_{eff} in the symmetric case, Eq.(6.43) is readily obtained.

Q.E.D.

Finally, as a basic consequence of Definition #4 and THM.2, the following Corollary can be stated concerning the covariance property of the Hamilton equations in standard form.

Corollary 1 to THM.2 - Covariance of the Hamilton equations for arbitrary point transformations.

The Hamilton equations (6.33)-(6.34) in standard form are covariant with respect to arbitrary point transformations

$$r^\mu \to q^\mu(r) \tag{6.48}$$

represented by a diffeomeorphism of class C^k with $k \geq 2$, which requires they are of the form

$$\frac{dq^\mu}{ds} = \frac{\partial \widetilde{H}_{eff}}{\partial P_{(q)\mu}} = \left[q^\mu, \widetilde{H}_{eff}\right], \tag{6.49}$$

$$\frac{dP_{(q)\mu}}{ds} = -\frac{\partial \widetilde{H}_{eff}}{\partial q^{\mu}} = \left[P_{(q)\mu}, \widetilde{H}_{eff}\right], \tag{6.50}$$

with $\widetilde{H}_{eff}$ denoting

$$\widetilde{H}_{eff}\left(q, P, [q]\right) \equiv H_{eff}\left(r, P, [r]\right) \tag{6.51}$$

and $P_{(q)\mu}$ being the transformed canonical momentum. As a consequence, Eqs.(6.49) and (6.50) satisfy also the covariance property with respect to arbitrary infinitesimal Lorentz transformations (Manifest Lorentz Covariance).

Proof - In fact, for an arbitrary point transformation of the type (6.48), the corresponding transformation for the momenta P_{ν} is

$$P_{(q)\mu} = \frac{\partial q^{\nu}}{\partial r^{\mu}} P_{\nu}, \tag{6.52}$$

which yields

$$\frac{\partial P_{(q)\nu}}{\partial P_{\mu}} = \frac{\partial q^{\mu}}{\partial r^{\nu}}, \tag{6.53}$$

$$\frac{\partial P_{(q)\mu}}{\partial P_{\nu}} = \frac{\partial r^{\nu}}{\partial q^{\mu}}. \tag{6.54}$$

Hence, it follows that

$$\frac{dq^{\mu}}{ds} = \frac{\partial \widetilde{H}_{eff}}{\partial P_{(q)\mu}} = \frac{\partial q^{\mu}}{\partial r^{\nu}}\frac{dr^{\nu}}{ds} = \frac{\partial q^{\mu}}{\partial r^{\nu}}\frac{\partial H_{eff}}{\partial P_{\nu}}, \tag{6.55}$$

$$\frac{dP_{(q)\mu}}{ds} = -\frac{\partial \widetilde{H}_{eff}}{\partial q^{\mu}} = \frac{\partial P_{(q)\mu}}{\partial P_{\nu}}\frac{dP_{\nu}}{ds} = -\frac{\partial r^{\nu}}{\partial q^{\mu}}\frac{\partial H_{eff}}{\partial r^{\nu}}, \tag{6.56}$$

which implies

$$\frac{\partial \widetilde{H}_{eff}}{\partial P_{(q)\mu}} = \frac{\partial q^{\mu}}{\partial r^{\nu}}\frac{\partial H_{eff}}{\partial P_{\nu}}, \tag{6.57}$$

$$\frac{\partial \widetilde{H}_{eff}}{\partial q^{\mu}} = \frac{\partial r^{\nu}}{\partial q^{\mu}}\frac{\partial H_{eff}}{\partial r^{\nu}}, \tag{6.58}$$

where $\widetilde{H}_{eff}$ is defined in Eq.(6.51) above. Therefore, the Hamilton equations in standard form for the Lagrangian coordinates q^{μ} and the canonical momenta $P_{(q)\mu}$ are respectively covariant [Eq.(6.57)] and controvariant [Eq.(6.58)] with respect to the point transformation (6.48). This is true also for arbitrary infinitesimal Lorentz transformations, which proves the MLC of Hamilton equations Eqs.(6.49) and (6.50) in standard form.

Q.E.D.

6.4 An example of non-local interaction: the classical EM RR problem

A crucial issue of the present investigation concerns the possible existence of physical systems subject to non-local interactions whose dynamics can be consistently described in terms of a variational action integral and which admit at the same time both Lagrangian and Hamiltonian formulations in standard form. In this section we prove that the EM RR problem for classical finite-size charged particles represents a physical example of non-local interactions of this kind. The reason behind the choice of considering extended particles is the necessity of avoiding the intrinsic divergences of the RR effect characteristic of the point-particle model.

In fact, consider the general form of the Hamilton action functional for the variational treatment of the dynamics of an extended charged particle in presence of an external EM field and with the inclusion of the RR self-interaction. This can be conveniently expressed as follows:

$$S_1\left(z,[z]\right) = S_M\left(z\right) + S_C^{(ext)}(z) + S_C^{(self)}(z,[z]), \tag{6.59}$$

where $S_M, S_C^{(ext)}$and $S_C^{(self)}$ are respectively the contributions from the inertial mass and the EM coupling with the external and the self fields. In particular, denoting by $j^{(self)\mu}(r)$ the particle 4-current density generated by the particle itself and observed at a 4-position r, the two coupling action integrals are provided by the following 4-scalars:

$$S_C^{(ext)}\left(z\right) = \frac{1}{c^2}\int_1^2 d^4r A^{(ext)\mu}\left(r\right) j_\mu^{(self)}(r), \tag{6.60}$$

$$S_C^{(self)}(z,[z]) = \frac{1}{c^2}\int_1^2 d^4r A^{(self)\mu}\left(r\right) j_\mu^{(self)}(r), \tag{6.61}$$

where $A_\mu^{(ext)}$ and $A_\mu^{(self)}$ denote the 4-vector potentials of the external and the self EM fields and z is a state to be suitably defined (see below). A clarification here is in order. The external EM 4-potential $A_\mu^{(ext)}\left(r\right)$ acting on the charged particle located at the 4-position r is assumed to be produced only by prescribed "external" sources, namely, excluding the particle itself, by the remaining possible EM sources belonging to the configuration space Γ_r. Within the framework of special relativity, both the inertial term and the coupling term with the external field carry only local dependencies, in the sense that they depend explicitly only on the local 4-position r. They provide the classical dynamics of charged particles in absence of any RR effect. On the other hand, the functional $S_C^{(self)}$ associated to the EM self-interaction contains both local and non-local contributions. In particular, since the state z of a finite-size particle must include a 4-position vector r, it follows that $S_C^{(self)}$ generally depends explicitly on two different 4-positions, r and $[r]$, to be properly defined (see below). The non-local property of $S_C^{(self)}$ represents a characteristic feature of RR phenomena.

From the relationship (6.59) it follows that the Hamilton action functional for the treatment of the RR admits the decomposition (6.13) introduced by THM.1, namely it can be written as the sum of two terms, carrying respectively only local and both local and non-local dependencies. In order to prove that the same functional admits also a Lagrangian and a Hamiltonian representation in standard form it is sufficient to show that the self-coupling functional is symmetric in z and $[z]$, in the sense defined in THM.1. For this purpose we need to determine explicitly the general expression of the 4-current and the self 4-potential for a rotating finite-size charged particle.

The first step consists in constructing a covariant representation for the 4-current density. We follow the approach presented by Nodvik (20). Thus, we consider an extended charged particle with charge and mass distributions having the same support $\partial\Omega$, to be identified with a smooth surface. Denoting by $r^{\mu}(s)$ the 4-vector position (with proper time s) of a reference point belonging to the internal open domain Ω and by ζ^{μ} a generic 4-vector of $\partial\Omega$, the displacement vector ξ^{μ} is defined as:

$$\xi^{\mu} \equiv \varsigma^{\mu} - r^{\mu}(s). \tag{6.62}$$

The particle model is prescribed by imposing the constraints of rigidity of $\partial\Omega$, namely for all ς^{μ} and $r^{\mu}(s)$ (20):

$$\begin{aligned} \xi^{\mu}\xi_{\mu} &= const., & (6.63)\\ \xi_{\mu}u^{\mu}(s) &= 0, & (6.64) \end{aligned}$$

where $u^{\mu}(s) \equiv \frac{d}{ds}r^{\mu}(s)$. In particular, we shall assume that mass and charge distributions are spherically symmetric and therefore characterized by a form factor $f\left(\xi^{2}\right) \equiv f\left(\xi^{\mu}\xi_{\mu}\right)$. This allows one to identify $r^{\mu}(s)$ as the center-point of $\partial\Omega$. The extended particle can in principle exhibit both translational and rotational degrees of freedom. In particular, the translational motion can be described in terms of $r^{\mu}(s)$. Instead, the rotational dynamics, which includes both space-time rotations associated to the so-called Thomas precession and pure spatial rotations, can be described in terms of the Euler angles $\alpha(s) \equiv \{\varphi(s), \vartheta(s), \psi(s)\}$. It follows that, in this case, the Lagrangian state z must be identified with the set of variables $z \equiv (r^{\alpha}(s), \alpha(s))$. In view of these definitions it is immediate to prove that the 4-current density for the finite-size particle can be written as follows:

$$j^{(self)\mu}(r) = qc\int_{-\infty}^{+\infty} ds \left\{ u^{\mu}\left[1 - \frac{du_{\alpha}}{ds}x^{\alpha}\right] - \frac{1}{c}\omega^{\mu\nu}x_{\nu} \right\} f(x^{2})\delta(x^{\alpha}u_{\alpha}), \tag{6.65}$$

where

$$x^{\mu} = r^{\mu} - r^{\mu}(s) \tag{6.66}$$

and $\omega^{\mu\nu} = \omega^{\mu\nu}(s)$ is the antisymmetric angular velocity tensor (20), which depends on s through the Euler angles $\alpha(s)$. The term $\left[1 + \frac{du_{\alpha}}{ds}x^{\alpha}\right]$ contains the acceleration of $r^{\mu}(s)$ and represents the contribution associated to the Thomas precession effect. This can be formally eliminated by using the properties of the Dirac-delta function, implying the identity:

$$\delta(x^\alpha u_\alpha(s)) = \frac{1}{\left|\frac{d[x^\alpha u_\alpha]}{ds}\right|}\delta(s-s_1) = \frac{1}{\left|1-\frac{du_\alpha}{ds}x^\alpha\right|}\delta(s-s_1), \tag{6.67}$$

where by definition $s_1 = s_1(r)$ is the root of the algebraic equation

$$u_\mu(s_1)\left[r^\mu - r^\mu(s_1)\right] = 0. \tag{6.68}$$

As a result, the 4-current can be equivalently expressed as

$$j^{(self)\mu}(r) = qc\int_{-\infty}^{+\infty} ds\left[u^\mu\delta(s-s_1) - \frac{1}{c}\omega^{\mu\nu}x_\nu\delta(x^\alpha u_\alpha)\right]f(x^2). \tag{6.69}$$

The second step consists in constructing a Green-function representation for the EM self-potential $A^{(self)\mu}$ in terms of the 4-current $j^{(self)\mu}(r)$. This technique is well-known. Thus, considering the Maxwell equations in flat space-time, in the Lorentz gauge $A^{(self)\beta},_\beta = 0$, the self 4-potential must satisfy the wave equation

$$\Box A^{(self)\mu} = \frac{4\pi}{c}j^{(self)\mu}(r), \tag{6.70}$$

where $\Box$ represents the D'Alembertian operator and $j^{(self)\mu}(r)$ is given by Eq.(6.69). The formal solution of Eq.(6.70) is

$$A^{(self)\mu}(r) = \frac{4\pi}{c}\int d^4r' G(r,r')j^{(self)\mu}(r'), \tag{6.71}$$

where $G(r,r')$ is the retarded Green's function corresponding to the prescribed charge density. By construction, it follows that $G(r,r')$ is symmetric with respect to r and r', and furthermore - since the particle is finite-size - both the 4-current and the self-potential are everywhere well-defined.

From these general results, it is immediate to prove the following theorem.

THM.3 - Symmetry properties of $S_C^{(self)}(z,[z])$

Given validity of Eq.(6.69) for the covariant expression of the current density for a finite-size charged particle and of Eq.(6.71) for the general expression of the corresponding EM self-potential, it follows that:

T3$_1$) The functional $S_C^{(self)}(z,[z])$, defined in Eq.(6.61) as an integral over the 4-volume element d^4r, can be written as a line integral of the form

$$S_C^{(self)}(z,[z]) = \int_{-\infty}^{+\infty} ds L_C^{(self)}(z,[z]), \tag{6.72}$$

where $L_C^{(self)}$ represents the Lagrangian of the coupling with the EM self-field. This is *defined as*

$$L_C^{(self)}(z,[z]) \equiv \frac{4\pi q}{c^2}\int_1^2 d^4r \left\{ \left[u^\mu \delta(s-s_1) - \frac{1}{c}\omega^{\mu\nu}x_\nu \delta(x^\alpha u_\alpha) \right] f(x^2) \int d^4r' G(r,r') j_\mu^{(self)}(r') \right\}. \tag{6.73}$$

T3$_2$) The functional $S_C^{(self)}(z,[z])$ contains both local and non-local dependencies in terms of the variational quantities $z \equiv z(s)$ and $[z] \equiv [z(s)]$. In particular, it is symmetric in these local and non-local variables, in the sense stated in THM.1, namely

$$S_C^{(self)}(z,[z]) = S_C^{(self)}([z],z). \tag{6.74}$$

T3$_3$) The functional $S_C^{(self)}(z,[z])$ contains at most only first-order derivatives of the variational functions $z(s)$.

Proof - T3$_1$) The proof of the first statement follows by noting that the action integral $S_C^{(self)}(z,[z])$ is a 4-scalar by definition. Hence, making explicit the expressions of $A^{(self)\mu}$ and $j_\mu^{(self)}$ in Eq.(6.61) according to the results in Eqs.(6.71) and (6.69), by exchanging the order of the integrations and invoking the symmetry property of the Green function, the conclusion can be easily reached. In particular, the variational Lagrangian is found to be of the general form given in Eq.(6.73).

T3$_2$) To prove the second statement we first notice that in Eq.(6.72) both z and z' are integration variables, while by definition the variational quantities are identified with $z(s)$ and $[z(s)] \equiv z'(s')$. These dependencies are carried respectively by the charge current densities $j^{(self)\mu}(r)$ and $j^{(self)\mu}(r')$. The result is then reached by noting that the functional carrying the self-coupling terms is symmetric with respect to the integrated quantities, and in particular with respect to $j^{(self)\mu}(r)$ and $j^{(self)\mu}(r')$. Hence, exchanging $\left(z, j^{(self)\mu}(r)\right)$ with $\left(z', j^{(self)\mu}(r')\right)$ does not affect the form of the functional, with the consequence that Eq.(6.74) is identically satisfied.

T3$_3$) The proof of the statement is an immediate consequence of the representation for the current density $j^{(self)\mu}(r)$ given in Eq.(6.69). In fact, the term proportional to the acceleration $\frac{du_\alpha}{ds}$ in Eq.(6.65) and which is associated to the Thomas precession, does not appear in Eq.(6.69), thanks to the property of the Dirac-delta function indicated above in Eq.(6.67).

Q.E.D.

An immediate consequence of THM.3 is that, thanks to THMs.1 and 2, the variational treatment of the dynamics of finite-size charged particles subject to the EM RR effect admits both Lagrangian and Hamiltonian representations in standard form. In particular, in this case, it follows that the following identification must be introduced:

$$L_b \equiv L_C^{(self)}, \tag{6.75}$$

where L_b is the Lagrangian defined above in THM.1.

6.5 Hamiltonian theory for the RR problem

In this section, based on THMs.1-3 and the theory developed in Ref.(19), we proceed constructing the Hamiltonian formulation for the RR problem. For this purpose, it is convenient to recall the explicit form of the EM self 4-potential obtained in Ref.(19). As shown in the previous Chapter, in the external domain (with respect to $\partial\Omega$) $A_{\mu}^{(self)}(r)$ is found to admit the following integral representation:

$$A_{\mu}^{(self)}(r) = 2q \int_{1}^{2} dr'_{\mu} \delta(\widehat{R}^{\alpha}\widehat{R}_{\alpha}). \tag{6.76}$$

Here $\widehat{R}^{\alpha} = r^{\alpha} - r^{\alpha}(s')$, with r^{α} and $r'^{\alpha} \equiv r^{\alpha}(s')$ denoting respectively the generic 4-position and the 4-position of the center of the charge distribution at proper time s'. As a fundamental consequence of the finite extension of the particle and the restrictions on the domain of validity of Eq.(6.76), the resulting variational functional and Faraday tensor for the self-field turn out to be completely different from the point-particle treatment. In particular, the action integral becomes now a non-local functional with respect to the 4-position r. As shown in Ref.(19) and in the previous Chapter, this can be written as a line integral in terms of a variational Lagrangian $L_1(r,[r])$ as follows:

$$S_1(r,[r]) = \int_{-\infty}^{+\infty} ds L_1(r,[r]). \tag{6.77}$$

Here $L_1(r,[r])$ is defined as:

$$L_1(r,[r]) = L_M(r) + L_C^{(ext)}(r) + L_C^{(self)}(r,[r]), \tag{6.78}$$

where

$$L_M(r,u) = \frac{1}{2} m_o c \frac{dr_{\mu}}{ds}\frac{dr^{\mu}}{ds}, \tag{6.79}$$

$$L_C^{(ext)}(r) = \frac{q}{c}\frac{dr^{\mu}}{ds}\overline{A}_{\mu}^{(ext)}(r), \tag{6.80}$$

are the local contributions respectively from the inertial and the external EM field coupling terms, with $\overline{A}_{\mu}^{(ext)}$ denoting the surface-averaged external EM potential (see Ref.(19)). On the other hand, $L_C^{(self)}$ represents the non-local contribution arising from the EM self-field coupling, which is provided by

$$L_C^{(self)}(r,[r]) = \frac{2q^2}{c}\frac{dr^{\mu}}{ds}\int_{1}^{2} dr'_{\mu}\delta(\widetilde{R}^{\mu}\widetilde{R}_{\mu} - \sigma^2), \tag{6.81}$$

where the 4-scalar $\sigma^2 \equiv \xi^{\mu}\xi_{\mu}$ is the radius of the surface distribution with respect to the center $r^{\mu}(s)$ and $\widetilde{R}^{\mu}$ is defined as

$$\widetilde{R}^{\alpha} \equiv r^{\alpha}(s) - r^{\alpha}(s'). \tag{6.82}$$

Notice that $\widetilde{R}^{\alpha}$ represents the displacement bi-vector between the actual position $r^{\alpha}(s)$ of the charge center at proper time s and the retarded position $r^{\alpha}(s')$ of the same point at the retarded proper time s'. It is immediate to verify that the representation of $S_C^{(self)}$ in terms of $L_C^{(self)}$ given in Eq.(6.81) satisfies the hypothesis of THM.1, and therefore the solution admits a Lagrangian representation in standard form. According to THM.1, this is obtained by setting

$$L_{eff} \equiv L_M(r) + L_C^{(ext)}(r) + 2L_C^{(self)}(r,[r]), \tag{6.83}$$

with $L_M(r)$, $L_C^{(ext)}(r)$ and $L_C^{(self)}$ respectively given by Eqs.(6.79)-(6.81). Then, the corresponding E-L equation is provided by the following covariant 4-vector, second-order delay-type ODE:

$$m_o c \frac{du_\mu(s)}{ds} = \frac{q}{c}\overline{F}_{\mu\nu}^{(ext)}(r(s))\frac{dr^\nu(s)}{ds} + \frac{q}{c}\overline{F}_{\mu k}^{(self)}\left(r(s), r(s')\right)\frac{dr^k(s)}{ds}, \tag{6.84}$$

where

$$u^\mu(s) \equiv \frac{dr^\mu(s)}{ds}. \tag{6.85}$$

Here the notation is as follows. Denoting by $F_{\mu\nu} \equiv F_{\mu\nu}^{(ext)} + F_{\mu\nu}^{(self)}$ the total Faraday tensor, $F_{\mu\nu}^{(ext)}$ and $F_{\mu\nu}^{(self)}$ are respectively the "external" and "self" Faraday tensors generated by $A_\nu^{(ext)}$ and $A_\nu^{(self)}$, which carry the contributions due to the external sources with respect to the charged particle and the particle EM self-interaction. In particular, the 4-tensor $\overline{F}_{\mu\nu}^{(ext)}(r(s))$ denotes the surface-average of the Faraday tensor associated to the external EM field, to be identified with

$$\overline{F}_{\mu\nu}^{(ext)} \equiv \partial_\mu \overline{A}_\nu^{(ext)} - \partial_\nu \overline{A}_\mu^{(ext)}, \tag{6.86}$$

with $\overline{A}_\nu^{(ext)}(r(s))$ only generated by external sources with respect to the single-particle whose dynamics is described by Eq.(6.84). Similarly, $\overline{F}_{\mu k}^{(self)}$ is the surface-average of the Faraday tensor contribution carried by the EM self 4-potential. In the parameter-free representation this is given by

$$\overline{F}_{\mu k}^{(self)}(r,[r]) = -4q\int_1^2 \left[dr'_\mu \frac{\partial}{\partial r^k}\delta\left(\widetilde{R}^\alpha \widetilde{R}_\alpha - \sigma^2\right) - dr'_k \frac{\partial}{\partial r^\mu}\delta\left(\widetilde{R}^\alpha \widetilde{R}_\alpha - \sigma^2\right)\right]. \tag{6.87}$$

As pointed out in Ref.(19), $\overline{F}_{\mu k}^{(self)}$ can also be parametrized in terms of the particle proper time s, by letting $r \equiv r(s)$ and $[r] \equiv r(s')$ in the previous equation, which also implies $dr'_\mu \equiv ds' \frac{dr'_\mu}{ds'}$. This means that the non-locality in Eq.(6.87) can be interpreted as non-locality in the particle proper time.

The remarkable feature of Eq.(6.87) is that the RR self-force (see the second term in the rhs of Eq.(6.84)) contains non-local effects only through the retarded particle 4-position and not through the 4-velocity. This feature is fundamental for the subsequent fluid treatment, since it permits the evaluation in the standard way of the velocity moments, retaining the exact form of the RR self-interaction.

The system of Eqs.(6.84) and (6.85) defines a delay-type ODE problem of the form

$$\left\{ \begin{array}{c} \frac{d\mathbf{y}}{ds} = \mathbf{X}_H\left(\mathbf{y},[r]\right), \\ \mathbf{y}\left(s_0\right) = \mathbf{y}_0, \\ \mathbf{y}\left(s_0'\right) = \mathbf{y}_{s_0'}, \quad \forall s_0' \in I_{s_0, s_0 - s_{ret}}, \end{array} \right. \tag{6.88}$$

with s_0 and s_{ret} denoting respectively the initial particle proper time and the causal retarded proper time (see Ref.(19)), and $\mathbf{X}_H$ the Hamiltonian vector field

$$\mathbf{X}_H\left(\mathbf{y},[r]\right) \equiv \left\{ \frac{\partial H_{eff}\left(r,P,[r]\right)}{\partial P_\mu}, -\frac{\partial H_{eff}\left(r,P,[r]\right)}{\partial r^\mu} \right\}. \tag{6.89}$$

Denoting by $\mathbf{y}\left(s\right) = \chi\left(\mathbf{y}_0, \left\{\mathbf{y}_{s_0'}, \forall s_0' \in I_{s_0, s_0 - s_{ret}}\right\}, s - s_0\right)$ the formal solution of the problem (6.88), in the reminder we shall assume that the map

$$\mathbf{y}_0 \rightarrow \mathbf{y}\left(s\right) \tag{6.90}$$

is a diffeomeorphism of class C^k, with $k \geq 1$.

Based on these results, the Hamiltonian formulation is provided by the following theorem.

THM.4 - Non-local variational and effective Hamiltonian functions for the non-rotating particle

Given validity of THMs.1-3, it follows that:

$T4_1$) The RR equation (6.84) for a non-rotating and spherically-symmetric charged particle admits the non-local Hamiltonian system $\{\mathbf{y} \equiv (r^\mu, p_\mu), H_1\}$. *Here p_μ and $H_1 \equiv H_1\left(r,p,[r]\right)$ are respectively the canonical momentum (6.23) defined with respect to the variational Lagrangian L_1 given in Eq.(6.78), and the corresponding non-local variational Hamiltonian (6.22) defined as the Legendre transformation of L_1. In particular, the variational non-local Hamiltonian (6.22) is identified with*

$$H_1\left(r,p,[r]\right) \equiv \frac{1}{2m_o c}\left(p_\mu - \frac{q}{c}A_\mu\right)\left(p^\mu - \frac{q}{c}A^\mu\right), \tag{6.91}$$

where A_μ is the total EM 4-potential

$$A_\mu\left(r,[r]\right) \equiv \overline{A}_\mu^{(ext)}\left(r\right) + \overline{A}_\mu^{(self)}\left(r,[r]\right), \tag{6.92}$$

and from Eq.(6.81) $\overline{A}_\mu^{(self)}$ is the functional

$$\overline{A}_\mu^{(self)}\left(r,[r]\right) \equiv 2q\int_1^2 dr_\mu' \delta(\widetilde{R}^\mu \widetilde{R}_\mu - \sigma^2). \tag{6.93}$$

T4$_2$) There exist P_μ and H_{eff}, defined respectively by Eqs.(6.36) and (6.35), such that

$$H_{eff}\left(r,P,[r]\right) \equiv \frac{1}{2m_o c}\left(P_\mu - \frac{q}{c}A_{(eff)\mu}\right)\left(P^\mu - \frac{q}{c}A^\mu_{(eff)}\right), \tag{6.94}$$

with $A_{(eff)\mu}$ the non-local effective EM 4-potential

$$A_{(eff)\mu}\left(r,P\right) \equiv \overline{A}^{(ext)}_\mu\left(r\right) + 2\overline{A}^{(self)}_\mu\left(r,[r]\right) \tag{6.95}$$

and $\overline{A}^{(self)}_\mu$ defined in Eq.(6.93).

T4$_3$) The effective and variational Hamiltonian functions H_{eff} and H_1 coincide when expressed in terms of the 4-velocity $\frac{dr^\mu(s)}{ds}$.

Proof - The proof of T4$_1$ and T4$_2$ follows immediately by applying THMs.1 and 2 with the variational Lagrangian L_1 given by Eq.(6.78). In particular, this yields

$$p_\mu = m_o c\frac{dr_\mu(s)}{ds} + \frac{q}{c}\left[\overline{A}^{(ext)}_\mu + \overline{A}^{(self)}_\mu\right] \tag{6.96}$$

and

$$P_\mu = m_o c\frac{dr_\mu(s)}{ds} + \frac{q}{c}\left[\overline{A}^{(ext)}_\mu + 2\overline{A}^{(self)}_\mu\right]. \tag{6.97}$$

The corresponding Legendre transformations then provide respectively Eq.(6.91) and Eq.(6.94). Finally, by direct substitution of Eq.(6.96) into Eq.(6.91) and Eq.(6.97) into Eq.(6.94), one obtains that

$$H_{eff} = H_1 = \frac{m_o c}{2}\frac{dr_\mu(s)}{ds}\frac{dr^\mu(s)}{ds}, \tag{6.98}$$

which proves also the last statement.

Q.E.D.

We remark that the Hamilton equation in standard form expressed in terms of H_{eff} and P_μ are differential equations of delay-type, as a consequence of the non-local dependencies appearing in H_{eff} which are characteristic of the RR phenomenon. In this case, for the well-posedness of the solution the initial conditions in the interval $I = [s_0 - s_{ret}, s_0]$ must be defined, with s_0 the initial proper time and s_{ret} a suitable retarded time. However, if the assumption of inertial motion in the proper time interval $I_0 = [-\infty, s_0]$ holds, then the mapping

$$T_{s_0,s} : \mathbf{y}_0 \equiv \mathbf{y}\left(s_0\right) \rightarrow \mathbf{y}\left(s\right) \equiv T_{s_0,s}\mathbf{y}_0, \tag{6.99}$$

with $\mathbf{y} = (r^\mu, P_\mu)$, defines a classical dynamical system (see Ref.(19)), and this dynamical system is Hamiltonian.

6.6 A Hamiltonian asymptotic approximation for the RR equation

In this section a detailed comparison of the present approach for extended particles with the customary point-particle treatments leading to the LAD and LL equations is carried out. For this purpose, asymptotic approximations of the exact RR self-force (6.87) are investigated.

The issue has been partially discussed in the previous Chapter (see also Ref.(19)). As pointed out there, an asymptotic approximation of the exact RR equation (6.84) can be obtained in validity of the *short delay-time ordering*, namely requiring

$$0 < \epsilon \equiv \left| \frac{s_{ret}}{s} \right| \ll 1, \tag{6.100}$$

where $s_{ret} = s - s'$, with s and s' denoting respectively the present and retarded particle proper times. This permits two different possible strategies, respectively based on Taylor expansions performed with respect to s (*present-time expansion*) or s' (*retarded-time expansion*). In particular, adopting the present-time expansion for the RR self-force (6.87), the delay-type ODE (6.84) can be reduced, in principle, to an infinite-order differential equation. Instead, by truncating the Taylor expansion to first-order in ϵ, ignoring mass-renormalization terms and taking the point-particle limit $\sigma \to 0$, in this way the customary expression for the LAD equation is recovered (see THM.3 of Ref.(19)).

