SCUOLA INTERNAZIONALE SUPERIORE DI STUDI AVANZATI

Area of Mathematics

Variational aspects of Liouville equations and systems

Ph.D. Thesis

Supervisor:

Prof. Andrea Malchiodi

Candidate:

Aleks Jevnikar

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Chapter 1

Introduction

1.1 The Toda system

The Toda system

$$-\Delta u_i(x) = \sum_{j=1}^{N} a_{ij} e^{u_j(x)}, \qquad x \in \Sigma,\ i = 1, \dots, N, \tag{1.1}$$

where Δ is the Laplace operator and $A = (a_{ij})_{ij}$ the *Cartan matrix* of $SU(N+1)$,

$$A = \begin{pmatrix} 2 & -1 & 0 & \dots & \dots & 0 \\ -1 & 2 & -1 & 0 & \dots & 0 \\ 0 & -1 & 2 & -1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & -1 & 2 & -1 \\ 0 & \dots & \dots & 0 & -1 & 2 \end{pmatrix},$$

plays an important role in geometry and mathematical physics. In geometry it appears in the description of holomorphic curves in $\mathbb{CP}^n$, see [11], [16], [24], [50]. In mathematical physics, it is a model for non-abelian Chern-Simons vortices, which might have applications in high-temperature superconductivity and which appear in a much wider variety compared to the Yang-Mills framework, see e.g. [92], [93] and [98] for further details and an up-to-date set of references.

The existence of abelian Chern-Simons vortices has been quite deeply investigated in the literature, see e.g. [14], [18], [79], [87], [91]. The study of the non-abelian case is more recent, and we refer for example to [38], [55], [58], [66], [80], [95].

We will be interested in the following problem on a compact surface Σ. For the sake of simplicity, we will assume that $Vol_g(\Sigma) = 1$.

$$\begin{cases} -\Delta u_1 = 2\rho_1 \left(\frac{h_1 e^{u_1}}{\int_\Sigma h_1 e^{u_1} dV_g} - 1 \right) - \rho_2 \left(\frac{h_2 e^{u_2}}{\int_\Sigma h_2 e^{u_2} dV_g} - 1 \right) - 4\pi \sum_{j=1}^m \alpha_{1,j} (\delta_{p_j} - 1), \\ -\Delta u_2 = 2\rho_2 \left(\frac{h_2 e^{u_2}}{\int_\Sigma h_2 e^{u_2} dV_g} - 1 \right) - \rho_1 \left(\frac{h_1 e^{u_1}}{\int_\Sigma h_1 e^{u_1} dV_g} - 1 \right) - 4\pi \sum_{j=1}^m \alpha_{2,j} (\delta_{p_j} - 1). \end{cases} \tag{1.2}$$

Here ρ_1, ρ_2 are real parameters, h_1, h_2 are smooth positive functions and $\alpha_{i,j} \geq 0$. The above system arises specifically from gauged self-dual Schrödinger equations, see e.g. Chapter 6 in [98]: the Dirac deltas represent *vortices* of the wave function, namely points where the latter vanishes.

To describe the history and the main features of the problem, we first desingularize the equation using a simple change of variables. Consider indeed the fundamental solution $G_p(x)$ of the Laplace equation on Σ with pole at p, i.e. the unique solution to

$$-\Delta G_p(x) = \delta_p - \frac{1}{|\Sigma|} \quad \text{on } \Sigma, \qquad \text{with} \quad \int_\Sigma G_p(x)\, dV_g(x) = 0. \tag{1.3}$$

By the substitution

$$u_i(x) \mapsto u_i(x) + 4\pi \sum_{j=1}^{m} \alpha_{i,j} G_{p_j}(x), \qquad h_i(x) \mapsto \widetilde{h}_i(x) = h_i(x) e^{-4\pi \sum_{j=1}^m \alpha_{i,j} G_{p_j}(x)} \tag{1.4}$$

problem (1.2) transforms into an equation of the type

$$\begin{cases} -\Delta u_1 = 2\rho_1 \left(\frac{\widetilde{h}_1 e^{u_1}}{\int_\Sigma \widetilde{h}_1 e^{u_1} dV_g} - 1 \right) - \rho_2 \left(\frac{\widetilde{h}_2 e^{u_2}}{\int_\Sigma \widetilde{h}_2 e^{u_2} dV_g} - 1 \right), \\ -\Delta u_2 = 2\rho_2 \left(\frac{\widetilde{h}_2 e^{u_2}}{\int_\Sigma \widetilde{h}_2 e^{u_2} dV_g} - 1 \right) - \rho_1 \left(\frac{\widetilde{h}_1 e^{u_1}}{\int_\Sigma \widetilde{h}_1 e^{u_1} dV_g} - 1 \right), \end{cases} \tag{1.5}$$

where the functions $\widetilde{h}_j$ satisfy

$$\widetilde{h}_i > 0 \quad \text{on } \Sigma \setminus \{p_1, \dots, p_m\}; \qquad \widetilde{h}_i(x) \simeq d(x, p_j)^{2\alpha_{i,j}}, \text{ near } p_j, \ i = 1, \ 2. \tag{1.6}$$

Problem (1.5) is variational, and solutions can be found as critical points of the Euler-Lagrange functional $J_\rho : H^1(\Sigma) \times H^1(\Sigma) \to \mathbb{R}$ ($\rho = (\rho_1, \rho_2)$) given by

$$J_\rho(u_1, u_2) = \int_\Sigma Q(u_1, u_2)\, dV_g + \sum_{i=1}^{2} \rho_i \left(\int_\Sigma u_i dV_g - \log \int_\Sigma \widetilde{h}_i e^{u_i} dV_g \right), \tag{1.7}$$

where $Q(u_1, u_2)$ is defined as:

$$Q(u_1, u_2) = \frac{1}{3} \left(|\nabla u_1|^2 + |\nabla u_2|^2 + \nabla u_1 \cdot \nabla u_2 \right). \tag{1.8}$$

The main difficulties in attacking (1.5) are mainly of two kinds: compactness issues and the Morse-structure of the functional, which we are going to describe below.

As many geometric problems, also (1.5) presents loss of compactness phenomena, as its solutions might blow-up. To describe the general phenomenon it is first convenient to discuss the case of the scalar counterpart of (1.5), namely a Liouville equation in the form

$$-\Delta u = 2\rho \left(\frac{\widetilde{h}\, e^u}{\int_\Sigma \widetilde{h}\, e^u dV_g} - 1 \right), \tag{1.9}$$

where $\rho \in \mathbb{R}$ and $\widetilde{h}$ behaves as in (1.6) near the singularities. Equation (1.9) rules the change of Gaussian curvature under conformal deformation of the metric, see [1], [19], [20], [59] and [86]. More precisely, letting $\tilde{g} = e^{2v} g$, the Laplace-Beltrami operator of the deformed metric is given by $\Delta_{\tilde{g}} = e^{-2v} \Delta_g$ and the change of the Gauss curvature is ruled by

$$-\Delta_g v = K_{\tilde{g}} e^{2v} - K_g,$$

where K_g and $K_{\tilde{g}}$ are the Gauss curvatures of (Σ, g) and of $(\Sigma, \tilde{g})$ respectively. Another motivation for the study of (1.9) is in mathematical physics as it models the mean field equation of Euler flows, see [15] and [56]. This equation has been very much studied in the literature; there are by now many results regarding existence, compactness of solutions, bubbling behavior, etc. We refer the interested reader to [6], [17], [29], [30], [31], [32], [35], [36], [67] and the reviews [68], [93].

Concerning (1.9) it was proved in [13], [60] and [61] that for the regular case a blow-up point $\overline{x}_R$ for a sequence $(u_n)_n$ of solutions relatively to $(\rho_n)_n$, i.e. there exists a sequence $x_n \to \overline{x}_R$ such that $u_n(x_n) \to +\infty$ as $n \to +\infty$, satisfies the following quantization property:

$$\lim_{r \to 0} \lim_{n \to +\infty} \rho_n \frac{\int_{B_r(\overline{x}_R)} \widetilde{h}\, e^{u_n}\, dV_g}{\int_\Sigma \widetilde{h}\, e^{u_n}\, dV_g} = 4\pi. \tag{1.10}$$

Somehow, each blow-up point has a quantized local mass. Furthermore, the limit profile of solutions is close to a *bubble*, namely a function $U_{\lambda,p}$ defined as

$$U_{\lambda,p}(y) = \log \left(\frac{4\lambda}{\left(1 + \lambda\, d(p, y)^2\right)^2} \right),$$

where $y \in \Sigma$, $d(p, y)$ stands for the geodesic distance and λ is a large parameter. In other words, the limit function is the logarithm of the conformal factor of the stereographic projection from S^2 onto $\mathbb{R}^2$, composed with a dilation.

For the singular case instead, it was proven in [7] and [5] that if blow-up occurs at a singular point $\overline{x}_S$ with weight $-4\pi\alpha$ then one has

$$\lim_{r\to 0}\lim_{n\to+\infty}\rho_n \frac{\int_{B_r(\overline{x}_S)} \widetilde{h}\, e^{u_n}\, dV_g}{\int_\Sigma \widetilde{h}\, e^{u_n}\, dV_g} = 4\pi(1+\alpha), \tag{1.11}$$

whereas (1.10) still holds true if blow-up occurs at a regular point.

This behaviour helps to explain the blow-up feature for system (1.5), which inherits some character from the scalar case. Consider first the regular case, that is, (1.2) with $\alpha_{i,j}=0$. Here a sequence of solutions can blow-up in three different ways: one component blows-up and the other does not; one component blows-up faster than the other; both components blow-up at the same rate.

It was proved in [49] and [51] that the quantization values for the two components are respectively $(4\pi,0)$ or $(0,4\pi)$ in the first case, $(8\pi,4\pi)$ or $(4\pi,8\pi)$ in the second case and $(8\pi,8\pi)$ in the third one. Notice that, by the results in [33], [39] and [75], all the five alternatives may indeed happen. See also [27] and [28] for further analysis in this direction.

When singular sources are present a similar phenomenon happens, which has been investigated in the paper [63]. If blow-up occurs at a point p with singular weights α_1,α_2 (we may allow them to vanish), the corresponding blow-up values would be

$$\begin{aligned} &(4\pi(1+\alpha_1),0); \qquad (0,4\pi(1+\alpha_2)); \qquad (4\pi(1+\alpha_1),4\pi(2+\alpha_1+\alpha_2)); \\ &(4\pi(2+\alpha_1+\alpha_2),4\pi(1+\alpha_2)); \qquad (4\pi(2+\alpha_1+\alpha_2),4\pi(2+\alpha_1+\alpha_2)). \end{aligned} \tag{1.12}$$

Other (finitely-many) blow-up values are indeed allowed, as more involved situations are not yet excluded (or known to exist). Consider a point p at which (1.2) has singular weights $\alpha_1=\alpha_1(p), \alpha_2=\alpha_2(p)$ in the first and the second component of the equation. We give then the following two definitions.

Definition 1.1.1 *Given a couple of non-negative numbers* (α_1,α_2) *we let* $\Gamma_{\alpha_1,\alpha_2}$ *be the subset of an ellipse in* $\mathbb{R}^2$ *defined by the equation*

$$\Gamma_{\alpha_1,\alpha_2} := \left\{(\sigma_1,\sigma_2)\ :\ \sigma_1,\sigma_2\ge 0, \sigma_1^2-\sigma_1\sigma_2+\sigma_2^2 = 2(1+\alpha_1)\sigma_1+2(1+\alpha_2)\sigma_2\right\}.$$

We then let $\Lambda_{\alpha_1,\alpha_2}\subseteq\Gamma_{\alpha_1,\alpha_2}$ *be the set constructed via the following rules:*

1. *the points* $(0,0)$, $(2(1+\alpha_1),0), (0,2(1+\alpha_2)), (2(1+\alpha_1),2(2+\alpha_1+\alpha_2))$, $(2(2+\alpha_1+\alpha_2),2(1+\alpha_2))$, $(2(2+\alpha_1+\alpha_2),2(2+\alpha_1+\alpha_2))$ *belong to* $\Lambda_{\alpha_1,\alpha_2}$*;*

2. *if* $(a,b)\in\Lambda_{\alpha_1,\alpha_2}$ *then also any* $(c,d)\in\Gamma_{\alpha_1,\alpha_2}$ *with* $c=a+2m$*,* $m\in\mathbb{N}\cup\{0\}$*,* $d\ge b$ *belongs to* $\Lambda_{\alpha_1,\alpha_2}$*;*

3. *if* $(a,b)\in\Lambda_{a_1,\alpha_2}$ *then also any* $(c,d)\in\Gamma_{\alpha_1,\alpha_2}$ *with* $d=b+2n$*,* $n\in\mathbb{N}\cup\{0\}$*,* $c\ge a$ *belongs to* $\Lambda_{\alpha_1,\alpha_2}$*.*

Definition 1.1.2 *Given* $\Lambda_{\alpha_1,\alpha_2}$ *as in Definition 1.1.1, we set*

$$\Lambda_0 = 2\pi\left\{(2p,2q)+\sum_{j=1}^m n_j(a_j,b_j)\ :\ p,q\in\mathbb{N}\cup\{0\}, n_j\in\{0,1\}, (a_j,b_j)\in\Lambda_{\alpha_{1,j},\alpha_{2,j}}\right\};$$

$$\Lambda_i = 4\pi\left\{n+\sum_{j=1}^m(1+\alpha_{i,j})n_j,\ n\in\mathbb{N}\cup\{0\},\ n_j\in\{0,1\}\right\}, \qquad i=1,2.$$

We finally set

$$\Lambda = \Lambda_0\cup(\Lambda_1\times\mathbb{R})\cup(\mathbb{R}\times\Lambda_2)\subseteq\mathbb{R}^2.$$

Remark 1.1.3 *Observe that in the regular case, namely for* $\alpha_{i,j} = 0$ *for all* i, j*, the set* Λ *is reduced to the standard critical set*

$$\Lambda = (4\pi\mathbb{N} \times \mathbb{R}) \cup (\mathbb{R} \times 4\pi\mathbb{N}).$$

From the local quantization results in [63] and [49] for the singular and for the regular case respectively, and some standard analysis (see in particular Section 1 in [13] and [10]) one finds the following global compactness result.

Theorem 1.1.4 *([10],[49],[63]) For* (ρ_1, ρ_2) *in a fixed compact set of* $\mathbb{R}^2 \setminus \Lambda$ *the family of solutions to* (1.5) *is uniformly bounded in* $C^{2,\beta}$ *for some* $\beta > 0$.

Remark 1.1.5 *There is actually an improvement of the latter result; in fact, Prof. Wei and Prof. Zhang recently informed us that under the assumption* $\alpha_{i,j} \leq C$ *for some positive constant* C*, the corresponding blow-up values are just those stated in* (1.12)*, as one would expect.*

Remark 1.1.6 *The set of lines* $\Lambda_1 \times \mathbb{R}$, $\mathbb{R} \times \Lambda_2$ *refer to the case of blowing-up solutions in which one component remains bounded, so it is not quantized. The quantization of the blowing-up component was obtained in [6] for the singular scalar case.*

Instead, the set Λ_0 *refers to couples* (u_1, u_2) *for which both components blow-up. Observe that* $\Lambda_{\alpha_1,\alpha_2}$ *is finite, and it coincides with the five elements* $(4\pi, 0)$, $(0, 4\pi)$, $(8\pi, 4\pi)$, $(4\pi, 8\pi)$, $(8\pi, 8\pi)$ *when both* α_1 *and* α_2 *vanish. Then,* Λ_0 *is a discrete set.*

In particular, Λ *is a closed set in* $\mathbb{R}^2$ *with zero Lebesgue measure.*

Let us now show how we can study the sub-levels of the functional and conclude existence of solutions via min-max methods. We present here the strategy for the scalar case (1.9); to make the argument clear let us assume that there are no singular sources in the equation or, in other words, let us assume $\widetilde{h} = h$ in (1.9) to be a *positive* smooth function. We recall next the classical Moser-Trudinger inequality, in its weak form

$$\log \int_\Sigma e^{u-\overline{u}} \, dV_g \leq \frac{1}{16\pi} \int_\Sigma |\nabla u|^2 \, dV_g + C; \qquad u \in H^1(\Sigma), \tag{1.13}$$

where C is a constant depending only on Σ and the metric g. The main tool in the variational study of this kind of problems is the so-called Chen-Li inequality, see [23]. In the scalar case, it implies that a suitable *spreading* of the term e^u yields a better constant in the Moser-Trudinger inequality, which in turn might imply a lower bound on the Euler functional $\widetilde{J}_\rho$ of (1.9)

$$\widetilde{J}_\rho(u) = \frac{1}{2} \int_\Sigma |\nabla u|^2 dV_g + 2\rho \left(\int_\Sigma u \, dV_g - \log \int_\Sigma h \, e^u dV_g \right), \qquad u \in H^1(\Sigma). \tag{1.14}$$

The consequence of this fact is that if $\rho < 4(k+1)\pi$, $k \in \mathbb{N}$, and if $\widetilde{J}_\rho(u)$ is large negative (i.e. when lower bounds fail) e^u accumulates near at most k points of Σ, see e.g. [36]. This suggests to introduce the family of unit measures Σ_k which are supported in at most k points of Σ, known as *formal barycenters* of Σ of order k

$$\Sigma_k = \left\{ \sum_{j=1}^k t_j \delta_{x_j} : \sum_{j=1}^k t_j = 1, \ x_j \in \Sigma \right\}. \tag{1.15}$$

One can show that, for any integer k, Σ_k is not contractible and that its homology is mapped injectively into that of the low sub-levels of $\widetilde{J}_\rho$. This allows to prove existence of solutions via suitable min-max schemes for every $\rho \notin 4\pi\mathbb{N}$.

The values $4\pi\mathbb{N}$ are *critical* and the existence problem becomes subtler due to a loss of compactness, see [21], [34] and [78] for discussion in this framework. To solve equation (1.9) (or equation (1.5)) in this case, one always needs geometry conditions, see [34], [99]. For example, for equation (1.9) with $\rho_1 = 4\pi$ and $\rho_2 \in (0, 4\pi]$, in [99] the author gave an existence result under suitable conditions on the Gaussian curvature $K(x)$ of Σ, namely $K(x)$ should satisfy

$$4\pi - \rho_2 - K(x) > 0 \qquad \text{for any } x \in \Sigma.$$

We return now to the Toda system; as we observed, a basic tool for studying functionals like J_ρ is the Moser-Trudinger inequality, see (1.13). Its analogue for the Toda system has been obtained in [50] and reads as

$$4\pi \sum_{i=1}^{2} \left(\log \int_\Sigma h_i e^{u_i} dV_g - \int_\Sigma u_i dV_g \right) \leq \int_\Sigma Q(u_1, u_2)\, dV_g + C \qquad \forall\, u_1, u_2 \in H^1(\Sigma), \tag{1.16}$$

for some $C = C(\Sigma)$. This inequality immediately allows to find a global minimum of J_ρ provided both ρ_1 and ρ_2 are less than 4π. For larger values of the parameters ρ_i J_ρ is unbounded from below and the problem becomes more challenging.

Concerning the regular Toda system, a first existence result in this direction was presented in [69] for $\rho_1 \in (4k\pi, 4(k+1)\pi)$, $k \in \mathbb{N}$ and $\rho_2 < 4\pi$. When one of the two parameters is small, the system (1.5) resembles the scalar case (1.9) and one can adapt the above argument to this framework as well. When both parameters exceed the value 4π, the description of the low sub-levels becomes more involved due to the interaction of the two components u_1 and u_2.

The first variational approach to understand this interaction was given in [71], where the authors obtained an existence result for $(\rho_1, \rho_2) \in (4\pi, 8\pi)^2$. This was done in particular by showing that if both components of the system concentrate near the same point and with the same *rate*, then the constants in the left-hand side of (1.16) can be nearly doubled.

The study of more general non-coercive regimes is the topic of Chapters 2, 3, see the next subsections.

1.1.1 A general existence result on compact surfaces of positive genus: Chapter 2

In the Chapter 2 we use min-max theory to find a critical point of J_ρ in a general non-coercive regime. Our main result, which is obtained in [9], is the following:

Theorem 1.1.7 *Let $\alpha_{i,j} \geq 0$ and let $\Lambda \subset \mathbb{R}^2$ be as in Definition 1.1.2. Let Σ be a compact surface neither homeomorphic to S^2 nor to $\mathbb{RP}^2$, and assume that $(\rho_1, \rho_2) \notin \Lambda$. Then the singular Toda system* (1.2) *is solvable.*

Let us point out that $\Lambda \subseteq \mathbb{R}^2$ is an explicit set formed by an union of straight lines and discrete points, see Remark 1.1.6. In particular it is a closed set with zero Lebesgue measure.

Up to our knowledge, *there is no previous existence result in the literature* for the singular Toda system. Our result is hence the first one in this direction, and is generic in the choice of parameters ρ_1 and ρ_2. In the regular case there are some previous existence results, see [49], [66], [69] and [71], some of which have a counterpart in [35] and [36] for the scalar case (1.9) (see also [37] for a higher order problem and [7], [4], [17] and [70] for the singular case). However, these require an upper bound either on one of the ρ_i's or both: hence our result covers most of the unknown cases also for the regular problem.

When both ρ_1 and ρ_2 are larger than 4π the description of the sub-levels becomes more involved, since the two components u_1 and u_2 interact in a non-trivial way. See [71] on this respect. We obtain here a *partial* topological characterization of the low energy levels of J_ρ, which is however sufficient for our purposes. This strategy has been used in [3] and in [2] for the singular scalar equation and for a model in electroweak theory respectively, while here the general non-abelian case is treated for the first time.

First, we construct two disjoint simple non-contractible curves γ_1, γ_2 which do not intersect singular points, and define global retractions Π_1, Π_2 of Σ onto these two curves. Such curves do not exist for $\Sigma = S^2$ or $\mathbb{RP}^2$, and hence our arguments do not work in those cases.

Combining arguments from [23], [69] and [71] we prove that if $\rho_1 < 4(k+1)\pi$ and $\rho_2 < 4(l+1)\pi$, $k, l \in \mathbb{N}$, then either $\widetilde{h}_1 e^{u_1}$ is close to Σ_k or $\widetilde{h}_2 e^{u_2}$ is close to Σ_l in the distributional sense. Then we can map continuously (and naturally) $\widetilde{h}_1 e^{u_1}$ to Σ_k or $\widetilde{h}_2 e^{u_2}$ to Σ_l; using then the retractions Π_i one can restrict himself to targets in $(\gamma_1)_k$ or $(\gamma_2)_l$ only. This alternative can be expressed naturally in terms of the *topological join* $(\gamma_1)_k * (\gamma_2)_l$. Roughly speaking, given two topological spaces A and B, the join $A * B$

is the formal set of segments joining elements of A with elements of B. More precisely, the topological join of two sets A, B is defined as the family of elements of the form

$$\frac{\{(a,b,s):\ a \in A,\ b \in B,\ s \in [0,1]\}}{R}, \tag{1.17}$$

where R is an equivalence relation such that

$$(a_1, b, 1) \overset{R}{\sim} (a_2, b, 1) \quad \forall a_1, a_2 \in A, b \in B \qquad \text{and} \qquad (a, b_1, 0) \overset{R}{\sim} (a, b_2, 0) \quad \forall a \in A, b_1, b_2 \in B.$$

The elements of the join are usually written as formal sums $(1-s)a + sb$.
In this way, we are able to define a global projection Ψ from low sub-levels of J_ρ onto $(\gamma_1)_k * (\gamma_2)_l$.

We can also construct a reverse map Φ_λ (where λ is a large parameter) from $(\gamma_1)_k * (\gamma_2)_l$ into arbitrarily low sub-levels of J_ρ using suitable test functions. Moreover, we show that the composition of both maps is homotopic to the identity map. Finally, $(\gamma_1)_k * (\gamma_2)_l$ is homeomorphic to a sphere of dimension $2k+2l-1$ see Remark 2.1.2: in particular it is not contractible, and this allows us to apply a min-max argument.

In this step a compactness property is needed, like the Palais-Smale's. The latter is indeed not known for this problem, but there is a way around it using a monotonicity method from [88]. For that, compactness of solutions comes to rescue, and here we use the results of [49] and [63], see Theorem 1.1.4. This is the reason why we assume $(\rho_1, \rho_2) \notin \Lambda$.

1.1.2 The case of compact surfaces of arbitrary genus: Chapter 3

In Chapter 3 we focus on the regular Toda system, namely

$$\begin{cases} -\Delta u_1 = 2\rho_1 \left(\frac{h_1 e^{u_1}}{\int_\Sigma h_1 e^{u_1} dV_g} - 1 \right) - \rho_2 \left(\frac{h_2 e^{u_2}}{\int_\Sigma h_2 e^{u_2} dV_g} - 1 \right), \\ -\Delta u_2 = 2\rho_2 \left(\frac{h_2 e^{u_2}}{\int_\Sigma h_2 e^{u_2} dV_g} - 1 \right) - \rho_1 \left(\frac{h_1 e^{u_1}}{\int_\Sigma h_1 e^{u_1} dV_g} - 1 \right), \end{cases} \tag{1.18}$$

where ρ_1, ρ_2 are real parameters and h_1, h_2 two positive smooth functions. Notice that in the above equation, differently from equation (1.2), we do not have the presence of singular terms in the right-hand side. We prove here the following result, see [48], which for the first time applies to surfaces of arbitrary genus when both parameters ρ_i are supercritical and one of them also arbitrarily large.

Theorem 1.1.8 *Let h_1, h_2 be two positive smooth functions and let Σ be any compact surface. Suppose that $\rho_1 \in (4k\pi, 4(k+1)\pi), k \in \mathbb{N}$ and $\rho_2 \in (4\pi, 8\pi)$. Then problem* (1.18) *has a solution.*

Remark 1.1.9 *Theorem 1.1.8 is new when Σ is a sphere and $k \geq 3$. As we already discussed, the case of surfaces with positive genus was covered in [9]. The case of $\Sigma \simeq S^2$ and $k = 1$ was covered in [71], while for $k = 2$ it was covered in [62]. In the latter paper the authors indeed computed the Leray-Schauder degree of the equation for the range of ρ_i's in Theorem 1.1.8. It turns out that the degree of* (1.18) *is zero for the sphere when $k \geq 3$: since solutions do exist by Theorem 1.1.8, it means that either they are degenerate, or that degrees of multiple ones cancel, so a global degree counting does not detect them. A similar phenomenon occurs for* (1.9) *on the sphere, when $\rho > 12\pi$, see [22]. Even for positive genus, we believe that our approach could be useful in computing the degree of the equation, as it happened in [67] for the scalar equation* (1.9)*. More precisely we speculate that the degree should be computable as $1 - \chi(Y)$, where the set Y is given in* (3.41)*. This is verified for example in the case of the sphere thanks to Lemma 3.4.4.*

Other results on the degree of the system, but for different ranges of parameters, are available in [72].

As described above, in the situation of Theorem 1.1.8 it is natural to characterize low sub-levels of the Euler-Lagrange energy J_ρ by means of the topological join $\Sigma_k * \Sigma_1$ (notice that $\Sigma_1 \simeq \Sigma$), see (1.17). However, differently from [9], we crucially take into account the interaction between the two components u_1 and u_2. As one can see from (1.8), the quadratic energy Q penalizes situations in which the gradients of the two components are aligned, and we would like to make a quantitative description of this effect. Our proof uses four new main ingredients.

• A refinement of the projection from low-energy sub-levels onto the topological join $\Sigma_k * \Sigma_1$ from [9], see Section 3.2, which uses the scales of concentration of the two components, and which extends some construction in [71]. Having to deal with arbitrarily high values of ρ_1, differently from [71] we also need to take into account of the stratified structure of Σ_k and to the closeness in measure sense to its substrata.

• A new, scaling invariant improved Moser-Trudinger inequality for system (1.18), see Proposition 3.2.5. This is inspired from another one in [4] for singular Liouville equations, i.e. of the form (1.9) but with Dirac masses on the right-hand side. The link between the two problems arises in the situation when one of the two components in (1.18) is much more concentrated than the other: in this case the measure associated to its exponential function resembles a Dirac delta compared to the other one. The above improved inequality gives extra constraints to the projection on the topological join, see Proposition 3.2.7 and Corollary 3.2.8.

• A new set of test functions showing that the characterisation of low energy levels of J_ρ is sharp, as a subset Y of $\Sigma_k * \Sigma_1$. We need indeed to build test functions modelled on a set which contains $\Sigma_{k-1} * \Sigma_1$, and the stratified nature of Σ_{k-1} makes it hard to obtain uniform upper estimates on such functions.

• A new topological argument showing the non-contractibility of the above set Y, which we use then crucially to develop our min-max scheme. The fact that Y is simply connected and has Euler characteristic equal to 1 forces us to use rather sophisticated tools from algebraic topology.

We expect that our approach might extend to the case of general physical parameters ρ_1, ρ_2, including the positive genus case and the *singular Toda system* with $\alpha_{i,j} \geq 0$, in which Dirac masses (corresponding to ramification or vortex points) appear in the right-hand side of (1.18), see also [8] for some results with this approach.

1.2 A mean field equation

The second topic of the thesis is the following class of mean field equations with two parameters on a compact surface Σ, namely a Liouville-type equation:

$$-\Delta u = \rho_1 \left(\frac{h_1 \, e^u}{\int_\Sigma h_1 \, e^u dV_g} - 1 \right) - \rho_2 \left(\frac{h_2 \, e^{-u}}{\int_\Sigma h_2 \, e^{-u} dV_g} - 1 \right), \tag{1.19}$$

where ρ_1, ρ_2 are real parameters and h_1, h_2 are two smooth positive functions. We recall that we are always assuming $Vol_g(\Sigma) = 1$, for the sake of simplicity.

This equation arises in mathematical physics as a mean field equation of the equilibrium turbulence with arbitrarily signed vortices. The mean field limit was first studied by Joyce and Montgomery [53] and by Pointin and Lundgren [83] by means of different statistical arguments. Later, many authors adopted this model, see for example [25], [64], [77] and the references therein. The case $\rho_1 = \rho_2$ plays also an important role in the study of constant mean curvature surfaces, see [96], [97].

Equation (1.19) has a variational structure with associated functional $I_\rho : H^1(\Sigma) \to \mathbb{R}$, with $\rho = (\rho_1, \rho_2)$, defined by

$$I_\rho(u) = \frac{1}{2} \int_\Sigma |\nabla u|^2 \, dV_g - \rho_1 \left(\log \int_\Sigma h_1 \, e^u \, dV_g - \int_\Sigma u \, dV_g \right) - \rho_2 \left(\log \int_\Sigma h_2 \, e^{-u} \, dV_g + \int_\Sigma u \, dV_g \right). \tag{1.20}$$

In [82] the authors derived a Moser-Trudinger inequality for e^u and e^{-u} simultaneously, namely

$$\log \int_\Sigma e^{u - \overline{u}} \, dV_g + \log \int_\Sigma e^{-u + \overline{u}} \, dV_g \leq \frac{1}{16\pi} \int_\Sigma |\nabla u|^2 \, dV_g + C, \tag{1.21}$$

with C depending only of Σ. By this result, solutions to (1.19) can be found immediately as global minima of the functional I_ρ whenever both ρ_1 and ρ_2 are less than 8π. For $\rho_i \geq 8\pi$ the existence problem becomes subtler and there are very few results.

The blow-up behavior of solutions of equation (1.19) is not yet developed in full generality; this analysis was carried out in [52], [81] and [82] under the assumption that $h_1 = h_2$, see in particular Theorem 1.1, Corollary 1.2 and Remark 4.5 in the latter paper. The following quantization property for a blow-up point $\overline{x}$ and a sequence $(u_n)_n$ of solutions relatively to $(\rho_{1,n}, \rho_{2,n})_n$ was obtained:

$$\lim_{r \to 0} \lim_{n \to +\infty} \rho_{1,n} \frac{\int_{B_r(\overline{x})} h \, e^{u_n} \, dV_g}{\int_\Sigma h \, e^{u_n} \, dV_g} \in 8\pi\mathbb{N}, \qquad \lim_{r \to 0} \lim_{n \to +\infty} \rho_{2,n} \frac{\int_{B_r(\overline{x})} h \, e^{-u_n} \, dV_g}{\int_\Sigma h \, e^{-u_n} \, dV_g} \in 8\pi\mathbb{N}. \tag{1.22}$$

As for the Toda system, the case of multiples of 8π may indeed occur, see [40] and [42].

Let now define the set $\widetilde{\Lambda}$ by

$$\widetilde{\Lambda} = (8\pi\mathbb{N} \times \mathbb{R}) \cup (\mathbb{R} \times 8\pi\mathbb{N}) \subseteq \mathbb{R}^2.$$

Combining (1.22) and some standard analysis (see the argument before Theorem 1.1.4) one finds the following result.

Theorem 1.2.1 *([52],[10]) Let (ρ_1, ρ_2) be in a fixed compact set of $\mathbb{R}^2 \setminus \widetilde{\Lambda}$ and assume $h_1 = h_2$. Then the set of solutions to* (1.19) *is uniformly bounded in $C^{2,\beta}$ for some $\beta > 0$.*

Remark 1.2.2 *It seems that above condition $h_1 = h_2$ can be relaxed and that the compactness result holds true for any choice of h_1, h_2. This follows from an improvement of the quantization property* (1.22) *and it is an ongoing project we have with Prof. Jun-cheng Wei and Wen Yang.*

Before introducing our main results we collect here some known existence results. The first one is given in [52] and treats the case $\rho_1 \in (8\pi, 16\pi)$ and $\rho_2 < 8\pi$. Via a blow-up analysis the authors proved existence of solutions on a smooth, bounded, non simply-connected domain Σ in $\mathbb{R}^2$ with homogeneous Dirichlet boundary condition. Later, this result is generalized in [100] to any compact surface without boundary by using variational methods. The strategy is carried out in the same spirit as in [68] and [69] for the Liouville equation (1.9) and the Toda system (1.5), respectively. The proof relies on some improved Moser-Trudinger inequalities obtained in [23]. The idea is that, in a certain sense, one can recover the topology of low sub-levels of the functional I_ρ just from the behaviour of e^u. Indeed the condition $\rho_2 < 8\pi$ guarantees that e^{-u} does not affect the variational structure of the problem.

The doubly supercritical regime, namely $\rho_i > 8\pi$, has to be attacked with a different strategy and is the topic of Chapters 4, 5 and 5.3, see the next subsections.

1.2.1 A first existence result in a doubly supercritical case: Chapter 4

In Chapter 4 for the first time we consider a doubly supercritical case, namely when both parameters ρ_i are greater than 8π. Via a min-max scheme we obtain an existence result without any geometry and topology conditions. Our main theorem is stated in [45] and is the following:

Theorem 1.2.3 *Let h_1, h_2 be two smooth positive functions. Assume that $\rho_1, \rho_2 \in (8\pi, 16\pi)$. Then there exists a solution to equation* (1.19).

The method to prove this existence result relies on a min-max scheme introduced by Malchiodi and Ruiz in [71] for the study of Toda systems; roughly speaking, the idea is that the role of e^{u_2} is played here by e^{-u}. Such a scheme is based on study of the topological properties of the low sub-levels of I_ρ.

We shall see that on low sub-levels at least one of the functions e^u or e^{-u} is very concentrated around some point of Σ. Moreover, both e^u and e^{-u} can concentrate at two points that could eventually coincide, but in this case the *scale of concentration* must be different. Roughly speaking, if e^u and e^{-u} concentrate around the same point at the same rate, then I_ρ is bounded from below. The same phenomenon is present in the regular Toda system (1.18), where the role of $-u$ is played by u_2, see [71]. We next make this statement more formal.

First, following the argument in [71], we define a continuous *rate of concentration* $\sigma = \sigma(f)$ of a positive function $f \in \Sigma$, normalized in L^1. Somehow the smaller is σ, the higher is the rate of concentration of f. Moreover we define a continuous *center of mass* $\beta = \beta(f) \in \Sigma$. This can be done when $\sigma \leq \delta$ for some fixed δ, therefore we have a map $\psi : H^1(\Sigma) \to \overline{\Sigma}_\delta$,

$$\psi(u) = \big(\beta(f_1), \sigma(f_1)\big), \quad \psi(-u) = \big(\beta(f_2), \sigma(f_2)\big),$$

where we have set

$$f_1 = \frac{e^u}{\int_\Sigma e^u \, dV_g}, \quad f_2 = \frac{e^{-u}}{\int_\Sigma e^{-u} \, dV_g}.$$

Here $\overline{\Sigma}_\delta$ is the topological cone over Σ, where we make the identification to a point when $\sigma \geq \delta$ for some $\delta > 0$ fixed, see (1.26). We point out that the argument presented here, involving the topological cone, is equivalent to the construction based on the topological join (1.17) in the spirit of Chapter 3.

The improvement of the Moser-Trudinger inequality discussed above is made rigorous in the following way: if $\psi(f_1) = \psi(f_2)$, then $I_\rho(u)$ is bounded from below, see Proposition 4.2.6. The proof is based on local versions of the Moser-Trudinger inequality on small balls and on annuli with small internal radius (see [71] for the argument for the regular Toda system (1.18)). We point out that our improved inequality is scaling invariant, differently from those proved by Chen-Li and Zhou (see [23] and [100]).

Using this fact, for $L > 0$ large we can introduce a continuous map:

$$I_\rho^{-L} \quad \xrightarrow{(\psi,\psi)} \quad X := \big(\overline{\Sigma}_\delta \times \overline{\Sigma}_\delta\big) \setminus \overline{D},$$

where $\overline{D}$ is the diagonal of $\overline{\Sigma}_\delta \times \overline{\Sigma}_\delta$ and I_ρ^{-L} is the sub-level of the functional, see the notation in Section 1.3. On the other hand, it is also possible to do the converse, namely to map (a retraction of) the set X into appropriate sub-levels of I_ρ. We next construct a family of test functions parametrized on (a suitable subset of) X on which I_ρ attains arbitrarily low values, see Proposition 4.3.4. Letting

$$X \quad \xrightarrow{\phi} \quad I_\rho^{-L}$$

the corresponding map, it turns out that the composition of these two maps is homotopic to the identity on X, see Proposition 4.3.7.

Exploiting the fact that X is not contractible, we are able to introduce a min-max argument to find a critical point of I_ρ. In this framework, an essential point is to use the *monotonicity argument* introduced by Struwe in [88] jointly with a compactness result stated in Theorem 4.1.1, since it is not known whether the Palais-Smale condition holds or not.

1.2.2 Existence and multiplicity results: Chapter 5

In Chapter 5 we consider more generic non-coercive regimes for the equation

$$-\Delta u = \rho_1 \left(\frac{h\, e^u}{\int_\Sigma h\, e^u \, dV_g} - 1 \right) - \rho_2 \left(\frac{h\, e^{-u}}{\int_\Sigma h\, e^{-u} \, dV_g} - 1 \right), \tag{1.23}$$

on a compact surface Σ, where ρ_1, ρ_2 are real parameters and h is a smooth positive function. Notice that we consider here just one potential h, differently from equation (1.19). The reason is that in this case we are allowed to use the general compactness result in Theorem 1.2.1.

The chapter is divided into three parts; the purpose of the first two parts (Sections 5.1 and 5.2) is to give both a general existence result and to address the multiplicity issue. We start by suitably adapting the argument presented for the Toda system in [9], see Subsection 1.1.1, to get the following existence result (still part of the paper [9]), see Section 5.1.

