ISAS - INTERNATIONAL SCHOOL FOR ADVANCED STUDIES

Thesis submitted for the degree of
Doctor Philosophiae

Singular Perturbations of the Laplacian and Connections with Models of Random Media

Candidate
Alessandro Teta

Supervisor
Prof. **G.F. Dell'Antonio**

Academic Year 1988/89

TRIESTE

SISSA - SCUOLA INTERNAZIONALE SUPERIORE STUDI AVANZATI
TRIESTE
Strada Costiera 11

Thesis submitted for the degree of
Doctor Philosophiae

Singular Perturbations of the Laplacian and Connections with Models of Random Media

Candidate
Alessandro Teta

Supervisor
Prof. **G.F. Dell'Antonio**

Academic Year 1988/89

Acknowledgments

I would like to thank Prof. G.F. Dell'Antonio (University of Roma), under whose patient guidance this thesis has been undertaken.
I am also indebted to Prof. R. Figari (University of Napoli) for his constant support and helpful suggestions.
Moreover I acknowledge Prof. S. Albeverio (Ruhr-Universität Bochum), Dr. M. Balzano (University of Cassino), Dr. J. Brasche (Ruhr-Universität Bochum) and Prof. G. Dal Maso (S.I.S.S.A. of Trieste) for helpful discussions on specific aspects of the work and Prof. S. Albeverio for the kind hospitality offered at the Ruhr-Universität Bochum, during the preparation of the thesis.
A special thanks is due to Prof. G. Giberti (University of Napoli) for having introduced me into the misteries of the world processing.

Contents

Introduction

The aim of this thesis is the study of a class of singular perturbations of the Laplacian in $L^2(R^3)$.
Roughly speaking with singular perturbations of the Laplacian we mean Schrödinger operators of the type $-\Delta + U(x)$, where the potential $U(x)$ is supported by a set of Lebesgue measure zero.
More precisely a singular perturbation of the Laplacian supported by a given set of Lebesgue measure zero can be characterized as a selfadjoint extension of the Laplacian restricted to smooth functions vanishing on the chosen set.
In different fields these operators were referred to with different denominations such as hamiltonians with δ-interactions, zero- range potentials, contact potentials etc..
In what follows we will give a description of such Schrödinger operators using a quadratic form method based on a renormalization technique. We will consider the cases in which the support of the potential is a discrete set of points, a regular curve or a regular surface, with a special emphasis on the second case which is not sufficiently studied in the literature.
Moreover we will study some applications of point interactions, i.e. interactions supported by a discrete set of points, to the description of some models of random media.

We start with some historical remarks.
The investigation of solid state physicists and nuclear physicists was focused, in the early days of quantum mechanics, on the search of solvable and realistic models for short range interactions (as opposed to the Coulomb interaction which was the only solvable, realistic long range interaction known).
The lack of a minimal length pushed toward the investigation of a zero-range interaction as a general model for short range interactions at low energies (when the wavelengths associated with the interacting particles are much bigger than the range of the interaction among them).
The first appearance in the physical literature of a Schrödinger operator with point interactions was due to Kronig and Penney ([43]), who considered a peri-

odic linear array of point interactions as a model for a one dimensional crystal.
The first applications of point interactions in three dimensions were due to Bethe and Peierls ([16]), Thomas ([72]) and Fermi ([23]) who studied point interactions as a realistic potential for the two-body nuclear interaction at low energy.
In subsequent works applications to the three-body problem in quantum mechanics, i.e. a three particle system with a two-body δ- interaction (see e.g. [49],[29],[71]) and to the many-body problem in quantum statistical mechanics (see e.g. [40]) were also studied.
As an aside it should be noticed that such N-body ($N \geq 3$) problem is significantly more complicated than the ones we will consider in this thesis. As a matter of fact a general definition of a perturbation of the Laplacian in $L^2(R^{3N})$ supported by the submanifold where the coordinates of at least two particles coincide is not yet available.
Point interactions (often called Fermi pseudopotentials) are by now widely used in many branches of physics (see e.g. [17],[20],[48],[64]).
Nevertheless it should be remarked that in almost all the applications one can find in the physical textbooks the hamiltonian with point interactions is only formally defined as

$$-\Delta + \sum_{i=1}^{N} \frac{1}{\alpha_i} \delta_{y_i}$$

where $y_i \in R^3$, δ_{y_i} is the Dirac measure in y_i and $-1/4\pi\alpha_i$ has the meaning of a scattering length.
As a consequence it is easily realized that all the computations (e.g. of the scattering data) can be performed using only the first order perturbation theory, i.e the Born approximation, since the subsequent orders give rise to divergent terms.
In particular there is the still persistent convinction in the physics community that point interactions are only a useful trick which cannot give rise to a well defined (i.e. selfadjoint) and non trivial (i.e. different from $-\Delta$) hamiltonian (see e.g. the remark in [40] pag. 282).
Nevertheless the difficulty arising from the above mentioned study of the three-body problem stimulated some efforts to give a rigorous mathematical meaning to such hamiltonians.
The first success was obtained in 1961 by Berezin and Faddeev ([15]).
They gave the rigorous definition of the Schrödinger operator with point interaction in the origin using the Krein's theory to characterize the selfadjoint extensions

of the Laplacian restricted to a class of smooth functions vanishing in a neighborhood of the origin.
By now many different techniques can be employed to give a rigorous definition, as selfadjoint operator in $L^2(R^3)$, of the Schrödinger operator $-\Delta_{\alpha,Y}$ with point interactions located at the set $Y = \{y_1, ..., y_N\}$, $y_j \in R^3$, with strength $\alpha = \{\alpha_1, ..., \alpha_N\}$, $\alpha_j \in R$ (see [7] for a complete account of the subject).
Perhaps the easiest way is to define it as the unique selfadjoint operator in $L^2(R^3)$ whose resolvent equals

$$(-\Delta_{\alpha,Y} - k^2)^{-1} = G_k + \sum_{j,l=1}^{N} [\, \Gamma_{\alpha,Y}(k)^{-1} \,]_{jl} \tilde{G}_k(\cdot - y_l) G_k(\cdot - y_j) \tag{.1}$$

where $k \in C$ with $Im\ k > 0$ (C is the complex plane), $\Gamma_{\alpha,Y}(k)$ is the $N \times N$ matrix

$$[\Gamma_{\alpha,Y}(k)]_{jl} = (\alpha_j - \frac{ik}{4\pi})\delta_{jl} - \tilde{G}_k(y_j - y_l) \tag{.2}$$

and

$$G_k = (-\Delta - k^2)^{-1} \tag{.3}$$

$$G_k(x - x') = (-\Delta - k^2)^{-1}(x - x') = \frac{e^{ik|x-x'|}}{4\pi|x - x'|} \tag{.4}$$

$$\tilde{G}_k(x) = \begin{cases} G_k(x) & x \neq 0 \\ 0 & x = 0 \end{cases} \tag{.5}$$

The real parameters $\alpha_1, ..., \alpha_N$ are related to the scattering length $a_{\alpha,Y}$ associated to $-\Delta_{\alpha,Y}$ via the formula (see [7] pag. 136)

$$a_{\alpha,Y} = -\frac{1}{4\pi} \sum_{j,l=1}^{N} [\, \Gamma_{\alpha,Y}(0) \,]_{jl}^{-1}$$

So if the interparticle distances are sufficiently large we can consider $-1/4\pi\alpha_j$ as the scattering length associated to the point interaction placed in y_j.
The resolvent given by (.1) is a simple expression (in fact a finite rank perturbation of the free resolvent) and so it can be easily handled to get information on

the operator $-\Delta_{\alpha,Y}$ (e.g. action, domain, spectrum etc.).
First one can show that the action of the operator on functions vanishing on an arbitrary neighborhood of Y reduces to the action of the Laplacian.
This means that $-\Delta_{\alpha,Y}$ really defines a hamiltonian with zero-range potentials located at Y.
Moreover, looking at the singularities of the free Green's function $G_k(x-x')$ appearing in (.1), one can see that the domain $D(-\Delta_{\alpha,Y})$ is not contained in the form domain of the Laplacian $H^1(R^3)$ ($H^m(R^n)$ indicates the standard Sobolev space of order m in R^n).
As a consequence we have that point interactions in three dimensions cannot be defined as small form perturbation of the Laplacian.
The same is true in two dimensions while in one dimension the situation is considerably simplified by the fact that the free Green's function belongs to $H^1(R)$.
In this last case the domain is made of elements of $D(-\Delta)$ satisfying a certain boundary condition at the points where the interaction is located and in fact point interactions can be defined as small form perturbation of the Laplacian.
In dimensions four the free Green's function is not square integrable and this prevents the construction of a non trivial perturbation of the Laplacian supported by a discrete set of points.
We remark that the above considerations can be generalized to the case of a perturbation supported by a manifold in R^n.
In fact if the codimension is greater than three the δ-interaction cannot be defined, for codimensions three or two we have the genuine δ- interaction (i.e. it is not a small form perturbation of the Laplacian) and for codimension one the δ-interaction merely corresponds to a boundary condition (i.e. it is a small form perturbation of the Laplacian).
In this thesis we always consider perturbations in $L^2(R^3)$ since they are particularly relevant in the physical applications.
Using the resolvent (.1) one can also investigate the spectral properties of $-\Delta_{\alpha,Y}$.
It is found that the spectrum doesn't exhibit pathologies: the non negative part of the spectrum is purely absolutely continuous while in the negative part there can be at most a finite number of eigenvalues of finite multiplicity.
In particular, exploiting the properties of the matrix $\Gamma_{\alpha,Y}(k)$, such eigenvalues and the corresponding eigenfunctions, together with all the relevant scattering quantities, can be explicitly computed.

Finally we mention that from (.1), for $|k|/inf_j \alpha_j \ll 1$ and $inf_j\ \alpha_j\ inf_{j\neq l}\ |y_j - y_l| \gg 1$, one can explicitly derive a complete low energy, small scattering length expansion which is free of divergences.
As we already mentioned, in the formal treatment of point interactions only the first order term of such expansion is available.
In recent years some problems, like the analysis of some kind of δ- shell interactions in quantum mechanics ([33],[11] and references therein), the motion of a quantum particle interacting with a polymer ([22]), the study of polymer measures in quantum field theory ([3] and references therein), some models of antenna in classical electrodynamics ([34],[74]), have motivated a certain interest in the rigorous construction of more general singular perturbations of the Laplacian.
In particular the existence of such operators has been proved, under some very weak conditions on the support of the potential, with a great variety of techniques.
Using non-standard analysis, which is a natural framework for justifying the formal manipulations usually involved by δ function computations, in [5] a proof is given for the existence of a perturbation of the Laplacian supported by a brownian path.
Other methods are used in [18], where the definition is obtained studying the non trivial selfadjoint extensions of the Laplacian restricted to smooth functions vanishing on the support of the perturbation, and in [4], where the singularly perturbed hamiltonian is given by resolvent limit of smooth approximating operators.
The latter approach is rather general and allows to obtain, as particular cases, perturbations supported by points, submanifold of R^n and brownian paths which are particularly useful in applications.
The specific case of a local δ-interaction (see discussion below) supported by a smooth curve seems to be only treated in [46], where the operator is characterized, via an application of embeddings theorems, in the space of Fourier transforms.
If the classical Dirichlet capacity of the support is different from zero then the definition of the operator can be also achieved using standard techniques like quadratic forms method or potential theory (see e.g. references given in [18]).
A more abstract result is given in [45], where the notion of singular bilinear form defining a perturbation of a positive and selfadjoint operator in a Hilbert space is introduced and conditions for the construction of the perturbed operator are given.

Somehow related questions are discussed in [67], where sufficient conditions for the existence of perturbations of a linear differential operator supported by submanifold of codimension greater than zero are given, and in [59], where existence and uniqueness of the solution of a Schrödinger type equation with a potential given by a distribution are proved.
It should be emphasized that, except for some special case like point interactions ([7]) or δ-sphere interactions ([11]), starting from the above general definitions it is difficult to get information about the properties of the operator, e.g. detailed structure of the domain, action of the operator, analysis of the spectrum, scattering theory etc..
In some cases it can also happen that the defined interaction has a non local character, in the sense that it is defined by a sort of generalized boundary condition connecting different points of the support of the interaction (see the discussion at the end of section 1.3).

Let us now give a survey of the material exposed in this thesis.

In chapter 1 we develop a new unified approach, based on the theory of quadratic forms and on a renormalization technique, for the description of Schrödinger operators with δ-like interactions supported by particular sets of Lebesgue measure zero in R^3.
More precisely given a set $\mathcal{E} \subset R^3$, where $\mathcal{E}$ can be a finite set of points, a regular curve or a regular surface, we define a quadratic form $F_{\mathcal{E}}$ satisfying the following properties :
i) it is closed and lower bounded,
ii) $D(F_{\mathcal{E}}) \supseteq D(F_0)$, where $D(F_0) = H^1(R^3)$ and $F_0(u,u) = \int_{R^3} |\nabla u|^2 dx$,
iii) $F_{\mathcal{E}}(u,v) = F_0(u,v)$ for any $u \in D(F_0)$ which vanishes in a neighborhood of $\mathcal{E}$ and for any $v \in D(F_0)$.
When these conditions are satisfied we say that $F_{\mathcal{E}}$ defines a perturbation of the Laplacian supported by $\mathcal{E}$ (cfr. definition given in [3],[5]).
Once the quadratic form is defined we provide a complete characterization of the domain and action of the selfadjoint operator associated to $F_{\mathcal{E}}$ and also an explicit formula for the resolvent.
Moreover it is clear from the construction that the interaction we are defining has a local character, being completely specified by an assigned function on $\mathcal{E}$ which

is a measure of the strength of the interaction.
In order to illustrate the idea of the construction of the quadratic form we start with the case in which $\mathcal{E}$ is a finite set of points (section 1.1).
As we have already mentioned, the domain of $-\Delta_{\alpha,Y}$ contains functions which do not belong to $H^1(R^3)$ and have singularities in $y_1, ..., y_N$ of the type $|\cdot - y_j|^{-1}$, $j = 1, ..., N$.
Using the analogy with electrostatics we can define, for each $u \in D(-\Delta_{\alpha,Y})$, N point charges $q_u = \{q_u^1, ..., q_u^1\}$ depending on the singularity of u in Y and the corresponding potential

$$\check{G}_{i\sqrt{\lambda}} q_u = \sum_{j=1}^{N} q_u^j G_{i\sqrt{\lambda}}(\cdot - y_j) \ , \quad for \ \ \lambda > 0.$$

Then each $u \in D(-\Delta_{\alpha,Y})$ has the property that $u - \check{G}_{i\sqrt{\lambda}} q_u$ has finite energy, i.e. belongs to $H^1(R^3)$.
This suggests that the quadratic form $F_{\alpha,Y}$, associated to $-\Delta_{\alpha,Y}$, can be decomposed into the sum $F_{\alpha,Y} = \mathcal{F}_Y^\lambda + \Phi_{\alpha,Y}^\lambda$, where $\mathcal{F}_Y^\lambda$ is essentially the energy associated to the regularized potential $u - \check{G}_{i\sqrt{\lambda}} q_u$ and $\Phi_{\alpha,Y}^\lambda$ is a sort of renormalized energy of the system of the point charges q_u located at Y.
In fact we will prove that $F_{\alpha,Y}$ really defines a (local) perturbation of the Laplacian supported by Y.
The same idea is then applied, in section 1.2, to the more interesting case of a perturbation of the Laplacian supported by a regular curve C.
One has only to replace the point charges q_u with a linear charge ξ_u distributed on C and consider the corresponding potential $\check{G}_{i\sqrt{\lambda}} \xi_u$ produced by ξ_u.
Again, for a given strength β , the resulting quadratic form is essentially the sum of the energy $\mathcal{F}_C^\lambda$ associated to $u - \check{G}_{i\sqrt{\lambda}} \xi_u$ and of an extra term $\Phi_{\beta,C}^\lambda$ due to the renormalized energy of ξ_u.
The proof that $F_{\beta,Y}$ defines a (local) perturbation of the Laplacian supported by C requires some work due to the fact that the space of the possible linear charges distributed on C is infinite dimensional.
Finally, in section 1.3, we will briefly consider the well known case of a perturbation supported by a regular surface (in this case the δ- interaction is shown to be described by a boundary condition on the surface).
We will show, by simple algebraic manipulations, that the corresponding quadratic form can be written in terms of a surface charge, in analogy with the previous

cases.
Moreover we discuss possible generalizations of our method in the direction of the results of [4].

Chapter 2 is devoted to the study of the spectrum of the Schrödinger operator $-\Delta_{\beta,C}$ with δ-interaction of strength β supported by the smooth curve C in R^3 of finite length.
Such analysis, at the best of our knowledge, have never been developed in the literature.
The methods used and the results obtained are a direct generalization of the work done for point interactions in [7].
Here the basic object of the analysis is the matrix $\Gamma_{\alpha,Y}(k)$ appearing in (.1), which completely characterizes the interaction.
In the case of a curve some technical complications arise from the fact that the basic object characterizing the interaction is now an unbounded operator, acting on the infinite dimensional space of the linear charges distributed on the curve.
In section 2.1 we show that the resolvent of $-\Delta_{\beta,C}$ is a smooth perturbation of the free resolvent.
Then, using the Weyl theorem and the limiting absorption principle, we obtain that the essential spectrum reduces to the absolutely continuous spectrum and coincides with the non negative real axis.
In section 2.2 we prove the absence of positive embedded eigenvalues and characterize the negative eigenvalues with the corresponding eigenfunctions.
Moreover we show that if there is a ground state than it is non degenerate and the corresponding eigenfunction is strictly positive.
The stationary scattering theory, with the explicit calculation of the scattering wave functions, the scattering amplitude and the scattering operator, is described in section 2.3, where also the connection with the time dependent scattering theory is outlined.
In section 2.4 we apply the preceding analysis to the special case of a perturbation of constant strength supported by a circle.
Using the expansion in Fourier series the model can be explicitly solved and additional information on the properties of the operator can be obtained (e.g. the behaviour when the radius is going to zero, the occurrence of bound states etc.).

In chapter 3 we are concerned with the study of two different kinds of applications of point interactions to the analysis of models of random media.
By a random medium we mean a macroscopic medium whose response to an external field can be only statistically given because the precise microscopic structure of the medium is not available.
Tipically such response is described by coefficients (e.g. thermal conductivity, dielectric tensor etc.) and/or boundary conditions (e.g. electrostatic of conductors, scattering of sound waves etc.) for the field equations.
As a consequence the description of the medium is naturally reduced, from a mathematical point of view, to the study of some kind of p.d.e. with random coefficients and/or with boundary conditions on randomly placed surfaces.
In the physical applications one is generally interested in the macroscopic behaviour of the medium. Then the main object of the mathematical theory is to describe the conditions under which the solutions of the field equations exhibit only a weak dependence on the specific microscopic realization of the medium (deterministic behaviour of the medium).
A typical result in this direction is the derivation of an effective equation, i.e. a deterministic equation which describes the behaviour of the system with high probability.
Another interesting question is the investigation of the fluctuations around the limit behaviour given by the effective equation, which are useful to estimate the deviation of the real medium from the effective medium.
The first kind of application of point interactions to the study of a random medium, given in sections 3.1, 3.2, concerns the low energy neutron scattering from liquids or amorphous substances ([64,[48]).
The interaction between the neutrons and nuclei of the target can be described with a good approximation by a point interaction with a prescribed scattering length.
Since the positions of the nuclei are only statistically known then the problem is reduced to the study of the following (formally written) Schrödinger equation

$$-\Delta\psi_N + \sum_{j=1}^{N} \frac{1}{\alpha_j}\delta_{y_j(\omega)}\,\psi_N = E\psi_N$$

where $-1/4\pi\alpha_j$ is the scattering length of the nucleus in $y_j(\omega)$ and ω is an element of some probability space.