As remarked in the Introduction to this Chapter, the resulting asymptotic approximation (given by the LAD equation) is non-variational and therefore non-Hamiltonian. In addition, contrary to the exact RR equation obtained here, the LAD equation, as well as the related LL approximation, both fail in the transient time intervals occurring when the external EM field acting on the particle is turned on and off. To elucidate this point, let us consider the dynamics of a charged particle which is in inertial motion in the past for all $s < s_0$ and from $s = s_0$ is subject to the action of an external EM field. Then, by construction, it is immediate to show that in the transient time interval $I_0 = [s_0, s_0 + s_{ret}]$ the exact RR self-force (6.84) is manifestly identically zero. In fact, in the case of inertial motion in the past (namely $u_\mu(s') = const.$) the RR self-force vanishes in such a time interval (see THM.1 in Ref.(19)). In contrast, both the LAD and LL equations predict incorrectly a non-vanishing RR self-force. The same kind of inconsistency (for the LAD and LL equations) arises when the analogous transient time interval corresponding to the turning-off of the external EM field is considered (7).

Therefore, the issue arises whether an alternative asymptotic approximation can be determined (for the exact RR equation) which simultaneously:

1) overcomes this deficiency, by taking into account consistently relativistic finite delay-time effects characteristic of the RR phenomenon;

2) is variational and admits a standard Hamiltonian formulation.

In this section we propose a solution to this problem, by performing a retarded-time expansion, which provides an alternative to the LAD and LL equations.

The Hamiltonian approximation

For definiteness, let us assume that the external force acting on the particle is non-vanishing only in a finite proper-time interval $I \equiv [s_0, s_1]$. Then, in validity of the ordering (6.100), we require that the external EM force is slowly varying in the sense that, denoting $r' \equiv r^\mu(s')$ and $r \equiv r^\mu(s)$,

$$\overline{F}^{(ext)}_{\mu\nu}(r') - \overline{F}^{(ext)}_{\mu\nu}(r) \sim O(\epsilon), \tag{6.101}$$

$$\left(\overline{F}^{(ext)}_{\mu\nu}(r') - \overline{F}^{(ext)}_{\mu\nu}(r)\right)_{,h} \sim O(\epsilon), \tag{6.102}$$

$$\left(\overline{F}^{(ext)}_{\mu\nu}(r') - \overline{F}^{(ext)}_{\mu\nu}(r)\right)_{,hk} \sim O(\epsilon). \tag{6.103}$$

Then, the retarded-time Hamiltonian approximation of the RR equation is obtained by performing a Taylor expansion in a neighborhood of s'. The result is summarized by the following theorem.

THM.5 - First-order, short delay-time Hamiltonian approximation (retarded-time expansion).

Given validity of the asymptotic ordering (6.100) and the smoothness assumptions (6.101)-(6.103) for the external EM field, neglecting corrections of order ϵ^n, with $n \geq 1$ (first-order approximation), the following results hold:

$T5_1$) The vector field

$$G_\mu \equiv \frac{q}{c}\overline{F}^{(self)}_{\mu k}(r(s), r(s'))\frac{dr^k(s)}{ds} \tag{6.104}$$

appearing in Eq.(6.84) can be approximated in a neighborhood of s' as

$$g_\mu(r(s')) = \left\{-m_{oEM}c\frac{d}{ds'}u_\mu(s') + g'_\mu(r(s'))\right\}, \tag{6.105}$$

to be referred to as retarded-time Hamiltonian approximation, in which the first term on the rhs identifies a retarded mass-correction term, $m_{oEM} \equiv \frac{q^2}{2c^2\sigma}$ denoting the leading-order EM mass. Finally, g'_μ is the 4-vector

$$g'_\mu(r(s')) = \frac{2}{3}\frac{q^2}{c}\left[\frac{1}{4}\frac{d^2}{ds'^2}u_\mu(s') - u_\mu(s')u^k(s')\frac{d^2}{ds'^2}u_k(s')\right]. \tag{6.106}$$

$T5_2$) The corresponding RR equation, obtained replacing G_μ with the asymptotic approximation g_μ (6.105), is variational, Lagrangian and admits a standard Lagrangian form. Let us denote with $r'_0 \equiv r_0(s')$ the extremal particle world-line at the retarded proper time s'. Then, in this approximation the corresponding asymptotic variational Lagrangian and effective Lagrangian functions coincide. Both are defined in terms of the asymptotic approximation $L^{(self)}_{C,asym}(r, r'_0)$, replacing $L^{(self)}_C$. To leading-order in ϵ,

this is found to be

$$L_{C,asym}^{(self)}(r, r_0') = g_\mu \left(r_0'\right) r^\mu. \tag{6.107}$$

T5$_3$) The asymptotic approximation given by Eq.(6.105) is also Hamiltonian. The asymptotic variational and effective Hamiltonian functions coincide and are given by

$$H_{1,asym} = p_\mu \frac{dr^\mu}{ds} - L_{1,asym} \tag{6.108}$$

with

$$L_{1,asym}(r, r_0') = L_M(r) + L_C^{(ext)}(r) + L_{C,asym}^{(self)}(r, r_0'), \tag{6.109}$$

and now

$$p_\mu = \frac{\partial L_{1,asym}}{\partial \frac{dr_\mu(s)}{ds}}. \tag{6.110}$$

Proof - T5$_1$) The proof can be carried out starting from Eq.(6.84) and performing explicitly the Taylor expansion in a neighborhood of $s' \equiv s - s_{ret}$. For a generic analytic function $f(s)$, this yields the power series of the form

$$f(s) = \sum_{k=0}^{\infty} \frac{(s-s')^k}{k!} \frac{d^k f(s')}{ds^k}. \tag{6.111}$$

In particular, for the 4-vectors $\frac{dr_\mu(s)}{ds}$ and $\widetilde{R}^k$ one obtains respectively the asymptotic approximations

$$\frac{dr_\mu(s)}{ds} \cong \frac{dr_\mu(s')}{ds'} + (s-s')\frac{d^2 r_\mu(s')}{ds'^2} + \frac{(s-s')^2}{2}\frac{d^3 r_\mu(s')}{ds'^3} + O\left(\epsilon^3\right) \tag{6.112}$$

and

$$\widetilde{R}^k \cong (s-s')\frac{dr^k(s')}{ds'} + \frac{(s-s')^2}{2}\frac{d}{ds'}u^k(s') + \frac{(s-s')^3}{6}\frac{d^2}{ds'^2}u^k(s') + O\left(\epsilon^4\right), \tag{6.113}$$

while for the time delay $s - s' \equiv s_{ret}$ the leading-order expression

$$s - s' \cong \sigma + O\left(\epsilon^2\right) \tag{6.114}$$

holds. By substituting these expansions in Eq.(6.87), the asymptotic solution given by Eq.(6.105) can be recovered.

T5$_2$)-T5$_3$) The proof follows by first noting that $L_{C,asym}^{(self)}$ contributes to the Euler-Lagrange equations only in terms of the local dependence in terms of r. Then, in this approximation the canonical momentum becomes

$$p_\mu = m_0 c \frac{dr_\mu(s)}{ds} + \frac{q}{c}\overline{A}_\mu^{(ext)}(r) = P_\mu, \tag{6.115}$$

while the asymptotic Hamiltonian reduces to

$$H_{1,asym}\left(r,p,r_0'\right)=\frac{1}{2m_oc}\left(p_\mu-\frac{q}{c}\overline{A}_\mu^{(ext)}(r)\right)\left(p^\mu-\frac{q}{c}\overline{A}_\mu^{(ext)\mu}(r)\right)+g_\mu\left(r_0'\right)r^\mu. \quad (6.116)$$

The corresponding Lagrangian and Hamiltonian equations manifestly coincide with Eq.(6.84) once the approximation (6.105) is invoked for the vector field G_μ.

Q.E.D.

Discussion and comparisons with point-particle treatments

The asymptotic Hamiltonian approximation, here pointed out for the first time (see THM.5), preserves the basic physical properties of the exact RR force (6.84). In fact in both cases, the RR force:

1) is non-local, depending on the past history of the finite-size charged particle;

2) admits a variational formulation;

3) is both Lagrangian and Hamiltonian;

4) satisfies the Einstein Causality Principle and, when applicable, the Newton Principle of Determinacy (see also Ref.(19));

5) describes correctly the transient time intervals in which the external force is turned on and off (sudden force).

For these reasons, physical comparisons based on the retarded-time Hamiltonian asymptotic approximation are meaningful. In particular, here we remark that the present approach departs in several ways with respect to point-particle treatments based on the LAD and LL equations. More precisely:

1) The same type of asymptotic ordering is imposed, which is based on the short delay-time ordering (6.100). However, in contrast with the LAD and LL equations, the expansion adopted in THM.5 and leading to the retarded-time Hamiltonian approximation can *only* be performed based on the knowledge of the exact RR force for finite-size particles.

2) Unlike the LAD and LL equations, the asymptotic Hamiltonian approximation carries the information of the past dynamical history of the charged particle through the retarded time s'. Therefore, the dynamical equation written adopting the approximation (6.105) is still a delay-type second-order ODE. The construction of its general solution becomes trivial in this case, since the self-force is considered as an explicit source term evaluated at proper time s'.

3) The asymptotic approximation provided by Eq.(6.105) cannot be regarded as a point-particle limit. In fact, the retarded mass-correction term would diverge in this limit.

4) The exact RR equation satisfies identically by construction the kinematic constraint $u_\mu u^\mu = 1$. The same constraint is satisfied to leading-order in ϵ also both by the retarded and present-time asymptotic expansions (and hence also the LAD equation).

5) The variational principle introduced in THM.5 is subject to the constraint that the past history is considered prescribed in terms of the extremal world-line. This requirement is consistent with the initial conditions for the RR equation, which is a

delay-type ODE depending only on the past history of the particle. This requires that the world-line trajectory is prescribed in the past, namely in the time interval $I = [-\infty, s_0]$. Since, however, the initial proper time s_0 is arbitrary, it follows that $r(s)$ can be considered prescribed also in the time interval $I' = [-\infty, s']$. In particular, if for all $s < s_0$ the motion is assumed to be inertial, the initial-value problem associated to the RR equation written in terms of the retarded asymptotic self-force (6.105) is well-posed, in the sense of the standard Newton Principle of Determinism, as discussed in the previous Chapter (see in particular THM.4 presented there and dealing with the existence and uniqueness of solutions for the exact RR equation).

6) One might think that the same type of constrained variational principle, of the kind adopted in THM.5, could be adopted also for the exact RR equation. However, this belief is wrong. In fact, since the variational functional (6.78) is symmetric with respect to the local and non-local world-line trajectories, there is no distinction between past and future. Since future cannot be prescribed, such a constrained variational principle for the exact equation is forbidden. On the contrary, the extremal RR equation (6.84) is obtained by imposing also the Einstein Causality Principle, and therefore it depends only on the past history.

7) Despite some formal similarities between the retarded-time Hamiltonian approximation versus the corresponding LAD and LL equations, the latter cannot be recovered even in the framework of some kind of constrained variational principle. In fact this would require to consider prescribed for example, second or higher-order proper-time derivatives of the particle position vector (namely the acceleration and its derivatives). This viewpoint is manifestly unacceptable, because it would amount to constraint the present state of the particle at proper time s.

8) The previous argument justifies, in turn, the introduction of the short delay-time asymptotic approximation given in THM.5. This is performed directly on the RR force, namely the 4-vector G_μ entering the RR equation itself. In this way the variational character of the RR problem is preserved. It follows that the corresponding variational functional as well as the Lagrangian and Hamiltonian functions for the asymptotic RR equation are constructed only "a posteriori".

9) Another advantage of the new representation (6.105) with respect to the customary LAD and LL equations is that it permits the approximate treatment of the solution also in the transient time intervals after the turning-on or the turning-off of the external EM field. In particular, in contrast to the LAD and LL equations, it predicts a vanishing RR self-force in the turning-on transient phase $I_0 = [s_0, s_0 + s_{ret}]$.

10) Finally, it should be remarked that the retarded asymptotic self-force (6.105) *cannot* be trivially obtained from the corresponding local asymptotic representation performed at proper time s and leading to the LAD equation by simply exchanging s with s' (or by a further Taylor expansion). Indeed, the relationship between the two can only be established based on the exact form of the self-force.

6.7 Collisionless relativistic kinetic theory for the EM RR effect - Canonical formalism

In this section the relativistic classical statistical mechanics (CSM) for a collisionless plasma with the inclusion of the EM RR effect is constructed. In particular we shall prove that the mathematical formalism introduced in the previous sections to deal with symmetric non-local interactions allows one to obtain a convenient formulation for the kinetic theory describing such a system and for the corresponding fluid representation. The derivation is based on the property of a symmetric non-local system represented by a finite-size charged particle of being Hamiltonian with respect to P_μ and H_{eff}.

In view of the peculiar features of the non-local RR phenomenon and the related delay-type differential Hamiltonian equations, it is instructive to adopt here an axiomatic formulation of the CSM for relativistic systems with the inclusion of such an effect. We shall assume that the latter are represented by a system of classical finite-size charged particles subject only to the action of a mean-field external EM force and a non-local self-interaction. We intend to show that, using the Hamiltonian representation in standard form given above, the explicit form of the relativistic Vlasov kinetic equation can be obtained for the kinetic distribution function describing the statistical dynamics of such a system. Therefore, the problem is reduced to a Vlasov-Maxwell description for a continuous distribution of relativistic charged particles.

For definiteness, let us consider the non-local Hamiltonian dynamical system in standard form $\{\mathbf{y}, H_{eff}\}$ given above. This is characterized by the superabundant state vector $\mathbf{y} = (r^\mu, P_\mu)$ spanning the extended 8th-dimensional phase-space Γ and with essential state variables $\mathbf{y}_1(\mathbf{y})$ spanning the 6th-dimensional reduced phase-space Γ_1. Introducing the *global proper time* $\widehat{s}$, $\Gamma_1(\widehat{s})$ is defined as

$$\Gamma_1(\widehat{s}) \equiv \left\{\mathbf{y} : \mathbf{y} \in \Gamma,\ |u| = 1,\ s(y) = \widehat{s},\ ds(y) = \sqrt{g_{\mu\nu} dr^\mu dr^\nu}\right\}, \tag{6.117}$$

where $|u| \equiv \sqrt{u^\alpha u_\alpha}$ and $s(y)$ is the world-line proper time uniquely associated to any $\mathbf{y}$. By assumption, $\Gamma_1(\widehat{s})$ is an invariant set, i.e., $\Gamma_1(\widehat{s}) = \Gamma_1$ for any $\widehat{s} \in \mathbb{R}$. Next, let us consider the Hamiltonian flow $T_{s_0,s}$ defined in Eq.(6.99). By construction the dynamical system is autonomous, namely the flow is of the form

$$T_{s_0,s}\mathbf{y}_0 \equiv \chi(\mathbf{y}_0, s - s_0). \tag{6.118}$$

The existence of the dynamical system $T_{s_0,s}$ for the state $\mathbf{y}(s)$ has been proved in Ref.(19) (see also previous Chapter). This requires that in the proper time interval $I_0 = [-\infty, s_0]$ the motion of each charged particle is inertial, namely the external EM field vanishes in the same interval. As a result of Eq.(6.99), any point in the phase-space Γ spanned by $\mathbf{y}$ or $\mathbf{y}_0$ is associated to a unique phase-space trajectory, namely such that $\mathbf{y} = \mathbf{y}(s)$, for any $\mathbf{y} \in \Gamma$. Due to (6.99) there exists necessarily $\mathbf{y}_0 \equiv \mathbf{y}(s_0)$ which is mapped in $\mathbf{y}(s)$. Viceversa, for any $s \in \mathbb{R}$ there exists a unique $\mathbf{y} = \mathbf{y}(s)$. However, we notice here that for the axiomatic formulation of the CSM for the RR problem the assumption of existence of the dynamical system $T_{s_0,s}$ is not a necessary condition. In fact, it is immediate to prove that the minimal requirement is actually provided only by the existence of the diffeomeorphism (6.90) defined above.

Now, for a prescribed $\widehat{s}_0 \in \mathbb{R}$ let us consider the set $B(\widehat{s}_0) \subseteq \Gamma_1$, with $B(\widehat{s}_0)$ an ensemble of states $\mathbf{y}_0$, each one prescribed at the initial proper time $s_0 = \widehat{s}_0$. Its image generated at any $s = \widehat{s} \in \mathbb{R}$ by the flow $T_{s_0,s}$, for each trajectory, is

$$B \equiv B(s) \equiv T_{s_0,s} B(s_0), \tag{6.119}$$

where s and s_0 denote now the *global proper times* $\widehat{s}$ and $\widehat{s}_0$.

We introduce the following axioms.

AXIOM #1: Probability on $K(\Gamma)$.

Let $K(\Gamma_1)$ be a family of subsets of Γ_1 which are L-measurable. We define the probability of $B(s) \in K(\Gamma_1)$ as the function

$$P(B) : K(\Gamma_1) \rightarrow [0,1] \tag{6.120}$$

such that it satisfies the constraints

$$P(\Gamma_1) = 1, \tag{6.121}$$

$$P(\varnothing) = 0, \tag{6.122}$$

$$P(\cup_{i\in N} B_i) = \sum_{i=0}^{\infty} P(B_i), \tag{6.123}$$

with $\{B_i \in K(\Gamma_1), i \in N\}$ being an arbitrary family of separate sets of $K(\Gamma_1)$.

AXIOM #2: Probability density.

For any $B(s) \in K(\Gamma_1)$ and for any state $\mathbf{y} \equiv (r^\mu, P_\mu)$ there exists a unique probability density $\rho(\mathbf{y}) > 0$ on Γ_1 such that

$$P(B(s)) = \int_\Gamma d\mathbf{y} \rho(\mathbf{y}) \delta(|u|-1) \delta(s - s(y)) \delta_{B(s)}(\mathbf{y}), \tag{6.124}$$

where $d\mathbf{y} = dr^\mu dP_\mu$ is the canonical measure on Γ and $\delta_{B(s)}(\mathbf{y})$ is the characteristic function of $B(s)$. Furthermore, $s(y)$ is a particle world-line proper time, while $s \equiv s_0 + \Delta s$, with Δs an *invariant proper time interval* independent of s_0. We notice that $s(y)$ can be equivalently parametrized in terms of the observer's coordinate time r^0, namely:

$$ds(y) \equiv dr^0 \sqrt{g_{\mu\nu} \frac{dr^\mu}{dr^0} \frac{dr^\nu}{dr^0}}. \tag{6.125}$$

AXIOM #3: Equiprobability.

Then, the equiprobability condition requires that, for all $B(s_0)$ and for all $s, s_0 \in I \subseteq \mathbb{R}$,

$$P(B(s)) = P(B(s_0)). \tag{6.126}$$

We remark that in the integral (6.124) the two Dirac-delta functions can be interpreted as physical realizability conditions, required to reduce the dimension of the volume element $d\mathbf{y}$ defined on the extended phase-space Γ.

We can now introduce the following theorem, concerning the validity of the Liouville equation for $\rho\left(\mathbf{y}\right)$.

THM.6 - Relativistic Liouville equation for $\rho\left(\mathbf{y}\right)$.

Given a Hamiltonian system $\left\{\mathbf{y}, H_{eff}\right\}$ and imposing the validity of Axioms #1-#3, it follows that the probability density $\rho\left(\mathbf{y}\left(s\right)\right)$ is a constant of motion, namely for any $s, s_0 \in \mathbb{R}$ (to be intended now as world-line proper times) and for any $\mathbf{y}_0 \in \Gamma$

$$\rho\left(\mathbf{y}\left(s\right)\right) = \rho\left(\mathbf{y}_0\right), \tag{6.127}$$

to be referred to as the integral Liouville equation. This can also be written equivalently as

$$\frac{d}{ds}\rho\left(\mathbf{y}\left(s\right)\right) = 0, \tag{6.128}$$

to be referred to as the differential Liouville equation. As a consequence, introducing the kinetic distribution function (KDF) $f\left(\mathbf{y}\right)$

$$f\left(\mathbf{y}\right) \equiv \rho\left(\mathbf{y}\right) N, \tag{6.129}$$

with N being the total number of particles in the configuration space of $B \subseteq K\left(\Gamma\right)$, it follows that also $f\left(\mathbf{y}\right)$ satisfies the Liouville equation (6.128).

Proof - We first notice that, from Axiom #1, by changing the integration variables we can write Eq.(6.124) as

$$\begin{aligned} P\left(B\left(s\right)\right) &= \int_{\Gamma} d\mathbf{y}\rho\left(\mathbf{y}\right)\delta\left(\left|u\right|-1\right)\delta\left(s-s\left(y\right)\right)\delta_{B(s)}\left(\mathbf{y}\right) = \\ &= \int_{\Gamma} d\mathbf{y}_0 \left|\frac{\partial \mathbf{y}\left(s\right)}{\partial \mathbf{y}_0}\right| \rho\left(\mathbf{y}\left(s\right)\right)\delta\left(\left|u\right|-1\right)\delta\left(s-s\left(y\right)\right)\delta_{B(s_0)}\left(\mathbf{y}\left(s_0\right)\right), \end{aligned} \tag{6.130}$$

with $\left|\frac{\partial \mathbf{y}(s)}{\partial \mathbf{y}_0}\right|$ being the Jacobian of the variable transformation from $\mathbf{y}\left(s\right)$ to $\mathbf{y}_0$. On the other hand, since the system $\left\{\mathbf{y}, H_{eff}\right\}$ is Hamiltonian, it follows identically that $\left|\frac{\partial \mathbf{y}(s)}{\partial \mathbf{y}_0}\right| = 1$. Hence, invoking Axiom #2 we can write

$$\int_{\Gamma} d\mathbf{y}_0 \left[\begin{array}{c} \rho\left(\mathbf{y}\left(s\right)\right)\delta\left(\left|u\right|-1\right)\delta\left(s-s\left(y\right)\right)+ \\ -\rho\left(\mathbf{y}_0\right)\delta\left(\left|u_0\right|-1\right)\delta\left(s_0-s\left(y_0\right)\right) \end{array}\right]\delta_{B(s_0)}\left(\mathbf{y}\left(s_0\right)\right) = 0, \tag{6.131}$$

from which it must be that

$$\rho\left(\mathbf{y}\left(s\right)\right)\delta\left(\left|u\right|-1\right)\delta\left(s-s\left(y\right)\right) = \rho\left(\mathbf{y}_0\right)\delta\left(\left|u_0\right|-1\right)\delta\left(s_0-s\left(y_0\right)\right). \tag{6.132}$$

On the other hand, by construction it follows that

$$\delta\left(|u|-1\right) = \frac{1}{\left|\frac{d|u|}{d|u_0|}\right|}\delta\left(|u_0|-1\right) = \delta\left(|u_0|-1\right), \tag{6.133}$$

$$\delta\left(s-s\left(y\right)\right) = \frac{1}{\left|\frac{ds}{ds_0}\right|}\delta\left(s_0-s\left(y_0\right)\right) = \delta\left(s_0-s\left(y_0\right)\right). \tag{6.134}$$

In fact, by definition the 4-velocity is normalized to 1 at all proper times, so that $\left|\frac{d|u|}{d|u_0|}\right| = 1$. Furthermore, $s \equiv s_0 + \Delta s$, with Δs being independent of the initial value s_0, and hence $\left|\frac{ds}{ds_0}\right| = 1$ too.

Finally, because of these conclusions, from Eq.(6.132) it follows that

$$\rho\left(\mathbf{y}\left(s\right)\right) = \rho\left(\mathbf{y}_0\right), \tag{6.135}$$

which represents the Liouville equation in integral form. By differentiating with respect to s the equivalent differential representation follows at once. An analogous equation holds manifestly also for the KDF $f\left(\mathbf{y}\right)$.

Q.E.D.

We conclude noting that, formally, the Liouville equation for non-local Hamiltonian systems in standard form is analogous to that characterizing local Hamiltonian systems. Such an equation can be viewed as a *Vlasov equation* for a relativistic collisionless plasma, in which each particle is subject only to the action of a mean-field EM interaction, generated respectively by the external and the self EM Faraday tensors. By definition, in this treatment the latter do not include retarded binary EM interactions. It follows that, in terms of the Lagrangian equation (6.128), the probability density $\rho\left(\mathbf{y}\left(s\right)\right)$ is parametrized in terms of the single-particle phase-space trajectory $\left\{\mathbf{y}\left(s\right), s \in I\right\}$. Hence, it advances in (proper) time s by means of the canonical state $\mathbf{y}\left(s\right)$ as determined by the Hamiltonian equations of motion (6.88).

Vlasov-Maxwell description

To define a well-posed problem, the relativistic Vlasov equation (6.128) must be coupled to the Maxwell equations, which determine the total EM field produced by all the relevant sources. In particular, in order to determine the external Faraday tensor $F_{\mu\nu}^{(ext)}$, the corresponding EM 4-potential $A_{\nu}^{(ext)}$ must be determined. In the Lorentz gauge, this is prescribed requiring it to be a solution of the Maxwell equations

$$\Box A^{(ext)\mu} = \frac{4\pi}{c} j^{(ext)\mu}(r), \tag{6.136}$$

where $j^{(ext)\mu}(r)$ is identified with the total current density

$$j^{(ext)\mu}(r) \equiv q\int d^4u\delta\left(|u|-1\right)u^{\mu} f\left(\mathbf{y}\right) + j^{(coils)\mu}(r). \tag{6.137}$$

Here, the first term is the Vlasov 4-current density, namely the velocity moment of $f(\mathbf{y})$ carrying the non-local phase-space contributions which yield the collective field produced by the plasma. The second term, instead, is produced by possible prescribed sources located outside the plasma domain. Therefore, in the Vlasov-Maxwell description the total EM 4-potential acting on a single particle must be considered as represented by $A_\nu = A_\nu^{(ext)} + A_\nu^{(self)}$, where $A_\nu^{(self)}$ is given by Eq.(6.76) and $A_\nu^{(ext)}$ is the solution of Eq.(6.136).

Therefore, the dynamical evolution of the KDF along a single-particle phase-space trajectory depends both explicitly, via $A_\nu^{(self)}$, and implicitly, via the 4-current $j^{(ext)\mu}(r)$, on the whole Faraday tensor $F_{\mu\nu} \equiv F_{\mu\nu}^{(ext)} + F_{\mu\nu}^{(self)}$. In this way contributions which are non-local both in configuration and phase-space are consistently included in the theory.

6.8 Kinetic theory: non-canonical representation

In this Section we present the equivalent representation of the kinetic theory developed in the previous Section adopting non-canonical variables. For definiteness, let us introduce an arbitrary non-canonical phase-space diffeomorphism from Γ to $\Gamma_{\mathbf{w}}$, with $\Gamma_{\mathbf{w}}$ denoting a transformed phase-space having the same dimension of Γ,

$$\mathbf{y} \equiv (r^\mu, P_\mu) \rightarrow \mathbf{w} \equiv \mathbf{w}(\mathbf{y}), \tag{6.138}$$

where, for example, $\mathbf{w}$ can be identified with the non-canonical state $\mathbf{y}_{nc} \equiv (r^\mu, p_\mu)$ defined in Eq.(6.162) or with $\mathbf{y}_u \equiv (r^\mu, u_\mu)$. In the second case the transformation, following from Eq.(6.97), is realized by

$$\begin{aligned} r^\mu &= r^\mu, && (6.139)\\ u_\mu &= P_\mu - \frac{q}{c}\left[\overline{A}_\mu^{(ext)} + 2\overline{A}_\mu^{(self)}\right]. && (6.140) \end{aligned}$$

The transformed RR equation in the variables $\mathbf{y}_u$ becomes therefore:

$$\begin{aligned} \frac{dr^\mu}{ds} &= u^\mu, && (6.141)\\ \frac{du_\mu}{ds} &= F_\mu, && (6.142) \end{aligned}$$

where $F_\mu = \frac{\partial p_\mu}{\partial r^\nu} u^\nu - \frac{\partial u_\mu}{\partial P_\nu}\frac{\partial H_{eff}}{\partial r^\nu}$. Denoting now by

$$f_1(\mathbf{w}(s)) = \left|\frac{\partial \mathbf{y}(s)}{\partial \mathbf{w}(s)}\right| f(\mathbf{y}(\mathbf{w}(s))) \tag{6.143}$$

the KDF mapped onto the transformed phase-space $\Gamma_{\mathbf{w}}$ by the KDF $f(\mathbf{y}(s))$, the differential Liouville-Vlasov equation (6.128) requires

$$\frac{d}{ds}\left[\left|\frac{\partial \mathbf{w}(s)}{\partial \mathbf{w}_0}\right| f_1(\mathbf{w}(s))\right] = 0, \tag{6.144}$$

where $\mathbf{w}_0 \equiv \mathbf{w}(s_0)$. At the same time, Eq.(6.128) also implies, thanks to the chain rule:

$$\frac{d}{ds} f(\mathbf{y}(\mathbf{w}(s))) = 0, \tag{6.145}$$

which for consistency delivers the well-known differential identity

$$\frac{d}{ds}\left[\left|\frac{\partial \mathbf{y}(s)}{\partial \mathbf{w}(s)}\right|\left|\frac{\partial \mathbf{w}(s)}{\partial \mathbf{w}_0}\right|\right] = 0. \tag{6.146}$$

From Eq.(6.144) it follows

$$\frac{d}{ds} f_1(\mathbf{w}(s)) + f_1(\mathbf{w}(s))\frac{d}{ds}\ln\left(\left|\frac{\partial \mathbf{w}(s)}{\partial \mathbf{w}_0}\right|\right) = 0. \tag{6.147}$$

This equation can be represented, for example, in terms of $\mathbf{w} \equiv \mathbf{y}_u$. In this case, due to the chain rule

$$\frac{d}{ds} f_1(\mathbf{w}(s)) = u^{\mu}\frac{\partial f_1(\mathbf{y}_u)}{\partial r^{\mu}} + F_{\mu}\frac{\partial f_1(\mathbf{y}_u)}{\partial u_{\mu}}, \tag{6.148}$$

while, thanks to Liouville theorem

$$\frac{d}{ds}\ln\left(\left|\frac{\partial \mathbf{w}(s)}{\partial \mathbf{w}_0}\right|\right) = \frac{\partial F_{\mu}}{\partial u_{\mu}}. \tag{6.149}$$

As an application of the result, it follows that, if the LL approximation is introduced for the 4-vector F_{μ}, namely Eqs.(6.141) and (6.142) are replaced with asymptotic equations of the form

$$\frac{dr^{\mu}_{LL}}{ds} = u^{\mu}_{LL}, \tag{6.150}$$

$$\frac{du^{\mu}_{LL}}{ds} = F^{\mu}_{LL}, \tag{6.151}$$

where F^{μ}_{LL} is the total EM force in this approximation, then Eq.(6.147) recovers the expression reported in Ref.(17). This provides the connection with the exact canonical theory here developed. We remark, however, that since the LL equation is only asymptotic, the mapping between the canonical state $\mathbf{y} \equiv (r^{\mu}, P_{\mu})$ and $\mathbf{y}_{LL} \equiv \left(r^{\mu}_{LL}, u_{LL\mu}\right)$ is also intrinsically asymptotic. Therefore, Eqs.(6.150) and (6.151) remain necessarily non-variational and non-canonical.