Theorem 1.2.4 *Let h be a smooth positive function. Suppose Σ is not homeomorphic to S^2 nor $\mathbb{RP}^2$, and that $\rho_i \notin 8\pi\mathbb{N}$ for $i = 1, 2$. Then (1.23) has a solution.*

The second part is devoted to the multiplicity problem of equation (1.23) and is part of the paper [47]. The goal is to present the first multiplicity result for this class of equations, see Section 5.2.

Theorem 1.2.5 *Let $\rho_1 \in (8k\pi, 8(k+1)\pi)$ and $\rho_2 \in (8l\pi, 8(l+1)\pi)$, $k, l \in \mathbb{N}$ and let Σ be a compact surface with genus $g(\Sigma) > 0$. Then, for a generic choice of the metric g and of the function h it holds*

$$\#\big\{\textit{solutions of } (1.23)\big\} \geq \binom{k + g(\Sigma) - 1}{g(\Sigma) - 1} \binom{l + g(\Sigma) - 1}{g(\Sigma) - 1}.$$

Here, by generic choice of (g, h) we mean that it can be taken in an open dense subset of $\mathcal{M}^2 \times C^2(\Sigma)^+$, where $\mathcal{M}^2$ stands for the space of Riemannian metrics on Σ equipped with the C^2 norm, see Proposition 5.2.4.

The proof is carried out by means of the Morse theory in the same spirit of [3] and [8], where the problem of prescribing conformal metrics on surfaces with conical singularities and the Toda system are considered, respectively. The argument is based on the analysis developed in Chapter 2, paper [9] (see also Section 1.1.1): in particular we will exploit the topological descriptions of the low sub-levels of I_ρ to get a lower bound on the number of solutions to (1.23). It will turn out indeed that the high sub-levels of I_ρ are contractible, while the low sub-levels carry some non trivial topology. In fact, we will describe the topology of the sub-levels by means of that of a bouquet $\mathcal{B}_N$ of N circles, where $\mathcal{B}_N$ is defined as $\mathcal{B}_N = \cup_{i=1}^N \mathcal{S}_i$, where $\mathcal{S}_i$ is homeomorphic to S^1 and $\mathcal{S}_i \cap \mathcal{S}_j = \{c\}$, and c is called the center of the bouquet, see Figure 5.2. We will finally apply the weak Morse inequalities to deduce the estimate on the number of solutions by means of the latter change of topology. Somehow, one expects that the more the topology of the surface Σ is involved, the higher is the number of solutions. In fact, we will exploit the genus of Σ to describe the topology of low sub-levels of I_ρ by means of some bouquet of circles, see Lemma 5.2.5 and Proposition 5.2.7. In this way we will capture the topological informations of Σ and provide a better bound on the number of solutions to (1.23).

Aim of the last part of the chapter (Section 5.3) is to present, differently from Chapter 4 and the first two parts of Chapter 5, an approach to problem (1.23) based on the associated Leray-Schauder degree. This argument is stated in the note [46] and yields new existence results, see Theorem 1.2.6.

Regarding the regular one-parameter case, namely the classic Liouville equation (1.9), combining the local quantization (1.10) with some further analysis, see for example [10], [13], we have that the set of solutions is uniformly bounded in $C^{2,\alpha}$, for any fixed $\alpha \in (0,1)$, provided $\rho \notin 4\pi\mathbb{N}$. It follows that one can define the Leray-Schauder degree associated to problem (1.9) with $\rho \in (4k\pi, 4(k+1)\pi), k \in \mathbb{N}$. In [60] it was shown that the degree is 1 when $\rho < 4\pi$. By the homotopic invariance of the degree, it is easy to see that the same is independent of the function h, the metric of Σ and it is constant on each interval $(4k\pi, 4(k+1)\pi)$. In fact it depends only on $k \in \mathbb{N}$ and the topological structure of Σ, as was proved in [22], where the authors provide the degree-counting formula

$$\deg(\rho) = \frac{1}{k!}(-\chi(\Sigma)+1)\cdots(-\chi(\Sigma)+k), \tag{1.24}$$

where $\chi(\Sigma)$ denotes the Euler characteristic of Σ. The proof of this result is carried out by analyzing the jump values of the degree after ρ crossing the critical thresholds. Later, this result was rephrased in [67] with a Morse theory point of view.

On the other hand, concerning the mean field equation with two parameters (1.23), by the compactness property in Theore 1.2.1, the associated degree can still be defined for $\rho_i \notin 8\pi\mathbb{N}, i = 1, 2$. However, this strategy has not been yet investigated and the existence results mostly rely on a variational approach.

Goal of Section 5.3 is to attack the problem with a different point of view and for the first time analyze the associated Leray-Schauder degree. This is done in the spirit of [72], where the Toda system (1.18) was analyzed. More precisely, we study its parity and we observe that when both parameters stay in the same interval, i.e. $\rho_i \in (8k\pi, 8(k+1)\pi)$, $k \in \mathbb{N}$ for $i = 1, 2$, the degree is always odd. The main result is the following.

Theorem 1.2.6 *Let $h > 0$ be a smooth function and suppose $\rho_i \in (8k\pi, 8(k+1)\pi)$, $k \in \mathbb{N}$ for $i = 1, 2$. Then problem* (1.23) *has a solution.*

Observe that we recover the result of [45] (see Chapter 4 and Subsection 1.2.1) and some cases of [9] (see Chapter 2 and Subsection 1.1.1): when Σ is homeomorphic to S^2 the above theorem yields a new existence result.

Remark 1.2.7 *Concerning the Leray-Schauder degree associated to equation* (1.23)*, in an ongoing project we have with Prof. Jun-cheng Wei and Wen Yang we provide a degree counting formula for parameters $\rho_1 \in (0, 8\pi) \cup (8\pi, 16\pi)$ and $\rho_2 \notin 8\pi\mathbb{N}$ by computing the degree contributed by the blow-up solutions.*

1.3 Notation

In this section we collect some useful notation we will use through the thesis.

Given points $x, y \in \Sigma$, $d(x,y)$ will stand for the metric distance between x and y on Σ. Similarly, for any $p \in \Sigma$, $\Omega, \Omega' \subseteq \Sigma$, we set:

$$d(p,\Omega) = \inf\left\{d(p,x) \ : x \in \Omega\right\}, \qquad d(\Omega,\Omega') = \inf\left\{d(x,y) \ : \ x \in \Omega, \ y \in \Omega'\right\}.$$

The symbol $B_s(p)$ stands for the open metric ball of radius s and centre p, while $A_p(r_1, r_2)$ is the open annulus of radii r_1, r_2 and centre p. For the complement of a set Ω in Σ we will write Ω^c.

Given a function $u \in L^1(\Sigma)$ and $\Omega \subset \Sigma$, the average of u on Ω is denoted by the symbol

$$\fint_\Omega u \, dV_g = \frac{1}{|\Omega|} \int_\Omega u \, dV_g.$$

We denote by $\overline{u}$ the average of u in Σ: since we are assuming $|\Sigma| = 1$, we have

$$\overline{u} = \int_\Sigma u \, dV_g = \fint_\Sigma u \, dV_g.$$

The sub-levels of the functional J_ρ will be indicated as

$$J_\rho^a := \left\{u = (u_1, u_2) \in H^1(\Sigma) \times H^1(\Sigma) : \ J_\rho(u_1, u_2) \leq a\right\}.$$

As it is mentioned in the introduction, some useful information arising from Moser-Trudinger type inequalities and their improvements are the concentration of e^u when u belongs to a low sub-level. To express this rigorously, we denote $\mathcal{M}(\Sigma)$ the set of all Radon measures on Σ, and introduce a norm by using duality versus Lipschitz functions, that is, we set:

$$\mathbf{d}(\nu_1, \nu_2) = \sup_{\|f\|_{Lip(\Sigma)} \leq 1} \left| \int_\Sigma f \, d\nu_1 - \int_\Sigma f \, d\nu_2 \right|; \qquad \nu_1, \nu_2 \in \mathcal{M}(\Sigma). \tag{1.25}$$

This is known as the *Kantorovich-Rubinstein* distance.

Given $\delta > 0$, we define the topological cone:

$$\overline{\Sigma}_\delta = \frac{\Sigma \times (0, +\infty)}{\Sigma \times [\delta, +\infty)}, \tag{1.26}$$

where the equivalence relation identifies $\Sigma \times [\delta, +\infty)$ to a single point.

Given $q \in \mathbb{N}$ and a topological space X, we will denote by $H_q(X)$ its q-th homology group with coefficient in $\mathbb{Z}$. For a subspace $A \subseteq X$ we write $H_q(X, A)$ for the q-th relative homology group of (X, A). We will denote by $\widetilde{H}_q(X)$ the reduced q-th homology group, i.e. $H_0(X) = \widetilde{H}_0(X) \oplus \mathbb{Z}$ and $H_q(X) = \widetilde{H}_q(X)$ for all $q > 0$.

The q-th Betti number of X will be indicated by $\beta_q(X)$, namely $\beta_q(X) = \text{rank } (H_q(X))$, while $\widetilde{\beta}_q(X)$ will correspond to the rank of the reduced homology group.

Throughout the paper the letter C will stand for large constants which are allowed to vary among different formulas or even within the same lines. When we want to stress the dependence of the constants on some parameter (or parameters), we add subscripts to C, as C_δ, etc. We will write $o_\alpha(1)$ to denote quantities that tend to 0 as $\alpha \to 0$ or $\alpha \to +\infty$; we will similarly use the symbol $O_\alpha(1)$ for bounded quantities.

Chapter 2

The Toda system: a general existence result

We consider here the following problem on a compact surface Σ:

$$\begin{cases} -\Delta u_1 = 2\rho_1 \left(\frac{h_1 e^{u_1}}{\int_\Sigma h_1 e^{u_1} dV_g} - 1 \right) - \rho_2 \left(\frac{h_2 e^{u_2}}{\int_\Sigma h_2 e^{u_2} dV_g} - 1 \right) - 4\pi \sum_{j=1}^m \alpha_{1,j}(\delta_{p_j} - 1), \\ -\Delta u_2 = 2\rho_2 \left(\frac{h_2 e^{u_2}}{\int_\Sigma h_2 e^{u_2} dV_g} - 1 \right) - \rho_1 \left(\frac{h_1 e^{u_1}}{\int_\Sigma h_1 e^{u_1} dV_g} - 1 \right) - 4\pi \sum_{j=1}^m \alpha_{2,j}(\delta_{p_j} - 1), \end{cases} \tag{2.1}$$

where ρ_1, ρ_2 are real parameters, h_1, h_2 are smooth positive functions and $\alpha_{i,j} \geq 0$. For an introduction to this topic see Section 1.1 and Subsection 1.1.1. The arguments of this chapter are collected in the paper [9]. We will give here the following existence result in a general non-coercive regime.

Theorem 2.0.1 *Let $\alpha_{i,j} \geq 0$ and let $\Lambda \subset \mathbb{R}^2$ be as in Definition 1.1.2. Let Σ be a compact surface neither homeomorphic to S^2 nor to $\mathbb{RP}^2$, and assume that $(\rho_1, \rho_2) \notin \Lambda$. Then the above Toda system is solvable.*

The plan of this chapter is the following: in Section 2.1 we construct a family of test functions with low energy modelled on the topological join of $(\gamma_1)_k$ and $(\gamma_2)_l$, see (1.15) and (1.17). In Section 2.2 we derive suitable improved Moser-Trudinger inequalities to construct projections from low sub-levels of the associated energy functional J_ρ into $(\gamma_1)_k * (\gamma_2)_l$. In Section 2.3 we prove our existence theorem using the min-max argument. In Section 2.4 we present some topological properties of the barycenter set Σ_k, which we will use through Chapters 2 and 3.

2.1 The test functions

We begin this section with an easy topological result, which will be essential in our analysis:

Lemma 2.1.1 *Let Σ be a compact surface not homeomorphic to S^2 nor $\mathbb{RP}^2$. Then, there exist two simple closed curves $\gamma_1, \gamma_2 \subseteq \Sigma$ satisfying (see Figure 2.1)*

1. *γ_1, γ_2 do not intersect each other nor any of the singular points p_j, $j = 1 \dots m$;*
2. *there exist global retractions $\Pi_i : \Sigma \to \gamma_i$, $i = 1, 2$.*

PROOF. The result is quite evident for the torus. For the Klein bottle, consider its fundamental square $ABAB^{-1}$. We can take γ_1 as the segment B, and γ_2 a segment parallel to B and passing by the center of the square. The retractions are given by just freezing one cartesian component of the point in the square.

Observe that we can assume that p_i do not intersect those curves.

For any other Σ under the conditions of the lemma, Dyck's Theorem implies that it is the connected sum of a torus and another compact surface, $\Sigma = \mathbb{T}^2 \# M$. Then, one can modify the retractions of the torus so that they are constant on M. ■

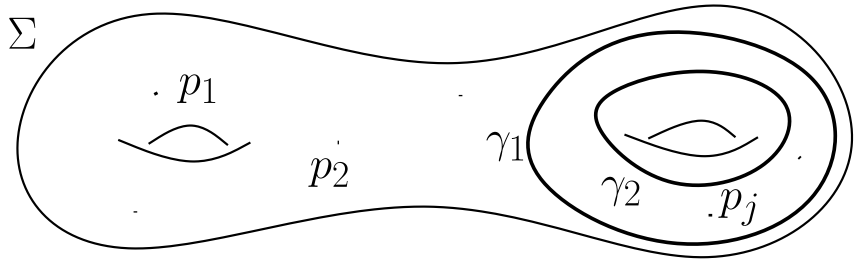

Figure 2.1: The curves γ_i.

Remark 2.1.1 *Observe that each curve γ_i generates a free subgroup in the first co-homology group of Σ. Then, Lemma 2.1.1 cannot hold for S^2 or $\mathbb{RP}^2$.*

For $\rho_1 \in (4k\pi, 4(k+1)\pi)$ and $\rho_2 \in (4l\pi, 4(l+1)\pi)$ we would like to build a family of test functions modelled on the topological join $(\gamma_1)_k * (\gamma_2)_l$, involving the formal barycenters of the curves γ_1, γ_2, see (1.15).

Remark 2.1.2 *Since each γ_i is homeomorphic to S^1, it follows from Proposition 3.2 in [4] that $(\gamma_1)_k$ is homeomorphic to S^{2k-1} and $(\gamma_2)_l$ to S^{2l-1} (the homotopy equivalence was found before in [54]). As it is well-known, the join $S^m * S^n$ is homeomorphic to S^{m+n+1} (see for example [43]), and therefore $(\gamma_1)_k * (\gamma_2)_l$ is homeomorphic to the sphere $S^{2k+2l-1}$.*

Let $\zeta = (1-s)\sigma_2 + s\sigma_1 \in (\gamma_1)_k * (\gamma_2)_l$, where:

$$\sigma_1 := \sum_{i=1}^{k} t_i \delta_{x_i} \in (\gamma_1)_k \qquad \text{and} \qquad \sigma_2 := \sum_{j=1}^{l} s_j \delta_{y_j} \in (\gamma_2)_l.$$

Our goal is to define a test function modelled uniformly on any $\zeta \in (\gamma_1)_k * (\gamma_2)_l$, depending on a positive parameter λ and belonging to low sub-levels of J for large λ, that is a map

$$\Phi_\lambda : (\gamma_1)_k * (\gamma_2)_l \to J_\rho^{-L}; \qquad L \gg 0.$$

For any $\lambda > 0$, we define the parameters

$$\lambda_{1,s} = (1-s)\lambda; \qquad \lambda_{2,s} = s\lambda.$$

We introduce $\Phi_\lambda(\zeta) = \varphi_{\lambda,\zeta}$ whose components are defined by

$$\begin{pmatrix} \varphi_1(x) \\ \varphi_2(x) \end{pmatrix} = \begin{pmatrix} \log \sum_{i=1}^{k} t_i \left(\frac{1}{1+\lambda_{1,s}^2 d(x,x_i)^2} \right)^2 - \frac{1}{2} \log \sum_{j=1}^{l} s_j \left(\frac{1}{1+\lambda_{2,s}^2 d(x,y_j)^2} \right)^2 \\ -\frac{1}{2} \log \sum_{i=1}^{k} t_i \left(\frac{1}{1+\lambda_{1,s}^2 d(x,x_i)^2} \right)^2 + \log \sum_{j=1}^{l} s_j \left(\frac{1}{1+\lambda_{2,s}^2 d(x,y_j)^2} \right)^2 \end{pmatrix}. \tag{2.2}$$

Notice that when $s = 0$ we have that $\lambda_{2,s} = 0$, and therefore, as $\sum_{j=1}^{l} s_j = 1$, the second terms in both rows are constant, independent of σ_2; a similar consideration holds when $s = 1$. These arguments imply that the function Φ_λ is indeed well defined on $(\gamma_1)_k * (\gamma_2)_l$.

We have then the following result.

Proposition 2.1.3 *Suppose $\rho_1 \in (4k\pi, 4(k+1)\pi)$ and $\rho_2 \in (4l\pi, 4(l+1)\pi)$. Then one has*

$$J_\rho(\varphi_{\lambda,\zeta}) \to -\infty \quad \text{as } \lambda \to +\infty \qquad \text{uniformly in } \zeta \in (\gamma_1)_k * (\gamma_2)_l.$$

Proof. We define $v_1, v_2 : \Sigma \to \mathbb{R}$ as follows;

$$v_1(x) = \log \sum_{i=1}^{k} t_i \left(\frac{1}{1+\lambda_{1,s}^2 d(x,x_i)^2} \right)^2, \qquad v_2(x) = \log \sum_{j=1}^{l} s_j \left(\frac{1}{1+\lambda_{2,s}^2 d(x,y_j)^2} \right)^2.$$

With this notation the components of $\varphi(x)$ are given by

$$\begin{pmatrix} \varphi_1(x) \\ \varphi_2(x) \end{pmatrix} = \begin{pmatrix} v_1(x) - \frac{1}{2}\, v_2(x) \\ -\frac{1}{2}\, v_1(x) + v_2(x) \end{pmatrix}.$$

We first prove two estimates on the gradients of v_1 and v_2.

$$|\nabla v_i(x)| \leq C\lambda_{i,s}, \qquad \text{for every } x \in \Sigma \text{ and } s \in [0,1], \quad i = 1,2, \tag{2.3}$$

where C is a constant independent of λ, $\zeta \in (\gamma_1)_k * (\gamma_2)_l$, and

$$|\nabla v_i(x)| \leq \frac{4}{d_{\,i,min}(x)}, \qquad \text{for every } x \in \Sigma, \quad i = 1,2, \tag{2.4}$$

where $d_{1,min}(x) = \min_{i=1,\dots,k} d(x, x_i)$ and $d_{2,min}(x) = \min_{j=1,\dots,l} d(x, y_j)$.

We show the inequalities just for v_1, as for v_2 the proof is similar. We have that

$$\nabla v_1(x) = -2\lambda_{1,s}^2 \frac{\sum_{i=1}^k t_i \big(1 + \lambda_{1,s}^2 d^2(x,x_i)\big)^{-3} \nabla\big(d^2(x,x_i)\big)}{\sum_{j=1}^k t_j \big(1 + \lambda_{1,s}^2 d^2(x,x_j)\big)^{-2}}.$$

Using the estimate $\big|\nabla\big(d^2(x,x_i)\big)\big| \leq 2d(x,x_i)$ and the following inequality

$$\frac{\lambda_{1,s}^2 d(x,x_i)}{1 + \lambda_{1,s}^2 d^2(x,x_i)} \leq C\lambda_{1,s}, \qquad i = 1,\dots,k,$$

with C a fixed constant, we obtain (2.3). For proving (2.4) we observe that if $\lambda_{1,s} = 0$ the inequality is trivially satisfied. If instead $\lambda_{1,s} > 0$ we have

$$\begin{aligned}
|\nabla v_1(x)| \;&\leq\; 4\lambda_{1,s}^2 \frac{\sum_{i=1}^k t_i \big(1 + \lambda_{1,s}^2 d^2(x,x_i)\big)^{-3} d(x,x_i)}{\sum_{j=1}^k t_j \big(1 + \lambda_{1,s}^2 d^2(x,x_j)\big)^{-2}} \leq 4\lambda_{1,s}^2 \frac{\sum_{i=1}^k t_i \big(1 + \lambda_{1,s}^2 d^2(x,x_i)\big)^{-2} \frac{d(x,x_i)}{\lambda_{1,s}^2 d^2(x,x_i)}}{\sum_{j=1}^k t_j \big(1 + \lambda_{1,s}^2 d^2(x,x_y)\big)^{-2}} \\
&\leq\; 4 \frac{\sum_{i=1}^k t_i \big(1 + \lambda_{1,s}^2 d^2(x,x_i)\big)^{-2} \frac{1}{d_{\,1,min}(x)}}{\sum_{j=1}^k t_j \big(1 + \lambda_{1,s}^2 d^2(x,x_j)\big)^{-2}} = \frac{4}{d_{\,1,min}(x)},
\end{aligned}$$

which proves (2.4).

We consider now the Dirichlet part of the functional J_ρ. Taking into account the definition of φ_1, φ_2 we have

$$\begin{aligned}
\int_\Sigma Q(\varphi_1,\varphi_2)\, dV_g \;&=\; \frac{1}{3}\int_\Sigma \big(|\nabla\varphi_1|^2 + |\nabla\varphi_2|^2 + \nabla\varphi_1 \cdot \nabla\varphi_2\big)\, dV_g \\
&=\; \frac{1}{3}\int_\Sigma \left(|\nabla v_1|^2 + \frac{1}{4}|\nabla v_2|^2 - \nabla v_1 \cdot \nabla v_2\right) dV_g + \frac{1}{3}\int_\Sigma \left(|\nabla v_2|^2 + \frac{1}{4}|\nabla v_1|^2 - \nabla v_2 \cdot \nabla v_1\right) dV_g + \\
&+\; \frac{1}{3}\int_\Sigma \left(-\frac{1}{2}|\nabla v_1|^2 - \frac{1}{2}|\nabla v_2|^2 + \frac{5}{4}(\nabla v_1 \cdot \nabla v_2)\right) dV_g \\
&=\; \frac{1}{4}\int_\Sigma |\nabla v_1|^2\, dV_g + \frac{1}{4}\int_\Sigma |\nabla v_2|^2\, dV_g - \frac{1}{4}\int_\Sigma \nabla v_1 \cdot \nabla v_2\, dV_g.
\end{aligned}$$

We first observe that the part involving the mixed term $\nabla v_1 \cdot \nabla v_2$ is bounded by a constant depending only on Σ. Indeed, we introduce the sets

$$A_i = \left\{ x \in \Sigma : d(x,x_i) = \min_{j=1}^{k} d(x,x_j) \right\}. \tag{2.5}$$

Using then (2.4) we have

$$\begin{aligned}
\int_\Sigma \nabla v_1 \cdot \nabla v_2\, dV_g \;&\leq\; \int_\Sigma |\nabla v_1||\nabla v_2|\, dV_g \leq 16 \int_\Sigma \frac{1}{d_{1,min}(x)\, d_{2,min}(x)}\, dV_g(x) \\
&\leq\; 16 \sum_{i=1}^k \int_{A_i} \frac{1}{d(x,x_i)\, d_{2,min}(x)}\, dV_g(x).
\end{aligned}$$

We take now $\delta > 0$ such that

$$\delta = \frac{1}{2} \min \left\{ \min_{i \in \{1,\dots k\}, j \in \{1,\dots l\}} d(x_i, y_j), \quad \min_{m,\ n \in \{1,\dots k\}, m \neq n} d(x_m, x_n) \right\}$$

and we split each A_i into $A_i = B_\delta(x_i) \cup (A_i \setminus B_\delta(x_i)), i = 1, \dots k$. By a change of variables and exploiting the fact that $d_{2,min}(x) \geq \frac{1}{C}$ in $B_\delta(x_i)$ we obtain

$$\sum_{i=1}^{k} \int_{B_\delta(x_i)} \frac{1}{d(x, x_i)\, d_{2,min}(x)} \, dV_g(x) \leq C.$$

Using the same argument for the part $A_i \setminus B_\delta(x_i)$ with some modifications and exchanging the role of $d_{1,min}$ and $d_{2,min}$ we finally deduce that

$$\int_\Sigma \nabla v_1 \cdot \nabla v_2 \, dV_g \leq C. \tag{2.6}$$

We want now to estimate the remaining part of the Dirichlet energy. For convenience we treat the cases $s = 0$ and $s = 1$ separately. Consider first the case $s = 0$: we then have $\nabla v_2(x) = 0$ and we get

$$\int_\Sigma Q(\varphi_1, \varphi_2) \, dV_g = \frac{1}{4} \int_\Sigma |\nabla v_1(x)|^2 \, dV_g(x).$$

We divide now the integral into two parts;

$$\frac{1}{4} \int_\Sigma |\nabla v_1(x)|^2 \, dV_g(x) = \frac{1}{4} \int_{\bigcup_i B_{\frac{1}{\lambda}}(x_i)} |\nabla v_1(x)|^2 \, dV_g(x) + \frac{1}{4} \int_{\Sigma \setminus \bigcup_i B_{\frac{1}{\lambda}}(x_i)} |\nabla v_1(x)|^2 \, dV_g(x).$$

From (2.3) we deduce that

$$\int_{\bigcup_i B_{\frac{1}{\lambda}}(x_i)} |\nabla v_1(x)|^2 \, dV_g(x) \leq C.$$

Using then (2.4) for the second part of the integral, recalling the definition (2.5) of the sets A_i, one finds that

$$\begin{aligned}
\frac{1}{4} \int_{\Sigma \setminus \bigcup_i B_{\frac{1}{\lambda}}(x_i)} |\nabla v_1(x)|^2(x) \, dV_g \quad &\leq \quad 4 \int_{\Sigma \setminus \bigcup_i B_{\frac{1}{\lambda}}(x_i)} \frac{1}{d_{1,min}^2(x)} \, dV_g(x) + C \\
&\leq \quad 4 \sum_{i=1}^{k} \int_{A_i \setminus B_{\frac{1}{\lambda}}(x_i)} \frac{1}{d_{1,min}^2(x)} \, dV_g(x) + C \\
&\leq \quad 8k\pi \big(1 + o_\lambda(1)\big) \log \lambda + C,
\end{aligned}$$

where $o_\lambda(1) \to 0$ as $\lambda \to +\infty$. Therefore we have

$$\int_\Sigma Q(\varphi_1, \varphi_2) \, dV_g \leq 8k\pi \big(1 + o_\lambda(1)\big) \log \lambda + C. \tag{2.7}$$

Reasoning as in [68], Proposition 4.2 part (ii), it is possible to show that

$$\int_\Sigma v_1 \, dV_g = -4\big(1 + o_\lambda(1)\big) \log \lambda; \qquad \log \int_\Sigma e^{v_1} \, dV_g = -2\big(1 + o_\lambda(1)\big) \log \lambda$$

$$\log \int_\Sigma e^{-\frac{1}{2} v_1} \, dV_g = 2\big(1 + o_\lambda(1)\big) \log \lambda,$$

and clearly

$$\int_\Sigma v_2 \, dV_g = O(1); \qquad \log \int_\Sigma e^{v_2} \, dV_g = O(1); \qquad \log \int_\Sigma e^{-\frac{1}{2} v_2} \, dV_g = O(1).$$

Therefore we get

$$\int_\Sigma \varphi_1\, dV_g = -4\big(1+o_\lambda(1)\big)\log\lambda; \qquad \log\int_\Sigma e^{\varphi_1}\, dV_g = -2\big(1+o_\lambda(1)\big)\log\lambda;$$

$$\int_\Sigma \varphi_2\, dV_g = 2(1+o_\lambda(1))\log\lambda; \qquad \log\int_\Sigma e^{\varphi_2}\, dV_g = -2(1+o_\lambda(1))\log\lambda.$$

Inserting the latter equalities in the expression of the functional J_ρ and using the fact that $\widetilde{h}_i \geq \frac{1}{C}, i = 1,2$ outside a small neighbourhood of the singular points (which are avoided by the curves γ_1, γ_2), we obtain

$$J_\rho(\varphi_1,\varphi_2) \leq \big(8k\pi - 2\rho_1 + o_\lambda(1)\big)\log\lambda + C,$$

where C is independent of λ and σ_1, σ_2.

For the case $s = 1$, by the same argument we have that

$$J_\rho(\varphi_1,\varphi_2) \leq \big(8l\pi - 2\rho_2 + o_\lambda(1)\big)\log\lambda + C.$$

We consider now the case $s \in (0,1)$. By (2.6) the Dirichlet part can be estimated by

$$\int_\Sigma Q(\varphi_1,\varphi_2)\, dV_g \leq \frac{1}{4}\int_\Sigma |\nabla v_1(x)|^2\, dV_g(x) + \frac{1}{4}\int_\Sigma |\nabla v_2(x)|^2\, dV_g(x) + C.$$

For a general s one can just substitute λ with $\lambda_{1,s}$ in (2.7) (and similarly for the v_2), to get the following estimate

$$\int_\Sigma Q(\varphi_1,\varphi_2)\, dV_g \leq 8k\pi\big(1+o_\lambda(1)\big)\log\big(\lambda_{1,s}+\delta_{1,s}\big) + 8l\pi\big(1+o_\lambda(1)\big)\log\big(\lambda_{2,s}+\delta_{2,s}\big) + C, \tag{2.8}$$

where $\delta_{1,s} > \delta > 0$ as $s \to 1$ and $\delta_{2,s} > \delta > 0$ as $s \to 0$, for some fixed δ. The same argument as for $s = 0, 1$ leads to

$$\int_\Sigma v_1\, dV_g = -4\big(1+o_\lambda(1)\big)\log\big(\lambda_{1,s}+\delta_{1,s}\big) + O(1); \qquad \int_\Sigma v_2\, dV_g = -4\big(1+o_\lambda(1)\big)\log\big(\lambda_{2,s}+\delta_{2,s}\big) + O(1),$$

therefore we obtain

$$\int_\Sigma \varphi_1\, dV_g = -4\big(1+o_\lambda(1)\big)\log\big(\lambda_{1,s}+\delta_{1,s}\big) + 2\big(1+o_\lambda(1)\big)\log\big(\lambda_{2,s}+\delta_{2,s}\big) + O(1), \tag{2.9}$$

$$\int_\Sigma \varphi_2\, dV_g = 2\big(1+o_\lambda(1)\big)\log\big(\lambda_{1,s}+\delta_{1,s}\big) - 4\big(1+o_\lambda(1)\big)\log\big(\lambda_{2,s}+\delta_{2,s}\big) + O(1). \tag{2.10}$$

We consider now the exponential term. We have

$$\int_\Sigma e^{\varphi_1}\, dV_g = \sum_{i=1}^k t_i \int_\Sigma \frac{1}{\big(1+\lambda_{1,s}^2 d(x,x_i)^2\big)^2}\left(\sum_{j=1}^l s_j \frac{1}{\big(1+\lambda_{2,s}^2 d(x,y_j)^2\big)^2}\right)^{-\frac{1}{2}} dV_g(x).$$

Clearly it is enough to estimate the term

$$\int_\Sigma \frac{1}{\big(1+\lambda_{1,s}^2 d(x,\overline{x})^2\big)^2}\left(\sum_{j=1}^l s_j \frac{1}{\big(1+\lambda_{2,s}^2 d(x,y_j)^2\big)^2}\right)^{-\frac{1}{2}} dV_g(x)$$

with $\overline{x} \in \{x_1,\dots x_k\}$. Letting $\delta = \dfrac{\min_j\{d(\overline{x},y_j)\}}{2}$ we divide the domain into two regions as follows: $\Sigma = B_\delta(\overline{x}) \cup (\Sigma \setminus B_\delta(\overline{x}))$. When we integrate in $B_\delta(\overline{x})$ we perform a change of variables for the part involving $\lambda_{1,r}$ and observing that $\frac{1}{C} \leq d(x,y_j) \leq C, j = 1,\dots,l$, for every $x \in B_\delta(\overline{x})$, we deduce

$$\int_{B_\delta(\overline{x})} \frac{1}{\big(1+\lambda_{1,s}^2 d(x,\overline{x})^2\big)^2}\left(\sum_{j=1}^l s_j \frac{1}{\big(1+\lambda_{2,s}^2 d(x,y_j)^2\big)^2}\right)^{-\frac{1}{2}} dV_g(x) = \frac{\big(\lambda_{2,s}+\delta_{2,s}\big)^2}{\big(\lambda_{1,s}+\delta_{1,s}\big)^2}\big(1+O(1)\big).$$

On the other hand for the integral over $\Sigma \setminus B_\delta(\overline{x})$ we use that $\frac{1}{C} \le d(x, \overline{x}) \le C$ to get that this part is a higher-order term and can be absorbed by the latter estimate. Recall now that $\widetilde{h}_1$ stays bounded away from zero in a neighbourhood of the curve γ_1 (see the beginning of the section). Therefore, since the contribution of the integral outside a neighbourhood of γ_1 is negligible, we can conclude that

$$\log \int_\Sigma \widetilde{h}_1 e^{\varphi_1}\, dV_g = 2\log\bigl(\lambda_{2,s} + \delta_{2,s}\bigr) - 2\log\bigl(\lambda_{1,s} + \delta_{1,s}\bigr) + O(1). \tag{2.11}$$

Similarly we have that

$$\log \int_\Sigma \widetilde{h}_2 e^{\varphi_2}\, dV_g = 2\log\bigl(\lambda_{1,s} + \delta_{1,s}\bigr) - 2\log\bigl(\lambda_{2,s} + \delta_{2,s}\bigr) + O(1). \tag{2.12}$$

Using the estimates (2.8), (2.9), (2.10), (2.11) and (2.12) we finally obtain

$$J_\rho(\varphi_1, \varphi_2) \le \bigl(8k\pi - 2\rho_1 + o_\lambda(1)\bigr)\log\bigl(\lambda_{1,s} + \delta_{1,s}\bigr) + \bigl(8l\pi - 2\rho_2 + o_\lambda(1)\bigr)\log\bigl(\lambda_{2,s} + \delta_{2,s}\bigr) + O(1).$$

Recalling that $\rho_1 > 4k\pi, \rho_2 > 4l\pi$ and observing that $\max\limits_{s\in[0,1]}\{\lambda_{1,s}, \lambda_{2,s}\} \to +\infty$ as $\lambda \to \infty$, we conclude the proof. ■

2.2 Moser-Trudinger inequalities and topological join

In this section we are going to give an improved version of the Moser-Trudinger inequality (1.16), where the constant 4π can be replaced by an integer multiple under the assumption that the integral of $\widetilde{h}_i e^{u_i}$ is distributed on different sets with positive mutual distance. The improved inequality implies that if $J_\rho(u_1, u_2)$ attains very low values, then $\widetilde{h}_i e^{u_i}$ has to concentrate near a given number (depending on ρ_i) of points for some $i \in \{1, 2\}$. As anticipated in the introduction, we will see that this induces a natural map from low sub-levels of J_ρ to the topological join of some sets of barycenters. This extends some analysis from [49] and [69], where the authors considered the case $\rho_2 < 4\pi$, and from [71], where both parameters belong to the range $(4\pi, 8\pi)$. We start with a covering lemma:

Lemma 2.2.1 *Let $\delta > 0$, $\theta > 0$, $k, l \in \mathbb{N}$ with $k \ge l$, $f_i \in L^1(\Sigma)$ be non-negative functions with $\|f_i\|_{L^1(\Sigma)} = 1$ for $i = 1, 2$ and $\{\Omega_{1,i}, \Omega_{2,j}\}_{i\in\{0,\dots,k\}, j\in\{0,\dots,l\}} \subset \Sigma$ such that*

$$d(\Omega_{1,i}, \Omega_{1,i'}) \ge \delta \qquad \forall\, i,\ i' \in \{0, \dots, k\} \text{ with } i \ne i';$$

$$d(\Omega_{2,j}, \Omega_{2,j'}) \ge \delta \qquad \forall\, j,\ j' \in \{0, \dots, l\} \text{ with } j \ne j',$$

and

$$\int_{\Omega_{1,i}} f_1 dV_g \ge \theta \qquad \forall\, i \in \{0, \dots, k\};$$

$$\int_{\Omega_{2,j}} f_2 dV_g \ge \theta \qquad \forall\, j \in \{0, \dots, l\}.$$

Then, there exist $\overline{\delta} > 0$, $\overline{\theta} > 0$, independent of f_i, and $\{\Omega_n\}_{n=1}^k \subset \Sigma$ such that

$$d(\Omega_n, \Omega_n') \ge \overline{\delta} \qquad \forall\, n,\ n' \in \{0, \dots, k\} \text{ with } n \ne n'$$

and

$$|\Omega_n| \ge \overline{\theta} \qquad \forall\, n \in \{0, \dots, k\};$$

$$\int_{\Omega_n} f_1 dV_g \ge \overline{\theta} \qquad \forall\, n \in \{0, \dots, k\};$$

$$\int_{\Omega_n} f_2 dV_g \ge \overline{\theta} \qquad \forall\, n \in \{0, \dots, l\}.$$

PROOF. We set $\overline{\delta} = \frac{\delta}{8}$ and consider the open cover $\left\{B_{\overline{\delta}}(x)\right\}_{x\in\Sigma}$ of Σ; by compactness, $\Sigma \subset \bigcup_{h=1}^{H} B_{\overline{\delta}}(x_h)$ for some $\{x_h\}_{h=1}^{H} \subset \Sigma$, $H = H\left(\overline{\delta}, \Sigma\right)$.
We choose $\{y_{1,i}, y_{2,j}\}_{i\in\{0,\dots,k\},j\in\{0,\dots,l\}} \subset \{x_h\}_{h=1}^{H}$ such that

$$\int_{B_{\overline{\delta}}(y_{1,i})} f_1 dV_g = \max\left\{\int_{B_{\overline{\delta}}(x_h)} f_1 dV_g \,:\, B_{\overline{\delta}}(x_h) \cap \Omega_{1,i} \neq \emptyset\right\};$$

$$\int_{B_{\overline{\delta}}(y_{2,j})} f_2 dV_g = \max\left\{\int_{B_{\overline{\delta}}(x_h)} f_2 dV_g \,:\, B_{\overline{\delta}}(x_h) \cap \Omega_{2,j} \neq \emptyset\right\}$$

Since $d(y_{1,i}, \Omega_{1,i}) < \overline{\delta}$, we have that $d(y_{1,i}, y_{1,i'}) \geq 6\overline{\delta}$ for $i \neq i'$. Analogously, $d(y_{2,j}, y_{2,j'}) \geq 6\overline{\delta}$ if $j \neq j'$.
In particular, this implies that for any $i \in \{0, \dots, k\}$ there exists at most one $j(i)$ such that $d(y_{2,j(i)}, y_{1,i}) < 3\overline{\delta}$.
We relabel the index i so that for $i = 1, \dots l$ such $j(i)$ exists, and we relabel the index j so that $j(i) = i$.
We now define:

$$\Omega_n := \begin{cases} B_{\overline{\delta}}(y_{1,n}) \cup B_{\overline{\delta}}(y_{2,n}) & \text{if } n \in \{0, \dots, l\} \\ B_{\overline{\delta}}(y_{1,n}) & \text{if } n \in \{l+1, \dots, k\}. \end{cases}$$

In other words, we make unions of balls $B_{\overline{\delta}}(y_{1,n}) \cup B_{\overline{\delta}}(y_{2,n})$ if they are close to each other: for separate balls, we make arbitrary unions. If $k > l$, the remaining balls are considered alone.