So we have a model of random medium whose response is taken into account via the random coefficients $(1/\alpha_j)\delta_{y_j(\omega)}$ and one can ask under what conditions a deterministic behaviour appears.
Our result is that if the nuclei are independently and identically distributed and $0 < c_1 < |\alpha_j|/N < c_2 < +\infty$ then in the limit $N \to +\infty$ the solution of the above equation converges to the solution of the effective equation

$$-\Delta\psi + U\psi = E\psi$$

where the effective potential U represents the scattering length per unit volume of the system of the scatterers.
Moreover the fluctuations of ψ_N around the limit ψ can also be characterized.
The result was well known since long time by the nuclear physicists ([64]) on the basis of a Born approximation of the formal δ-interaction and then neglecting the multiple scattering effects.
The treatment we use here is not subject to these limitations since in the rigorous definition of point interactions any order of approximation in the Neumann series is properly taken into account and, in particular, the multiple scattering effects are not neglected.
This fact seems to have some non trivial consequences on the physical theory (see the discussion in section 3.2).
We observe that the convergence result can also be interpreted in the opposite direction, in the sense that any given reasonable potential for the Schrödinger equation can be reconstructed by many randomly distributed point interactions, whose strength and distribution law are uniquely determined by the potential itself.
In this sense point interactions can be considered as a sort of elementary potentials in quantum mechanics.
In the last two sections of the chapter we study the connections between point interactions and boundary value problems.
In particular in section 3.3 we consider the Schrödinger operator with δ-interaction supported by N spheres centered in $Y = \{y_1, ..., y_N\}$ of strength $\gamma = \{\gamma^1, ..., \gamma^N\}$.
As it is pointed out in section 1.3 such operator is nothing but the Laplacian defined on functions satisfying a boundary condition of the type

$$\frac{\partial u}{\partial n_j^+} - \frac{\partial u}{\partial n_j^-} = \gamma^j u$$

on the surface of each sphere, where $\partial/\partial n_j^+$ (resp. $\partial/\partial n_j^-$) indicates the derivative in the outward normal direction to the surface evaluated from the exterior (resp. the interior) of the surface.
When the radius r of the spheres goes to zero and if the strength is renormali zed according to $\gamma^j(r) = -(r + 4\pi\alpha_j r^2)^{-1}$, we prove the resolvent convergence to the Schrödinger operator with point interactions located at Y of strength $\alpha = \{\alpha_1, ..., \alpha_N\}$.
We observe that such result can be taken as a new definition of point interactions.
The treatment is then generalized in section 3.4 for the analysis of another model of random medium, in which randomness enters via boundary conditions imposed on many randomly placed obstacles.
In fact we consider the Laplacian with boundary conditions of the above type on the surface of N independently and identically distributed spheres with radius N^{-1} .
For $N \to +\infty$ and $\gamma_N^j = -N(1 + 4\pi\alpha_j)^{-1}$, $1 + 4\pi\alpha_j \neq 0$, we prove resolvent convergence to $-\Delta + U$, where again U is the scattering length per unit volume of the system of the obstacles.
The key ingredient of the proof, following the idea of [25], is the use of point interactions as a good approximation for the converging sequence of operators (see the discussion at the end of section 3.4).
In section 3.3, 3.4 it is also sketched how the above results can be extended to the exterior Robin (or third kind) boundary value problem for the Laplacian.
We stress that the sequence of boundary value problems under consideration is not uniformly bounded from below and moreover the effective limit potential can also have a negative part.
At the best of our knowledge this seems to be the first result on the asymptotic behaviour of such kind of boundary value problems.

Finally in the appendix we give the derivation of the time-dependent propagator associated to the Schrödinger operator with one point interaction in R^3.

Chapter 1

Quadratic forms for singular perturbations of the Laplacian

1.1 Point interactions

In the introduction we have defined the Schrödinger operator with point interactions via its resolvent. Here we give an alternative definition via the construction of the corresponding quadratic form.

Let $Y = \{y_1, \ldots, y_N\}$ be a set of N distinct points of R^3 and $\alpha = \{\alpha_1, \ldots, \alpha_N\}$, with $\alpha_j \in R, j = 1, \ldots, N$.
For each positive λ let us define the following quadratic form in $L^2(R^3)$

$$D(F_{\alpha,Y}) = \{u \in L^2(R^3) | \ \exists q_u \in R^N \quad s.t. \quad u - \breve{G}_{i\sqrt{\lambda}} q_u \in H^1(R^3)\} \tag{1.1}$$

$$F_{\alpha,Y}(u,u) = \mathcal{F}_Y^\lambda(u,u) + \Phi_{\alpha,Y}^\lambda(q_u, q_u) \tag{1.2}$$

where

$$\mathcal{F}_Y^\lambda(u,u) = \int_{R^3} |\nabla(u - \breve{G}_{i\sqrt{\lambda}} q_u)|^2 dx + \lambda \int_{R^3} (u - \breve{G}_{i\sqrt{\lambda}} q_u)^2 dx - \lambda \int_{R^3} u^2 dx \tag{1.3}$$

$$\Phi_{\alpha,Y}^\lambda(q_u, q_u) = \sum_{l,j=1}^{N} [\Gamma_{\alpha,Y}(i\sqrt{\lambda})]_{lj} q_u^l q_u^j \tag{1.4}$$

and $\breve{G}_k a$, $k \in C$ with $Im\ k > 0$, indicates the potential in R^3 produced by the charge distribution a

$$\breve{G}_k a(x) = \int G_k(x - x') a(x') \mu(dx') \qquad x \in R^3$$

Depending on the context, the measure μ can be the sum of N Dirac measures or the projection of the Lebesgue measure in R^3 on a smooth curve or surface.

It easy to check that $\Phi^{\lambda}_{\alpha,Y}$ can be obtained as a renormalized energy of the point charges q_u

$$\Phi^{\lambda}_{\alpha,Y}(q_u,q_u) = -\sum_{l=1}^{N} q_u^l \lim_{|x-y_l|\to 0}[\sum_{j=1}^{N} q_u^j G_{i\sqrt{\lambda}}(x-y_j) - \frac{q_u^l}{4\pi|x-y_l|}] + \sum_{l=1}^{N} \alpha_l (q_u^l)^2$$

Moreover the point charges q_u^j corresponding to a given $u \in L^2(R^3)$ are uniquely determined.
In fact for $q_u \neq q_u'$, from the definition (1.1), one would have $\breve{G}_{i\sqrt{\lambda}}(q_u - q_u') \in H^1(R^3)$, which is absurd.
In particular one has $q_u = 0$ for any $u \in H^1(R^3)$.
It is also possible to give an explicit formula for the point charges q_u^j associated to u

$$q_u^j = \frac{8\pi}{3} \lim_{r\to 0} \frac{r}{|B_r(y_j)|} \int_{B_r(y_j)} u\, dx$$

where $|B_r(y_j)|$ is the volume of the sphere $B_r(y_j)$ of radius r centered in y_j. The proof is easily obtained using the Hölder inequality and the continuous embedding of $H^1(R^3)$ into $L^6(R^3)$ (see e.g. [32]).

Remark. Notice that the domain $D(F_{\alpha,Y})$ is independent of $\lambda > 0$. Moreover for any $\lambda, \lambda' > 0$

$$\begin{aligned}
&\mathcal{F}_Y^{\lambda}(u,u) - \mathcal{F}_Y^{\lambda'}(u,u) = \\
&= \int_{R^3} [-\Delta(\breve{G}_{i\sqrt{\lambda'}} q_u - \breve{G}_{i\sqrt{\lambda}} q_u)](2u - \breve{G}_{i\sqrt{\lambda'}} q_u - \breve{G}_{i\sqrt{\lambda}} q_u) dx \quad + \\
&+ \int_{R^3} [\lambda(\breve{G}_{i\sqrt{\lambda}} q_u)^2 - \lambda'(\breve{G}_{i\sqrt{\lambda'}} q_u)^2 - 2\lambda u(\breve{G}_{i\sqrt{\lambda}} q_u) + 2\lambda' u(\breve{G}_{i\sqrt{\lambda'}} q_u)] dx \quad = \\
&= (\lambda' - \lambda) \int_{R^3} (\breve{G}_{i\sqrt{\lambda}} q_u)(\breve{G}_{i\sqrt{\lambda'}} q_u) dx = \sum_{l,j=1}^{N} q_u^l q_u^j [G_{i\sqrt{\lambda}}(y_l - y_j) - G_{i\sqrt{\lambda'}}(y_l - y_j)] = \\
&= -\Phi^{\lambda}_{\alpha,Y}(q_u,q_u) \; + \; \Phi^{\lambda'}_{\alpha,Y}(q_u,q_u)
\end{aligned}$$

where we have used the equation $-\Delta(\breve{G}_{i\sqrt{\lambda'}}q_u - \breve{G}_{i\sqrt{\lambda}}q_u) = \lambda\breve{G}_{i\sqrt{\lambda}}q_u - \lambda'\breve{G}_{i\sqrt{\lambda'}}q_u$, the resolvent identity and an integration by parts.
We conclude that $F_{\alpha,Y}$ is in fact independent of the choice of $\lambda > 0$; the positive constant λ has the only purpose to provide a regularization of the behaviour of the Green's function at infinity.

The following proposition shows that $F_{\alpha,Y}$ defines a perturbation of the Laplacian supported by Y

Proposition 1.1.1 *$F_{\alpha,Y}$ is a quadratic form in $L^2(R^3)$ closed and bounded below. Moreover*

$$D(F_{\alpha,Y}) \supset D(F_0) \tag{1.5}$$

$$F_{\alpha,Y}(u,v) = F_0(u,v) \tag{1.6}$$

for any $u, v \in D(F_0)$.

Proof. The existence of a lower bound is a consequence of the fact that ([7] pag. 116) $\Gamma_{\alpha,Y}(i\sqrt{\lambda})$ is a symmetric matrix whose eigenvalues are all strictly increasing in λ, so that there exists $\lambda_0(\alpha, Y) > 0$ such that $\Gamma_{\alpha,Y}(i\sqrt{\lambda})$ defines a scalar product in R^N for all $\lambda > \lambda_0(\alpha, Y)$. To prove that the form is closed it is more convenient to consider

$$F^{\lambda}_{\alpha,Y}(u,u) = F_{\alpha,Y}(u,u) + \lambda \int_{R^3} u^2 dx \tag{1.7}$$

for any $\lambda > \lambda_0(\alpha, Y)$. For any sequence $u_n \in D(F^{\lambda}_{\alpha,Y})$ converging to u in $L^2(R^3)$ and such that $\lim_{n,m} F^{\lambda}_{\alpha,Y}(u_n - u_m, u_n - u_m) = 0$ one has

$$\lim_{n,m} \|w_n - w_m\|_{H^1(R^3)} = 0 \tag{1.8}$$

$$\lim_{n,m} \Phi^{\lambda}_{\alpha,Y}(q_{u_n} - q_{u_m}, q_{u_n} - q_{u_m}) = 0 \tag{1.9}$$

where $w_n = u_n - \breve{G}_{i\sqrt{\lambda}}q_{u_n}$. Then there exist $w \in H^1(R^3)$ and $q \in R^N$ such that

$$\lim_{n} \|w_n - w\|_{H^1(R^3)} = 0 \tag{1.10}$$

$$\lim_n \|\breve{G}_{i\sqrt{\lambda}} q_{u_n} - \breve{G}_{i\sqrt{\lambda}} q\|_2 = 0 \tag{1.11}$$

(By $\|\cdot\|_p$, $0 < p \leq +\infty$, we indicate the L^p-norm in R^3).
Formulas (1.10),(1.11) and the uniqueness of the strong limit give $u = w + \breve{G}_{i\sqrt{\lambda}} q$, i.e. $u \in D(F^{\lambda}_{\alpha,Y})$, moreover

$$\lim_n F^{\lambda}_{\alpha,Y}(u - u_n, u - u_n) = 0 \tag{1.12}$$

and the closure of $F_{\alpha,Y}$ is proved. Finally the inclusion relation (1.5) and the equality (1.6) can be easily checked.

Q.E.D.

As it is well known, each quadratic form in a Hilbert space which is closed and bounded below uniquely defines a selfadjoint operator which is bounded below (see e. g. [42]).
In the present case, using the explicit form of $F_{\alpha,Y}$, it is not hard to reconstruct the domain and the action of the associated selfadjoint operator $-\Delta_{\alpha,Y}$

$$\begin{aligned} D(-\Delta_{\alpha,Y}) = \{u \in D(F_{\alpha,Y}) |\ & u - \breve{G}_{i\sqrt{\lambda}} q_u \in H^2(R^3)\ , \\ & (u - \breve{G}_{i\sqrt{\lambda}} q_u)|_Y = \Gamma_{\alpha,Y}(i\sqrt{\lambda}) q_u\} \end{aligned} \tag{1.13}$$

where $f|_A$ indicates the restriction of the function f to the set A and

$$(-\Delta_{\alpha,Y} + \lambda)u = (-\Delta + \lambda)(u - \breve{G}_{i\sqrt{\lambda}} q_u) \tag{1.14}$$

Moreover formula (.1) for the resolvent can be easily obtained. The proof of (1.13), (1.14), (.1) can be carried out along the same line as the proof of the propositions 1.2.4, 1.2.5 of the next section and is omitted here.

Remark. If one defines $r_j = |x - y_j|$,$\forall x \in R^3, j = 1, \ldots, N$, then it can be verified that the boundary condition satisfied by $u \in D(-\Delta_{\alpha,Y})$ at the points $y_1, \ldots, y_N$ can be written in the form extensively used in the literature (see [7],[70])

$$\lim_{r_j \to 0} [\frac{\partial (r_j u)}{\partial r_j} - 4\pi \alpha_j (r_j u)] = 0 \quad j = 1, \ldots, N.$$

Remark. We finally observe that a perturbation of the Laplacian supported by points in dimension two can be constructed following exactly the same line of the three dimensional case, the only difference being the logarithmic singularity of the two dimensional Green's function for $-\Delta + \lambda$. Formulas for the domain, the action and the resolvent of the associated operator can also be obtained.
The one dimensional case is quite easy in the sense that no renormalization is required for the energy of the point charges.
An analogous situation will occur for perturbation of the Laplacian supported by a surface in R^3.

1.2 Perturbations supported by curves

The construction of a perturbation of the Laplacian supported by a curve is more delicate. The essential reason is that the space of the linear charges distributed on the curve is infinite-dimensional, so that in the definition of the renormalized energy of the linear charge distribution one has to face problems of domain and closure.

Let C be a curve in R^3 of class C^1 and, for a chosen initial point and orientation, let $y = y(s)$, $s \in I$, be a parametric representation of C, where I can be a finite interval (closed curve) or the whole real line (infinite open curve).
Typically s will be the arc-length of C relative to the chosen initial point. Moreover we assume that there exists a positive constant ϵ_0 such that the following two conditions are satisfied

C- 1

$$|y(s) - y(s')| \geq |s - s'|(1 - c|s - s'|^\tau) \qquad whenever \ \ |s - s'| \leq \epsilon_0$$

where τ, c are positive constants satisfying $c\epsilon_0^\tau < 1$.

C- 2

$$|y(s) - y(s')| > \epsilon_0(1 - c\epsilon_0^\tau) + K \log \frac{|s - s'|}{\epsilon_0} \qquad whenever \ \ |s - s'| > \epsilon_0$$

where K is a positive constant.

Conditions C-1, C-2 guarantee in particular that the curve cannot have multiple points; moreover one can verify that if C-1, C-2 are fulfilled then

$$sup_{s\in I} \int_I e^{-\sqrt{\lambda}|y(s)-y(s')|} ds' \ < \ +\infty \tag{1.15}$$

for $\lambda > K^{-2}$.

As a preliminary step in the construction of the quadratic form we define

$$a_{i\sqrt{\lambda}}(s) = -\int_{|s-s'|>\epsilon} G_{i\sqrt{\lambda}}(y(s)-y(s'))ds' \; +$$

$$+\int_{|s-s'|<\epsilon}[\frac{1}{4\pi|s-s'|} - G_{i\sqrt{\lambda}}(y(s)-y(s'))]ds' - \frac{1}{2\pi}\log 2\epsilon \qquad (1.16)$$

for any fixed $\lambda > 0$ and $\epsilon < \epsilon_0$.
We observe that $a_{i\sqrt{\lambda}}(s)$ depends only on the geometry of C and it is a constant in the particular cases of a straight line and of a circumference of radius r

$$a^{s.l.}_{i\sqrt{\lambda}} = \frac{1}{2\pi}(\kappa + \log\frac{\sqrt{\lambda}}{2})$$

$$a^{circ.}_{i\sqrt{\lambda}} = \frac{1}{2\pi}\int_0^{\frac{\pi}{2}}[\frac{1}{t} - \frac{\exp(-2\sqrt{\lambda}r\sin t)}{\sin t}]dt - \frac{1}{2\pi}\log 2\pi r$$

where κ is the Euler's constant. Moreover one can verify that $a_{i\sqrt{\lambda}}(s)$ is in fact independent of the choice of $\epsilon < \epsilon_0$ and it is a continuous and bounded function of $s \in I$ for any $\lambda > 0$. We shall need the following technical lemma

Lemma 1.2.1

$$\lim_{\lambda\to+\infty} inf_{s\in I}\; a_{i\sqrt{\lambda}}(s) = +\infty \qquad (1.17)$$

Proof. The first integral in the r.h.s. of (1.16) can be estimated through C-2 ; one obtains the bound

$$\int_{|s-s'|>\epsilon} G_{i\sqrt{\lambda}}(y(s)-y(s'))ds' \; < \; \frac{1}{4\pi\epsilon_0(1-c\epsilon_0^\tau)}\; sup_{s\in I}\int_I e^{-\sqrt{\lambda}|y(s)-y(s')|}ds' \qquad (1.18)$$

Moreover using condition C-1 one has

$$\int_{|s-s'|<\epsilon}[\frac{1}{4\pi|s-s'|} - G_{i\sqrt{\lambda}}(y(s)-y(s'))]ds' \geq$$

$$\geq \int_{|s-s'|<\epsilon}\frac{1}{4\pi|s-s'|}\{1 - \frac{\exp[-\sqrt{\lambda}|s-s'|(1-c|s-s'|^\tau)]}{1-c|s-s'|^\tau}\}ds' \equiv$$

$$\equiv J(\sqrt{\lambda}) \qquad (1.19)$$

For λ sufficiently large, the last integral is positive and satisfies

$$\frac{dJ(\sqrt{\lambda})}{d\sqrt{\lambda}} = 2\int_0^{\epsilon} e^{-\sqrt{\lambda}\varsigma(1-c\varsigma^r)}d\varsigma \ > \ \frac{1}{\sqrt{\lambda}} \tag{1.20}$$

The proof of the lemma then follows from (1.15), (1.18) and (1.20).

Q.E.D.

We now introduce a class of admissible linear charge distributions on C together with an expression for the (renormalized) energy for such charge distributions.

For a given continuous, bounded from below and real valued function β defined on C and for each positive number λ let us define

$$D(\Phi^{\lambda}_{\beta,C}) = \{\xi \in L^2(I) |\ \Phi^{\lambda}_{\beta,C}(\xi,\xi) < +\infty\} \tag{1.21}$$

$$\begin{aligned}\Phi^{\lambda}_{\beta,C}(\xi,\xi) \ &= \ \frac{1}{2}\int_{I\times I}(\xi(s)-\xi(s'))^2 G_{i\sqrt{\lambda}}(y(s)-y(s'))dsds' + \\ &+ \int_I (\xi(s))^2(a_{i\sqrt{\lambda}}(s)+\beta(s))ds \end{aligned} \tag{1.22}$$

The following proposition shows that, for λ sufficiently large, (1.21) and (1.22) define a positive and closed quadratic form in $L^2(I)$.

Proposition 1.2.2 *There exists $\lambda_0(\beta,C) > 0$ such that, for any $\lambda > \lambda_0(\beta,C)$, $D(\Phi^{\lambda}_{\beta,C})$ is a Hilbert space w.r. to the scalar product $\Phi^{\lambda}_{\beta,C}(\cdot,\cdot)$.*

Proof. The existence of a $\lambda_0(\beta,C) > 0$ such that $\Phi^{\lambda}_{\beta,C}(\cdot,\cdot)$ defines a positive definite scalar product for $\lambda > \lambda_0(\beta,C)$ is a consequence of Lemma 1. In order to prove completeness one has only to mimic the Riesz-Fisher proof of the completeness of L^2. Given a Cauchy sequence ξ_n in $D(\Phi^{\lambda}_{\beta,C})$, it is sufficient to prove convergence for a subsequence. Pick a subsequence, still denoted by ξ_n, such that $\Phi^{\lambda}_{\beta,C}(\xi_n-\xi_{n+1},\xi_n-\xi_{n+1}) < 2^{-n}$. Then

$$\Theta(s) = \sum_{n=1}^{\infty} |\xi_n(s)-\xi_{n+1}(s)|$$

is the monotone limit of

$$\Theta_N(s) = \sum_{n=1}^{N} |\xi_n(s) - \xi_{n+1}(s)|$$

and by the monotone convergence theorem $\Theta \in L^2(I)$ and $\Theta < +\infty$ a.e.. Moreover $(\Theta_N(s) - \Theta_N(s'))^2 G_{i\sqrt{\lambda}}(y(s) - y(s'))$ converges a.e. in $I \times I$ to $(\Theta(s) - \Theta(s'))^2 G_{i\sqrt{\lambda}}(y(s) - y(s'))$ and

$$\Phi^{\lambda}_{\beta,C}(\Theta_N, \Theta_N) \leq \sum_{n=1}^{N} \Phi^{\lambda}_{\beta,C}(|\xi_{n+1} - \xi_n|, |\xi_{n+1} - \xi_n|) < const.$$

so by the Fatou lemma we get $\Theta \in D(\Phi^{\lambda}_{\beta,C})$. Thus $\xi_1 + \sum_{n=1}^{\infty}(\xi_{n+1} - \xi_n)$ is absolutely convergent to a sum ξ with $|\xi - \xi_1| \leq \Theta$, so that $\xi \in D(\Phi^{\lambda}_{\beta,C})$. It is now straightforward to show that $\lim_n \Phi^{\lambda}_{\beta,C}(\xi - \xi_n, \xi - \xi_n) = 0$.