6.9 Fluid moment equations

We now proceed to compute explicitly the relativistic fluid moment equations which follow from the Liouville equation. To this aim, the relativistic Liouville equation is conveniently written as a PDE (Eulerian form)

$$u^{\mu}\frac{\partial f(\mathbf{y})}{\partial r^{\mu}} + G^{\mu}(\mathbf{y})\frac{\partial f(\mathbf{y})}{\partial u_{\mu}} = 0, \tag{6.152}$$

where $G^{\mu}(\mathbf{y})$ is defined by Eq.(6.84), or as an ODE (Lagrangian form):

$$\frac{dr^{\mu}}{ds}\frac{\partial f(\mathbf{y}(s))}{\partial r^{\mu}} + \frac{du_{\mu}}{ds}\frac{\partial f(\mathbf{y}(s))}{\partial u_{\mu}} = 0, \tag{6.153}$$

with $\mathbf{y}(s)$ being the phase-space trajectory of a particle. Then, the relativistic fluid equations related to the Liouville equation are defined as the following integrals over the momentum space:

$$\int d^4u\delta(|u|-1)\,G\left[u^{\mu}\frac{\partial f(\mathbf{y})}{\partial r^{\mu}} + G^{\mu}(\mathbf{y})\frac{\partial f(\mathbf{y})}{\partial u_{\mu}}\right] = 0. \tag{6.154}$$

Similarly, the corresponding fluid fields are defined as

$$\int d^4u\delta(|u|-1)\,Gf(\mathbf{y}), \tag{6.155}$$

with $G = 1, u^{\mu}, u^{\mu}u^{\nu}, ...$ and u^{μ} is the 4-velocity. In particular, we shall denote

$$n(r) \equiv \int d^4u\delta(|u|-1)\,f(\mathbf{y}), \tag{6.156}$$

$$N^{\mu}(r) = n(r)\,U^{\mu}(r) \equiv \int d^4u\delta(|u|-1)\,u^{\mu}f(\mathbf{y}), \tag{6.157}$$

$$T^{\mu\nu}(r) \equiv \int d^4u\delta(|u|-1)\,u^{\mu}u^{\nu}f(\mathbf{y}), \tag{6.158}$$

to be referred to as *the number density, the 4-flow and the stress-energy tensor.*

It is immediate to prove that the corresponding moment equations are as follows.

Continuity equation

For $G = 1$ the Liouville equation provides the continuity equation

$$\partial_{\mu}N^{\mu}(r) = 0. \tag{6.159}$$

Energy-momentum equation

For $G = u^{\nu}$ the Liouville equation provides the energy-momentum equation

$$\partial_{\mu}T^{\mu\nu}(r) = F^{\nu\mu}_{(tot)}(r)\,N_{\mu}(r), \tag{6.160}$$

where, from Eq.(6.84) we have that

$$F_{(tot)}^{\nu\mu}(r) \equiv \frac{q}{m_o c^2}\left[\overline{F}^{(ext)\mu\nu} + \overline{F}^{(self)\nu\mu}\right] \tag{6.161}$$

is the total EM force, with $\overline{F}^{(self)\nu\mu}$ containing the retarded non-local contributions arising from the EM RR effect.

We remark the following properties.

1) As a consequence of the Hamiltonian formulation in standard form, the fluid equations obtained from the kinetic equation with the inclusion of the RR effect are formally the same as in the usual treatment for local systems.

2) The contribution of the RR effect to the fluid equations is contained explicitly in the source term in the rhs of Eq.(6.160), and also implicitly in the definition of the fluid fields. In fact, by assumption, the KDF is a function of the effective Hamiltonian state $\mathbf{y} \equiv (r^\mu, P_\mu)$, which depends on the retarded self-potential. Hence, the fluid fields defined by Eqs.(6.156)-(6.158) must be interpreted as the fluid fields of the plasma which is emitting self-radiation and is therefore subject to the RR effect.

The implicit contribution of the RR self-force

It is worth discussing the features of the theory in connection with the implicit contribution of the RR effect contained in the definition of the fluid fields. In particular, here we show that such contribution can be made explicit and an analytical asymptotic estimation of it can be give provide some suitable assumptions are imposed on the physical system. This concerns the case in which the contribution of the self-potential is small in comparison with the external EM potential in the KDF. In these circumstances, the exact KDF can be Taylor expanded as follows:

$$f(\mathbf{y}) \simeq f(\mathbf{y}_{nc}) + (\mathbf{y} - \mathbf{y}_{nc}) \left.\frac{\partial f(\mathbf{y})}{\partial \mathbf{y}}\right|_{\mathbf{y}=\mathbf{y}_{nc}} + ..., \tag{6.162}$$

where $\mathbf{y}_{nc} \equiv (r^\mu, p_\mu)$ is the state which is canonical in absence of the EM self-field. It is clear that, by construction, only the canonical momenta are involved in this expansion, since the configuration state is left unchanged by the presence of the self-force. Therefore, from the form of the previous expansion it follows that the first term of the series, namely $f(\mathbf{y}_{nc})$, does not contain any contribution from the RR self-field. Consider, for simplicity, the Taylor series to first order. Then, the corresponding fluid fields can be decomposed as follows:

$$n(r) \simeq n_0(r) + n_1(r), \tag{6.163}$$

$$N^\mu(r) \simeq N_0^\mu(r) + N_1^\mu(r), \tag{6.164}$$

$$T^{\mu\nu}(r) \simeq T_0^{\mu\nu}(r) + T_1^{\mu\nu}(r), \tag{6.165}$$

where

$$n_0(r) \equiv \int d^4u\delta(|u|-1) f(\mathbf{y}_{nc}), \tag{6.166}$$

$$\begin{aligned} n_1(r) &\equiv \int d^4u\delta(|u|-1)(\mathbf{y}-\mathbf{y}_{nc}) \left.\frac{\partial f(\mathbf{y})}{\partial \mathbf{y}}\right|_{\mathbf{y}=\mathbf{y}_{nc}} = \\ &= \frac{2q}{c}\overline{A}_\mu^{(self)} \int d^4u\delta(|u|-1) \left.\frac{\partial f(\mathbf{y})}{\partial P_\mu}\right|_{P_\mu=p_\mu}, \end{aligned} \tag{6.167}$$

and similar definitions hold for the other two fluid fields.

To illustrate the procedure, let us consider, for example, the case of a relativistic Maxwellian distribution of the form (23)

$$f_M(\mathbf{y}) \equiv \frac{1}{(2\pi\hbar)^3} \exp\left[\frac{\mu - P^\mu U_\mu}{T}\right], \tag{6.168}$$

where μ, P^μ, U_μ and T are respectively the chemical potential, the canonical momentum and the fluid 4-velocity and temperature. Then, in terms of the previous expansion, we obtain for the density

$$n_0(r) \equiv \frac{4\pi m^2 cT}{(2\pi\hbar)^3} K_2\left(\frac{mc^2}{T}\right) \exp\left[\frac{\mu}{T} - \frac{q}{c}\frac{\overline{A}_\mu^{(ext)} U^\mu}{T}\right], \tag{6.169}$$

$$n_1(r) \equiv -\frac{2q}{c}\frac{\overline{A}_\mu^{(self)} U^\mu}{T} n_0(r), \tag{6.170}$$

with $K_2\left(\frac{mc^2}{T}\right)$ being the modified Bessel function of the second kind. As can be seen, the effect of the RR self-field appears only in $n_1(r)$ through the integral over the non-local dependencies contained in the potential $\overline{A}_\mu^{(self)}$. It follows that for a Maxwellian KDF the 4-flow $N^\mu(r)$ can be written as

$$N^\mu(r) \simeq [n_0(r) + n_1(r)] U^\mu(r), \tag{6.171}$$

while the expansion terms of the stress-energy tensor $T^{\mu\nu}(r)$ are given by

$$T_0^{\mu\nu}(r) \equiv \frac{1}{c^2} n_0 e U^\mu U^\nu - p_0 \Delta^{\mu\nu}, \tag{6.172}$$

$$T_1^{\mu\nu}(r) \equiv \frac{1}{c^2} n_1 e U^\mu U^\nu - p_1 \Delta^{\mu\nu}. \tag{6.173}$$

Here the notation is as in Ref.(23). Thus, $\Delta^{\mu\nu}$ is the projector operator $\Delta^{\mu\nu} \equiv \eta^{\mu\nu} -$

$c^{-2}U^{\mu}U^{\nu}$, e is the energy per particle

$$e = mc^2 \frac{K_3\left(\frac{mc^2}{T}\right)}{K_2\left(\frac{mc^2}{T}\right)} - T \tag{6.174}$$

and from the definition of the pressure as $p = nT$ it follows that

$$p_0(r) = n_0(r)T, \tag{6.175}$$

$$p_1(r) = n_1(r)T = -\frac{2q}{c}\overline{A}_{\mu}^{(self)}U^{\mu}n_0(r). \tag{6.176}$$

Finally, let us consider how the fluid equations are modified from the introduction of the series expansion (6.162). Substituting the relations (6.163)-(6.165) into the moment equations, for the continuity equation we get

$$\partial_{\mu}N_0^{\mu}(r) = -\partial_{\mu}N_1^{\mu}(r), \tag{6.177}$$

and for the momentum equation

$$\partial_{\mu}T_0^{\mu\nu} = F_{(tot)}^{\nu\mu}N_{\mu} - \partial_{\mu}T_1^{\mu\nu}. \tag{6.178}$$

In this way, on the lhs we have isolated the terms of the "unperturbed fluid", namely the physical observables corresponding to a charged fluid in absence of RR. On the other hand, the asymptotic contributions of the RR effect have been isolated on the rhs, which allows one to interpret them as source terms due to extra forces acting on the unperturbed fluid. In particular, the presence of the RR acts like a non-conservative collisional operator, if we interpret it as a sort of retarded scattering of the fluid (and therefore, of the single particles at the kinetic level) with itself.

6.10 Lagrangian formulation of the fluid equations

An important issue concerns the treatment of the non-local contributions appearing in the fluid equations both in the definitions of the fluid fields and in the source term in the momentum equation. This requires, in particular, the explicit representation of the self-potential $\overline{A}_{\mu}^{(self)}$ and the EM self-force $\overline{F}_{\mu k}^{(self)}$ defined respectively in Eqs.(6.93) and (6.87). In fact, in the previous sections these non-local contributions have been written in a parameter-free representation (integral form), so that they do not depend on the retarded particle velocity. This allowed us to perform the velocity integrals in a straightforward way, only in terms of local 4-velocities, in agreement with the formalism adopted for the Hamiltonian formulation in standard form.

To treat these non-local terms it is first convenient to represent the fluid moment equations in Lagrangian form, describing the dynamics of fluid elements along their respective Lagrangian path (LP). By substituting the definition (6.157) in Eq.(6.159) we obtain the corresponding Lagrangian form of the continuity equation, given by

$$\frac{D}{Ds}n + n\partial_\mu U^\mu = 0, \tag{6.179}$$

where $\frac{D}{Ds} \equiv U^\mu (r(s)) \partial_\mu$ is the convective Lagrangian derivative along the LP of the fluid element parametrized in terms of the arc-length s, and $U^\mu (r(s)) = \frac{dr^\mu(s)}{ds}$. Similarly, writing the stress-energy tensor $T^{\mu\nu}(r)$ as $T^{\mu\nu}(r) = nU^\mu U^\nu + P^{\mu\nu}(r)$, with $P^{\mu\nu}(r) \equiv T^{\mu\nu}(r) - nU^\mu U^\nu$, the energy-momentum equation (6.160) can be represented in Lagrangian form as follows:

$$n\frac{D}{Ds}U^\nu = nF^{\nu\mu}_{(tot)}U_\mu - \partial_\mu P^{\mu\nu}. \tag{6.180}$$

Analogous results can be given for the asymptotic equations (6.177) and (6.178).

With the introduction of the LPs, the parametrization of the non-local contributions can be easily reached in terms of the LP arc-length s. Consider, for example, the self-potential $\overline{A}^{(self)}_\mu$. This can be expressed as

$$\overline{A}^{(self)}_\mu (r, [r]) \equiv 2q \int_1^2 ds' \frac{dr'_\mu}{ds'} \delta(\widetilde{R}^\mu \widetilde{R}_\mu - \sigma^2), \tag{6.181}$$

where by definition now $\frac{dr'_\mu}{ds'} = U^\mu (r(s'))$ is defined along a fluid element LP. Then, by expressing the Dirac-delta function as

$$\delta(\widetilde{R}^\mu \widetilde{R}_\mu - \sigma^2) = \frac{1}{\left|2\widetilde{R}^\alpha U_\alpha\right|} \delta\left(s' - s + s_{ret}\right), \tag{6.182}$$

it follows that $\overline{A}^{(self)}_\mu$ can be equivalently written in the integrated form as

$$\overline{A}^{(self)}_\mu (r, [r]) = q \left[\frac{U_\mu (r(s'))}{\left|\widetilde{R}^\alpha U_\alpha (r(s'))\right|} \right]_{s'=s-s_{ret}}, \tag{6.183}$$

with $\widetilde{R}^\alpha$ being the displacement vector defined along a LP. In particular, in agreement with the Einstein Causality Principle, the retarded time $s_{ret} = s - s'$ is the positive root of the delay-time equation

$$\widetilde{R}^\mu \widetilde{R}_\mu - \sigma^2 = 0. \tag{6.184}$$

An analogous derivation can be carried out also for the self-force $\overline{F}^{(self)}_{\mu k}$, giving the following result

$$\overline{F}_{\mu k}^{(self)}(r,[r]) = -2q\left\{\frac{1}{\left|\widetilde{R}^{\alpha}U_{\alpha}(s')\right|}\frac{D}{Ds'}X_{\mu k}\left(r\left(s'\right)\right)\right\}_{s'=s-s_{ret}}, \tag{6.185}$$

where

$$X_{\mu k}\left(r\left(s'\right)\right) \equiv \left[\frac{U_{\mu}(r\left(s'\right))\widetilde{R}_{k} - U_{k}(r\left(s'\right))\widetilde{R}_{\mu}}{\widetilde{R}^{\alpha}U_{\alpha}(r\left(s'\right))}\right]. \tag{6.186}$$

Again, this expression must be intended as a parametrization defined along a fluid element LP.

We conclude by commenting on the following notable aspects of the theory presented here.

1) The fluid equations with the inclusion of the non-local effect related to the EM RR have been derived in a closed analytical form in both Eulerian and Lagrangian formulations. In particular, it follows that the fluid dynamics of the non-local kinetic system is intrinsically non-local too.

2) Non-local contributions of the RR appear both in explicit and implicit contributions, through the definitions of the fluid fields as velocity moments of the KDF.

3) From the point of view of the fluid description, it follows that the natural setting for the treatment of the non-local fluid equations is given by the Lagrangian formulation and the concept of LPs. This is a consequence of the fact that the exact moment equations are of delay-type. In fact, in order to properly deal with the non-local contributions of the RR the parametrization of the retarded effects in terms of the arc-length of the corresponding LPs is needed. It follows that the dynamics of a generic fluid element along its LP is related to the EM RR effect produced at the retarded time along the LP itself.

6.11 Asymptotic approximation

In the previous sections we derived an exact formulation for both kinetic and fluid theories describing systems of relativistic charged particles subject to the EM RR self-interaction. In particular, we have pointed out that the kinetic and fluid equations are of delay-type, and therefore intrinsically non-local, due to the characteristic feature of the RR effect of being a non-local retarded effect. The retarded proper time is determined by Eq.(6.184) in agreement with the causality principle. Notice that this equation has formally the same expression for the single-particle or the kinetic dynamics and for the fluid equations in Lagrangian form (see also Ref.(19)). By inspecting Eq.(6.184) it is easy to realize that the order of magnitude of the delay-time is approximately $s_{ret} \sim \sigma/c$, and therefore very small for classical elementary particles. The smallness of the retarded time may represent a serious problem for the practical implementation of the exact theory presented here. In fact, the retarded time associated to the RR can be orders of magnitude smaller than any other characteristic time for most of relevant physical situations. The question is of primary importance, for example, for the actual numerical integration of the exact fluid equations.

In view of these considerations, in this section we provide asymptotic estimations of the non-local terms appearing in the moment equations, which allow one to overcome the difficulty connected with the finite delay-time intervals carried by the RR phenomenon. This requires to introduce a suitable asymptotic expansion of the exact non-local terms by means of approximations in which the self-interaction contributions are all expressed only through local quantities. The result has potential interest also in relation to the use of Eulerian integration schemes for the fluid equations with the inclusion of the RR effect.

Specifically, the present analysis requires to develop an asymptotic approximation which involves the treatment of the delay-time s_{ret}. This is accomplished within the short delay-time ordering approximation given by Eq.(6.100). In the following we shall work adopting the Lagrangian representation form for the fluid equations. To perform the asymptotic expansion, we assume that both the external EM field acting on each fluid element and the macroscopic fluid fields associated to the kinetic system are smooth function of the coordinate 4-position vector r^{α}, namely they are of class C^k, with $k \geq 2$. The result of the asymptotic approximation for the terms associated to the RR self-interaction is provided by the following theorem.

THM.7 - First-order, short delay-time asymptotic approximation (present-time expansion).

Given validity of the asymptotic ordering (6.100) and the smoothness assumptions for the external EM and the fluid fields, neglecting corrections of order ϵ^n, with $n \geq 1$ (first-order approximation), it follows that:

T7$_1$) The retarded self-potential $\overline{A}_{\mu}^{(self)}$ defined in Eq.(6.183) can be expanded in a neighborhood of s as follows:

$$\overline{A}_{\mu}^{(self)} = \left. \overline{A}_{\mu}^{(self)} \right|_s [1 + O(\epsilon)], \tag{6.187}$$

where the present-time leading-order contribution $\left. \overline{A}_{\mu}^{(self)} \right|_s$ is given by

$$\left. \overline{A}_{\mu}^{(self)} \right|_s = q \left[\frac{1}{\sigma} U_{\mu}(r(s)) - \frac{D}{Ds} U_{\mu}(r(s)) \right], \tag{6.188}$$

with $\frac{D}{Ds}$ being the convective derivative along a fluid element Lagrangian path.

T7$_2$) Concerning Eq.(6.180), let us define the vector field K_{μ} as follows:

$$K_{\mu} \equiv \frac{q}{m_o c^2} \overline{F}_{\mu\nu}^{(self)} U^{\nu}, \tag{6.189}$$

with $\overline{F}_{\mu\nu}^{(self)}$ defined in Eq.(6.185). Then, in a neighborhood of s, K_{μ} can be expanded *as follows:*

$$K_\mu = K_\mu|_s \left[1 + O(\epsilon)\right], \tag{6.190}$$

where the present-time leading-order contribution $K_\mu|_s$ *is given by*

$$K_\mu|_s = \left\{-\frac{1}{\sigma}\frac{q^2}{2m_o c^2}\frac{D}{Ds}U_\mu\left(r\left(s\right)\right) + g_\mu\right\}, \tag{6.191}$$

with g_μ *denoting the 4-vector*

$$g_\mu = \frac{2}{3}\frac{q^2}{m_o c^2}\left[\frac{D^2}{Ds^2}U_\mu - U_\mu(s)U^k(s)\frac{D^2}{Ds^2}U_k\right]. \tag{6.192}$$

Proof - The proof of T7$_1$) and T7$_2$) can be reached by introducing a Taylor expansion in terms of the retarded time s' for the relevant quantities appearing in Eqs.(6.183) and (6.185). In particular, for the 4-velocity $U_\mu\left(r\left(s'\right)\right)$ and the displacement vector $\widetilde{R}^k$ we obtain respectively

$$U_\mu\left(r\left(s'\right)\right) \cong U_\mu\left(r\left(s\right)\right) - (s-s')\frac{D}{Ds}U_\mu\left(r\left(s\right)\right) + \frac{(s-s')^2}{2}\frac{D^2}{Ds^2}U_\mu\left(r\left(s\right)\right) + O\left(\epsilon^3\right) \tag{6.193}$$

and

$$\widetilde{R}^k \cong (s-s')U^k - \frac{(s-s')^2}{2}\frac{D}{Ds}U^k + \frac{(s-s')^3}{6}\frac{D^2}{Ds^2}U^k + O\left(\epsilon^4\right), \tag{6.194}$$

while for the time delay $s - s' \equiv s_{ret}$ we get

$$s - s' \cong \sigma + O\left(\epsilon^2\right). \tag{6.195}$$

By substituting these expansions in Eqs.(6.183) and (6.185), after straightforward calculations the asymptotic solutions (6.187) and (6.190) follow identically.

Q.E.D.

We notice that the asymptotic expansion of the self-potential illustrated in THM.7 is required to reduce the non-local dependencies which are implicit in the definition of the fluid fields through the KDF. On the other hand, within the approximation obtained in THM.7 for the 4-vector K_μ, the RR equation (6.180) reduces to a local third-order ordinary differential equation. In particular, Eq.(6.185) in THM.7 represents the analogue of the LAD equation for the single-particle dynamics, which contains the first derivative of the particle 4-acceleration. In view of this similarity, the asymptotic solution (6.190) can be further simplified adopting a second reduction-step of the same kind of that which leads to the LL form of the self-force for single charged particles (11). This is obtained by assuming that the RR effect is only a small correction to the motion of the fluid. As a consequence, an iterative approximation can be adopted which permits to represent the self-force in terms of the instantaneous fluid forces. The latter include both the external EM field and the pressure forces. In particular, according to this method, to leading-order for the fluid 4-acceleration we have

$$\frac{D}{Ds}U^{\nu}=F^{\nu\mu}_{(ext)}U_{\mu}-\frac{1}{n}\partial_{\mu}P^{\mu\nu}, \tag{6.196}$$

where, for brevity we have introduced the notation

$$F^{\nu\mu}_{(ext)}\equiv\frac{q}{m_oc^2}\overline{F}^{(ext)\nu\mu}. \tag{6.197}$$

The iteration gives

$$\begin{aligned}\frac{D^2}{Ds^2}U^{\nu} &= \partial_l F^{\nu\mu}_{(ext)}U_{\mu}U^l+F^{\nu\mu}_{(ext)}\left(F_{(ext)\mu l}U^l-\frac{1}{n}\partial_l P^l_{\mu}\right)+\\ &+\frac{1}{n}\partial_{\mu}P^{\mu\nu}U^l\partial_l\ln n-\frac{1}{n}U^l\partial_l\partial_{\mu}P^{\mu\nu}.\end{aligned} \tag{6.198}$$

Substituting this expansion in Eq.(6.190) and invoking the symmetry property of the Faraday tensor provides for the first-order term $K_{\mu}|_s$ the following approximation:

$$K_{\mu}|_s\simeq\frac{q^2}{m_oc^2}\left\{-\frac{1}{2\sigma}\left[\frac{q}{m_oc^2}\overline{F}^{(ext)}_{\mu\nu}U^{\nu}-\frac{1}{n}\partial_{\nu}P^{\nu}_{\mu}\right]+\frac{2q}{3m_oc^2}h^{(1)}_{\mu}+\frac{2}{3}h^{(2)}_{\mu}\right\}, \tag{6.199}$$

where the first term on the rhs represents the mass-renormalization contribution, and $h^{(1)}_{\mu}$ denotes the 4-vector

$$h^{(1)}_{\mu}=\partial_l\overline{F}^{(ext)}_{\mu\nu}U^{\nu}U^l-\frac{q}{m_oc^2}\overline{F}^{(ext)}_{\mu\nu}\overline{F}^{(ext)\nu l}U_l+\frac{q}{m_oc^2}\left(\overline{F}^{(ext)}_{kl}U^l\right)\left(\overline{F}^{(ext)k\nu}U_{\nu}\right)U_{\mu}, \tag{6.200}$$

while $h^{(2)}_{\mu}$ is given by

$$\begin{aligned}h^{(2)}_{\mu} &= -\frac{q}{m_oc^2}\frac{1}{n}\overline{F}^{(ext)}_{\mu\beta}\partial_l P^{l\beta}+\frac{1}{n}\partial_{\nu}P^{\nu}_{\mu}U^l\partial_l\ln n-\frac{1}{n}U^l\partial_l\partial_{\nu}P^{\nu}_{\mu}+ \\ &+\frac{q}{m_oc^2}\frac{1}{n}U_{\mu}U^k\overline{F}^{(ext)}_{k\beta}\partial_l P^{l\beta}-\frac{1}{n}U_{\mu}U^k\partial_{\nu}P^{\nu}_{k}U^l\partial_l\ln n+\frac{1}{n}U_{\mu}U^kU^l\partial_l\partial_{\nu}P^{\nu}_{k}.\end{aligned} \tag{6.201}$$

Eq.(6.199) represents the fluid analogue of the LL approximation of the self-force holding for single particle dynamics, with the mass-renormalization term retained. In particular here we notice that:

1) Eq.(6.199) provides a local approximation of the fluid self-force carrying the contribution of the RR effect. In contrast to Eq.(6.190), thanks to the iterative reduction procedure only second-order derivatives of the position vector appear in this approximation.

2) For consistency, Eq.(6.199) must be evaluated adopting the asymptotic expansion (6.187) also for the evaluation of the self-potential entering the definition of the fluid fields through the canonical momenta P_{μ} in the KDF.

3) Moreover, consistent with the approximation in which the RR self-potential is

small with respect to the external EM potential, also the asymptotic approximation (6.162) can be adopted, which allows one to treat explicitly in an asymptotic way all the implicit RR contributions.

4) Finally, collecting together the analytical approximations provided by Eqs.(6.162), (6.187) and (6.199), the fluid equations are reduced to a set of asymptotic local second-order PDEs. This provides a convenient representation also for Eulerian implementation schemes of the same equations.

The detail comparison of Eqs.(6.198)-(6.201) with the literature is discussed in the next section.

Retarded-time asymptotic expansion

Despite the previous considerations, it is worth pointing out that, formally also for the fluid equations, an analogous result to THM.7 can be given. This is based on performing a Taylor expansion of the fluid RR force based on the retarded-time approximation. In this case, it is found that Eq.(6.183) is approximated as

$$\overline{A}_{\mu}^{(self)} = \left.\overline{A}_{\mu}^{(self)}\right|_{s'}\left[1+O(\epsilon)\right], \tag{6.202}$$

where the retarded-time leading-order contribution $\left.\overline{A}_{\mu}^{(self)}\right|_{s'}$ is simply given by

$$\left.\overline{A}_{\mu}^{(self)}\right|_{s'} = \frac{q}{\sigma}U_{\mu}\left(r\left(s'\right)\right), \tag{6.203}$$

while Eq.(6.185) for the self-force, written in terms of K_{μ} defined in Eq.(6.189), becomes

$$K_{\mu} = \left.K_{\mu}\right|_{s'}\left[1+O(\epsilon)\right], \tag{6.204}$$

where the retarded-time leading-order contribution $\left.K_{\mu}\right|_{s'}$ is now given by

$$\left.K_{\mu}\right|_{s} = \left\{-\frac{q^{2}}{2\sigma m_{o}c^{2}}\frac{D}{Ds'}U_{\mu}\left(r\left(s'\right)\right)+g_{\mu}'\left(r\left(s'\right)\right)\right\}, \tag{6.205}$$

with g_{μ}' denoting here the 4-vector

$$g_{\mu} = \frac{2}{3}\frac{q^{2}}{m_{o}c^{2}}\left[\frac{1}{4}\frac{D^{2}}{Ds'^{2}}U_{\mu}\left(r\left(s'\right)\right)-U_{\mu}\left(r\left(s'\right)\right)U^{k}\left(r\left(s'\right)\right)\frac{D^{2}}{Ds'^{2}}U_{k}\left(r\left(s'\right)\right)\right]. \tag{6.206}$$

This alternative expansion has the distinctive advantage (with respect to the present-time expansion) of retaining all the physical properties of the exact fluid equations for the treatment of RR delay-time effects. This alternative formulation is relevant for comparisons with the point-particle treatment.