It is easy to check that those sets satisfy the theses of Lemma 2.2.1. ■

To show the improved Moser-Trudinger inequality, we will need a *localized* version of the inequality (1.16), which was proved in [71].

Lemma 2.2.2 *([71]) Let $\delta > 0$ and $\Omega \Subset \widetilde{\Omega} \subset \Sigma$ be such that $d\left(\Omega, \partial\widetilde{\Omega}\right) \geq \delta$.*
Then, for any $\varepsilon > 0$ there exists $C = C(\varepsilon, \delta)$ such that for any $u = (u_1, u_2) \in H^1(\Sigma) \times H^1(\Sigma)$

$$\log \int_{\Omega} e^{u_1 - \fint_{\widetilde{\Omega}} u_1 dV_g} dV_g + \log \int_{\Omega} e^{u_2 - \fint_{\widetilde{\Omega}} u_2 dV_g} dV_g \leq \frac{1}{4\pi} \int_{\widetilde{\Omega}} Q(u_1, u_2) dV_g + \varepsilon \int_{\Sigma} Q(u_1, u_2) dV_g + C.$$

Here comes the improved inequality: basically, if the *mass* of both $\widetilde{h}_1 e^{u_1}$ and $\widetilde{h}_2 e^{u_2}$ is spread respectively on at least $k+1$ and $l+1$ different sets, then the logarithms in (1.16) can be multiplied by $k+1$ and $l+1$ respectively.
Notice that this result was given in [69] in the case $l = 0$ and in [71] in the case $k = l = 1$.

Lemma 2.2.3 *Let $\delta > 0$, $\theta > 0$, $k, l \in \mathbb{N}$ and $\{\Omega_{1,i}, \Omega_{2,j}\}_{i\in\{0,\dots,k\},j\in\{0,\dots,l\}} \subset \Sigma$ be such that*

$$d(\Omega_{1,i}, \Omega_{1,i'}) \geq \delta \qquad \forall\, i,\ i' \in \{0, \dots, k\} \text{ with } i \neq i';$$

$$d(\Omega_{2,j}, \Omega_{2,j'}) \geq \delta \qquad \forall\, j,\ j' \in \{0, \dots, l\} \text{ with } j \neq j'.$$

Then, for any $\varepsilon > 0$ there exists $C = C(\varepsilon, \delta, \theta, k, l, \Sigma)$ such that any $u = (u_1, u_2) \in H^1(\Sigma) \times H^1(\Sigma)$ satisfying

$$\int_{\Omega_{1,i}} \widetilde{h}_1 e^{u_1} dV_g \geq \theta \int_{\Sigma} \widetilde{h}_1 e^{u_1} dV_g \qquad \forall\, i \in \{0, \dots, k\};$$

$$\int_{\Omega_{2,j}} \widetilde{h}_2 e^{u_2} dV_g \geq \theta \int_{\Sigma} \widetilde{h}_2 e^{u_2} dV_g \qquad \forall\, j \in \{0, \dots, l\}$$

verifies

$$(k+1) \log \int_{\Sigma} \widetilde{h}_1 e^{u_1 - \overline{u_1}} dV_g + (l+1) \log \int_{\Sigma} \widetilde{h}_2 e^{u_2 - \overline{u_2}} dV_g \leq \frac{1+\varepsilon}{4\pi} \int_{\Sigma} Q(u_1, u_2) dV_g + C.$$

Proof. In the proof we assume that $\overline{u_1} = \overline{u_2} = 0$. After relabelling the indexes, we can suppose $k \geq l$ and apply Lemma 2.2.1 with $f_i = \dfrac{\widetilde{h}_i e^{u_i}}{\int_\Sigma \widetilde{h}_i e^{u_i} dV_g}$ to get $\{\Omega_j\}_{j=0}^k \subset \Sigma$ with

$$d(\Omega_i, \Omega_j) \geq \overline{\delta} \qquad \forall\, i, j \in \{0, \dots, k\} \text{ with } i \neq j$$

and

$$\int_{\Omega_i} \widetilde{h}_1 e^{u_1} dV_g \geq \overline{\theta} \int_\Sigma \widetilde{h}_1 e^{u_1} dV_g \qquad \forall\, i \in \{0, \dots, k\};$$
$$\int_{\Omega_j} \widetilde{h}_2 e^{u_2} dV_g \geq \overline{\theta} \int_\Sigma \widetilde{h}_2 e^{u_2} dV_g \qquad \forall\, j \in \{0, \dots, l\}.$$

Notice that:

$$\log \int_\Sigma \widetilde{h}_i e^{u_i} dV_g = ⨍_{\widetilde{\Omega}_j} u_i dV_g + \log \int_\Sigma \widetilde{h}_1 e^{u_i - ⨍_{\widetilde{\Omega}_j} u_i dV_g} dV_g, \; i = 1, 2.$$

The average on $\widetilde{\Omega}_j$ can be estimated by Poincaré inequality:

$$⨍_{\widetilde{\Omega}_j} u_i dV_g \leq \frac{1}{\left|\widetilde{\Omega}_j\right|} \int_\Sigma |u_i| dV_g \leq C \left(\int_\Sigma |\nabla u_i|^2 dV_g \right)^{1/2} \leq C + \varepsilon \int_\Sigma |\nabla u_i|^2 dV_g, \; i = 1, \; 2. \tag{2.13}$$

We now apply, for any $j \in \{0, \dots, k\}$ Lemma 2.2.2 with , $\Omega = \Omega_j$ and $\widetilde{\Omega} = \widetilde{\Omega}_j := \left\{ x \in \Sigma : d(x, \Omega_j) < \frac{\overline{\delta}}{2} \right\}$: for $j \in \{0, \dots, l\}$ we get

$$\begin{aligned}
& \log \int_\Sigma \widetilde{h}_1 e^{u_1 - ⨍_{\widetilde{\Omega}_j} u_1 dV_g} dV_g + \log \int_\Sigma \widetilde{h}_2 e^{u_2 - ⨍_{\widetilde{\Omega}_j} u_2 dV_g} dV_g \\
\leq \; & 2 \log \frac{1}{\overline{\theta}} + \log \int_{\Omega_j} \widetilde{h}_1 e^{u_1 - ⨍_{\widetilde{\Omega}_j} u_1 dV_g} dV_g + \log \int_{\Omega_j} \widetilde{h}_2 e^{u_2 - ⨍_{\widetilde{\Omega}_j} u_2 dV_g} dV_g \\
\leq \; & C + \log \int_{\Omega_j} e^{u_1 - ⨍_{\widetilde{\Omega}_j} u_1 dV_g} dV_g + \log \int_{\Omega_j} e^{u_2 - ⨍_{\widetilde{\Omega}_j} u_2 dV_g} dV_g \\
\leq \; & C + \frac{1}{4\pi} \int_{\widetilde{\Omega}_j} Q(u_1, u_2) dV_g + \varepsilon \int_\Sigma Q(u_1, u_2) dV_g, \; j = 1, \dots l.
\end{aligned} \tag{2.14}$$

For $j \in \{l+1, \dots, k\}$ we have

$$\begin{aligned}
& \log \int_\Sigma \widetilde{h}_1 e^{u_1 - ⨍_{\widetilde{\Omega}_j} u_1 dV_g} dV_g \leq \log \frac{1}{\overline{\theta}} + \left\| \widetilde{h}_1 \right\|_{L^\infty(\Sigma)} + \log \int_{\Omega_j} e^{u_1 - ⨍_{\widetilde{\Omega}_j} u_1 dV_g} dV_g \\
\leq \; & C - \log \int_{\Omega_j} e^{u_2 - ⨍_{\widetilde{\Omega}_j} u_2 dV_g} dV_g + \frac{1}{4\pi} \int_{\widetilde{\Omega}_j} Q(u_1, u_2) dV_g + \varepsilon \int_\Sigma Q(u_1, u_2) dV_g.
\end{aligned} \tag{2.15}$$

The exponential term on the second component can be estimated by using Jensen's inequality:

$$\begin{aligned}
& \log \int_{\Omega_j} e^{u_2 - ⨍_{\widetilde{\Omega}_j} u_2 dV_g} dV_g = \log |\Omega_j| + \log ⨍_{\Omega_j} e^{u_2 - ⨍_{\widetilde{\Omega}_j} u_2 dV_g} dV_g \\
\geq \; & \log |\Omega_j| \geq -C.
\end{aligned} \tag{2.16}$$

Putting together (2.16) and (2.17), we have:

$$\log \int_\Sigma \widetilde{h}_1 e^{u_1 - ⨍_{\widetilde{\Omega}_j} u_1 dV_g} dV_g \leq \frac{1}{4\pi} \int_{\widetilde{\Omega}_j} Q(u_1, u_2) dV_g + \varepsilon \int_\Sigma Q(u_1, u_2) dV_g + C, \; j = l+1 \dots k. \tag{2.17}$$

Summing over all $j \in \{0, \dots, k\}$ and taking into account (2.14), (2.17), we obtain the result, renaming ε appropriately. ∎

We will now use a technical result that gives sufficient conditions to apply Lemma 2.2.3. Its proof can be found for instance in [36, 69].

Lemma 2.2.4 *([69], [71]) Let $f \in L^1(\Sigma)$ be a non-negative function with $\|f\|_{L^1(\Sigma)} = 1$ and let $m \in \mathbb{N}$ be such that there exist $\varepsilon > 0$, $r > 0$ with*

$$\int_{\bigcup_{j=0}^m B_r(x_j)} f dV_g < 1 - \varepsilon \qquad \forall \{x_j\}_{j=0}^m \subset \Sigma.$$

Then there exist $\overline{\varepsilon} > 0$, $\overline{r} > 0$, not depending on f, and $\{\overline{x}_j\}_{j=1}^m \subset \Sigma$ satisfying

$$\int_{B_{\overline{r}}(\overline{x}_j)} f dV_g > \overline{\varepsilon} \qquad \forall\, j \in \{1, \dots, m\},$$

$$B_{2\overline{r}}(\overline{x}_i) \cap B_{2\overline{r}}(\overline{x}_j) = \emptyset \qquad \forall\, i, j \in \{1, \dots, m\}, i \neq j.$$

Now we have enough tools to obtain information on the structure of very low sub-levels of J_ρ:

Lemma 2.2.5 *Suppose $\rho_1 \in (4k\pi, 4(k+1)\pi)$ and $\rho_2 \in (4l\pi, 4(l+1)\pi)$. Then, for any $\varepsilon > 0$, $r > 0$, there exists $L = L(\varepsilon, r) > 0$ such that for any $u \in J_\rho^{-L}$ there are either some $\{x_i\}_{i=1}^k \subset \Sigma$ verifying*

$$\frac{\int_{\bigcup_{i=1}^k B_r(x_i)} \widetilde{h}_1 e^{u_1} dV_g}{\int_\Sigma \widetilde{h}_1 e^{u_1} dV_g} \geq 1 - \varepsilon$$

or some $\{y_j\}_{j=1}^l \subset \Sigma$ verifying

$$\frac{\int_{\bigcup_{j=1}^l B_r(y_j)} \widetilde{h}_2 e^{u_2} dV_g}{\int_\Sigma \widetilde{h}_2 e^{u_2} dV_g} \geq 1 - \varepsilon.$$

Proof. Suppose by contradiction that the statement is not true, that is there are $\varepsilon_1, \varepsilon_2 > 0$, $r_1, r_2 > 0$, and $\{u_n = (u_{1,n}, u_{2,n})\}_{n\in\mathbb{N}} \subset H^1(\Sigma) \times H^1(\Sigma)$ such that $J_\rho(u_{1,n}, u_{2,n}) \underset{n\to+\infty}{\longrightarrow} -\infty$ and

$$\frac{\int_{\bigcup_{i=1}^k B_{r_1}(x_i)} \widetilde{h}_1 e^{u_{1,n}} dV_g}{\int_\Sigma \widetilde{h}_1 e^{u_{1,n}} dV_g} < 1 - \varepsilon_1; \qquad \frac{\int_{\bigcup_{j=1}^l B_{r_2}(y_j)} \widetilde{h}_2 e^{u_{2,n}} dV_g}{\int_\Sigma \widetilde{h}_2 e^{u_{2,n}} dV_g} < 1 - \varepsilon_2, \qquad \forall \{x_i\}_{i=1}^k, \{y_j\}_{j=1}^l \subset \Sigma.$$

Then, we may apply twice Lemma 2.2.4 with $f = \dfrac{\widetilde{h}_i e^{u_i}}{\int_\Sigma \widetilde{h}_i e^{u_i} dV_g}$, $\widetilde{\varepsilon} = \varepsilon_i$, $\widetilde{r} = r_i$ and find $\overline{\varepsilon}_1, \overline{\varepsilon}_2 > 0$, $\overline{r}_1, \overline{r}_2 > 0$ and $\{\overline{x}_i\}_{i=0}^k, \{\overline{y}_j\}_{j=0}^l$ with

$$\int_{B_{\overline{r}_1}(\overline{x}_i)} \widetilde{h}_1 e^{u_1} dV_g \geq \overline{\varepsilon}_1 \int_\Sigma \widetilde{h}_1 e^{u_1} dV_g \qquad \forall\, i \in \{0, \dots, k\};$$

$$\int_{B_{\overline{r}_2}(\overline{y}_j)} \widetilde{h}_2 e^{u_2} dV_g \geq \overline{\varepsilon}_2 \int_\Sigma \widetilde{h}_2 e^{u_2} dV_g \qquad \forall\, j \in \{0, \dots, l\},$$

and

$$B_{2\overline{r}_1}(\overline{x}_i) \cap B_{2\overline{r}_1}(\overline{x}_j) = \emptyset \qquad \forall\, i, j \in \{0, \dots, k\} \text{ with } i \neq j;$$

$$B_{2\overline{r}_2}(\overline{y}_j) \cap B_{2\overline{r}_2}(\overline{y}_j) = \emptyset \qquad \forall\, i, j \in \{0, \dots, l\} \text{ with } i \neq j.$$

Hence, we obtain an improved Moser-Trudinger inequality for $u_n = (u_{1,n}, u_{2,n})$ applying Lemma 2.2.3 with $\widetilde{\delta} := 2\min\{\overline{r}_1, \overline{r}_2\}$, $\widetilde{\theta} := \min\{\overline{\varepsilon}_1, \overline{\varepsilon}_2\}$ and $\Omega_{1,i} := B_{\overline{r}_1}(\overline{x}_i)$, $\Omega_{2,j} := B_{\overline{r}_2}(\overline{y}_j)$.
Moreover, Jensen's inequality gives

$$\int_\Sigma \widetilde{h}_i e^{u_{i,n} - \overline{u_{i,n}}} dV_g = \int_\Sigma e^{\log \widetilde{h}_i + u_{i,n} - \overline{u_{i,n}}} dV_g \geq e^{\int_\Sigma \log \widetilde{h}_i dV_g},$$

so, choosing

$$\widetilde{\varepsilon} \in \left(0, \min\left\{\frac{4\pi(k+1)}{\rho_1} - 1, \frac{4\pi(l+1)}{\rho_2} - 1\right\}\right)$$

we get

$$
\begin{array}{rcl}
-\infty & \underset{n\to+\infty}{\longleftarrow} & J_\rho(u_{1,n},u_{2,n}) \\
& \geq & \left(\dfrac{4\pi(k+1)}{1+\widetilde{\varepsilon}}-\rho_1\right)\log\displaystyle\int_\Sigma \widetilde{h}_1 e^{u_{1,n}-\overline{u_{1,n}}}dV_g \\
& + & \left(\dfrac{4\pi(l+1)}{1+\widetilde{\varepsilon}}-\rho_2\right)\log\displaystyle\int_\Sigma \widetilde{h}_2 e^{u_{2,n}-\overline{u_{2,n}}}dV_g - C \\
& \geq & \left(\dfrac{4\pi(k+1)}{1+\widetilde{\varepsilon}}-\rho_1\right)\displaystyle\int_\Sigma \log\widetilde{h}_1 dV_g + \left(\dfrac{4\pi(l+1)}{1+\widetilde{\varepsilon}}-\rho_2\right)\displaystyle\int_\Sigma \log\widetilde{h}_2 dV_g - C \\
& \geq & -C
\end{array}
$$

that is a contradiction. ■

Recall the distance $\mathbf{d}$ defined in (1.25). An immediate consequence of the previous lemma is that at least one of the two $\widetilde{h}_i e^{u_i}$'s (once normalized in L^1) has to be very close respectively to the sets of k-barycenters or l-barycenters over Σ:

Proposition 2.2.6 *Suppose $\rho_1 \in (4k\pi, 4(k+1)\pi)$ and $\rho_2 \in (4l\pi, 4(l+1)\pi)$. Then, for any $\varepsilon > 0$, there exists $L > 0$ such that any $u \in J_\rho^{-L}$ verifies either*

$$
\mathbf{d}\left(\frac{\widetilde{h}_1 e^{u_1}}{\int_\Sigma \widetilde{h}_1 e^{u_1} dV_g}, \Sigma_k\right) < \varepsilon \qquad \text{or} \qquad \mathbf{d}\left(\frac{\widetilde{h}_2 e^{u_2}}{\int_\Sigma \widetilde{h}_2 e^{u_2} dV_g}, \Sigma_l\right) < \varepsilon.
$$

Proof. We apply Lemma 2.2.5 with $\widetilde{\varepsilon} = \dfrac{\varepsilon}{4}$, $\widetilde{r} = \dfrac{\varepsilon}{2}$; it is not restrictive to suppose that the first alternative occurs and that $\displaystyle\int_\Sigma \widetilde{h}_1 e^{u_1} dV_g = 1$. Hence we get L and $\{x_i\}_{i=1}^k$ and we define, for such an $u = (u_1, u_2) \in J_\rho^{-L}$,

$$
\sigma_1(u) = \sum_{i=1}^k t_i(u)\delta_{x_i} \in \Sigma_k \qquad \text{where } t_i(u) = \int_{B_{\widetilde{r}}(x_i)\setminus\bigcup_{j=1}^{i-1} B_{\widetilde{r}}(x_j)} \widetilde{h}_1 e^{u_1} dV_g + \frac{1}{k}\int_{\Sigma\setminus\bigcup_{j=1}^{k} B_{\widetilde{r}}(x_j)} \widetilde{h}_1 e^{u_1} dV_g.
$$

Then, for any $\phi \in Lip(\Sigma)$,

$$
\begin{aligned}
&\left|\int_{\Sigma\setminus\bigcup_{i=1}^k B_{\widetilde{r}}(x_i)} \left(\frac{\widetilde{h}_1 e^{u_1}}{\int_\Sigma \widetilde{h}_1 e^{u_1} dV_g} - \sigma_1(u)\right)\phi dV_g\right| = \\
&= \int_{\Sigma\setminus\bigcup_{i=1}^k B_{\widetilde{r}}(x_i)} \widetilde{h}_1 e^{u_1}\phi dV_g \leq \int_{\Sigma\setminus\bigcup_{i=1}^k B_{\widetilde{r}}(x_i)} \widetilde{h}_1 e^{u_1} dV_g \|\phi\|_{L^\infty(\Sigma)} < \widetilde{\varepsilon}\|\phi\|_{L^\infty(\Sigma)}
\end{aligned}
$$

and

$$
\begin{aligned}
&\left|\int_{\bigcup_{i=1}^k B_{\widetilde{r}}(x_i)} \left(\frac{\widetilde{h}_1 e^{u_1}}{\int_\Sigma \widetilde{h}_1 e^{u_1} dV_g} - \sigma_1(u)\right)\phi\, dV_g\right| \\
= &\left|\int_{\bigcup_{i=1}^k B_{\widetilde{r}}(x_i)} \widetilde{h}_1 e^{u_1}\phi\, dV_g - \sum_{i=1}^k\left(\int_{B_{\widetilde{r}}(x_i)\setminus\bigcup_{j=1}^{i-1} B_{\widetilde{r}}(x_j)} \widetilde{h}_1 e^{u_1} dV_g + \frac{1}{k}\int_{\Sigma\setminus\bigcup_{j=1}^{k} B_{\widetilde{r}}(x_j)} \widetilde{h}_1 e^{u_1} dV_g\right)\phi(x_i)\right| \\
= &\left|\int_{\bigcup_{i=1}^k \left(B_{\widetilde{r}}(x_i)\setminus\bigcup_{j=1}^{i-1} B_{\widetilde{r}}(x_j)\right)} \widetilde{h}_1 e^{u_1}(\phi - \phi(x_i))dV_g - \int_{\Sigma\setminus\bigcup_{j=1}^{k} B_{\widetilde{r}}(x_j)} \widetilde{h}_1 e^{u_1} dV_g \phi(x_i)\right| \\
\leq\ &\widetilde{r}\|\nabla\phi\|_{L^\infty(\Sigma)}\int_{\bigcup_{i=1}^k \left(B_{\widetilde{r}}(x_i)\setminus\bigcup_{j=1}^{i-1} B_{\widetilde{r}}(x_j)\right)} \widetilde{h}_1 e^{u_1} dV_g + \|\phi\|_{L^\infty(\Sigma)}\int_{\Sigma\setminus\bigcup_{j=1}^{k} B_{\widetilde{r}}(x_j)} \widetilde{h}_1 e^{u_1} dV_g \\
<\ &\widetilde{r}\|\nabla\phi\|_{L^\infty(\Sigma)} + \widetilde{\varepsilon}\|\phi\|_{L^\infty(\Sigma)}.
\end{aligned}
$$

Hence we can conclude the proof:

$$\begin{aligned}
& \mathbf{d}\left(\frac{\widetilde{h}_1 e^{u_1}}{\int_\Sigma \widetilde{h}_1 e^{u_1} dV_g}, \Sigma_k\right) \leq \mathbf{d}\left(\frac{\widetilde{h}_1 e^{u_1}}{\int_\Sigma \widetilde{h}_1 e^{u_1} dV_g}, \sigma_1(u)\right) = \sup_{\|\phi\|_{Lip(\Sigma)}=1} \left| \int_\Sigma \left(\frac{\widetilde{h}_1 e^{u_1}}{\int_\Sigma \widetilde{h}_1 e^{u_1} dV_g} - \sigma_1(u)\right) \phi \, dV_g \right| \\
= & \sup_{\|\phi\|_{Lip(\Sigma)}=1} \left| \int_{\Sigma \setminus \bigcup_{i=1}^k B_{\widetilde{r}}(x_i)} \left(\frac{\widetilde{h}_1 e^{u_1}}{\int_\Sigma \widetilde{h}_1 e^{u_1} dV_g} - \sigma_1(u)\right) \phi \, dV_g \right| \\
+ & \sup_{\|\phi\|_{Lip(\Sigma)}=1} \left| \int_{\bigcup_{i=1}^k B_{\widetilde{r}}(x_i)} \left(\frac{\widetilde{h}_1 e^{u_1}}{\int_\Sigma \widetilde{h}_1 e^{u_1} dV_g} - \sigma_1(u)\right) \phi \, dV_g \right| \\
< & \sup_{\|\phi\|_{Lip(\Sigma)}=1} 2\widetilde{\varepsilon}\|\phi\|_{L^\infty(\Sigma)} + \widetilde{r}\|\nabla\phi\|_{L^\infty(\Sigma)} \leq 2\widetilde{\varepsilon} + \widetilde{r} = \varepsilon,
\end{aligned}$$

as desired. ■

When a measure is close in the Lip' sense to an element in Σ_l, it is then possible to map it continuously to a nearby element in this set, see Proposition 2.4.1. With the previous estimates, recalling the definition of ψ_l in the latter proposition, it is now easy to define a projection map in the following form:

Proposition 2.2.7 *Suppose $\rho_1 \in (4k\pi, 4(k+1)\pi)$, $\rho_2 \in (4l\pi, 4(l+1)\pi)$ and let Φ_λ be as in (2.2). Then for L sufficiently large there exists a continuous map*

$$\Psi : J_\rho^{-L} \to (\gamma_1)_k * (\gamma_2)_l$$

such that the composition

$$(\gamma_1)_k * (\gamma_2)_l \xrightarrow{\Phi_\lambda} J_\rho^{-L} \xrightarrow{\Psi} (\gamma_1)_k * (\gamma_2)_l$$

is homotopically equivalent to the identity map on $(\gamma_1)_k * (\gamma_2)_l$ provided that λ is large enough.

The rest of this section is devoted to the proof of this proposition.

By Proposition 2.2.6 we know that either $\psi_k\left(\frac{\widetilde{h}_1 e^{u_1}}{\int_\Sigma \widetilde{h}_1 e^{u_1} dV_g}\right)$ or $\psi_l\left(\frac{\widetilde{h}_2 e^{u_2}}{\int_\Sigma \widetilde{h}_2 e^{u_2} dV_g}\right)$ is well defined (or both), since either $\mathbf{d}\left(\frac{\widetilde{h}_1 e^{u_1}}{\int_\Sigma \widetilde{h}_1 e^{u_1} dV_g}, \Sigma_k\right) < \varepsilon$ or $\mathbf{d}\left(\frac{\widetilde{h}_2 e^{u_2}}{\int_\Sigma \widetilde{h}_2 e^{u_2} dV_g}, \Sigma_l\right) < \varepsilon$ (or both).

We then set

$$d_1 = \mathbf{d}\left(\frac{\widetilde{h}_1 e^{u_1}}{\int_\Sigma \widetilde{h}_1 e^{u_1} dV_g}, \Sigma_k\right); \qquad d_2 = \mathbf{d}\left(\frac{\widetilde{h}_2 e^{u_2}}{\int_\Sigma \widetilde{h}_2 e^{u_2} dV_g}, \Sigma_l\right),$$

and consider a function $\widetilde{s} = \widetilde{s}(d_1, d_2)$ defined as

$$\widetilde{s}(d_1, d_2) = f\left(\frac{d_1}{d_1 + d_2}\right), \tag{2.18}$$

where f is such that

$$f(z) = \begin{cases} 0 & \text{if } z \in [0, 1/4], \\ 2z - \frac{1}{2} & \text{if } z \in (1/4, 3/4), \\ 1 & \text{if } z \in [3/4, 1]. \end{cases} \tag{2.19}$$

Consider the global retractions $\Pi_1 : \Sigma \to \gamma_1$ and $\Pi_2 : \Sigma \to \gamma_2$ given in Lemma 2.1.1, and define:

$$\Psi(u_1, u_2) = (1 - \widetilde{s})(\Pi_1)_* \psi_k\left(\frac{\widetilde{h}_1 e^{u_1}}{\int_\Sigma \widetilde{h}_1 e^{u_1} dV_g}\right) + \widetilde{s}(\Pi_2)_* \psi_l\left(\frac{\widetilde{h}_2 e^{u_2}}{\int_\Sigma \widetilde{h}_2 e^{u_2} dV_g}\right), \tag{2.20}$$

where $(\Pi_i)_*$ stands for the push-forward of the map Π_i. Notice that when one of the two ψ's is not defined the other necessarily is, and the map is well defined by the equivalence relation.

In what follows, we are going to need the following auxiliary lemma:

Lemma 2.2.8 *Given $n \in \mathbb{N}$, define χ_λ as $\chi_\lambda(x) = \sum_{i=1}^{n} t_i \left(\frac{\lambda}{1+\lambda^2 d(x,x_i)^2} \right)^2$. Take a L^∞ function τ : $\Sigma \to \mathbb{R}$ satisfying:*

i) $\tau(x) > m > 0$ *for all* $x \in B(x_i, \delta)$.

ii) $|\tau(x)| \leq M$ *for all* $x \in \Sigma$.

Then, there exist constants $c > 0$, $C > 0$ depending only on Σ, m, M, such that for every $\lambda > 0$,

$$c_0 \min \left\{ 1, \frac{1}{\lambda} \right\} < \mathbf{d} \left(\frac{\tau \, \chi_\lambda}{\int_\Sigma \tau \, \chi_\lambda \, dV_g}, \Sigma_n \right) < \frac{C_0}{\lambda}.$$

Proof. We show the proof for $n = 1$; the general case uses the same ideas and will be skipped. We also assume $\lambda > 1$. First of all, observe that

$$C > \int_\Sigma \chi_\lambda(x) \, dV_g(x) > c > 0$$

for some positive constants c, C.

For the upper estimate, it suffices to show that for any f Lipschitz, $\|f\|_{Lip(\Sigma)} \leq 1$,

$$\int_\Sigma \tau(x) \left(\frac{\lambda}{1+\lambda^2 d(x,x_0)^2} \right)^2 (f(x) - f(x_0)) \, dV_g(x) \leq \frac{C}{\lambda}.$$

Indeed, by ii),

$$\int_{(B_\delta(x_0))^c} \tau(x) \left(\frac{\lambda}{1+\lambda^2 d(x,x_0)^2} \right)^2 dV_g(x) \leq \frac{C}{\lambda^2},$$

and using geodesic coordinates x centered at x_0, we find

$$\left| \int_{B_\delta(x_0)} \tau(x) \left(\frac{\lambda}{1+\lambda^2 d(x,x_0)^2} \right)^2 (f(x) - f(x_0)) \, dV_g(x) \right|$$

$$\leq C \int_{B_{\delta\lambda}(0))} \tau \left(x_0 + \frac{y}{\lambda} \right) \left(\frac{1}{1+|y|^2} \right)^2 \left| f \left(x_0 + \frac{y}{\lambda} \right) - f(x_0) \right| dy$$

$$\leq C \int_{\mathbb{R}^2} \left(\frac{1}{1+y^2} \right)^2 \left| \frac{y}{\lambda} \right| dy \leq \frac{C}{\lambda}.$$

We now prove the estimate from below. Given $p \in \Sigma$, we estimate $\mathbf{d}(\chi_\lambda, \ \delta_p)$. Define the Lipschitz function $f(x) = d(x,p)$. We now show that:

$$\min_{p \in \Sigma} \int_\Sigma \tau(x) \left(\frac{\lambda}{1+\lambda^2 d(x,x_0)^2} \right)^2 d(x,p) \, dV_g(x) \geq \frac{c}{\lambda}.$$

As above, the integral in the exterior of $B_\delta(x_0)$ is negligible. Moreover, in the same coordinates as above, and taking into account i), we obtain:

$$\int_{B_\delta(x_0)} \tau(x) \left(\frac{\lambda}{1+\lambda^2 |x-x_0|^2} \right)^2 d(x,p) \, dV_g(x) \sim \int_{B_{\delta\lambda}(0)} \tau(x) \left(x_0 + \frac{y}{\lambda} \right) \left(\frac{1}{1+|y|^2} \right)^2 \left| x_0 - p + \frac{y}{\lambda} \right| dy$$

$$\geq \frac{m}{\lambda} \int_{B_\delta(0)} \left(\frac{1}{1+|y|^2} \right)^2 |y + \lambda(x_0 - p)| \, dy.$$

It suffices to show that we cannot choose p_λ so that

$$\int_{B_\delta(0)} \left(\frac{1}{1+|y|^2} \right)^2 |y + \lambda(x_0 - p_\lambda)| \, dx \to 0 \text{ as } \lambda \to +\infty. \tag{2.21}$$

Indeed, if $\lambda|x_0 - p_\lambda| \to +\infty$, the expression (2.21) diverges. If not, we can assume that $\lambda(x_0 - p_\lambda) \to z \in \mathbb{R}^2$. Then, (2.21) converges to

$$\int_{B_\delta(0)} \left(\frac{1}{1+|y|^2}\right)^2 |y+z| \, dx > 0.$$

which concludes the proof. ■

From the previous lemma we deduce the following.

Proposition 2.2.9 *Let φ_i be defined by (2.2). Then there exist constants $c > 0$, $C > 0$ such that for every $\lambda > 1$ and every $s \in (0,1)$ one has*

$$c_0 \min\left\{1, \frac{1}{\lambda_{1,s}}\right\} \leq \mathbf{d}\left(\frac{\widetilde{h}_1 e^{\varphi_1}}{\int_\Sigma \widetilde{h}_1 e^{\varphi_1} \, dV_g}, \Sigma_k\right) \leq \frac{C_0}{\lambda_{1,s}}; \qquad c_0 \min\left\{1, \frac{1}{\lambda_{2,s}}\right\} \leq \mathbf{d}\left(\frac{\widetilde{h}_2 e^{\varphi_2}}{\int_\Sigma \widetilde{h}_2 e^{\varphi_2} \, dV_g}, \Sigma_l\right) \leq \frac{C_0}{\lambda_{2,s}}.$$

Proof. Clearly, it suffices to prove the estimates for φ_1 in the case $\lambda_{1,s} > 1$. By the normalization, it suffices to prove it to the function $\varsigma = \varphi_1 - 2\log(\lambda_{1,s} \max\{1, \ \lambda_{2,s}\})$.

Observe now that we can write $e^\varsigma = \chi_{\lambda_{1,s}}(x)\, \tau(x)$, with:

$$\tau(x) = \widetilde{h}_1(x) \left[\sum_{j=1}^{l} s_j \left(\frac{\max\{1, \ \lambda_{2,s}\}^2}{1 + \lambda_{2,s}^2 d(x, y_j)^2}\right)^2\right]^{-1/2}.$$

It suffices to show that τ satisfy the conditions of Lemma 2.2.8 to conclude. ■

We are now in position to prove that the composition $\Psi \circ \Phi_\lambda$ is homotopic to the identity, where Ψ is as in (2.20) and $\Phi_\lambda(\zeta) = \varphi_{\lambda,\zeta}$ is as in (2.2). Take $\zeta = (1-s)\sigma_1 + s\sigma_2 \in (\gamma_1)_k * (\gamma_2)_l$, with

$$\sigma_1 = \sum_{i=1}^{k} t_i \delta_{x_i}, \qquad \sigma_2 = \sum_{j=1}^{l} s_j \delta_{y_j}.$$

Set $d_1 = \mathbf{d}\left(\frac{\widetilde{h}_1 e^{\varphi_1}}{\int_\Sigma \widetilde{h}_1 e^{\varphi_1} dV_g}, \Sigma_k\right)$, $d_2 = \mathbf{d}\left(\frac{\widetilde{h}_2 e^{\varphi_2}}{\int_\Sigma \widetilde{h}_2 e^{\varphi_2} dV_g}, \Sigma_l\right)$. By the previous proposition and the definition of $\lambda_{1,s}, \lambda_{2,s}$, there exist constants $0 < c_0 < C_0$ such that

$$c_0 \min\left\{1, \frac{1}{\lambda(1-s)}\right\} \leq d_1 \leq \frac{C_0}{\lambda(1-s)}, \quad c_0 \min\left\{1, \frac{1}{\lambda s}\right\} \leq d_2 \leq \frac{C_0}{\lambda s}.$$

Observe then that at least one between d_1 and d_2 must be smaller than $\frac{2C_0}{\lambda}$. Given $\delta > 0$ sufficiently small, we have:

$$s < \delta \Rightarrow \begin{cases} \frac{d_1}{d_1+d_2} \leq \dfrac{\frac{C_0}{\lambda(1-s)}}{\frac{c_0}{\lambda(1-s)} + \frac{c_0}{\lambda s}} = \dfrac{C_0}{c_0} s & \text{if } \lambda\, s \geq 1; \\ \frac{d_1}{d_1+d_2} \leq \dfrac{\frac{C_0}{\lambda(1-s)}}{c_0 + \frac{c_0}{\lambda(1-s)}} \leq \dfrac{C_0}{c_0}\dfrac{1}{\lambda} & \text{if } \lambda\, s \leq 1. \end{cases}$$

In any case, by choosing λ, δ adequately, we obtain that $\widetilde{s} = 0$. This fact is important, since the projection $\psi_l\left(\frac{\widetilde{h}_2 e^{\varphi_2}}{\int_\Sigma \widetilde{h}_2 e^{\varphi_2} dV_g}\right)$ could not be well defined.

Analogously, we have that if $s > (1-\delta)$, then the projection $\psi_k\left(\frac{\widetilde{h}_1 e^{\varphi_1}}{\int_\Sigma \widetilde{h}_1 e^{\varphi_1} dV_g}\right)$ could not be well defined, but $\widetilde{s} = 1$. Moreover, if $\delta \leq s \leq (1-\delta)$, then $d_i \leq \frac{C}{\delta\lambda}$, and hence both projections $\psi_k\left(\frac{\widetilde{h}_1 e^{\varphi_1}}{\int_\Sigma \widetilde{h}_1 e^{\varphi_1} dV_g}\right)$, $\psi_l\left(\frac{\widetilde{h}_2 e^{\varphi_2}}{\int_\Sigma \widetilde{h}_2 e^{\varphi_2} dV_g}\right)$ are well defined.

Letting $\widetilde{\zeta}_\lambda = \Psi \circ \Phi_\lambda(\zeta) = (1 - \widetilde{s}_\lambda)\widetilde{\sigma}_{1,\lambda} + \widetilde{s}_\lambda \widetilde{\sigma}_{2,\lambda}$, we consider the following homotopy:

$$H_1 : (0,1] \times ((\gamma_1)_k * (\gamma_2)_l) \to ((\gamma_1)_k * (\gamma_2)_l)\,,$$

$$H_1(\mu, (1-s)\sigma_1 + s\sigma_2) = (1 - s_{\mu,\lambda})\widetilde{\sigma}_{1,\frac{\lambda}{\mu}} + s_{\mu,\lambda}\widetilde{\sigma}_{2,\frac{\lambda}{\mu}}\,,$$

where $s_{\mu,\lambda} = (1-\mu) f(s) + \mu \widetilde{s}_\lambda$, and f is given by (2.19). Observe that $H_1(1,\cdot) = \Psi \circ \Phi_\lambda$.

Suppose now μ tends to zero. Then, as λ is fixed, $\frac{\lambda}{\mu} \to +\infty$, and hence $\dfrac{\widetilde{h}_i e^{\varphi_{i,\frac{\lambda}{\mu}}}}{\int_\Sigma \widetilde{h}_i e^{\varphi_{i,\frac{\lambda}{\mu}}} dV_g} \rightharpoonup \sigma_i$. Proposition 2.4.1 implies that $\psi_k\left(\dfrac{\widetilde{h}_1 e^{\varphi_1}}{\int_\Sigma \widetilde{h}_1 e^{\varphi_1} dV_g}\right) \to \sigma_1$, $\psi_l\left(\dfrac{\widetilde{h}_2 e^{\varphi_2}}{\int_\Sigma \widetilde{h}_2 e^{\varphi_2} dV_g}\right) \to \sigma_2$. Since Π_i are retractions, we conclude that $\widetilde{\sigma}_{i,\frac{\lambda}{\mu}} \to \sigma_i$. In other words,

$$\lim_{\mu \to 0} H_1(\mu, (1-s)\sigma_1 + s\sigma_2) = (1 - f(s))\sigma_1 + f(s)\sigma_2.$$

We now define:

$$H_2 : [0,1] \times ((\gamma_1)_k * (\gamma_2)_l) \to ((\gamma_1)_k * (\gamma_2)_l)\,,$$

$$H_2(\mu, (1-s)\sigma_1 + s\sigma_2) = [1 - (\mu f(s) + (1-\mu)s)]\sigma_1 + (\mu f(s) + (1-\mu)s)\sigma_2.$$

The concatenation of H_1 and H_2 gives the desired homotopy.

2.3 Min-max scheme

We now introduce the variational scheme which yields existence of solutions: this remaining part follows the ideas of [35] (see also [67]).