Q.E.D.

By proposition 1.2.2 we get that $\Phi^{\lambda}_{\beta,C}$, for $\lambda > \lambda_0(\beta, C)$, defines a positive and selfadjoint operator $\Gamma_{\beta,C}(i\sqrt{\lambda})$, which acts on smooth ξ as follows

$$[\Gamma_{\beta,C}(i\sqrt{\lambda})\xi](s) = \int_I [\xi(s) - \xi(s')] G_{i\sqrt{\lambda}}(y(s) - y(s')) ds' + \xi(s)[\beta(s) + a_{i\sqrt{\lambda}}(s)] \tag{1.23}$$

Now we want to recall two useful properties of the potential

$$\breve{G}_{i\sqrt{\lambda}}\xi(x) = \int_I \xi(s) G_{i\sqrt{\lambda}}(x - y(s)) ds \quad x \in R^3 \tag{1.24}$$

produced by the linear charge $\xi \in D(\Phi^{\lambda}_{\beta,C})$.
We observe first that the map $\xi \mapsto \breve{G}_{i\sqrt{\lambda}}\xi$, $\lambda > \lambda_0(\beta, C)$, is a linear bounded map from $D(\Phi^{\lambda}_{\beta,C})$ to $L^2(R^3)$, and its norm converges to zero for $\lambda \to +\infty$.
The proof, based on an application of Fubini theorem, is straightforward.
Next it can be shown that $\breve{G}_{i\sqrt{\lambda}}\xi \notin H^1(R^3)$. We give here a sketch of the proof.

The regularity conditions C-1, C-2 allow us to define, for any $\delta > 0$ sufficiently small, a neighborhood of C

$$C_\delta = \{x \in R^3 |\ \exists!\ y(s_x) \in C \ \ s.t.\ \ x \ \textit{lies on the normal plane to } C \ \textit{in } y(s_x) \textit{ and } |x - y(s_x)| < \delta\} \tag{1.25}$$

Then an integration by parts yields

$$\int_{R^3\setminus C_\delta} |\nabla \breve{G}_{i\sqrt{\lambda}}\xi|^2 dx + \lambda \int_{R^3\setminus C_\delta} \breve{G}_{i\sqrt{\lambda}}(\xi^2)\ \breve{G}_{i\sqrt{\lambda}}1\ dx =$$
$$= -\frac{1}{2}\int_{I\times I} dsds'(\xi(s) - \xi(s'))^2 \int_{R^3\setminus C_\delta} dx\ \nabla_x G_{i\sqrt{\lambda}}(x - y(s)) \cdot \nabla_x G_{i\sqrt{\lambda}}(x - y(s')) +$$
$$+ \int_{\partial C_\delta} (\breve{G}_{i\sqrt{\lambda}}1 - \frac{1}{2\pi}\log\frac{1}{\delta})\frac{\partial \breve{G}_{i\sqrt{\lambda}}(\xi^2)}{\partial n} d\Sigma + \frac{1}{2\pi}\log\frac{1}{\delta}\int_{\partial C_\delta} \frac{\partial \breve{G}_{i\sqrt{\lambda}}(\xi^2)}{\partial n} d\Sigma \tag{1.26}$$

where $\breve{G}_{i\sqrt{\lambda}}1(x) = \int_I G_{i\sqrt{\lambda}}(x - y(s))ds$, $x \in R^3$. The second term in the l.h.s. of (1.26) remains bounded in the limit $\delta \to 0$ and the first two terms in the r.h.s. of (1.26) converge to $-\Phi^\lambda_{\beta,C}(\xi,\xi) + \int_I (\xi(s))^2\beta(s)ds$, which is finite by hypothesis.
The assertion is then proved, since the last term in (1.26) diverges logarithmically.
This procedure shows, in particular, that $\Phi^\lambda_{\beta,C}(\xi,\xi)$ can be considered as a renormalized energy of the linear charge distribution (cfr. introduction).

Now we have all the ingredients to define, for $\lambda > \lambda_0(\beta, C)$, the following quadratic form in $L^2(R^3)$

$$D(F_{\beta,C}) = \{u \in L^2(R^3) |\ \exists \xi_u \in D(\Phi^\lambda_{\beta,C})\ \ s.t.\ \ u - \breve{G}_{i\sqrt{\lambda}}\xi_u \in H^1(R^3)\} \tag{1.27}$$

$$F_{\beta,C}(u,u) = \mathcal{F}^\lambda_C(u,u) + \Phi^\lambda_{\beta,C}(\xi_u,\xi_u) \tag{1.28}$$

where

$$\mathcal{F}^\lambda_C(u,u) = \int_{R^3} |\nabla(u - \breve{G}_{i\sqrt{\lambda}}\xi_u)|^2 dx + \lambda\int_{R^3}(u - \breve{G}_{i\sqrt{\lambda}}\xi_u)^2 dx - \lambda\int_{R^3} u^2 dx \tag{1.29}$$

Given $u \in L^2(R^3)$ then the corresponding linear charge ξ_u is uniquely determined, in particular one has $\xi_u = 0$ for $u \in H^1(R^3)$ (cfr. section 1.1). Moreover reasoning as in remark at pag. 13 it is not hard to show that $F_{\beta,C}$ is in fact independent of the choice of $\lambda > 0$.

The following proposition shows that $F_{\beta,C}$ defines a perturbation of the Laplacian supported by C

Proposition 1.2.3 *$F_{\beta,C}$ is a closed quadratic form in $L^2(R^3)$ bounded below; one has*

$$D(F_{\beta,C}) \supset D(F_0) \tag{1.30}$$

$$F_{\beta,C}(u,v) = F_0(u,v) \tag{1.31}$$

for any $u, v \in D(F_0)$.

Proof. The existence of a lower bound for $F_{\beta,C}$ is a consequence of the positivity of $\Phi^{\lambda}_{\beta,C}$ for $\lambda > \lambda_0(\beta, C)$. In order to prove that it is closed we consider

$$F^{\lambda}_{\beta,C}(u,u) = F_{\beta,C}(u,u) + \lambda \int_{R^3} u^2 dx \tag{1.32}$$

for any $\lambda > \lambda_0(\beta, C)$. For any sequence $u_n \in D(F^{\lambda}_{\beta,C})$ converging to $u \in L^2(R^3)$ and such that $\lim_{n,m} F^{\lambda}_{\beta,C}(u_n - u_m, u_n - u_m) = 0$, one has

$$\lim_{n,m} \|z_n - z_m\|_{H^1(R^3)} = 0 \tag{1.33}$$

$$\lim_{n,m} \Phi^{\lambda}_{\beta,C}(\xi_{u_n} - \xi_{u_m}, \xi_{u_n} - \xi_{u_m}) = 0 \tag{1.34}$$

where $z_n = u_n - \check{G}_{i\sqrt{\lambda}}\xi_{u_n}$. Thus there exist $z \in H^1(R^3)$ and $\xi \in D(\Phi^{\lambda}_{\beta,C})$ such that

$$\lim_{n} \|z_n - z\|_{H^1(R^3)} = 0 \tag{1.35}$$

$$\lim_{n} \Phi^{\lambda}_{\beta,C}(\xi - \xi_{u_n}, \xi - \xi_{u_n}) = 0 \tag{1.36}$$

By the continuity of the potential of a linear charge and the uniqueness of the strong limit we get $u = z + \breve{G}_{i\sqrt{\lambda}}\xi$, which means $u \in D(F^{\lambda}_{\beta,C})$. Moreover by (1.35),(1.36) we have $\lim_n F^{\lambda}_{\beta,C}(u-u_n, u-u_n) = 0$ and the closure of $F^{\lambda}_{\beta,C}$ is proved. Finally the inclusion relation (1.30) and the equality (1.31) can be easily verified.

Q.E.D.

The selfadjoint operator $-\Delta_{\beta,C}$ associated to $F_{\beta,C}$ is by definition the Schrödinger operator with δ-interaction supported by C of strength β; its domain and action are completely characterized as follows

Proposition 1.2.4

$$\begin{aligned} D(-\Delta_{\beta,C}) \;=\; & \{u \in D(F_{\beta,C}) \mid \xi_u \in D(\Gamma_{\beta,C}(i\sqrt{\lambda}))\ ,\ u - \breve{G}_{i\sqrt{\lambda}}\xi_u \in H^2(R^3)\ , \\ & (u - \breve{G}_{i\sqrt{\lambda}}\xi_u)|_C = \Gamma_{\beta,C}(i\sqrt{\lambda})\xi_u\} \end{aligned} \tag{1.37}$$

$$(-\Delta_{\beta,C} + \lambda)u = (-\Delta + \lambda)(u - \breve{G}_{i\sqrt{\lambda}}\xi_u) \tag{1.38}$$

Proof. Let $u \in D(-\Delta_{\beta,C})$ then, by definition, there exists $g \in L^2(R^3)$ such that

$$F_{\beta,C}(u,v) = (v,g) \tag{1.39}$$

for any $v \in D(F_{\beta,C})$. (By $(.,.)$ we indicate the scalar product in $L^2(R^3)$). In particular for $v \in H^1(R^3)$ one has $\xi_v = 0$ and (1.39) becomes

$$\int_{R^3} \nabla v \cdot \nabla(u - \breve{G}_{i\sqrt{\lambda}}\xi_u)dx + \lambda \int_{R^3} v(u - \breve{G}_{i\sqrt{\lambda}}\xi_u)dx = \int_{R^3} v(g + \lambda u)dx \tag{1.40}$$

which gives $u - \breve{G}_{i\sqrt{\lambda}}\xi_u \in H^2(R^3)$. Thus an integration by parts yields

$$(-\Delta + \lambda)(u - \breve{G}_{i\sqrt{\lambda}}\xi_u) = (-\Delta_{\beta,C} + \lambda)u \tag{1.41}$$

For an arbitrary $v \in D(F_{\beta,C})$ equality (1.39) can be written as

$$\int_I \xi_v(s)(u - \breve{G}_{i\sqrt{\lambda}}\xi_u)(y(s)) = \Phi^{\lambda}_{\beta,C}(\xi_v, \xi_u) \tag{1.42}$$

where (1.40), (1.41) have been used and orders of integration have been interchanged. Equation (1.42) gives now $\xi_u \in D(\Gamma_{\beta,C}(i\sqrt{\lambda}))$ and $(\Gamma_{\beta,C}(i\sqrt{\lambda})\xi_u)(s) = (u - \breve{G}_{i\sqrt{\lambda}}\xi_u)(y(s))$. Conversely given u belonging to the r.h.s. of (1.37) it is easily checked, following the same line of reasoning, that $u \in D(-\Delta_{\beta,C})$ proving (1.38).

Q.E.D.

Remark. For $u \in D(-\Delta_{\beta,C})$, with a smooth ξ_u, let us denote by $\bar{u}(y(s),\delta)$ the mean value of u over the circle of (a sufficiently small) radius δ, centered in $y(s) \in C$ and orthogonal to C in $y(s)$; then one can define two continuous function on C

$$\phi_0(s) = 2\pi \lim_{\delta\to 0} \frac{\bar{u}(y(s),\delta)}{\log 1/\delta} \tag{1.43}$$

$$\phi_1(s) = \lim_{\delta\to 0}[\bar{u}(y(s),\delta) - \frac{\phi_0(s)}{2\pi}\log\frac{1}{\delta}] \tag{1.44}$$

and it can be verified that the boundary condition $(u - \breve{G}_{i\sqrt{\lambda}}\xi_u)(y(s)) = (\Gamma_{\beta,C}(i\sqrt{\lambda})\xi_u)(s)$ is equivalent to

$$\phi_1(s) = \alpha(s)\phi_0(s) \tag{1.45}$$

We remark that the last equation, for a fixed $s \in I$, essentially coincides with the boundary condition defining a point interaction in dimension two (see [7] pag. 98).

It is also possible to give explicitly the resolvent of $-\Delta_{\beta,C}$

Proposition 1.2.5 *For $\lambda > \lambda_0(\beta, C)$ and $g \in L^2(R^3)$ we have*

$$(-\Delta_{\beta,C} + \lambda)^{-1}g = G_{i\sqrt{\lambda}}g + \breve{G}_{i\sqrt{\lambda}}[\Gamma_{\beta,C}(i\sqrt{\lambda})^{-1}(G_{i\sqrt{\lambda}}g|_C)] \tag{1.46}$$

Proof. The r.h.s. of (1.46) defines a bounded linear operator from $L^2(R^3)$ onto $D(-\Delta_{\beta,C})$; using (1.38) one immediately proves the proposition.

Q.E.D.

In conclusion we observe that, starting from (1.46), one can investigate spectral properties of $-\Delta_{\beta,C}$ (such as the location of the point spectrum and the absence of the singular continuous spectrum) and study the scattering theory for the pair $(-\Delta_{\beta,C}, -\Delta)$. We will come back to these questions in the next chapter.

1.3 Perturbations supported by surfaces and generalizations

We start describing how the idea of the preceding sections can be applied to a simpler situation.
It is well known that a perturbation of the Laplacian supported by a regular closed surface S can be defined by the following quadratic form

$$D(F_{\gamma,S}) = H^1(R^3) \tag{1.47}$$

$$F_{\gamma,S}(u,u) = \int_{R^3} |\nabla u|^2 dx - \int_S \gamma u_S^2 \, d\Sigma \tag{1.48}$$

where γ is a smooth, real valued function defined on S and u_S denotes the trace of u on S. If one defines the surface charge σ_u associated to $u \in H^1(R^3)$

$$\sigma_u = \gamma u_S \tag{1.49}$$

and, for any $\lambda > 0$, the potential produced by σ_u

$$\breve{G}_{i\sqrt{\lambda}}\sigma_u(x) = \int_S \sigma_u(\varsigma) G_{i\sqrt{\lambda}}(x-\varsigma) d\Sigma(\varsigma) \quad x \in R^3 \tag{1.50}$$

then a simple calculation shows that $F_{\gamma,S}$ can be written in a form analogous to the previous cases

$$F_{\gamma,S}(u,u) = \mathcal{F}_S^\lambda(u,u) + \Phi_{\gamma,S}^\lambda(\sigma_u,\sigma_u) \tag{1.51}$$

where

$$\mathcal{F}_S^\lambda(u,u) = \int_{R^3} |\nabla(u - \breve{G}_{i\sqrt{\lambda}}\sigma_u)|^2 dx + \lambda \int_{R^3} (u - \breve{G}_{i\sqrt{\lambda}}\sigma_u)^2 dx - \lambda \int_{R^3} u^2 dx \tag{1.52}$$

$$\Phi_{\gamma,S}^\lambda(\sigma_u,\sigma_u) = \int_S \frac{\sigma_u^2}{\gamma} d\Sigma - \int_S \sigma_u(\breve{G}_{i\sqrt{\lambda}}\sigma_u) d\Sigma \tag{1.53}$$

$\Phi^{\lambda}_{\gamma,S}$ is clearly a bounded quadratic form in $L^2(S)$, thus it defines a bounded selfadjoint operator $\Gamma_{\gamma,S}(i\sqrt{\lambda})$ in $L^2(S)$.
Using the methods of the preceding sections one can reconstruct the domain and the action of the selfadjoint operator $-\Delta_{\gamma,S}$ defined by $F_{\gamma,S}$ and one can calculate the resolvent. The results are summarized below

$$D(-\Delta_{\gamma,S}) = \{u \in H^1(R^3) |\ u - \check{G}_{i\sqrt{\lambda}}\sigma_u \in H^2(R^3)\ , \\ \partial u/\partial n^+ - \partial u/\partial n^- = \sigma_u \ \ on\ \ S\} \tag{1.54}$$

$$(-\Delta_{\gamma,S} + \lambda)u = (-\Delta + \lambda)(u - \check{G}_{i\sqrt{\lambda}}\sigma_u) \tag{1.55}$$

$$(-\Delta_{\gamma,S} + \lambda)^{-1}g = G_{i\sqrt{\lambda}}g + \check{G}_{i\sqrt{\lambda}}[\Gamma_{\gamma,S}(i\sqrt{\lambda})^{-1}(G_{i\sqrt{\lambda}}g|_S)] \tag{1.56}$$

for any $g \in L^2(R^3)$.
It is a simple exercise to show that, when S is a sphere, (1.56) reduces to the resolvent given in [11].

In conclusion we want to compare our method for constructing singular perturbations of the Laplacian with a more general one developed in [4] and essentially based on resolvent limits of smooth approximating operators.
The claim is that our results can be generalized, with only minor changes, to the cases treated in [4], i.e. even in this more general situation we can explicitly write the quadratic form and characterize the domain and the action of the associated operator.
Following [4] we consider a Radon probability measure μ in R^3 (the generalization to arbitrary dimension is straightforward) and a positive μ-measurable function γ , bounded and bounded away of zero.
Moreover we assume

$$\int\int G_{i\sqrt{\lambda}}(x-y)\mu(dx)\mu(dy) < +\infty \tag{1.57}$$

$$\gamma^{-1}(x) - \int G_{i\sqrt{\lambda}}(x-y)\mu(dy) > \delta \quad x \in supp\ \mu \tag{1.58}$$

for some δ and λ positive.
Then (see propositions 2.3, 2.6 in [4]) the formulas

$$(\breve{G}_{i\sqrt{\lambda}}\sigma)(x) = \int G_{i\sqrt{\lambda}}(x-y)\sigma(y)\mu(dy) \tag{1.59}$$

$$(\Gamma_{\gamma,\mu}(i\sqrt{\lambda})\sigma)(x) = \gamma^{-1}(x)\sigma(x) - \int G_{i\sqrt{\lambda}}(x-y)\sigma(y)\mu(dy) \tag{1.60}$$

respectively define a continuous map from $L^\infty(\mu)$ to $L^2(R^3)$ and a continuous map from $L^\infty(\mu)$ onto $L^1(\mu)$, whose inverse exists and is continuous with norm less than δ^{-1}.
We observe that if μ is the restriction of the Lebesgue measure in R^3 to a regular surface S then the conditions (1.59),(1.60) are certainly satisfied.
Using (1.59),(1.60) we can now define the quadratic form in $L^2(R^3)$

$$D(F_{\gamma,\mu}) = \{u \in L^2(R^3) \mid \exists \sigma_u \in L^\infty(\mu) \ \ s.t. \ \ u - \breve{G}_{i\sqrt{\lambda}}\sigma_u \in H^1(R^3)\} \tag{1.61}$$

$$F_{\gamma,\mu}(u,u) = \mathcal{F}^\lambda_\mu(u,u) + \Phi^\lambda_{\gamma,\mu}(\sigma_u,\sigma_u) \tag{1.62}$$

where

$$\mathcal{F}^\lambda_\mu(u,u) = \int_{R^3} |\nabla(u - \breve{G}_{i\sqrt{\lambda}}\sigma_u)|^2 dx + \lambda \int_{R^3} (u - \breve{G}_{i\sqrt{\lambda}}\sigma_u)^2 dx - \lambda \int_{R^3} u^2 dx \tag{1.63}$$

$$\Phi^\lambda_{\gamma,\mu}(\sigma_u,\sigma_u) = \int \sigma_u(x)(\Gamma_{\gamma,\mu}(i\sqrt{\lambda})\sigma_u)(x)\mu(dx) \tag{1.64}$$

Clearly (1.64) defines a norm in $L^\infty(\mu)$ so that, following the line of reasoning of the preceding sections, it is not hard to see that $F_{\gamma,\mu}$ defines a perturbation of the Laplacian supported by *supp* μ of strength γ. Moreover the associated operator is given by

$$\begin{aligned} D(-\Delta_{\gamma,\mu}) = \{u \in D(F_{\gamma,\mu}) \mid u - \breve{G}_{i\sqrt{\lambda}}\sigma_u \in H^2(R^3)\ , \\ (u - \breve{G}_{i\sqrt{\lambda}}\sigma_u)|_{supp\mu} = \Gamma_{\gamma,\mu}(i\sqrt{\lambda})\sigma_u\} \end{aligned} \tag{1.65}$$

$$(-\Delta_{\gamma,\mu}+\lambda)u=(-\Delta+\lambda)(u-\breve{G}_{i\sqrt{\lambda}}\sigma_u) \tag{1.66}$$

If one wants to define a perturbation of the Laplacian supported by sets of lower codimension, e.g. points, regular curves or even very irregular sets like brownian paths, one has to introduce a weaker condition on μ

$$\int\int[\int G_{i\sqrt{\lambda}}(x-z)G_{i\sqrt{\lambda}}(z-y)dz]\mu(dx)\mu(dy)<+\infty \tag{1.67}$$

Again we observe that if μ is the restriction of the Lebesgue measure in R^3 to a regular curve C or if it is the Dirac measure supported by a discrete set then the condition (1.67) is certainly satisfied.
Under the assumption (1.67) it is still true that

$$(\breve{G}_{i\sqrt{\lambda}}\xi)(x)=\int G_{i\sqrt{\lambda}}(x-y)\xi(y)\mu(dy) \tag{1.68}$$

defines a continuous map from $L^\infty(\mu)$ to $L^2(R^3)$ (proposition 2.3 of [4]). In order to obtain a continuous and invertible operator analogous to (1.60) now one has to compensate the singular behaviour of the electrostatic potential of a charge distributed on *supp* μ.
This can be done by introducing a suitable non local interaction, i.e. given a positive, bounded and bounded away of zero μ-measurable function α, it can be proved that

$$(\Gamma_{\alpha,\mu}(i\sqrt{\lambda})\xi)(x)=\alpha(x)\xi(x)+\int[G_i(x-y)-G_{i\sqrt{\lambda}}(x-y)]\xi(y)\mu(dy) \tag{1.69}$$

defines a continuous map from $L^\infty(\mu)$ onto $L^1(\mu)$ whose inverse exists and it is continuous (proposition 3.2 in [4]).
Proceding as in the previous case the quadratic form defining a perturbation of the Laplacian supported by *supp* μ can now be constructed and the corresponding operator characterized.