6.12 Discussion and comparisons with literature

In this section we analyze in detail the physical properties of the kinetic and fluid theory developed for the EM RR problem, providing also a comparison with the literature. This concerns, in particular, the recent paper by Berezhiani et al. (17), where an analogous research program is presented for the relativistic hydrodynamics with RR based on the LL solution of the self-force.

Kinetic theory

Let us start by considering the kinetic theory. The solution here obtained has the following key features:

1) The kinetic theory adopts the Hamiltonian formulation of the RR problem developed here. The result is based on the exact analytical solution for the EM self-potential of finite-size charged particles, obtained in Ref.(19).

2) The kinetic theory is developed here for systems of charged particles subject to an external mean-field EM interaction and the RR self-interaction produced by the same particles. Due to the non-local property of the RR interaction, the formulation of kinetic theory is non-trivial. For this purpose, in contrast to previous literature, an axiomatic formulation of CSM is adopted. Its key element is the introduction of a suitable definition for the Lorentz-invariant probability-measure in the particle extended phase-space. As a consequence, the corresponding Liouville-Vlasov kinetic equation with the inclusion of the exact RR effect is achieved in Hamiltonian form, namely in such a way to preserve the phase-space canonical measure. For comparison, instead, previous literature approaches dealt with measure non-preserving phase-space dynamics.

3) In particular, the kinetic theory has been developed within the canonical formalism representing the KDF in terms of the canonical state $\mathbf{y} \equiv (r^{\mu}, P_{\mu})$. For reference, the connection with the corresponding non-canonical treatment is provided in Section 6.8. This in turn implies that non-local contributions associated to the self-potential (6.93) enter implicitly in the definition of the corresponding fluid moments (6.156)-(6.158). This is made possible only within the framework of the present exact formulation, in which the analytical solution for the self-potential is by construction non-divergent. This feature departs from recent approaches where instead non-Hamiltonian formulations were adopted, based on the LL point-like approximation of the RR self-force. In such a case in fact, the explicit dependence of the KDF in terms of the EM self-potential cannot be retained.

4) Both the RR equation for single-particle dynamics and the kinetic equation for the KDF are of delay-type, reflecting the characteristic nature of the RR phenomenon. This property is completely missing from the previous literature on the subject, exclusively based on the LL local asymptotic approximation.

Fluid theory

For what concerns the fluid treatment, we notice that:

1) Both the fluid fields and the fluid moment equations retain the standard form (available in the absence of RR effects) and can be equivalently represented in Eulerian or Lagrangian form. This follows from the exact representation here adopted both for the RR self-potential and the RR self-force. In both cases the only non-local dependencies are those associated to the position 4-vector.

2) The exact fluid equations with the inclusion of the RR effect are delay-type PDEs. Because of this feature, their natural representation appears to be the Lagrangian form. In fact, the integration along the LPs must be in principle performed taking into account the retarded RR interaction.

3) From the exact theory presented here it follows that each fluid equation of a given order does not depend on fluid fields of higher orders. For example, the momentum equation contains only second-order tensor fields, identified respectively with the plasma stress-energy tensor and the EM Faraday tensor. This result contrasts with the treatment given in Ref.(17) where instead the asymptotic formulation based on the LL equation leads to moment equations involving higher-order tensor fields (for comparison, see also the related discussion in Section 6.8).

4) If a kinetic closure is chosen, then the fluid moments appearing in the fluid equations are all uniquely determined. In particular, the stress-energy tensor is prescribed in terms of the KDF. This implies that both implicit and explicit contributions of the RR effect appear in the resulting equations, carried respectively by the fluid fields and the EM self-force in the momentum equation. Remarkably, kinetic closure is achieved prescribing solely the pressure contribution carried by the stress-energy tensor. Instead, in the approach of Ref.(17) the closure conditions involve generally also the specification of higher-order moments of the KDF.

5) An important feature of the exact fluid equations here obtained is that they can in principle be exactly implemented numerically adopting a Lagrangian scheme.

6) A remarkable aspect of the present theory is that the relevant asymptotic expansions are performed only "a posteriori" after integration over the velocity space. This means that the approximations involved are introduced only on the configuration space-variables (i.e., the fluid fields) and not on the phase-space KDF. In particular, a convenient approximation is the one obtained in the short delay-time ordering, which reduces the non-local dependencies to local terms. As a consequence, the introduction of higher-order moments is ruled out by construction.

Comparison with point-particle treatments

The relevant comparison here is represented by Ref.(17). Such an approach is based on the adoption of the LL equation for the single-particle dynamics for the construction of the relativistic Vlasov-Maxwell description. The corresponding moment equations can be in principle adopted for the construction of a closed set of fluid equations. This requires however the specification of suitable closure-conditions. Let us briefly point out the novel features of the current treatment for what concerns the adoption of the finite-size particle model in the construction of the kinetic and fluid descriptions. In detail:

1) Both in the kinetic and fluid treatments the RR force is taken into account by means of a non-local interaction. This is an intrinsic feature of the assumed finite extension of the charged particle. In the fluid treatment, in particular, as shown above, the RR force can be parametrized in terms of the past Lagrangian fluid velocity and position. This permits to treat consistently the causal delay-time effects due to the finite-size of the particles.

2) In validity of the asymptotic ordering given by Eq.(6.100), an asymptotic retarded-time Hamiltonian approximation of the RR force based on a retarded-time expansion has been given for the fluid equations. This approximation preserves the basic physical features of the solution based on the exact form of the RR self-force.

3) If the present-time asymptotic expansion is performed on the exact fluid moment equations, the resulting expression of the fluid RR force obtained adopting the finite-size charge model appears different from that given in Ref.(17).

These conclusions enable us to carry out a detailed comparison with the literature, emphasizing the basic differences between kinetic and fluid treatments based on finite-size and point particles.

A) Kinetic theory.

The kinetic equation adopted in Ref.(17) is based on the LL equation (see therein Eqs.7 and 8). This means that the RR force in this approximation is non-conservative, non-variational and therefore non-Hamiltonian. In addition the LL equation: 1) does not retain finite delay-time effects characteristic of the RR phenomenon; 2) is not valid in the case of strong EM fields, where the iterative reduction scheme on which it is based, may fail; 3) ignores mass-renormalization effects (which are incompatible with the point-particle model). In contrast, the treatment of the relativistic Vlasov kinetic equation obtained here (see the Eulerian and Lagrangian equations (6.152) and (6.153)) is qualitatively different. In fact, even if the resulting RR equation remains a second-order ODE, it is conservative, variational, Hamiltonian and applies for arbitrary external EM fields. Further remarkable aspects are related to the adoption of the finite-size charge model, in which the charge and mass distributions have the same support. As a consequence, in this case the self 4-potential is everywhere well-defined, contrary to the point particle model. In addition, this is prescribed analytically, a feature which allows one to treat consistently the RR delay-time effects.

B) Fluid theory.

The fluid treatment here obtained is provided by the Eulerian Eqs.(6.159)-(6.160) or the equivalent Lagrangian equations (6.179)-(6.180). The latter, considered as fluid equations, are manifestly not closed. However, the Hamiltonian formulation achieved here and holding for finite-size particles allows one to achieve a physically consistent kinetic closure condition, by prescribing uniquely the pressure tensor $P_{\mu\nu}$ in Eq.(6.180). We stress that in our treatment no higher-order moments need to be specified. In contrast, the corresponding Euler equation reported in Ref.(17) (see Eqs.11 and 12 therein) actually depends also on a third-order tensor moment, which must be prescribed (see comments in Sec.IIIA of Ref.(17)). Let us now consider the asymptotic fluid treatments based on the present theory. These can be achieved invoking either the present-time or the retarded-time asymptotic expansions (see previous Section). The first expansion is mostly relevant for comparisons with Ref.(17) (given in THM.7) and enables one to achieve a local approximation of the delay-time effects carried by the RR force. However, remarkably, the resulting asymptotic fluid equations (6.198)-(6.201) remain qualitatively different from the corresponding ones given in Ref.(17). In particular: 1) no higher-order moments appear after performing the Taylor expansion and the iteration scheme discussed after THM.7; 2) a non-vanishing mass-correction contribution is now included (see first term on the rhs of Eq.(6.199)). Finally, we mention that the retarded-time asymptotic expansion given by Eqs.(6.202)-(6.206) provides a novel approximation which retains basic properties of the exact solution. In particular: 1) it only applies for finite-size particles; 2) it relies on the Hamiltonian formulation of the RR problem and of the Vlasov-Maxwell treatment; 3) it permits to retain transient-time and delay-time effects; 4) it takes into account retarded mass-correction effects; 5) in this approximation the natural fluid description is Lagrangian.

6.13 Conclusions

In this Chapter, novel results have been obtained concerning the kinetic and fluid descriptions of relativistic collisionless plasmas with the inclusion of EM RR effects.

Relevant consequences of the variational form of the EM RR equation previously achieved for classical finite-size charged particles have been investigated. It has been shown that the non-local RR problem admits both Lagrangian and Hamiltonian representations in standard form, defined respectively in terms of effective Lagrangian and Hamiltonian functions. A notable novel feature of the theory concerns the development of a Hamiltonian retarded-time expansion of the RR force, which applies in validity of the short delay-time asymptotic ordering. On such a basis, the axiomatic formulation of classical statistical mechanics for relativistic collisionless plasmas with the inclusion of non-local RR effects has been presented. As a major result, the kinetic theory for such a system has been formulated in standard Hamiltonian form. The Liouville-Vlasov equation has been proved to hold in the extended phase-space, subject to non-local RR self-interactions. Remarkably, the non-local effects have been proved to enter the relativistic kinetic equation only through the retarded particle 4-position. As a consequence, the corresponding fluid moment equations can be determined in standard way by integration over the space of canonical momenta and cast both in Eulerian and Lagrangian forms. It has been pointed out that the exact relativistic fluid equations are intrinsically of delay-type and contain both implicit and explicit non-local contributions associated to the RR effect. The issue concerning the problem of fluid closure conditions has been discussed. In contrast with previous literature, it is found that in the present approach the closure conditions remain the standard ones, i.e., as in the absence of RR effects. Hence, the specification of higher-order moments of the KDF, for a given moment equation, is not required. Finally, appropriate approximations have been obtained for the fluid equations by employing "a posteriori" the relevant asymptotic expansions applicable in the short delay-time ordering. This allows one to reduce the exact non-local equations either to a set of local PDEs or to retarded PDEs still retaining finite delay-time effects.

The theory developed here has potential wide-ranging applications which concern the study of relativistic astrophysical plasmas for which RR emission processes are important. This involves, for example, plasmas in accretion disks, relativistic jets and active galactic nuclei. Other possible applications are also suggested for the case of laboratory plasmas generated in the presence of pulsed-laser sources.

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Chapter 7

Hamiltonian structure of classical N-body systems of finite-size particles subject to EM interactions

7.1 Introduction

In classical physics the formulation of the Hamiltonian mechanics of N-body systems composed of interacting particles is still incomplete. This includes, in particular, the case of charged particles acted on by an externally-prescribed EM field as well as binary and self EM interactions. Indeed, based on general relativity (or special relativity, as appropriate in the case of a flat Minkowski space-time) as well as quantum mechanics, common prerequisites for a dynamical theory for such systems should be:

Prerequisite #1: its covariance with respect to arbitrary local coordinate transformations. In the context of special relativity this requirement reduces to the condition of covariance with respect to the Lorentz group.

Prerequisite #2: the inclusion of both retarded and local interactions.

Prerequisite #3: the consistency with the Einstein causality principle.

Prerequisite #4: the validity of the Hamilton variational principle, yielding a set of equations of motion for all the N particles of the N-body system.

Prerequisite #5: the existence of a *Hamiltonian structure*.

As clarified below, all of these statements should be regarded as intrinsic properties of classical N-body systems which are characterized by non-local, i.e., retarded causal interactions, like those associated with EM fields (1). In particular, requirements #4 and #5 involve the assumptions that the equations of motion of a generic N-body system of this type should admit both Lagrangian and Hamiltonian variational formulations, obtained by means of a Hamilton variational principle, as well as a *standard*

Hamiltonian form, i.e., a set $\{\mathbf{x}, H_N\}$ with the following properties:

a) $\mathbf{x} = \left(\mathbf{x}^{(i)}, i = 1, N\right)$ is a super-abundant canonical state, with $\mathbf{x}^{(i)}$ denoting an appropriate i-th particle canonical state;

b) H_N (to be referred to as *system Hamiltonian*) is a suitably regular function. In view of prerequisites #2 and #3, we expect H_N to be prescribed in terms of a *non-local phase-function* of the form $H_N(\mathbf{x}, [\mathbf{x}])$, $\mathbf{x}$ and $[\mathbf{x}]$ denoting respectively local and non-local dependences;

c) for all particles $i = 1, N$ belonging to the N-body system, the variational equations of motion must admit the *standard Hamiltonian form* expressed in terms of the Poisson brackets with respect to the system Hamiltonian, namely:

$$\frac{d\mathbf{x}^{(i)}}{ds_{(i)}} = \left[\mathbf{x}^{(i)}, H_N\right]. \tag{7.1}$$

Here the notation is standard. Thus, $\left(s_{(1)}, ...s_{(N)}\right)$ and $[\eta, \xi] \equiv [\eta, \xi]_{(\mathbf{x})}$ are respectively the particles proper times and the *local Poisson brackets* (PBs). The latter are defined in terms of the super-abundant canonical state $\mathbf{x}$ as

$$[\eta, \xi] = \left(\frac{\partial \eta}{\partial \mathbf{x}}\right)^T \cdot \underline{\underline{\mathbf{J}}} \cdot \left(\frac{\partial \xi}{\partial \mathbf{x}}\right), \tag{7.2}$$

with all components of $\mathbf{x}$ to be considered independent (i.e., $\mathbf{x}$ as *unconstrained*). Furthermore, $\underline{\underline{\mathbf{J}}}$ is the canonical Poisson matrix (2), while $\eta(\mathbf{x})$ and $\xi(\mathbf{x})$ denote two arbitrary smooth phase-functions.

Evidently, the above prerequisites should be regarded as overriding conditions for the transition from classical to quantum theory of the N-body dynamics to be possible. However, despite notable efforts (see for example Dirac, 1949 (3)) the solution to the problem of fitting them together has remained still incomplete to date, at least in the case of systems of charged particles subject to EM interactions.

From the point of view of classical physics the reason is related to the nature of EM interactions occurring in N-body systems. These can be carried respectively both by external sources (unary interactions, due to prescribed external EM fields) and by the particles themselves of the system (internal interactions). The latter include both binary EM interactions acting between any two arbitrary charged particles and the self EM interaction, usually known as the EM radiation-reaction (RR; Dirac (4), Pauli (5), Feynman (6)). As pointed out in Refs.(7) and (8), a rigorous treatment of the EM self-interaction consistent with prerequisites #1-#5 can only be achieved for extended classical particles, i.e., particles characterized by mass and charge distributions with finite support. In particular, a convenient mathematical model is obtained by assuming that these classical particles are non-rotating and their mass and charge distributions are quasi-rigid in their rest frames, according to the model adopted in the present investigation (7, 8, 9, 10). In Refs.(7, 8) the dynamics of single extended particles (1-body problem) in the presence of their EM self-fields was systematically investigated in the context of classical electrodynamics, by means of a variational approach based on a Hamilton variational principle. As a result, the Hamiltonian description for isolated particles subject to the combined action of the external and the self EM interaction has been established.

However, fundamental issues still remain unanswered regarding the analogous formulation of a consistent dynamical theory holding for classical N-body systems of finite-size charges subject to only EM interactions (*EM-interacting N-body systems*). In fact, it is well known that traditional formulations of the relativistic dynamics of classical charged particles are unsatisfactory, at least because of the following main reasons.

The first one is related to the approximations usually adopted in classical electrodynamics for the treatments of RR phenomena. In most of previous literature charged particles are regarded as point-like and the so-called short delay-time ordering

$$\epsilon \equiv \frac{t - t'}{t} \ll 1 \tag{7.3}$$

is assumed to hold, with t and t' denoting respectively the "present" and "retarded" coordinate times, both defined with respect to a suitable Laboratory frame. This motivates the introduction of asymptotic approximations, both for the EM self 4-potential and the corresponding self-force, which are based on power-series expansions in terms of the dimensionless parameter ϵ and are performed in a neighborhood of the present coordinate time t (see related discussion in Ref.(8)). Nevertheless, previous approaches of this type have lead in the past *to intrinsically non-variational and therefore non-Hamiltonian equations of motion* (7, 8, 11). These are exemplified by the well-known LAD and LL RR equations, due respectively to Lorentz, Abraham and Dirac (Lorentz, 1985 (12); Abraham, 1905 (13) and Dirac (4)) and Landau and Lifschitz (14). Both features make these treatments incompatible with the physical prerequisites indicated above.

The second motivation arises in reference to the so-called "no-interaction" theorem proposed by Currie (15, 16). According to its claim, isolated N-body systems, formed by at least two point particles, which are subject to mutual EM binary interactions and in which the canonical coordinates are identified with the space parts of the particle position 4-vectors, *cannot define a Hamiltonian system with manifestly covariant canonical equations of motion.* The correctness of such a statement has been long questioned (see for example Fronsdal, 1971 (17) and Komar, 1978-1979 (18, 19, 20, 21)). In particular, the interesting question has been posed whether *the "no-interaction" theorem can actually be eluded in physically realizable classical systems.* Interestingly, Currie approach is based on the well-known generator formalism formulated originally by Dirac (DGF; Dirac, 1949 (3)). Therefore, related preliminary questions concern the conditions of validity of DGF itself and, in particular, whether such an approach actually applies *at all* to *EM-interacting N-body systems.*

7.2 Goals of the investigation

Put all the previous motivations in perspective, in this Chapter a systematic solution to these issues is presented, for the case of EM-interacting N-body systems of classical finite-size charged particles. The theory is developed in the framework of classical electrodynamics and special relativity (i.e., assuming a flat Minkowski space-time) and is shown to satisfy all the prerequisites indicated above (#1-#5). The reference publications for the contents presented in this Chapter are Refs.(9, 10).

Starting point is the determination of both Lagrangian and Hamiltonian equations of motion for classical charged particles belonging to an EM-interacting N-body system. By construction the latter can be considered as a system of smooth hard sphere, namely in which hard collisions occurring between the particles, when their boundaries $\partial\Omega_{(i)}$ (see below) come into contact, conserve each particle angular momentum. In particular, for simplicity, in the following all extended particles will be considered as acted upon only by EM interactions, thus ignoring the effect of hard collisions on the N-body dynamics. As in Refs.(7, 8), where the 1-body problem was investigated, the derivation is based on the variational formulation of the problem. This requires, in particular, the determination of the appropriate variational functionals required for the description of both binary and self EM interactions. The resulting action functional is found to be expressed as a line integral in terms of a suitable non-local variational Lagrangian, the non-locality being associated both to the finite-extension of the charge distributions and to delay-time effects arising in binary EM interactions. Based on the Hamilton variational principle expressed in superabundant variables, the resulting variational equations of motion (for the N-body system) are then proved to be necessarily delay-type ODEs. As a consequence, based on the definition of suitable effective Lagrangian and Hamiltonian functions, a manifestly-covariant representation of these equations in both standard Lagrangian and Hamiltonian forms is reached. The main goal of this Chapter is then to show that these features allow a Hamiltonian structure $\{\mathbf{x}, H_N\}$ to be properly defined in terms of a suitable superabundant canonical state $\mathbf{x}$ and a non-local system Hamiltonian function H_N. The result follows by noting that the Hamiltonian equations in standard form admit also a representation in terms of local PBs, defined with respect to the super-abundant canonical state $\mathbf{x}$.

A notable development concerns the construction of an approximation of the Hamilton equations in standard form. This holds in validity of both the short delay-time and large-distance orderings, namely under the same asymptotic conditions usually invoked in the literature for the asymptotic treatment of the RR problem. Based on the analogous approach developed in Ref.(8), it is shown that a suitable *N-body Hamiltonian approximation* [of the exact problem] can actually be reached, which preserves its Hamiltonian structure. In particular it is proved that, unlike in the LAD and LL equations, the asymptotic approximation obtained in this way keeps the variational character of the exact theory, retaining the standard Lagrangian and Hamiltonian forms of the N-body dynamical equations as well as the delay-time contributions arising from the various EM interactions.

Further interesting conclusions are drawn concerning the validity of DGF in the present context. This refers, in particular, to the so-called instant-form representation of Poincarè generators for infinitesimal transformations of the inhomogeneous Lorentz group. It is pointed out that DGF in its original formulation only applies to *local Hamiltonian systems* and therefore is inapplicable to (the treatment of) the EM-interacting N-body systems considered here. For definiteness, the correct set of Poincarè generators, corresponding to the exact non-local Hamiltonian structure $\{\mathbf{x}, H_N\}$ determined here, together with their instant-form representation, are also provided. This permits to develop a modified formulation of DGF, denoted as *non-local generator formalism*, which overcomes the previous limitations and is applicable also to the treatment of non-local Hamiltonians in terms of *essential* (i.e., *constrained*) canonical variables.

Finally, on the same ground, the Currie "no-interaction" theorem is proved to be violated in any case by the Hamiltonian structure, i.e., both by its *exact realization* $\{\mathbf{x}, H_N\}$ and its *asymptotic approximation*. Counter-examples which overcome the limitations stated by the "no-interaction" theorem are explicitly provided. In particular, the purpose of this discussion is to prove that indeed a standard Hamiltonian formulation for the N-body system of EM-mutually-interacting charged particles can be consistently obtained. The main cause of the failure of the Currie theorem is identified in the conditions of validity of DGF on which the proof of the theorem itself is based.

7.3 N-body EM current density and self 4-potential

For the sake of clarity, the key points of the formulation presented in Refs.(7, 8) are first recalled, concerning the definition of extended chargee particle and of the corresponding 4-current and self 4-potential within the framework of N-body system. We shall assume that each finite-size particle is characterized by a positive constant rest mass $m_0^{(i)}$ and a non-vanishing constant charge $q^{(i)}$, for $i = 1, N$, both distributed on the same support $\partial\Omega_{(i)}$ (particle boundary). More precisely, for the i-th particle, the mass and charge distributions can be defined as follows. Assuming that initially in a time interval $[-\infty, t_o]$ the i-th particle is at rest with respect to an inertial frame (particle rest-frame $\mathcal{R}_o$ where the external forces acting on the particle vanish identically), we shall assume that:

1) In this frame there exists a point, hereafter referred to as *center of symmetry (COS)*, whose position 4-vector $r^{\mu}_{(i)COS} \equiv (ct, \mathbf{r}_o)$ spans the Minkowski space-time $\mathcal{M}^4 \subseteq \mathbb{R}^4$ with metric tensor $\eta_{\mu\nu} \equiv$diag$(+1 - 1 - 1 - 1)$. With respect to the COS the support $\partial\Omega_{(i)}$ is a stationary spherical surface of radius $\sigma_{(i)} > 0$ of equation $(\mathbf{r} - \mathbf{r}_o)^2 = \sigma^2_{(i)}$.

2) The i-th particle is *quasi-rigid*, i.e., its mass and charge distributions are stationary and spherically-symmetric on $\partial\Omega_{(i)}$.

3) Mass and charge densities do not possess pure spatial rotations. Therefore, introducing for each particle the Euler angles $\alpha(s_{(i)}) \equiv \{\varphi, \vartheta, \psi\}_{(s_{(i)})}$ which define its spatial orientation (see definitions in Ref.(22)), the condition of *vanishing spatial*

rotation is obtained imposing that $\alpha(s_{(i)}) = const.$ holds identically. As a consequence, only the translational motion of charged particles need to be taken into account.

In addition, as stated before, hard collisions occurring between the particles are considered ignorable. As a consequence, for all particles the equations of motion (7.1) are assumed to hold identically, namely for all $s_{(i)} \in I \equiv \mathbb{R}$.

For each extended particle the covariant expressions for the corresponding charge and mass current densities readily follow (see Ref.(7)). In particular, these can be expressed in integral form respectively as:

$$j^{(i)\mu}(r) = \frac{q^{(i)}c}{4\pi\sigma_{(i)}^2}\int_{-\infty}^{+\infty} ds u^{(i)\mu}(s)\delta(|x_{(i)}| - \sigma_{(i)})\delta(s - s_{1(i)}), \tag{7.4}$$

$$j^{(i)\mu}_{mass}(r) = \frac{m_o^{(i)}c}{4\pi\sigma_{(i)}^2}\int_{-\infty}^{+\infty} ds u^{(i)\mu}(s)\delta(|x_{(i)}| - \sigma_{(i)})\delta(s - s_{1(i)}), \tag{7.5}$$

where by definition $s_{1(i)}$ is the root of the algebraic equation

$$u_\mu^{(i)}(s_{1(i)})\left[r^\mu - r^{(i)\mu}\left(s_{1(i)}\right)\right] = 0. \tag{7.6}$$

Here $u^{(i)\mu}(s_{(i)}) \equiv \frac{dr^{(i)\mu}(s_{(i)})}{ds_{(i)}}$ is the 4-velocity of the COS for the i-th particle, while

$$x_{(i)}^\mu = r^\mu - r^{(i)\mu}\left(s_{(i)}\right). \tag{7.7}$$

Finally, following the equivalent derivations given in Refs.(7, 8), the non-divergent EM self 4-potential $A_\mu^{(self)(i)}(r)$ for the single (namely, i-th) extended particle can be readily obtained as well. It is sufficient to report here the solution for $A_\mu^{(self)(i)}(r)$ which is valid in the external domain with respect to the spherical shell of the same particle. For a generic displacement 4-vector $X^{(i)\mu} \in M^4$ of the form

$$X^{(i)\mu} = r^\mu - r^{(i)\mu}\left(s_{(i)}\right), \tag{7.8}$$

which is subject to the constraint

$$X^{(i)\mu}u_\mu^{(i)}(s_{(i)}) = 0, \tag{7.9}$$

this sub-domain is defined by the inequality

$$X^{(i)\mu}X_\mu^{(i)} \leq -\sigma_{(i)}^2. \tag{7.10}$$

In such a set, $A_\mu^{(self)(i)}(r)$ is expressed in integral form by the equation

$$A_\mu^{(self)(i)}(r) = 2q^{(i)}\int_1^2 dr'_\mu \delta(\widehat{R}^{(i)\alpha}\widehat{R}_\alpha^{(i)}), \tag{7.11}$$

where $\widehat{R}^{(i)\alpha}$ is the bi-vector

$$\widehat{R}^{(i)\alpha} = r^{\alpha} - r^{(i)\prime\alpha}, \tag{7.12}$$

with $r^{(i)\prime\alpha} \equiv r^{(i)\prime\alpha}(s'_{(i)})$ being the i-th particle COS 4-vector evaluated at the retarded proper time $s'_{(i)}$, obtained as the causal root of the equation $\widehat{R}^{(i)\alpha}\widehat{R}^{(i)}_{\alpha} = 0$.

7.4 The non-local N-body action integral

In this section we formulate the N-body Hamilton action functional suitable for the variational treatment of a system of N finite-size charged particles subject to external, binary and self EM interactions. In such a case, in analogy to Ref.(8), the action integral can be conveniently expressed in hybrid super-abundant variables as follows:

$$S_N(r,u,[r]) = \sum_{i=1,N}\left[S_M^{(i)}(r,u) + S_C^{(ext)(i)}(r) + S_C^{(self)(i)}(r,[r]) + S_C^{(bin)(i)}(r,[r])\right] \tag{7.13}$$

(*non-local action integral*). Here r and u represent *local* dependences with respect to the 4-vector position and the 4-velocity, while $[r]$ stands for *non-local* dependences with respect to the 4-vector position. In particular, the latter are included only via the functionals produced by the EM-coupling with the self and binary EM fields for the i-th particle, namely $S_C^{(self)(i)}$ and $S_C^{(bin)(i)}$. Instead, $S_M^{(i)}$ and $S_C^{(ext)(i)}$ identify for each particle the functionals produced by the inertial mass and by the EM-coupling with the external EM field. We stress that the functionals $S_M^{(i)}(r,u)$, $S_C^{(ext)(i)}(r)$ and $S_C^{(self)(i)}(r,[r])$ are formally analogous to the case of a 1-body problem treated in the previous Chapters (see also Refs.(7, 8)) and can be represented as line-integrals (see below). We now proceed evaluating explicitly the new contribution $S_C^{(bin)(i)}(r,[r])$.