By Proposition 2.1.3, given any $L > 0$, there exists λ so large that $J_\rho(\varphi_{\lambda,\zeta}) < -L$ for any $\zeta \in (\gamma_1)_k * (\gamma_2)_l$. We choose L so large that Proposition 2.2.7 applies: we then have that the following composition

$$(\gamma_1)_k * (\gamma_2)_l \quad \xrightarrow{\Phi_\lambda} \quad J_\rho^{-L} \xrightarrow{\Psi} \quad (\gamma_1)_k * (\gamma_2)_l$$

is homotopic to the identity map. In this situation it is said that the set J_ρ^{-L} *dominates* $(\gamma_1)_k * (\gamma_2)_l$ (see [43], page 528). Since $(\gamma_1)_k * (\gamma_2)_l$ is not contractible, this implies that

$$\Phi_\lambda((\gamma_1)_k * (\gamma_2)_l) \text{ is not contractible in } J_\rho^{-L}.$$

Moreover, we can take λ larger so that $\Phi_\lambda((\gamma_1)_k * (\gamma_2)_l) \subset J_\rho^{-2L}$.

Define the topological cone with base $(\gamma_1)_k * (\gamma_2)_l$ via the equivalence relation

$$\mathcal{C} = \frac{(\gamma_1)_k * (\gamma_2)_l \times [0,1]}{(\gamma_1)_k * (\gamma_2)_l \times \{0\}} :$$

notice that, since $(\gamma_1)_k * (\gamma_2)_l \simeq S^{2k+2l-1}$, then $\mathcal{C}$ is homeomorphic to a Euclidean ball of dimension $2k + 2l$.

We now define the min-max value:

$$m = \inf_{\xi \in \Gamma} \max_{u \in \mathcal{C}} J(\xi(u)),$$

where

$$\Gamma = \{\xi : \mathcal{C} \to H^1(\Sigma) \times H^1(\Sigma) : \ \xi(\zeta) = \varphi_{\lambda,\zeta} \ \forall\, \zeta \in \partial\mathcal{C}\}. \tag{2.22}$$

Observe that $t\Phi_\lambda : \mathcal{C} \to H^1(\Sigma) \times H^1(\Sigma)$ belongs to Γ, so this is a non-empty set. Moreover,

$$\sup_{\zeta \in \partial\mathcal{C}} J_\rho(\xi(\zeta)) = \sup_{\zeta \in (\gamma_1)_k * (\gamma_2)_l} J_\rho(\varphi_{\lambda,\zeta}) \leq -2L.$$

We now show that $m \geq -L$. Indeed, $\partial\mathcal{C}$ is contractible in $\mathcal{C}$, and hence in $\xi(\mathcal{C})$ for any $\xi \in \Gamma$. Since $\partial\mathcal{C}$ is not contractible in J_ρ^{-L}, we conclude that $\xi(\mathcal{C})$ is not contained in J_ρ^{-L}. Being this valid for any arbitrary $\xi \in \Gamma$, we conclude that $m \geq -L$.

From the above discussion, the functional J_ρ satisfies the geometrical properties required by min-max theory, see [90]. However, we cannot directly conclude the existence of a critical point, since it is not known whether the Palais-Smale condition holds or not. The conclusion needs a different argument, which has been used intensively (see for instance [35], [37]), so we will be sketchy.

We take $\tilde{\nu} > 0$ such that

$$[\rho_1 - \tilde{\nu}, \rho_1 + \tilde{\nu}] \times [\rho_2 - \tilde{\nu}, \rho_2 + \tilde{\nu}] \subset \mathbb{R}^2 \setminus \Lambda,$$

where Λ is the set defined as in Definition 1.1.2.

Consider now $\nu \ll \tilde{\nu}$ and the parameter $\mu \in [1-\nu, 1+\nu]$. It is clear that the min-max scheme described above works uniformly for any μ in this range and $\mu\rho = (\mu\rho_1, \mu\rho_2)$. In other words, for any $L > 0$, there exists λ large enough so that

$$\sup_{\zeta \in \partial \mathcal{C}} J_{\mu\rho}(\xi(\zeta)) < -2L; \qquad m_\mu := \inf_{\xi \in \Gamma} \sup_{\zeta \in \mathcal{C}} J_{\mu\rho}(\xi(\zeta)) \geq -L, \quad \mu\rho = (\mu\rho_1, \mu\rho_2). \tag{2.23}$$

In this way, we are led to a problem depending on the parameter μ that satisfies a uniform min-max structure. In this framework, the following lemma is well-known, usually taking the name *monotonicity trick*. This technique was first used by Struwe in [88]; a first abstract version was made in [44] (see also [35], [65]).

Lemma 2.3.1 *There exists $\Upsilon \subset [1-\nu, 1+\nu]$ satisfying:*

1. $\big|[1-\nu, 1+\nu] \setminus \Upsilon\big| = 0.$

2. *For any $\mu \in \Upsilon$, the functional $J_{\mu\rho}$ possesses a bounded Palais-Smale sequence $(u_{1,n}, u_{2,n})_n$ at level m_μ.*

Proof. We give here the idea of the proof for the reader's convenience. Recalling the definition of the functional J_ρ given in (1.7), observe that for $\mu' \geq \mu$ we get

$$\frac{J_{\mu\rho}(u)}{\mu} - \frac{J_{\mu'\rho}(u)}{\mu'} = \left(\frac{1}{\mu} - \frac{1}{\mu'}\right) \int_\Sigma Q(u_1, u_2)\, dV_g \geq 0.$$

Therefore, it follows also that

$$\frac{m_\mu}{\mu} - \frac{m_{\mu'}}{\mu'} \geq 0.$$

In other words, the function $\mu \mapsto \frac{m_\mu}{\mu}$ is non-increasing and hence is almost everywhere differentiable. We define Υ to be the set where the latter function is differentiable. Using Struwe's monotonicity argument, see [88], one can see that at the points where $\frac{m_\mu}{\mu}$ is differentiable $J_{\mu\rho}$ admits a bounded Palais-Smale sequence at level m_μ. ■

Conclusion. Consider first $\mu \in \Upsilon$. Passing to a subsequence, the bounded Palais-Smale sequence can be assumed to converge weakly. Standard arguments show that the weak limit is indeed strong and that it is a critical point of $J_{\mu\rho}$.

Consider now $\mu_n \in \Upsilon$, $\mu_n \to 1$, and let $(u_{1,n}, u_{2,n})$ denote the corresponding solutions. It is then sufficient to apply the compactness result in Theorem 1.1.4, which yields convergence of $(u_{1,n}, u_{2,n})$ to a solution of (1.5).

2.4 Appendix: on the topology of Barycenter Spaces

In this appendix we collect some useful properties of the barycenter space concerning its CW structure and the existence of a projection map, see the next subsections.

2.4.1 CW structure of Barycenter Spaces

In this subsection we show that barycenter spaces of CW-complexes are again CW. The notation here is independent of the rest of the thesis, and the proofs use arguments from algebraic topology. The argument presented here was introduced by Prof. Sadok Kallel and is stated in the Appendix of [9].

We adopt the notation $\mathcal{B}_n$ for barycenter and Sym^{*n} for symmetric join, see [54]. We also need the notation

$$\Delta_{k-1} = \Big\{(t_1, \dots, t_k)\ \in [0,1]^k \mid \sum t_i = 1\Big\}$$

for the $(k-1)$-dimensional complex. This we view as a CW-complex with faces being subcomplexes. For $k < n$, we write as $\Delta_{k-1} \hookrightarrow \Delta_{n-1}$ the standard face inclusion given by adjoining trivial coordinate entries $(t_1, \dots, t_k) \mapsto (t_1, \dots, t_k, 0, \dots, 0)$. Similarly for based X, with basepoint x_0, we embed $X^k \hookrightarrow X^n$ by adjoining basepoints.

Proposition 2.4.1 *If X is a based connected CW-complex, then $\mathcal{B}_n(X)$ can be equipped with a CW structure so that all vertical projections in the following diagram are cellular maps and all horizontal maps are subcomplex inclusions*

$$\begin{tikzcd}
\Delta_{k-1} \times X^k \arrow[r, hook] \arrow[d] & \Delta_{n-1} \times X^n \arrow[d] \\
\mathcal{B}_k(X) \arrow[r, hook] & \mathcal{B}_n(X).
\end{tikzcd}$$

The proof uses standard facts about CW complexes which we now review.

(1) If (X, A) is a relative CW complex, then the quotient space X/A is a CW complex with a vertex corresponding to A.

(2) More generally if A is a subcomplex of a CW complex X, Y is a CW complex, and $f : A \longrightarrow Y$ is a cellular map, then the *pushout* $Y \cup_f X$ has an induced CW complex structure that contains Y as a subcomplex and has one cell for each cell of X that is not in A. We represent this construction by a diagram

$$\begin{tikzcd}
A \arrow[r, hook, "i"] \arrow[d, "f"] & X \arrow[d] \\
Y \arrow[r] & X \cup_f Y
\end{tikzcd}$$

with the understanding that all maps arriving at $X \cup_f Y$ are cellular with respect to the induced cell structure there.

(3) A finite group, or more generally a discrete group G acts *cellularly* on X means that: (i) if σ is an open cell of X then $g\sigma$ is again an open cell in X for all $g \in G$, and (ii) if $g \in G$ fixes an open cell σ, that is $g\sigma = \sigma$, then it fixes σ pointwise (i.e. $gx = x$ for all $x \in \sigma$). A CW-complex is a *cellular G-space* if G acts cellularly on X. If a finite group G acts cellulary on X, then X/G is a CW-complex. Furthermore, if $f : X \longrightarrow Y$ is a G-equivariant cellular map between cellular G-spaces, then the induced map $X/G \longrightarrow Y/G$ is cellular with respect to the induced CW-structures.

Properties (1) and (2) can be found in ([73], Chapter 10.2). Property (3) follows from Proposition 1.15 and Ex. 1.17 of [94] (Chapter 2). Throughout we endow X with a CW-structure so that the permutation action of $\mathfrak{S}_n$ on X^n is cellular, and so that x_0 is a 0-cell or vertex.

Proof of Proposition 2.4.1. We recall the definition of the barycenter spaces. Given X a space, then its n-th barycenter space is the quotient space

$$\mathcal{B}_n(X) := \coprod_{k=1}^{n} \Delta_{k-1} \times_{\mathfrak{S}_k} X^k /_{\sim}$$

where $\Delta_{k-1}\times_{\mathfrak{S}_k} X^k$ is the quotient of $\Delta_{k-1}\times X^k$ by the symmetric group $\mathfrak{S}_k$ acting diagonally, and where $\sim$ is the equivalence relation generated by:

$$\begin{array}{ll}(i) & [t_1,\dots,t_{i-1},0,t_{i+1},\dots,t_n;x_1,\dots,x_i,\dots x_n]\\ & \sim [t_1,\dots,t_{i-1},t_{i+1},\dots,t_n;x_1,\dots,\hat{x}_i,\dots x_n]\end{array}$$

(here $\hat{x}_i$ means the i-th entry has been suppressed), and by

$$\begin{array}{ll}(ii) & [t_1,\dots,t_i,\dots,t_j,\dots,t_n;x_1,\dots,x_i,\dots,x_j,\dots,x_n]\\ & \sim [t_1,\dots,t_{i-1},t_i+t_j,t_{i+1},\dots,\hat{t}_j,\dots,t_n;x_1,\dots,x_i,\dots,\hat{x}_j,\dots,x_n] \quad \text{if} \quad x_i=x_j.\end{array}$$

An intermediate construction is to consider the symmetric join $\mathrm{Sym}^{*n}(X)$ which is the quotient of $\coprod_{k=1}^n \Delta_{k-1}\times_{\mathfrak{S}_k} X^k$ by the equivalence relation (i) only. There are quotient projections

$$\Delta_{n-1}\times X^n \longrightarrow \Delta_{n-1}\times_{\mathfrak{S}_n} X^n \longrightarrow \mathrm{Sym}^{*n}(X)\longrightarrow \mathcal{B}_n(X)$$

and it is convenient to write an equivalence class in $\Delta_{n-1}\times_{\mathfrak{S}_n} X^n$ or any of its images in $\mathrm{Sym}^{*n}X$ and $\mathcal{B}_n(X)$ by

$$\sum_{i=1}^n t_ix_i := [t_1,\dots,t_n;x_1,\dots,x_n].$$

Addition means the sum is abelian and this reflects the symmetric group action. The relation (i) means the entry $0x_i$ is suppressed, and relation (ii) means that $t_ix+t_jx=(t_i+t_j)x$.

To show that $\mathcal{B}_n(X)$ is CW, we proceed by induction. When $n=1$, $\mathcal{B}_1X=X$ so there is nothing to prove. For the general case, write

$$\mathcal{B}_nX=\mathcal{B}_{n-1}X\cup(\Delta_{n-1}\times_{\mathfrak{S}_n} X^n)/_\sim$$

and write $X^n_{fat}\subset X^n$ the fat diagonal consisting of all n-tuples $(x_1,\dots,x_n)$ with $x_i=x_j$ for some $i\neq j$. Denote by

$$W_n=(\partial\Delta_{n-1}\times_{\mathfrak{S}_n} X^n)\bigcup(\Delta_{n-1}\times_{\mathfrak{S}_n} X^n_{fat})$$

the subspace of $\Delta_{n-1}\times_{\mathfrak{S}_n} X^n$ consisting of all classes $\sum t_ix_i$ with $t_i=0$ for some i or $x_i=x_j$ for some $i\neq j$. Then W_n is a CW subcomplex of X^n because the $\mathfrak{S}_n$-equivariant decomposition of X^n can always be arranged so that Δ_{fat} is a subcomplex. There is a well-defined quotient map $f: W_n\longrightarrow \mathcal{B}_{n-1}$ sending

$$\begin{array}{lll}\sum t_jx_j & \longmapsto & \sum_{j\neq i} t_jx_j \quad \text{if } t_i=0\\ \sum t_jx_j & \longmapsto & t_1x_1+\dots+(t_i+t_j)x_i+\dots+\widehat{t_jx_j}+\dots+t_nx_n \quad \text{if } x_i=x_j\end{array}$$

and we have the pushout diagram

$$(*)\quad \begin{tikzcd} W_n \arrow[r, hook] \arrow[d, "f"] & \Delta_{n-1}\times_{\mathfrak{S}_n} X^n \arrow[d] \\ \mathcal{B}_{n-1}X \arrow[r] & \mathcal{B}_n(X). \end{tikzcd}$$

If we can show that f is cellular, then by property (2) and induction, $\mathcal{B}_n(X)$ will be CW as desired.

The map f has two restrictions f_1 and f_2 on the pieces $\partial\Delta_{n-1}\times_{\mathfrak{S}_n} X^n$ and $\Delta_{n-1}\times_{\mathfrak{S}_n} X^n_{fat}\subset W_n$ respectively. To see that f_1 is cellular, write $\partial\Delta_{n-1}$ as a union of faces $F_i=\{(t_1,\dots,t_n),t_i=0\}$ each homeomorphic to Δ_{n-2}. Write $X^n_i=\{(x_1,\dots,x_n)\in X^n\mid x_i=x_0\}$ where $x_0\in X$ is the basepoint. The maps $F_i\times X^n\longrightarrow F_i\times X^n_i$, $(t_1,\dots,t_n;x_1,\dots,x_n)\mapsto(t_1,\dots,t_n;x_1,\dots,x_0,\dots,x_n)$; which for a given i replaces x_i by x_0, are cellular and so is their union

$$\bigcup_i F_i\times X^n\longrightarrow\bigcup_i F_i\times X^n_i.$$

This map is $\mathfrak{S}_n$-equivariant and so passes to a cellular map between quotients

$$\begin{tikzcd}
\left(\bigcup_i F_i \times X^n\right)/_{\mathfrak{S}_n} \arrow[r] \arrow[d, equal] & \left(\bigcup_i F_i \times X^n_i\right)/_{\mathfrak{S}_n} \arrow[d, equal] \\
\partial\Delta_{n-1} \times_{\mathfrak{S}_n} X^n \arrow[r, "g"] & \Delta_{n-2} \times_{\mathfrak{S}_{n-1}} X^{n-1}.
\end{tikzcd}$$

The restriction f_1 is now the composite of cellular maps

$$\partial\Delta_{n-1} \times_{\mathfrak{S}_n} X^n \xrightarrow{g} \Delta_{n-2} \times_{\mathfrak{S}_{n-1}} X^{n-1} \longrightarrow \mathcal{B}_{n-1}(X)$$

thus it is cellular. We proceed the same way for the restriction f_2. Write $X^n_{fat} = \bigcup_{i<j} X^n_{ij}$ where $X^n_{ij} = \{(x_1, \ldots, x_n) \in X^n \mid x_i = x_j, i < j\}$. Each X^n_{ij} is identified with X^{n-1}. There are maps $\tau_{ij} : \Delta_{n-1} \times X^n_{ij} \longrightarrow F_i \times X^n_i$ sending

$$\begin{aligned}
&(t_1, \ldots, t_n, x_1, \ldots, x_n) \\
&\longmapsto (t_1, \ldots, t_{i-1}, 0, t_{i+1}, \ldots, t_{j-1}, t_i + t_j, t_{j+1}, \ldots, t_n; x_1, \ldots, x_{i-1}, x_0, x_{i+1}, \ldots, x_n)
\end{aligned}$$

which are cellular being the product of cellular maps (i.e it can be checked that the map $\Delta_{n-1} \longrightarrow \partial\Delta_{n-1}$ sending $(t_1, \ldots, t_n) \longrightarrow (t_1, \ldots, t_{i-1}, 0, t_{i+1}, \ldots, t_{j-1}, t_i + t_j, t_{j+1}, \ldots, t_n)$ sends faces to faces and hence is cellular). The map $\bigcup \tau_{ij}$ is not $\mathfrak{S}_n$-equivariant, but the composite

$$\bigcup_{i<j} \Delta_{n-1} \times X^n_{ij} \longrightarrow \bigcup_i F_i \times X^n_i \longrightarrow \left(\bigcup_i F_i \times X^n_i\right)/_{\mathfrak{S}_n}$$

factors through the $\mathfrak{S}_n$-quotient. More precisely, we have the diagram

$$\begin{tikzcd}
\left(\bigcup_{i<j} \Delta_{n-1} \times X^n_{ij}\right)/_{\mathfrak{S}_n} \arrow[r] \arrow[d, equal] & \left(\bigcup_i F_i \times X^n_i\right)/_{\mathfrak{S}_n} \arrow[d, equal] & \\
\Delta_{n-1} \times_{\mathfrak{S}_n} X^n_{fat} \arrow[r, "\tau"] & \Delta_{n-2} \times_{\mathfrak{S}_{n-1}} X^{n-1} \arrow[r] & \mathcal{B}_{n-1}(X)
\end{tikzcd}$$

with all maps in this diagram cellular. The bottom composite f_2 must therefore be cellular.

In conclusion, the map $f = f_1 \cup f_2$ in the diagram (*) is cellular and this completes the proof. ■

Example 2.4.2 We take a special look at $\mathcal{B}_2(X)$. Consider $\mathrm{Sym}^{*2} X$ which consists of elements of the form $t_1 x + t_2 y$ with $t_1 + t_2 = 1$ and the identification $0x + 1y = y$. By using the order on the t_i's in $I = [0, 1]$, this can be written as

$$\begin{aligned}
\mathrm{Sym}^{*2}(X) &= \{(t_1, t_2, x_1, x_2) \mid t_1 \leq t_2, t_1 + t_2 = 1\}/_\sim \\
&= J \times (X \times X)/_\sim
\end{aligned}$$

where $J = \{0 \leq t_1 \leq t_2 \leq 1, t_1 + t_2 = 1\}$ is a copy of the one-simplex, and the identification $\sim$ is such that $(0, 1, x, y) \sim (0, 1, x', y)$ and $(\frac{1}{2}, \frac{1}{2}, x, y) \sim (\frac{1}{2}, \frac{1}{2}, y, x)$. Note that $(0, 1)$ and $(\frac{1}{2}, \frac{1}{2})$ are precisely the faces or endpoints of J. This is saying that $\mathrm{Sym}^{*2} X$ is precisely the double mapping cylinder

$$\begin{tikzcd}
X^2 \times \{(0,1)\} \sqcup X^2 \times \{(\frac{1}{2}, \frac{1}{2})\} \arrow[r, hook] \arrow[d, "p_2 \sqcup \pi"] & X^2 \times J \arrow[d] \\
X \sqcup \mathrm{SP}^2 X \arrow[r] & \mathrm{Sym}^{*2} X
\end{tikzcd}$$

where p_2 is the projection onto the second factor $X^2 \longrightarrow X$, and π is the $\mathbb{Z}_2$-quotient map $X^2 \longrightarrow \mathrm{SP}^2 X$ (see [54]). Both maps p_2 and π are cellular (property (3)). This gives $\mathrm{Sym}^{*2}(X)$ a CW-structure according to property (3). We can now consider the pushout diagram

$$\begin{tikzcd}
J \times X \arrow[r] \arrow[d] & \mathrm{Sym}^{*2} X \arrow[d] \\
X \arrow[r] & \mathcal{B}_2 X
\end{tikzcd} \tag{2.24}$$

where the left vertical map $J \times X \longrightarrow X$ is projection hence cellular, while the top map $J \times X \longrightarrow \mathrm{Sym}^{*2} X$, $((t_1, t_2), x) \mapsto t_1 x + t_2 x$, is a subcomplex inclusion. By property (2), $\mathcal{B}_2(X)$ is CW.

2.4.2 A projection onto the Barycenter Space

Recall the definition of the distance $\mathbf{d}$ given by (1.25). When a measure is $\mathbf{d}$-close to an element in Σ_l, see (1.15), it is then possible to map it continuously to a nearby element in this set. The next proposition collects some properties of this map, which has been proved in [37], but we give here a much shorter and self-consistent proof, using the results of Subsection 2.4.1 (here $\Sigma_l = \mathcal{B}_l(\Sigma)$).

Proposition 2.4.1 *Given $l \in \mathbb{N}$, for ε_l sufficiently small there exists a continuous retraction*

$$\psi_l : \{\nu \in \mathcal{M}(\Sigma),\ \mathbf{d}(\nu, \Sigma_l) < 2\varepsilon_l\} \to \Sigma_l.$$

Here continuity is referred to the distance $\mathbf{d}$. In particular, if $\nu_n \rightharpoonup \nu$ in the sense of measures, with $\nu \in \Sigma_l$, then $\psi_l(\nu_n) \to \nu$.

Furthermore, the following property holds: given any $\varepsilon > 0$ there exists $\varepsilon' \ll \varepsilon$, ε' depending on l and ε such that if $\mathbf{d}(\nu, \Sigma_{l-1}) > \varepsilon$ then there exist l points $x_1, \dots, , x_l$ such that

$$d(x_i, x_j) > 2\varepsilon' \quad \text{for } i \neq j; \qquad \int_{B_{\varepsilon'}(x_i)} \nu > \varepsilon' \quad \text{for all } i = 1, \dots, l.$$

Proof. Observe that the inclusion $Lip(\Sigma) \subset C(\Sigma)$ is compact: therefore, $\mathcal{M}(\Sigma) = C(\Sigma)' \subset Lip(\Sigma)'$ is also compact. Of course, the set $\Sigma_l \subset \mathcal{M}(\Sigma)$, and then it is inside $Lip(\Sigma)'$. Since Σ_l is a CW complex it follows that it is a Euclidean Neighbourhood Retract (ENR) (see Appendix E of [12]). Therefore, there exists a neighbourhood $V \supset \Sigma_l$ in the Lip' topology, and a continuous retraction $\psi_l : V \to \Sigma_l$.

Now, if $\nu_n \rightharpoonup \nu \in \Sigma_l$ in the sense of measures, by compactness, $\nu_n \to \nu$ in Lip', and by continuity, $\psi_l(\nu_n) \to \psi_l(\nu)$. But, since ψ_l is a retraction, $\psi_l(\nu) = \nu$.

The last property of the statement of the proposition is proved in Lemma 2.3 in [37] (together with the proof of Lemma 3.10). ∎

Chapter 3

The Toda system on compact surfaces of arbitrary genus

We are interested here in the regular Toda system on a compact surface Σ, namely

$$\begin{cases} -\Delta u_1 = 2\rho_1 \left(\frac{h_1 e^{u_1}}{\int_\Sigma h_1 e^{u_1} dV_g} - 1 \right) - \rho_2 \left(\frac{h_2 e^{u_2}}{\int_\Sigma h_2 e^{u_2} dV_g} - 1 \right), \\ -\Delta u_2 = 2\rho_2 \left(\frac{h_2 e^{u_2}}{\int_\Sigma h_2 e^{u_2} dV_g} - 1 \right) - \rho_1 \left(\frac{h_1 e^{u_1}}{\int_\Sigma h_1 e^{u_1} dV_g} - 1 \right), \end{cases} \tag{3.1}$$

where ρ_1, ρ_2 are real parameters and h_1, h_2 two smooth positive functions. We prove here the first existence result for surfaces of arbitrary genus when both parameters ρ_i are supercritical and one of them also arbitrarily large. For an introduction concerning this problem see Section 1.1 and Subsection 1.1.2. The following result is stated in [48].

Theorem 3.0.1 *Let h_1, h_2 be two positive smooth functions and let Σ be any compact surface. Suppose that $\rho_1 \in (4k\pi, 4(k+1)\pi), k \in \mathbb{N}$ and $\rho_2 \in (4\pi, 8\pi)$. Then the above Toda system has a solution.*

The chapter is organized as follows. In Section 3.1 we recall some improved versions of the Moser-Trudinger inequality, first some which rely on the *macroscopic* spreading of the components u_1, u_2 and then some refined ones, which are scaling invariant. In Section 3.2 we derive a new - still scaling invariant - improved version of the Moser-Trudinger inequality for systems, and we use it to find a characterization of low energy levels of the associated energy functional J_ρ by means of a subset Y of the topological join $\Sigma_k * \Sigma_1$, see (1.17). In Section 3.3 we construct then suitable test functions which show the optimality of the above characterization. In Section 3.4 we finally introduce the variational method to prove the existence of solutions.

3.1 Preliminaries

In the next two subsections we will recall and discuss some improved versions of the Moser-Trudinger inequality (1.16) which hold under suitable assumptions on the components of the system. The first type of inequality relies on the spreading of the (exponentials of the) components over the surface (see Section 2.2). The second one, from [71], relies instead on comparing the scales of concentration of the two components.

3.1.1 Macroscopic improved inequalities

The first kind of improved inequality was already introduced in Section 2.2. We repeat here the argument for the reader's convenience, as we will need it later on. Basically, if the *mass* of both e^{u_1} and e^{u_2} is *spread* respectively on at least $k+1$ and $l+1$ different sets, then the logarithms in (1.16) can be multiplied by $k+1$ and $l+1$ respectively. The proof relies on localizing (1.16) by using cut-off functions near the regions of volume concentration.

Lemma 3.1.1 *([9]) Let $\delta > 0$, $\theta > 0$, $k, l \in \mathbb{N}$ and $\{\Omega_{1,i}, \Omega_{2,j}\}_{i\in\{0,\dots,k\}, j\in\{0,\dots,l\}} \subset \Sigma$ be such that*

$$d(\Omega_{1,i}, \Omega_{1,i'}) \geq \delta \qquad \forall\, i,\ i' \in \{0,\dots,k\} \text{ with } i \neq i';$$

$$d(\Omega_{2,j}, \Omega_{2,j'}) \geq \delta \qquad \forall\, j,\ j' \in \{0,\dots,l\} \text{ with } j \neq j'.$$

Then, for any $\varepsilon > 0$ there exists $C = C\,(\varepsilon, \delta, \theta, k, l, \Sigma)$ such that any $(u_1, u_2) \in H^1(\Sigma) \times H^1(\Sigma)$ satisfying

$$\int_{\Omega_{1,i}} e^{u_1}\, dV_g \geq \theta \int_\Sigma e^{u_1}\, dV_g \qquad \forall\, i \in \{0,\dots,k\};$$

$$\int_{\Omega_{2,j}} e^{u_2}\, dV_g \geq \theta \int_\Sigma e^{u_2}\, dV_g \qquad \forall\, j \in \{0,\dots,l\}$$

verifies

$$4\pi(k+1)\log \int_\Sigma e^{u_1 - \overline{u}_1}\, dV_g + 4\pi(l+1)\log \int_\Sigma e^{u_2 - \overline{u}_2}\, dV_g \leq (1+\varepsilon)\int_\Sigma Q(u_1,u_2)\, dV_g + C.$$

As one can see, larger constants in the left-hand side of (1.16) can be helpful in obtaining lower bounds on the functional J_ρ even when the coefficients ρ_1, ρ_2 exceed the threshold value $(4\pi, 4\pi)$. A consequence of this fact is that when the energy $J_\rho(u_1, u_2)$ is large negative, then e^{u_1}, e^{u_2} are forced to *concentrate* near certain points in Σ whose number depends on ρ_1, ρ_2.

In Section 2.2, using the improved inequality from Lemma 2.2.3, the following result was proven.

Proposition 3.1.2 *([9]) Suppose $\rho_1 \in (4k\pi, 4(k+1)\pi)$ and $\rho_2 \in (4l\pi, 4(l+1)\pi)$. Then, for any $\varepsilon > 0$, there exists $L > 0$ such that any $(u_1, u_2) \in J_\rho^{-L}$ verifies either*

$$\mathbf{d}\left(\frac{e^{u_1}}{\int_\Sigma e^{u_1}\, dV_g}, \Sigma_k\right) < \varepsilon \qquad \text{or} \qquad \mathbf{d}\left(\frac{e^{u_2}}{\int_\Sigma e^{u_2}\, dV_g}, \Sigma_l\right) < \varepsilon.$$

This alternative can be expressed naturally in terms of the topological join of $\Sigma_k * \Sigma_l$, see (1.17). Indeed, for $\rho_1 \in (4k\pi, 4(k+1)\pi)$ and $\rho_2 \in (4l\pi, 4(l+1)\pi)$ we can define a continuous map Ψ from the low sub-levels J_ρ^{-L} onto this set, see Proposition 2.2.7:

$$\Psi : J_\rho^{-L} \to \Sigma_k * \Sigma_l,$$

$$\Psi(u_1, u_2) = (1 - \widetilde{s})\psi_k\left(\frac{e^{u_1}}{\int_\Sigma e^{u_1}\, dV_g}\right) + \widetilde{s}\,\psi_l\left(\frac{e^{u_2}}{\int_\Sigma e^{u_2}\, dV_g}\right), \tag{3.2}$$

where $\widetilde{s}$ is defined as in (2.18). Notice that we consider a slight modification of the map Ψ introduced in Proposition 2.2.7, where the retractions Π_i from Lemma 2.1.1 are involved. With a little abuse of notation we continue to denote by Ψ the above modified map.

3.1.2 Scaling-invariant improved inequalities

In [71] the authors set up a tool to deal with situations to which Lemma 2.2.3 does not apply, for example in cases when both e^{u_1}, e^{u_2} are concentrated around only one point. They provided a definition of the *center* and the *scale of concentration* of such functions, to obtain new improved inequalities in terms of these (see Section 4.2 for a brief summary of this argument). We are interested here in measures concentrated around possibly multiple points. We need therefore a localized version of the argument in [71], which applies to measures supported in a ball and sufficiently concentrated around its center.

Given $x_0 \in \Sigma$ and $r > 0$ small, consider the set

$$\mathcal{A}_{x_0,r} = \left\{ f \in L^1(B_r(x_0)) \ :\ f > 0 \ \text{ a.e. and } \int_{B_r(x_0)} f\, dV_g = 1 \right\},$$

endowed with the topology inherited from $L^1(\Sigma)$.

Fix a constant $R > 1$ and let $R_0 = 3R$. Define $\sigma : B_r(x_0) \times \mathcal{A}_{x_0,r} \to (0, +\infty)$ such that:

$$\int_{B_{\sigma(x,f)}(x) \cap B_r(x_0)} f \, dV_g = \int_{\left(B_{R_0\sigma(x,f)}(x)\right)^c \cap B_r(x_0)} f \, dV_g. \tag{3.3}$$

It is easy to check that $\sigma(x,f)$ is uniquely determined and continuous (both in $x \in B_r(x_0)$ and in $f \in L^1$). Moreover, see (3.2) in [71], σ satisfies:

$$d(x,y) \leq R_0 \max\{\sigma(x,f), \sigma(y,f)\} + \min\{\sigma(x,f), \sigma(y,f)\}. \tag{3.4}$$

We now define $T : B_r(x_0) \times \mathcal{A}_{x_0,r} \to \mathbb{R}$ as

$$T(x,f) = \int_{B_{\sigma(x,f)}(x) \cap B_r(x_0)} f \, dV_g.$$

Lemma 3.1.3 *([71], with minor adaptations) If $\bar{x} \in \overline{B_r(x_0)}$ is such that $T(\bar{x},f) = \max_{y \in \overline{B_r(x_0)}} T(y,f)$, then $\sigma(\bar{x},f) < 3\sigma(x,f)$ for any other $x \in \overline{B_r(x_0)}$.*

As a consequence of the previous lemma and of a covering argument, one can obtain the following:

Lemma 3.1.4 *([71], with minor adaptations) There exists a fixed $\tau > 0$ such that*

$$\max_{x \in \overline{B_r(x_0)}} T(x,f) > \tau > 0 \qquad \textit{for all } f \in \mathcal{A}_{x_0,r}.$$

Let us define $\sigma : \mathcal{A}_{x_0,r} \to \mathbb{R}$ by

$$\sigma(f) = 3 \min\left\{\sigma(x,f) : \; x \in \overline{B_r(x_0)}\right\},$$

which is obviously a continuous function.

Given τ as in Lemma 3.1.4, consider the set:

$$S(f) = \left\{x \in \overline{B_r(x_0)} \;:\; T(x,f) > \tau, \; \sigma(x,f) < \sigma(f)\right\}. \tag{3.5}$$

If $\bar{x} \in \overline{B_r(x_0)}$ is such that $T(\bar{x},f) = \max_{x \in \overline{B_r(x_0)}} T(x,f)$, then Lemmas 3.1.3 and 3.1.4 imply that $\bar{x} \in S(f)$. Therefore, $S(f)$ is a non-empty set for any $f \in \mathcal{A}_{x_0,r}$. Moreover, recalling (3.3) and the notation before it, from (3.4) we have that:

$$diam(S(f)) \leq (R_0 + 1)\sigma(f). \tag{3.6}$$

We will now restrict ourselves to a class of functions in $L^1(B_r(x_0))$ which are almost entirely concentrated near the center x_0. In this case one expects $\sigma(f)$ to be small and points in $S(f)$ to be close to x_0: see Remark 3.1.5 for precise estimates in this spirit. Given $\varepsilon > 0$ small, let us introduce the class of functions

$$\mathcal{C}_{\varepsilon,r}(x_0) = \left\{f \in \mathcal{A}_{x_0,r} : \int_{B_\varepsilon(x_0)} f \, dV_g > 1 - \varepsilon\right\}. \tag{3.7}$$

Remark 3.1.5 *For this class of functions we claim that $T(x,f) \leq \varepsilon$ when $d(x,x_0) > 2\varepsilon$. In fact, if $\sigma(x,f) \leq d(x,x_0) - \varepsilon$ then we are done, since*

$$T(x,f) = \int_{B_{\sigma(x,f)}(x) \cap B_r(x_0)} f \, dV_g \leq \int_{B_\varepsilon(x_0)^c \cap B_r(x_0)} f \, dV_g \leq \varepsilon.$$

If this is not the case, i.e. $\sigma(x,f) > d(x,x_0) - \varepsilon$, then using $d(x,x_0) > 2\varepsilon$ we obtain

$$\begin{aligned} R_0\sigma(x,f) &> R_0(d(x,x_0) - \varepsilon) > \frac{R_0}{2}\, d(x,x_0) \\ &> d(x,x_0) + \varepsilon. \end{aligned}$$

Similarly as before we get

$$T(x,f) = \int_{\left(B_{R_0\sigma(x,f)}(x)\right)^c \cap B_r(x_0)} f \, dV_g \leq \int_{B_\varepsilon(x_0)^c \cap B_r(x_0)} f \, dV_g \leq \varepsilon.$$

Being τ universal, ε can be taken so small that $(T(x,f) - \tau)^+ = 0$ outside $B_{2\varepsilon}(x_0)$, $\forall f \in \mathcal{C}_{\varepsilon,r}(x_0)$.

By the Nash embedding theorem, we can assume that $\Sigma \subset \mathbb{R}^N$ isometrically, $N \in \mathbb{N}$. Take an open tubular neighborhood $\Sigma \subset U \subset \mathbb{R}^N$ of Σ, and $\delta > 0$ small enough so that:

$$co\left[B_{(R_0+1)\delta}(x) \cap \Sigma\right] \subset U \qquad \forall\, x \in \Sigma, \tag{3.8}$$

where co denotes the convex hull in $\mathbb{R}^N$.

For $f \in \mathcal{C}_{\varepsilon,r}(x_0)$ we define now

$$\eta(f) = \frac{\displaystyle\int_\Sigma (T(x,f)-\tau)^+ \left(\sigma(f)-\sigma(x,f)\right)^+ x \, dV_g}{\displaystyle\int_\Sigma (T(x,f)-\tau)^+ \left(\sigma(f)-\sigma(x,f)\right)^+ dV_g} \in \mathbb{R}^N,$$

which is well-defined, see Remark 3.1.5. The map η yields a sort of center of mass in $\mathbb{R}^N$ of the measure induced by f. Observe that the integrands become non-zero only on the set $S(f)$. However, whenever $\sigma(f) \le \delta$, (3.6) and (3.8) imply that $\eta(f) \in U$, and so we can define:

$$\beta : \{f \in \mathcal{A}_{x_0,r} : \ \sigma(f) \le \delta\} \to \Sigma, \quad \beta(f) = P \circ \eta(f),$$

where $P : U \to \Sigma$ is the orthogonal projection.

We finally define the map $\psi : \mathcal{C}_{\varepsilon,r}(x_0) \to \Sigma \times (0,r)$, which will be the main tool of this subsection.

$$\psi(f) = (\beta, \sigma). \tag{3.9}$$

Roughly, this map expresses the center of mass of f and its scale of concentration around this point.

In [71] it was proved that if both components (u_1,u_2) of the Toda system concentrate around the same point in Σ, with the same scale of concentration, then the constants in the left-hand side of (1.16) can be nearly doubled.

Remark 3.1.6 *The core of the argument of the improved inequality in [71] consists in proving that*

$$\psi\left(\frac{e^{u_1}}{\int_{B_r(x)} e^{u_1}\, dV_g}\right) = \psi\left(\frac{e^{u_2}}{\int_{B_r(y)} e^{u_2}\, dV_g}\right)$$

implies the existence of $\sigma > 0$ and of two balls $B_\sigma(z_1)$, $B_\sigma(z_2)$ such that

$$\frac{\displaystyle\int_{B_\sigma(z_i)} e^{u_i}\, dV_g}{\displaystyle\int_\Sigma e^{u_i}\, dV_g} \ge \gamma_0, \quad \frac{\displaystyle\int_{(B_{R\sigma}(z_i))^c \cap B_r(z_i)} e^{u_i}\, dV_g}{\displaystyle\int_\Sigma e^{u_i}\, dV_g} \ge \gamma_0, \quad \textit{for } i = 1,2 \qquad \textit{with} \qquad d(z_1,z_2) \lesssim \sigma, \tag{3.10}$$

for some fixed positive constant γ_0. Once this is achieved, the improved inequality is obtained by scaling arguments and Kelvin inversions (see Section 3 in [71] for full details).