Remark. It should be emphasized that the above construction leads to a non local δ-interaction.
If μ is the Dirac measure supported by a discrete set or if it is the restriction

of the Lebesgue measure in R^3 to the smooth curve considered in section 1.2, such non locality means that different points of the set are connected by the boundary condition defining the operator domain and this can be unsatisfactory from the physical point of view.
Such difficulty can be avoided for regular sets using our renormalization technique but for irregular sets like brownian paths, which are nowhere differentiable and recurrent, our method cannot be applied and the problem of finding an interaction which is local in a reasonable sense is still open.

Chapter 2

Spectral analysis for the Schrödinger operator with an interaction supported by a curve

2.1 The continuous spectrum

Throughout this chapter we will consider perturbations of the Laplacian in $L^2(R^3)$ supported by a curve C satisfying the conditions imposed in section 1.2 and, moreover, of finite length L (the case of infinite length with a sufficiently fast decaying of the interaction at infinity requires some additional technical complications).
We will find that, in spite of its rather intricate construction, the operator is well behaved, i.e. the non negative part of the spectrum is purely absolutely continuous while in the negative part there can be at most a finite number of eigenvalues of finite multiplicity.
Moreover it has a non trivial scattering theory which can be explicitly described.

We start with some preliminaries.
It is convenient for later use to introduce the analytic continuation of the operator $\Gamma_{\beta,C}(i\sqrt{\lambda})$ defined in section 1.2

$$\begin{aligned}\Gamma_{\beta,C}(k) &= \Gamma_{\beta,C}(i\sqrt{\lambda}) + (\tilde{G}_{i\sqrt{\lambda}} - \tilde{G}_k) = \\ &= [1 + (\tilde{G}_{i\sqrt{\lambda}} - \tilde{G}_k)\Gamma_{\beta,C}(i\sqrt{\lambda})^{-1}]\Gamma_{\beta,C}(i\sqrt{\lambda}) \end{aligned} \tag{2.1}$$

where $k \in \mathcal{C}$, $\lambda > \lambda_0(\beta, C)$ and $\tilde{G}_{i\sqrt{\lambda}} - \tilde{G}_k$ is an integral operator of Hilbert-Schimdt type in $L^2(I)$ with kernel given by

$$(\tilde{G}_{i\sqrt{\lambda}} - \tilde{G}_k)(s, s') = \frac{e^{-\sqrt{\lambda}|y(s)-y(s')|} - e^{ik|y(s)-y(s')|}}{4\pi|y(s) - y(s')|} \tag{2.2}$$

It is easly seen that the definition (2.1) is independent of the choice of $\lambda > \lambda_0(\beta, C)$; moreover $D(\Gamma_{\beta,C}(k)) = D(\Gamma_{\beta,C}(i\sqrt{\lambda}))$, $\Gamma_{\beta,C}(k)$ is closed and satisfies $\Gamma_{\beta,C}(k)^* = \Gamma_{\beta,C}(-\bar{k})$.

Now following the proposition 1.2.4 we can characterize the Schrödinger operator $-\Delta_{\beta,C}$ in $L^2(R^3)$ with δ-interaction supported by C of strength β in terms of a complex k

$$D(-\Delta_{\beta,C}) = \{u \in L^2(R^3) | \ \exists \xi_u \in D(\Gamma_{\beta,C}) \ , \ u - \breve{G}_k \xi_u \in H^2(R^3) \ , \\ (u - \breve{G}_k \xi_u)|_C = \Gamma_{\beta,C}(k)\xi_u\} \quad (2.3)$$

$$(-\Delta_{\beta,C} - k^2)u = (-\Delta - k^2)(u - \breve{G}_k \xi_u) \quad (2.4)$$

where $Im\ k > 0$.

Since we know that $\Gamma_{\beta,C}(k)$ is invertible at least for $Re\ k = 0$ and $Im\ k > \sqrt{\lambda_0(\beta, C)}$ then a direct application of the Fredholm alternative ([55] pag 201) gives us that $\Gamma_{\beta,C}(k)$ has a bounded inverse in $L^2(I)$ for each $k \in C \backslash \mathcal{A}$, where $\mathcal{A}$ is a discrete set.
Moreover the points of $\mathcal{A}$ in the upper half complex plane can lie only on the imaginary axis, between 0 and $\sqrt{\lambda_0(\beta, C)}$, otherwise, reasoning as for iii) of proposition 2.2.1 of the next section, we could exhibit a complex eigenvalue of $-\Delta_{\beta,C}$.
Using the invertibility of $\Gamma_{\beta,C}(k)$ it is also possible to write the resolvent of $-\Delta_{\beta,C}$ as function defined in the complex plane

$$(-\Delta_{\beta,C} - k^2)^{-1} g = G_k g + \breve{G}_k[\Gamma_{\beta,C}(k)^{-1}(G_k g|_C)] \quad (2.5)$$

where $k^2 \in \rho(-\Delta_{\beta,C})$, $Im\ k > 0$ and $g \in L^2(R^3)$.

Now we turn to the study of the continuous spectrum which is based, via the Weyl theorem and the limiting absorption principle, on the analysis of the resolvent.
The key result, given by the following proposition, shows that the resolvent of $-\Delta_{\beta,C}$ is a smooth perturbation of the free resolvent

Proposition 2.1.1 *The bounded operator in $L^2(R^3)$ defined by*

$$R^{\lambda}_{\beta,C} = (-\Delta_{\beta,C} + \lambda)^{-1} - (-\Delta + \lambda)^{-1} \tag{2.6}$$

is a trace class operator for $\lambda > \lambda_0(\beta, C)$.

Proof. The action of the operator $R^{\lambda}_{\beta,C}$ is explicitly given, for any $g \in L^2(R^3)$, by

$$R^{\lambda}_{\beta,C}\, g = \check{G}_k[\Gamma_{\beta,C}(i\sqrt{\lambda})^{-1}(G_{i\sqrt{\lambda}}g|_C)] \tag{2.7}$$

Then the first step of the proof is to consider the approximating operator

$$(1_\delta R^{\lambda}_{\beta,C} 1_\delta)g = 1_\delta \check{G}[\Gamma_{\beta,C}(i\sqrt{\lambda})^{-1}(G_{i\sqrt{\lambda}}(1_\delta g))|_C] \tag{2.8}$$

where 1_δ is, for $\delta > 0$ sufficiently small, the characteristic function of a small neighborhood C_δ of the curve C (see section 1.2).
Clearly one has

$$R^{\lambda}_{\beta,C} = s - \lim_{\delta \to 0} 1_\delta R^{\lambda}_{\beta,C} 1_\delta \tag{2.9}$$

An application of the Fubini theorem shows that $1_\delta R^{\lambda}_{\beta,C} 1_\delta$ is an integral operator in $L^2(R^3)$ with integral kernel explicitly given by

$$(1_\delta R^{\lambda}_{\beta,C} 1_\delta)(x, x') = 1_\delta(x) \int_I ds G_{i\sqrt{\lambda}}(x - y(s))[\Gamma_{\beta,C}(i\sqrt{\lambda})^{-1}(1_\delta(x') G_{i\sqrt{\lambda}}(x' - \cdot))] \tag{2.10}$$

Our next point is to show that (2.10) defines the integral kernel of a trace class operator. The proof is based on the application of a standard criterion which is recalled here for the convenience of the reader (see e.g. the lemma in [57] , pag. 65)

Let μ be a Baire measure on a locally compact Hausdorff space X and let K be a continuous function on $X \times X$. Suppose that

$$\int \phi(x)\phi(y)K(x, y)d\mu(x)d\mu(y) \geq 0$$

for any ϕ continuous with compact support and

$$\int K(x,x)d\mu(x) < +\infty$$

Then there is a trace class operator A on $L^2(X, d\mu)$ with integral kernel K . Moreover

$$Tr(A) = \int K(x,x)d\mu(x)$$

Using the explicit expression (2.10) it is easy to check continuity of the kernel in $R^3 \times R^3$ and positivity of the corresponding operator; furthermore we have the following uniform bound for the trace

$$\int_{R^3} dx(1_\delta R^\lambda_{\beta,C} 1_\delta)(x,x') \leq \frac{L}{8\pi\sqrt{\lambda}} \|\Gamma_{\beta,C}(i\sqrt{\lambda})^{-1}\|_{L^2(I)} \tag{2.11}$$

Applying the above criterion, we conclude that $1_\delta R^\lambda_{\beta,C} 1_\delta$ is a trace class operator.
Moreover, due to the uniform bound (2.11), there exists a subsequence of $1_\delta R^\lambda_{\beta,C} 1_\delta$ weakly convergent as $\delta \to 0$ in the topology of the trace and this, together with (1.9), implies our thesis.

Q.E.D.

We are now in position to characterize the continuous spectrum of $-\Delta_{\beta,C}$

Proposition 2.1.2

$$\sigma_{ess}(-\Delta_{\beta,C}) = \sigma_{ac}(-\Delta_{\beta,C}) = [0, +\infty) \tag{2.12}$$

$$\sigma_{sing}(-\Delta_{\beta,C}) = \emptyset \tag{2.13}$$

Proof. The Weyl theorem on the stability of the essential spectrum (see e.g. [58] pag. 112) asserts that if A and B are selfadjoint operators and the difference of their resolvents is compact than $\sigma_{ess}(A) = \sigma_{ess}(B)$.
In our case, taking $A = -\Delta_{\beta,C}$, $B = -\Delta$ and using the proposition 2.1.1,

we get $\sigma_{ess}(-\Delta_{\beta,C}) = \sigma_{ess}(-\Delta) = [0, +\infty)$.
The basic tool for proving the absence of the singular continuous spectrum is the limiting absorption principle.
A simplified version of the principle is the following (see e.g. [58] pag. 139)

Let A be a selfadjoint operator and (a, b) a bounded interval of R. If $(\phi, (A - E)^{-1}\phi)$ is bounded, as function of $E \in \mathcal{C}$, on $M = \{x + i\epsilon \mid \epsilon \in (0, 1)$, $x \in (a, b)\}$ for each ϕ in a dense set than A has purely absolutely continuous spectrum in (a, b).

Using the explicit expression of the resolvent (2.5) and the above principle one immediately proves that $\sigma_{sing}(-\Delta_{\beta,C}) = \emptyset$ and this completes the proof.

Q.E.D.

2.2 The point spectrum

The result of the analysis of the point spectrum, summarized in the proposition below, shows a strict analogy with the corresponding characterization of eigenvalues and eigenfunctions of the Schrödinger operator with point interactions (see [7] pag 116).
Nevertheless in the present case the proofs cannot be carried out along the same line since, now, the basic object of the analysis, i.e. the operator $\Gamma_{\beta,C}$, acts on an infinite-dimensional space and is unbounded.

Proposition 2.2.1

i) $\sigma_p(-\Delta_{\beta,C}) \subset (-\infty, 0)$

ii) $-\Delta_{\beta,C}$ *has a finite number of eigenvalues counting multiplicity and for* $Im\ k > 0$ *it results*

$k^2 \in \sigma_p(-\Delta_{\beta,C})$ *iff zero is an eigenvalue of* $\Gamma_{\beta,C}(k)$

and the multiplicity of the eigenvalue k^2 *equals the multiplicity of the eigenvalue zero of* $\Gamma_{\beta,C}(k)$.

iii) If $E_0 = k_0^2 < 0$ *is an eigenvalue of* $-\Delta_{\beta,C}$ *then the corresponding eigenfunctions* ψ_0 *are of the form*

$$\psi_0 = \breve{G}_{k_0}\xi_0 \ , \quad Im\ k_0 > 0 \tag{2.14}$$

where ξ_0 *is an eigenfunction of* $\Gamma_{\beta,C}(k_0)$ *corresponding to the eigenvalue zero.*

iv) If $-\Delta_{\beta,C}$ *has a ground state it is non degenerate and the correspond ing eigenfunction can be chosen to be strictly positive.*

Proof. Assertion *i)* means that $-\Delta_{\beta,C}$ cannot have positive eigenvalues and the proof is essentially based on a unique continuation theorem for the Schrödinger equation (see e.g. the proof of theorem XIII.57 of [58]).
Suppose there exists $E > 0$ and $\psi \in D(-\Delta_{\beta,C})$ such that $-\Delta_{\beta,C}\psi = E\psi$.

Enclose C in a large ball $B_r = \{x \in R^3 \mid |x| < r\}$, $r > 0$. Then in $R^3 \setminus B_r$ the eigenvalue equation reads $-\Delta\psi = E\psi$.
Expanding ψ in spherical harmonics and reasoning as in [58] pag 225 one obtains $\psi = 0$ in $R^3 \setminus B_r$.
Now fix $x_0 \in R^3 \setminus B_r$, $y \in B_r \setminus C$ and a smooth curve γ of length L_γ joining x_0 and y and such that $dist\ (\gamma, C) > 0$.
Choose an integer n and r_0, $0 < r_0 < \frac{1}{3}\ dist\ (\gamma, C)$, so that $\frac{1}{2}r_0 n \geq L_\gamma > \frac{1}{2}r_0(n-1)$ and $x_1, ..., x_n = y$ on γ ; then the length of γ between x_i and x_{i-1} , $i = 1, ..., n-1$, is $\leq \frac{1}{2}r_0$. In particular $|x_i - x_{i-1}| \leq \frac{1}{2}r_0$.
If we cover γ with balls $B_{r_0}(x_i)$ of radius r_0 centered in x_i , $i = 0, ..., n$, then the proof of assertion $i)$ follows from the following fact (see [58] pag 243): $\psi = 0$ in a small neighborhood of x_i and $|\Delta\psi| \leq E|\psi|$ in $B_{r_0}(x_i)$ imply $\psi = 0$ in $B_{r_0}(x_i)$.

We now prove assertions $ii)$ and $iii)$. Let $\xi_0 \in D(\Gamma_{\beta,C})$ be a solution of

$$\Gamma_{\beta,C}(k_0)\xi_0 = 0 \tag{2.15}$$

for some $E = k_0^2 < 0$, $Im\ k_0 > 0$.
Then one has $\psi_0 = \breve{G}_{k_0}\xi_0 \in D(-\Delta_{\beta,C})$; moreover

$$\begin{aligned} -\Delta_{\beta,C}\breve{G}_{k_0}\xi_0 &= (-\Delta - k^2)(\breve{G}_{k_0}\xi_0 - \breve{G}_k\xi_0) + k^2\breve{G}_k\xi_0 = \\ &= E_0\breve{G}_{k_0}\xi_0 \end{aligned} \tag{2.16}$$

where $k^2 \in \rho(-\Delta_{\beta,C})$, $Im\ k > 0$.
Hence ψ_0 is an eigenfunction of $-\Delta_{\beta,C}$ corresponding to the eigenvalue E_0.
On the other hand let $E_0 = k_0^2 < 0$, $Im\ k_0 > 0$, be an eigenvalue of $-\Delta_{\beta,C}$ and ψ_0 a corresponding eigenfunction. Then there exists $\xi_0 \in D(\Gamma_{\beta,C})$ such that

$$w = \psi_0 - \breve{G}_k\xi_0 \quad \in H^2(R^3) \ , \quad k^2 \in \rho(-\Delta_{\beta,C}) \ , \ Im\ k > 0 \tag{2.17}$$

$$w|_C = \Gamma_{\beta,C}(k)\xi_0 \tag{2.18}$$

Using (2.4) and the eigenvalue equation we obtain

$$(-\Delta - k^2)w = (k_0^2 - k^2)\psi_0 \tag{2.19}$$

Solving (2.19) for w and then applying $-\Delta - k_0^2$ we get

$$(-\Delta - k_0^2)w = (k_0^2 - k^2)\breve{G}_k \xi_0 \tag{2.20}$$

From the above equation and (2.17) we find that

$$\psi_0 = \breve{G}_{k_0} \xi_0 \tag{2.21}$$

Moreover using (2.18) and (2.21) we finally get

$$\Gamma_{\beta,C}(k_0)\xi_0 = 0 \tag{2.22}$$

Assertions *ii*) and *iii*) are then proved if we observe that, by the Fredholm theorem, there exists only a finite multiplicity of solutions of $\Gamma_{\beta,C}(k)\xi = 0$ for each $k \in \mathcal{A}$ and $\mathcal{A} \bigcap \{k \in \mathcal{C} \mid Re\ k = 0\ ,\ 0 < Im\ k \leq \sqrt{\lambda_0(\beta, C)}\}$ is finite.

In order to prove assertion *iv*) concerning the properties of the ground state it is useful to recall the following notions

A bounded operator A in $L^2(M, d\mu)$ (where $\langle M, \mu \rangle$ is a σ-finite measure space) is called positivity preserving if Af is positive whenever $f \in L^2(M, d\mu)$ is positive.
A is called positivity improving if Af is strictly positive whenever f is positive (see e.g. [58] pag. 201).
Moreover we say that the unit contraction operates on the quadratic form F in $L^2(M, d\mu)$ if
$u \in D(F)\ ,\ v = (0 \vee u) \wedge 1 \Rightarrow v \in D(F)\ ,\ F(v,v) \leq F(u,u)$
(see e.g. [30] pag. 5).

First we note that the unit contraction operates on the quadratic form $\Phi^{\lambda}_{\beta,C}$, $\lambda > \lambda_0(\beta, C)$ and so $\Gamma_{\beta,C}(i\sqrt{\lambda})^{-1}$ is positivity preserving for $\lambda > \lambda_0(\beta, C)$ ([30] theorem 1.4.1).

Taking into account that $G_{i\sqrt{\lambda}}$, $\lambda > 0$, is positivity improving we get that the whole resolvent of $-\Delta_{\beta,C}$ is positivity improving for $\lambda > \lambda_0(\beta, C) = -inf\ \sigma(-\Delta_{\beta,C})$.
As it is well known (see e.g. [58] pag. 204) this is a sufficient condition for the validity of assertion iv).

Q.E.D.

2.3 Stationary scattering theory

The aim of this section is to describe the stationary scattering theory for the pair $(-\Delta_{\beta,C}, -\Delta)$.

First we calculate the generalized eigenfunctions from the resolvent in the usual way

$$\psi_{\beta,C}(k\omega, x) = \lim_{\epsilon\to 0} \lim_{|x'|\to+\infty} 4\pi|x'|e^{-i(k+i\epsilon)|x'|}[-\Delta_{\beta,C} - (k+i\epsilon)^2]^{-1}(x,x') =$$
$$= e^{ik\omega x} + \int_I G_k(x - y(s))(\Gamma_{\beta,C}(k)^{-1}e^{ik\omega y(\cdot)})(s)ds \tag{2.23}$$

where $\omega \in S^2$ (S^2 is the unit sphere in R^3) is defined by $\omega = -x'/|x'|$, $k \notin \mathcal{A}$, $k \geq 0$, $x \in R^3$.
It easily seen that $\psi_{\beta,C}(k\omega, x)$ satisfies

$$-\Delta_{\beta,C}\psi_{\beta,C}(k\omega,\cdot) = k^2\psi_{\beta,C}(k\omega,\cdot) \tag{2.24}$$

in the distributional sense.
Using the generalized eigenfunctions one can compute all the relevant scattering quantities. The on-shell scattering amplitude is given by

$$f_{\beta,C}(k,\omega,\omega') = \lim_{|x|\to+\infty} |x|e^{-ik|x|}[\psi_{\beta,C}(k\omega',x') - e^{ik\omega' x}] =$$
$$= \frac{1}{4\pi}\int_I e^{-ik\omega y(s)}(\Gamma_{\beta,C}(k)^{-1}e^{ik\omega' y(\cdot)})(s)ds \tag{2.25}$$

where again $\omega \in S^2$ is defined by $\omega = x/|x|$ and $k \notin \mathcal{A}$, $k \geq 0$.
The unitary on-shell scattering operator $S_{\beta,C}(k)$ in $L^2(S^2)$ has integral kernel

$$S_{\beta,C}(k)(\omega,\omega') = \delta(\omega-\omega') - \frac{k}{i8\pi^2}\int_I e^{-ik\omega y(s)}(\Gamma_{\beta,C}(k)^{-1}e^{ik\omega' y(\cdot)})(s)ds \tag{2.26}$$

$k \notin \mathcal{A}$, $k \geq 0$, $\omega, \omega' \in S^2$.
Finally the low-energy limits of the scattering operator and the scattering amplitude are respectively given by

$$n - \lim_{k\to 0} S_{\beta,C}(k) = 1 \tag{2.27}$$

$$-\lim_{k\to 0} f_{\beta,C}(k,\omega,\omega') = -\frac{1}{4\pi}\int_I (\Gamma_{\beta,C}(0)^{-1}1)(s)ds \tag{2.28}$$

for $0 \notin \mathcal{A}$ and $1(\cdot)$ the unit function on I.
The limit (2.28) defines the scattering length $a_{\beta,C}$ associated to $-\Delta_{\beta,C}$.