$S_C^{(bin)(i)}(r,[r])$: EM coupling with the binary-interaction field

The action integral $S_C^{(bin)(i)}(r,[r])$ containing the coupling between the EM field generated by particle j, for $j = 1,N$, and the electric 4-current of particle i is of critical importance. Its evaluation is similar to that of the action integral of the self-interaction. For the sake of clarity, in this subsection we present the relevant results, while the details of the mathematical derivation can be found in Ref.(9). According to the standard approach (14), $S_C^{(bin)(i)}(r,[r])$ is defined as the 4-scalar

$$S_C^{(bin)(i)}(r,[r]) = \sum_{j=1,N i\neq j} S_C^{(bin)(ij)}(r,[r]), \tag{7.14}$$

where $S_C^{(bin)(ij)}(r,[r])$ is defined as

$$S_C^{(bin)(ij)}(r,[r]) = \int_1^2 d\Omega \frac{1}{c^2} A^{(self)(i)\mu}(r) j_{\mu}^{(j)}(r)\,, \tag{7.15}$$

with $A^{(self)(i)\mu}(r)$ being the EM 4-potential generated by particle i at 4-position r, whose expression is given by Eq.(7.11). In addition, $j_\mu^{(j)}(r)$ is the 4-current carried by particle j evaluated at the same 4-position and given by Eq.(7.4), while $d\Omega$ is the invariant 4-volume element. In particular, in an inertial frame S_I with Minkowski metric tensor $\eta_{\mu\nu}$, this can be represented as $d\Omega = cdtdxdydz$, where (x, y, z) are orthogonal Cartesian coordinates. As shown in Ref.(9), an explicit evaluation of the action integral (7.15) yields the following representation:

$$S_C^{(bin)(ij)}(r,[r]) = \frac{2q^{(i)}q^{(j)}}{c}\int_1^2 dr_\mu^{(i)}\left(s_{(i)}\right)\int_1^2 dr^{(j)\mu}(s_{(j)})\delta(\widetilde{R}^{(ij)\alpha}\widetilde{R}_\alpha^{(ij)} - \sigma_{(j)}^2), \tag{7.16}$$

where $s_{(i)}$ and $s_{(j)}$ are respectively the proper times of particles i and j, while $\widetilde{R}^{(ij)\alpha}$ denotes

$$\widetilde{R}^{(ij)\alpha} \equiv r^{(j)\alpha}\left(s_{(j)}\right) - r^{(i)\alpha}(s_{(i)}). \tag{7.17}$$

It is worth pointing out the following basic properties of the functional $S_C^{(bin,i)(ij)}$. First, it is a non-local functional in the sense that it contains a coupling between the "past" and the "future" of the particles of the N-body system. In fact it can be equivalently represented as

$$S_C^{(bin)(ij)}(r,[r]) = \frac{2q^{(i)}q^{(j)}}{c}\int_{-\infty}^{+\infty} ds_{(i)}\frac{dr_\mu^{(i)}\left(s_{(i)}\right)}{ds_{(i)}}\int_{-\infty}^{+\infty} ds_{(j)}\frac{dr^\mu(s_{(j)})}{ds_{(j)}}\delta(\widetilde{R}^{(ij)\alpha}\widetilde{R}_\alpha^{(ij)} - \sigma_{(j)}^2). \tag{7.18}$$

Furthermore, the N-body system functional (7.14) is symmetric, namely it fulfills the property

$$\sum_{i,j=1,N} S_C^{(bin)(ij)}(r_A,[r_B]) = \sum_{i,j=1,N} S_C^{(bin)(ji)}(r_B,[r_A]), \tag{7.19}$$

where r_A and r_B are two arbitrary curves of the N-body system.

The non-local N-body variational Lagrangian

Let us now provide a line-integral representation of the *Hamilton functional* S_N in the form

$$S_N = \sum_{i=1,N}\int_{-\infty}^{+\infty} ds_{(i)}L_1^{(i)}(r,u,[r]) \equiv \sum_{i=1,N}\int_{-\infty}^{+\infty}\Upsilon_{(i)}(r,u,[r]), \tag{7.20}$$

where $\Upsilon_{(i)}(r,[r],u)$ and $L_1^{(i)}(r,[r],u)$ are respectively the i-th particle non-local contributions to the fundamental Lagrangian differential form and to the corresponding *non-local variational Lagrangian*. Invoking Eq.(7.18) and recalling also the results of previous Chapters, $L_1^{(i)}(r,[r],u)$ can be written as

$$L_1^{(i)}(r,u,[r]) = L_M^{(i)}(r,u) + L_C^{(ext)(i)}(r) + L_C^{(self)(i)}(r,[r]) + L_C^{(bin)(i)}(r,[r]), \tag{7.21}$$

where $L_M^{(i)}(r,u), L_C^{(ext)(i)}(r)$ and $L_C^{(self)(i)}(r,[r]), L_C^{(bin)(i)}(r,[r])$ denote respectively the local and non-local terms. In particular, the first one is the contribution carried by the inertial term, while $L_C^{(ext)(i)}$, $L_C^{(self)(i)}$ and $L_C^{(bin)(i)}$ identify respectively the external, self and binary EM-field-coupling Lagrangians. These are defined as follows:

$$
\begin{aligned}
L_M^{(i)}(r,u) &\equiv m_o^{(i)} c u_\mu^{(i)} \left[\frac{dr^{(i)\mu}}{ds_{(i)}} - \frac{1}{2} u^{(i)\mu} \right], &(7.22)\\
L_C^{(ext)(i)}(r) &\equiv \frac{q^{(i)}}{c} \frac{dr^{(i)\mu}}{ds_{(i)}} \overline{A}_\mu^{(ext)(i)}(r^{(i)}(s_{(i)}), \sigma_{(i)}), &(7.23)\\
L_C^{(self)(i)}(r,[r]) &\equiv \frac{q^{(i)}}{c} \frac{dr^{(i)\mu}}{ds_{(i)}} \overline{A}_\mu^{(self)(i)}, &(7.24)\\
L_C^{(bin)(i)}(r,[r]) &\equiv \sum_{j=1,N} L_C^{(bin)(ij)}(r,[r]) = \frac{q^{(i)}}{c} \frac{dr^{(i)\mu}}{ds_{(i)}} \sum_{j=1,N} \overline{A}_\mu^{(bin)(ij)} \left(\sigma_{(j)}\right), &(7.25)
\end{aligned}
$$

where in the last equation $i \neq j$. Here, $\overline{A}_\mu^{(ext)(i)}$, $\overline{A}_\mu^{(self)(i)}$ and $\overline{A}_\mu^{(bin)(ij)}$ denote the surface-averages performed on the i-th particle boundary $\partial\Omega_{(i)}$ respectively of the external, self and binary EM 4-potentials. In particular, $\overline{A}_\mu^{(self)(i)}$ and $\overline{A}_\mu^{(bin)(ij)}$ are defined as

$$
\begin{aligned}
\overline{A}_\mu^{(self)(i)} &\equiv 2q^{(i)} \int_1^2 dr_\mu^{(i)\prime} \delta(\widetilde{R}^{(i)\mu} \widetilde{R}_\mu^{(i)} - \sigma_{(i)}^2), &(7.26)\\
\overline{A}_\mu^{(bin)(ij)} \left(\sigma_{(j)}\right) &\equiv 2q^{(j)} \int_{-\infty}^{+\infty} ds_{(j)} \frac{dr^\mu(s_{(j)})}{ds_{(j)}} \delta(\widetilde{R}^{(ij)\alpha} \widetilde{R}_\alpha^{(ij)} - \sigma_{(j)}^2). &(7.27)
\end{aligned}
$$

In addition, $\widetilde{R}^{(i)\mu}$ is the bi-vector

$$
\widetilde{R}^{(i)\mu} \equiv r^{(i)\mu}\left(s_{(i)}\right) - r^{(i)\mu}(s'_{(i)}), \tag{7.28}
$$

with $s_{(i)}$ and $s'_{(i)}$ denoting respectively "present" and "retarded" proper times of the i-th particle.

7.5 Non-local N-body variational principle and standard Lagrangian form

Let us now proceed constructing the explicit form of the N-body relativistic equations of motion for each extended charged particle *in the presence of EM interactions* (i.e., including *external, binary and self EM interactions*). This is achieved by adopting for the N-body problem a *synchronous variational principle* (23, 24) which can be expressed in terms of the super-abundant hybrid (i.e., generally non-Lagrangian) variables

$$f^{(i)}(s_{(i)}) \equiv \left[r^{(i)\mu}(s_{(i)}), u_{\mu}^{(i)}(s_{(i)})\right], \tag{7.29}$$

and for a suitable functional class of variations $\{f\}$. The latter is identified with the set of real functions of class $C^k(\mathbb{R})$, with $k \geq 2$, and fixed endpoints which are prescribed for each particle $i = 1, N$ at suitable proper times $s_{(i)1}$ and $s_{(i)2}$, with $s_{(i)1} < s_{(i)2}$, i.e.,

$$\{f\} \equiv \left\{ \begin{array}{c} f^{(i)}(s_{(i)}) : \ f^{(i)}(s_{(i)}) \in C^k(\mathbb{R}); \\ f^{(i)}(s_{(i)j}) = f_j^{(i)}; \\ i = 1, N; \ j = 1, 2 and k \geq 2 \end{array} \right\}. \tag{7.30}$$

It follows that by construction the variational derivatives of the Hamilton functional S_N (see Eq.(7.20)) are performed in terms of synchronous variations, i.e., by keeping constant the i-th particle proper time $s_{(i)}$. The result is expressed by the following theorem.

THM.1 - N-body hybrid synchronous Hamilton variational principle
Given validity of the prerequisites #1-#5 for the N-body system, let us assume that:

1. *The Hamilton action $S_N(r, u, [r])$ is defined by Eq.(7.20).*

2. *The real functions $f^{(i)}(s_{(i)})$ in the functional class $\{f\}$ [see Eq.(7.30)] are identified with the super-abundant variables (7.29) which are subject to synchronous variations $\delta f^{(i)}(s_{(i)}) \equiv f^{(i)}(s_{(i)}) - f_1^{(i)}(s_{(i)})$. The latter belong to the functional class of synchronous variations $\{\delta f^{(i)}\}$, with*

$$\delta f_k^{(i)}(s_{(i)}) \ = f_k^{(i)}(s_{(i)}) - f_{1k}^{(i)}(s_{(i)}), \tag{7.31}$$

   for $k = 1, 2$, $\forall f^{(i)}(s_{(i)}), f_1^{(i)}(s_{(i)}) \in \{f\}$.

3. *The extremal curves $f^{(i)}(s_{(i)}) \in \{f\}$ for S_N, which are solutions of the E-L equations*

$$\frac{\delta S_N(r, u, [r])}{\delta f^{(i)}(s_{(i)})} = 0, \tag{7.32}$$

   exist for arbitrary variations $\delta f^{(i)}(s_{(i)})$ (hybrid synchronous Hamilton variational principle).

4. *If the curves $r^{(i)\mu}(s_{(i)})$, for $i = 1, N$ are all extremal, each line element $ds_{(i)}$ satisfies the constraint*

$$ds_{(i)}^2 = \eta_{\mu\nu} dr^{(i)\mu}(s_{(i)}) dr^{(i)\nu}(s_{(i)}). \tag{7.33}$$

5. *The 4-vector field $A_{\mu}^{(ext)}(r)$ is suitably smooth in the whole Minkowski space-time M^4.*

6. *The E-L equations for the extremal curves $r^{(i)\mu}(s_{(i)})$ are determined consistently with the Einstein causality principle.*

7. *All the synchronous variations $\delta f_k^{(i)}(s_{(i)})$ (k=1,2 and $i = 1, N$) are considered as being independent.*

It then follows that the E-L equations for $u^{(i)\mu}$ and $r^{(i)\mu}$ following from the synchronous hybrid Hamilton variational principle (7.32) give respectively

$$\frac{\delta S_N}{\delta u_{\mu}^{(i)}} = m_o^{(i)} c dr^{(i)\mu} - m_o^{(i)} c u^{(i)\mu} ds_{(i)} = 0, \tag{7.34}$$

$$\frac{\delta S_N}{\delta r^{(i)\mu}(s_{(i)})} = -m_o^{(i)} c du_{\mu}^{(i)}(s_{(i)}) + \frac{q^{(i)}}{c} F_{\mu\nu}^{(tot)(i)} dr^{(i)\nu}(s_i) = 0, \tag{7.35}$$

where $F_{\mu\nu}^{(tot)(i)}$ is the total Faraday tensor acting on particle i and given by

$$F_{\mu\nu}^{(tot)(i)} \equiv \overline{F}_{\mu\nu}^{(ext)(i)} + \overline{F}_{\mu\nu}^{(self)(i)} + \overline{F}_{\mu\nu}^{(bin)(i)}, \tag{7.36}$$

where all quantities are intended as surface-averages on the i-th particle shell-surface $\partial\Omega_{(i)}$. Eqs.(7.34) and (7.35) are hereon referred to as N-body equations of motion. In particular:

1) $\overline{F}_{\mu\nu}^{(ext)(i)}\left(r^{(i)}\right) \equiv \partial_{\mu}\overline{A}_{\nu}^{(ext)} - \partial_{\nu}\overline{A}_{\mu}^{(ext)}$ is the antisymmetric Faraday tensor of the external EM field evaluated on the extremal curve $r^{(i)\mu} = r^{(i)\mu}\left(s_{(i)}\right)$.

2) $\overline{F}_{\mu\nu}^{(self)(i)}\left(r^{(i)}, \left[r^{(i)}\right]\right) \equiv \overline{F}_{\mu\nu}^{(self)(i)}\left(r^{(i)}\left(s_{(i)}\right), r^{(i)}\left(s'_{(i)}\right)\right)$ is the non-local antisymmetric Faraday tensor produced by the EM self-field of the i-th particle and acting on the same particle. This is given by

$$\overline{F}_{\mu\nu}^{(self)(i)} = 2\left[\partial_{\mu}\overline{A}_{\nu}^{(self)} - \partial_{\nu}\overline{A}_{\mu}^{(self)}\right], \tag{7.37}$$

namely

$$\overline{F}_{\mu\nu}^{(self)(i)} = -\left[\frac{2q^{(i)}}{\left|\widetilde{R}^{(i)\alpha}u_{\alpha}^{(i)}(s'_{(i)})\right|}\frac{d}{ds'_{(i)}}\left\{\frac{u_{\mu}^{(i)}(s'_{(i)})\widetilde{R}_{\nu}^{(i)} - u_{\nu}^{(i)}(s'_{(i)})\widetilde{R}_{\mu}^{(i)}}{\widetilde{R}^{(i)\alpha}u_{\alpha}^{(i)}(s'_i)}\right\}\right]_{s'_{(i)}=s_{(i)}-s_{(i)ret}}, \tag{7.38}$$

where the delay-time $s_{(i)ret}$ is the positive (causal) root of the 1-particle delay-time equation

$$\widetilde{R}^{(i)\alpha}\widetilde{R}_{\alpha}^{(i)} - \sigma_{(i)}^2 = 0. \tag{7.39}$$

3) $\overline{F}_{\mu\nu}^{(bin)(i)}$ is the non-local antisymmetric Faraday tensor produced on particle i by *the action of all the remaining particles, i.e.*

$$\overline{F}_{\mu\nu}^{(bin)(i)} \equiv \sum_{j=1,N i\neq j} \overline{F}_{\mu\nu}^{(bin)(ij)} \left(r^{(i)}, \left[r^{(j)}\right], \sigma_{(i)}, \sigma_{(j)}\right), \tag{7.40}$$

where

$$\begin{aligned}\overline{F}_{\mu\nu}^{(bin)(ij)} \left(r^{(i)}, \left[r^{(j)}\right], \sigma_{(i)}, \sigma_{(j)}\right) &= \left[H_{\mu\nu}^{(ij)}\left(s_{(i)}, s_{(j)}\right)\right]_{s_{(j)}=s_{(i)}^{(A)}\left(\sigma_{(i)}\right)} + \\ &+ \left[H_{\mu\nu}^{(ij)}\left(s_{(i)}, s_{(j)}\right)\right]_{s_{(j)}=s_{(ij)}^{(B)}\left(\sigma_{(j)}\right)}. \end{aligned} \tag{7.41}$$

Here the notation is as follows. $H_{\mu\nu}^{(ij)}$ *is defined as*

$$H_{\mu\nu}^{(ij)}\left(s_{(i)}, s_{(j)}\right) = -\frac{q^{(j)}}{\left|\widetilde{R}^{(ij)\alpha} u_{\alpha}^{(j)}(s_{(j)})\right|} \frac{d}{ds_{(j)}} \left\{ \frac{u_{\mu}^{(j)}(s_{(j)})\widetilde{R}_{\nu}^{(ij)} - u_{\nu}^{(i)}(s_{(j)})\widetilde{R}_{\mu}^{(ij)}}{\widetilde{R}^{(ij)\alpha} u_{\alpha}^{(j)}(s_{(j)})} \right\}, \tag{7.42}$$

while the delay-time $s_{(j)} = s_{(i)}^{(A)}\left(\sigma_{(i)}\right)$ *and* $s_{(j)} = s_{(ij)}^{(B)}\left(\sigma_{(j)}\right)$ *are respectively the positive (causal) roots of the 2-particle delay-time equations*

$$\widetilde{R}^{(i)\alpha}\widetilde{R}_{\alpha}^{(i)} - \sigma_{(i)}^2 = 0, \tag{7.43}$$
$$\widetilde{R}^{(ij)\alpha}\widetilde{R}_{\alpha}^{(ij)} - \sigma_{(j)}^2 = 0. \tag{7.44}$$

Therefore, $s_{(i)}^{(A)}$ *and* $s_{(ij)}^{(B)}$ *depend respectively on* $\sigma_{(i)}$ *and* $\sigma_{(j)}$.

Proof - The proof is analogous to the corresponding 1-body problem detailed in THM.1 of Ref.(7). Indeed, since the Dirac-deltas $\delta(\widetilde{R}^{(i)\mu}\widetilde{R}_{\mu}^{(i)} - \sigma_{(i)}^2)$ and $\delta(\widetilde{R}^{(ij)\alpha}\widetilde{R}_{\alpha}^{(ij)} - \sigma_{(j)}^2)$ are independent of particle 4-velocities, the variations with respect to $u_{\mu}^{(i)}$ deliver necessarily the E-L equation (7.34). To prove also Eq.(7.35), we notice that the synchronous variations of the functionals $S_M^{(i)}(r,u)$, $S_C^{(ext)(i)}(r)$ and $S_C^{(self)(i)}(r,[r])$ necessarily coincide with those of the 1-body problem. Therefore, it is sufficient to inspect the variational derivative of the non-local binary-interaction functional $S_C^{(bin)(i)}(r,[r])$. Its variation with respect to $\delta r^{(i)\mu}(s_{(i)})$ takes the form

$$\delta S_C^{(bin)(i)} = \sum_{j=1,N i\neq j} \left\{ [\delta A + \delta B]_{(ij)} + [\delta A + \delta B]_{(ji)} \right\}, \tag{7.45}$$

where

$$\begin{aligned}\delta A_{(ij)} &\equiv -\tfrac{2q^{(i)}q^{(j)}}{c}\eta_{\mu\nu}\int_1^2 \delta r^{(i)\mu} d\left[\int_1^2 dr^{(j)\nu}\delta(\widetilde{R}^{(ij)\alpha}\widetilde{R}_{\alpha}^{(ij)} - \sigma_{(i)}^2)\right], \\ \delta B_{(ij)} &\equiv \tfrac{2q^{(i)}q^{(j)}}{c}\eta_{\alpha\beta}\int_1^2 dr^{(j)\beta}\int_1^2 dr^{(i)\alpha}\delta r^{(i)\mu}\tfrac{\partial}{\partial r^{(i)\mu}}\delta(\widetilde{R}^{(ij)\alpha}\widetilde{R}_{\alpha}^{(ij)} - \sigma_{(i)}^2),\end{aligned} \tag{7.46}$$

and the second term $[\delta A + \delta B]_{(ji)}$ follows by exchanging the particle indices. Then, using the chain rule and integrating by parts, after elementary algebra Eqs.(7.41) and (7.42) follow. In agreement with the Einstein causality principle the positive roots of the delay-time equations (7.43) and (7.44) must be selected.

Q.E.D.

A few comments are here in order regarding the implications of THM.1.

1. **Coordinate-time parametrization of the N-body equations of motion**

   It is important to stress that for each i-th particle, its equations of motion [in particular the E-L Eqs.(7.34) and (7.35)] can be parametrized in terms of the single coordinate time t rather than the corresponding particle proper time $s_{(i)}$. This is obtained introducing the representations in terms of the single coordinate (i.e., Laboratory) time $t \in I \in \mathbb{R}$, namely letting for all $i = 1, N$

$$\begin{aligned} r^{(i)\mu}(t) &\equiv (ct, \mathbf{r}^{(i)}), && (7.47)\\ ds_{(i)} &= \frac{cdt}{\gamma_{(i)}}, && (7.48)\end{aligned}$$

   with $\gamma_{(i)}$ and $\beta_{(i)}$ denoting the usual relativistic factors

$$\begin{aligned} \gamma_{(i)} &= \left(1-\beta^2_{(i)}\right)^{-1/2}, && (7.49)\\ \beta_{(i)} &= \mathbf{v}^{(i)}/c. && (7.50)\end{aligned}$$

   This implies also that the 4-velocity can be represented as $u^{(i)\mu} \equiv \frac{1}{c}\gamma^{-1}_{(i)} v^{(i)\mu}$ with $v^{(i)\mu} \equiv (c, \mathbf{v}^{(i)})$. Hence, equations (7.34) and (7.35) become respectively

$$m_o^{(i)} c dr^{(i)\mu} - m_o^{(i)} c v^{(i)\mu} dt = 0, \tag{7.51}$$

$$-m_o^{(i)} cd\left[\frac{\gamma_{(i)}}{c} v^{(i)}_\mu (s_{(i)})\right] + \frac{q^{(i)}}{c} F^{(tot)(i)}_{\mu\nu} dr^{(i)\nu}(s_i) = 0. \tag{7.52}$$

2. **Delay-time effects**

   Delay-time effects which appear both in the EM RR and binary interactions are due to the extended size of the charged particles. In particular, the delay-time characterizing the self-interaction acting on particle i depends only on the radius of the charge distribution of the same particle. Instead, the delay-time appearing in the binary interaction experienced by particle i depends either on the radius $\sigma_{(i)}$ of particle i or on the radii $\sigma_{(j)}$ of all the remaining particles. In the case of N-body system of like particles, such that $\sigma_{(i)} = \sigma_{(j)} = \sigma$, the two terms on the r.h.s. of Eq.(7.45) coincide yielding a single delay-time contribution in Eq.(7.41). Explicit evaluation of delay times involves the construction of the positive (causal) roots of the equations (7.43) and (7.44). Based on the coordinate-time parametrization (7.48), these can be solved explicitly for the *coordinate* *delay-time* $t_{(i)ret} \equiv t'_{(i)} - t$. The causal roots are in the two cases respectively

$$t_{(i)ret}(t) = \frac{1}{c}\sqrt{\left[\mathbf{r}^{(i)}(t) - \mathbf{r}^{(i)}(t - t_{(i)ret}(t))\right]^2 + \sigma_{(i)}^2} > 0, \tag{7.53}$$

$$t_{(ij)ret}(t) = \frac{1}{c}\sqrt{\left[\mathbf{r}^{(i)}(t) - \mathbf{r}^{(j)}(t - t_{(ij)ret}(t))\right]^2 + \sigma_{(j)}^2} > 0. \tag{7.54}$$

Notice that the same roots can also be equivalently represented in terms of the corresponding particle proper times ($s_{(i)}$ for $i = 1, N$). For this purpose it is sufficient to introduce for the i-th particle proper time the parametrization $s_{(i)} \equiv s_{(i)}(t)$ which is determined in terms of the coordinate time t by means of the equations (7.33). It follows, in particular, that the proper delay-times corresponding to (7.53) and (7.54) become respectively $s_{(i)ret}(s_{(i)}) \equiv s(t_{(i)ret}(t))$ and $s_{(ij)ret}(s_{(j)}) \equiv s(t_{(ij)ret}(t))$.

Along the lines of the approach given in Ref.(8), it is immediate to show that the hybrid-variable variational principle given in THM.1 can be given an equivalent Lagrangian formulation. An elementary consequence is provided by the following proposition.

Corollary to THM.1 - Standard Lagrangian form of the N-body equations of motion

Given validity of THM.1, let us introduce the non-local real function

$$L_{eff,N} = \sum_{i=1,N} L_{eff}^{(i)}(r, u, [r]), \tag{7.55}$$

where $L_{eff,N}$ is denoted as N-body effective Lagrangian, while $L_{eff}^{(i)}(r, u, [r])$ is defined as

$$L_{eff}^{(i)}(r, u, [r]) \equiv L_M^{(i)}(r, u) + L_C^{(ext)(i)}(r) + 2L_C^{(self)(i)}(r, [r]) + L_{eff}^{(bin)(i)}(r, [r]). \tag{7.56}$$

Here $L_M^{(i)}$, $L_C^{(ext)(i)}$ and $L_C^{(self)(i)}$ coincide with the variational Lagrangians defined above (see Eqs.(7.22)-(7.24)), while $L_{eff}^{(bin)(i)}$ is given by

$$L_{eff}^{(bin)(i)}(r, [r]) \equiv \sum_{j=1,N i\neq j} \frac{2q^{(i)}q^{(j)}}{c} \frac{dr_\mu^{(i)}(s_{(i)})}{ds_{(i)}} \int_{-\infty}^{+\infty} ds_{(j)} \frac{dr^\mu(s_{(j)})}{ds_{(j)}} K^{(ij)}, \tag{7.57}$$

with $K^{(ij)}$ being the sum of Dirac-deltas

$$K^{(ij)} \equiv \delta(\widetilde{R}^{(ij)\alpha}\widetilde{R}_\alpha^{(ij)} - \sigma_{(j)}^2) + \delta(\widetilde{R}^{(ij)\alpha}\widetilde{R}_\alpha^{(ij)} - \sigma_{(i)}^2). \tag{7.58}$$

Then, the E-L equations (7.34) and (7.35) coincide with the E-L equations in standard form (see Refs.(7, 8)) determined in terms of the N-body effective Lagrangian $L_{eff,N}$.

In particular, the E-L equations in standard form for the i-th particle become

$$\frac{\partial L_{eff,N}}{\partial u_{\mu}^{(i)}(s_{(i)})} = 0, \tag{7.59}$$

$$F_{\mu}^{(i)}(r) L_{eff,N} = 0, \tag{7.60}$$

where

$$F_{\mu}^{(i)}(r) \equiv \frac{d}{ds_{(i)}} \frac{\partial}{\partial \frac{dr^{(i)\mu}(s_{(i)})}{ds_{(i)}}} - \frac{\partial}{\partial r^{(i)\mu}(s_{(i)})} \tag{7.61}$$

denotes the E-L differential operator.

Proof - The proof is based on THM.2 of Ref.(7) and by noting that the E-L differential operator acts only on local quantities. Hence, the equivalence of Eqs.(7.59)-(7.60) with the corresponding E-L equations (7.34) and (7.35) follows by elementary algebra and in view of the identity $F_{\mu}^{(i)}(r) L_{eff,N} = F_{\mu}^{(i)}(r) L_{eff}^{(i)}$.

Q.E.D.

To conclude this Section a final remark is necessary regarding the functional setting of the N-body equations of motion given above.

We first notice that the E-L equations (7.34) and (7.35), and the equivalent Lagrange equations in standard form (7.60), imply for all $i = 1, N$ the second-order delay-type ODEs

$$m_o^{(i)} c \frac{d^2 r^{(i)\mu}(s_{(i)})}{ds_{(i)}^2} = \frac{q^{(i)}}{c} F_{\nu}^{(tot)(i)\mu} dr^{(i)\nu}(s_i). \tag{7.62}$$

Let us introduce the Lagrangian state $\mathbf{w} \equiv \left(\left(r^{(i)}, u^{(i)} \right), i = 1, N \right)$, with $u^{(i)\mu} \equiv \frac{dr^{(i)\mu}}{ds_{(i)}}$, spanning the N-body phase-space $\Gamma_N \equiv \prod_{i=1,N} \Gamma_{1(i)}$, where $\Gamma_{1(i)} = M_{(i)}^{(4)} \times U_{(i)}^{(4)}$ and $U_{(i)}^{(4)} \equiv \mathbb{R}^4$ indicate respectively the corresponding 1-body phase and velocity spaces, the latter endowed with a metric tensor $\eta_{\mu\nu}$. To define the initial conditions, let us make use of the coordinate-time parametrization (7.48), denoting $\widehat{\mathbf{w}}(t) \equiv \left(\left(\widehat{r}^{(i)}(t), \widehat{u}^{(i)}(t) \right), i = 1, N \right)$ and $\widehat{r}^{(i)}(t) \equiv r^{(i)}(s_{(i)}(t))$, $\widehat{u}^{(i)}(t) \equiv u^{(i)}(s_{(i)}(t))$. Then, a well-posed problem for Eqs.(7.62) is obtained prescribing the *initial history set* $\{\widehat{\mathbf{w}}\}_{t_0} \subset \Gamma_N$. For an arbitrary *coordinate initial time* $t_0 \in I \equiv \mathbb{R}$, this is defined as the ensemble of initial states

$$\{\widehat{\mathbf{w}}\}_{t_0} \equiv \left\{ \left(\left(\widehat{r}^{(i)}(t), \widehat{u}^{(i)}(t) \right) \in C^{(k-1)}(I), i = 1, N \right), \forall t \in \left[t_0 - t_{ret}^{\max}(t_0), t_0 \right], k \geq 2 \right\}. \tag{7.63}$$

Here, for a given initial (coordinate) time $t_0 \in I$, $t_{ret}^{\max}(t_0)$ denotes the maximum (for all particles) of the delay-times $t_{(i)ret}(t_0)$ and $t_{(ij)ret}(t_0)$, namely

$$t_{ret}^{\max}(t_0) = \max \left\{ t_{(i)ret}(t_0), t_{(ij)ret}(t_0), \forall i, j = 1, N \right\}. \tag{7.64}$$

Solutions of Eqs.(7.62) fulfilling the initial conditions defined by the history set $\{\widehat{\mathbf{w}}\}_{t_0}$ are sought in the functional class of smooth 4-vector solutions of the form $r^{(i)\mu} \equiv$

$r^{(i)\mu}(s_{(i)})$, with $s_{(i)} = s_{(i)}(t)$ and $t \in I_0 \equiv (t_0, \infty)$, which belong to the functional class

$$\{\mathbf{r}(s)\} \equiv \left\{ r^{(i)\mu} \middle| r^{(i)\mu} \equiv r^{(i)\mu}(s_{(i)}) \in C^{(k)}(I), s_{(i)}(t) \in C^{(k)}(I), k \geq 2, \forall t \in I_0 \right\}, \quad (7.65)$$

for $i = 1, N$. In the following we shall assume that *in the setting defined by Eq.(7.65) with the history set (7.63), the ODEs (7.62) admit a unique global solution of class* $C^{(k-1)}(I_0)$*, with* $k \geq 2$.