Even when e^{u_1}, e^{u_2} are not necessarily concentrated near a single point, the assumptions of the next proposition still allow to obtain (3.10), and hence again nearly double constants in the left-hand side of (1.16).

Proposition 3.1.7 *([71], with minor changes) Let $\tilde\varepsilon > 0$ and $\delta' > 0$. Then there exist $R = R(\tilde\varepsilon)$ and ψ as in definition* (3.9) *such that: for any $(u_1,u_2) \in H^1(\Sigma) \times H^1(\Sigma)$ such that there exist $x,y \in \Sigma$ with*

$$\int_{B_r(x)} e^{u_1}\, dV_g \ge \delta' \int_\Sigma e^{u_1}\, dV_g, \qquad \int_{B_r(y)} e^{u_2}\, dV_g \ge \delta' \int_\Sigma e^{u_1}\, dV_g;$$

$$\frac{e^{u_1}}{\int_{B_r(x)} e^{u_1}\, dV_g} \in \mathcal{C}_{\varepsilon,r}(x), \qquad \frac{e^{u_2}}{\int_{B_r(y)} e^{u_2}\, dV_g} \in \mathcal{C}_{\varepsilon,r}(y)$$

and

$$\psi\left(\frac{e^{u_1}}{\int_{B_r(x)} e^{u_1}\, dV_g}\right) = \psi\left(\frac{e^{u_2}}{\int_{B_r(y)} e^{u_2}\, dV_g}\right), \tag{3.11}$$

the following inequality holds:

$$8\pi\left(\log\int_\Sigma e^{u_1-\overline{u}_1}\,dV_g+\log\int_\Sigma e^{u_2-\overline{u}_2}\,dV_g\right)\le(1+\tilde{\varepsilon})\int_\Sigma Q(u_1,u_2)\,dV_g+C, \tag{3.12}$$

for some $C=C(\tilde{\varepsilon},\delta',\Sigma)$.

Remark 3.1.8 (i) *Condition* (3.11) *can be relaxed. In fact, let* $C_1>1$ *and* $C_2>0$ *be two positive constants and define*

$$\psi\left(\frac{e^{u_1}}{\int_{B_r(x)}e^{u_1}\,dV_g}\right)=(\beta_1,\sigma_1),\qquad \psi\left(\frac{e^{u_2}}{\int_{B_r(y)}e^{u_2}\,dV_g}\right)=(\beta_2,\sigma_2).$$

Then, the result still holds true if

$$\frac{1}{C_1}\le\frac{\sigma_1}{\sigma_2}\le C_1,\qquad d(\beta_1,\beta_2)\le C_2\,\sigma_1.$$

In such case, the constant C *would also depend on* C_1 *and* C_2.

(ii) *In the right-hand side of* (3.12) *one can actually integrate* $Q(u_1,u_2)$ *only in any set compactly containing* $B_r(x)\cup B_r(y)$. *This can be seen using suitable cut-off functions, see the comments before Lemma 2.2.3.*

We can now improve this result for situations in which the first component of the system is concentrated around l points of Σ, $l\in\mathbb{N}$. The proof relies on combining the argument for Proposition 3.1.7 with the macroscopic improved inequality of Lemma 2.2.3 (see also Remark 3.1.8 (ii)).

Proposition 3.1.9 *Let* $\tilde{\varepsilon}>0$, $\delta'>0$ *and* $k\in\mathbb{N}$. *Then there exist* $R=R(\tilde{\varepsilon})$ *and* ψ *as in definition* (3.9) *such that: for any* $(u_1,u_2)\in H^1(\Sigma)\times H^1(\Sigma)$ *with the property that there exist* $\{x_i\}_{i\in\{1,\dots,k\}}\subset\Sigma$, $y\in\Sigma$ *with*

$$d(x_i,x_j)>4\delta'\qquad\forall\, i,\ j\in\{1,\dots,k\}\ \textit{with}\ i\neq j;$$

$$\int_{B_{\delta'}(x_i)}e^{u_1}\,dV_g\ge\delta'\int_\Sigma e^{u_1}\,dV_g\quad \textit{for}\ i=1,\dots,k;\qquad \int_{B_{\delta'}(y)}e^{u_2}\,dV_g\ge\delta'\int_\Sigma e^{u_2}\,dV_g,$$

such that

$$\frac{e^{u_1}}{\int_{B_{\delta'}(x_i)}e^{u_1}\,dV_g}\in\mathcal{C}_{\varepsilon,\delta'}(x_i)\quad\textit{for}\ i=1,\dots,k;\qquad \frac{e^{u_2}}{\int_{B_{\delta'}(y)}e^{u_2}\,dV_g}\in\mathcal{C}_{\varepsilon,\delta'}(y)$$

and

$$\psi\left(\frac{e^{u_1}}{\int_{B_{\delta'}(x_l)}e^{u_1}\,dV_g}\right)=\psi\left(\frac{e^{u_2}}{\int_{B_{\delta'}(y)}e^{u_2}\,dV_g}\right)\quad \textit{for some}\ l\in\{1,\dots,k\},$$

the following inequality holds:

$$4\pi(k+1)\log\int_\Sigma e^{u_1-\overline{u}_1}\,dV_g+8\pi\log\int_\Sigma e^{u_2-\overline{u}_2}\,dV_g\le(1+\tilde{\varepsilon})\int_\Sigma Q(u_1,u_2)\,dV_g+C,$$

for some $C=C(\tilde{\varepsilon},\delta',l,\Sigma)$.

In the next section we will derive a new improved inequality for the Toda system with scaling invariant features, see Proposition 3.2.5. The result is inspired by arguments developed in [4] for the singular Liouville equation where a Dirac delta is involved, see Remark 3.2.6, and for the first time this type of inequality is presented for a two-component problem.

3.2 A refined projection onto the topological join

Suppose that $\rho_1\in(4k\pi,4(k+1)\pi)$ and $\rho_2\in(4\pi,8\pi)$. Let Ψ by the map introduced in (3.2) from the low sub-levels of J_ρ onto the topological join $\Sigma_k*\Sigma_1$, see (1.17). We will need next to take also into account the fine structure of the measures e^{u_1} and e^{u_2}, as described in (3.9). For this reason we will modify the map Ψ so that the join parameter s in (1.17) will depend on the local centres of mass and the local scales defined in (3.9) and (3.13). We will see in the sequel that this will provide extra information for describing functions in the low sub-levels of J_ρ.

3.2.1 Construction

We start by defining the local centres of mass and the local scales of functions which are concentrated around l well separated points of Σ.

Let $l \geq 2$ and consider $0 < \varepsilon_l \ll \varepsilon_{l-1} \ll 1$ as given in Proposition 2.4.1 and suppose it holds $\mathbf{d}\left(\frac{e^{u_1}}{\int_\Sigma e^{u_1}\,dV_g}, \Sigma_l\right) < 2\varepsilon_l$ so that ψ_l is well-defined. Assume moreover $\mathbf{d}\left(\frac{e^{u_1}}{\int_\Sigma e^{u_1}\,dV_g}, \Sigma_{l-1}\right) > \varepsilon_{l-1}$. By the second part of Proposition 2.4.1 there exist $\varepsilon'_{l-1} \ll \varepsilon_{l-1}$ and l points $x_1^l, \dots, x_l^l$ such that

$$d\left(x_i^l, x_j^l\right) > 2\varepsilon'_{l-1} \quad \text{for } i \neq j; \qquad \int_{B_{\varepsilon'_{l-1}}(x_i^l)} e^{u_1}\,dV_g > \varepsilon'_{l-1}\int_\Sigma e^{u_1}\,dV_g \quad \text{for all } i = 1, \dots, l.$$

We localize then u_1 around the point x_i^l and define

$$f_{loc}^{x_i^l}(u_1) = \frac{e^{u_1}\chi_{B_{\varepsilon'_{l-1}}(x_i^l)}}{\displaystyle\int_{B_{\varepsilon'_{l-1}}(x_i^l)} e^{u_1}\,dV_g}.$$

Given $\varepsilon > 0$, by the second assertion of Proposition 2.4.1, taking ε_l sufficiently small one gets

$$\int_{B_\varepsilon(x_i^l)} f_{loc}^{x_i^l}(u_1)\,dV_g > 1 - \varepsilon; \qquad \text{for} \quad \mathbf{d}\left(\frac{e^{u_1}}{\int_\Sigma e^{u_1}\,dV_g}, \Sigma_l\right) < 2\varepsilon_l.$$

It follows that $f_{loc}^{x_i^l}(u_1) \in \mathcal{C}_{\varepsilon,\varepsilon'_{l-1}}(x_i^l)$, see (3.7), and hence the map ψ in (3.9) is well-defined on $f_{loc}^{x_i^l}(u_1)$. We then set

$$\left(\beta_{x_i^l}, \sigma_{x_i^l}\right) := \psi\left(f_{loc}^{x_i^l}(u_1)\right). \tag{3.13}$$

In this way, starting from a function with $\mathbf{d}\left(\frac{e^{u_1}}{\int_\Sigma e^{u_1}\,dV_g}, \Sigma_l\right) < 2\varepsilon_l$ and such that $\mathbf{d}\left(\frac{e^{u_1}}{\int_\Sigma e^{u_1}\,dV_g}, \Sigma_{l-1}\right) > \varepsilon_{l-1}$ we obtain, around each point x_i^l, a notion of local center of mass and scale of concentration.

When $l = 1$ we have to deal with just one point x_1^1 of Σ. We then apply the map ψ to the function $f_{loc}^{x_1^1}$ directly.

As we discussed above, we would like to map low energy sub-levels of J_ρ into the topological join $\Sigma_k * \Sigma_1$ taking the above scales into account. More precisely, the parameter s in (1.17) will depend on the local scale $\sigma_{x_i^l}$ only of the points nearby the center of mass of e^{u_2} (in case of ambiguity, we will define a sort of averaged scale).

To proceed rigorously, let $0 < \varepsilon_k \ll \varepsilon_{k-1} \ll \dots \ll \varepsilon_1 \ll 1$ be as before. We consider cut-off functions $\mathfrak{f}, \mathfrak{g}_l, \mathfrak{h}$ for $l = 1, \dots, k-1$ such that

$$\mathfrak{f}(t) = \begin{cases} 0 & t \geq 2\varepsilon_k, \\ 1 & t \leq \varepsilon_k, \end{cases} \qquad \mathfrak{g}_l(t) = \begin{cases} 0 & t \geq 2\varepsilon_l, \\ 1 & t \leq \varepsilon_l, \end{cases} \qquad l = 1, \dots, k-1, \tag{3.14}$$

$$\mathfrak{h}(t) = \begin{cases} 0 & t \geq \frac{\varepsilon'_{k-1}}{8}, \\ 1 & t \leq \frac{\varepsilon'_{k-1}}{16}. \end{cases} \tag{3.15}$$

We define now a *global scale* $\sigma_1(u_1) \in (0,1]$ for e^{u_1} in three steps. Suppose $\mathbf{d}\left(\frac{e^{u_2}}{\int_\Sigma e^{u_2}\,dV_g}, \Sigma_1\right) < 2\varepsilon_1$, so that $\psi(f_{loc}^z(u_2)) = (\beta_z, \sigma_z)$ is well-defined.

First of all, we define an *averaged scale* for e^{u_1} by recurrence in the following way. If we have $\mathbf{d}\left(\frac{e^{u_1}}{\int_\Sigma e^{u_1}\,dV_g}, \Sigma_1\right) < 2\varepsilon_1$, we set $C_1(u_1) = \sigma_{x_1^1}$. For $l \in \{2, \dots, k-1\}$, we define recursively

$$C_l(u_1) = \mathfrak{g}_{l-1}\left(\mathbf{d}\left(\frac{e^{u_1}}{\int_\Sigma e^{u_1}\,dV_g}, \Sigma_{l-1}\right)\right) C_{l-1}(u_1) + \left(1 - \mathfrak{g}_{l-1}\left(\mathbf{d}\left(\frac{e^{u_1}}{\int_\Sigma e^{u_1}\,dV_g}, \Sigma_{l-1}\right)\right)\right) \frac{1}{l}\sum_{i=1}^{l} \sigma_{x_i}^l.$$

Secondly, we interpolate between $C_{k-1}(u_1)$ and the local scale of the closest point to β_z among the $\beta_{x_i^k}$'s (provided they are well-defined), setting

$$B(u_1, u_2) = \mathfrak{h}\left(d(\beta_z, \left\{\beta_{x_1^k}, \dots, \beta_{x_k^k}\right\})\right)\sigma_x + \left(1 - \mathfrak{h}\left(d\left(\beta_z, \{\beta_{x_1^k}, \dots, \beta_{x_k^k}\}\right)\right)\right)\frac{1}{k}\sum_{i=1}^{k} \sigma_{x_i^k},$$

$$A(u_1,u_2) = \mathfrak{g}_{k-1}\left(\mathbf{d}\left(\frac{e^{u_1}}{\int_\Sigma e^{u_1}\,dV_g},\Sigma_{k-1}\right)\right)C_{k-1}(u_1) + \left(1-\mathfrak{g}_{k-1}\left(\mathbf{d}\left(\frac{e^{u_1}}{\int_\Sigma e^{u_1}\,dV_g},\Sigma_{k-1}\right)\right)\right)B(u_1,u_2),$$

where $x = x_j^k$ was chosen so that it realizes the minimum of $d\left(\beta_z,\{\beta_{x_1^k},\dots,\beta_{x_k^k}\}\right)$: notice that since $d(x_j^k,x_l^k)\geq 2\varepsilon'_{k-1}$ for $j\neq l$, by (3.15) the point realizing the latter minimum is unique if $\mathfrak{h}\neq 0$.

As a third and final step, to check whether e^{u_1} is $\mathbf{d}$-close to Σ_k, we set

$$\sigma_1(u_1) = \mathfrak{f}\left(\mathbf{d}\left(\frac{e^{u_1}}{\int_\Sigma e^{u_1}\,dV_g},\Sigma_k\right)\right)A(u_1,u_2) + \left(1-\mathfrak{f}\left(\mathbf{d}\left(\frac{e^{u_1}}{\int_\Sigma e^{u_1}\,dV_g},\Sigma_k\right)\right)\right).$$

We define next the *global scale* $\sigma_2(u_2)\in(0,1]$ of e^{u_2}. We will be interested here in functions concentrated near just one point of Σ. Therefore we just need the single local scale $C_1(u_2)=\sigma_z$ if $\psi(f^z_{loc}(u_2)) = (\beta_z,\sigma_z)$ is well-defined. Moreover, we have to check the $\mathbf{d}$-closeness of e^{u_2} to Σ_1. Hence the scale reads

$$\sigma_2(u_2) = \mathfrak{g}_1\left(\mathbf{d}\left(\frac{e^{u_2}}{\int_\Sigma e^{u_2}\,dV_g},\Sigma_1\right)\right)\sigma_z + \left(1-\mathfrak{g}_1\left(\mathbf{d}\left(\frac{e^{u_2}}{\int_\Sigma e^{u_2}\,dV_g},\Sigma_1\right)\right)\right).$$

We can now specify the join parameter s in (1.17). Fix a constant $M\gg 1$ and consider the function

$$F_M(t) = \begin{cases} 0 & t\leq 1/M,\\ \dfrac{t}{1+t} & t\in\left[\frac{2}{M},M\right],\\ 1 & t\geq 2M.\end{cases}$$

We then define

$$s(u_1,u_2) = F_M\left(\frac{\sigma_1(u_1)}{\sigma_2(u_2)}\right). \tag{3.16}$$

We now pass to considering the maps ψ_k and ψ_1 which are needed in the projection onto the join $\Sigma_k * \Sigma_1$, see (3.2). As mentioned in the introduction of this section, it is convenient to modify these maps in such a way that they take into account the local centres of mass defined in (3.9) and (3.13). More precisely, when e^{u_1} is concentrated in k well separated points of Σ, we rather consider the local centres of mass $\beta_{x_i^l}$ in (3.13) than the supports of the map ψ_k in Proposition 2.4.1.

Suppose $\mathbf{d}\left(\frac{e^{u_1}}{\int_\Sigma e^{u_1}\,dV_g},\Sigma_k\right) < 2\varepsilon_k$ so that ψ_k is well-defined and suppose $\mathbf{d}\left(\frac{e^{u_1}}{\int_\Sigma e^{u_1}\,dV_g},\Sigma_{k-1}\right) > \varepsilon_{k-1}$ so that $\beta_{x_i^k}$ are defined for $i=1,\dots,k$. Let

$$\psi_k\left(\frac{e^{u_1}}{\int_\Sigma e^{u_1}\,dV_g},\Sigma_k\right) = \sum_{i=1}^k t_i\delta_{y_i}, \qquad t_i\in[0,1],\ y_i\in\Sigma.$$

Observe that, by construction and by the second statement in Proposition 2.4.1, $d(\beta_{x_i^k},y_i)\to 0$ as $\varepsilon_k\to 0$. Hence there exists a geodesic γ_i joining y_i and $\beta_{x_i^k}$ in unit time. We then perform an interpolation in the following way:

$$\widetilde{\psi}_k\left(\frac{e^{u_1}}{\int_\Sigma e^{u_1}\,dV_g}\right) = \begin{cases} \sum_{i=1}^k t_i\delta_{y_i} & \text{if } \mathbf{d}\left(\frac{e^{u_1}}{\int_\Sigma e^{u_1}\,dV_g},\Sigma_{k-1}\right)\leq\varepsilon_{k-1},\\ \sum_{i=1}^k t_i\delta_{\gamma_i\left(\frac{1}{\varepsilon_{k-1}}\mathbf{d}\left(\frac{e^{u_1}}{\int_\Sigma e^{u_1}\,dV_g},\Sigma_{k-1}\right)-1\right)} & \text{if } \mathbf{d}\left(\frac{e^{u_1}}{\int_\Sigma e^{u_1}\,dV_g},\Sigma_{k-1}\right)\in(\varepsilon_{k-1},2\varepsilon_{k-1}),\\ \sum_{i=1}^k t_i\delta_{\beta_{x_i^k}} & \text{if } \mathbf{d}\left(\frac{e^{u_1}}{\int_\Sigma e^{u_1}\,dV_g},\Sigma_{k-1}\right)\geq 2\varepsilon_{k-1}.\end{cases} \tag{3.17}$$

For a function u_2 with $\mathbf{d}\left(\frac{e^{u_2}}{\int_\Sigma e^{u_2}\,dV_g},\Sigma_1\right) < 2\varepsilon_1$, letting $\psi_1\left(\frac{e^{u_2}}{\int_\Sigma e^{u_2}\,dV_g}\right) = \delta_z$ we let

$$\widetilde{\psi}_1\left(\frac{e^{u_2}}{\int_\Sigma e^{u_2}\,dV_g}\right) = \delta_{\beta_z}. \tag{3.18}$$

With these maps and this join parameter we finally define the refined projection $\widetilde{\Psi}: J_\rho^{-L}\to\Sigma_k*\Sigma_1$ as

$$\widetilde{\Psi}(u_1,u_2) = (1-s)\widetilde{\psi}_k\left(\frac{e^{u_1}}{\int_\Sigma e^{u_1}\,dV_g}\right) + s\,\widetilde{\psi}_1\left(\frac{e^{u_2}}{\int_\Sigma e^{u_2}\,dV_g}\right). \tag{3.19}$$

3.2.2 A new improved Moser-Trudinger inequality

Using the improved geometric inequality in [4] for the singular Liouville equation we can provide a dilation-invariant improved inequality for system (1.18). Before stating the main result we prove some auxiliary lemmas; we first recall our notation on annuli in Section 1.3.

Lemma 3.2.1 *Let $\gamma_0 > 0, \tau_0 > 0, z \in \Sigma$ and $r_2 > r_1 > 0$ (both small) be such that*

$$\frac{\displaystyle\int_{A_z(r_1,r_2)} e^{u_2}\,dV_g}{\displaystyle\int_\Sigma e^{u_2}\,dV_g} > \gamma_0 \qquad \text{and} \qquad \sup_{y\in A_z(r_1,r_2)} \frac{\displaystyle\int_{B_{\tau_0 d(y,z)}(y)} e^{u_2}\,dV_g}{\displaystyle\int_{A_z(r_1,r_2)} e^{u_2}\,dV_g} < 1-\tau_0. \tag{3.20}$$

Then, for any $\varepsilon > 0$ there exist $C = C(\varepsilon,\tau_0,\gamma_0)$, $\widetilde{\tau}_0 = \widetilde{\tau}_0(\tau_0,\gamma_0)$, $\widetilde{r}_1 \in \left[\frac{r_1}{C},\frac{r_1}{4}\right]$, $\widetilde{r}_2 \in [4r_2, Cr_2]$ and $\widetilde{u}_2 \in H^1(\Sigma)$ such that

a) $\widetilde{u}_2$ is constant in $B_{\widetilde{r}_1}(z)$ and on $\partial B_{\widetilde{r}_2}(z)$;

b) $\displaystyle\int_{A_z(\widetilde{r}_1,\widetilde{r}_2)} |\nabla\widetilde{u}_2|^2\,dV_g \le \int_{A_z(\widetilde{r}_1,\widetilde{r}_2)} |\nabla u_2|^2\,dV_g + \varepsilon\int_\Sigma |\nabla u_2|^2\,dV_g;$

c) $\displaystyle\sup_{y\in A_z(\widetilde{r}_1,\widetilde{r}_2)} \frac{\displaystyle\int_{B_{\widetilde{\tau}_0 d(y,z)}(y)} e^{\widetilde{u}_2}\,dV_g}{\displaystyle\int_{A_z(\widetilde{r}_1,\widetilde{r}_2)} e^{\widetilde{u}_2}\,dV_g} < 1-\widetilde{\tau}_0.$

Proof. First of all, we modify u_2 so it becomes constant in $B_{\widetilde{r}_1}(z)$ and on $\partial B_{\widetilde{r}_2}(z)$. Take $\varepsilon > 0$: we can find $C = C(\varepsilon)$ and properly chosen $\widetilde{r}_1 \in \left[\frac{r_1}{C},\frac{r_1}{4}\right]$, $\widetilde{r}_2 \in [4r_2, Cr_2]$ such that

$$\int_{A_z(\widetilde{r}_1,2\widetilde{r}_1)} |\nabla u_2|^2\,dV_g \le \varepsilon\int_\Sigma |\nabla u_2|^2\,dV_g, \qquad \int_{A_z(\widetilde{r}_2/2,\widetilde{r}_2)} |\nabla u_2|^2\,dV_g \le \varepsilon\int_\Sigma |\nabla u_2|^2\,dV_g.$$

We denote by $\overline{u}_2(\widetilde{r}_1)$ and $\overline{u}_2(\widetilde{r}_2)$ the following averages;

$$\overline{u}_2(\widetilde{r}_1) = \fint_{A_z(\widetilde{r}_1,2\widetilde{r}_1)} u_2\,dV_g, \qquad \overline{u}_2(\widetilde{r}_2) = \fint_{A_z(\widetilde{r}_2/2,\widetilde{r}_2)} u_2\,dV_g. \tag{3.21}$$

Let now χ be a cut-off function, with values in $[0,1]$, such that

$$\chi = \begin{cases} 0 & \text{in } B_{\widetilde{r}_1}(z),\\ 1 & \text{in } A_z(2\widetilde{r}_1,\widetilde{r}_2/2),\\ 0 & \text{in } (B_{\widetilde{r}_2}(z))^c \end{cases}$$

and define

$$\widetilde{u}_2 = \begin{cases} \chi(d(x,z))u_2 + (1-\chi(d(x,z))\overline{u}_2(\widetilde{r}_1)) & \text{in } B_{2\widetilde{r}_1}(z),\\ u_2 & \text{in } A_z(2\widetilde{r}_1,\widetilde{r}_2/2),\\ \chi(d(x,z))u_2 + (1-\chi(d(x,z))\overline{u}_2(\widetilde{r}_2)) & \text{in } (B_{\widetilde{r}_2/2}(z))^c. \end{cases} \tag{3.22}$$

By Poincaré's inequality the Dirichlet energy of $\widetilde{u}_2$ is bounded by

$$\int_{A_z(\widetilde{r}_1,2\widetilde{r}_1)} |\nabla\widetilde{u}_2|^2\,dV_g \le \widetilde{C}\varepsilon\int_\Sigma |\nabla u_2|^2\,dV_g, \qquad \int_{A_z(\widetilde{r}_2/2,\widetilde{r}_2)} |\nabla\widetilde{u}_2|^2\,dV_g \le \widetilde{C}\varepsilon\int_\Sigma |\nabla u_2|^2\,dV_g,$$

where $\widetilde{C}$ is a universal constant. Hence one gets

$$\int_{A_z(\widetilde{r}_1,\widetilde{r}_2)} |\nabla\widetilde{u}_2|^2\,dV_g \le \int_{A_z(\widetilde{r}_1,\widetilde{r}_2)} |\nabla u_2|^2\,dV_g + 2\widetilde{C}\varepsilon\int_\Sigma |\nabla u_2|^2\,dV_g.$$

We are left with proving that there exists $\widetilde{\tau}_0 = \widetilde{\tau}_0(\tau_0, \gamma_0)$ such that

$$\sup_{y \in A_z(\widetilde{r}_1, \widetilde{r}_2)} \frac{\displaystyle\int_{B_{\widetilde{\tau}_0 d(y,z)}(y)} e^{\widetilde{u}_2}\, dV_g}{\displaystyle\int_{A_z(\widetilde{r}_1, \widetilde{r}_2)} e^{\widetilde{u}_2}\, dV_g} < 1 - \widetilde{\tau}_0. \tag{3.23}$$

If this is not the case, there exist $(u_{2,n})_n \subset H^1(\Sigma)$ verifying (3.20), $(\widetilde{r}_{1,n})_n \subset \left[\frac{r_1}{C}, \frac{r_1}{4}\right]$, $(\widetilde{r}_{2,n})_n \subset [4r_2, Cr_2]$, cut-off functions $(\chi_n)_n$ and $(\widetilde{u}_{2,n})_n \subset H^1(\Sigma)$ defined in analogous way as $\widetilde{u}_2$ in (3.22), such that

$$\frac{e^{\widetilde{u}_{2,n}}}{\displaystyle\int_{A_z(\widetilde{r}_{1,n}, \widetilde{r}_{2,n})} e^{\widetilde{u}_{2,n}}\, dV_g} \rightharpoonup \delta_{\bar{x}} \tag{3.24}$$

in the sense of measures, for some $\bar{x} \in A_z\left(\frac{r_1}{C}, Cr_2\right)$. We distinguish between three situations.

Case 1. Suppose first that $\bar{x} \in A_z(r_1, 2r_2)$. By the choices of the cut-off functions and (3.22), as $\widetilde{u}_{2,n}$ coincides with $u_{2,n}$ on $A_z(r_1/2, 2r_2)$, it follows that

$$\frac{e^{u_{2,n}}}{\displaystyle\int_{A_z(r_1, 2r_2)} e^{u_{2,n}}\, dV_g} = \frac{e^{\widetilde{u}_{2,n}}}{\displaystyle\int_{A_z(r_1, 2r_2)} e^{\widetilde{u}_{2,n}}\, dV_g} \rightharpoonup \delta_{\bar{x}}. \tag{3.25}$$

Case 1.1. Let $\bar{x} \in A_z(r_1, \frac{3}{2}r_2)$. To get a contradiction to (3.25), we prove that there exists $\bar{\tau}_0 = \bar{\tau}_0(\tau_0, \gamma_0)$ such that

$$\sup_{y \in A_z\left(r_1, \frac{3}{2}r_2\right)} \int_{B_{\bar{\tau}_0 d(y,z)}(y)} e^{u_{2,n}}\, dV_g \le (1 - \bar{\tau}_0) \int_{A_z(r_1, 2r_2)} e^{u_{2,n}}\, dV_g. \tag{3.26}$$

Let $\bar{\tau}_0 = \tau_0/2$. If $B_{\bar{\tau}_0 d(y,z)}(y) \subseteq A_z(r_1(1-\tau_0), r_2(1+\tau_0))$ we can use directly the second part of the assumption (3.20) on $u_{2,n}$ to get the bound on the left-hand side of (3.26) (taking $\bar{\tau}_0$ sufficiently small). Moreover, by the first part of (3.20) on $u_{2,n}$ we deduce

$$\int_{A_z(r_1, r_2)} e^{u_{2,n}}\, dV_g \ge \gamma_0 \int_\Sigma e^{u_{2,n}}\, dV_g \ge \gamma_0 \int_{A_z(r_1, 2r_2)} e^{u_{2,n}}\, dV_g.$$

Given then $B_r(y) \subseteq A_z(r_2, 2r_2)$, since $B_r(y) \cap A_z(r_1, r_2) = \emptyset$, by the first inequality in (3.20) it follows that

$$\int_{B_r(y)} e^{u_{2,n}}\, dV_g \le (1 - \gamma_0) \int_{A_z(r_1, 2r_2)} e^{u_{2,n}}\, dV_g \qquad \text{for any } B_r(y) \subseteq A_z(r_2, 2r_2). \tag{3.27}$$

Now, if $B_{\bar{\tau}_0 d(y,z)}(y) \subseteq A_z(r_2, 2r_2)$ we exploit (3.27) to deduce the bound on the left-hand side of (3.26) taking a possibly smaller $\bar{\tau}_0$. This concludes the proof of the claim (3.26).

Case 1.2. Suppose $\bar{x} \in A_z(\frac{5}{4}r_2, 2r_2)$. Using again (3.27) we obtain a contradiction to (3.25).

Case 2. Consider now $\bar{x} \in A_z(r_1/2, r_2)$: reasoning exactly as in Case 1 we get a contradiction.

Case 3. We are left with the case $\bar{x} \in (A_z(r_1/2, 2r_2))^c$: notice that differently from the previous two cases, the cut-off functions χ_n might not be identically equal to 1 near $\bar{x}_0$. For this choice of $\bar{x}$ and by (3.24) one gets

$$\frac{\displaystyle\int_{A_z(r_1, r_2)} e^{\widetilde{u}_{2,n}}\, dV_g}{\displaystyle\int_{A_z(\widetilde{r}_{1,n}, \widetilde{r}_{2,n})} e^{\widetilde{u}_{2,n}}\, dV_g} \to 0. \tag{3.28}$$

Using the definition of $\widetilde{u}_{2,n}$ in $A_z(\widetilde{r}_{2,n}/2, \widetilde{r}_{2,n})$ given by (3.22) and applying Young's inequality with $1/p = \chi_n$ and $1/q = 1 - \chi_n$ we have

$$e^{\widetilde{u}_{2,n}} = e^{\chi_n u_{2,n}} e^{(1-\chi_n)\overline{u}_2(\widetilde{r}_{2,n})} \le \chi_n e^{u_{2,n}} + (1-\chi_n) e^{\overline{u}_{2,n}(\widetilde{r}_{2,n})} \qquad \text{in } A_z(\widetilde{r}_{2,n}/2, \widetilde{r}_{2,n}). \tag{3.29}$$

Recall the notation in (3.21): by Jensen's inequality it follows that

$$e^{\overline{u}_{2,n}(\widetilde{r}_{2,n})} \leq \fint_{A_z(\widetilde{r}_{2,n}/2,\, \widetilde{r}_{2,n})} e^{u_{2,n}}\, dV_g.$$

Therefore, integrating (3.29) one can show that

$$\int_{A_z(\widetilde{r}_{2,n}/2, \widetilde{r}_{2,n})} e^{\widetilde{u}_{2,n}}\, dV_g \leq 2 \int_{A_z(\widetilde{r}_{2,n}/2, \widetilde{r}_{2,n})} e^{u_{2,n}}\, dV_g.$$

Similarly we get

$$\int_{A_z(\widetilde{r}_{1,n}, 2\widetilde{r}_{1,n})} e^{\widetilde{u}_{2,n}}\, dV_g \leq 2 \int_{A_z(\widetilde{r}_{1,n}, 2\widetilde{r}_{1,n})} e^{u_{2,n}}\, dV_g.$$

In conclusion we have

$$\int_{A_z(\widetilde{r}_{1,n}, \widetilde{r}_{2,n})} e^{\widetilde{u}_{2,n}}\, dV_g \leq 2 \int_{\Sigma} e^{u_{2,n}}\, dV_g.$$

This, together with (3.28), implies that

$$\frac{\displaystyle\int_{A_z(r_1,r_2)} e^{u_{2,n}}\, dV_g}{\displaystyle\int_{\Sigma} e^{u_{2,n}}\, dV_g} \leq 2 \frac{\displaystyle\int_{A_z(r_1,r_2)} e^{\widetilde{u}_{2,n}}\, dV_g}{\displaystyle\int_{A_z(\widetilde{r}_{1,n}, \widetilde{r}_{2,n})} e^{\widetilde{u}_{2,n}}\, dV_g} \to 0,$$

which is in contradiction with (3.20). Therefore we are done. ■

Lemma 3.2.2 *Under the same assumptions of Lemma 3.2.1, let $\widetilde{u}_2 \in H^1(\Sigma)$ be the function given there. Then, property* c) *can be extended to the following one: there exists $\bar{\tau}_0 > 0$ such that*

$$\sup_{y \in B_{\widetilde{r}_2}(z),\, y \neq z} \frac{\displaystyle\int_{B_{\bar{\tau}_0 d(y,z)}(y)} e^{\widetilde{u}_2}\, dV_g}{\displaystyle\int_{B_{\widetilde{r}_2}(z)} e^{\widetilde{u}_2}\, dV_g} < 1 - \bar{\tau}_0. \tag{3.30}$$

Proof. By property c) of Lemma 3.2.1 we just have to show (3.30) for $y \in B_{\widetilde{r}_1}(z)$. Observe that, by definition, $\widetilde{u}_2$ is constant in $B_{\widetilde{r}_1}(z)$. Therefore, for any $B_{\widetilde{\tau}_0 d(y,z)}(y) \subseteq B_{\widetilde{r}_1}(z)$, which implies $d(y,z) \leq \widetilde{r}_1$, we have

$$\int_{B_{\widetilde{\tau}_0 d(y,z)}(y)} e^{\widetilde{u}_2}\, dV_g = \frac{\widetilde{\tau}_0^2 d(y,z)^2}{\widetilde{r}_1^2} \int_{B_{\widetilde{r}_1}(z)} e^{\widetilde{u}_2}\, dV_g \leq \widetilde{\tau}_0^2 \int_{B_{\widetilde{r}_1}(z)} e^{\widetilde{u}_2}\, dV_g \leq \widetilde{\tau}_0^2 \int_{B_{\widetilde{r}_2}(z)} e^{\widetilde{u}_2}\, dV_g,$$

and we conclude that (3.30) holds true for $\widetilde{\tau}_0$ small enough. For the same choice of $\widetilde{\tau}_0$ we are left with the case $B := B_{\widetilde{\tau}_0 d(y,z)}(y) \cap (B_{\widetilde{r}_1}(z))^c \neq \emptyset$. The integral over B will be bounded by the integral over a larger ball with center shifted onto $\partial B_{\widetilde{r}_1}(z)$. Using normal coordinates at z consider the shift of center $y \mapsto \widetilde{r}_1 \frac{y}{d(y,z)}$. Then we have, using the property c);

$$\int_B e^{\widetilde{u}_2}\, dV_g \leq \int_{B_{\widetilde{\tau}_0 \widetilde{r}_1}\left(\widetilde{r}_1 \frac{y}{d(y,z)}\right)} e^{\widetilde{u}_2}\, dV_g \leq (1 - \widetilde{\tau}_0) \int_{B_{\widetilde{r}_2}(z)} e^{\widetilde{u}_2}\, dV_g.$$

Therefore, we get

$$\int_{B_{\widetilde{\tau}_0 d(y,z)(y)}} e^{\widetilde{u}_2}\, dV_g \leq \widetilde{\tau}_0^2 \int_{B_{\widetilde{r}_2}(z)} e^{\widetilde{u}_2}\, dV_g + \int_B e^{\widetilde{u}_2}\, dV_g \leq \widetilde{\tau}_0^2 \int_{B_{\widetilde{r}_2}(z)} e^{\widetilde{u}_2}\, dV_g + (1 - \widetilde{\tau}_0) \int_{B_{\widetilde{r}_2}(z)} e^{\widetilde{u}_2}\, dV_g.$$

Taking $\bar{\tau}_0$ possibly smaller we obtain the conclusion. ■

We recall here the improved geometric inequality stated in Proposition 4.1 of [4], with $k = 1$ and $\alpha = 1$.

Proposition 3.2.3 *([4]) Let $p \in \Sigma$ and let $r > 0$, $\tau_0 > 0$. Then, for any $\varepsilon > 0$ there exists $C = C(\varepsilon, r)$ such that*

$$\log \int_{B_r(p)} d(x,p)^2 e^{2v}\, dV_g \leq \frac{1+\varepsilon}{8\pi} \int_{B_r(p)} |\nabla v|^2\, dV_g + C,$$

for every function $v \in H_0^1(B_r(p))$ such that

$$\sup_{y \in B_r(p);\, y \neq p} \frac{\displaystyle\int_{B_{\tau_0 d(y,p)}(y)} d(x,p)^2 e^{2v}\, dV_g}{\displaystyle\int_{B_r(p)} d(x,p)^2 e^{2v}\, dV_g} < 1 - \tau_0.$$

We state now the new improved Moser-Trudinger inequality.