Remark. We observe that the time-dependent scattering theory for $(-\Delta_{\beta,C}, -\Delta)$ can also be developed following the line of appendix E of [7]. In particular by the invariance principle (see e.g. [57] pag 27) and the trace class property of $R^{\lambda}_{\beta,C}$ we immediately get existence and asymptotic completeness of wave operators together with existence and unitariety of the associated scattering operator.
Moreover using the eigenfunction expansion of $-\Delta_{\beta,C}$ and standard techniques (abelian limits and so on) we can also establish the usual correspondence between time-dependent and time-independent scattering theory, i.e.

$$(\Omega_+(-\Delta_{\beta,C}, -\Delta)g)(x) = s - \lim_{R\to 0} \frac{1}{(2\pi)^{3/2}} \int_{|x|<R} k^2 dk \int_{S^2} d\omega \psi^+_{\beta,C}(k\omega, x)\hat{g}(k) \tag{2.29}$$

and analogously for $\Omega_-(-\Delta_{\beta,C}, -\Delta)$, where

$$\psi^-_{\beta,C}(k\omega, x) = \psi_{\beta,C}(k\omega, x) \ , \ \psi^+_{\beta,C}(k\omega, x) = \overline{\psi_{\beta,C}}(-k\omega, x) \tag{2.30}$$

and $\hat{g}$ is the Fourier transform of g.
Then it can be shown that the scattering operator is unitarily equivalent to the direct integral of the on-shell scattering operator $S_{\beta,C}(k)$ in $L^2(S^2)$ defined by (2.26).

2.4 An example of solvable model

Here we will briefly discuss the special case in which C is the circle given by $y = y(\varsigma) = (r\cos\varsigma, r\sin\varsigma, 0)$, $\varsigma \in [-\pi, \pi]$, and β is a constant.
This case can be interesting because it is an example of solvable model, in the sense that all the relevant quantities (e.g. eigenvalues, resonances, scattering amplitude etc.) can be explicitly computed and additional information on the properties of the operator can be obtained.

Using the expansion in Fourier series the quadratic form $\Phi^{\lambda}_{\beta,C}$ can be written as

$$\Phi^{\lambda}_{\beta,C}(\xi,\xi) = \sum_{m=-\infty}^{+\infty} |\xi_m|^2 [r^2 \int_{-\pi}^{\pi} (1-\cos 2m\varsigma) \frac{e^{-2\sqrt{\lambda} r \sin\varsigma}}{4\pi r \sin\varsigma} d\varsigma +$$
$$+ r\beta + \frac{r}{2\pi}\log\frac{1}{2\pi r} + \frac{r}{2\pi}\int_0^{\pi/2} (\frac{1}{\varsigma} - \frac{e^{-2\sqrt{\lambda} r \sin\varsigma}}{\sin\varsigma}) d\varsigma] \tag{2.31}$$

where, for each $h \in L^2([-\pi,\pi])$, we define $h_m = \frac{1}{\sqrt{2\pi}}\int_{-\pi}^{\pi} h(\varsigma) e^{-im\varsigma} d\varsigma$.
From (2.31) it results that in the space of the (discrete) Fourier transforms the operator $\Gamma_{\beta,C}(k)$ is diagonalized so it can be characterized by

$$D(\Gamma_{\beta,C}(k)) = \{\xi \in L^2(I) \mid \Gamma_{\beta,C}(k)_m \xi_m \in l^2\} \tag{2.32}$$

$$(\Gamma_{\beta,C}(k)\xi)(\phi) = \frac{1}{\sqrt{2\pi}} \sum_{m=-\infty}^{+\infty} \Gamma_{\beta,C}(k)_m \xi_m e^{im\phi} \tag{2.33}$$

where

$$\Gamma_{\beta,C}(k)_m = r\beta + \frac{r}{2\pi}[\log\frac{1}{2\pi r} + \int_0^{\pi/2} (\frac{1}{\varsigma} - \frac{e^{2ikr\sin\varsigma}}{\sin\varsigma}\cos 2m\varsigma) d\varsigma] \tag{2.34}$$

Using the above expression we can also write the resolvent as

$$(g, (-\Delta_{\beta,C} - k^2)^{-1} f) = (g, G_k f) + \; r^2 \sum_{m=-\infty}^{+\infty} \frac{(G_k g)^*_m (G_k f)_m}{\Gamma_{\beta,C}(k)_m} \tag{2.35}$$

where $k^2 \in \rho(-\Delta_{\beta,C})$, $Im\, k > 0$, $f, g \in L^2(R^3)$.

For small values of the radius only the term $m = 0$ is relevant in the series expansion in (2.35); moreover

$$\Gamma_{\beta,C}(k)_0 \simeq r\beta + \frac{r}{2\pi}\log\frac{1}{8r} - \frac{ikr^2}{2} \tag{2.36}$$

This means that in the limit $r \to 0$ we have resolvent convergence to point interaction of strength $\alpha \in R$ placed at the origin iff the parameter β is explicitly dependent on r according to the formula

$$\beta(r) = -\frac{1}{2\pi}\log\frac{1}{8r} + 2\pi r\alpha \tag{2.37}$$

As we will see in a moment the divergent term in (2.37) is determined by imposing that $-\Delta_{\beta,C}$ has a zero-energy resonance without bound states while the coefficient of α is simply the measure of the shrinking circle. We remark that exactly the same situation occurs if one tries to obtain a point interaction as the limit of an interaction supported by a shrinking sphere ([11]).

Another question which can be easily investigated is the occurrence of bound states. By proposition 2.2.1 and symmetry properties of $-\Delta_{\beta,C}$ we have that $E_0 < 0$ is the ground state iff it solves the equation

$$\beta + \frac{1}{2\pi}\log\frac{1}{8r} + \frac{1}{2\pi}\int_0^{\pi/2}\frac{1 - e^{-2\sqrt{-E_0}r\sin\varsigma}}{\sin\varsigma}d\varsigma = 0 \tag{2.38}$$

Now the integral in (2.38) is a function $F(x)$, where $x = 2\sqrt{-E_0}\, r$, satisfying

$$F(0) = 0\,, \quad \lim_{x\to+\infty} F(x) = +\infty\,, \quad F'(x) > 0 \quad \forall x \geq 0 \tag{2.39}$$

So we conclude that $-\Delta_{\beta,C}$ has at least one bound state iff

$$\beta + \frac{1}{2\pi}\log\frac{1}{8r} < 0 \tag{2.40}$$

(while $= 0$ leads to the occurrence of a zero-energy resonance).
In particular the condition (2.40) implies that for $\beta \geq 0$ (i.e. "repulsive"

interaction) we have at least one bound state for a sufficiently large r while for $\beta < 0$ (i.e. "attractive" interaction) we have no bound state for a sufficiently small r.

We conclude considering the scattering amplitude which, as usual, can be explicitly computed using the Fourier expansion

$$f_{\beta,C}(k,\omega,\omega') = \frac{r^2}{4\pi} \sum_{m=-\infty}^{+\infty} \frac{(e_k)^*_m (e_{k'})_m}{\Gamma_{\beta,C}(k)_m} \tag{2.41}$$

where $(e_k)_m = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} e^{ik\omega y(\varsigma) - im\varsigma} d\varsigma$.
By (2.41) we have that the scattering length is given by

$$a_{\beta,C} = -\frac{r}{2} \frac{1}{\beta + (1/2\pi)\log(1/8r)} \tag{2.42}$$

Formula (2.42) shows that for $r \to 0$ point interaction is obtained keeping the scattering length constant.
Finally it is an easy exercise to show that the scattering amplitude has a particularly simple form in the special case of an incident plane wave propagating along the z-axis (i.e. $\omega = (0,0)$, $\omega' = (\theta', \phi')$)

$$f_{\beta,C}(k,\theta') = \frac{r^2}{2\Gamma_{\beta,C}(k)_0} J_0(kr\sin\theta') \tag{2.43}$$

where $J_0(\cdot)$ is the Bessel function of zero order ([1]).

Chapter 3

Point interactions and models of random media

3.1 Limit theorems for many randomly distributed point interactions

In this chapter we turn to the study of some interesting properties of the simplest perturbations of the Laplacian, i.e. point interactions, in connection with the analysis of some models of random media.
In particular in this section we start giving a rigorous mathematical meaning to the idea that the effects on a quantum particle of a generic potential can be viewed as the sum of suitable normalized and distributed point interactions, whose strength and distribution are uniquely determined by the potential itself.
In the next section we will study the physical application of such mathematical result to the low energy neutron scattering (see introduction).

Let us introduce first some notation. Let V be any positive density distribution such that

$$V(x) \geq 0 \ , \ \int_{R^3} V(x)dx = 1 \ , \ \|V\|_2 < +\infty \tag{3.1}$$

In particular V belongs to the Rollnik class, i.e.

$$\|V\|_R^2 = \int_{R^3} \frac{|V(x)||V(y)|}{|x-y|^2} dxdy \leq c\|V\|_1^{1/3}\|V\|_2^{2/3} \tag{3.2}$$

(see for example [65]).
Now we consider N points $y_1, ..., y_N \in R^3$ identically and independently distributed according to the common density law $V(x)$. In other words

on the set of all configurations $Y^{(N)} = \{y_1, ..., y_N\}$ of N points of R^3 we introduce the probability measure $P^{(N)} = \{V(x)dx\}^{\otimes N}$.
Moreover for any given real function $\alpha(x)$, $x \in R^3$, such that $0 < c_1 < |\alpha|(x) < c_2 < +\infty$, continuous apart from a set of $V(x)dx$-measure zero we define $\alpha^{(N)} = \{\alpha_1, ..., \alpha_N\}$, where $\alpha_i = \alpha(y_i)$.
We notice that any potential $U \in L^1(R^3) \cap L^2(R^3)$ can be written as the ratio V/α of a density distribution in $L^2(R^3)$ with a function α satisfying the assumptions stated above. In fact it is enough to define

$$V = \frac{|U|(x)}{\|U\|_1} \tag{3.3}$$

$$\alpha = \frac{(sgn\ U)(x)}{\|U\|_1} \tag{3.4}$$

Consider the Schrödinger operator $-\Delta_{N\alpha^{(N)},Y^{(N)}}$ with point interactions in $Y^{(N)}$ of strength $N\alpha^{(N)}$. For $N\ inf_i \alpha_i/|k| \gg 1$ we can approximate the resolvent of $-\Delta_{N\alpha^{(N)},Y^{(N)}}$ with the corresponding Neumann series truncated at the first order (Born approximation)

$$(-\Delta_{N\alpha^{(N)},Y^{(N)}} - k^2)^{-1}(x, x') \simeq G_k(x - x') + \frac{1}{N}\sum_{i=1}^{N} \bar{G}_k(x - y_i)\frac{1}{\alpha_i}G_k(x' - y_i) \tag{3.5}$$

then by the law of the large numbers we have

$$\lim_N \frac{1}{N}\sum_{i=1}^{N} \bar{G}_k(x - y_i)\frac{1}{\alpha_i}G_k(x' - y_i) = \int_{R^3} \bar{G}_k(x - z)\frac{V(z)}{\alpha(z)}G_k(x' - z)dz \tag{3.6}$$

so at least in the above approximation we have

$$\lim_N(-\Delta_{N\alpha^{(N)},Y^{(N)}} - k^2)^{-1} = (-\Delta - \frac{V}{\alpha} - k^2)^{-1} \tag{3.7}$$

The claim is that the same convergence result is true up to any order in the Neumann series, i.e. we have the following proposition ([24])

Proposition 3.1.1 *$\forall \epsilon > 0$, $\forall f \in L^2(R^3)$ and $k = i\sqrt{\lambda}$, λ positive large enough*

$$\lim_N P^{(N)}(\{Y^{(N)} \mid \|(-\Delta_{N\alpha^{(N)},Y^{(N)}} - k^2)^{-1} f - (-\Delta - \frac{V}{\alpha} - k^2)^{-1} f\|_2 > \epsilon\}) = 0 \tag{3.8}$$

In other words we have strong resolvent convergence uniformly on a set of configurations $Y^{(N)}$ of measure increasing to 1 as N goes to infinity. Moreover we mention that the fluctuations around the limit operator can be completely characterized

Proposition 3.1.2 *For any $f, g \in L^2(R^3)$ and $k = i\sqrt{\lambda}$, λ positive large enough, the random variable*

$$N^{1/2}(f, [(-\Delta_{N\alpha^{(N)},Y^{(N)}} - k^2)^{-1} - (-\Delta - \frac{V}{\alpha} - k^2)^{-1}]g) \tag{3.9}$$

converges in distribution, when N goes to infinity, to the gaussian random variable of mean zero and variance

$$(A_k f\ A_k g, \frac{1}{\alpha^2}\ A_k f\ A_k g)_V - (A_k f, \frac{1}{\alpha}\ A_k g)_V^2 \tag{3.10}$$

where A_k is a short hand notation for $(-\Delta - \frac{V}{\alpha} - k^2)^{-1}$ and $(f, g)_V = \int_{R^3} f(x)g(x)V(x)dx$.

Here we will prove only proposition 3.1.1 while for the proof of proposition 3.1.2, which is rather long and technical, the reader is referred to [25], [26].

Proof of proposition 3.1.1 For any $f, g \in L^2(R^3)$ we can write (see (.1) in the introduction)

$$(f, (-\Delta_{N\alpha^{(N)},Y^{(N)}} + \lambda)^{-1} g) = (f, G_{i\sqrt{\lambda}} g) + \sum_{j=1}^{N} Q_N^j (G_{i\sqrt{\lambda}} f)(y_j) \tag{3.11}$$

where Q_N^j are the solutions of the linear system

$$(N\alpha_j + \frac{\sqrt{\lambda}}{4\pi})Q_N^j - \sum_{l=1}^{N} \tilde{G}_{i\sqrt{\lambda}}(y_l - y_j)Q_N^l = (G_{i\sqrt{\lambda}}g)(y_j), \quad j = 1, ..., N \quad (3.12)$$

The first step is the investigation of the uniform solvability (with respect to a sufficiently large N) of the above linear system.
We notice first that

$$\|\frac{\tilde{G}_{i\sqrt{\lambda}}}{N}\|_{H.S.} = \{\frac{1}{N^2}\sum_{l,j=1}^{N} \frac{\exp(-2\sqrt{\lambda}|y_l - y_j|)}{16\pi^2|y_l - y_j|^2}\}^{1/2} \quad (3.13)$$

where $\|A\|_{H.S.}$ denotes the Hilbert-Schmidt norm of a matrix A and where

$$(\tilde{G}_{i\sqrt{\lambda}})_{lj} = \tilde{G}_{i\sqrt{\lambda}}(y_l - y_j) \quad (3.14)$$

By the law of large numbers and by our assumptions on the density distribution V, we then have

$$\|\frac{\tilde{G}_{i\sqrt{\lambda}}}{N}\|_{H.S.} \leq c(\lambda)\|V\|_R \quad (3.15)$$

with $\lim_{\lambda\to+\infty} c(\lambda) = 0$, $\|V\|_R < +\infty$.
This implies that the matrix (see the definition (.2) in the introduction)

$$\frac{1}{N}\Gamma_{N\alpha^{(N)},Y^{(N)}}(i\sqrt{\lambda}) \quad (3.16)$$

has a bounded inverse, and then (3.12) is solvable, for an appropriate choice of λ and N sufficiently large.

In order to analyze the properties of the Q_N^l when N is very large, we consider the integral equation corresponding to (3.12) in the continuum

$$\alpha(x)Q(x) - \int_{R^3} G_{i\sqrt{\lambda}}(x-y)V(y)Q(y)dy = (G_{i\sqrt{\lambda}}g)(x) \quad (3.17)$$

which is solved, for λ large enough, by

$$\alpha(x)Q(x) = [(-\Delta - \frac{V}{\alpha} + \lambda)^{-1}g](x) \tag{3.18}$$

Notice that, by our assumption on V and α , the potential $-V/\alpha$ is Kato-small with respect to the Laplacian (see e.g. [56]) and so the hamiltonian $-\Delta - V/\alpha$ is selfadjoint on the operator domain of the Laplacian $H^2(R^3)$. Since $H^2(R^3)$ is embedded into the space of holder continuous function of order less than 1/2 (see e.g. [42]), in particular from (3.18) we get the continuity $V(x)dx$-almost everywhere of the function $Q(x)$.

Now, comparing (3.17) with (3.12), we are in position to estimate the difference $NQ_N^j - Q(y_j)$

$$\sum_{j=1}^{N} [(\alpha_l + \frac{\sqrt{\lambda}}{4\pi N})\delta_{lj} - (\frac{\tilde{G}_{i\sqrt{\lambda}}}{N})_{lj}][N^{1/2}Q_N^j - Q(y_j)/N^{1/2}] =$$
$$= (O^1)_l + (O^2)_l \tag{3.19}$$

where

$$(O^1)_l = \frac{1}{N^{3/2}}\sum_{j=1}^{N}\tilde{G}_{i\sqrt{\lambda}}(y_l - y_j)Q(y_j) - \frac{1}{N^{1/2}}(G_{i\sqrt{\lambda}}VQ)(y_l) \tag{3.20}$$

$$(O^2)_l = -\frac{\sqrt{\lambda}}{4\pi N^{3/2}}Q(y_l) \tag{3.21}$$

By direct computation

$$E(\sum_{l=1}^{N}(O^1)_l^2) = \frac{N-1}{N^2}(1, G_{i\sqrt{\lambda}}^2 VQ^2)_V - \frac{N-2}{N^2}\|G_{i\sqrt{\lambda}}VQ\|_V^2 \tag{3.22}$$

$$E(\sum_{l=1}^{N}(O^2)_l^2) = \frac{\lambda}{16\pi^2 N^2}\|Q\|_V^2 \tag{3.23}$$

where E means expectation value, $\|f\|_V^2 = (f,f)_V$ and $(G_{i\sqrt{\lambda}}^2 f)(x) = \int_{R^3}[G_{i\sqrt{\lambda}}(x-y)]^2 f(y)dy$. From (3.22) and (3.23) we infer

$$\lim_N E(\sum_{l=1}^N (O^n)_l^2) = 0 \qquad n = 1,2 \tag{3.24}$$

which together with the invertibility of the matrix $\frac{1}{N}\Gamma_{N\alpha^{(N)},Y^{(N)}}(i\sqrt{\lambda})$ gives us that for any $g \in L^2(R^3)$ and λ sufficiently large

$$\lim_N \{\frac{1}{N}\sum_{l=1}^N [NQ_N^l - Q(y_l)]^2\}^{1/2} = 0 \tag{3.25}$$

on a set of configurations $Y^{(N)}$ of measure going to 1 as N goes to $+\infty$.

The final step of the proof of the proposition follows now easily from

$$\begin{aligned}
&(f, [(-\Delta_{N\alpha^{(N)},Y^{(N)}} + \lambda)^{-1} - (-\Delta - \frac{V}{\alpha} + \lambda)^{-1}]g) = \\
&= \sum_{l=1}^N Q_N^l (G_{i\sqrt{\lambda}} f)(y_l) - (f, (-\Delta + \lambda)^{-1} \frac{V}{\alpha} (-\Delta - \frac{V}{\alpha} + \lambda)^{-1} g) = \\
&= \sum_{l=1}^N [Q_N^l - Q(y_l)/N](G_{i\sqrt{\lambda}} f)(y_l) + \frac{1}{N}\sum_{l=1}^N Q(y_l)(G_{i\sqrt{\lambda}} f)(y_l) - \\
&- \int_{R^3} (G_{i\sqrt{\lambda}} f)(x) Q(x) V(x) dx \leq \\
&\leq sup_x |G_{i\sqrt{\lambda}} f|(x) \{\frac{1}{N}\sum_{l=1}^N [NQ_N^l - Q(y_l)]^2\}^{1/2} + \\
&+ |\theta(Y^{(N)}) - E[\theta(Y^{(N)})]|
\end{aligned} \tag{3.26}$$

where

$$\theta(Y^{(N)}) = \frac{1}{N}\sum_{l=1}^N Q(y_l)(G_{i\sqrt{\lambda}} f)(y_l) \tag{3.27}$$

Again by direct computation

$$\begin{aligned}
&E|\theta - E\theta|^2 = \\
&= \frac{1}{N}\{\int_{R^3} (G_{i\sqrt{\lambda}} f)^2(x) Q^2(x) V(x) dx - [\int_{R^3} (G_{i\sqrt{\lambda}} f)(x) Q(x) V(x) dx]^2\} \leq \\
&\leq \frac{c}{N} sup_x |G_{i\sqrt{\lambda}} f|^2(x) \; [\|Q\|_V^2 + (1, |Q|)_V^2]
\end{aligned} \tag{3.28}$$

From (3.25)-(3.28), taking into account that $sup_x|G_{i\sqrt{\lambda}}f|(x) \leq c\|f\|_2$ and our assumptions on V, we conclude that for every $g \in L^2(R^3)$, on a set of configurations $Y^{(N)}$ of measure going to 1 as N goes to infinity

$$\lim_N \{ \frac{|(f, [(-\Delta_{N\alpha^{(N)},Y^{(N)}} + \lambda)^{-1} - (-\Delta - V/\alpha + \lambda)^{-1}]g)|}{\|f\|_2} \} = 0 \qquad (3.29)$$

concluding the proof of proposition 3.1.1.