7.6 N-body non-local Hamiltonian theory

Based on THM.1 and its Corollary, an equivalent non-local Hamiltonian formulation can be given for the hybrid and Lagrangian-variable approaches stated in THM.1 and Corollary. The strategy is similar to that developed in Ref.(8). Thus, first we proceed constructing the intermediate set of hybrid variables $\mathbf{y} \equiv (r, p) \equiv \left(r^{(i)\mu}, p_\mu^{(i)}, i = 1, N \right)$ and the related *non-local variational Hamiltonian* $H_1^{(i)} = H_1^{(i)}(r, p, [r])$, identified, as usual, with the Legendre transformation of the corresponding *non-local variational Lagrangian* $L_1^{(i)}$. Hence, for all $i = 1, N$:

$$H_1^{(i)} = p_\mu^{(i)} \frac{dr^{(i)\mu}}{ds_{(i)}} - L_1^{(i)}, \quad (7.66)$$

while $p_\mu^{(i)}$ is the i-th particle conjugate momentum defined in terms of $L_1^{(i)}$ as

$$p_\mu^{(i)} \equiv \frac{\partial L_1^{(i)}}{\partial \frac{dr^{(i)\mu}}{ds_{(i)}}}. \quad (7.67)$$

From THM.1 it follows that

$$p_\mu^{(i)} = m_o^{(i)} c u_\mu^{(i)} + \frac{q^{(i)}}{c} A_\mu^{(tot)(i)}, \quad (7.68)$$

where $A_\mu^{(tot)(i)}$ is given by

$$A_\mu^{(tot)(i)}(r, [r]) = \overline{A}_\mu^{(ext)(i)} + \overline{A}_\mu^{(self)(i)} + \sum_{j=1,N i\neq j} \overline{A}_\mu^{(bin)(ij)}, \quad (7.69)$$

according to the definitions given in Eqs.(7.26)-(7.27). As a consequence, $H_1^{(i)}$ becomes simply

$$H_1^{(i)}(r, p, [r]) = \frac{1}{2m_o^{(i)} c} \left[p_\mu^{(i)} - \frac{q^{(i)}}{c} A_\mu^{(tot)(i)} \right] \left[p^{(i)\mu} - \frac{q^{(i)}}{c} A^{(tot)(i)\mu} \right]. \quad (7.70)$$

This permits us to represent the Hamilton action functional in terms of the hybrid state $\mathbf{y}$, yielding

$$S_{H_N}(r,p,[r]) = \sum_{i=1,N} S_{H_1^{(i)}}, \tag{7.71}$$

where

$$S_{H_1^{(i)}} \equiv \int_{s_{(i)1}}^{s_{(i)2}} ds_{(i)} \left[p_\mu^{(i)} \frac{dr^{(i)\mu}}{ds_{(i)}} - H_1^{(i)} \right] \tag{7.72}$$

represents the i-th particle contribution. Of course, as an alternative, analogous dynamical variables can be defined also in terms of the effective Lagrangian $L_{eff}^{(i)}$ [see Eq.(7.56)]. This yields the notion of *effective Hamiltonian* $H_{eff}^{(i)}$ and of the corresponding state $\mathbf{x} \equiv (r,P) \equiv \left(r^{(i)\mu}, P_\mu^{(i)}, i=1,N \right)$, which will be shown below to identify a (super-abundant) canonical state (see Corollary to THM.2). Thus, $H_{eff}^{(i)}$ - to be considered a non-local function of the form $H_{eff}^{(i)} = H_{eff}^{(i)}(r,P,[r])$ - is prescribed in terms of the Legendre transformation with respect to $L_{eff}^{(i)}$, namely letting:

$$H_{eff}^{(i)} \equiv P_\mu^{(i)} \frac{dr^{(i)\mu}}{ds_{(i)}} - L_{eff}^{(i)}, \tag{7.73}$$

while $P_\mu^{(i)}$ denotes the *effective canonical momentum*

$$P_\mu^{(i)} \equiv \frac{\partial L_{eff}^{(i)}}{\partial \frac{dr^{(i)\mu}}{ds_{(i)}}}. \tag{7.74}$$

From the Corollary to THM.1 it follows immediately that

$$P_\mu^{(i)} = m_o^{(i)} c u_\mu^{(i)} + \frac{q^{(i)}}{c} A_{(eff)\mu}^{(tot)(i)}, \tag{7.75}$$

where $A_{(eff)\mu}^{(tot)(i)}$ is given by

$$A_{(eff)\mu}^{(tot)(i)} = \overline{A}_\mu^{(ext)(i)} + 2\overline{A}_\mu^{(self)(i)} + \sum_{j=1,N i\neq j} \overline{A}_{(eff)\mu}^{(bin)(ij)}, \tag{7.76}$$

and

$$\overline{A}_{(eff)\mu}^{(bin)(ij)} = 2q^{(j)} \int_{-\infty}^{+\infty} ds_{(j)} \frac{dr^\mu(s_{(j)})}{ds_{(j)}} K^{(ij)}, \tag{7.77}$$

with $K^{(ij)}$ being given by Eq.(7.58). Finally, $H_{eff}^{(i)}$ becomes

$$H_{eff}^{(i)} = \frac{1}{2m_o^{(i)} c} \left[P_\mu^{(i)} - \frac{q^{(i)}}{c} A_{(eff)\mu}^{(tot)(i)} \right] \left[P^{(i)\mu} - \frac{q^{(i)}}{c} A_{(eff)}^{(tot)(i)\mu} \right]. \tag{7.78}$$

Therefore, by direct comparison with Eq.(7.70) it follows identically that

$$H_{eff}^{(i)} \equiv H_1^{(i)}. \tag{7.79}$$

Then the following theorem, casting the Hamilton variational principle of THM.1 in terms of the state $\mathbf{x}$, holds.

THM.2 - N-body non-local Hamiltonian variational principle

Given validity of THM.1 with Corollary and the definitions (7.66)-(7.72) as well as (7.73)-(7.77), let us assume that the curves $f^{(i)}(s_{(i)}) \equiv \mathbf{y}^{(i)} = (r^{(i)\mu}, p_\mu^{(i)})_{(s_{(i)})}$ belong to the functional class $\{f\}$ of C^2-functions subject to the boundary conditions

$$\mathbf{y}^{(i)}(s_{(i)k}) = \mathbf{y}_k^{(i)}, \tag{7.80}$$

for $k = 1, 2$, $s_{(i)1}, s_{(i)2} \in I \subseteq \mathbb{R}$ and with $s_{(i)1} < s_{(i)2}$. Then the following proposition holds:

The modified Hamilton variational principle

$$\delta S_{H_N} = 0 \tag{7.81}$$

subject to independent synchronous variations $\delta f^{(i)}(s_{(i)}) \equiv \left(\delta r^{(i)\mu}(s_{(i)}), \delta p_\mu^{(i)}(s_{(i)})\right)$ performed in the functional class indicated above, yields, for all $i = 1, N$, the E-L equations

$$\frac{\delta S_{H_1^{(i)}}}{\delta p_\mu^{(i)}} = 0, \tag{7.82}$$

$$\frac{\delta S_{H_1^{(i)}}}{\delta r^{(i)\mu}} = 0. \tag{7.83}$$

These equations coincide identically with the N-body variational equations of motion (7.34) and (7.35). Hence, the set $\{\mathbf{y}, H_N\} \equiv \left\{\mathbf{y}^{(i)}, H_1^{(i)}, i = 1, N\right\}$ defines a non-local Hamiltonian system.

Proof - The proof can be reached, after elementary algebra, by invoking the symmetry properties of the variational functional S_{H_N}, namely

$$S_{H_N}(r_A, p, [r_B]) = S_{H_N}(r_B, p, [r_A]), \tag{7.84}$$

where again r_A and r_B are two N-body arbitrary curves of the functional class $\{\mathbf{y}\}$. It follows that the variational derivative in the E-L equation Eq.(7.83) becomes

$$\frac{\delta S_{H_1^{(i)}}}{\delta r^{(i)\mu}} \equiv \left.\frac{\delta S_{H_1^{(i)}}}{\delta r^{(i)\mu}}\right|_{[r]} + \sum_{j=1,N} \left.\frac{\delta S_{H_1^{(j)}}}{\delta \left[r^{(i)\mu}\right]}\right|_r = 0, \tag{7.85}$$

where the summation is performed only on the non-local contributions. As a consequence, the E-L equations (7.82) and (7.83) yield

$$\frac{\delta S_{H_1^{(i)}}}{\delta p_\mu^{(i)}} = m_o^{(i)} c \frac{dr^{(i)\mu}}{ds_{(i)}} - \left[p_\mu^{(i)} - \frac{q^{(i)}}{c} A_\mu^{(tot)(i)} \right] = 0, \tag{7.86}$$

$$\frac{\delta S_{H_1^{(i)}}}{\delta r^{(i)\mu}} = -\frac{dp_\mu^{(i)}}{ds_{(i)}} + \frac{q^{(i)}}{c} \frac{dr^{(i)\nu}(s_i)}{ds_{(i)}} \left[\frac{\partial A_\mu^{(tot)(i)}}{\partial r^{(i)\nu}} + F_{\mu\nu}^{(tot)(i)} \right] = 0. \tag{7.87}$$

Taking into account the definitions given by Eq.(7.68) the equivalence with Eqs.(7.34) and (7.35) is immediate.

Q.E.D.

Let us now pose the problem of the construction of the corresponding N-body Hamiltonian equations in standard form, as suggested by the results of Ref.(8). The non-local Hamiltonian system $\{\mathbf{y}, H_N\}$ is said to admit a *standard Hamiltonian form* $\left\{\mathbf{x}, H_{eff}^{(1)}, ...H_{eff}^{(N)}\right\}$ if the N-body equations of motion can be cast, for all $i = 1, N$, in the form

$$\frac{dr^{(i)\mu}}{ds_{(i)}} = \frac{\partial H_{eff}^{(i)}}{\partial P_\mu^{(i)}}, \tag{7.88}$$

$$\frac{dP_\mu^{(i)}}{ds_{(i)}} = -\frac{\partial H_{eff}^{(i)}}{\partial r^{(i)\mu}}, \tag{7.89}$$

in terms of a suitably-defined *effective particle Hamiltonian* $H_{eff}^{(i)}$, to be identified with Eq.(7.78). In particular, a N-body system with state $\mathbf{x} \equiv \left\{\mathbf{x}^{(i)}, i = 1, N\right\}$ is said to be endowed with a *Hamiltonian structure* $\{\mathbf{x}, H_{N,eff}\}$ if, for all particles belonging to the N-body system, the equations of motion for the i-th canonical particle state $\mathbf{x}^{(i)}$ can be represented in the PBs notation (7.2) in terms of a single Hamiltonian function $H_{N,eff}$, i.e., for all $i = 1, N$

$$\frac{d\mathbf{x}^{(i)}}{ds_{(i)}} = \left[\mathbf{x}^{(i)}, H_{N,eff}\right], \tag{7.90}$$

with $H_{N,eff}$ denoting a still to be determined, appropriate *system effective Hamiltonian*. Extending the treatment holding for the 1-body problem, here it is proved that the Hamiltonian structure $\{\mathbf{x}, H_{N,eff}\}$ holds also in the case of EM-interacting N-body systems. The following proposition holds.

Corollary to THM.2 - Standard Hamiltonian form and Hamiltonian structure of the N-body equations of motion

Given validity of THM.2 and the definitions given by Eqs.(7.73)-(7.78), it follows that:

TC2$_1$) *The non-local Hamiltonian system $\{\mathbf{x}, H_N\}$ admits a standard Hamiltonian form defined in terms of the set $\left\{\mathbf{x}, H_{eff}^{(1)}, ...H_{eff}^{(N)}\right\}$, with $H_{eff}^{(i)}$ the i-th particle effec-*

tive Hamiltonian [given by Eq.(7.78)]; *furthermore* $\mathbf{x} \equiv \left\{ \mathbf{x}^{(i)}, i = 1, N \right\}$ *is the superabundant canonical state, while*

$$\mathbf{x}^{(i)} \equiv \left(r^{(i)\mu}, P_{\mu}^{(i)} \right), \tag{7.91}$$

$$r^{(i)\mu} = \left(r^{(i)0}, \mathbf{r}^{(i)} \right), \tag{7.92}$$

$$P^{(i)\mu} \equiv \left(P^{(i)0}, \mathbf{P}^{(i)} \right), \tag{7.93}$$

are respectively the i-th particle canonical state, 4-position and effective canonical momentum [defined by Eq.(7.75)]. *As a consequence, Eqs.(7.82) and (7.83) can be cast in the standard Hamiltonian form (7.88) and (7.89).*

TC2$_2$) *The equations (7.88) and (7.89) admit also the equivalent representation (7.90) and hence the set $\{\mathbf{x}, H_{N,eff}\}$ defines a Hamiltonian structure, with $H_{N,eff}$ being the effective N-body Hamiltonian function*

$$H_{N,eff} \equiv \sum_{i=1,N} H_{eff}^{(i)}. \tag{7.94}$$

TC2$_3$) *Introducing the system Hamiltonian*

$$H_N \equiv \sum_{i=1,N} H^{(i)} \tag{7.95}$$

defined in terms of the variational i-th particle variational Hamiltonian $H^{(i)}$ [see Eq.(*7.70*)], *it follows identically that*

$$H_N = H_{N,eff}. \tag{7.96}$$

Proof - TC2$_1$) The proof follows from straightforward algebra. The first equation manifestly reproduces Eq.(7.82), because of the definition of $H_{eff}^{(i)}$ given above. Similarly, in the second equation the partial derivative of $H_{eff}^{(i)}$ recovers the correct form of the total EM force expressed in terms of $A_{(eff)\mu}^{(tot)(i)}$. TC2$_2$) To prove the existence of the Hamiltonian structure, it is sufficient to notice that $\frac{\partial H_{eff}^{(i)}}{\partial P_{\mu}^{(i)}} = \left[r^{(i)\mu}, H_{N,eff} \right]$ and $\frac{\partial H_{eff}^{(i)}}{\partial r^{(i)\mu}} = -\left[P_{\mu}^{(i)}, H_{N,eff} \right]$. TC2$_3$) By construction [see Eqs.(7.70) and (7.78)] for all $i = 1, N$ the effective and variational Hamiltonians coincide [see Eq.(7.79)]. This implies the validity of Eq.(7.96) too, namely H_N identifies also the system Hamiltonian. It follows that for all particles $i = 1, N$ the canonical equations of motion (7.90) recover the standard Hamiltonian form expressed in terms of the PBs with respect to the the system Hamiltonian, i.e., Eqs.(7.1), so that $\{\mathbf{x}, H_N\}$ identifies the Hamiltonian structure of the EM-interacting N-body system.

Q.E.D.

7.7 General implications of the non-local N-body theory

Let us now comment on the general implications of the previous theorems.

Remark #1 - Difference form of the Hamilton equations of motion. The canonical equations (7.1) imply the following difference equations, i.e., the infinitesimal canonical transformation generated by H_N:

$$d\mathbf{x}^{(i)} = ds_{(i)}\left[\mathbf{x}^{(i)}, H_N\right]. \tag{7.97}$$

Remark #2 - Coordinate-time representation of the Hamilton equations of motion. Introducing the coordinate-time parametrization (7.48), Eqs.(7.97) become

$$d\mathbf{x}^{(i)} = \frac{cdt}{\gamma_{(i)}}\left[\mathbf{x}^{(i)}, H_N\right], \tag{7.98}$$

with $\gamma_{(i)}$ denoting again the relativistic factor (7.49), while $[\cdot,\cdot]$ are the local PBs evaluated with respect to the super-abundant canonical state $\mathbf{x}$. These yield explicitly

$$dr^{(i)\mu} = \frac{dt}{\gamma_{(i)}}\frac{1}{m_o^{(i)}}\left(P_\mu^{(i)} - \frac{q^{(i)}}{c}A_{(eff)\mu}^{(tot)(i)}\right) \equiv \frac{dt}{\gamma_{(i)}}cu_\mu^{(i)} \equiv dtv_\mu^{(i)}, \tag{7.99}$$

$$\begin{aligned} dP_\mu^{(i)} &= \frac{dt}{\gamma_{(i)}}\frac{q^{(i)}}{m_o^{(i)}c}\frac{\partial A_{(eff)\nu}^{(tot)(i)}}{\partial r^{(i)\mu}}\left(P^{(i)\nu} - \frac{q^{(i)}}{c}A_{(eff)}^{(tot)(i)\nu}\right) \equiv \\ &\equiv \frac{cdt}{\gamma_{(i)}}\frac{q^{(i)}}{c}\frac{\partial A_{(eff)\nu}^{(tot)(i)}}{\partial r^{(i)\mu}}u^{(i)\nu} \equiv dt\frac{q^{(i)}}{c}\frac{\partial A_{(eff)\nu}^{(tot)(i)}}{\partial r^{(i)\mu}}v^{(i)\nu}. \end{aligned} \tag{7.100}$$

Remark #3 - Well-posedness of the N-body equations of motion. All the equations of motion indicated above [see THMs.1 and 2 and their Corollaries] are equivalent to each other and are manifestly Lorentz-covariant. Then, a well-posed problem for the Hamiltonian equations in standard form (7.90) can be obtained in analogy to the problem defined by Eqs.(7.62),(7.63). This is achieved, first, by prescribing the appropriate initial history set $\{\widehat{\mathbf{x}}\}_{t_0} \subset \Gamma_N$. For an arbitrary coordinate initial time $t_0 \in I \equiv \mathbb{R}$, this is defined as the ensemble of initial states

$$\{\widehat{\mathbf{x}}\}_{t_0} \equiv \left\{\widehat{\mathbf{x}}(t) \equiv \left(\left(\widehat{r}^{(i)}(t), \widehat{p}^{(i)}(t)\right) \in C^{(k-1)}(I), i = 1, N\right), \forall t \in \left[t_0 - t_{ret}^{\max}(t_0), t_0\right], k \geq 2\right\} \tag{7.101}$$

(*canonical history set*), where $t_{ret}^{\max}(t_0)$ is the maximum delay-time at t_0 [see Eq.(7.64)]. Furthermore, in analogy with Eq.(7.65), solutions of Eqs.(7.90) fulfilling the initial conditions defined by the history set $\{\widehat{\mathbf{x}}\}_{t_0}$ are sought in the functional class (7.30) by identifying

$$f^{(i)}(s_{(i)}) \equiv \left[r^{(i)\mu}(s_{(i)}), p_\mu^{(i)}(s_{(i)})\right]. \tag{7.102}$$

In the following we shall assume that *in the setting defined by Eq.(7.102) with the canonical history set (7.101), the ODEs (7.90) admit a unique global solution of class* $C^{(k-1)}(I_0)$ *with* $k \geq 2$.

Remark #4 - Extremant and extremal curves. In all cases indicated above the solutions of N-body E-L equations of motion (*extremal curves*) and of the Lagrangian and Hamiltonian equations in standard form given by the Corollaries to THMs.1 and 2 (*extremant curves*), satisfy identically, for all $i = 1, N$, the kinematic constraints

$$u_\mu^{(i)} u^{(i)\mu} = 1 \tag{7.103}$$

(*velocity constraints*) and

$$ds_{(i)}^2 = \eta_{\mu\nu} dr^{(i)\mu} dr^{(i)\nu} \tag{7.104}$$

(*line-element constraints*). The first constraint implies that the time-components of the 4-velocity depend on the corresponding space components, while the second constraint requires that the particles' proper times are uniquely related to the corresponding coordinate times. In particular, we shall denote as *extremant canonical curves*

$$\mathbf{x}(s_{(1)}, ...s_{(N)}) \equiv \left\{ \mathbf{x}^{(1)}(s_{(1)}), ...\mathbf{x}^{(N)}(s_{(N)}) \right\}, \tag{7.105}$$

with $\mathbf{x}^{(i)} = \mathbf{x}^{(i)}(s_{(i)})$ for all $i = 1, N$, arbitrary particular solutions of the canonical equations (7.127).

Remark #5 - Unconstrained varied functions. By assumption, both the varied functions $f^{(i)} = \left[r^{(i)\mu}, u_\mu^{(i)} \right]_{(s_{(i)})}$, $\mathbf{y}^{(i)} = (r^{(i)\mu}, p_\mu^{(i)})_{(s_{(i)})}$ and $\mathbf{x}^{(i)} = (r^{(i)\mu}, P_\mu^{(i)})_{(s_{(i)})}$ entering respectively THMs.1 and 2 as well as the Corollary of THM.2 are *unconstrained*, namely they are solely subject to the requirement that end points and boundary values are kept fixed (and therefore do *not* fulfill the previous kinematic constraints). This implies, in particular, that all of the (8) components of $f^{(i)}$, $\mathbf{y}^{(i)}$ and $\mathbf{x}^{(i)}$ must be considered independent. On the other hand, both the extremal and extremant curves satisfy all of the required kinematic constraints, so that only (6) of them are actually independent for each particle.

Remark #6 - Non-local Hamiltonian structure and unconstrained canonical state. Thanks to proposition TC2$_3$ of the Corollary to THM.2 the Hamiltonian structure $\{\mathbf{x}, H_{N,eff}\}$ coincides with $\{\mathbf{x}, H_N\}$, H_N denoting the non-local system Hamiltonian defined by Eq.(7.95). We remark, however, that in the PBs given by Eq.(7.1) the partial derivatives must be evaluated with respect to the unconstrained states $\mathbf{x}^{(i)}$ and not $\mathbf{y}^{(i)}$ indicated above. This means that H_N must be considered a function of $\mathbf{x}$. It is immediate to prove that the same Hamiltonian structure $\{\mathbf{x}, H_N\}$ holds provided the super-abundant canonical state $\mathbf{x} \equiv \left(\mathbf{x}^{(i)}, i = 1, N\right)$ is considered unconstrained. In fact, as shown by the Corollary to THM.2, in such a case the canonical equations in standard form (7.88) and (7.89) admit a PB-representation of the form (7.90). For this purpose let us make use of the coordinate-time parametrization $\mathbf{x} \equiv \widehat{\mathbf{x}}(t)$, denoting $\widehat{\mathbf{x}}(t) = \mathbf{x}_o + d\mathbf{x}$, with $\mathbf{x}_o \equiv \widehat{\mathbf{x}}(t_o)$ and $d\mathbf{x} \equiv \left(d\mathbf{x}^{(1)}, ...d\mathbf{x}^{(N)}\right)$. Furthermore let us require that the initial history set $\{\widehat{\mathbf{x}}\}_{t_0}$ is prescribed. Then, a necessary and sufficient condition for the equations of motion to admit the standard Hamiltonian form (7.1) is that the fundamental local PBs for the state $\mathbf{x} \equiv \widehat{\mathbf{x}}(t)$, defined with respect to the same state $\mathbf{x}_o \equiv \widehat{\mathbf{x}}(t_o)$, namely

$$\left[r^{(i)\mu}, r^{(j)\nu}\right]_{(\mathbf{x}_o)} = 0, \tag{7.106}$$

$$\left[P_\mu^{(i)}, P_\nu^{(j)}\right]_{(\mathbf{x}_o)} = 0, \tag{7.107}$$

$$\left[r^{(i)\mu}, P_\nu^{(j)}\right]_{(\mathbf{x}_o)} = \delta^{ij}\delta_\nu^\mu, \tag{7.108}$$

are identically satisfied for all $i, j = 1, N$ and $\mu, \nu = 0, 3$. This is realized only when the super-abundant variables $\mathbf{x} \equiv \left(\mathbf{x}^{(i)}, i = 1, N\right)$ *are considered independent*.

Remark #7 - The canonical flow is not a dynamical system. A final issue concerns the properties of the flow generated by the canonical problem (7.90) and (7.101)-(7.102) (*canonical flow*). In the N-body phase-space Γ_N this is an ensemble $C^{(k-1)}-$homeomorphism (with $k \geq 2$) of the type

$$\{\widehat{\mathbf{x}}\}_{t_0} \leftrightarrow \widehat{\mathbf{x}}(t), \tag{7.109}$$

which maps an arbitrary history set $\{\widehat{\mathbf{x}}\}_{t_0} \subset \Gamma_N$ onto a state $\widehat{\mathbf{x}}(t)$ crossed at a later coordinate time t (i.e., at $t > t_0$). *This map does not generally define a dynamical system.* In fact, unless there is a subset on non-vanishing measure of Γ_N in which $\{\widehat{\mathbf{x}}\}_{t_0}$ reduces to the initial *instant set*

$$\{\widehat{\mathbf{x}}\}_{t_0} \equiv \{\widehat{\mathbf{x}}(t_o) = \mathbf{x}_0\}, \tag{7.110}$$

the flow (7.109) is not a bijection in Γ_N. To prove the statement it is sufficient to notice that - in the case of a non-local Hamiltonian structure $\{\mathbf{x}, H_{N,eff}\}$ - if the history set is left unspecified and only the initial state $\widehat{\mathbf{x}}(t_o)$ is prescribed, the image of the initial state is obviously generally non-unique [and hence it may not coincide with $\widehat{\mathbf{x}}(t)$]. In fact, while the same initial state $\widehat{\mathbf{x}}(t_o)$ may be produced by different history sets, for example $\{\widehat{\mathbf{x}}\}_{t_0}$ and $\{\widehat{\mathbf{x}}'\}_{t_0}$, the same history sets will generally give rise to different images $\widehat{\mathbf{x}}(t)$ and $\widehat{\mathbf{x}}'(t)$.

We emphasize that for N-body systems subject to EM interactions the instant set (7.110) can be realized only for special initial conditions, i.e., for example, if for all $t \leq t_0$ all the particles of the system are in inertial motion with respect to an inertial Lorentz frame. Since, unlike the external EM field, binary and self EM interactions cannot be "turned off", it follows that the set of initial conditions (7.110) has necessarily null measure in Γ_N.

7.8 The N-body Hamiltonian asymptotic approximation

In this section we want to develop asymptotic approximations for the equations of motion of EM-interacting N-body systems. This involves different asymptotic conditions to be imposed on both the self and binary interactions. In particular:

1) For the RR self-interaction of each particle i with itself this is provided by the *short delay-time ordering*, namely the requirement that the dimensionless parameters $\epsilon_{(i)} \equiv \frac{(s_{(i)}-s'_{(i)})}{s_{(i)}}$, for $i=1,N$ are all infinitesimal of the same order ϵ, i.e. $\epsilon \sim \epsilon_{(i)} \ll 1$, $s_{(i)}-s'_{(i)}$ denoting the i-th proper-time difference between observation (s) and emission (s') of self-radiation.

2) For the binary EM interactions the Minkowski distance $\left|\widetilde{R}^{(ij)\alpha}\right|$ between two arbitrary particles of the system is much larger than their radii, in the sense that for all $i,j=1,N$, with $i \neq j$, the *large-distance ordering* $0 < \frac{\sigma_{(i)}}{\left|\widetilde{R}^{(ij)\alpha}\right|} \sim \frac{\sigma_{(j)}}{\left|\widetilde{R}^{(ij)\alpha}\right|} \lesssim \epsilon$ holds.

The fundamental issue arises whether an approximation can be found for the N-body problem which:

1) is consistent with the orderings 1) and 2);

2) recovers the variational, Lagrangian and Hamiltonian character of the exact theory (see THMs.1 and 2);

3) preserves both the standard Lagrangian and Hamiltonian forms of the equations of motion;

4) retains finite delay-time effects characteristics of both the RR and binary EM interactions, consistent with the prerequisites #1-#5 indicated above.