Remark 3.2.4 *In what follows, the number r is supposed to be small but not tending to zero, while σ could be arbitrarily small.*

Proposition 3.2.5 *Let $r > 0, \gamma_0 > 0$ and $\tau_0 > 0$. For any $\varepsilon > 0$ there exists $C = C(\varepsilon, r, \tau_0, \gamma_0)$ such that, if for some $\sigma \in \left(0, \frac{r}{C^2}\right)$ and $z \in \Sigma$ it holds*

$$\frac{\displaystyle\int_{B_{\sigma/2}(z)} e^{u_1}\, dV_g}{\displaystyle\int_{\Sigma} e^{u_1}\, dV_g} > \gamma_0, \qquad \frac{\displaystyle\int_{A_z(C\sigma, \frac{r}{C})} e^{u_2}\, dV_g}{\displaystyle\int_{\Sigma} e^{u_2}\, dV_g} > \gamma_0 \tag{3.31}$$

and

$$\sup_{y \in A_z(C\sigma, \frac{r}{C})} \frac{\displaystyle\int_{B_{\tau_0 d(y,z)}(y)} e^{u_2}\, dV_g}{\displaystyle\int_{A_z(C\sigma, \frac{r}{C})} e^{u_2}\, dV_g} < 1 - \tau_0, \tag{3.32}$$

then

$$4\pi \log \int_{\Sigma} e^{u_1 - \overline{u}_1}\, dV_g + 8\pi \log \int_{\Sigma} e^{u_2 - \overline{u}_2}\, dV_g \leq \int_{B_r(z)} Q(u_1, u_2)\, dV_g + \varepsilon \int_{\Sigma} Q(u_1, u_2)\, dV_g + C.$$

Proof. Taking r sufficiently small we may suppose that we have the Euclidean flat metric in the ball $B_{Cr}(z)$. Suppose for simplicity that $\overline{u}_1 = \overline{u}_2 = 0$ and that $z = 0$. Observe that we can write

$$\log \int_{B_r(0)} e^{u_2}\, dV_g = \log \int_{B_r(0)} |x|^2 e^{2\left(\frac{u_2}{2} - \log|x|\right)}\, dV_g.$$

We wish to apply Proposition 3.2.3 to $\frac{u_2}{2} - \log|x|$, so we need to modify this function in such a way that it becomes constant outside a given ball. Moreover, it will be useful to also replace it with a constant inside a smaller ball. In this process we should not lose the volume-spreading property (3.32). By Lemma 3.2.1 this can be done and we let $C = C(\varepsilon, \tau_0, \gamma_0)$, $\widetilde{r}_1 \in \left[\sigma, \frac{C\sigma}{4}\right]$, $\widetilde{r}_2 \in \left[\frac{4r}{C}, r\right]$ and $\widetilde{u}_2 \in H^1(\Sigma)$ be as in the statement of the lemma. By property a) in Lemma 3.2.1 and by Lemma 3.2.2 we are in position to apply Proposition 3.2.3 to $(\widetilde{u}_2 - \widetilde{u}_2(\widetilde{r}_2)) \in H_0^1(B_{\widetilde{r}_2}(0))$ and get

$$\begin{aligned}
\log \int_{\Sigma} e^{u_2}\, dV_g &\leq \log \int_{A_0(C\sigma, \frac{r}{C})} e^{u_2}\, dV_g + C \;=\; \log \int_{A_0(C\sigma, \frac{r}{C})} |x|^2 e^{2\left(\frac{u_2}{2} - \log|x|\right)}\, dV_g + C \\
&\leq \log \int_{B_{\widetilde{r}_2}(0)} |x|^2 e^{2\widetilde{u}_2}\, dV_g + C \;=\; \log \int_{B_{\widetilde{r}_2}(0)} |x|^2 e^{2(\widetilde{u}_2 - \widetilde{u}_2(\widetilde{r}_2))}\, dV_g + \widetilde{u}_2(\widetilde{r}_2) + C \\
&\leq \frac{1+\varepsilon}{8\pi} \int_{A_0(\widetilde{r}_1, \widetilde{r}_2)} |\nabla \widetilde{u}_2|^2\, dV_g + \widetilde{u}_2(\widetilde{r}_2) + C \\
&\leq \frac{1+\varepsilon}{8\pi} \int_{A_0(\widetilde{r}_1, \widetilde{r}_2)} \left|\nabla\left(\frac{u_2}{2} - \log|x|\right)\right|^2 dV_g + \varepsilon \int_{\Sigma} |\nabla u_2|^2\, dV_g + \widetilde{u}_2(\widetilde{r}_2) + C \\
&\leq \frac{1}{8\pi} \int_{A_0(\sigma, r)} \left|\nabla\left(\frac{u_2}{2} - \log|x|\right)\right|^2 dV_g + \varepsilon \int_{\Sigma} Q(u_1, u_2)\, dV_g + \widetilde{u}_2(\widetilde{r}_2) + C, \qquad (3.33)
\end{aligned}$$

where in the first row we exploited (3.31), while in the last one we used the definition of $\widetilde{r}_1, \widetilde{r}_2$. Observe that by the definition (3.22) of $\widetilde{u}_2$ we have

$$\widetilde{u}_2(\widetilde{r}_2) = \fint_{A_z(\widetilde{r}_2/2,\widetilde{r}_2)} \left(\frac{u_2}{2} - \log|x|\right) dV_g.$$

Applying Hölder's and Poincaré's inequalities one gets

$$\begin{aligned} \fint_{A_z(\widetilde{r}_2/2,\widetilde{r}_2)} \left(\frac{u_2}{2} - \log|x|\right) dV_g &\le \fint_{A_z(\widetilde{r}_2/2,\widetilde{r}_2)} |u_2|\, dV_g + \widetilde{C}_r \le C_r \|u_2\|_{L^2(\Sigma)} + \widetilde{C}_r \\ &\le C_r \left(\int_\Sigma |\nabla u_2|^2\, dV_g\right)^{1/2} + \widetilde{C}_r \le \varepsilon \int_\Sigma |\nabla u_2|^2\, dV_g + \frac{\widetilde{C}_r C_r}{\varepsilon}. \end{aligned} \tag{3.34}$$

Inserting the latter estimate into (3.33) we deduce

$$\log \int_\Sigma e^{u_2}\, dV_g \le \frac{1}{8\pi} \int_{A_0(\sigma,r)} \left|\nabla\left(\frac{u_2}{2} - \log|x|\right)\right|^2 dV_g + \varepsilon \int_\Sigma Q(u_1,u_2)\, dV_g + C. \tag{3.35}$$

Using the integration by parts we get

$$\int_{A_0(\sigma,r)} \left|\nabla\left(\frac{u_2}{2} - \log|x|\right)\right|^2 dV_g = \frac{1}{4} \int_{A_0(\sigma,r)} |\nabla u_2|^2\, dV_g - 2\pi \log\sigma + 2\pi \fint_{\partial B_\sigma(0)} u_2\, dS_g - 2\pi \fint_{\partial B_r(0)} u_2\, dS_g.$$

Observe now that by the L^1 embedding of H^1 and the trace inequalities, there exists $C > 0$ such that

$$\left| \fint_{B_\sigma(0)} u_2\, dV_g - \fint_{\partial B_\sigma(0)} u_2\, dS_g \right| \le C \left(\int_{B_\sigma(0)} |\nabla u_2|^2\, dV_g \right)^{1/2},$$

where C is independent of σ since the latter inequality is dilation invariant. Therefore, reasoning as in (3.34) we obtain

$$\int_{A_0(\sigma,r)} \left|\nabla\left(\frac{u_2}{2} - \log|x|\right)\right|^2 dV_g \le \frac{1}{4} \int_{A_0(\sigma,r)} |\nabla u_2|^2\, dV_g - 2\pi\log\sigma + 2\pi \overline{u}_2(\sigma) + \varepsilon \int_\Sigma |\nabla u_2|^2\, dV_g + C,$$

where $\overline{u}_2(\sigma) = \fint_{B_\sigma(0)} u_2\, dV_g$. Finally, by the fact that

$$\frac{1}{4}|\nabla u_2|^2 = Q(u_1,u_2) - \frac{1}{12}|\nabla(u_2 + 2u_1)|^2,$$

we get

$$\begin{aligned} \int_{A_0(\sigma,r)} \left|\nabla\left(\frac{u_2}{2} - \log|x|\right)\right|^2 dV_g \ &\le \int_{A_0(\sigma,r)} Q(u_1,u_2)\, dV_g - \frac{1}{12} \int_{A_0(\sigma,r)} |\nabla(u_2+2u_1)|^2\, dV_g + \\ &- \ 2\pi\log\sigma + 2\pi\overline{u}_2(\sigma) + \varepsilon \int_\Sigma |\nabla u_2|^2\, dV_g + C. \end{aligned} \tag{3.36}$$

We claim now that for any $\tilde{\varepsilon} > 0$ one has

$$\int_{A_0(\sigma,r)} |\nabla(u_2+2u_1)|^2\, dV_g \ge 2\pi \left(\frac{2}{\tilde{\varepsilon}} (\overline{u}_2(\sigma) + 2\overline{u}_1(\sigma)) + \frac{1}{\tilde{\varepsilon}^2} \log\sigma \right) - \varepsilon \int_\Sigma Q(u_1,u_2)\, dV_g - C. \tag{3.37}$$

Letting $v(x) = u_2(x) + 2u_1(x)$ we have to prove

$$\int_{A_0(\sigma,r)} |\nabla v|^2\, dV_g \ge 2\pi \left(\frac{2}{\tilde{\varepsilon}} \overline{v}(\sigma) + \frac{1}{\tilde{\varepsilon}^2} \log\sigma \right),$$

where $\overline{v}(\sigma) = \overline{u}_2(\sigma) + 2\overline{u}_1(\sigma)$. Choose $k \in \mathbb{N}$ such that

$$\int_{A_0(2^k\sigma, 2^{k+1}\sigma)} |\nabla v|^2\, dV_g \le \varepsilon \int_\Sigma |\nabla v|^2\, dV_g,$$

and define

$$\begin{cases} \tilde{u}(x) = \overline{v}(\sigma) & \text{if } x \in B_{2^k\sigma}(0), \\ \Delta\tilde{u}(x) = 0 & \text{if } x \in A_0(2^k\sigma, 2^{k+1}\sigma), \\ \tilde{u}(x) = v(x) & \text{if } x \notin B_{2^{k+1}\sigma}(0). \end{cases}$$

Then there exists a universal constant C_0 such that

$$\begin{aligned} \int_{A_0(2^k\sigma,r)} |\nabla\tilde{u}|^2\, dV_g &\leq \int_{A_0(\sigma,r)} |\nabla v|^2\, dV_g + C_0\varepsilon \int_\Sigma |\nabla v|^2\, dV_g \\ &\leq \int_{A_0(\sigma,r)} |\nabla v|^2\, dV_g + C_0\varepsilon \int_\Sigma Q(u_1,u_2)\, dV_g. \end{aligned}$$

Solving the Dirichlet problem in $A_0(2^k\sigma, r)$ with constant data $\overline{v}(\sigma)$ on $\partial B_{2^k\sigma}(0)$ one gets

$$\begin{cases} w(x) = A\log\sigma & \text{if } |x| > 2^k\sigma, \\ w(2^k\sigma) = A\log(2^k\sigma) = \overline{v}(\sigma) & \text{if } |x| = 2^k\sigma, \end{cases}$$

for some constant A. We have that

$$\int_{A_0(2^k\sigma,r)} |\nabla w|^2\, dV_g = 2\pi A^2 \log\frac{1}{2^k\sigma} - C = 2\pi \frac{\overline{v}(\sigma)^2}{\log\frac{1}{2^k\sigma}} - C.$$

Moreover

$$\int_{A_0(2^k\sigma,r)} |\nabla w|^2\, dV_g \leq \int_{A_0(2^k\sigma,r)} |\nabla\tilde{u}|^2\, dV_g.$$

Finally, using Young's inequality

$$\overline{v}(\sigma)\log\frac{1}{\sigma} \leq \frac{1}{2}\left(\tilde{\varepsilon}\overline{v}(\sigma)^2 + \frac{1}{\tilde{\varepsilon}}\left(\log\frac{1}{\sigma}\right)^2\right),$$

we end up with

$$\frac{\overline{v}(\sigma)^2}{\log\frac{1}{\sigma}} \geq \left(\frac{2}{\tilde{\varepsilon}}\overline{v}(\sigma) + \frac{1}{\tilde{\varepsilon}^2}\log\sigma\right).$$

Therefore we conclude

$$\begin{aligned} 2\pi\left(\frac{2}{\tilde{\varepsilon}}\overline{v}(\sigma) + \frac{1}{\tilde{\varepsilon}^2}\log\sigma\right) - C &\leq 2\pi\frac{\overline{v}(\sigma)^2}{\log\frac{1}{\sigma}} - C = \int_{A_0(2^k\sigma,r)} |\nabla w|^2\, dV_g \\ &\leq \int_{A_0(2^k\sigma,r)} |\nabla\tilde{u}|^2\, dV_g \leq \int_{A_0(\sigma,r)} |\nabla v|^2\, dV_g + C_0\varepsilon\int_\Sigma Q(u_1,u_2)\, dV_g, \end{aligned}$$

which proves the claim (3.37).

Inserting (3.37) into (3.36) we have

$$\begin{aligned} \int_{A_0(\sigma,r)} \left|\nabla\left(\frac{u_2}{2} - \log|x|\right)\right|^2 dV_g &\leq \int_{A_0(\sigma,r)} Q(u_1,u_2)\, dV_g - \frac{1}{12}2\pi\left(\frac{2}{\tilde{\varepsilon}}(\overline{u}_2(\sigma) + 2\overline{u}_1(\sigma)) + \frac{1}{\tilde{\varepsilon}^2}\log\sigma\right) + \\ &- 2\pi\log\sigma + 2\pi\overline{u}_2(\sigma) + \varepsilon\int_\Sigma Q(u_1,u_2)\, dV_g + C. \end{aligned}$$

Choosing $\tilde{\varepsilon} = 1/6$ we obtain

$$\begin{aligned} \int_{A_0(\sigma,r)} \left|\nabla\left(\frac{u_2}{2} - \log|x|\right)\right|^2 dV_g &\leq \int_{A_0(\sigma,r)} Q(u_1,u_2)\, dV_g - 4\pi\overline{u}_1(\sigma) - 8\pi\log\sigma + \\ &+ \varepsilon\int_\Sigma Q(u_1,u_2)\, dV_g + C. \end{aligned} \tag{3.38}$$

We use then (3.38) in (3.35) to get

$$8\pi\log\int_\Sigma e^{u_2}\, dV_g \leq \int_{A_0(\sigma,r)} Q(u_1,u_2)\, dV_g - 4\pi\overline{u}_1(\sigma) - 8\pi\log\sigma + \varepsilon\int_\Sigma Q(u_1,u_2)\, dV_g + C. \tag{3.39}$$

For the first component we consider the scalar local Moser-Trudinger inequality, see for example Proposition 2.3 of [71], namely

$$\begin{aligned} \log \int_{B_{r/2}(0)} e^{u_1}\, dV_g &\leq \frac{1}{16\pi} \int_{B_r(0)} |\nabla u_1|^2\, dV_g + \bar{u}_1(r) + \varepsilon \int_\Sigma |\nabla u_1|^2\, dV_g + C \\ &\leq \frac{1}{4\pi} \int_{B_r(0)} Q(u_1,u_2)\, dV_g + \bar{u}_1(r) + \varepsilon \int_\Sigma Q(u_1,u_2)\, dV_g + C. \end{aligned}$$

Performing a dilation to $B_\sigma(0)$ one gets

$$4\pi \log \int_{B_{\sigma/2}(0)} e^{u_1}\, dV_g \leq \int_{B_\sigma(0)} Q(u_1,u_2)\, dV_g + 4\pi \overline{u}_1(\sigma) + 8\pi \log \sigma + \varepsilon \int_\Sigma Q(u_1,u_2)\, dV_g + C.$$

We then use the assumption (3.31) and we obtain

$$4\pi \log \int_\Sigma e^{u_1}\, dV_g \leq \int_{B_\sigma(0)} Q(u_1,u_2)\, dV_g + 4\pi \overline{u}_1(\sigma) + 8\pi \log \sigma + \varepsilon \int_\Sigma Q(u_1,u_2)\, dV_g + C. \tag{3.40}$$

Summing equations (3.39) and (3.40) we deduce

$$4\pi \log \int_\Sigma e^{u_1}\, dV_g + 8\pi \log \int_\Sigma e^{u_2}\, dV_g \leq \int_{B_r(z)} Q(u_1,u_2)\, dV_g + \varepsilon \int_\Sigma Q(u_1,u_2)\, dV_g + C,$$

which concludes the proof. ■

Remark 3.2.6 *The above result is inspired by the work [4] (see in particular Proposition 4.1 there) where the singular Liouville equation is considered. The authors derive a geometric inequality by means of the angular distribution of the conformal volume near the singularities. Somehow the singular equation can be seen as the limit case of the regular one. Roughly speaking, when one component is much more concentrated with respect to the other one, its effect resembles that of a Dirac delta.*

3.2.3 Lower bounds on the functional J_ρ.

We are going to exploit the improved inequality stated in Proposition 3.2.5 to derive new lower bounds of the energy functional J_ρ defined in (1.7), see Proposition 3.2.7. This will give us some extra constraints for the map from the low sub-levels of J_ρ onto the topological join $\Sigma_k * \Sigma_1$, see (1.17).

Given a small $\delta > 0$, our aim is to describe the low sub-levels of the functional J_ρ by means of the set

$$Y := (\Sigma_k * \Sigma_1) \setminus S \subseteq \Sigma_k * \Sigma_1, \tag{3.41}$$

where

$$S = \left\{ \left(\nu, \delta_z, \frac{1}{2} \right) \in \Sigma_k * \Sigma_1 : \nu = \sum_{i=1}^k t_i \delta_{x_i} \ ; \ d(x_i,x_j) \geq \delta \ \forall i \neq j, \ \delta \leq t_i \leq 1-\delta \ \forall i \ ; \ z \in supp\,(\nu) \right\}. \tag{3.42}$$

We will show that there is a lower bound for J_ρ whenever $\widetilde{\Psi}$, which is defined in (3.19), has image inside S, see Proposition 3.2.7.

Consider $\mathcal{C}_{\varepsilon,r}(x_0)$ as given in (3.7), $f \in \mathcal{C}_{\varepsilon,r}(x_0)$ and ψ defined in (3.9). Before stating the next main result we recall some properties of the map ψ, see Proposition 3.1 in [70] (with minor adaptations).

Fact. Let $\psi(f) = (\beta, \sigma)$. Then, given $R > 1$ there exists $p \in \Sigma$ with the following properties:

$$\begin{gathered} d(p,\beta) \leq C'\sigma \text{ for some } C' = C'(R); \\ \int_{B_\sigma(p) \cap B_r(x_0)} f\, dV_g > \tau, \qquad \int_{(B_{R\sigma}(p))^c \cap B_r(x_0)} f\, dV_g > \tau, \end{gathered} \tag{3.43}$$

where τ depends only on R and Σ.

Recall also the distance $\mathbf{d}$ between measures in (1.25), the numbers $\varepsilon_i > 0$ in Proposition 2.4.1, the projections $\widetilde{\psi}_k$, $\widetilde{\psi}_1$ in (3.17), (3.18) and the definition of the parameter s in the topological join given by (3.16).

Proposition 3.2.7 *Suppose that $\rho_1 \in (4k\pi, 4(k+1)\pi)$, $\rho_2 \in (4\pi, 8\pi)$ and that $\mathbf{d}\left(\frac{e^{u_1}}{\int_\Sigma e^{u_1}\, dV_g}, \Sigma_k\right) < 2\varepsilon_k$, $\mathbf{d}\left(\frac{e^{u_2}}{\int_\Sigma e^{u_2}\, dV_g}, \Sigma_1\right) < \varepsilon_1$. Let*

$$\widetilde{\psi}_k\left(\frac{e^{u_1}}{\int_\Sigma e^{u_1}\, dV_g}\right) = \sum_{i=1}^k t_i \delta_{x_i}, \qquad \widetilde{\psi}_1\left(\frac{e^{u_2}}{\int_\Sigma e^{u_2}\, dV_g}\right) = \delta_{\beta_z}.$$

There exist $\delta > 0$ and $L > 0$ such that, if the following properties hold true:

1) $d(x_i, x_j) \geq \delta$ $\forall i \neq j$ and $t_i \in [\delta, 1-\delta]$ $\forall i = 1, \dots, k$;

2) $s(u_1, u_2) = 1/2$;

3) $\beta_z = x_l$ for some $l \in \{1, \dots, k\}$;

then

$$J_\rho(u_1, u_2) \geq -L.$$

Proof. Suppose w.l.o.g. that $\overline{u}_1 = \overline{u}_2 = 0$. We first observe that exploiting the assumption $s(u_1, u_2) = 1/2$ we deduce $\sigma_1(u_1) = \sigma_2(u_2)$. Secondly, it is not difficult to show that from property 1) it follows $\mathbf{d}\left(\frac{e^{u_1}}{\int_\Sigma e^{u_1}\, dV_g}, \Sigma_{k-1}\right) \geq 2\varepsilon_{k-1}$. Therefore, by the definition of $\widetilde{\psi}_k$ we deduce that $x_i = \beta_{x_i^k}$ for $i = 1, \dots, k$, where the $\beta_{x_i^k}$ are the local centres of mass given by (3.13). Hence we get

$$\widetilde{\psi}_k\left(\frac{e^{u_1}}{\int_\Sigma e^{u_1}\, dV_g}\right) = \sum_{i=1}^k t_i \delta_{\beta_{x_i^k}}.$$

Recalling that we have set (see Subsection 3.2.1)

$$\sigma_2(u_2) = \mathfrak{g}_1\left(\mathbf{d}\left(\frac{e^{u_2}}{\int_\Sigma e^{u_2}\, dV_g}, \Sigma_1\right)\right)\sigma_z + \left(1 - \mathfrak{g}_1\left(\mathbf{d}\left(\frac{e^{u_2}}{\int_\Sigma e^{u_2}\, dV_g}, \Sigma_1\right)\right)\right),$$

using the fact that $\mathbf{d}\left(\frac{e^{u_2}}{\int_\Sigma e^{u_2}\, dV_g}, \Sigma_1\right) < \varepsilon_1$, by the definition of $\mathfrak{g}_1$ in (3.14), $\sigma_2(u_2)$ reduces to σ_z. We recall now also the definition of $\sigma_1(u_1)$, namely

$$\sigma_1(u_1) = \mathfrak{f}\left(\mathbf{d}\left(\frac{e^{u_1}}{\int_\Sigma e^{u_1}\, dV_g}, \Sigma_k\right)\right) A(u_1, u_2) + \left(1 - \mathfrak{f}\left(\mathbf{d}\left(\frac{e^{u_1}}{\int_\Sigma e^{u_1}\, dV_g}, \Sigma_k\right)\right)\right),$$

where $A(u_1, u_2)$ is defined in Subsection 3.2.1. The assumption $\mathbf{d}\left(\frac{e^{u_1}}{\int_\Sigma e^{u_1}\, dV_g}, \Sigma_k\right) < 2\varepsilon_k$ implies that $\mathfrak{f}\left(\mathbf{d}\left(\frac{e^{u_1}}{\int_\Sigma e^{u_1}\, dV_g}, \Sigma_k\right)\right) > 0$. As before, using property 1) we obtain from $\mathbf{d}\left(\frac{e^{u_1}}{\int_\Sigma e^{u_1}\, dV_g}, \Sigma_{k-1}\right) \geq 2\varepsilon_{k-1}$ that $\mathfrak{g}_{k-1}\left(\mathbf{d}\left(\frac{e^{u_1}}{\int_\Sigma e^{u_1}\, dV_g}, \Sigma_{k-1}\right)\right) = 0$ and hence $A(u_1, u_2) = B(u_1, u_2)$ (see the notation before (3.16)). Moreover, the condition 3) implies that $\mathfrak{h}\big(d(\beta_z, \{\beta_{x_1^k}, \dots, \beta_{x_k^k}\})\big) = 1$. Therefore $B(u_1, u_2) = \sigma_{x_l^k}$. Hence one finds

$$\sigma_{u_1} = \mathfrak{f}\left(\mathbf{d}\left(\frac{e^{u_1}}{\int_\Sigma e^{u_1}\, dV_g}, \Sigma_k\right)\right)\sigma_{x_l^k} + \left(1 - \mathfrak{f}\left(\mathbf{d}\left(\frac{e^{u_1}}{\int_\Sigma e^{u_1}\, dV_g}, \Sigma_k\right)\right)\right).$$

We distinguish between two cases.

Case 1. Suppose first that $\mathfrak{f}\left(\mathbf{d}\left(\frac{e^{u_1}}{\int_\Sigma e^{u_1}\, dV_g}, \Sigma_k\right)\right) = 1$. In this case we obtain $\sigma_{x_l^k} = \sigma_1(u_1) = \sigma_2(u_2) = \sigma_z$. By this fact and by property 3) we get $(\beta_{x_l^k}, \sigma_{x_l^k}) = (\beta_z, \sigma_z)$. Let $r = \delta/4$: from (3.43) and the definition of β_z, $\beta_{x_i^k}$, there exists $\widetilde{\gamma}_0 > 0$ such that

$$\int_{B_r\left(\beta_{x_i^k}\right)} e^{u_1}\, dV_g \geq \widetilde{\gamma}_0 \int_\Sigma e^{u_1}\, dV_g \quad \text{for} \quad i = 1, \dots, k; \qquad \int_{B_r(\beta_z)} e^{u_2}\, dV_g \geq \widetilde{\gamma}_0 \int_\Sigma e^{u_2}\, dV_g. \tag{3.44}$$

Therefore, we are in position to apply Proposition 3.1.9 and get

$$4(k+1)\pi \log \int_\Sigma e^{u_1}\, dV_g + 8\pi \log \int_\Sigma e^{u_2}\, dV_g \leq (1+\varepsilon) \int_\Sigma Q(u_1, u_2)\, dV_g + C_r.$$

The conclusion then follows from the expression of J_ρ and from the upper bounds on ρ_1, ρ_2.

Case 2. Suppose now that $\mathfrak{f}\left(\mathbf{d}\left(\frac{e^{u_1}}{\int_\Sigma e^{u_1}\,dV_g}, \Sigma_k\right)\right) < 1$: we deduce immediately that $\mathbf{d}\left(\frac{e^{u_1}}{\int_\Sigma e^{u_1}\,dV_g}, \Sigma_k\right) \in (\varepsilon_k, 2\varepsilon_k)$.

Given $\varepsilon > 0$, let $R = R(\varepsilon)$ be such that Proposition 3.1.7 holds true. Let $C' = C'(R)$ and $\tau = \tau(R)$ be as in (3.43). Take $\tau_0 = \tau/100, \gamma_0 = \widetilde{\gamma}_0\tau$, where γ_0 is given as in (3.44), and let $C = C(\varepsilon, r, \tau_0, \gamma_0)$ be the constant obtained in Proposition 3.2.5. We then define $\widetilde{C} = \max\{C', C\}$. Moreover, observe that by construction $\sigma_{x_l^k} \leq \sigma_1(u_1) = \sigma_2(u_2) = \sigma_z$.

If $\sigma_{x_l^k} \leq \sigma_z \leq \widetilde{C}^8\sigma_{x_l^k}$ we still can apply Proposition 3.1.9 as before, see Remark 3.1.8. Consider now the case $\widetilde{C}^8\sigma_{x_l^k} \leq \sigma_z$. We distinguish between two situations.

Case 2.1. If r is as in Case 1, suppose that

$$\int_{B_{\widetilde{C}^4\sigma_{x_l^k}}(\beta_z)} e^{u_2}\,dV_g > \tau_0 \int_{B_r(\beta_z)} e^{u_2}\,dV_g \left(> \widetilde{\gamma}_0\tau_0 \int_\Sigma e^{u_2}\,dV_g : \text{ see (3.44)}\right). \tag{3.45}$$

By the fact that $\widetilde{C}^4\sigma_{x_l^k} \ll \sigma_z$, from (3.43) we also get

$$\int_{\left(B_{R\widetilde{C}^4\sigma_{x_l^k}}(\beta_z)\right)^c \cap B_r(\beta_z)} e^{u_2}\,dV_g > \tau_0 \int_{B_r(\beta_z)} e^{u_2}\,dV_g > \widetilde{\gamma}_0\tau_0 \int_\Sigma e^{u_2}\,dV_g. \tag{3.46}$$

The conditions on the local scale of u_1, given by $(\beta_{x_l^k}, \sigma_{x_l^k}) = \psi\big(f_{loc}^{x_l^k}(u_1)\big)$, yield by (3.43) the existence of $p \in \Sigma$ such that

$$\int_{B_{\sigma_{x_l^k}}(p)} e^{u_1}\,dV_g > \tau \int_{B_r\left(\beta_{x_l^k}\right)} e^{u_1}\,dV_g > \widetilde{\gamma}_0\tau \int_\Sigma e^{u_1}\,dV_g,$$

$$\int_{\left(B_{R\,\sigma_{x_l^k}}(p)\right)^c \cap B_r\left(\beta_{x_l^k}\right)} e^{u_1}\,dV_g > \tau \int_{B_r\left(\beta_{x_l^k}\right)} e^{u_1}\,dV_g > \widetilde{\gamma}_0\tau \int_\Sigma e^{u_1}\,dV_g.$$

The latter formulas, together with (3.45) and (3.46) imply an improved Moser-Trudinger inequality, see Remarks 3.1.6 and 3.1.8:

$$8\pi\left(\log\int_\Sigma e^{u_1}\,dV_g + \log\int_\Sigma e^{u_2}\,dV_g\right) \leq (1+\varepsilon)\int_{B_r(\beta_z)} Q(u_1,u_2)\,dV_g + C_0(\varepsilon, r, \tau, \widetilde{\gamma}_0). \tag{3.47}$$

Case 2.2. Suppose now that the second situation occurs, namely

$$\int_{B_{\widetilde{C}^4\sigma_{x_l^k}}(z)} e^{u_2}\,dV_g \leq \tau_0 \int_{B_r(\beta_z)} e^{u_2}\,dV_g. \tag{3.48}$$

The goal is to apply the improved inequality stated in Proposition 3.2.5. Take $\sigma = (C')^2\sigma_{x_l^k}$ and $A_{\beta_z}(C\sigma, \frac{r}{C})$ as the annulus on which we will test the conditions (3.31) and (3.32). We start by considering (3.31). Observe that

$$\int_{B_{\sigma/2}(z)} e^{u_1}\,dV_g > \gamma_0 \int_\Sigma e^{u_1}\,dV_g$$

follows from (3.43) and (3.44) by the choice of σ and γ_0. Similarly, using the volume concentration of u_2 in $(B_{R\sigma_z}(p))^c \cap B_r(\beta_z)$ in (3.43) and (recalling the definition of $\widetilde{C}$) $C\sigma \ll R\sigma_z$ we get

$$\int_{A_{\beta_z}(C\sigma, \frac{r}{C})} e^{u_2}\,dV_g > \gamma_0 \int_\Sigma e^{u_2}\,dV_g$$

by taking ε_1 sufficiently small in Proposition 3.2.7. We are left by proving condition (3.32), i.e.

$$\sup_{y \in A_{\beta_z}(C\sigma, \frac{r}{C})} \frac{\displaystyle\int_{B_{\tau_0 d(y,z)}(y)} e^{u_2}\,dV_g}{\displaystyle\int_{A_{\beta_z}(C\sigma, \frac{r}{C})} e^{u_2}\,dV_g} < 1 - \tau_0.$$

If this is not the case, then there exists $y \in A_{\beta_z}(C\sigma, \frac{r}{C})$ such that

$$\int_{B_{\tau_0 d(y,z)}(y)} e^{u_2}\, dV_g \geq (1-\tau_0) \int_{A_{\beta_z}(C\sigma,\frac{r}{C})} e^{u_2}\, dV_g.$$

Using the assumption (3.48) and $\sigma < \widetilde{C}^4 \sigma_{x_l^k}$ we get

$$\begin{aligned}\int_{B_{\tau_0 d(y,z)}(y)} e^{u_2}\, dV_g &\geq (1-\tau_0)\int_{A_{\beta_z}(C\sigma,\frac{r}{C})} e^{u_2}\, dV_g \geq (1-\tau_0)\int_{A_{\beta_z}(C\sigma,\frac{r}{C})} e^{u_2}\, dV_g \\ &= (1-\tau_0)\int_{B_r(\beta_z)} e^{u_2}\, dV_g - (1-\tau_0)\int_{B_{C\sigma}(\beta_z)} e^{u_2}\, dV_g \geq (1-2\tau_0)\int_{B_r(\beta_z)} e^{u_2}\, dV_g.\end{aligned}$$

Moreover, by the property of the local scale of u_2 given by $(\beta_z, \sigma_z) = \psi(f_{loc}^z(u_2))$, see (3.43), we have

$$\int_{B_{\sigma_z(p)}} e^{u_2}\, dV_g > \tau \int_{B_r(\beta_z)} e^{u_2}\, dV_g; \qquad \int_{(B_{R\sigma_z}(p))^c \cap B_r(\beta_z)} e^{u_2}\, dV_g > \tau \int_{B_r(\beta_z)} e^{u_2}\, dV_g.$$

Notice that by the choice of τ_0 the three properties above cannot hold simultaneously. Hence, we have a contradiction. Finally, we are in position to apply Proposition 3.2.5 and deduce that

$$4\pi \log \int_\Sigma e^{u_1}\, dV_g + 8\pi \log \int_\Sigma e^{u_2}\, dV_g \leq \int_{B_r(\beta_z)} Q(u_1,u_2)\, dV_g + \varepsilon \int_\Sigma Q(u_1,u_2)\, dV_g + C.$$

Observe that by the latter formula and by (3.47), in both *Case 2.1* and *Case 2.2* we can assert that

$$4\pi \log \int_\Sigma e^{u_1}\, dV_g + 8\pi \log \int_\Sigma e^{u_2}\, dV_g \leq \int_{B_r(\beta_z)} Q(u_1,u_2)\, dV_g + \varepsilon \int_\Sigma Q(u_1,u_2)\, dV_g + C. \tag{3.49}$$

Recall that under Case 2 we have $\mathbf{d}\left(\frac{e^{u_1}}{\int_\Sigma e^{u_1}\, dV_g}, \Sigma_k\right) > \varepsilon_k$. By the second part of Proposition 2.4.1 (applied with $l = k+1$) there exist $\bar{\varepsilon}_k > 0$, depending only on ε_k, and $k+1$ points $\bar{x}_1, \dots, \bar{x}_{k+1}$ such that

$$d(\bar{x}_i, \bar{x}_j) > 2\bar{\varepsilon}_k \quad \text{for } i \neq j; \qquad \int_{B_{\bar{\varepsilon}_k}(\bar{x}_i)} e^{u_1}\, dV_g > \bar{\varepsilon}_k \int_\Sigma e^{u_1}\, dV_g \quad \text{for all } i = 1, \dots, k+1.$$

Without loss of generality we can assume $\delta < \bar{\varepsilon}_k/8$. By this the choice of δ there exist k points $\bar{y}_1, \dots \bar{y}_k$ such that

$$d(\bar{y}_i, \bar{y}_j) > \bar{\varepsilon}_k \quad \text{for } i \neq j; \qquad d(\bar{y}_i, \beta_{x_i^k}) > \delta \quad \text{for all } i = 1, \dots, k;$$
$$\int_{B_{\bar{\varepsilon}_k}(\bar{y}_i)} e^{u_1}\, dV_g > \bar{\varepsilon}_k \int_\Sigma e^{u_1}\, dV_g \quad \text{for all } i = 1, \dots, k.$$

We perform then a local Moser-Trudinger inequality for u_1 in each region, see (3.40), and summing up we have (recall that $r = \delta/4$)

$$4k\pi \log \int_\Sigma e^{u_1}\, dV_g \leq \int_{\left(B_r\left(\beta_{x_l^k}\right)\right)^c} Q(u_1,u_2)\, dV_g + \varepsilon \int_\Sigma Q(u_1,u_2)\, dV_g + C_r, \tag{3.50}$$

where the average was estimated using Hölder's and Poincaré's inequalities as in (3.34). By summing equations (3.49) and (3.50) we deduce

$$4(k+1)\pi \log \int_\Sigma e^{u_1}\, dV_g + 8\pi \log \int_\Sigma e^{u_2}\, dV_g \leq (1+\varepsilon)\int_\Sigma Q(u_1,u_2)\, dV_g + C,$$

so we conclude as in Case 1. ■

By Proposition 3.2.7 we obtain the following corollary.

Corollary 3.2.8 *Let S be as in* (3.42) *and let $Y = (\Sigma_k * \Sigma_1) \setminus S$. Then, for $\widetilde{L} > 0$ large $\widetilde{\Psi}$ (defined in* (3.19)*) maps the low sub-levels $J_\rho^{-\widetilde{L}}$ into the set Y.*

3.3 Test functions

We show that the lower bound in Proposition 3.2.7 is optimal, see also Corollary 3.2.8. In fact, we will construct suitable test functions modelled on Y on which J_ρ attains arbitrarily negative values.

To describe our construction, let us recall the test functions employed for the scalar case (1.9). When $\rho > 4\pi$, as mentioned in the Introduction, the energy I_ρ in (1.14) is unbounded below. One can see that using test functions of the type

$$\varphi_{\lambda,z}(x) = \log\left(\frac{\lambda}{1+\lambda^2 d(x,z)^2}\right)^2, \tag{3.51}$$

for a given point $z \in \Sigma$ and for $\lambda > 0$, as $\lambda \to +\infty$ these satisfy the properties

$$e^{\varphi_{\lambda,z}} \rightharpoonup \delta_z \qquad \text{and} \qquad I_\rho(\varphi_{\lambda,z}) \to -\infty \qquad (\rho > 4\pi), \tag{3.52}$$

holding uniformly in $z \in \Sigma$. More in general, if $\rho \in (4k\pi, 4(k+1)\pi)$, a natural family of test functions can be modelled on Σ_k, see [36, 37]. In fact, setting

$$\varphi_{\lambda,\nu}(x) = \log\sum_{i=1}^{k} t_i\left(\frac{\lambda}{1+\lambda^2 d(x,x_i)^2}\right)^2; \qquad \nu = \sum_{i=1}^{k} t_i\delta_{x_i}, \tag{3.53}$$

similarly to (3.52), for $\lambda \to +\infty$ one has uniformly in $\nu \in \Sigma_k$

$$\mathbf{d}(e^{\varphi_{\lambda,\nu}}, \nu) \to 0 \qquad \text{and} \qquad I_\rho(\varphi_{\lambda,\nu}) \to -\infty \qquad (\rho \in (4k\pi, 4(k+1)\pi)).$$

When dealing with the energy functional J_ρ in (1.7) one can expect to interpolate between the $\varphi_{\lambda,\nu}$ for the component u_1 and the $\varphi_{\lambda,z}$ for u_2 when $\rho_1 \in (4k\pi, 4(k+1)\pi)$, $\rho_2 \in (4\pi, 8\pi)$. Therefore, the topological join $\Sigma_k * \Sigma_1$ represents a natural object to parametrize globally this family, with the join parameter s playing the role of interpolation parameter. However, as mentioned in the Subsection 1.1.2, the cross term in the quadratic energy penalizes gradients pointing in the same direction. By this reason, not all elements in $\Sigma_k * \Sigma_1$ will give rise to test functions with low energy. It will turn out that the subset Y of $\Sigma_k * \Sigma_1$, see (3.41), will be the right one to look at.

3.3.1 A convenient deformation of $Y \cap \left\{s = \frac{1}{2}\right\}$.

We construct here a continuous deformation of $Y \cap \left\{s = \frac{1}{2}\right\}$, which is relatively open in the join $\Sigma_k * \Sigma_1$, onto some closed subset: see Corollary 3.3.6. This will allow us to build test functions depending on a compact space of parameters, which is easier. Before doing this, we recall some facts from Section 3 of [67].

There exists a deformation retract $H_0(\mathfrak{t}, \cdot)$ of a neighborhood (with respect to the metric induced by $\mathbf{d}$ in (1.25)) of Σ_{k-1} in Σ_k onto Σ_{k-1}. To see this, one can take a positive δ_1 small enough and consider a non-increasing continuous function $F_0 : (0, +\infty) \to (0, +\infty)$ such that

$$F_0(t) = \frac{1}{t} \text{ for } t \in (0, \delta_1]; \qquad\qquad F_0(t) = \frac{1}{2\delta_1} \text{ for } t > 2\delta_1. \tag{3.54}$$

We then define $F : \Sigma_k \setminus \Sigma_{k-1} \to \mathbb{R}$ as

$$F\left(\sum_{i=1}^{k} t_i\delta_{x_i}\right) = \underbrace{\sum_{i\neq j} F_0(d(x_i,x_j))}_{F_1((x_i)_i)} + \underbrace{\sum_{i=1}^{k} \frac{1}{t_i(1-t_i)}}_{F_2((t_i)_i)}. \tag{3.55}$$

Notice that F is well defined on $\Sigma_k\setminus\Sigma_{k-1}$, as it is invariant under permutation of the couples $(t_i, x_i)_{i=1,\dots,k}$. Observe also that it tends to $+\infty$ as its argument approaches Σ_{k-1}. Moreover, the gradient of F with respect to the metric of $\Sigma^k \times T_0$ (where T_0 is the simplex containing the k-tuple $T := (t_i)_i$) tends to $+\infty$ in norm as $\sum_{i=1}^{k} t_i\delta_{x_i}$ tends to Σ_{k-1}. It follows that, sending L to $+\infty$, we get a deformation retract of

$F_L := \{F \geq L\} \cup \Sigma_{k-1}$ onto Σ_{k-1} for L sufficiently large. We then obtain H_0 by a reparametrization of the (positive) gradient flow of F.