Q.E.D.

We conclude this section with some remarks on possible generalizations of Proposition 3.1.1 to more singular situations.

Remark. First we note that the result should be true for a potential in the Rollnik class, which seems very natural in our context, being equivalent to require that the system of the charges $Q(x)$ has finite energy.
The proof requires some additional technicalities due to the fact that the operator domain of the limit hamiltonian doesn't coincide any more with the operator domain of the Laplacian, so that the charges $Q(x)$ might not be continuous.

Remark. Second one can consider the case in which the support of the limit potential is a set of Lebesgue measure zero (cfr. sections 1.2, 1.3).
The case of a regular surface is quite easy and the proof can be carried out just as in the case of the potential treated above.
The situation is slightly more complicated if the support of the potential is a regular curve since the limit Γ operator (cfr. section 1.2) is now unbounded. Moreover if β is the "potential on the curve" of finite length L then it turns out that the point interactions are to be distributed independently and uniformly on the curve, with strength given by $(N/2\pi L)\log N + N\beta(y_i)$ (i.e. the renormalization of the strength is different from the previous cases).
Finally we observe that, at least in principle, the result should be true even in the general case of singular perturbations of the Laplacian mentioned at the end of the section 1.3, the difficulties being only of technical nature.

3.2 Physical applications to neutron optics

As we pointed out in the introduction the investigation of point interactions begun in the early thirties among physicists attempting to find solvable and realistic models for short range interactions in non relativistic quantum mechanichs, mainly nuclear interactions at low energies (see e.g. [16],[23],[17] and [50] for the physical motivations).
In particular zero-range interactions are still widely used by the nuclear theorists in the study of the low energy neutron scattering from condensed matter, which is the central subject of the so called neutron optics ([41],[13],[37], [48],[61],[62],[64]).
Although the rigorous treatment of point interactions has by now a long history no attempt to exploit the entire power of a complete series expansion has been made.
The aim of this section, part of which will have an heuristic character, is to show how the correct definition of point interactions and the methods developed in the last section to study many randomly distributed point interactions can be used to introduce a rigorous model for such scattering processes.
The question might have some interest because, as we will see in a moment, this seems to be one of the rare occasions in which a rigorous mathematical treatment could clarify some aspects of the physical theory and make available a complete computational scheme.
We start describing briefly the present status of the physical theory under consideration here.
Neutron optics is the branch of nuclear physics devoted to the study of the coherent elastic scattering of slow neutrons from condensed matter, producing typical optical phenomena like reflection, rifraction, diffraction and so on.
The experimental evidence shows that, with a good approximation, the motion of the neutrons can be described by a "macroscopic" wave function Ψ , which is called the coherent wave, satisfying a one-body Schrödinger equation with an effective potential U , called optical potential.
One can also describe absorption or diffuse scattering considering a complex optical potential and take the influence of the spin into account but

here, for the sake of simplicity, we will treat only the purely coherent elastic scattering.
From a theoretical point of view the first task is to prove that the effective Schrödinger equation with the optical potential can be derived from the microscopic dynamics of the neutrons scattered by a large number of atoms and moreover to give an explicit expression for the optical potential. This is the content of the so called theory of dispersion.
It is claimed in the literature (see e.g. [62],[64]) that the first question, i.e. the existence of an optical potential, can be solved in complete generality without specifying the nature of the interaction between the neutron and the system. Only when the specific form of the optical potential is required then one has to introduce the dynamics.
In particular it is found that, according to the "elementary" theory ([61]), the optical potential is given by the scattering length per unit volume associated to the system.
This result is obtained by simply averaging the point interactions in the Born approximation, i.e. neglecting the multiple scattering of the neutrons.
It should be remarked that the Born approximation is the only available inasmuch as one has a formal theory of point interactions and this lead the physicists to the convinction that point interactions cannot be used to describe multiple scattering processes.
A more "advanced" theory ([61],[62],[64]) is based on the consideration of the so called local field, which essentially gives corrections due to the multiple scattering.
The problem is reduced to the solution of a set of equations (the quantum mechanical analogue of Ewald's equations in optics) plus a phenomenological constitutive equation relating the scattered wave and the local field.
The result of this rather intricate theory is an optical potential equal to the scattering length per uniy volume multiplied by a factor essentially depending on the pair correlation function of the system of the scatterers, which is defined by

$$\frac{g(r)}{4\pi r^2}dr = Pr\ \{|y_i - y_j| \in (r, r+dr)\ \mid\ y_i\} \tag{3.30}$$

where $y_1, ..., y_N$ are the positions of the nuclei.

When $y_1, ..., y_N$ are independently distributed the factor reduces to 1 and the result of the "elementary" theory is recovered.
We are now in position to discuss a possible application of the result of the preceding section to the problem of the theory of dispersion.
First of all we observe that the assumptions of our model correspond to a scattering process in the long wavelength limit.
In fact, in the limit $N \to +\infty$ we have considered, both the scattering length of each scatterer and the interparticle distance are going to zero while the wavelength of the incident particle is kept constant.
So our result seems to be suitable for a rigorous description of cold neutron scattering from independently distributed nuclei.
Under these conditions we recover (proposition 3.1.1) the result of the above mentioned "elementary" theory since the optical potential we have found is just the scattering length per unit volume of the medium $-V/\alpha$ (in fact $-1/N\alpha_i$ is the scattering length of the scatterer in y_i and V is the density of the scatterers).
The result seems interesting because it has been obtained using the correctly defined Schrödinger operator with point interactions as hamiltonian of the incident neutron, without considering any Born approximation.
In particular this means that we have explicitly taken the multiple scattering into account.
This fact can be also easily recognized if one writes the scattering wave function ([7]) associated to the hamiltonian $-\Delta_{N\alpha^{(N)},Y^{(N)}}$ considered in the last section

$$\psi_{N\alpha^{(N)},Y^{(N)}}(k\omega, x) = e^{ik\omega x} + \sum_{j=1}^{N} \frac{e^{ik|x-y_j|}}{4\pi|x-y_j|} Q^j \tag{3.31}$$

where $k \geq 0$, $\omega \in S^2$ and the "charges" Q^j satisfy the set of linear equations

$$(N\alpha_j + \frac{ik}{4\pi})Q^j = e^{ik\omega y_j} + \sum_{j' \neq j} \frac{e^{ik|y_j - y_{j'}|}}{4\pi|y_j - y_{j'}|} Q^{j'} \tag{3.32}$$

The formula (3.31) can be physically interpreted as follows:
the total scattered wave is the sum of the incident plane wave plus a scattered spherical wave from each scatterer proportional to the "charge" $Q^{j'}$.

Such charge again depends on the scattered waves from all other nuclei, as shown by (3.32), so that it is evident that $\psi_{N\alpha^{(N)},Y^{(N)}}$ embodies all possible multiple scattering effects in the collision of the neutron with the system of nuclei.
Such considerations lead to the conclusion that the above referred assertion that the result of the "elementary" theory neglects multiple scattering doesn't seem correct.
From our point of view the word "elementary" should only indicate that the scatterers are supposed independently distributed.
In this framework the rôle of a more sophisticated theory of dispersion should be the analysis of the low energy scattering of neutrons from a system of correlated scatterers.
This study is still in progress but we clearly expect to obtain the optical potential of the above mentioned "advanced" theory of dispersion in the long wavelength limit.
This result would be an indirect confirmation of our guess that the real difference between "elementary" and "advanced" theory is determined by the consideration of correlation effects and not of multiple scattering effects (which are already present in the "elementary" theory) and moreover that the correctly defined point interactions give a good physical model of interaction for any low energy scatterig process in neutron optics.
A result which can be, at least partially, considered in the same direction is given in [21],[6], where a rigorous analysis of the kinematical theory of neutron scattering is developed. The kinematical theory ([48],[64]), contrary to the above described dynamical theory, is devoted to the study of the scattering of the thermal neutrons from small samples of condensed matter.
Here the expression "thermal neutrons" means that the wavelength of the incoming particles is of the same order of magnitude as the interparticle distance in the sample and "small samples" means that the multiple scattering effects are not taken into account.
The scattering process is again described in the physical literature using the formal point interactions in the first Born approximation and the result obtained for the scattering cross section is

$$\frac{d\sigma}{d\Omega}(q) \simeq \frac{1}{\alpha^2} \sum_{i,j=1}^{N} e^{iq(y_i - y_j)} \simeq \frac{1}{\alpha^2}[1 + V\hat{g}(q)] \tag{3.33}$$

where q is the transferred momentum and $\hat{g}$ is the Fourier transform of the pair correlation function.
The formula (3.33) shows why the kinematical theory is particularly interesting in the study of the structure of the materials, since the pair correlation function of the material is related to directly measurable quantities.
The situation can be modelized as follows: let y_i be distributed according to an homogeneous point process in R^3 of smooth density V and pair correlation function g and let us consider the Schrödinger operator with point interactions in $y_1, ..., y_N$ of strength $1/N^{1/2}\alpha$, $\alpha \in R$; then in the limit $N \to \infty$ the corresponding scattering cross section converges to the r.h.s. of (3.33).
This result shows that, from a mathematical point of view, the difference between the kinematical and the dynamical theory lies in a different scaling law for the scattering length of each scatterer as N goes to infinity.
In conclusion we note that it should be possible to extend the methods of the last section and of [21],[6] to describe situations in which each scatterer is considered harmonically bounded around some equilibrium position so that inelastic scattering is allowed ([23],[47],[48],[64]).

3.3 Point interactions and boundary value problems

This section is devoted to show the relation between point interactions and two particular boundary value problems for the Laplace equation ([28]). Let $S_r = \cup_{j=1}^N S_r^j$ be the union of N disjoint spherical surfaces $S_r^j = \{x \in R^3 \mid |x - y_j| = r\}$ of radius r and centers $y_j \in R^3$ and let $\gamma_r = \{\gamma_r^1, ..., \gamma_r^N\}$ be a smooth function defined on S_r.
Then we consider the following boundary value problem in R^3

$$\begin{aligned} &(-\Delta + \lambda)u_r = f && in\ R^3 \setminus S_r \\ &\frac{\partial u_r}{\partial n_j^+} - \frac{\partial u_r}{\partial n_j^-} = \gamma_r^j u_r && on\ S_r^j,\ j = 1, ..., N \end{aligned} \tag{3.34}$$

where $\lambda > 0$ and $f \in L^2(R^3)$.
By a simple integration by parts it is easily realized that the quadratic form associated to the problem (3.34) coincides with (1.47)), (1.48) of chapter 1 so that the solution of (3.34) is nothing but $(-\Delta_{\gamma_r, S_r} + \lambda)^{-1} f$, where $-\Delta_{\gamma_r, S_r}$ is the Schrödinger operator with δ-interaction supported by S_r of strength γ_r defined in section 1.3 .
We note that problems of the type (3.34) arise e.g. in non relativistic Quantum Mechanics in the study of some kinds of δ-shell interactions ([11],[33] and references therein).
The second boundary value problem we consider here is the exterior Robin problem

$$\begin{aligned} &(-\Delta + \lambda)v_r = f && in\ R^3 \setminus \cup_{j=1}^n B_r^j \\ &\frac{\partial v_r}{\partial n_j^+} - \gamma_r^j v_r = 0 && on\ S_r^j,\ j = 1, ..., N \end{aligned} \tag{3.35}$$

where now $f \in L^2(R^3 \setminus \cup_{j=1}^N B_r^j)$ and $B_r^j = \{x \in R^3 \mid |x - y_j| \leq r\}$.
In the physical applications the Robin problem is useful to describe some situations in electrochemistry, in the computation of the skin effect, in

problems of heat propagation and, in electrostatics, to study the potential of a conductor covered by a thin dielectric film ([44],[54]).

Our claim is that if we fix

$$\gamma_r^j = -(r + 4\pi\alpha_j r^2)^{-1} \qquad j = 1, ..., N \tag{3.36}$$

where $\alpha_j \in R$, then the solutions of the problems (3.34), (3.35) are both converging for $r \to 0$ to

$$u = (-\Delta_{\alpha,Y} + \lambda)^{-1} f \tag{3.37}$$

where $\alpha = \{\alpha_1, ..., \alpha_n\}$, $Y = \{y_1, ..., y_N\}$.

The physical meaning of the position (3.36) can be understood if one thinks of (3.34), (3.35) as scattering problems.
If for each scatterer S_r^j we consider the s-wave effective range expansion ([51] pag 309), then condition (3.36) means that in the limit $r \to 0$ we obtain point interactions in Y of strength α from problems (3.34), (3.35) if we keep the scattering length of each scatterer to be the constant value $(4\pi\alpha_j)^{-1}$.
We observe that the result for the problem (3.34) in the case $N = 1$ is proved in [11] while for the problem (3.35) it is announced in [24], where an heuristic argument for its validity is also given.

We start with problem (3.34).
In the Hilbert space $\oplus_{j=1}^N L^2(S_r^j)$ of the n-tuples $\sigma_r = ({\sigma_r}^1, ..., {\sigma_r}^N)$, with ${\sigma_r}^j \in L^2(S_r^j)$, endowed with the usual scalar product

$$(\sigma_r, \tau_r)_r = \sum_{j=1}^N \int_{S_r^j} {\sigma_r}^j \tau_r^j \, d\Sigma^j \tag{3.38}$$

and the corresponding norm

$$\|\sigma_r\|_r = [\sum_{j=1}^N \int_{S_r^j} ({\sigma_r}^j)^2 d\Sigma^j \,]^{1/2} , \tag{3.39}$$

we introduce the following bounded and selfadjoint operator

$$(\Gamma_{\gamma_r,S_r}(i\sqrt{\lambda})\sigma_r)^j = -\frac{\sigma_r{}^j}{\gamma_r^j} - \check{G}_{i\sqrt{\lambda}}\sigma_r{}^j|_{S_r^j} - \sum_{l=1,l\neq j}^{N} \check{G}_{i\sqrt{\lambda}}\sigma_r{}^l|_{S_r^j} \tag{3.40}$$

Then, as we know from section 1.3, the solution of (3.34) can be represented in the form

$$u_r = G_{i\sqrt{\lambda}}f + \sum_{j=1}^{N} \check{G}_{i\sqrt{\lambda}}\sigma_r{}^j \tag{3.41}$$

where the surface charges $\sigma_r{}^j$ are the solutions in $L^2(S_r^j)$ of the following system of integral equations

$$(\Gamma_{\gamma_r,S_r}(i\sqrt{\lambda})\sigma_r)^j = G_{i\sqrt{\lambda}}f|_{S_r^j} \tag{3.42}$$

(The dependence of the charges on f and λ has been dropped to semplify the notation).
Our first goal is to establish the uniform (with respect to a sufficiently small r) solvability of (3.42), i.e. to obtain an estimate of the lower bound for the operator defined by (3.40).
A simple calculation gives

$$\int_{S_r^j} \sigma_r{}^j \check{G}_{i\sqrt{\lambda}}\sigma_r{}^j|_{S_r^j}\, d\Sigma^j \leq \frac{\sinh\sqrt{\lambda}r}{\sqrt{\lambda}} e^{-\sqrt{\lambda}r}\|\sigma_r\|_r^2 \tag{3.43}$$

$$|\sum_{j,l=1,l\neq j}^{N} \int_{S_r^j} \sigma_r{}^j \check{G}_{i\sqrt{\lambda}}\sigma_r{}^l|_{S_r^j}\, d\Sigma^j\, | \leq \frac{4\pi\sinh\sqrt{\lambda}r\; e^{-\sqrt{\lambda}r}}{\sqrt{\lambda}(1-r/d)}\|\tilde{G}_{i\sqrt{\lambda}}\|_{H.S.}\|\sigma_r\|_r^2 \tag{3.44}$$

where $d = inf_{l\neq j}|y_l - y_j|$. Using (3.43), (3.44) we get

$$\begin{aligned}(\sigma_r, \Gamma_{\gamma_r,S_r}(i\sqrt{\lambda})\sigma_r)_r &\geq r^2[\, inf_j(4\pi\alpha_j) + \sqrt{\lambda} - 4\pi\frac{\sinh\sqrt{\lambda}r}{\sqrt{\lambda}r}\|\tilde{G}_{i\sqrt{\lambda}}\|_{H.S.}\,]\|\sigma_r\|_r^2 + \\ &+ o(r^3)\|\sigma_r\|_r^2\end{aligned} \tag{3.45}$$

From (3.45) we conclude that there exist $r_0 > 0$ sufficiently small and $\lambda(r_0) > 0$ such that $r^{-2}\Gamma_{\gamma_r,S_r}(i\sqrt{\lambda})$ is strictly positive (and then invertible)

whenever $r < r_0$ and $\lambda > \lambda(r_0)$.
This fact is the basic ingredient for the proof of our first result

Proposition 3.3.1 *If γ_r^j is given by* (3.36) *then*

$$\lim_{r\to 0} r^{-\eta}\, sup_{f\in L^2(R^3)} \frac{\|u-u_r\|_2}{\|f\|_2} = 0 \qquad \forall \eta \in [0,1/2) \tag{3.46}$$

Proof. Let us fix $f, g \in L^2(R^3)$. Then using the representation (3.41), (3.42) of the solution of (3.34) and the explicit expression of $(-\Delta_{\alpha,Y}+\lambda)^{-1}f$ (see (.1) in the introduction) we separate the contributions to the difference $u - u_r$ as follows

$$\begin{aligned} |(g, u-u_r)| &\leq \sum_{j=1}^{N} |(g, G_{i\sqrt{\lambda}}(\cdot - y_j)q^j - \breve{G}_{i\sqrt{\lambda}}\bar{\sigma}_r{}^j)| + \\ &+ |(g, \breve{G}_{i\sqrt{\lambda}}(\sigma_r - \bar{\sigma}_r))| \end{aligned} \tag{3.47}$$

where

$$q^j = \sum_{l=1}^{N} [\Gamma_{\alpha,Y}(i\sqrt{\lambda})^{-1}]_{jl} G_{i\sqrt{\lambda}} f(y_l) \tag{3.48}$$

$$\bar{\sigma}_r{}^j = \frac{\sqrt{\lambda}}{4\pi r \sinh\sqrt{\lambda} r} q^j \tag{3.49}$$

The surface charges $\bar{\sigma}_r$ have been chosen in order to satisfy the equality

$$\breve{G}_{i\sqrt{\lambda}}\bar{\sigma}_r{}^j(x) = G_{i\sqrt{\lambda}}(x-y_j)q^j \qquad \forall x \in R^3 \setminus B_r^j, \;\; j = 1, ..., N \tag{3.50}$$

As a consequence of (3.50), the first term of (3.47) gives a non zero contribution only for a region of integration whose measure is going to zero as $r \to 0$.
Using the Schwartz inequality and the standard potential estimate (see e.g. [32])

$$\|G_{i\sqrt{\lambda}} f\|_\infty \leq c\|f\|_2 \tag{3.51}$$

one gets

$$\sum_{j=1}^{N} | \int_{B_r^j} g(x)[\ G_{i\sqrt{\lambda}}(x-y_j)q^j - (\breve{G}_{i\sqrt{\lambda}}\bar{\sigma}_r{}^j)(x)\ ]dx| \leq$$
$$\leq (\sum_{j=1}^{N} |q^j|)[\ sup_j |G_{i\sqrt{\lambda}}(1_r^j g)(y_j)| + \frac{e^{-\sqrt{\lambda}r}}{4\pi r} sup_j \int_{B_r^j} |g(x)|dx\ ] \leq$$
$$\leq cr^{1/2}\|G_{i\sqrt{\lambda}}f\|_\infty \|g\|_2 \leq$$
$$\leq cr^{1/2}\|f\|_2\|g\|_2 \tag{3.52}$$

where 1_r^j is the characteristic function of B_r^j.
Now we turn to the second term in (3.47). An application of the Fubini theorem and the Schwartz inequality yield

$$|(g, \breve{G}_{i\sqrt{\lambda}}(\sigma_r - \bar{\sigma}_r))| = |(\sigma_r - \bar{\sigma}_r, G_{i\sqrt{\lambda}}g|_{S_r})_r| \leq$$
$$\leq \|G_{i\sqrt{\lambda}}g|_{S_r}\|_r \|\sigma_r - \bar{\sigma}_r\|_r \tag{3.53}$$

Again by the estimate (3.51)

$$\|G_{i\sqrt{\lambda}}g|_{S_r}\|_r \leq cr\|g\|_2 \tag{3.54}$$

and moreover

$$\|\sigma_r - \bar{\sigma}_r\|_r \leq \|\Gamma_{\gamma_r,S_r}(i\sqrt{\lambda})^{-1}\|_r \|G_{i\sqrt{\lambda}}f|_{S_r} - \Gamma_{\gamma_r,S_r}(i\sqrt{\lambda})\bar{\sigma}_r\|_r \tag{3.55}$$

So using the lower bound (3.45) we find

$$|(g, \breve{G}_{i\sqrt{\lambda}}(\sigma_r - \bar{\sigma}_r))| \leq c\|g\|_2\ sup_j\ sup_{\varsigma\in S_r^j} |G_{i\sqrt{\lambda}}f(\varsigma) - (\Gamma_{\gamma_r,S_r}(i\sqrt{\lambda})\bar{\sigma}_r)(\varsigma)| \tag{3.56}$$

Taking the explicit expression of $\bar{\sigma}_r$ and of γ_r^j into account we have

$$|(g, \breve{G}_{i\sqrt{\lambda}}(\sigma_r - \bar{\sigma}_r))| \leq$$
$$\leq c\|g\|_2[\ sup_j\ sup_{\varsigma\in S_r^j} |G_{i\sqrt{\lambda}}f(\varsigma) - G_{i\sqrt{\lambda}}f(y_j)| + o(r)|q^j| + cr \sum_{l=1,l\neq j}^{N} |q^l|\ ] \leq$$
$$\leq cr^\eta \|g\|_2\|f\|_2 \qquad \forall\eta \in [0,1/2) \tag{3.57}$$

where again (3.51) and the Sobolev inequality (see e.g. [42])

$$sup_{x,x'\in R^3} \frac{|G_{i\sqrt{\lambda}}f(x) - G_{i\sqrt{\lambda}}f(x')|}{|x-x'|^\eta} \leq c\|f\|_2 \qquad \forall \eta \in [0,1/2) \tag{3.58}$$

have been used.
Finally by (3.52), (3.57) we obtain the proof of the proposition.