In this regard, a fundamental result is the discovery pointed out in Ref.(8) of an asymptotic Hamiltonian approximation of this type for single extended particles subject to the EM self-interaction. This refers to the retarded-time Taylor expansion of the Faraday tensor contribution carried by the RR self-force. More precisely, in the case of a single particle, this is obtained by Taylor-expanding the RR self-force

$$G_{\mu}^{(i)} \equiv \frac{q^{(i)}}{c} \overline{F}_{\mu k}^{(self)(i)} \left(r^{(i)} \left(s_{(i)} \right), r^{(i)} \left(s'_{(i)} \right) \right) \frac{dr^{(i)k}(s_{(i)})}{ds_{(i)}} \tag{7.111}$$

for $i=1$ (see Eq.(7.38)) *in the neighborhood of the retarded proper-time* $s'_{(i)}$. Here we claim that an analogous conclusion can be drawn also for the corresponding N-body problem, by introducing the same expansion to all charged particles and invoking the large-distance ordering for the binary interaction. For this purpose, we shall assume that the external force acting on each charged particle is slowly varying in the sense that, denoting $r' \equiv r^{(i)\mu}\left(s'_{(i)}\right)$ and $r \equiv r^{(i)\mu}\left(s_{(i)}\right)$,

$$\overline{F}_{\mu\nu}^{(ext)}\left(r'\right) - \overline{F}_{\mu\nu}^{(ext)}\left(r\right) \sim O\left(\epsilon\right), \tag{7.112}$$

$$\left(\overline{F}_{\mu\nu}^{(ext)}\left(r'\right) - \overline{F}_{\mu\nu}^{(ext)}\left(r\right)\right)_{,h} \sim O\left(\epsilon\right), \tag{7.113}$$

$$\left(\overline{F}_{\mu\nu}^{(ext)}\left(r'\right) - \overline{F}_{\mu\nu}^{(ext)}\left(r\right)\right)_{,hk} \sim O\left(\epsilon\right). \tag{7.114}$$

Then, the following proposition holds.

THM.3 - N-body asymptotic Hamiltonian approximation.

Given validity of THM.2 and the short delay-time and large-distance asymptotic orderings as well as the smoothness assumptions (7.112)-(7.114) for the external EM field, neglecting corrections of order ϵ^n, with $n \geq 1$ (first-order approximation), the following results hold:

T3$_1$) The vector fields (7.111) describing the RR self-force are approximated in a neighborhood of $s'_{(i)}$ as

$$g_\mu^{(i)}\left(r^{(i)}\left(s'_{(i)}\right)\right) = \left\{-m_{oEM}^{(i)} c \frac{d}{ds'_{(i)}} u_\mu^{(i)}\left(s'_{(i)}\right) + g_\mu^{(i)\prime}\left(r^{(i)}\left(s'_{(i)}\right)\right)\right\}, \tag{7.115}$$

to be referred to as retarded-time Hamiltonian approximation for the self-force, in which the first term on the r.h.s. identifies a retarded mass-correction term, $m_{oEM}^{(i)} \equiv \frac{q^{(i)2}}{2c^2\sigma_{(i)}}$ denoting the leading-order EM mass. Finally, $g_\mu^{(i)\prime}$ are the 4-vectors

$$g_\mu^{(i)\prime}\left(r^{(i)}\left(s'_{(i)}\right)\right) = \frac{2}{3}\frac{q^{(i)2}}{c}\left[\frac{1}{4}\frac{d^2}{ds'^2_{(i)}} u_\mu^{(i)}\left(s'_{(i)}\right) - u_\mu^{(i)}(s'_{(i)}) u^{(i)k}(s'_{(i)}) \frac{d^2}{ds'^2_{(i)}} u_k^{(i)}\left(s'_{(i)}\right)\right]. \tag{7.116}$$

T3$_2$) The tensor fields $\overline{F}_{\mu\nu}^{(bin)(ij)}$ for all $i,j = 1,N$, with $i \neq j$, appearing in the binary EM interaction (see Eq.(7.41)) are approximated by the leading-order (point-particle) terms:

$$\overline{F}_{\mu\nu}^{(bin)(ij)} \cong \overline{F}_{\mu\nu}^{(bin)(ij)}\left(r^{(i)}, \left[r^{(j)}\right], \sigma_{(i)} = 0, \sigma_{(j)} = 0\right). \tag{7.117}$$

T3$_3$) The corresponding asymptotic N-body equations of motion obtained replacing $G_\mu^{(i)}$ and $\overline{F}_{\mu\nu}^{(bin)(ij)}$ with the asymptotic approximations (7.115) and (7.117) are variational, Lagrangian and admit a standard Lagrangian form. Denoting with $r_0^{(i)\prime} \equiv r_0\left(s'_{(i)}\right)$ the extremal i-th particle world-line at the retarded proper time $s'_{(i)}$, the i-th particle asymptotic variational Lagrangian functions become:

$$L_{1,asym}^{(i)}(r,u,[r]) = L_M^{(i)}(r,u) + L_C^{(ext)(i)}(r) + L_{C,asym}^{(self)(i)}(r^{(i)}, r_0^{(i)\prime}) + L_{C,asym}^{(bin)(i)}(r,[r]). \tag{7.118}$$

Here $L_M^{(i)}$ and $L_C^{(ext)(i)}$ remain unchanged (see Eqs.(7.22) and (7.23)), while the non-local terms $L_{C,asym}^{(self)(i)}(r,[r])$ and $L_{C,asym}^{(bin)(i)}(r,[r])$ are respectively

$$L_{C,asym}^{(self)(i)}(r^{(i)}, r_0^{(i)\prime}) = g_\mu^{(i)}\left(r_0^{(i)\prime}\right) r^{(i)\mu}, \tag{7.119}$$

$$L_{C,asym}^{(bin)(i)}(r,[r]) = \frac{q^{(i)}}{c}\frac{dr^{(i)\mu}}{ds_{(i)}} \sum_{j=1,N} \overline{A}_\mu^{(bin)(ij)}\left(\sigma_{(j)} = 0\right), \tag{7.120}$$

for $i \neq j$ and where, from Eq.(7.27), one obtains

$$\overline{A}_{\mu}^{(bin)(ij)}\left(\sigma_{(j)}=0\right) \equiv 2q^{(j)}\int_{-\infty}^{+\infty} ds_{(j)}\frac{dr^{\mu}(s_{(j)})}{ds_{(j)}}\delta(\widetilde{R}^{(ij)\alpha}\widetilde{R}_{\alpha}^{(ij)}). \tag{7.121}$$

Similarly, the effective particle Lagrangians are, for $i=1,N$:

$$L_{eff,asym}^{(i)}(r,u,[r]) \equiv L_{M}^{(i)}(r,u)+L_{C}^{(ext)(i)}(r)+L_{C,asym}^{(self)(i)}(r^{(i)},r_{0}^{(i)\prime})+2L_{C,asym}^{(bin)(i)}(r,[r]). \tag{7.122}$$

T3$_4$) The N-body equations obtained imposing the asymptotic approximations given by Eqs.(7.115) and (7.117) are also Hamiltonian. The asymptotic variational and effective Hamiltonian functions are given respectively by

$$\begin{aligned} H_{1,asym}^{(i)} &= p_{\mu}^{(i)}\frac{dr^{(i)\mu}}{ds_{(i)}}-L_{1,asym}^{(i)}, && (7.123)\\ H_{eff,asym}^{(i)} &\equiv P_{\mu}^{(i)}\frac{dr^{(i)\mu}}{ds_{(i)}}-L_{eff,asym,}^{(i)} && (7.124)\end{aligned}$$

with $L_{1,asym}^{(i)}$ and $L_{eff,asym,}^{(i)}$ defined by Eqs.(7.118) and (7.122), while now

$$\begin{aligned} p_{\mu}^{(i)} &\equiv \frac{\partial L_{1,asym}^{(i)}}{\partial\frac{dr_{\mu}^{(i)}(s_{(i)})}{ds_{(i)}}}, && (7.125)\\ P_{\mu}^{(i)} &\equiv \frac{\partial L_{eff,asym}^{(i)}}{\partial\frac{dr^{(i)\mu}}{ds_{(i)}}}. && (7.126)\end{aligned}$$

Proof - T3$_1$) The proof is analogous to that given in THM.5 of Ref.(8). T3$_2$) To prove the validity of Eq.(7.117), let us recall the definition of $\overline{F}_{\mu\nu}^{(bin)(ij)}$ given by Eq.(7.41). Then, for each particle, imposing the large-distance ordering and neglecting corrections of order ϵ^{n}, with $n\geq 1$, the leading-order contribution is given by Eq.(7.117), which depends on a single delay-time determined by the positive root of the equation $\widetilde{R}^{(ij)\alpha}\widetilde{R}_{\alpha}^{(ij)}=0$. T3$_3$) The proof follows by first noting that the function $L_{C,asym}^{(self)(i)}$ contributes to the i-th particle E-L equations only in terms of the local dependence in terms of $r^{(i)}$. Second, in the large-distance ordering, the asymptotic approximation for the N-body Lagrangian carrying the binary interactions yields a symmetric functional, as the exact one. Therefore, the N-body asymptotic equations are necessarily variational and Lagrangian. Straightforward algebra shows that the E-L equations determined with respect to the asymptotic variational Lagrangian (7.118) coincide with the asymptotic approximations proved by propositions T3$_1$) and T3$_2$). In a similar way it is immediate to prove the validity of Eq.(7.122), which shows that the same asymptotic equations admit a standard Lagrangian form. T3$_4$) Finally, the equivalent N-body variational and standard Hamiltonian formulations follow by performing

Legendre transformations on the corresponding asymptotic variational and effective Lagrangian functions. It follows that the asymptotic N-body equations of motion can also be represented in the standard Hamiltonian form in terms of $H_{eff,asym}^{(i)}$.

Q.E.D.

It is worth pointing out the unique features of THM.3. These are related, in particular, to the asymptotic expansion performed on the RR self-force alone. In most of the previous literature, the short delay-time expansion is performed with respect to the particle present proper-time. This leads unavoidably to local asymptotic equations (analogous to the LAD and LL equations) which are intrinsically non-variational and therefore non-Lagrangian and non-Hamiltonian. In contrast, the short delay-time expansion adopted here approximates the non-local RR vector field in a manner that meets the goals indicated at the beginning of the section. The remarkable consequence is that the asymptotic N-body equations of motion retain the representation in standard Hamiltonian form characteristic of the corresponding exact equations. Finally, we stress that in all cases both for the exact and asymptotic formulations, the variational and effective Lagrangian and Hamiltonian functions are always non-local functions of the particle states. The non-locality is intrinsic and arises even in the 1-body systems, being due to the functional form of the EM 4-potential generated by each extended particle.

7.9 On the validity of the Dirac generator formalism

A seminal approach in relativistic dynamics is the Dirac generator formalism developed originally by Dirac (Dirac, 1949 (3)) to describe the dynamics of interacting N-body systems in the Minkowski space-time. Dirac's primary goal is actually to determine the underlying *dynamical system*, exclusively based on DGF. In his words "*In setting up such a new dynamical system one is faced at the outset by the two requirements of special relativity and of Hamiltonian equations of motion*". It thus "*...becomes a matter of great importance to set up* (in this way) *new dynamical systems and see if they will better describe the atomic world*" (quoted from Ref.(3)).

DGF is couched on the Lie algebra of Poincarè generators for classical N-body systems. The basic hypothesis behind Dirac approach is that these systems must have a Hamiltonian structure $\{\mathbf{z}, K_N\}$ of some sort, with $\mathbf{z} = \left\{\mathbf{z}^{(1)}, ...\mathbf{z}^{(N)}\right\}$ and K_N being a suitable canonical state and a Hamiltonian function of the system. In particular, the canonical states $\mathbf{z}^{(i)}$ of all particles $i = 1, N$ must satisfy, by assumption, covariant Hamilton equations of motion of the form

$$\frac{d\mathbf{z}^{(i)}}{ds_{(i)}} = \left[\mathbf{z}^{(i)}, K_N\right], \tag{7.127}$$

with $s_{(i)}$ denoting the i-th particle proper time. However, it must be stressed that certain aspects of Dirac theory remain "a priori" undetermined. This concerns the functional settings both of the canonical state $\mathbf{z}$ and of the Hamiltonian function K_N. In particular, the system state $\mathbf{z}=\left\{\mathbf{z}^{(1)},...\mathbf{z}^{(N)}\right\}$ remains in principle unspecified, so that it might be identified either with a set of *super-abundant* or *essential* canonical variables. Thus, for example, in the two cases the i-the particle state $\mathbf{z}^{(i)}$ might be prescribed respectively either as: a) the ensemble of two 4-vectors $\mathbf{z}^{(i)}=\left(r^{(i)\mu},\pi^{(i)\mu}\right)$, with $r^{(i)\mu}=\left(r^{(i)0},\mathbf{r}^{(i)}\right)$ and $\pi^{(i)\mu}=\left(\pi^{(i)0},\pi^{(i)}\right)$ being respectively the particle 4-position and its conjugate 4-momentum; b) the ensemble of the corresponding two 3-vectors obtained taking only the space parts of the same 4-vectors $r^{(i)\mu}$ and $\pi^{(i)\mu}$, namely in terms of $\mathbf{z}'^{(i)}=\left(\mathbf{r}^{(i)},\pi^{(i)}\right)$. Thus, depending on the possible prescription, the very definitions of the PBs entering Eqs.(7.127) and DGF change.

Furthermore, it remains "a priori" unspecified whether K_N is actually intended as a local or a non-local function of the canonical state $\mathbf{z}$. In Ref.(3), however, certain restrictions on the nature of the set $\{\mathbf{z},K_N\}$ are actually implied. These will be discussed below, leaving aside for the moment further discussions on this important issue.

Provided the Hamiltonian structure $\{\mathbf{z},K_N\}$ exists, any set of smooth dynamical variables η, ξ and ζ depending locally on the canonical state $\mathbf{z}$ necessarily fulfills the following laws

$$[\eta,\xi]=-[\xi,\eta], \tag{7.128}$$
$$[\eta,\xi+\zeta]=[\eta,\xi]+[\eta,\zeta], \tag{7.129}$$
$$[\xi,\eta\zeta]=[\xi,\eta]\zeta+\eta[\xi,\zeta], \tag{7.130}$$
$$[[\xi,\eta],\zeta]+[[\eta,\zeta],\xi]+[[\zeta,\xi],\eta]=0. \tag{7.131}$$

DGF relies on the Lie transformation formalism and is based on the representation of the Lorentz transformation group in terms of the corresponding generator algebra. This is defined as the set of phase-functions (Poincarè algebra generators) $\{F\}$ given by

$$F=-\widehat{p}^{\mu}a_{\mu}+\frac{1}{2}\widehat{M}^{\mu\nu}b_{\mu\nu}, \tag{7.132}$$

with a_{μ}, $b_{\mu\nu}$ being suitable real constant infinitesimals and $\widehat{p}^{\mu}$, $\widehat{M}^{\mu\nu}=-\widehat{M}^{\nu\mu}$ appropriate local phase-functions obeying the PBs (*Lorentz conditions*)

$$[\widehat{p}_{\mu},\widehat{p}_{\nu}] = 0, \tag{7.133}$$
$$\left[\widehat{M}_{\mu\nu},\widehat{p}_{\alpha}\right] = -\eta_{\mu\alpha}\widehat{p}_{\nu}+\eta_{\nu\alpha}\widehat{p}_{\mu}, \tag{7.134}$$
$$\left[\widehat{M}_{\mu\nu},\widehat{M}_{\alpha\beta}\right] = -\eta_{\mu\alpha}\widehat{M}_{\nu\beta}+\eta_{\nu\alpha}\widehat{M}_{\mu\beta}-\eta_{\mu\beta}\widehat{M}_{\alpha\nu}+\eta_{\nu\beta}\widehat{M}_{\alpha\mu}. \tag{7.135}$$

Hence, F as given by Eq.(7.132) generate respectively infinitesimal 4-translations (for $b_{\mu\nu}\equiv 0$ and $a_{\mu}\neq 0$) and 4-rotations (for $b_{\mu\nu}\neq 0$ and $a_{\mu}\equiv 0$, corresponding either to Lorentz-boosts or spatial rotations) via infinitesimal canonical transformations of the type

$$\mathbf{z}\rightarrow\mathbf{z}'+\delta_{o}\mathbf{z}, \tag{7.136}$$

with $\delta_o \mathbf{z} \sim O(\delta)$, $\delta > 0$ denoting a suitable infinitesimal. $\delta_o \mathbf{z}$ is determined identifying it with

$$\delta_o \mathbf{z} \equiv [\mathbf{z}, F], \tag{7.137}$$

to be referred to as the *local variation of* $\mathbf{z}$. Hence, according to DGF an arbitrary dynamical variable ξ depending *locally* and *smoothly* on the canonical state $\mathbf{z}$ transforms in terms of the law

$$\xi \to \xi' = \xi + \delta_o \xi, \tag{7.138}$$
$$\delta_o \xi(\mathbf{z}) \equiv [\xi, F], \tag{7.139}$$

where $\delta_o \xi(\mathbf{z}) = [\xi(\mathbf{z}') - \xi(\mathbf{z})]\left[1 + O(\delta)\right]$. It is immediate to determine an admissible representation for the generators $\{F\} \equiv \left\{\widehat{p}^{\mu}, \widehat{M}^{\mu\nu}\right\}$. Let us first consider a relativistic 1-body system represented by a single particle in the absence of external forces. For definiteness, let us assume that $\forall s_{(1)} \in I \equiv \mathbb{R}$ the particle 4-velocity is constant. Then $\widehat{p}^{\mu}$ and $\widehat{M}^{\mu\nu}$are manifestly given by:

$$\begin{aligned} \widehat{p}_{\mu} &= \pi_{\mu}^{(1)}, && (7.140)\\ \widehat{M}_{\mu\nu} &= q_{\mu}^{(1)}\pi_{\nu}^{(1)} - q_{\nu}^{(1)}\pi_{\mu}^{(1)}, && (7.141) \end{aligned}$$

where respectively the 4-vectors $q_{\mu}^{(1)}$ and $\pi_{\mu}^{(1)}$ are to be identified with the 1-body coordinates and momenta. In the case of the corresponding N-body problem for relativistic interacting particles, in Dirac paper three different realizations of $\left\{\widehat{p}^{\mu}, \widehat{M}^{\mu\nu}\right\}$ were originally proposed, which are referred to as the instant, point and front forms. All of them follow by imposing the velocity kinematic constraints (7.103). In particular, the instant form is realized by prescribing the reference frame in such a way to set $r_0^{(i)} = 0$, for all $i = 1, N$, namely describing each particle position only in terms of the space components $\mathbf{r}^{(i)} \equiv r_l^{(i)}$ of its position 4-vector, for $l = 1, 3$. In detail, recalling Eq.(7.92) and introducing the notation

$$\pi_{\mu}^{(i)} = (\pi_0^{(i)}, \pi^{(i)}), \tag{7.142}$$

according to Dirac the instant form (for N-body systems of interacting particles) is obtained by imposing the velocity kinematic constraints (7.103) on the *free-particle* canonical momenta $\pi_{free,\mu}^{(i)} \equiv m_o^{(i)} c u_{\mu}^{(i)} = \left(\pi_{free,0}^{(i)}, \pi_{free}^{(i)}\right)$ [i.e., in the absence of an external EM field] such that

$$\pi_{free,0}^{(i)} = \sqrt{m_o^{(i)2} c^2 + \pi_{free}^{(i)2}}, \tag{7.143}$$

and then introducing a suitable *interaction 4-potential* $V_{\mu} \equiv (V_0, \mathbf{V})$ taking into account all the particle interactions. Letting $l, m = 1, 3$, this yields the N-body *Dirac constrained instant-form generators* $\left(\widehat{p}_0, \widehat{p}_l, \widehat{M}_{lm}, \widehat{N}_{l0}\right)$ (3), represented in terms of the constrained states $\mathbf{z}'^{(i)} = \left(\mathbf{r}^{(i)}, \pi^{(i)}\right)$ (for $i = 1, N$):

$$\widehat{p}_0 = \sum_{i=1,N} p_0^{(i)} = \sum_{i=1,N} \sqrt{m_o^{(i)2}c^2 + \pi_{free}^{(i)2}} + V_0, \tag{7.144}$$

$$\widehat{p}_l = \sum_{i=1,N} \pi_l^{(i)}, \tag{7.145}$$

$$\widehat{M}_{lm} = \sum_{i=1,N} \left[r_l^{(i)} \pi_m^{(i)} - r_m^{(i)} \pi_l^{(i)} \right], \tag{7.146}$$

$$\widehat{N}_{l0} = \sum_{i=1,N} r_l^{(i)} \sqrt{m_o^{(i)2}c^2 + \pi_{free}^{(i)2}} + V_l, \tag{7.147}$$

with $V_l \equiv V_0 \sum_{i=1,N} r_l^{(i)}$ and V_0 denoting the time-component of a suitable *interaction potential* 4-vector $V_\mu \equiv (V_0, \mathbf{V})$. Here both $\widehat{p}_0$ and $\widehat{N}_{l0}$ are still expressed in terms of the free-particle canonical momentum $\pi_{free}^{(i)}$, while $\widehat{N}_{l0}$ differs from $\widehat{M}_{l0}$ because of the imposed kinematic constraint. Therefore, if interactions occur, their contribution show up only in $\widehat{p}_0$ and $\widehat{N}_{l0}$. In Ref.(3) the interaction-dependent Poincarè generators were called "Hamiltonians".

Nevertheless, for the validity of the transformation laws (7.138) as well as of the Lorentz conditions (7.135), Eqs.(7.144)-(7.147) are actually to be cast in terms of the 4-momenta of the interacting system $\pi^{(i)}$ (rather than the free particle momenta $\pi_{free}^{(i)}$). This means that in general $\pi^{(i)}$ should be considered as suitably-prescribed functions of $\pi_{free}^{(i)}$ and of the interaction 4-potential V_μ. For definiteness, let us consider the case of an isolated N-body system subject only to *binary interactions* occurring between point particles of the same system. In such a case the interaction potential 4-vector V_μ is necessarily *separable* (31), i.e., such that

$$V_\mu \equiv (V_0, \mathbf{V}) = \sum_{i=1,N} V_\mu^{(i)}, \tag{7.148}$$

with $V_\mu^{(i)}$ denoting the *i-th particle interaction potential 4-vector.* Then, assuming that $V_\mu^{(i)}$ are only position-dependent, in view of Eqs.(7.144) and (7.147), for each particle the canonical 4-momentum of interacting particles $\pi_\mu^{(i)}$ must depend linearly on $V_\mu^{(i)}$ and $\pi_{free,\mu}^{(i)}$, namely it takes the form

$$\pi_\mu^{(i)} = \pi_{free,\mu}^{(i)} - V_\mu^{(i)}, \tag{7.149}$$

which implies in turn that necessarily $\pi_0^{(i)} = \sqrt{m_o^{(i)2}c^2 + \left(\pi^{(i)} - \mathbf{V}^{(i)}\right)^2} + V_0^{(i)}$. As a consequence, Eqs.(7.144)-(7.147) are actually replaced with

$$\widehat{p}_0 = \sum_{i=1,N} p_0^{(i)} = \sum_{i=1,N} \sqrt{m_o^{(i)2}c^2 + \left(\pi^{(i)} - \mathbf{V}^{(i)}\right)^2} + V_0, \tag{7.150}$$

$$\widehat{p}_l = \sum_{i=1,N} \pi_l^{(i)}, \tag{7.151}$$

$$\widehat{M}_{lm} = \sum_{i=1,N} \left[r_l^{(i)}\pi_m^{(i)} - r_m^{(i)}\pi_l^{(i)}\right], \tag{7.152}$$

$$\widehat{N}_{l0} = \sum_{i=1,N} r_l^{(i)} \sqrt{m_o^{(i)2}c^2 + \left(\pi^{(i)} - \mathbf{V}^{(i)}\right)^2} + V_{0i}, \tag{7.153}$$

where (7.151) and (7.152) retain their free-particle form. In order that the Poincarè generators $\widehat{p}_0$ and $\widehat{p}_l$ commute [in accordance with Eqs.(7.135), with PBs now defined in terms of the constrained state $\mathbf{z}'$], then it follows necessarily that the 4-vectors $V_\mu^{(i)}$, for all $i = 1, N$, must be *local* functions of the 4-positions of the particles of the N-body system, namely

$$V_\mu^{(i)} = V_\mu^{(i)}(r_l^{(1)}, ...r_l^{(N)}). \tag{7.154}$$

The explicit proof of this statement is given below [see in particular the inequality (7.166)]. Hence, by construction, in the Dirac approach the "Hamiltonians" $\widehat{p}_0$ and $\widehat{N}_{l0}$ are *necessarily local functions* too. It follows that DGF *applies only to local Hamiltonian systems.*

It is worth noting that the same conclusion follows directly also from Dirac's claim (see the quote from his paper given above) that the N-body system should generate a dynamical system. In fact, in the customary language of analytical mechanics the latter is intended as a parameter-dependent map of the phase-space Γ_N onto itself. This means that, when the canonical state $\mathbf{z}$ is parametrized in terms of the coordinate time t, there should exist a homeomorphism in Γ_N of the form:

$$\widehat{\mathbf{z}}(t_o) = \mathbf{z}_0 \leftrightarrow \widehat{\mathbf{z}}(t), \tag{7.155}$$

with $t, t_o \in I \equiv \mathbb{R}$. Therefore, if the previous statement by Dirac is taken for granted, in view of the discussion reported above, it implies again that K_N must only be a local function of the system canonical state $\mathbf{z}$.

Let us now analyze, for comparison, the implications of the theory developed in the present work for EM-interacting N-body systems.

1. **Non-local Hamiltonian structure**

The first issue is related to the Hamiltonian structure $\{\mathbf{z}, K_N\}$ which characterizes these systems. According to the Corollary of THM.2 this should be identified with $\{\mathbf{x}, H_N\}$. Thus, the super-abundant canonical state $\mathbf{z}$ should coincide with $\mathbf{x} \equiv \left\{\mathbf{x}^{(i)}, i = 1, N\right\}$ spanning the $8N$-dimensional phase-space $\Gamma_N \equiv \Pi_{i=1,N}\Gamma_1^{(i)}$ (with $\Gamma_1^{(i)} \subset$

$\mathbb{R}^8$), while K_N is identified with the non-local Hamiltonian $H_N \equiv H_N(r, P, [r])$. In particular, $\mathbf{x}^{(i)} = \left(r^{(i)\mu}, P_\mu^{(i)}\right)$ is the corresponding i-th particle canonical state, with $r^{(i)\mu}$ and $P_\mu^{(i)}$ denoting respectively its position and canonical momentum 4-vectors. In the present case and in contrast to DGF, it follows that:

1. *Property #1*: the system Hamiltonian $K_N \equiv H_N$ must be necessarily *non-local*.

2. *Property #2*: the super-abundant state $\mathbf{x} \equiv \left\{\mathbf{x}^{(i)}, i = 1, N\right\}$ is canonical, namely it satisfies the canonical equations (7.1) in terms of the local PBs defined with respect to the same state. This occurs if $\mathbf{x}$ is considered unconstrained, i.e., when the i-th particle state $\mathbf{x}^{(i)}$ is identified with $\mathbf{x}^{(i)} = \left(r^{(i)\mu}, P_\mu^{(i)}\right)$. Hence, the Hamiltonian structure $\{\mathbf{x}, H_N\}$ holds in the unconstrained $8N$-dimensional phase-space $\Gamma_N \equiv \Pi_{i=1,N}\Gamma_1^{(i)}$, with $\Gamma_1^{(i)} \subset \mathbb{R}^8$. As a consequence, also the PBs (including the fundamental PBs (7.108) and the Lorentz conditions (7.135)) are defined with respect to the unconstrained state $\mathbf{x}$.

3. *Property #3*: only the extremal or extremant canonical curves $\mathbf{x}(s_{(1)}, ..s_{(N)})$ [see Eq.(7.105)] and not the varied functions satisfy identically the kinematic constraints (7.103) and (7.104).

4. *Property #4*: the 4-potential V_μ must be necessarily a non-local function. In particular, for binary EM interactions it must be a separable function, i.e., of the form (7.148):

$$V_\mu(r, [r]) = \sum_{i=1,N} V_\mu^{(i)}(r, [r]), \tag{7.156}$$

$$V_\mu^{(i)}(r, [r]) \equiv \frac{q^{(i)}}{c} A_{(eff)\mu}^{(tot)(i)}, \tag{7.157}$$

with $A_{(eff)\mu}^{(tot)(i)}$ being defined by Eq.(7.76).

2. **Conditions of validity of Dirac instant-form generators**

A further issue is related to the representation of the Poincarè generators and, in particular, to the instant form representation given by Dirac and usually adopted in the literature. The latter is based on Eqs.(7.144)-(7.147), rather than on Eqs.(7.150)-(7.153), in which $\pi^{(i)2}$ replaces $\pi_{free}^{(i)2}$ under the square root on the r.h.s. of Eqs.(7.144) and (7.147). On the other hand, based on the non-local Hamiltonian structure $\{\mathbf{x}, H_N\}$ expressed in terms of the unconstrained super-abundant canonical state $\mathbf{x}$ [see Eqs.(7.91), (7.92) and (7.93)], an admissible realization for $\left\{\widehat{p}^\mu, \widehat{M}^{\mu\nu}\right\}$ can be determined which holds for an arbitrary $N \geq 1$. In fact, it is immediate to verify that the phase-functions

$$\widehat{p}_\mu = \sum_{i=1,N} P_\mu^{(i)}, \tag{7.158}$$

$$\widehat{M}_{\mu\nu} = \sum_{i=1,N} \left[r_\mu^{(i)} P_\nu^{(i)} - r_\nu^{(i)} P_\mu^{(i)} \right], \tag{7.159}$$

satisfy identically the PBs (7.135) expressed in terms of the same state $\mathbf{x}$. In view of Property #2 this requires that the canonical generators $\left\{ \widehat{p}^\mu, \widehat{M}^{\mu\nu} \right\}$ defined by Eq.(7.159) *must be considered independent.* Hence, no constraints (on them) can possibly arise by imposing the validity of the PBs (7.135).