We introduce now the set $\widetilde{Y}_{\frac{1}{2}} \subseteq Y \cap \left\{s = \frac{1}{2}\right\} \subseteq \Sigma_k * \Sigma_1$ defined as

$$\widetilde{Y}_{\frac{1}{2}} = \left\{\left(\nu, \delta_z, \frac{1}{2}\right) : \nu \in \Sigma_{k-1}\right\} \cup \left\{\left(\nu, \delta_z, \frac{1}{2}\right) : \nu \in \Sigma_k \setminus \Sigma_{k-1},\ z \notin supp\,(\nu)\right\}.$$

The next result holds true.

Lemma 3.3.1 *There exists a continuous deformation $\widetilde{H}(\mathtt{t}, \cdot)$ of the set $Y \cap \left\{s = \frac{1}{2}\right\}$ onto $\widetilde{Y}_{\frac{1}{2}}$.*

Proof. Let $\delta > 0$ be as in (3.42). Consider $0 < \widetilde{\delta} \ll \delta$ and let $\widetilde{f} : (0, +\infty) \to (0, +\infty)$ be a non-increasing continuous function given by

$$\widetilde{f}(t) = \begin{cases} \frac{1}{t^2} & \text{in } t \leq \widetilde{\delta}, \\ 0 & \text{in } t \geq 2\widetilde{\delta}. \end{cases}$$

Moreover, recall the deformation retract $H_0(\mathtt{t}, \cdot)$ of a neighborhood of Σ_{k-1} in Σ_k onto Σ_{k-1} constructed above. To define $\widetilde{H}$ we distinguish among four situations, fixing $\hat{\delta} \ll \widetilde{\delta}$ (in particular we take $\hat{\delta}$ so small that H_0 is well-defined on $3\hat{\delta}$-neighbourhood of Σ_{k-1} in the metric $\mathbf{d}$).

(i) $\mathbf{d}(\nu, \Sigma_{k-1}) \leq \hat{\delta}$. Recall that elements in $Y \cap \left\{s = \frac{1}{2}\right\}$ are triples of the form $\left(\nu, \delta_z, \frac{1}{2}\right)$ with $\nu \in \Sigma_k$. In this first case we project ν onto Σ_{k-1}, while δ_z remains fixed. If H_0 is the retraction described above, we simply define $\widetilde{H}$ to be

$$\widetilde{H}\left(\mathtt{t}, \nu, \delta_z, \frac{1}{2}\right) = \left(H_0(\mathtt{t}, \nu), \delta_z, \frac{1}{2}\right).$$

(ii) $\mathbf{d}(\nu, \Sigma_{k-1}) \in [\hat{\delta}, 2\hat{\delta}]$. Let

$$\nu_1(\mathtt{t}) = H_0(\mathtt{t}, \nu) = \sum_{i=1}^{k} t_i(\mathtt{t}) \delta_{x_i(\mathtt{t})}.$$

If $\widetilde{f}$ is as before, we introduce the following flow acting on the support of δ_z:

$$\frac{d}{d\mathtt{t}} z(\mathtt{t}) = \sum_{i=1}^{k} t_i(\mathtt{t}) f\big(d(z(\mathtt{t}), x_i(\mathtt{t}))\big) \nabla_z d\big(z(\mathtt{t}), x_i(\mathtt{t})\big). \tag{3.56}$$

To define $\widetilde{H}$ in this case we interpolate from a constant motion in z and (3.56) depending on $\mathbf{d}(\nu, \Sigma_{k-1})$:

$$\widetilde{H}\left(\mathtt{t}, \nu, \delta_z, \frac{1}{2}\right) = \left(\nu_1(\mathtt{t}), \delta_{z\left(\mathtt{t}\,\frac{\mathbf{d}(\nu, \Sigma_{k-1}) - \hat{\delta}}{\hat{\delta}}\right)}, \frac{1}{2}\right).$$

Notice that when $\mathbf{d}(\nu, \Sigma_{k-1}) = 2\hat{\delta}$ we get $z\left(\mathtt{t}\,\frac{\mathbf{d}(\nu, \Sigma_{k-1}) - \hat{\delta}}{\hat{\delta}}\right) = z(\mathtt{t})$ and this point never intersects the support of $\nu_1(\mathtt{t})$, unless $\nu_1(\mathtt{t}) \in \Sigma_{k-1}$. Therefore, as for case (i), $\widetilde{H}\left(1, \nu, \delta_z, \frac{1}{2}\right) \in \widetilde{Y}_{\frac{1}{2}}$.

(iii) $\mathbf{d}(\nu, \Sigma_{k-1}) \in [2\hat{\delta}, 3\hat{\delta}]$. In this case the evolution of ν interpolates between the projection onto Σ_{k-1} and staying fixed, i.e. we set

$$\nu_2(\mathtt{t}) = H_0\left(\mathtt{t}\,\frac{3\hat{\delta} - \mathbf{d}(\nu, \Sigma_{k-1})}{\hat{\delta}}, \nu\right)$$

and let $z(\mathtt{t})$ evolve according to (3.56) with $t_i(\mathtt{t}), x_i(\mathtt{t})$ given by $\sum_{i=i}^{k} t_i(\mathtt{t}) \delta_{x_i(\mathtt{t})} = \nu_2(\mathtt{t})$, so we define $\widetilde{H}$ as

$$\widetilde{H}\left(\mathtt{t}, \nu, \delta_z, \frac{1}{2}\right) = \left(\nu_2(\mathtt{t}), \delta_{z(\mathtt{t})}, \frac{1}{2}\right).$$

(iv) $\mathbf{d}(\nu, \Sigma_{k-1}) \geq 3\hat{\delta}$. The deformation $\widetilde{H}$ leaves now ν fixed, while we let $z(\mathtt{t})$ evolve by (3.56) with $t_i(\mathtt{t}) \equiv t_i$ and $x_i(\mathtt{t}) \equiv x_i$.

$$\widetilde{H}\left(\mathtt{t}, \nu, \delta_z, \frac{1}{2}\right) = \left(\nu, \delta_{z(\mathtt{t})}, \frac{1}{2}\right).$$

Observe that in this case, by the definition of $\widetilde{f}$ and by the choice of $\widetilde{\delta}$, the latter flow of z does not intersect the support of ν and $d(z, z(1)) = O(\widetilde{\delta})$. ■

We next *slice* the set $\widetilde{Y}_{\frac{1}{2}}$ in the second entry δ_z: for $p \in \Sigma$ we introduce $\widetilde{Y}_{\left(\frac{1}{2},p\right)} \subseteq \Sigma_k$ given by

$$\widetilde{Y}_{\left(\frac{1}{2},p\right)} = \left\{ \nu \in \Sigma_k : \left(\nu, \delta_p, \frac{1}{2} \right) \in \widetilde{Y}_{\frac{1}{2}} \right\}, \tag{3.57}$$

so that

$$\widetilde{Y}_{\frac{1}{2}} = \bigcup_{p \in \Sigma} \left(\widetilde{Y}_{\left(\frac{1}{2},p\right)}, \delta_p, \frac{1}{2} \right).$$

In Proposition 3.3.4 we will further deform $\widetilde{Y}_{\left(\frac{1}{2},p\right)}$ to some compact subset of Σ_k (depending on p).

Let $\delta_2 > 0$ be a small number, $p \in \Sigma$ and χ_{δ_2} a cut-off function such that

$$\chi_{\delta_2} = \begin{cases} 0 & \text{in } B_{\delta_2}(p), \\ 1 & \text{in } (B_{2\delta_2}(p))^c. \end{cases} \tag{3.58}$$

We start by proving the following lemmas (we are extending the notation in (1.15) to any subset of Σ).

Lemma 3.3.2 *Let $p \in \Sigma$ and let $\delta_2 > 0$ be as before. There exists $\delta_3 > 0$ sufficiently small such that the above defined map $H_0(\mathfrak{t}, \cdot)$ is a deformation retract of $\left\{ \nu \in \widetilde{Y}_{\left(\frac{1}{2},p\right)} : \int_\Sigma \chi_{\delta_2}\, d\nu \geq \delta_2, \mathbf{d}\left(\frac{\chi_{\delta_2}\nu}{\|\chi_{\delta_2}\nu\|}, \Sigma_{k-2} \right) \in (0, \delta_3) \right\} \cap \{\mathbf{d}(\nu, \Sigma_{k-1}) < \delta_3\}$ onto $(\Sigma \setminus \{p\})_{k-1}$ with the property that $\forall \mathfrak{t} \in [0,1]$ we have $p \notin supp\ H_0(\mathfrak{t}, \nu)$.*

Proof. Let δ_1 be as in (3.54). We can assume that $\delta_1 \leq \delta_2/16$. We first prove that $H_0(\mathfrak{t}, \cdot)$ has the property that as the $\mathbf{d}$-distance of ν from Σ_{k-1} tends to zero then the support of the measure $H_0(\mathfrak{t}, \nu)$ is contained in a shrinking neighborhood of the support of ν (uniformly in ν). We will then show that H_0 restricted to the particular set considered in the statement gives the desired deformation retract.

To prove the first assertion we endow Σ^k, which the k-tuple $X := (x_i)_i$ belongs to, with the product metric, and the simplex T_0, containing the k-tuple $T := (t_i)_i$, with its standard metric induced from $\mathbb{R}^k$. Then one can notice that, as the singularities of F_1 and F_2 behave like the inverse of the distance from the boundaries of their domains, there exists a constant C such that

$$\frac{1}{C} F_1(X)^2 - C \leq |\nabla_X F_1(X)| \leq C F_1(X)^2 + C; \qquad \frac{1}{C} F_2(T)^2 - C \leq |\nabla_T F_2(T)| \leq C F_2(T)^2 + C. \tag{3.59}$$

We now consider the evolution $\mathfrak{s} \mapsto \zeta(\nu, \mathfrak{s})$ with initial datum ν in a small neighborhood of Σ_{k-1}, where, we recall, F attains large values and its gradient does not vanish. If we evolve by the gradient of F then X evolves by the gradient of F_1 and T by the gradient of F_2. By the last formula we then have

$$\left| \frac{dX}{d\mathfrak{s}} \right| = |\nabla_X F_1| \leq C F_1(X)^2 + C.$$

On the other hand, still by (3.59), we have that

$$\frac{dF}{d\mathfrak{s}} = |\nabla_X F_1(X)|^2 + |\nabla_T F_2(T)|^2 \geq \frac{1}{C^2} F_1(X)^4 + \frac{1}{C^2} F_2(T)^4 - 2C.$$

Notice that this quantity is strictly positive if F is large enough, see (3.55), which allows to invert the function $\mathfrak{s} \mapsto F(\zeta(\nu, \mathfrak{s}))$. Therefore, if $\mathfrak{s}_\nu$ is the maximal time of existence for $\zeta(\nu, \mathfrak{s})$ we can write that

$$\int_0^{\mathfrak{s}_\nu} \left| \frac{dX}{d\mathfrak{s}} \right| d\mathfrak{s} = \int_{F(\nu)}^{\infty} \left| \frac{dX}{d\mathfrak{s}} \right| \frac{1}{\frac{dF}{d\mathfrak{s}}}\, dF.$$

By the above two inequalities we deduce that

$$\int_0^{\mathfrak{s}_\nu} \left| \frac{dX}{d\mathfrak{s}} \right| d\mathfrak{s} \leq \int_{F(\nu)}^{\infty} \frac{C F_1(X)^2 + C}{\frac{1}{C^2} F_1(X)^4 + \frac{1}{C^2} F_2(T)^4 - 2C}\, dF.$$

By elementary inequalities, recalling that $F = F_1(X) + F_2(T)$ we also find

$$\int_0^{\mathfrak{s}_\nu} \left| \frac{dX}{d\mathfrak{s}} \right| d\mathfrak{s} \leq \tilde{C} \int_{F(\nu)}^{\infty} \frac{1}{F^2 - \tilde{C}} \, dF.$$

Therefore, as ν approaches Σ_{k-1}, namely for $F(\nu)$ large, we find that the displacement of X becomes smaller and smaller. This gives us the claim stated at the beginning of the proof.

Next, we observe that by being $\nu \in \widetilde{Y}_{\left(\frac{1}{2},p\right)}$ and $\mathbf{d}\left(\frac{\chi_{\delta_2}\nu}{\|\chi_{\delta_2}\nu\|}, \Sigma_{k-2}\right) > 0$ by assumption, it follows the existence of at most one point of the support of ν in the ball $B_{\frac{3}{4}\delta_2}(p)$ which does not coincide with p. Moreover, by the above claim we have that the points outside $B_{\delta_2}(p)$ following the flow induced by F move by a distance of order $o_{\delta_3}(1)$, since $\mathbf{d}(\nu, \Sigma_{k-1}) < \delta_3$. Therefore, choosing δ_3 sufficiently small we get the existence of at most one point in the ball $B_{\frac{3}{4}\delta_2}(p)$, different from p, even while the flow is acting.

By the choice of F_1, see (3.54), (3.55), and by the choice $\delta_1 \leq \frac{\delta_2}{16}$, we deduce that the point inside $B_{\frac{3}{4}\delta_2}(p)$ it is not affected by the flow and in particular it does not collapse onto p: the proof is complete. ■

Lemma 3.3.3 *There exists a deformation retract $H(\mathfrak{t}, \cdot)$ of $\left\{\nu \in \widetilde{Y}_{\left(\frac{1}{2},p\right)} : \int_\Sigma \chi_{\delta_2} \, d\nu \geq \delta_2\right\}$ to the set:*

$$\mathcal{B} := \left(\Sigma \setminus B_{\delta_2}(p)\right)_k \cup \left\{ card\left((supp\ (\nu)) \setminus B_{\delta_2}(p)\right) \leq k - 2 \right\}.$$

Proof. Let us first consider a deformation retract which pushes points in $\Sigma \setminus \{p\}$ away from p. Define $H_1(\mathfrak{t}, \cdot)$, $\mathfrak{t} \in [0,1]$ as follows: if $\nu = \sum_{i=1}^k t_i \delta_{x_i}$, $x_i \neq p$, then (using normal coordinates around p)

$$H_1(\mathfrak{t}, \nu) = \sum_{i=1}^k t_i \delta_{x_{i,\mathfrak{t}}}, \qquad \text{where} \qquad x_{i,\mathfrak{t}} = \begin{cases} \frac{x_i}{|x_i|}\left((1-\mathfrak{t})|x_i| + \mathfrak{t}\,\delta_2\right) & \text{if } d(p, x_i) < \delta_2. \\ x_i & \text{if } d(p, x_i) \geq \delta_2. \end{cases}$$

We next introduce two cut-off functions $\chi_1^{\delta_3}, \chi_2^{\delta_3}$ as in Figure 1 ($\chi_2^{\delta_3}$ corresponds to the dashed graph).

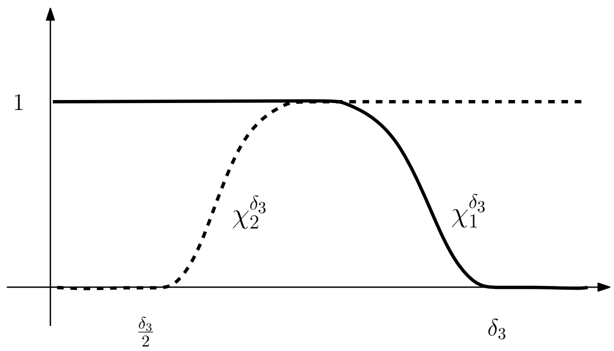

Figure 3.1: The cut-off functions $\chi_1^{\delta_3}, \chi_2^{\delta_3}$.

For $\{\mathbf{d}(\nu, \Sigma_{k-1}) < \delta_3\}$ we define the deformation retract $H_2(\mathfrak{t}, \cdot)$ as an *interpolation* between the homotopies H_0 and H_1, precisely

$$H_2(\mathfrak{t}, \nu) = H_1\left(\mathfrak{t}\,\chi_2^{\delta_3}\left(\mathbf{d}\left(\frac{\chi_{\delta_2}\nu}{\|\chi_{\delta_2}\nu\|}, \Sigma_{k-2}\right)\right), H_0\left(\mathfrak{t}\,\chi_1^{\delta_3}\left(\mathbf{d}\left(\frac{\chi_{\delta_2}\nu}{\|\chi_{\delta_2}\nu\|}, \Sigma_{k-2}\right)\right), \nu\right)\right).$$

The introduction of the cut-off functions makes the deformation retract continuous with respect to the topology induced by the $\mathbf{d}$-distance.

For $\mathbf{d}(\nu, \Sigma_{k-1})$ arbitrary we instead define H as

$$H(\mathfrak{t}, \nu) = H_1\left(\mathfrak{t}\,\chi_2^{\delta_3}\left(\mathbf{d}(\nu, \Sigma_{k-1})\right), H_2\left(\chi_1^{\delta_3}(\mathbf{d}(\nu, \Sigma_{k-1})), \nu\right)\right).$$

Again, notice that the cut-off functions in the first argument of H_1 give continuity in ν. ■

The main result of this subsection is the following proposition: we retract $\widetilde{Y}_{\left(\frac{1}{2},p\right)}$ to a set of measures $\Sigma_{k,p,\bar{\tau}}$ (see (3.60)) for which either the support is bounded away from p, or for which there are at most $k-2$ points not closest to p. As we will see, these conditions will be helpful to find suitable test functions with low Euler-Lagrange energy, see the next subsections.

Proposition 3.3.4 *There exist $\bar{\tau} \gg 1$ and a retraction $\mathcal{R}_p$ of $\widetilde{Y}_{\left(\frac{1}{2},p\right)}$ to the following set:*

$$
\begin{aligned}
\Sigma_{k,p,\bar{\tau}} &= \left\{ \nu = \sum_{i=1}^{k} t_i \delta_{x_i} \in \Sigma_k : d(x_i,p) \geq \frac{1}{\bar{\tau}} \ \forall i \right\} \cup \\
&\cup \left\{ \nu = \sum_{i=1}^{k} t_i \delta_{x_i} \in \Sigma_k : \mathit{card}\{x_j : \, d(x_j,p) > \min_i d(x_i,p)\} \leq k-2 \right\}.
\end{aligned} \tag{3.60}
$$

Proof. Recall first the definition (3.58) of χ_{δ_2}. We then extend the result in Lemma 3.3.3 to arbitrary values of $m_2(\nu) = \int_\Sigma \chi_{\delta_2}\, d\nu$, namely also for $m_2 < \delta_2$, finding a retraction onto $\mathcal{B}$. Consider normal coordinates around p. Define $m(\nu) = \Big\| \nu \Big(\chi_{\delta_2}(m_2(\nu)) + (1-\chi_{\delta_2}(|x|))(1-\chi_{\delta_2}(m_2(\nu))) \Big) \Big\|$ and let

$$
T(\nu) = \begin{cases} \dfrac{\nu\Big(\chi_{\delta_2}(m_2(\nu)) + (1-\chi_{\delta_2}(|x|))(1-\chi_{\delta_2}(m_2(\nu)))\Big)}{m(\nu)} & \text{if } \ m_2(\nu) < 2\delta_2, \\ \nu & \text{if } \ m_2(\nu) \geq 2\delta_2. \end{cases}
$$

We then define the retraction as

$$
\tilde{R}(\nu) = T\big(H(\chi_{\delta_2}(m_2(\nu)), \nu)\big).
$$

Let $\nu_H = H\big(\chi_{\delta_2}(m_2(\nu)), \nu\big)$. To have $\tilde{R}$ well-defined we need to ensure that whenever T is acting, namely for $m_2(\nu_H) < 2\delta_2$, we have $m(\nu_H) > 0$. Clearly, it is enough to show that

$$
\int_\Sigma (1-\chi_{\delta_2})\, d\nu_H > 0. \tag{3.61}
$$

We point out that

$$
m_2(\nu_H) + \int_\Sigma (1-\chi_{\delta_2})\, d\nu_H = 1.
$$

Therefore, by $m_2 < 2\delta_2$ we obtain

$$
\int_\Sigma (1-\chi_{\delta_2})\, d\nu_H > 1 - 2\delta_2.
$$

Finally, we construct a retraction of $\mathcal{B}$ onto $\Sigma_{k,p,\bar{\tau}}$. For $\nu \in \mathcal{B}$ with $\|(1-\chi_{\delta_2})\nu\| > 0$ we define a parameter $\tau = \tau(\nu) \in (0,+\infty]$ in the following way:

$$
\frac{1}{\tau} = \mathbf{d}\left(\frac{(1-\chi_{\delta_2})\nu}{\|(1-\chi_{\delta_2})\nu\|}, \delta_p \right). \tag{3.62}
$$

Consider normal coordinates around p. Let $\bar{\tau} \gg 1$ be such that $\frac{1}{\bar{\tau}} \ll \delta_2 \ll 1$ and let $f : \mathcal{B} \times \Sigma \to \mathbb{R}^+$ and $g : \mathbb{R}^+ \to \mathbb{R}^+$ be two smooth functions such that

$$
f(\nu,x) = \begin{cases} 0 & \text{if } \tau = +\infty, \\ \frac{x}{|x|}\frac{1}{\tau} & \text{if } \tau < +\infty \text{ and } |x| \leq \frac{1}{\bar{\tau}}, \\ 1 & \text{if } \tau < +\infty \text{ and } |x| \geq \frac{2}{\bar{\tau}}, \end{cases} \qquad g(t) = \begin{cases} t & \text{if } t \leq \frac{1}{\bar{\tau}}, \\ 1 & \text{if } t \geq \frac{2}{\bar{\tau}}. \end{cases}
$$

For $\nu = \sum_{i=1}^k s_i \delta_{y_i} \in \mathcal{B}$ with $\|(1-\chi_{\delta_2})\nu\| > 0$ we consider $(1-\chi_{\delta_2})\nu = \sum_{i=1}^k t_i \delta_{x_i}$ and then define

$$
\widetilde{\nu} = \frac{\sum_{i=1}^k t_i g(|x_i|) \delta_{x_i f(\nu,x_i)}}{\sum_{i=1}^k t_i g(|x_i|)}. \tag{3.63}
$$

Observe that for $d(x_i,p) \le \frac{1}{\bar\tau}$ $\forall i$, (3.63) reads as

$$\widetilde{\nu} = \frac{\sum_{i=1}^k t_i |x_i| \delta_{\frac{x_i}{|x_i|}\frac{1}{\bar\tau}}}{\sum_{i=1}^k t_i |x_i|},$$

while for $d(x_i,p) \ge \frac{2}{\bar\tau}$ $\forall i$, we obtain $\widetilde{\nu} = \sum_{i=1}^k t_i \delta_{x_i}$.

For a general $\nu \in \mathcal{B}$ the retraction is given by

$$\mathcal{R}_p(\nu) = (1-m_2)\widetilde{\nu} + \chi_{\delta_2}\nu. \tag{3.64}$$

Observe that when $\|(1-\chi_{\delta_2})\nu\| = 0$, τ is not defined. However, the map $\mathcal{R}_p(\nu)$ is well-defined since in this case we have $m_2 = 1$. Notice furthermore that $\mathcal{R}_p(\nu) \in \Sigma_k$ since $\|\mathcal{R}_p(\nu)\| = 1$ and since we do not increase the number of points in the support of ν, due to the fact that the map $\nu \mapsto \widetilde{\nu}$ does not affect the points x_i with $d(x_i,p) \ge \frac{2}{\bar\tau}$, which was chosen such that $\frac{2}{\bar\tau} \ll \delta_2$. ■

Remark 3.3.5 (i) *With the above definitions, letting δ_2 tend to zero, one shows that the map $\mathcal{R}_p$ is homotopic to the identity on its domain.*

(ii) *The parameter δ_2 is chosen so that $\delta_2 \ll \delta$.*

Combining Lemma 3.3.1 and Proposition 3.3.4 (applying its proof uniformly in $p \in \Sigma$) we obtain the following result; notice that by construction, the retraction $\mathcal{R}_p$ from Proposition 3.3.4 depends continuously on p.

Corollary 3.3.6 *There exist $\bar\tau \gg 1$ and a continuous deformation $\mathcal{R}$ of $Y \cap \left\{s = \frac{1}{2}\right\}$ onto the set*

$$\bigcup_{p \in \Sigma} \left\{ \left(\nu, \delta_p, \frac{1}{2}\right) : \nu \in \Sigma_{k,p,\bar\tau} \right\},$$

where $\Sigma_{k,p,\bar\tau}$ is as in (3.60).

In the next two subsections we perform the construction of test functions using the above deformations.

3.3.2 Test functions modelled on $\widetilde{Y}_{\left(\frac{1}{2},p\right)} * \delta_p$

In this subsection we introduce a class of test functions parametrized on $\widetilde{Y}_{\left(\frac{1}{2},p\right)} * \delta_p \subseteq Y$, see (3.57) and (3.41). The latter subset of Y is where the interaction between the two components of (1.18) is stronger, and hence where more refined energy estimates will be needed. The remainder of Y will be taken care of in the next subsection.

The retraction $\mathcal{R}_p$ defined in Proposition 3.3.4 will play a crucial role in the construction of the test functions. Indeed, starting from a measure in $\widetilde{Y}_{\left(\frac{1}{2},p\right)}$ we will consider, through the map $\mathcal{R}_p$, a configuration belonging to $\Sigma_{k,p,\bar\tau}$, see (3.60). When considering $\widetilde{Y}_{\left(\frac{1}{2},p\right)} * \delta_p$ and the corresponding join parameter s, our goal is to pass continuously from vector-valued functions (φ_1, φ_2) with $e^{\varphi_1} \simeq \hat{\nu} \in \Sigma_{k,p,\bar\tau}$ (in the distributional sense) to functions (φ_1,φ_2) with $e^{\varphi_2} \simeq \delta_p$. This needs to be done so that the energy $J_\rho(\varphi_1,\varphi_2)$ stays arbitrarily low.

As the formulas are rather involved, we first discuss the general ideas beyond them. Our construction relies on superpositions of *regular bubbles* and *singular bubbles*. Regular bubbles are functions as in (3.51) which (roughly) optimize inequality (1.13) in the scalar case. Singular bubbles instead are profiles of solutions to (1.9) when a Dirac mass is present in the right-hand side: this singular version of (1.9) *shadows* system (1.18) when one component has a higher concentration than the other.

From the computational point of view, regular (respectively singular) bubbles behave like logarithmic functions of the distance from a point truncated at a proper scale, with coefficient -4 (respectively -6). By this reason we sometimes substitute an expression as in (3.51) (or in the subsequent formula) with truncated logarithms.

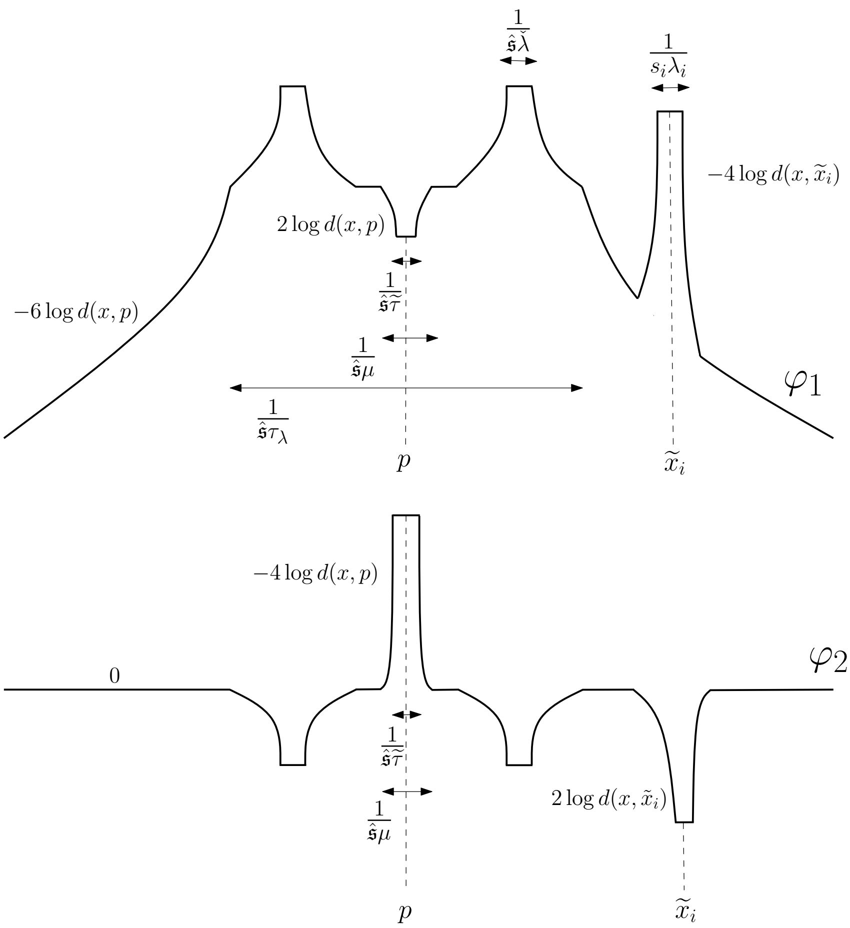

Figure 3.2: The test functions.

Another aspect of the construction is the following: at a scale where the function φ_i dominates, the gradient of the other component φ_j of (1.18) will behave like $-\frac{1}{2}\nabla\varphi_i$: the reason of this relies in the fact that this choice minimizes $Q(\varphi_1,\varphi_2)$, see (1.8), for φ_i fixed.

We introduce now the test functions (φ_1,φ_2) as in Figure 3.2, starting by motivating the definitions of the parameters involved. Consider $p\in\Sigma$ and $\nu\in\widetilde{Y}_{\left(\frac{1}{2},p\right)}$: recalling Proposition 3.3.4 and defining

$$\hat{\nu}:=\mathcal{R}_p(\nu)=\sum_{i=i}^{k}t_i\delta_{x_i}\in\Sigma_{k,p,\bar{\tau}}, \tag{3.65}$$

let τ be as given in (3.62). Consider parameters $\widetilde{\tau}\gg\mu\gg\lambda\gg1$ and let $\mathfrak{s}\geq1$ be a scaling parameter which will be used to deform one component into the other one: this will be chosen to depend on the join parameter s. Roughly speaking, φ_1 is made by a singular bubble at scale $\frac{1}{\hat{\mathfrak{s}}\tau_\lambda}$, where $\hat{\mathfrak{s}}$ is given by (3.68) (but one can think $\hat{\mathfrak{s}}=\mathfrak{s}$ for the moment) and

$$\tau_\lambda:=\min\{\tau,\lambda\}, \tag{3.66}$$

on top of which we add regular bubbles at scales $\frac{1}{s_i\lambda_i}$ centred at points $\widetilde{x}_i$ with $d(\widetilde{x}_i,p)\geq\frac{1}{\hat{\mathfrak{s}}\tau}$ for all i. The parameters s_i,λ_i are defined by (3.71) and (3.70) in order to get comparable integrals of e^{φ_1} near all points $\widetilde{x}_i$; we will discuss later why we take sometimes $\hat{\mathfrak{s}}\neq\mathfrak{s}$. The centres $\tilde{x}_i$ of the regular bubbles are defined as follows: letting $\bar{\delta}$ small but fixed, we set in normal coordinates at p:

$$\widetilde{x}_i=\frac{1}{\widetilde{s}_i}x_i,\qquad \widetilde{s}_i=\begin{cases}\hat{\mathfrak{s}} & \text{if } d(x_i,p)\leq\bar{\delta},\\ 1 & \text{if } d(x_i,p)\geq2\bar{\delta}.\end{cases} \tag{3.67}$$

We point out that for $d(x_i, p) \leq \bar{\delta}$ we get $\widetilde{x}_i = \frac{1}{\mathfrak{s}} x_i$, which gives continuity when x_i approaches the *plateau* $\{d(\cdot, p) \leq \frac{1}{\tau_\lambda}\}$. For $d(x_i, p) \geq \bar{\delta}$ instead the position of the points does not depend on $\mathfrak{s}$.

The effect of the increasing parameter $\mathfrak{s}$ depends on the starting configuration $\nu \in \widetilde{Y}_{(\frac{1}{2}, p)}$: in case we have points x_i on the plateau of the singular bubble, i.e. $d(x_i, p) \leq \frac{1}{\tau_\lambda}$ for some i, the support of the singular and regular bubbles of φ_1 shrinks; moreover, the points $\widetilde{x}_i$ approach p. On the other hand, φ_2 is (qualitatively) dilated by a factor $\frac{1}{\mathfrak{s}}$ so that e^{φ_2} loses concentration at the expense of e^{φ_1}.

In case we do not have points on the plateau, namely when $d(\widetilde{x}_i, p) \geq \frac{1}{\tau_\lambda}$ for all i, it is not convenient anymore to develop a singular bubble with center p as $\mathfrak{s}$ increases. To prevent this situation we give an upper bound on $\hat{\mathfrak{s}}$ depending on τ. For $\tau_1 \geq 1$ large but fixed we let $\hat{P} : (0, +\infty) \to (0, +\infty)$ be a non-decreasing continuous function defined by

$$\begin{cases} \hat{P}(t) = 1 & \text{for } t \leq \tau_1, \\ \hat{P}(t) \to +\infty & \text{for } t \to 2\tau_1. \end{cases}$$

If τ is as in (3.62), we then define $\hat{\mathfrak{s}} = \hat{\mathfrak{s}}(\mathfrak{s}, \tau)$ as

$$\hat{\mathfrak{s}} = \begin{cases} \min\{\mathfrak{s}, \hat{P}(\tau)\} & \text{if } \tau < 2\tau_1, \\ \mathfrak{s} & \text{if } \tau \geq 2\tau_1. \end{cases} \tag{3.68}$$

Notice that by construction of the retraction $\mathcal{R}_p$, see Proposition 3.3.4, when there are no points on the plateau $\{d(\cdot, p) \leq \frac{1}{\tau_\lambda}\}$ it follows that $\tau \leq C$ and therefore, taking $2\tau_1 > C$, we get $\hat{\mathfrak{s}} \leq \hat{P}(C) < +\infty$.

In this situation, namely for $\hat{\mathfrak{s}}$ bounded from above, the second component φ_2 remains fixed when we start to concentrate the first component φ_1. To do this we develop more and more concentrated bubbles around the points $\widetilde{x}_i$; we introduce a parameter $\check{\lambda} = \check{\lambda}(\tau)$ so that $\check{\lambda} \to +\infty$ even for $\tau \leq 2\tau_1$ when $\mathfrak{s}$ increases. Let $\check{P} : (0, +\infty) \to (0, +\infty)$ be a non-increasing continuous functions such that

$$\begin{cases} \check{P}(t) \to +\infty & \text{for } t \to 2\tau_1, \\ \check{P}(t) = 1 & \text{for } t \geq 4\tau_1. \end{cases}$$

We then let

$$\check{\lambda} = \check{\mathfrak{s}}\lambda, \qquad \check{\mathfrak{s}} = \begin{cases} \mathfrak{s} & \text{if } \tau \leq 2\tau_1, \\ \min\{\mathfrak{s}, \check{P}(\tau)\} & \text{if } \tau > 2\tau_1. \end{cases} \tag{3.69}$$

To have comparable integral of e^{φ_1} at each peak around $\widetilde{x}_i$ for $i = 1, \dots, k$, we impose the conditions

$$\begin{cases} \log \lambda_i - \log d(x_i, p) = \log \tau_\lambda + \log \check{\lambda} & \text{if } d(x_i, p) > \frac{1}{\tau_\lambda}, \\ \lambda_i = \check{\lambda} & \text{if } d(x_i, p) \leq \frac{1}{\tau_\lambda} \end{cases} \tag{3.70}$$

and

$$\log s_i + \log \widetilde{s}_i = 2 \log \hat{\mathfrak{s}}, \tag{3.71}$$

which determine λ_i and s_i.

Recall the definitions of $\hat{\nu}$ in (3.65): motivated by the above discussion, we define the functions (φ_1, φ_2) as follows (see Figure 3.2). The positive peaks of φ_1 are given by

$$v_1(x) = v_{1,1}(x) + v_{1,2}(x) = \log \sum_{i=1}^{k} t_i \frac{\max\left\{1, \min\left\{\left(\frac{4}{d(\widetilde{x}_i,p)} d(x, \widetilde{x}_i)\right)^{-4}, \left(\frac{4}{d(\widetilde{x}_i,p)} \frac{1}{s_i \lambda_i}\right)^{-4}\right\}\right\}}{\left((\hat{\mathfrak{s}}\tau_\lambda)^{-2} + d(x,p)^2\right)^3},$$

where

$$v_{1,1}(x) = \log \sum_{i=1}^{k} t_i \max\left\{1, \min\left\{\left(\frac{4}{d(\widetilde{x}_i, p)} d(x, \widetilde{x}_i)\right)^{-4}, \left(\frac{4}{d(\widetilde{x}_i, p)} \frac{1}{s_i \lambda_i}\right)^{-4}\right\}\right\},$$

$$v_{1,2}(x) = \log \frac{1}{\left((\hat{\mathfrak{s}}\tau_\lambda)^{-2} + d(x,p)^2\right)^3}.$$

The positive peak of φ_2 is instead defined by

$$v_2(x) = \log\left(\max\left\{1, \min\left\{(\hat{\mathfrak{s}}\mu\, d(x,p))^{-4}, \left(\frac{\mu}{\widetilde{\tau}}\right)^{-4}\right\}\right\}\right).$$

We finally set

$$\varphi_{\lambda,\widetilde{\tau},\mathfrak{s}}(x) = \begin{pmatrix} \varphi_1(x) \\ \varphi_2(x) \end{pmatrix} := \begin{pmatrix} v_1(x) - \frac{1}{2}v_2(x) \\ -\frac{1}{2}v_{1,1}(x) + v_2(x) \end{pmatrix}. \tag{3.72}$$

The main result of this subsection is the following proposition.

Proposition 3.3.7 *Suppose that $\rho_1 \in (4k\pi, 4(k+1)\pi)$, $\rho_2 \in (4\pi, 8\pi)$, let $\tilde{\Psi}$ be defined in* (3.19)*, and let $\varphi_{\lambda,\widetilde{\tau},\mathfrak{s}}$ be defined in* (3.72)*, with $p \in \Sigma$ and $\nu \in \widetilde{Y}_{(\frac{1}{2},p)}$. Then, for suitable values of $\widetilde{\tau} \gg \mu \gg \lambda \gg 1$ and for $\mathfrak{s} = 1$, $\tilde{\Psi}(\varphi_{\lambda,\widetilde{\tau},1})$ is valued into the second component of the join $\Sigma_k * \Sigma_1$. Moreover there is a value $\mathfrak{s}_{p,\nu} > 1$ of $\mathfrak{s}$, which depends continuously on p, ν such that $\tilde{\Psi}(\varphi_{\lambda,\widetilde{\tau},\mathfrak{s}_{p,\nu}})$ is valued into the first component of the join, and such that*

$$J_\rho(\varphi_{\lambda,\widetilde{\tau},\mathfrak{s}}) \to -\infty \quad \text{as } \lambda \to +\infty \quad \text{uniformly in } \mathfrak{s} \in [1, \mathfrak{s}_{p,\nu}] \text{ and in } p, \nu.$$

PROOF. As some of the estimates are rather technical, most of the proof is postponed to the Section 3.5.