Q.E.D.

We now briefly consider the exterior boundary value problem (3.35). Again the first question is to investigate whether it is uniformly solvable in the limit $r \to 0$. Following the standard methods of the potential theory we represent the solution as ([44],[36],[12])

$$v_r = G_{i\sqrt{\lambda}}f + \sum_{j=1}^{N} \check{G}_{i\sqrt{\lambda}}\sigma_r{}^j \tag{3.59}$$

where the surface charges $\sigma_r{}^j$ are determined by imposing the boundary conditions. The result is the following system of integral equations

$$(\hat{\Gamma}_{\gamma_r,S_r}(i\sqrt{\lambda})\sigma_r)^j \equiv \frac{\sigma_r{}^j}{2\gamma_r^j} - H_r^j(i\sqrt{\lambda})\sigma_r{}^j - \sum_{l=1,l\neq j}^{N} H_r^{jl}(i\sqrt{\lambda})\sigma_r{}^l =$$
$$= (-\frac{1}{\gamma_r^j}\frac{\partial G_{i\sqrt{\lambda}}f}{\partial n_j^+} + G_{i\sqrt{\lambda}}f)|_{S_r^j} \qquad j = 1,...,N \tag{3.60}$$

where $\forall \varsigma \in S_r^j$ we have posed

$$(H_r^j(i\sqrt{\lambda})\sigma_r{}^j)(\varsigma) = \int_{S_r^j}[\,-\frac{1}{\gamma_r^j}\frac{\partial G_{i\sqrt{\lambda}}(\varsigma-\varsigma')}{\partial n_j^+(\varsigma)} + G_{i\sqrt{\lambda}}(\varsigma-\varsigma')\,]\sigma_r{}^j(\varsigma')\,d\Sigma^j(\varsigma') \tag{3.61}$$

$$(H_r^{jl}(i\sqrt{\lambda})\sigma_r{}^l)(\varsigma) = \int_{S_r^l}[\,-\frac{1}{\gamma_r^j}\frac{\partial G_{i\sqrt{\lambda}}(\varsigma-\varsigma')}{\partial n_j^+(\varsigma)} + G_{i\sqrt{\lambda}}(\varsigma-\varsigma')\,]\sigma_r{}^l(\varsigma')\,d\Sigma^l(\varsigma') \tag{3.62}$$

It is easily recognized that $\hat{\Gamma}_{\gamma_r,S_r}(i\sqrt{\lambda})$ is a bounded and selfadjoint operator in $\oplus_{j=1}^N L^2(S_r^j)$ (see e.g. [54]). Moreover exploiting the spherical symmetry of the obstacles we can explicitly compute the normal derivative in (3.61), so, by a direct calculation, we get

$$\int_{S_r^j} \sigma_r{}^j(\frac{1}{\gamma_r^j} - H_r^j(i\sqrt{\lambda}))\sigma_r{}^j \, d\Sigma^j \geq r^2[\, inf_j \, (4\pi\alpha_j) + \frac{\sqrt{\lambda}}{2} \,]\|\sigma_r\|_r^2 + o(r^3)\|\sigma_r\|_r^2 \tag{3.63}$$

The estimate of the operator defined by (3.62) can be performed along the line of (3.44), the only difference being the presence of the normal derivative of the Green's function which is controlled by the absolute value of the gradient.

In conclusion we are in the same situation as for problem (3.34), i.e. there exist $r_0' > 0$ sufficiently small and $\lambda(r_0') > 0$ such that $r^{-2}\hat{\Gamma}_{\gamma_r,S_r}(i\sqrt{\lambda})$ is strictly positive, and then invertible, whenever $r < r_0'$ and $\lambda > \lambda(r_0')$.

In the analysis of the limit for $r \to 0$ of the exterior problem (3.35) another preliminary question must be faced.

In order to avoid difficulties arising in the study of a sequence of problems defined on varying domains (in our case $R^3 \setminus \bigcup_{j=1}^N B_r^j$ for $r \to 0$), it is useful to define an extension to all R^3 of the solution v_r , i.e. a function defined in all R^3 which reduces to v_r in $R^3 \setminus \bigcup_{j=1}^N B_r^j$.

Taking the representation (3.59) into account it is natural to choose the extension $\hat{v}_r$ obtained by simply taking the values in $\bigcup_{j=1}^N B_r^j$ of the potentials $G_{i\sqrt{\lambda}}f$, $\check{G}_{i\sqrt{\lambda}}\sigma_r{}^j$, which are naturally defined as continuous functions throughout the space.

Then the resulting extended Robin problem in R^3 can be written as

$$\begin{aligned} &(-\Delta + \lambda)\hat{v}_r = 1_r f && in \;\; R^3 \setminus S_r \\ &\frac{\partial \hat{v}_r}{\partial n_j^+} - \frac{\partial \hat{v}_r}{\partial n_j^-} = -\sigma_r{}^j && on \;\; S_r, \;\; j = 1, ..., N \end{aligned} \tag{3.64}$$

where now $f \in L^2(R^3)$, 1_r is the characteristic function of $R^3 \setminus \bigcup_{j=1}^N B_r^j$ and $\sigma_r{}^j$ is the solution of (3.60).

We remark that there is no way to obtain the problem (3.34) as the extension to R^3 of the problem (3.35), so that the two problems are to be

considered essentially different even if they can be treated with similar methods.
Now we can state the convergence result for the problem (3.64)

Proposition 3.3.2 *If* γ_r^j *is given by* (3.36) *then*

$$\lim_{r\to 0} r^{-\eta}\ sup_{f\in L^2(R^3)}\frac{\|u-\hat{v}_r\|_2}{\|f\|_2}=0 \qquad \forall\eta\in[0,1/2) \tag{3.65}$$

The proof is omitted here for the sake of brevity but it should now be clear that the situation is completely analogous to the previous case so that the proof can be carried out along the same line with only minor changes, due to the presence of the normal derivative of the Green's function.

Remark. In conclusion we observe that it should be possible to generalize the connection between point interactions and the boundary value problems (3.34), (3.35) to the case of shrinking obstacles S_r^j of arbitrary shape.
The main problem is to replace the expression (3.36) for γ_r^j with a more general one, independent of the shape of the obstacles. Such expression should be determined by imposing that in the limit $r\to 0$ each scatterer keeps a non vanishing scattering length.
The line of the proof should be the same but all the estimates are complicated by the fact that the potential of a uniform charge distribution on the obstacles is not explicitly computable any more.

3.4 Effective equations for two models of random media

Here, developing further the arguments of sections 3.1, 3.3, we analyse the asymptotic behaviour of the two boundary value problems introduced in the preceding section when there are many randomly distributed obstacles; in particular we will study the limit when the number N of the obstacles goes to infinity and their linear size is going to zero.
As we pointed out in the introduction, this is a particular case of the general problem of the study of random media.
The result we will find is that for $N \to +\infty$ the two media behave as a continuous medium whose response to an external field is described by an effective potential given by the density of scattering length associated to the system of obstacles.

Let us consider N points $y_1, ..., y_N \in R^3$ independently and identically distributed according to the continuous probability density law $V(x)$. Otherwise stated, the set of all configurations $Y^{(N)} = \{y_1, ..., y_N\}$ of N points in R^3 is equipped with the probability measure $P^{(N)} = \{V(x)dx\}^{\otimes N}$.
Centered in each point y_j , we consider the collection of spherical surfaces $S_N = \bigcup_{j=1}^N S_N^j$, where $S_N^j = \{x \in R^3 \mid |x - y_j| = 1/N\}$.
The technique of the proof will be based on the consideration of a set of configurations $Y^{(N)}$ satisfying the following two regularity conditions

$$A_1 \qquad inf_{j \neq l} |y_j - y_l| \geq cN^{-1+\nu} \qquad \forall \nu \in (0, 1/3)$$

$$A_2 \qquad \frac{1}{N^2} \sum_{j,l=1, j \neq l}^{N} \frac{1}{|y_j - y_l|^{3-\xi}} \leq c_\xi < +\infty \qquad \forall \xi > 0$$

It is not hard to prove, using independence of the y_j's, continuity of V and the law of large numbers ([52]), that the set of configurations $Y^{(N)}$ satisfying A_1 , A_2 has a probability going to 1 as N goes to infinity.
Moreover, as a consequence of condition A_1 , we have

$$S_N^j \bigcap S_N^l = \emptyset \qquad \forall j \neq l \quad and\ N\ sufficiently\ large \tag{3.66}$$

Since we are interested in the limit $N \to +\infty$ in the sequel we can always choose N sufficiently large in such a way that (3.66) holds.
Let us now consider the following boundary value problem

$$\begin{aligned} &(-\Delta + \lambda)u_N = f && in\ \ R^3 \setminus S_N \\ &\frac{\partial u_N}{\partial n_j^+} - \frac{\partial u_N}{\partial n_j^-} = \gamma_N^j u_N && on\ \ S_N^j,\ \ j = 1, ..., N \end{aligned} \tag{3.67}$$

where $\lambda > 0$, $f \in L^2(R^3)$,

$$\gamma_N^j = -N(1 + 4\pi\alpha_j)^{-1} \qquad j = 1, ..., N \tag{3.68}$$

and $\alpha_j = \alpha(y_j)$ ($\alpha(x)$, $x \in R^3$ is the function defined in section 3.1).
Moreover we define $\gamma_N = \{\gamma_N^1, ..., \gamma_N^N\}$ and $\alpha^{(N)} = \{\alpha_1, ..., \alpha_N\}$.
In the sequel we will always assume $1 + 4\pi\alpha_j \neq 0$.
It is evident from (3.67) that the case $1 + 4\pi\alpha_j = 0$ corresponds to a Dirichlet boundary condition imposed on S_N^j so that it should be treated separatly using, e.g., the methods of [25].
Our problem is to find the limit of the solution u_N of the problem (3.67) for $N \to +\infty$.

The first step, analogously to the case of a fixed number of obstacles, is the investigation of the uniform solvability of (3.67).
Introducing the Hilbert space $\bigoplus_{j=1}^N L^2(S_N^j)$ of the n-tuples $\sigma_N = ({\sigma_N}^1, ..., {\sigma_N}^N)$, with ${\sigma_N}^j \in L^2(S_N^j)$, equipped with the usual scalar product and norm

$$(\sigma_N, \tau_N)_N = \sum_{j=1}^N \int_{S_N^j} {\sigma_N}^j \tau_N^j \, d\Sigma^j \tag{3.69}$$

$$\|\sigma_N\|_N = \sum_{j=1}^N \int_{S_N^j} ({\sigma_N}^j)^2 \, d\Sigma^j \tag{3.70}$$

we represent the solution of (3.67) in the form

$$u_N = G_{i\sqrt{\lambda}} + \sum_{j=1}^{N} \breve{G}_{i\sqrt{\lambda}} {\sigma_N}^j \tag{3.71}$$

where the surface charges ${\sigma_N}^j$ are the solutions in $L^2(S_N^j)$ of

$$(\Gamma_{\gamma_N,S_N}(i\sqrt{\lambda})\sigma_N)^j = G_{i\sqrt{\lambda}} f|_{S_N^j} \qquad j = 1, ...N \tag{3.72}$$

(Again the dependence of the charges on f and λ has been dropped to semplify the notation).
Dividing the diagonal and off-diagonal contributions in $\Gamma_{\gamma_N,S_N}(i\sqrt{\lambda})$, we define

$$(\Gamma_{\gamma_N,S_N}(i\sqrt{\lambda})^{(1)}\sigma_N)^j = -\frac{{\sigma_N}^j}{\gamma_N^j} - \breve{G}_{i\sqrt{\lambda}} {\sigma_N}^j|_{S_N^j} \tag{3.73}$$

$$(\Gamma_{\gamma_N,S_N}(i\sqrt{\lambda})^{(2)}\sigma_N)^j = -\sum_{l=1,l\neq j}^{N} \breve{G}_{i\sqrt{\lambda}} {\sigma_N}^l|_{S_N^j} \tag{3.74}$$

Reasoning as for (3.44) and using the estimate (3.15), it is easily shown that

$$\|N\Gamma_{\gamma_N,S_N}(i\sqrt{\lambda})^{(2)}\|_N \leq c(\lambda) \tag{3.75}$$

where $c(\lambda)$ is independent on N and satisfies $\lim_{\lambda\to+\infty} c(\lambda) = 0$.
Since the operator $\breve{G}_{i\sqrt{\lambda}}$ is compact in $L^2(S_N^j)$ (see e.g. [54]), the question of the invertibility for $N\Gamma_{\gamma_N,S_N}(i\sqrt{\lambda})^{(1)}$ is reduced to find possible values of $\lambda > 0$ such that there exist non trivial solutions in $L^2(S_N^j)$ of the homogeneous equation

$$-\frac{N{\sigma_N}^j}{\gamma_N^j} - N\breve{G}_{i\sqrt{\lambda}} {\sigma_N}^j = 0 \tag{3.76}$$

A simple scaling argument shows that this is equivalent to find non trivial solutions in $L^2(S_1^j)$ of

$$(1+4\pi\alpha_j)\sigma^j - \breve{G}_{i\sqrt{\mu}}\sigma^j = 0 \tag{3.77}$$

where $\mu = N^{-2}\lambda$.
Equation (3.77) is just the eigenvalue equation for the Schrödinger operator with δ-interaction supported by S_1^j of strength $(1+4\pi\alpha_j)^{-1}$ (see [11]).
It easily seen ([11]) that there exists only a finite number μ_h , $h = 1, ..., m$, of values of μ such that (3.77) admits non trivial solutions.
This implies that the possible non trivial solutions of (3.76) are obtained only for $\lambda_h = \mu_h N^2$, $h = 1, ..., m$, which are all going to $+\infty$ for $N \to +\infty$.
So, taking also the estimate (3.75) into account, we can choose N_0 sufficiently large and $\lambda(N_0) > 0$ such that $N\Gamma_{\gamma_N, S_N}(i\sqrt{\lambda})$ is invertible in $\oplus_{j=1}^N L^2(S_N^j)$ whenever $\lambda > \lambda(N_0)$ and $N > N_0$.

This uniform invertibility is the key ingredient of the proof of the following proposition

Proposition 3.4.1 $\forall\epsilon > 0$, $\forall f \in L^2(R^3)$ *and* λ *positive large enough*

$$\lim_N P^{(N)}\left(\{Y^{(N)} \mid \|u_N - (-\Delta - V/\alpha + \lambda)^{-1} f\|_2 > \epsilon\} \right) = 0 \tag{3.78}$$

Proof. Introducing the operator $-\Delta_{N\alpha^{(N)}, Y^{(N)}}$ and using the proposition 3.1.1, we are reduced to prove

$$\lim_N \|u_N - (-\Delta_{N\alpha^{(N)}, Y^{(N)}} + \lambda)^{-1} f\|_2 = 0 \tag{3.79}$$

for each configuration $Y^{(N)}$ satisfying the regularity conditions A_1 , A_2.
Following the line of the proposition 3.3.1 we have, for each $g \in L^2(R^3)$

$$|(g, u_N - (-\Delta_{N\alpha^{(N)}, Y^{(N)}} + \lambda)^{-1} f)| \leq$$

$$\leq \sum_{j=1}^N |(g, G_{i\sqrt{\lambda}}(\cdot - y_j) Q_N^j - \breve{G}_{i\sqrt{\lambda}} \bar{\sigma}_N{}^j)| + |(g, \breve{G}_{i\sqrt{\lambda}}(\sigma_N - \bar{\sigma}_N))| \tag{3.80}$$

where

$$Q_N^j = \sum_{l=1}^{N} [\, \Gamma_{N\alpha^{(N)},Y^{(N)}}(i\sqrt{\lambda})^{-1} \,]_{jl}\, G_{i\sqrt{\lambda}} f(y_l) \tag{3.81}$$

$$\bar{\sigma}_N{}^j = \frac{\sqrt{\lambda}}{4\pi \sinh \sqrt{\lambda} N^{-1}} Q_N^j \tag{3.82}$$

The first term in the r.h.s. of (3.80) can be estimated just as in the case of a fixed N (see (3.52)), so that one has

$$\sum_{j=1}^{N} |(\, g, G_{i\sqrt{\lambda}}(\cdot - y_j) Q_N^j - \check{G}_{i\sqrt{\lambda}} \bar{\sigma}_N{}^j \,)| \leq c N^{-1/2} \|f\|_2 \|g\|_2 \tag{3.83}$$

For the second term in the r.h.s. of (3.80) , again following the line of (3.53)-(3.56) and using the uniform invertibility of $N\Gamma_{\gamma_N,S_N}(i\sqrt{\lambda})$, one gets

$$|(\, g, \check{G}_{i\sqrt{\lambda}}(\sigma_N - \bar{\sigma}_N) \,)| \leq c\|g\|_2 \; sup_j \; sup_{\varsigma \in S_N^j} \; |G_{i\sqrt{\lambda}} f(\varsigma) - (\Gamma_{\gamma_N,S_N}(i\sqrt{\lambda})\bar{\sigma}_N)(\varsigma)| \tag{3.84}$$

Taking the explicit expression of $\bar{\sigma}_N$ into account we have

$$\begin{aligned} &|(\, g, \check{G}_{i\sqrt{\lambda}}(\sigma_N - \bar{\sigma}_N) \,)| \leq \\ &\leq c\|g\|_2 [\; sup_j \; sup_{\varsigma \in S_N^j} \; |G_{i\sqrt{\lambda}} f(\varsigma) - G_{i\sqrt{\lambda}} f(y_j)| + c' \; sup_j |Q_N^j| + \\ &+ sup_j \; sup_{\varsigma \in S_N^j} \sum_{l=1, l\neq j}^{N} |Q_N^l| |G_{i\sqrt{\lambda}}(y_j - y_l) - G_{i\sqrt{\lambda}}(\varsigma - y_l)| \,] \end{aligned} \tag{3.85}$$

Now applying the Sobolev inequality (see e.g. [42])

$$sup_j \; sup_{\varsigma \in S_N^j} \; |G_{i\sqrt{\lambda}} f(\varsigma) - G_{i\sqrt{\lambda}} f(y_j)| \leq c N^{-\eta} \|f\|_2 \qquad \forall \eta \in [0, 1/2) \tag{3.86}$$

Moreover using the uniform invertibility of $N^{-1}\Gamma_{N\alpha^{(N)},Y^{(N)}}(i\sqrt{\lambda})$ (see (3.16)) and the standard potential estimate (3.51)

$$sup_j\ |Q_N^j| \leq [\ \sum_{j=1}^{N}(Q_N^j)^2\ ]^{1/2} \leq cN^{-1}[\ \sum_{j=1}^{N}(G_{i\sqrt{\lambda}}f(y_j))^2\ ^{1/2} \leq$$
$$\leq cN^{-1/2}\|f\|_2 \tag{3.87}$$

Finally using the estimate (3.87) on the charges Q_N^j , the Schwartz inequality and the regularity conditions A_1 , A_2

$$sup_j\ sup_{\varsigma\in S_N^j} \sum_{l=1,l\neq j}^{N} |Q_N^l||G_{i\sqrt{\lambda}}(y_j - y_l) - G_{i\sqrt{\lambda}}(\varsigma - y_l)| \leq$$
$$\leq cN^{-1}\ sup_j \sum_{l=1,l\neq j}^{N} |Q_N^l|\frac{1}{|y_j - y_l|^2} \leq$$
$$\leq cN^{-3/2}\|f\|_2\ (\sum_{j,l=1,j\neq l}^{N} \frac{1}{|y_j - y_l|^4}\ )^{1/2} \leq$$
$$\leq cN^{-1/2}\|f\|_2\ (\frac{1}{N^2}\sum_{j,l=1,j\neq l} N\frac{1}{|y_j - y_l|^{3-\xi}}\ )^{1/2}\ (\ sup_j\frac{1}{|y_j - y_l|^{1+\xi}}\ )^{1/2} \leq$$
$$\leq cN^{-1/2}\|f\|_2\ (\ sup_j\frac{1}{|y_j - y_l|}\ )^{(1+\xi)/2} \leq$$
$$\leq cN^{-\frac{\nu+\nu\xi-\xi}{2}}\ \|f\|_2 \tag{3.88}$$

where $\nu \in [0, 1/3)$ and $\xi > 0$.
Substituting (3.86),(3.87),(3.88) in (3.85) we get

$$|(\ g, \breve{G}_{i\sqrt{\lambda}}(\sigma_N - \bar{\sigma}_N)\ )| \leq cN^{-\beta}\|g\|_2\|f\|_2 \qquad \forall\beta \in [0, 1/6) \tag{3.89}$$

which, together with (3.83), completes the proof of the proposition.
Q.E.D.