Another possibility, however, lies in the adoption of a constrained formulation. This is obtained imposing the kinematic constraints (7.103) and identifying the canonical state with the constrained vector $\mathbf{x}' \equiv \left(\mathbf{x}'^{(i)}, i = 1, N\right)$ with $\mathbf{x}'^{(i)} = (\mathbf{r}^{(i)}, \mathbf{P}^{(i)})$. Recalling again Eqs.(7.92) and (7.93), here $\mathbf{r}^{(i)}$ and $\mathbf{P}^{(i)}$ denote respectively the space parts of the corresponding i-th particle 4-vectors.

To carry out a detailed comparison with Dirac, let us consider in particular the instant-form representation of $\left\{ \widehat{p}^\mu, \widehat{M}^{\mu\nu} \right\}$ as given by Eq.(7.159). In such a case the generators are represented by the set, defined for $l, m = 1, 3$:

$$\widehat{p}_0 = \sum_{i=1,N} P_0^{(i)}, \tag{7.160}$$

$$\widehat{p}_l = \sum_{i=1,N} P_l^{(i)}, \tag{7.161}$$

$$\widehat{M}_{lm} = \sum_{i=1,N} \left[r_l^{(i)} P_m^{(i)} - r_m^{(i)} P_l^{(i)} \right], \tag{7.162}$$

$$\widehat{N}_{l0} = \sum_{i=1,N} \left[r_l^{(i)} P_0^{(i)} \right], \tag{7.163}$$

where $\widehat{p}_0$, $\widehat{p}_l$, $\widehat{M}_{lm}$ and $\widehat{N}_{l0}$ must all be considered as independent. The corresponding *constrained* (representation of the) *instant-form generators*, with $\widehat{p}_0$ and $\widehat{N}_{l0}$ expressed in terms of the constrained state $\mathbf{x}'$, become therefore

$$\widehat{p}_0|_{\mathbf{x}'} = \sum_{i=1,N} \left[\sqrt{m_o^{(i)2} c^2 + \left(\mathbf{P}^{(i)} - \frac{q^{(i)}}{c} \mathbf{A}_{(eff)}^{(tot)(i)} \right)^2} + \frac{q^{(i)}}{c} A_{(eff)0}^{(tot)(i)} \right], \tag{7.164}$$

$$\widehat{N}_{l0}\Big|_{\mathbf{x}'} = \sum_{i=1,N} \left[r_l^{(i)} \sqrt{m_o^{(i)2} c^2 + \left(\mathbf{P}^{(i)} - \frac{q^{(i)}}{c} \mathbf{A}_{(eff)}^{(tot)(i)} \right)^2} + \frac{q^{(i)}}{c} r_l^{(i)} A_{(eff)0}^{(tot)(i)} \right] \tag{7.165}$$

where we have represented $A_{(eff)\mu}^{(tot)(i)} \equiv \left(A_{(eff)0}^{(tot)(i)}, \mathbf{A}_{(eff)}^{(tot)(i)} \right)$, with $A_{(eff)\mu}^{(tot)(i)}$ being defined by Eq.(7.76) setting $\overline{A}_{\mu}^{(ext)(i)}=0$. A characteristic obvious feature of the constrained representations given above is that of the non-local dependences arising both from binary and self EM interactions. Analogous conclusions can be drawn also for the so-called point and front forms of the same generators. This implies that the Lorentz conditions (7.135), with PBs now defined in terms of the same constrained state $\mathbf{x}'$, are generally violated. Indeed, due to the non-locality of $A_{(eff)\mu}^{(tot)(i)}$ in this case the PBs-inequalities

$$[\widehat{p}_{0\mathbf{x}'},\widehat{p}_{l}]_{(\mathbf{x}')}\neq 0 \tag{7.166}$$

hold. Hence, *if - consistent with DGF - the validity of the Lorentz conditions (7.135) is imposed, the constrained forms of the Poincaré generators are manifestly not applicable to the treatment of EM-interacting N-body systems.*

Nevertheless, it is immediate to prove that $\left.\widehat{p}_{0}\right|_{\mathbf{x}'}$ indeed generates the correct evolution equations for the constrained state $\mathbf{x}'$. In fact, denoting by $[\cdot,\cdot]_{(\mathbf{x}')}$ the local PBs evaluated with respect to the constrained state $\mathbf{x}'$, let us determine by means of the PBs

$$\delta_{o}\xi(\mathbf{x})\equiv[\xi,F]_{(\mathbf{x}')}\,, \tag{7.167}$$

the infinitesimal transformations $\delta_{o}\mathbf{r}^{(i)}$ and $\delta_{o}\mathbf{P}^{(i)}$ generated by $F=dt\left.\widehat{p}_{0}\right|_{\mathbf{x}'}$. It is immediate to prove that these yield respectively

$$\delta_{o}\mathbf{r}^{(i)\mu} = dt\left[\mathbf{r}^{(i)\mu},\left.\widehat{p}_{0}\right|_{\mathbf{x}'}\right]_{(\mathbf{x}')}\equiv dt\mathbf{v}^{(i)}, \tag{7.168}$$

$$\delta_{o}\mathbf{P}^{(i)} = dt\left[\mathbf{P}^{(i)},\left.\widehat{p}_{0}\right|_{\mathbf{x}'}\right]_{(\mathbf{x}')}\equiv dt\frac{q^{(i)}}{c}\nabla_{(i)}A_{(eff)\nu}^{(tot)(i)}v^{(i)\nu}, \tag{7.169}$$

where the r.h.s. of both equations coincide identically with the spatial parts of the canonical Eqs.(7.99) and (7.100). Hence, as expected, *the constrained state x' is indeed canonical. Eqs.(7.168) and (7.169) provide the Hamiltonian equations for x' in terms of the non-local Hamiltonian function $\left.\widehat{p}_{0}\right|_{\mathbf{x}'}$.* Again, a necessary and sufficient condition for Eqs.(7.99) and (7.100) to hold is that the fundamental PBs

$$\left[\mathbf{r}^{(i)},\mathbf{r}^{(j)}\right]_{(\mathbf{x}_{0}')} = 0, \tag{7.170}$$

$$\left[\mathbf{P}^{(i)},\mathbf{P}^{(j)}\right]_{(\mathbf{x}_{0}')} = 0, \tag{7.171}$$

$$\left[\mathbf{r}^{(i)},\mathbf{P}^{(j)}\right]_{(\mathbf{x}_{0}')} = \delta^{ij}\mathbf{1}, \tag{7.172}$$

are identically satisfied for all $i,j=1,N$. Here $\mathbf{x}_{0}'$ and $\mathbf{x}'$ are identified respectively with $\mathbf{x}_{o}'\equiv\widehat{\mathbf{x}}'(t_{o})$ and $\mathbf{x}'\equiv\widehat{\mathbf{x}}'(t)=\mathbf{x}_{o}'+d\mathbf{x}'$, with $d\mathbf{x}'\equiv\left(\delta_{o}\mathbf{x}^{(1)},..,\delta_{o}\mathbf{x}^{(N)}\right)$, while the previous PBs are evaluated with respect to the initial state $\mathbf{x}_{o}'$. Furthermore, also in this case the canonical initial history set $\{\widehat{\mathbf{x}}'\}_{t_{0}}$, to be defined in analogy with Eq.(7.101), is assumed prescribed.

7.10 Non-local generator formalism

A basic consequence of the previous considerations is that in the case of non-local phase-functions, such as H_N or $\left.\widehat{p}_0\right|_{\mathbf{x}'}$, the local transformation law (7.138) becomes inapplicable.

A suitably-modified formulation of DGF appropriate for the treatment of non-local phase-functions must therefore be developed. This can be immediately obtained. In fact, let us consider an arbitrary non-local function of the form $\xi = \xi(\mathbf{z}, [\mathbf{z}])$, with $\mathbf{z}$ and $[\mathbf{z}]$ denoting respectively local and non-local functional dependences with respect to the canonical state $\mathbf{z}$. Let us consider an arbitrary infinitesimal canonical transformation generated by F of the form $\mathbf{z} \rightarrow \mathbf{z}' = \mathbf{z} + [\mathbf{z}, F]$, with $\delta_o \mathbf{z}$ to be considered as infinitesimal (i.e., of $O(\Delta)$). Then, requiring that ξ is suitably smooth both with respect to $\mathbf{z}$ and $[\mathbf{z}]$, the corresponding infinitesimal variation of ξ can be approximated with

$$\delta\xi(\mathbf{z}, [\mathbf{z}]) \equiv [\xi(\mathbf{z} + \alpha\delta_o\mathbf{z}, [\mathbf{z} + \alpha\delta_o\mathbf{z}]) - \xi(\mathbf{z}, [\mathbf{z}])] \, [1 + O(\Delta)] \, , \tag{7.173}$$

$\delta\xi(\mathbf{z}, [\mathbf{z}])$ being the (Frechet) functional derivative of $\xi(\mathbf{z}, [\mathbf{z}])$, namely

$$\delta\xi(\mathbf{z}, [\mathbf{z}]) \equiv \lim_{\alpha\rightarrow 0} \frac{d}{d\alpha} \xi(\mathbf{z} + \alpha\delta_o\mathbf{z}, [\mathbf{z} + \alpha\delta_o\mathbf{z}]) \equiv \{\xi(\mathbf{z}, [\mathbf{z}]), F\} \, . \tag{7.174}$$

Here $\{\xi(\mathbf{z}, [\mathbf{z}]), F\}$ denotes the *non-local Poisson brackets* (NL-PBs) and generally also F can be considered a non-local function of the form $F(\mathbf{z}, [\mathbf{z}])$ [i.e., of a type analogous to ξ]. Such a definition reduces manifestly to (7.138) in case of local functions.

Let us prove that the transformation law (7.174) is indeed the correct one. To elucidate this point, let us consider the 4-scalar defined by the Dirac-delta $\xi(r, [r]) \equiv \delta\left(\widetilde{R}^{(i)\alpha}\widetilde{R}^{(i)}_{\alpha} - \sigma^2_{(i)}\right)$ entering the non-local Lagrangian and Hamiltonian functions in the EM self-interaction, where $\widetilde{R}^{(i)\alpha}$ denotes the bi-vector defined by Eq.(7.28). Let us consider, for example, the action of an arbitrary infinitesimal Lorentz transformation defined by $\delta_o r^{(i)\mu}$. In order that $\widetilde{R}^{(i)\alpha}\widetilde{R}^{(i)}_{\alpha}$ is left invariant by the transformation (Lorentz invariance) it must be

$$\delta\xi(r, [r]) \equiv \{\xi(r, [r]), F\} = 0, \tag{7.175}$$

with $F = -\widehat{p}^{\mu} a_{\mu}$. This means that the NL-PBs $\{\xi(r, [r]), F\}$ defined by Eq.(7.174), rather than the local PBs $[\xi(r, [r]), F]$, must vanish identically. In particular Eq.(7.174), contrary to the local variation (7.137), preserves the Lorentz invariance of 4-scalars and hence provides the correct transformations law. Hence, in particular, it follows that for an isolated N-body system with arbitrary $N > 1$:

$$\delta H_N(r, P, [r]) \equiv \{H_N(r, P, [r]), F\} = 0, \tag{7.176}$$

$$\delta \left.\widehat{p}_0\right|_{\mathbf{x}'} \equiv \left\{\left.\widehat{p}_0\right|_{\mathbf{x}'}, F\right\} = 0. \tag{7.177}$$

It is immediate to prove that Eq.(7.176) holds by construction for all Poincarè generators [see Eqs.(7.159) above], while - instead - generally

$$\delta_o H_N(r, P, [r]) \equiv [H_N(r, P, [r]), F] \neq 0, \tag{7.178}$$

$$\delta_o \left.\widehat{p}_0\right|_{\mathbf{x}'} \equiv \left[\left.\widehat{p}_0\right|_{\mathbf{x}'}, F\right]_{(\mathbf{x}')} \neq 0. \tag{7.179}$$

Hence, consistent with the results indicated above, we conclude that *the local transformation laws realized by the Lorentz conditions (7.135), which are a distinctive feature of DGF, become invalid in the case of non-local Hamiltonians.*

The *non-local generator formalism* is therefore formally achieved by imposing modified Lorentz conditions obtained from Eqs.(7.135), in which *the local PBs are replaced with the non-local PBs defined by Eq.(7.174).*

In particular, *the correct transformation laws for the constrained instant-form Poincarè generators* [see Eqs.(7.161),(7.162) and (7.164),(7.165)] *follow by imposing for the Hamiltonians* [see Eqs.(7.164) and (7.165)] *appropriate non-local transformation laws of the type (7.177), all defined with respect to the constrained state* $\mathbf{x}'$. Finally, it must be remarked that the non-local generator formalism does not affect the validity of the canonical equations of motion (7.168) and (7.169) as well as the fundamental PBs (7.172) indicated above for the constrained state $\mathbf{x}'$, which *remain unchanged.*

7.11 Counter-examples to the "no-interaction" theorem

An open problem in relativistic dynamics is related to the so-called *"no-interaction" theorem* due to Currie (Currie, 1963 (15)), derived by adopting the DGF, and in particular the instant form representation for the Poincarè generators (see previous Section) given in Ref.(3). According to this theorem, an isolated classical N-body system of mutually interacting particles which admits a Hamiltonian structure in which the coordinate variables of the individual particles coincide with the space parts 3-vectors of the particles 4-positions and the canonical equations of motion are Lorentz covariant, can only be realized by means of a collection of free particles. This requires, in particular that *"...it is impossible to set up a canonical theory of two interacting particles in which the individual particle positions are the space parts of 4-vectors"*. In other words, according to the theorem, it should be impossible to formulate - in terms of a Hamiltonian system - a covariant canonical theory for an isolated system of $N > 1$ classical particles subject to binary interactions (see also Ref.(20)). The validity of the theorem was confirmed by several other authors (see for example, Beard and Fong, 1969 (25), Kracklauer, 1976 (26), Martin and Sanz, 1978 (27), Mukunda and Sudarshan, 1981 (28), Balachandran *et al.*, 1982 (29)). Its original formulation obtained by Currie for the case of two interacting particles ($N = 2$) was subsequently extended to include the case $N = 3$ (Cannon and Jordan, 1964 (30)), first-class constraints (see Sudarshan and Mukunda, 1983 (31) and the corresponding Lagrangian proof given by Marmo *et al.*, 1984 (32)) and the treatment of curved space-time (De Bièvre, 1986 (33) and Li, 1989 (34)). Common assumptions to these approaches are that:

1. *Hypothesis #1:* Both DGF and the Dirac instant form realization of the Poincarè generators apply. In particular, the Poincarè generators in the instant form, corresponding to the constrained Hamiltonian structure $\{\mathbf{z}, K_N\}$, satisfy identically both to the commutation rules (7.135) and the kinematic constraints (7.103).

2. *Hypothesis #2:* K_N admits the Poincarè group of symmetry, i.e., it commutes with $\{F\}$.

3. *Hypothesis #3:* All particles, in a suitable proper-time interval, are not subject to the action of an external force (locally/globally isolated N-body system).

Nevertheless, the theorem has been long questioned (see for example Fronsdal, 1971 (17) and Komar, 1978-1979 (18, 19, 20, 21)). In particular, there remains the dilemma whether the "no-interaction" theorem actually applies at all for N-body systems subject only to non-local EM interactions. This refers in particular, to extended charged particles in the presence of binary and self EM forces. Another interesting question is whether restrictions placed by the "no-interaction" theorem actually exist for physically realizable classical systems. Several authors have advanced the conjecture that the limitations set by the Currie theorem might be avoided in the framework of constrained dynamics formulated adopting a super-abundant-variable canonical approach (see for example Komar, 1978 (20) and Marmo *et al.*, 1984 (32) and references indicated therein). In particular, to get a better understanding of interacting N-body systems, Todorov (35) and then Komar (18, 19, 20, 21) developed a manifestly covariant classical relativistic model for two particles, of an action-at-a-distance kind. In the Todorov-Komar model the dynamics is given in terms of two first-class constraints. An equivalent model was discovered by Droz-Vincent (36, 37) based on a two-time formulation of the classical relativistic dynamics. However, the precise identification of the Hamiltonian structure $\{\mathbf{z}, K_N\}$ pertaining to N-body systems subject to EM interactions has remained elusive to date.

Here we claim that counter-examples, escaping both the assumptions and the restrictions of the "no-interaction" theorem, can be achieved, based on the classical N-body system of extended charged particles formulated here. Starting from the Corollary to THM.2, the following theorem applies.

THM.4 - Standard Hamiltonian form of a locally-isolated 1-body system and a globally-isolated N-body system.

In validity of THM.2 and of the definitions given by Eqs.(7.73)-(7.78), the following propositions hold:

T4$_1$) The Hamiltonian structure $\{\mathbf{x}, H_N\}$ of the classical system formed by a single extended charged particle is preserved also in the particular case in which the external EM 4−potential is such that along the particle world-line $r^{(1)}(s_{(1)})$:

$$A_{\mu}^{(ext)}(r^{(1)}(s_{(1)})) = \begin{cases} \neq 0 & \forall s_{(1)} \in]-\infty, s_o] \\ 0 & \forall s_{(1)} \in]s_o, +\infty, [\end{cases} \tag{7.180}$$

(locally-isolated particle).

T4$_2$) The Hamiltonian structure $\{\mathbf{x}, H_N\}$ of the classical N-body system formed by extended charged particles is preserved also in the particular case in which the external EM 4−potential vanishes identically,

$$A_\mu^{(ext)}(r) \equiv 0 \tag{7.181}$$

(globally-isolated N-body system).

Proof - T4$_1$) The proof is an immediate consequence of the Corollary to THM.2. In fact in the absence of an external EM field, the effective EM 4-potential $A_{(eff)\mu}^{(tot)(1)}$ (see Eq.(7.76)) simply reduces to

$$A_{(eff)\mu}^{(tot)(1)} = 2\overline{A}_\mu^{(self)(1)}, \tag{7.182}$$

where, in view of the requirement (7.180), $\overline{A}_\mu^{(self)(1)}$ is non-vanishing also in the interval $]s_o, +\infty, [$. Hence, both the Lagrangian and Hamiltonian equations in standard form [see respectively Eqs.(7.59) and (7.88),(7.89)] are satisfied, with $H_{eff}^{(1)} \equiv H_{N,eff} \equiv H_N$ still defined by Eqs.(7.78) and (7.95). T4$_2$) The proof is similar. In this case, due to assumption (7.181), $A_{(eff)\mu}^{(tot)(i)}$ reduces to

$$A_{(eff)\mu}^{(tot)(i)} = 2\overline{A}_\mu^{(self)(i)} + \sum_{j=1,N,i\neq j} \overline{A}_{(eff)\mu}^{(bin)(ij)}. \tag{7.183}$$

Hence, also in this case both the Lagrangian and Hamiltonian equations in standard form still hold, with $H_{eff}^{(i)}$ and $H_{N,eff}$ defined by Eqs.(7.78) and (7.96).

Q.E.D.

It is clear that both propositions T4$_1$) and T4$_2$) indeed escape the "no interaction" theorem (avoiding also the limitations set by its assumptions #1-#4). In fact, concerning the Hamiltonian structure $\{\mathbf{x}, H_N\}$ associated to the classical N-body system of extended charged particles, from THM.4 it follows that:

- The effective Hamiltonian is a non-local function of the canonical state $\mathbf{x}$.
- The canonical particle equations of motion (7.90) satisfy the correct transformation laws with respect to the Poincarè group, since the non-local system Hamiltonian $H_N(r, P, [r])$ is by construction a Lorentz 4-scalar.
- The super-abundant canonical state $\mathbf{x} \equiv \left\{\mathbf{x}^{(i)}, i = 1, N\right\}$ is defined in terms of $\mathbf{x}^{(i)} \equiv \left(r^{(i)\mu}, P_\mu^{(i)}\right)_{(s_{(i)})}$, where $r^{(i)\mu}$ and $P_\mu^{(i)}$ are represented by Eqs.(7.92) and (7.93).
- The extremant curves $\left(\mathbf{x}^{(1)}(s_{(1)}), ...\mathbf{x}^{(N)}(s_{(N)})\right)$ solutions of Eqs.(7.127) satisfy identically the kinematic constraints (7.103) and (7.104). As a consequence, only

the space parts of the extremant 4-vectors $r^{(i)\mu}(s_{(1)})$ and $P_\mu^{(i)}(s_{(1)})$ are, for all $i = 1, N$, actually independent.

- In case of T4$_1$), the single-particle motion is non-inertial for all $s_{(1)} \in I \equiv \mathbb{R}$. Hence, the instant form of Dirac generators [see Eqs.(7.144) and (7.147)] becomes inapplicable even in the case of a 1-body system.

7.12 On the failure of the "no-interaction" theorem

The actual causes of the failure of the "no-interaction" theorem emerge clearly from the analysis of the conditions of validity of DGF and the Dirac instant-form generators. For systems of extended charged particles subject only to EM interactions the previous assumptions #1-#4 (common to all customary approaches (15, 25, 26, 27, 28, 29, 30, 31, 32, 33)) which characterize the underlying Hamiltonian structure $\{\mathbf{z}, K_N\}$ make it incompatible with the exact non-local Hamiltonian structure $\{\mathbf{x}, H_N\}$ determined here. In fact, in difference to $\{\mathbf{z}, K_N\}$, the Hamiltonian structure $\{\mathbf{x}, H_N\}$ is characterized by:

- Super-abundant canonical variables $\mathbf{x} \equiv \left\{\mathbf{x}^{(i)}, i = 1, N\right\}$, with $\mathbf{x}^{(i)} \equiv \left(r^{(i)\mu}, P_\mu^{(i)}\right)$ being the i-th particle canonical state.

- Extremant curves $\left(\mathbf{x}^{(1)}(s_{(1)}), ...\mathbf{x}^{(N)}(s_{(N)})\right)$ which satisfy identically the kinematic constraints discussed above. This is a characteristic property of the canonical extremant curves only. In fact, the same constraints are not satisfied by the super-abundant canonical state $\mathbf{x}$.

- Fundamental PBs (7.108) which are satisfied only by the unconstrained state $\mathbf{x}$. Hence, the non-local Hamiltonian structure $\{\mathbf{x}, H_N\}$ is warranted if all the canonical variables defining the state $\mathbf{x} \equiv \left\{\mathbf{x}^{(i)}, i = 1, N\right\}$ are considered independent. This means that, in order for the fundamental PBs (7.108) to be fulfilled, these constraints cannot be imposed "a priori" on the canonical state.

- Poincarè generators [see Eqs.(7.160)-(7.163)] which satisfy the commutation rules (7.135) when they are considered independent, as the super-abundant canonical variables $\mathbf{x} \equiv \left\{\mathbf{x}^{(i)}, i = 1, N\right\}$, and fulfilling the fundamental PBs. For this reason, the Poincarè generators are necessarily left unconstrained by imposing the validity of the same equations [i.e., Eqs.(7.135)].

- A *non-local* Hamiltonian of the form $H_N(r, P, [r])$. In particular, it follows that H_N for classical N-body systems of extended charged particles reduces to a local function only in the case of *a single isolated particle which exhibits inertial motion.* In view of THM.1 given in Ref.(7), this requires the external EM 4-potential acting on such a particle to vanish identically along the particle world-line, i.e., $A_\mu^{(ext)}(r(s_1)) = 0$ for all $s_1 \in I \equiv \mathbb{R}$.

In conclusion, contrary to the claim of the "no-interaction" theorem, a Lorentz covariant Hamiltonian formulation for the dynamics of N-body systems, with $N \geq 1$, actually exists also for mutually interacting charged particles subject to binary as well as self EM interactions. The result holds even in the presence of an external EM field, for extended classical particles described by the Hamiltonian structure $\{\mathbf{x}, H_N\}$ determined here.

One might conjecture that the validity of the "no-interaction" theorem could be restored by introducing a suitable asymptotic approximation for the N-body system dynamics. The latter is related, in particular, to the short delay-time and large-distance approximations (see Section 7.8), invoked here for the treatment of particle self and binary EM interactions. It is immediate to prove that also this route is necessarily unsuccessful. The reason lays in THM.3 and its consequences. In fact, as shown above, a Hamiltonian structure of the same type of $\{\mathbf{x}, H_N\}$ can be recovered for the asymptotic N-body equations of motion determined by the same theorem. This is identified with the set $\left\{\mathbf{x}, H_{N,eff}^{asym}\right\}$, where $H_{N,eff}^{asym} = \sum_{i=1,N} H_{eff,asym}^{(i)}$ and $H_{eff,asym}^{(i)}$ is given by Eq.(7.124). By construction, $\left\{\mathbf{x}, H_{N,eff}^{asym}\right\}$ inherits the same qualitative properties of the exact Hamiltonian structure $\{\mathbf{x}, H_N\}$. Therefore, in particular, in this approximation $\mathbf{x}$ satisfies the fundamental PBs (7.108) if it is unconstrained. In addition, since the same definition applies for the Poincarè generators and their representation in the instant form, the same conclusions on the validity of the "no-interaction" theorem follow.

7.13 Conclusions

A formidable open problem in classical mechanics is provided by the missing consistent Hamiltonian formulation for the dynamics of EM-interacting N-body systems. This critically affects both classical and quantum mechanics. In this investigation a solution to this fundamental issue has been reached exclusively within the framework of classical electrodynamics and special relativity. In particular, the Hamiltonian structure of classical N-body systems composed of EM-interacting finite-size charged particles has been explicitly determined and investigated.

Both local and non-local EM interactions have been retained. The former are due to externally-prescribed EM fields, while the latter include both binary and self EM interactions, both characterized by finite delay-time effects. Binary interactions occur between any two charges of the N-body systems, while self interactions ascribe to the so-called radiation-reaction phenomena due to action of the EM self-field on a finite-size particle. All of these contributions have been consistently dealt with in the derivation of the N-body dynamical equations of motion by means of a variational approach based on the hybrid synchronous Hamilton variational principle.

Both Lagrangian and Hamiltonian covariant differential equations have been obtained, which are intrinsically of delay-type. The same equations have also been proved to admit a representation in both standard Lagrangian and Hamiltonian forms, through the definition of effective non-local Lagrangian and Hamiltonian functions. The property of Hamilton equations of admitting a Poisson bracket representation has lead us to prove the existence of a non-local Hamiltonian structure $\{\mathbf{x}, H_N\}$ for the N-body system of EM-interacting particles. This has been shown to be determined by the non-local Hamiltonian function H_N and to hold for the superabundant canonical states $\mathbf{x}$. In particular the correct Hamiltonian equations of motion are obtained considering the same vector $\mathbf{x}$ as unconstrained, the relevant (kinematic) constraints being satisfied identically by the solution of the same equations.

A further interesting development concerns the asymptotic approximation determined for the Hamiltonian structure $\{\mathbf{x}, H_N\}$ of the full N-body problem. Here we have shown that consistent with the short delay-time and large-distance asymptotic orderings the latter can be preserved also by a suitable asymptotic Hamiltonian approximation. In particular, the perturbative expansion adopted here permits to retain consistently delay-time contributions, while preserving also the variational character and the standard Lagrangian and Hamiltonian forms of the N-body dynamical equations. As a basic consequence the very Hamiltonian structure of the N-body problem is warranted. This permits us to overcome the usual difficulties related to the adoption of non-variational and non-Hamiltonian approximations previously developed in the literature.

Two important applications of the theory have been pointed out.

The first one concerns the famous and widely cited (both in the context of classical and quantum mechanics) paper by Dirac (1949) on the generator formalism approach to the forms of the Poincarè generators for the inhomogeneous Lorentz group. Contrary to a widespread belief, we have found out that the Dirac approach is not valid in the case of N-body systems subject to retarded, i.e., non-local, interactions. In fact, the Lorentz conditions for the instant-form Poincarè generators are found to be satisfied only in the case of local Hamiltonians. Analogous conclusions can be drawn also for the so-called point and front-forms of the same generators. Due to the non-local character of the Hamiltonian structure $\{\mathbf{x}, H_N\}$ this means that the Dirac generator formalism expressed in terms of the essential (i.e., constrained) canonical state $\mathbf{x}'$ is not "*per se*" directly applicable to the treatment of EM-interacting N-body systems. However, as shown here, in the same variables its extension to non-local Hamiltonians can be readily achieved by suitably modifying the Lorentz conditions so to account for the non-local dependences of the Hamiltonian structure $\{\mathbf{x}, H_N\}$.

Second, the validity of the Currie "no-interaction" theorem, concerning the Hamiltonian description of the relativistic dynamics of isolated interacting particles, has been investigated. It has been proved that the set $\{\mathbf{x}, H_N\}$ violates the statements of the theorem. The cause of the failure of theorem (and its proof) lays precisely in the adoption of the Dirac generator formalism. Explicit counter-examples which overcome the limitations posed by the "no-interaction" theorem have been issued. Contrary to the claim of the "no-interaction" theorem, it has been demonstrated that a standard Hamiltonian formulation for the N-body system of charged particles subject to EM interactions can be consistently formulated.

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