The analysis of the asymptotic behaviour for $N \to +\infty$ of the problem (3.67) can be reproduced with only minor changes to study the corresponding exterior Robin boundary value problem

$$(-\Delta+\lambda)v_N = f \qquad in\ R^3 \setminus \bigcup_{j=1}^{N} B_N^j$$

$$\frac{\partial v_N}{\partial n_j^+} - \gamma_N^j v_N = 0 \qquad on\ S_N^j,\ \ j=1,...,N \tag{3.90}$$

where now $f \in L^2(R^3 \setminus \bigcup_{j=1}^N B_N^j)$, $B_N^j = \{x \in R^3 \mid |x-y_j| \leq 1/N\}$ and γ_N^j is given by (3.68).
As usual we represent the solution v_N as

$$v_N = G_{i\sqrt{\lambda}} f + \sum_{j=1}^{N} \check{G}_{i\sqrt{\lambda}} \sigma_N{}^j \tag{3.91}$$

where $\sigma_N{}^j$ are the solutions of the integral equations (3.60)-(3.62), with r and γ_r^j replaced by N^{-1} and γ_N^j.
Again we define the extension $\hat{v}_N$ of v_N to all R^3 by simply taking the natural extension in $\bigcup_{j=1}^N B_N^j$ of the potentials $G_{i\sqrt{\lambda}} f$, $\check{G}_{i\sqrt{\lambda}} \sigma_N{}^j$ and replacing the r.h.s. of the equation by $1_N f$, where now $f \in L^2(R^3)$ and 1_N is the characteristic function of $R^3 \setminus \bigcup_{j=1}^N B_N^j$.
Since all the other steps of the analysis can be carried out following the corresponding steps of the previous case, we simply state the convergence result omitting the details of the proof.

Proposition 3.4.2 $\forall \epsilon > 0$, $\forall f \in L^2(R^3)$, *and* λ *positive sufficiently large*

$$\lim_N P^{(N)}(\ \{Y^{(N)} \mid \|\hat{v}_N - (-\Delta - V/\alpha + \lambda)^{-1} f\|_2 > \epsilon\}\ ) = 0 \tag{3.92}$$

Remark. It should be emphasized that in the proof of the propositions (3.4.1), (3.4.2) the role of the point interactions (more precisely of $-\Delta_{N\alpha^{(N)},Y^{(N)}}$) is that of a good approximation for $N \to +\infty$ of the solutions u_N or v_N of the problems under consideration.

The idea of this procedure is taken from [25] where, in the corresponding Dirichlet problem the proof is divided into two parts: first the real Green's function is approximated by the potential of suitable chosen point charges and then for such potential the limit for $N \to +\infty$ is explicitely computed. The main difference between the two situations is that in the Dirichlet problem the effectiveness of the point charges approximation is proved using the maximum principle while in the present case we are forced to study the integral equations of the potential theory.
Even if slightly more intricate this last method seems more general and in principle it could be applied to a large variety of boundary value problems for p.d.e. (e.g. Neumann problem for the Laplace equation, Navier-Stokes equation and so on).

Appendix

Time-dependent propagator for the one-center point interaction in three dimensions

The aim of this appendix is to compute the explicit form of the time-dependent propagator for the Schrödinger operator $-\Delta_{\alpha,y}$ with a point interaction in $y \in R^3$ of strength $\alpha \in R$.
(In order to simplify the notation we fix y to be the origin o of R^3).
The derivation is straightforward and we report it because, as far as we know, it never appeared in the literature.
In recent papers (see e.g. [60]) only the simpler case of a point interaction in dimension one has been treated.

We start rewriting the integral kernel of the resolvent of $-\Delta_{\alpha,o}$ (see formula (.1) of the introduction in the case $N = 1$)

$$(-\Delta_{\alpha,o} - k^2)^{-1}(x, x') = G_k(x - x') + \frac{1}{\alpha - ik/4\pi} G_k(x) G_k(x') \tag{4.1}$$

where $k \in C$, $Im\, k > 0$, $x, x' \in R^3$.
Computing the inverse Laplace transform (see e.g. [1]), we get the semi-group integral kernel associated with $-\Delta_{\alpha,o}$

$$e^{-z(-\Delta_{\alpha,o})}(x, x') = G_0(x, x'; -iz) + \\ + \frac{1}{4\pi\sqrt{\pi z}|x||x'|} \left[e^{-\frac{(|x|+|x'|)^2}{4z}} - 4\pi\alpha \int_0^{+\infty} e^{-4\pi\alpha u} e^{-\frac{(u+|x|+|x'|)^2}{4z}} du \right] \tag{4.2}$$

where $Re\, z > 0$ and

$$G_0(x, x'; t) = \frac{1}{(4\pi i t)^{3/2}} e^{-\frac{|x-x'|^2}{4it}} \tag{4.3}$$

is the free propagator.
For $\alpha \geq 0$ the corresponding unitary group is obtained from (4.2) after the substitution $z \to it$. In particular, for $\alpha > 0$, we can integrate by parts and obtain the following alternative representation

$$e^{-it(-\Delta_{\alpha,o})}(x,x') = G_0(x,x';t) + \\ + \frac{1}{|x||x'|}\int_0^{+\infty} e^{-4\pi\alpha u}(u+|x|+|x'|)G_0(u+|x|+|x'|,0;t)du \quad (4.4)$$

For $\alpha < 0$ the convergence of the integral in (4.2) for real time may seem problematic. This fact is related to the existence of exactly one bound state for $-\Delta_{\alpha,o}$ (see e.g. [7] pag. 15).
It turns out that the explicit form of the propagator is more conveniently written in such a way to isolate the specific contribution of the bound state. More precisely we write

$$\int_0^{+\infty} e^{-4\pi\alpha u} e^{-\frac{(u+|x|+|x'|)^2}{4z}} du = \\ = \int_{-\infty}^{+\infty} e^{-4\pi\alpha u} e^{-\frac{(u+|x|+|x'|)^2}{4z}} du - \int_{-\infty}^{0} e^{-4\pi\alpha u} e^{-\frac{(u+|x|+|x'|)^2}{4z}} du = \\ = \sqrt{4\pi z}\; e^{z(4\pi\alpha)^2} e^{4\pi\alpha(|x|+|x'|)} - \int_0^{+\infty} e^{4\pi\alpha u} e^{-\frac{(u-|x|-|x'|)^2}{4z}} du$$

Using the last equality we can now substitute $z \to it$ and obtain the representation of the unitary group for $\alpha < 0$

$$e^{-it(-\Delta_{\alpha,o})}(x,x') = G_0(x,x';t) + \\ + \frac{1}{|x||x'|}\int_0^{+\infty} e^{4\pi\alpha u}(u-|x|-|x'|)G_0(u-|x|-|x'|,0;t)du + \\ + (-2\alpha)\,\frac{e^{4\pi\alpha(|x|+|x'|)}}{|x||x'|}\, e^{it(4\pi\alpha)^2} \quad (4.5)$$

We observe that the integral in (4.5) is exactly the same as in (4.4) (except for a change $\alpha \to -\alpha$, $|x|+|x'| \to -|x|-|x'|$) while the last term is the contribution due to the ground state.

Remark. We emphasize that the knowledge of the explicit form of the time-dependent propagator can be used to get more information on the motion of a particle subject to a point interaction.
Following the line of [60] one can describe the explicit time evolution of a wave packet, e.g. a gaussian wave packet, comparing it with the usual time-independent scattering theory and, more important, one can investigate the meaning of the semiclassical approximation for $-\Delta_{\alpha,o}$.

List of the main symbols

References

1. Abramowitz M., Stegun I.A., *Handbook of Mathematical Functions.* Dover, New York, 1972.

2. Albeverio S., Fenstad J.E., Hoegh-Krohn R., Singular perturbations and nonstandard analysis. *Trans. Amer. Math. Soc.* **252**(1979) 275-295.

3. Albeverio S., Fenstad J.E., Hoegh-Krohn R., Karwowski W., Lindstrom T., Perturbations of the Laplacian supported by null sets, with applications to polymer measure and quantum fields. *Phys. Lett.* **104A** (1984) 396-400.

4. Albeverio S., Fenstad J.E., Hoegh-Krohn R., Karwowski W., Lindstrom T., Schrödinger operators with potentials supported by null sets. Preprint BiBos, University of Bielefeld, in preparation.

5. Albeverio S., Fenstad J.E., Hoegh-Krohn R., Lindstrom T., *Non Standard Methods in Stochastic Analysis and Mathematical Physics.* Academic Press, New York-San Francisco-London, 1986.

6. Albeverio S., Figari R., Gesztesy F., Hoegh-Krohn R., Holden H., Kirsch W., Point interaction Hamiltonians for crystals with random defects. Preprint BiBos, University of Bielefeld, n. 348/88.

7. Albeverio S., Gesztesy F., Hoegh-Krohn R., Holden H., *Solvable Models in Quantum Mechanics.* Springer-Verlag, New York, 1988.

8. Albeverio S., Hoegh-Krohn R., Schrödinger operator with point interactions and short-range expansions. *Physica* **124A** (1984) 11-28.

9. Albeverio S., Hoegh-Krohn R., Streit L., Energy forms, Hamiltonians and distorted brownian paths. *J. Math. Phys.* **18** (1977) 907-917.

10. Albeverio S., Hoegh-Krohn R., Wu T.T., A class of exactly solvable three-body quantum mechanical problems and universal low-energy behavior. *Phys. Lett.* **83A** (1981) 105-109.

11. Antoine J.P., Gesztesy F., Shabani J., Exactly solvable models of sphere interactions in quantum mechanics. *J. Phys.* **A20** (1987) 3687-3712.

12. Asakura F., On the Green's function for $\Delta - \lambda^2$ with the boundary condition of the third kind in the exterior domain of a bounded obstacle. *J. Math. Kyoto Univ.* **18-3** (1978) 615-625.

13. Bacon G.E., *Neutron Diffraction.* Clarendon Press, Oxford, 1975.

14. Balzano M., Random relaxed Dirichlet problems. To appear in *Ann. Mat. Pura Appl.*.

15. Berezin F.A., Faddeev L.D., A remark on Schrödinger's equation with a singular potential. *Soviet. Math. Dokl.* **2** (1961) 372-372.

16. Bethe H., Peierls R., Quantum theory of the diplon. *Proc. Roy. Soc.* **148A** (1935) 146-156.

17. Blatt J.M., Weisskopf V.F., *Theoretical Nuclear Physics.* Springer-Verlag, New York-Heidelberg-Berlin, 1979.

18. Brasche J.F., Perturbations of self-adjoint operators supported by null sets, Dissertation zur Erlangung des Doktorgrades der Fakultät für Mathematik der Universität Bielefeld, 1988.

19. Brasche J.F., Teta A., Spectral analysis for the Schrödinger operator with an interaction supported by a regular curve. Preprint Ruhr-Universität Bochum, in preparation.

20. Demkov Y.N., Ostrowskii V.N., *The Use of Zero-Range Potentials in Atomic Physics.* (in Russian), Nauka, Moscow, 1975; transl. Plenum Press.

21. Dürr D., Figari R., Zanghi N., preprint in preparation.

22. Edwards S.F., Functional problems in the theory of polymers. In *Functional Integration and Its Applications*. A.M. Arthur ed., Clarendon Press, Oxford, 1975.

23. Fermi E., Sul moto dei neutroni nelle sostanze idrogenate. *Ricerca Scientifica* **7** (1936) 13-52.

24. Figari R., Holden H., Teta A., A law of large numbers and a central limit theorem for the Schrödinger operator with zero-range potentials. *J. Stat. Phys.* **51** (1988) 205-214.

25. Figari R., Orlandi E., Teta A., The Laplacian in regions with many small obstacles: Fluctuations around the limit operator. *J. Stat. Phys.* **41** (1985) 465-487.

26. Figari R., Orlandi E., Teta A., A central limit theorem for the Laplacian in regions with many small holes. In *Stochastic Processes-Mathematics and Physics II.* S. Albeverio, P. Blanchard, L. Streit ed., Lect. Notes in Mathematics No. 1250. Springer-Verlag, New York, 1987.

27. Figari R., Papanicolaou G., Rubinstein J., The point interaction approximation for diffusion in regions with many small holes. Preprint New York University, 1986.

28. Figari R., Teta A., On the connection between point interactions and boundary value problems. Preprint in preparation.

29. Flamand G., Mathematical theory of non-relativistic two- and three-particle systems with point interactions. In *Cargese Lecture in Theoretical Physics 1967.* F. Lurcat ed., Gordon and Breach, New York, 1967.

30. Fukushima M., *Dirichlet Forms and Markov Processes.* North-Holland/Kodansha, Amsterdam-Oxford-New York, 1980.

31. Gaveau B., Schulman L.S., Explicit time-dependent Schrödinger propagators. *J. Phys.* **A19** (1986) 1833-1846.

32. Gilbarg D., Trudinger N., *Elliptic Partial Differential Equations of Second Order.* Springer, New York, 1977.

33. Gottfried K., *Quantum Mechanics. Vol I: Fundamentals.* W.A. Benjamin Inc., New York-Amsterdam, 1966.

34. Grossmann A., Wu T.T., A class of potentials with extremely narrow resonances. *Chin. J. Phys.* **25** (1987) 129-139.

35. Grossmann A., Wu T.T., Fermi pseudopotentials in higher dimensions. *J. Math. Phys.* **25** (1984) 1742-1745.

36. Günter N., *Potential theory and its applications to basic problems of Mathematical Physics.* Ungar, New York, 1967.

37. Höhler G., *Neutron Physics.* Springer Tracts in Modern Physics, vol. 80, Springer-Verlag, Berlin, 1977.

38. Hruslov E.J., The method of orthogonal projections and the Dirichlet problem in domains with a fine-grained boundary. *Math. USSR Sb.* **17** (1972) 37-59.

39. Hruslov E.J., The asymptotic behavior of solutions of the second boundary value problems under fragmentation of the boundary of the domain. *Math. USSR Sbornik* **35** (1979) 266-282.

40. Huang K., *Statistical Mechanics.* Wiley, New York-London-Sidney, 1966.

41. Hughes D.J., *Neutron Optics.* Interscience, New York, 1954.

42. Kato T., *Perturbation Theory for Linear Operators.* Springer- Verlag, Berlin-Heidelberg-New York, 1980.

43. Kronig R.L., Penney W.G., Quantum mechanics of electrons in crystal lattices. *Proc. Roy. Soc.* **130A** (1931) 499-513.

44. Kellogg O.D., *Foundations of Potential Theory.* Springer-Verlag, Berlin, 1929.

45. Koshmanenko V.D., Singular perturbations defined by forms. Preprint BiBos, University of Bielefeld, n. 314/88.

46. Kurylev Y.V., Boundary conditions on a curve for a three-dimensional Laplace operator. *J. Soviet Math.* **22** (1983) no 1.

47. Landau L.D., Lifshits E.M., *Quantum Mechanics.* Pergamon Press, Oxford, 1965.

48. Lovesey S.W., *Theory of Neutron Scattering from Condensed Matter.* Clarendon Press, Oxford, 1984.

49. Milnos R.A., Faddeev L.D., On the point interaction for a three particle system in quantum mechanics. *Soviet Phys. Dokl.* **6** (1962) 1072-1074.

50. Mott N.F., Massey H.S.W., *The Theory of Atomic Collisions. Third edition.* Claredon Press, Oxford, 1987.

51. Newton R.G., *Scattering Theory of Wave and Particles.* Springer-Verlag, New York-Heidelberg-Berlin, 1982.

52. Ozawa S., On an elaboration of M. Kac's theorem concerning the eigenvalue of the Laplacian in a region with randomly distributed small obstacles. *Comm. Math. Phys.* **91** (1983) 473-487.

53. Papanicolau G., Varadhan S.R.S., Diffusions in regions with many small holes. In Lect. Notes in Control and Information No. 75, Springer, (1980), 190-206.

54. Ramm A.G., *Scattering by Obstacles.* D. Reidel Publishing Company, Dordrecht, 1986.

55. Reed M., Simon B., *Methods of Modern Mathematical Physics. I. Functional Analysis.* Academic Press, New York-London, 1972.

56. Reed M., Simon B., *Methods of Modern Mathematical Physics. II. Fourier Analysis, Self-Adjointness.* Academic Press, New York-San Francisco- London, 1975.

57. Reed M., Simon B., *Methods of Modern Mathematical Physics. III. Scattering Theory.* Academic Press, New York-San Francisco-London, 1979.

58. Reed M., Simon B., *Methods of Modern Mathematical Physics. IV. Analysis of Operators.* Academic Press, New York-San Francisco-London, 1978.

59. Rozanov Y.A., On the Schrödinger type equation with generalized potential. *Math. USSR Sbornik* **55** (1986) 475-484.

60. Schulman L.S., Application of the propagator for the delta-function potential. In *Path Integrals from meV to MeV, Bielefeld Encounters in Physics and Mathematics VII.* M.C. Gutzwiller, A. Inomata, J.R. Klauder, L. Streit ed., World Scientific, Singapore, 1986, 302-311.

61. Sears V.F., Dynamical theory of neutron diffraction. *Canad. J. Phys.* **56** (1978) 1261-1288.

62. Sears V.F., Fundamental aspects of neutron optics. *Phys. Rep.* **82 No.1** (1982) 1-29.

63. Sears V.F., Schrödinger equation for neutron optics. *Physica* **B 151** (1988) 156-159.

64. Sears V.F., *Neutron Optics.* Oxford University Press, Oxford, 1989.

65. Simon B., *Quantum Mechanics for Hamiltonians Defined as Quadratic Forms.* Princeton University Press, Princeton NJ, 1971.

66. Simon B., *Trace Ideals and Their Applications.* Cambridge University Press, Cambridge, 1979.

67. Svendsen E.C., The effect of submanifolds upon essential self-adjointness and deficiency indices. *J. Math. Anal. Appl.* **80** (1981) 551-565.

68. Teta A., Quadratic forms for singular perturbations of the Laplacian. Preprint S.I.S.S.A. 165 FM (Dec 88) (to appear in *Publ. of the R.I.M.S. Kyoto University*).

69. Thirring W., *A Course in Mathematical Physics 3. Quantum Mechanics of Atoms and Molecules.* Springer-Verlag, New York-Wien, 1981.

70. Thomas L.E., Birman-Schwinger bounds for the Laplacian with point interactions. *J. Math. Phys.* **20** (1979) 1848-1850.

71. Thomas L.E., Multiparticle Schrödinger Hamiltonians with point interactions. *Phys. Rev.* **D 30** (1984) 1233-1237.

72. Thomas L.H., The interaction between a neutron and a proton and the structure of H^3. *Phys. Rev.* **47** (1935) 903-909.

73. Weidmann J., *Linear Operators in Hilbert Spaces.* Springer-Verlag, New York-Heidelberg-Berlin, 1980.

74. Wu T.T., Fermi pseudopotentials and resonances in arrays. In *Resonances-Models and Phenomena. Proceedings Bielefeld, 1984.* S. Albeverio, L.S. Ferreira, L. Streit ed., Lect. Notes in Physics vol. 211. Springer-Verlag, Berlin-Heidelberg-New York-Tokyo, 1984, 293-306.