Concerning the first statement, when $\mathfrak{s} = 1$, by construction (see in particular Lemma 3.5.2) one can see that most of the integral of e^{φ_2} is concentrated in a ball centred at p with radius of order $\frac{1}{\widetilde{\tau}}$, while that of e^{φ_1} near at most k balls of larger scale. From the definitions of scales $\sigma_1(u_1), \sigma_2(u_2)$ in Subsection 3.2.1 it follows that for $\mathfrak{s} = 1$ the quantity $s(\varphi_1, \varphi_2)$ defined in (3.16) is equal to 1, provided we choose the parameters $\widetilde{\tau} \gg \mu \gg \lambda \gg 1$ properly. By the way $\tilde{\Psi}$ is defined, this implies our first statement.

As $\mathfrak{s}$ increases, see again Lemma 3.5.2, the scale $\sigma_1(\varphi_1)$ (as defined in Subsection 3.2.1) decreases while, depending on τ, the scale of $\sigma_2(\varphi_2)$ reaches some positive value bounded away from zero. In particular for $\tau \geq 2\tau_1$ (recall (3.68)), by the estimates in Lemma 3.5.2, for $\mathfrak{s} \simeq \log\widetilde{\tau} - 2\log\mu$ the scale $\sigma_2(\varphi_2)$ becomes of order 1. In any case, for $\mathfrak{s}$ sufficiently large $s(\varphi_1, \varphi_2) = 0$, so $\tilde{\Psi}$ maps the test function into the first component of the joint. As the scales $\sigma_1(\varphi_1)$, $\sigma_2(\varphi_2)$ vary continuously in φ_1 and φ_2, $\mathfrak{s}_{p,\nu}$ can be chosen to depend continuously in p and ν.

Regarding the energy estimates, the most delicate situation is when τ is large, i.e. when $\hat{\mathfrak{s}} = \mathfrak{s}$, see (3.68). In this case $\mathfrak{s}_{p,\nu} \simeq \log\widetilde{\tau} - 2\log\mu$ and the computations are worked-out in the Appendix. When τ instead is smaller than the fixed number $2\tau_1$ (see again (3.68)) the *singular* part of the first component of the test function (with slope $-6\log d(\cdot, p)$) has negligible contribution and the support of the measure $\hat{\nu}$ in (3.65) is bounded away from p by a fixed positive amount. In this case the interaction between the two components is negligible, and similar estimates as those in Proposition 2.1.3 can be applied. ■

We proceed now with parameterizing the above functions via the number s in the topological join. Ideally, one would like to have $\mathfrak{s}$ varying from 1 to $\mathfrak{s}_{p,\nu}$ as s decreases from 1 to 0. However, for this map to be well defined on the topological join, we will need to eliminate the dependence of the test function on the first (resp. second) component of the join when $s = 1$ (resp. $s = 0$). For this reason, we will need some extra deformations depending on s. The construction goes as follows, depending on three ranges of the join parameter s.

The case $s \in \left[\frac{1}{4}, \frac{3}{4}\right]$

Let $\varphi_{\lambda,\widetilde{\tau},\mathfrak{s}}$ be defined in (3.72), with $p \in \Sigma$ and $\nu \in \widetilde{Y}_{(\frac{1}{2},p)}$. We set

$$\Phi_\lambda(\nu, p, s) = \varphi_{\lambda,\widetilde{\tau},2(1-\mathfrak{s}_{p,\nu})s+\frac{3}{2}\mathfrak{s}_{p,\nu}-\frac{1}{2}}, \tag{3.73}$$

so that $\Phi_\lambda(\nu, p, \frac{1}{4}) = \varphi_{\lambda,\widetilde{\tau},\mathfrak{s}_{p,\nu}}$ and $\Phi_\lambda(\nu, p, \frac{3}{4}) = \varphi_{\lambda,\widetilde{\tau},1}$.

The case $s \in \left[0, \frac{1}{4}\right]$

Starting from test functions of the form $\varphi_{\lambda,\widetilde{\tau},\mathfrak{s}_{p,\nu}}$, the goal will be to eliminate the dependence on the second component of the join, namely on the measure δ_p. To this end, we divide the interval $\left[0, \frac{1}{4}\right]$ in several subintervals in which we perform different operations on the test functions. Moreover, we want J_ρ to attend arbitrarily low values while doing these procedures. Notice that in what follows, this range of the join parameter s will correspond to $\mathfrak{s} = \mathfrak{s}_{p,\nu}$ which is given in Proposition 3.3.7.

Step 1. Let $s \in \left[\frac{3}{16}, \frac{1}{4}\right]$. We flatten here the function v_2 in the second component of (3.72) by considering the following deformation:

$$\check{\varphi}^t_{\lambda,\widetilde{\tau}}(x) = \begin{pmatrix} \check{\varphi}^t_1(x) \\ \check{\varphi}^t_2(x) \end{pmatrix} := \begin{pmatrix} v_1(x) - \frac{1}{2}\,t\,v_2(x) \\ -\frac{1}{2}v_{1,1}(x) + t\,v_2(x) \end{pmatrix}, \qquad t \in [0,1].$$

We will then take

$$\Phi_\lambda(\nu,p,s) = \check{\varphi}^t_{\lambda,\widetilde{\tau}}(x), \qquad t = 16\left(s - \frac{3}{16}\right). \tag{3.74}$$

It is easy to see that J_ρ attends arbitrarily low values on this deformation by minor modifications in the proof of Proposition 3.3.7.

Step 2. Let $s \in \left[\frac{1}{8}, \frac{3}{16}\right]$. Starting from $s = \frac{3}{16}$ we deform the test functions introduced in (3.72) to the standard test functions of the form given as in (3.53). Roughly speaking, the idea is to modify the profile of the first component φ_1 (see Figure 3.2) by performing the following two continuous deformations: we first flatten the singular bubble $v_{1,2}$, see above (3.72). On the other hand we eliminate the dependence of the point p in the regular bubbles $v_{1,1}$. Therefore, we set

$$v^t_1(x) = v^t_{1,1}(x) + v^t_{1,2}(x),$$

where

$$v^t_{1,1}(x) = \log \sum_{i=1}^k t_i \max\left\{1, \min\left\{\left(\left(\frac{4}{d(\widetilde{x}_i,p)}\right)^t d(x,\widetilde{x}_i)\right)^{-4}, \left(\left(\frac{4}{d(\widetilde{x}_i,p)}\right)^t \frac{1}{s_i\lambda_i}\right)^{-4}\right\}\right\},$$

and $v^t_{1,2}(x) = t\,v_{1,2}(x)$. Finally, recalling that we have flattened v_2 in Step 1, we consider

$$\tilde{\varphi}^t_{\lambda,\widetilde{\tau}}(x) = \begin{pmatrix} \tilde{\varphi}^t_1(x) \\ \tilde{\varphi}^t_2(x) \end{pmatrix} := \begin{pmatrix} v^t_1(x) \\ -\frac{1}{2}v^t_{1,1}(x) \end{pmatrix}, \qquad t \in [0,1]. \tag{3.75}$$

We will then take

$$\Phi_\lambda(\nu,p,s) = \tilde{\varphi}^t_{\lambda,\widetilde{\tau}}(x), \qquad t = 16\left(s - \frac{1}{8}\right). \tag{3.76}$$

Concerning $\tilde{\varphi}^t_1$, its peaks around $\widetilde{x}_i$ for $i = 1,\dots,k$, are truncated at scale $\frac{1}{s_i\lambda_i}$, with s_i given by (3.71) and λ_i to be chosen in the following way in order to have comparable volume at any $\widetilde{x}_i$:

$$\begin{cases} \log\lambda_i + \log s_i - t\log d(\widetilde{x}_i,p) = (t+1)\log\hat{\mathfrak{s}} + \log\check{\lambda} + t\log\tau_\lambda & \text{if } d(x_i,p) > \frac{1}{\tau_\lambda}, \\ \lambda_i = \check{\lambda} & \text{if } d(x_i,p) \le \frac{1}{\tau_\lambda}. \end{cases} \tag{3.77}$$

Observe that for $t = 0$ we get again (3.70). The following result holds true.

Proposition 3.3.8 *Suppose that $\rho_1 \in (4k\pi, 4(k+1)\pi)$, $\rho_2 \in (4\pi, 8\pi)$. Let $\tilde{\varphi}^t_{\lambda,\widetilde{\tau}}$ be defined as in (3.75), with $p \in \Sigma$ and $\nu \in \widetilde{Y}_{(\frac{1}{2},p)}$. Then, one has*

$$J_\rho(\tilde{\varphi}^t_{\lambda,\widetilde{\tau}}) \to -\infty \quad \text{as } \lambda \to +\infty \quad \text{uniformly in } t \in [0,1] \text{ and in } p,\nu.$$

The most delicate case is when the set of the points on the plateau is not empty, i.e. for $I_1 \neq \emptyset$, see (3.109). We give the proof of the latter result just in this situation, skipping the case $I_1 = \emptyset$ where the singular bubble of the first component of the test function (with slope $-6\log d(\cdot,p)$) has negligible contribution and the estimates are rather easy. As observed in Case 1 of the proof of Proposition 3.3.7, see below (3.122), for $I_1 \neq \emptyset$ we deduce $\hat{\mathfrak{s}} = \mathfrak{s}$ and $\check{\lambda} \le C\lambda$. Moreover, for this range of the join parameter s, we have $\mathfrak{s} = \mathfrak{s}_{p,\nu} \gg 1$. The proof will follow from the estimates below, which are obtained exactly as Lemmas 3.5.1, 3.5.2, 3.5.3 by using (3.71) and (3.77).

Lemma 3.3.9 *For $t \in [0,1]$ we have that*

$$\fint_\Sigma \tilde{\varphi}^t_1\,dV_g = O(1), \qquad \fint_\Sigma \tilde{\varphi}^t_2\,dV_g = O(1).$$

Lemma 3.3.10 *Recalling the notation in* (3.102), *for* $t \in [0,1]$ *it holds that*

$$\int_\Sigma e^{\tilde\varphi_1^t}\, dV_g \simeq_C \hat{\mathfrak{s}}^{2+2t}\tau_\lambda^{2t}\check\lambda^2, \qquad \int_\Sigma e^{\tilde\varphi_2^t}\, dV_g \simeq_C 1.$$

Lemma 3.3.11 *Let* $I_1, I_2 \subseteq I$ *be as in* (3.109). *Then, for* $t \in [0,1]$ *we have*

$$\begin{aligned}\int_\Sigma Q(\tilde\varphi_1^t,\tilde\varphi_2^t)\, dV_g \;&\leq\; 8|I_1|\pi\big(\log\check\lambda - t\log\tau_\lambda + (1-t)\log\hat{\mathfrak{s}}\big) + \sum_{i\in I_2} 8\pi\big(\log s_i + \log\lambda_i - t\log d(\widetilde{x}_i,p)\big) + \\ &+\; 16t\pi\sum_{i\in I_2}\log d(\widetilde{x}_i,p) + 24t^2\pi\big(\log\tau_\lambda + \log\hat{\mathfrak{s}}\big) + C,\end{aligned}$$

for some $C = C(\Sigma)$.

Proof of Proposition 3.3.8. Using Lemmas 3.3.9, 3.3.10 and 3.3.11, the energy estimate we obtain is

$$\begin{aligned}J_\rho(\tilde\varphi_1^t,\tilde\varphi_2^t) \;&\leq\; 8|I_1|\pi\big(\log\check\lambda - t\log\tau_\lambda + (1-t)\log\hat{\mathfrak{s}}\big) + \sum_{i\in I_2} 8\pi\big(\log s_i + \log\lambda_i - t\log d(\widetilde{x}_i,p)\big) + \\ &+\; 16t\pi\sum_{i\in I_2}\log d(\widetilde{x}_i,p) + 24t^2\pi\big(\log\tau_\lambda + \log\hat{\mathfrak{s}}\big) - \rho_1\big((2+2t)\log\hat{\mathfrak{s}} + 2t\log\tau_\lambda + 2\log\check\lambda\big) + C,\end{aligned}$$

for some constant $C > 0$. Inserting the condition (3.77) we obtain

$$\begin{aligned}J_\rho(\tilde\varphi_1^t,\tilde\varphi_2^t) \;&\leq\; 8|I_1|\pi\big(\log\check\lambda - t\log\tau_\lambda + (1-t)\log\hat{\mathfrak{s}}\big) + \sum_{i\in I_2} 8\pi\big((t+1)\log\hat{\mathfrak{s}} + \log\check\lambda + t\log\tau_\lambda\big) + \\ &+\; 16t\pi\sum_{i\in I_2}\log d(\widetilde{x}_i,p) + 24t^2\pi\big(\log\tau_\lambda + \log\hat{\mathfrak{s}}\big) - \rho_1\big((2+2t)\log\hat{\mathfrak{s}} + 2t\log\tau_\lambda + 2\log\check\lambda\big) + C.\end{aligned}$$

Notice that for $t = 1$ we get exactly the estimate in (3.122) (recall that we have flattened v_2). The latter estimate can be rewritten as

$$\begin{aligned}J_\rho(\tilde\varphi_1^t,\tilde\varphi_2^t) \;&\leq\; \log\hat{\mathfrak{s}}\Big(8(1-t)|I_1|\pi + 8(t+1)|I_2|\pi + 24t^2\pi - (2+2t)\rho_1\Big) + \log\check\lambda\big(8(|I_1|+|I_2|)\pi - 2\rho_1\big) + \\ &+\; \log\tau_\lambda\,\big(8t|I_2|\pi - 8t|I_1|\pi + 24t^2\pi - 2t\rho_1\big) + 16t\pi\sum_{i\in I_2}\log d(\widetilde{x}_i,p) + C.\end{aligned}$$

As observed in Case 1 of the proof of Proposition 3.3.7, by construction of $\Sigma_{k,p,\bar\tau}$, see (3.60), it holds $|I_2| \leq k-2$ whenever $|I_1| \neq \emptyset$. Therefore, we conclude that the latter estimate is uniformly large negative in $t \in [0,1]$ since $\rho_1 > 4k\pi$ and by the fact that $\hat{\mathfrak{s}} = \hat{\mathfrak{s}}_{p,\nu} \gg \check\lambda \geq \tau_\lambda$. Observe that for $t = 0$ we get

$$J_\rho(\tilde\varphi_1^t,\tilde\varphi_2^t) \leq \log\hat{\mathfrak{s}}\big(8(|I_1|+|I_2|)\pi - 2\rho_1\big) + \log\check\lambda\big(8(|I_1|+|I_2|)\pi - 2\rho_1\big) + C,$$

which is the estimate one expects by considering standard bubbles as in (3.53), see for example part (i) of Proposition 4.2 in [69]. ■

Recall now the definition of $\hat\nu$ given in (3.65): $\hat\nu = \mathcal{R}_p(\nu) = \sum_{i=i}^k t_i\delta_{x_i} \in \Sigma_{k,p,\bar\tau}$. Notice that in the construction of the test functions (3.72), the points x_i are dilated according to (3.67), so deformed to the points $\widetilde{x}_i$. Observe that for $t = 0$ we obtain in (3.75) standard test functions as in (3.53). Roughly speaking, the first component resembles the form of $\varphi_{\lambda,\tilde\nu}$, see (3.53), where $\tilde\nu = \sum_{i=i}^k t_i\delta_{\widetilde{x}_i}$.

In what follows we will skip the energy estimates since they are quite standard for test functions as in (3.53), see for example part (i) of Proposition 4.2 in [69].

Step 3. Consider $s \in \big[\frac{1}{16},\frac{1}{8}\big]$. We will deform here the points $\widetilde{x}_i$ to the original points x_i. Observe that by construction, see (3.67), we have $d(x_i,\widetilde{x}_i) \leq 2\bar\delta$ for all i. Hence there exists a geodesic $\tilde\gamma_i$ joining $\widetilde{x}_i$ and x_i in unit time and we set $x_i^t = \tilde\gamma_i(t)$ with $t \in [0,1]$. Denoting by $\hat\varphi^t_{\lambda,\widetilde\tau} = (\hat\varphi_1^t,\hat\varphi_2^t)$ the corresponding test functions, we will then take

$$\Phi_\lambda(\nu,p,s) = \hat\varphi^t_{\lambda,\widetilde\tau}(x), \qquad t = 16\left(\frac{1}{8} - s\right). \tag{3.78}$$

Once we have deformed the points $\widetilde{x}_i$ to the original one x_i, i.e. for $t = 1$, we get test functions for which the first component has the form of $\varphi_{\lambda,\mathcal{R}_p(\nu)}$.

Step 4. Consider $s \in \left[0, \frac{1}{16}\right]$. In this step we eliminate the dependence on the map $\mathcal{R}_p$. Observe that $\mathcal{R}_p$ is homotopic to the identity map, see Remark 3.3.5, and let $\mathcal{H}_{\mathcal{R}_p} : \widetilde{Y}_{\left(\frac{1}{2},p\right)} \times [0,1] \to \widetilde{Y}_{\left(\frac{1}{2},p\right)}$ be a continuous map such that $\mathcal{H}_{\mathcal{R}_p}(\cdot,0) = \mathcal{R}_p$ and $\mathcal{H}_{\mathcal{R}_p}(\cdot,1) = \mathrm{Id}_{\widetilde{Y}_{\left(\frac{1}{2},p\right)}}$. We consider then the deformation $\nu_t = \mathcal{H}_{\mathcal{R}_p}(\nu,t)$ and letting $\bar{\varphi}^t_{\lambda,\widetilde{\tau}} = (\bar{\varphi}^t_1, \bar{\varphi}^t_2)$ be the corresponding test functions, we set

$$\Phi_\lambda(\nu,p,s) = \bar{\varphi}^t_{\lambda,\widetilde{\tau}}(x), \qquad t = 16\left(\frac{1}{16} - s\right). \tag{3.79}$$

Such a deformation will bring us to test functions which resemble the form of $\varphi_{\lambda,\nu}$.

The case $s \in \left[\frac{3}{4}, 1\right]$

The goal here will be to continuously deform the initial test functions in (3.72), with $\mathfrak{s} = \mathtt{1}$, to a configuration which does not depend on the measure ν, see (3.65). Furthermore, we want in this procedure J_ρ to attend arbitrarily low values. For this purpose we flatten v_1, see (3.72), by using the following deformation:

$$\varphi^t_{\lambda,\widetilde{\tau}}(x) = \begin{pmatrix} \varphi^t_1(x) \\ \varphi^t_2(x) \end{pmatrix} := \begin{pmatrix} t\, v_1(x) - \frac{1}{2} v_2(x) \\ -\frac{1}{2}\, t\, v_{1,1}(x) + v_2(x) \end{pmatrix}, \qquad t \in [0,1]. \tag{3.80}$$

We will then take

$$\Phi_\lambda(\nu,p,s) = \varphi^t_{\lambda,\widetilde{\tau}}(x), \qquad t = 4(1-s). \tag{3.81}$$

The next result holds true.

Proposition 3.3.12 *Suppose that $\rho_1 \in (4k\pi, 4(k+1)\pi)$, $\rho_2 \in (4\pi, 8\pi)$ and let $\varphi^t_{\lambda,\widetilde{\tau}}$ be defined as in* (3.80)*, with $p \in \Sigma$ and $\nu \in \widetilde{Y}_{\left(\frac{1}{2},p\right)}$. Then, one has*

$$J_\rho(\varphi^t_{\lambda,\widetilde{\tau}}) \to -\infty \quad \text{as } \lambda \to +\infty \quad \text{uniformly in } t \in [0,1] \text{ and in } p, \nu.$$

The latter result follows from the next estimates which are obtained similarly as in Lemmas 3.5.1, 3.5.2, 3.5.3, using the fact that $\mathfrak{s} = 1$.

Lemma 3.3.13 *For $t \in [0,1]$ we have that*

$$\fint_\Sigma \varphi^t_1 \, dV_g = O(1), \qquad \fint_\Sigma \varphi^t_2 \, dV_g = O(1).$$

Lemma 3.3.14 *Recalling the notation in* (3.102)*, there exists a constant $C_1(\tau_\lambda,\lambda)$ such that for $t \in [0,1]$*

$$\int_\Sigma e^{\varphi^t_1} \, dV_g \simeq_C \int_\Sigma e^{t v_1} \, dV_g = C_1(\tau_\lambda,\lambda), \qquad \int_\Sigma e^{\varphi^t_2} \, dV_g \simeq_C \int_\Sigma e^{v_2} \, dV_g \simeq_C \frac{\widetilde{\tau}^2}{\mu^4}.$$

Lemma 3.3.15 *For $t \in [0,1]$ we have that*

$$\int_\Sigma Q(\varphi^t_1, \varphi^t_2) \, dV_g \le 8\pi\big(\log\widetilde{\tau} - \log\mu\big) + C_2(\tau_\lambda,\lambda),$$

for some constant $C_2(\tau_\lambda,\lambda)$.

PROOF OF PROPOSITION 3.3.12. Exploiting Lemmas 3.3.13, 3.3.14 and 3.3.15 we deduce

$$\begin{aligned} J_\rho(\varphi^t_1, \varphi^t_2) &\le 8\pi\big(\log\widetilde{\tau} - \log\mu\big) - \rho_2\big(2\log\widetilde{\tau} - 4\log\mu\big) + \widetilde{C}_1(\tau_\lambda,\lambda) + C_2(\tau_\lambda,\lambda) \\ &\le \log\widetilde{\tau}(8\pi - 2\rho_2) + \log\mu(4\rho_2 - 8\pi) + \widetilde{C}_1(\tau_\lambda,\lambda) + C_2(\tau_\lambda,\lambda), \end{aligned}$$

for some constant $\widetilde{C}_1(\tau_\lambda,\lambda)$. The latter upper bound is large negative since $\rho_2 > 4\pi$ and by the choice of the parameters $\widetilde{\tau} \gg \mu \gg \lambda \ge \tau_\lambda$. ■

3.3.3 The global construction

In this subsection we will perform a global construction of a family of test functions modelled on Y, relying on the estimates of the previous subsection. More precisely, as Y is not compact, we will consider a compact retraction of it.

Letting $\left(\mathfrak{D}, \frac{1}{2}\right) \subseteq \left(\Sigma_k \times \Sigma_1, \frac{1}{2}\right)$ be the domain of the map $\mathcal{R}$ in Corollary 3.3.6, we extend it to $\{(\mathfrak{D}, s) : s \in (0,1)\}$ fixing the second component and considering the same action of $\mathcal{R}$ on the first one.

Secondly, we retract the set Y to a subset where the (extended) map $\mathcal{R}$ is well-defined or where $s \in \{0,1\}$. In order to do this, for $\nu = \sum_{i=1}^k t_i \delta_{x_i} \in \Sigma_k$ we let

$$\mathcal{D}(\nu) = \min_{i=1,\dots k,\, i \neq j} \big\{ d(x_i, x_j),\, t_i,\, 1 - t_i \big\}.$$

Moreover, recall the choices of δ, δ_2 given in (3.42) and (3.58) respectively. Observe that for $\mathcal{D}(\nu) \leq \delta$ we are in the domain of $\mathcal{R}$. Moreover, for $\mathcal{D}(\nu) > \delta$ and $d(p, supp\,(\nu)) \geq \delta_2$ the map $\mathcal{R}$ is still well-defined. The idea is then to retract the set Y to a subset where one of the above alternatives holds true or where $s \in \{0,1\}$. We define now the retraction of Y in three steps.

Step 1. Let $\mathcal{D}(\nu) \geq 2\delta$. In this situation we can deform a configuration (ν, δ_p, s) to a configuration $(\nu, \delta_{\widetilde{p}}, \widetilde{s}) \in Y$ (recall (3.41)) where either $d(\widetilde{p}, supp\,(\nu)) \geq \delta_2$ or $\widetilde{s} \in \{0,1\}$. Let

$$\Theta = (\Theta_1, \Theta_2) : [0, +\infty) \times [0,1] \setminus \left\{ \left(0, \frac{1}{2}\right) \right\} \to [0, +\infty) \times [0,1] \setminus \big((0, \delta_2) \times (0,1)\big)$$

be the radial projection as in Figure 3.3.

Observe now that by the fact that $\delta_2 \ll \delta$ (recall Remark 3.3.5), for $\mathcal{D}(\nu) \geq 2\delta$ we get the existence of a unique point $x_{j_p} \in \{x_1, \dots, x_k\}$ such that $d(p, x_{j_p}) \leq \delta_2$. To get then the above-described deformation we define, in normal coordinates around x_{j_p}, the following map:

$$(\nu, \delta_p, s) \mapsto \left(\nu, \delta_{\Theta_1\left(d(p, supp\,(\nu), s)\right) \frac{p}{|p|}}, \Theta_2\big(d(p, supp\,(\nu)), s\big) \right) \in \widetilde{\Upsilon}_\Theta,$$

where

$$\begin{aligned} \widetilde{\Upsilon}_\Theta &= \Big\{ (\nu, \delta_p, s) : \mathcal{D}(\nu) \geq 2\delta,\, d(p, supp\,(\nu)) \geq \delta_2 \Big\} \cup \\ &\cup \Big\{ (\nu, \delta_p, s) : \mathcal{D}(\nu) \geq 2\delta,\, d(p, supp\,(\nu)) \leq \delta_2,\, s \in \{0,1\} \Big\}. \end{aligned} \tag{3.82}$$

Step 2. Let $\mathcal{D}(\nu) \in [\delta, 2\delta]$. In this range we interpolate between the deformation Θ and the identity map. Consider the radial projection $\Theta^t = (\Theta_1^t, \Theta_2^t)$ given as in Figure 3.4, with $t = \frac{(\mathcal{D}(\nu) - \delta)}{\delta}$:

$$\Theta^t = (\Theta_1^t, \Theta_2^t) : [0, +\infty) \times [0,1] \setminus \left\{ \left(0, \frac{1}{2}\right) \right\} \to \Upsilon_t,$$

where

$$\Upsilon_t = [0, +\infty) \times [0,1] \setminus \left((0, t\delta_2) \times \left(\frac{1}{2}(1-t), \frac{1}{2}(1+t) \right) \right).$$

Observe that for $\mathcal{D}(\nu) = 2\delta$ one gets $\Theta^t = \Theta^1 = \Theta$, while for $\mathcal{D}(\nu) = \delta$ one deduces $\Theta^t = \Theta^0 = \mathrm{Id}$. We then set

$$(\nu, \delta_p, s) \mapsto \left(\nu, \delta_{\Theta_1^t\left(d(p, supp\,(\nu), s)\right) \frac{p}{|p|}}, \Theta_2^t\big(d(p, supp\,(\nu)), s\big) \right).$$

Step 3. Let us now introduce the set we obtain after the deformation performed in Step 2:

$$\widetilde{\Upsilon}_\delta = \big\{ (\nu, \delta_p, s) : \mathcal{D}(\nu) = t \in [\delta, 2\delta],\, (p, s) \in \Upsilon_t \big\},$$

which we will deform using the radial projection $\widetilde{\Theta}_\delta : \widetilde{\Upsilon}_\delta \to \hat{\Upsilon}_\delta$ given as in Figure 3.5, where $\hat{\Upsilon}_\delta$ is defined by (see Figure 3.6, where $\partial\hat{\Upsilon}_\delta$ is represented):

$$\begin{aligned} \hat{\Upsilon}_\delta &= \Big\{ (\nu, \delta_p, s) : \mathcal{D}(\nu) \in [\delta, 2\delta],\, d(p, supp\,(\nu)) \leq \delta_2,\, s \in \{0,1\} \Big\} \cup \Big\{ (\nu, \delta_p, s) : \mathcal{D}(\nu) = \delta \Big\} \cup \\ &\cup \Big\{ (\nu, \delta_p, s) : \mathcal{D}(\nu) \in [\delta, 2\delta],\, d(p, supp\,(\nu)) \geq \delta_2 \Big\}. \end{aligned} \tag{3.83}$$

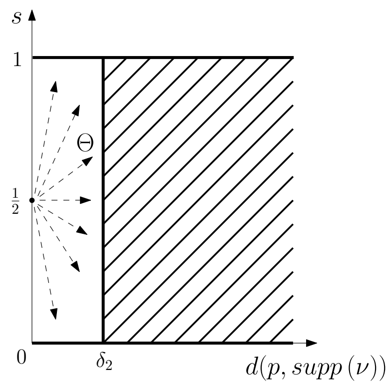

Figure 3.3: The radial projection Θ.

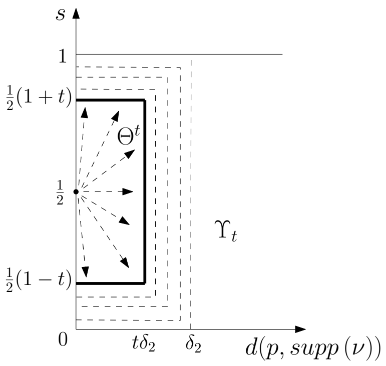

Figure 3.4: The radial projection Θ^t.

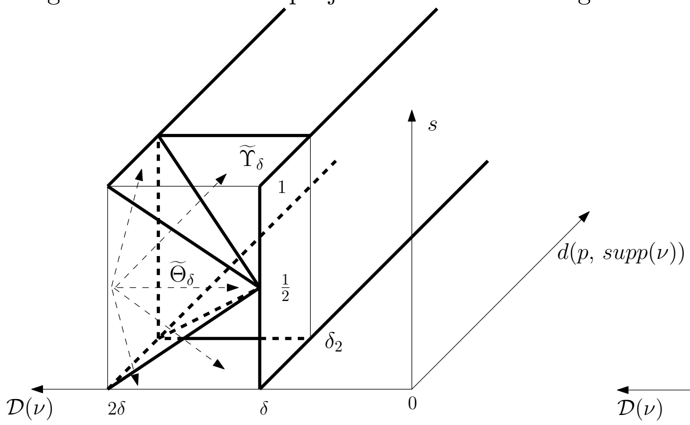

Figure 3.5: The radial projection $\widetilde{\Theta}_\delta$.

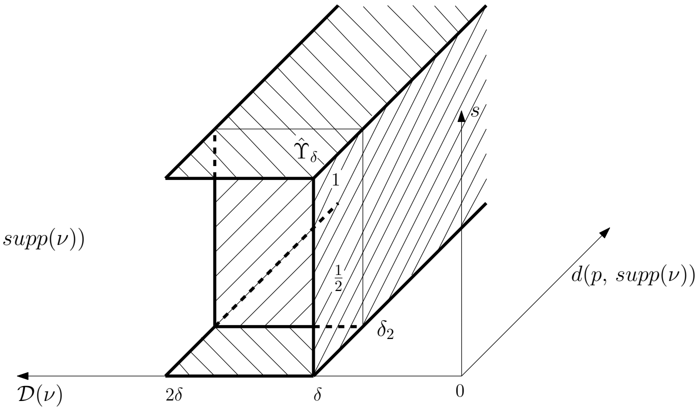

Figure 3.6: The set $\partial\hat{\Upsilon}_\delta$.

Construction of the test functions. Observing that for $\mathcal{D}(\nu) \leq \delta$ we are already in the domain of $\mathcal{R}$ and recalling the sets (3.82), (3.83), we have found a retraction $\mathcal{F} : Y \to Y_{\mathcal{R}}$, where

$$\begin{aligned} Y_{\mathcal{R}} &= \Big\{(\nu, \delta_p, s) : \mathcal{D}(\nu) \leq \delta\Big\} \cup \widetilde{\Upsilon}_\delta \cup \widetilde{\Upsilon}_\Theta \qquad (3.84)\\ &= \Big\{(\nu, \delta_p, s) : \mathcal{D}(\nu) \leq \delta\Big\} \cup \Big\{(\nu, \delta_p, s) : \mathcal{D}(\nu) \geq \delta,\, d(p, supp\,(\nu)) \geq \delta_2\Big\} \cup \\ &\cup \;\Big\{(\nu, \delta_p, s) : \mathcal{D}(\nu) \geq \delta,\, d(p, supp\,(\nu)) \leq \delta_2,\, s \in \{0,1\}\Big\}, \end{aligned}$$

on which the map $\mathcal{R}$ is well-defined or where $s \in \{0, 1\}$.

Remark 3.3.16 *By the way the retraction $\mathcal{F}$ is constructed, it is clear that we have indeed a deformation retract of the set Y onto $Y_{\mathcal{R}}$, i.e. there exists a continuous map $\mathcal{F}_t : Y \times [0,1] \to Y$ such that $\mathcal{F}_0 = Id_Y$, $\mathcal{F}_1 = \mathcal{F} : Y \to Y_{\mathcal{R}}$ and $\mathcal{F}_1(\xi) = \xi$ for all $\xi \in Y_{\mathcal{R}}$.*

We finally call $\Phi_\lambda = \Phi_\lambda(\nu, p, s)$ the test functions in the Subsections 3.3.2, 3.3.2 and 3.3.2 (see (3.73), (3.74), (3.76), (3.78), (3.79) and (3.81)) using as parameters $(\nu, p, s) \in Y_{\mathcal{R}}$ (where we use the identification $p \simeq \delta_p$). By the estimates obtained in Subsection 3.3.2 the next result holds true.

Proposition 3.3.17 *Suppose that $\rho_1 \in (4k\pi, 4(k+1)\pi)$, $\rho_2 \in (4\pi, 8\pi)$. Then, we have*

$$J_\rho\big(\Phi_\lambda(\nu, p, s)\big) \to -\infty \quad \text{as } \lambda \to +\infty \quad \text{uniformly in } (\nu, p, s) \in Y_{\mathcal{R}}.$$

The definition of Φ_λ reflects naturally the join element (ν, p, s) in the sense that, once composed with the map $\widetilde{\Psi}$ in (3.19) we obtain a map homotopic to the identity on $Y_{\mathcal{R}}$, see the next section.

3.4 Proof of Theorem 1.1.8

In this section we introduce the variational scheme that we will use to prove Theorem 1.1.8. As we already observed, the case of surfaces with positive genus was obtained in [9]. Therefore, for now on we will consider the case when Σ is homeomorphic to S^2. We will first analyze the topological structure of the set Y in (3.41) and then introduce a suitable min-max scheme.

3.4.1 On the topology of Y when Σ is a sphere

In this subsection we will use the notation $\simeq$ for a homotopy equivalence and $\cong$ for an isomorphism. Consider the topological join $X = S^2_k * S^2$ (observe that S^2_1 is homeomorphic to S^2) and recall the definition of its subset S given in (3.42), that is

$$S = \left\{ \left(\nu, \delta_y, \frac{1}{2}\right) \in S^2_k * S^2 \ : \ \nu \in S^2_k \setminus (S^2_{k-1})^\delta, \ y \in supp(\nu) \right\},$$

where we have set

$$(S^2_{k-1})^\delta = \left\{ \nu \in S^2_k \ : \ \nu = \sum_{i=1}^k t_i \delta_{x_i} \ ; \ d(x_i, x_j) < \delta \text{ for some } i \neq j \right\} \cup$$
$$\cup \left\{ \nu \in S^2_k \ : \ \nu = \sum_{i=1}^k t_i \delta_{x_i} \ ; \ t_i < \delta \text{ for some } i \right\} \cup \left\{ \nu \in S^2_k \ : \ \nu = \sum_{i=1}^k t_i \delta_{x_i} \ ; \ t_i > 1 - \delta \text{ for some } i \right\}.$$

Notice that S is a smooth manifold of dimension $3k-1$, with boundary of dimension $3k-2$.

The key point of this subsection is to prove that the complementary subspace $Y = (S^2_k * S^2) \setminus S$ is not contractible, see Proposition 3.4.6. Before we do so, we establish some properties of Y and S. Below, U_δ will represent an open neighborhood of S not meeting $(S^2_{k-1})^\delta * S^2$ with the property that $\overline{U}_\delta$ is a manifold with boundary $\partial \overline{U}_\delta$, where both U_δ and $\overline{U}_\delta$ deformation retract onto S and such that $\overline{U}_\delta \setminus S$ deformation retracts onto $\partial \overline{U}_\delta$ (see Figure 3.7).

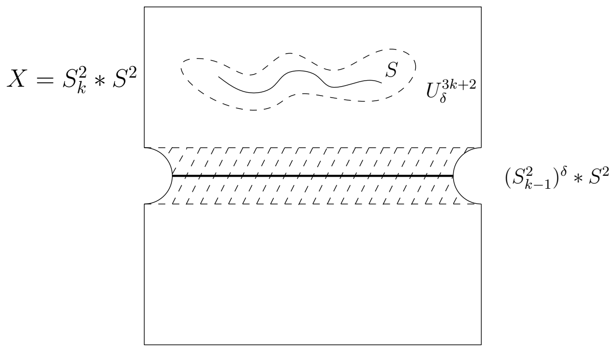

Figure 3.7: Here $X = S^2_k * S^2$ is the ambient, $(S^2_{k-1})^\delta * S^2$ is a neighborhood of $S^2_{k-1} * S^2$ in X, S misses this neighborhood and U_δ is a neighborhood of S in that complement.

For a metric space $\mathcal{X}$, throughout this subsection we use the notation for the k-tuples in $\mathcal{X}$

$$F(\mathcal{X}, k) := \{(x_1, \ldots, x_k) \in \mathcal{X}^k \mid x_i \neq x_j, i \neq j\}$$

and $B(\mathcal{X}, n)$ to denote its quotient by the permutation action of the symmetric group. These are respectively the ordered and unordered k-th *configuration spaces* of $\mathcal{X}$.

Lemma 3.4.1 *S is up to homotopy equivalence a degree-k covering of $B(S^2, k)$. Its homological dimension is at most k and its mod-2 homology is completely described by*

$$H_*(S) \cong H_*(S^2) \otimes H_*(B(\mathbb{R}^2, k-1)).$$

PROOF. The barycentric set S^2_k is a suitable quotient of

$$\Delta_{k-1} \times_{\mathfrak{S}_k} (S^2)^k,$$

with $\mathfrak{S}_k$ acting diagonally by permutations and $\Delta_{k-1} = \{(t_0, \ldots, t_k)\ t_i \in [0,1], \sum t_i = 1\}$. The identification occurs when $x_i = x_j$ for some $i \neq j$ or when $t_i = 0$ for some i. When this happens we are identifying points in S^2_{k-1}. This means that if $\dot{\Delta}_{k-1}$ is the open simplex, then

$$S^2_k \setminus S^2_{k-1} = \dot{\Delta}_{k-1} \times_{\mathfrak{S}_k} F(S^2, k), \tag{3.85}$$

where $F(S^2, k)$ is the configuration space of k distinct points on S^2. The action of $\mathfrak{S}_k$ on $F(S^2, k)$ is free, so we have a bundle

$$\dot{\Delta}_{k-1} \times_{\mathfrak{S}_k} F(S^2, k) \to B(S^2, k),$$

where $B(S^2, k) := F(S^2, k)/\mathfrak{S}_k$ is the configuration of k-unordered points on S^2. The preimages, being copies of the simplex, are contractible so that necessarily

$$S^2_k \setminus S^2_{k-1} \simeq B(S^2, k).$$

In fact $\{\frac{1}{k}\}$ maps to $\dot{\Delta}_{k-1}$ with image $(\frac{1}{k}, \ldots, \frac{1}{k})$ and the induced map

$$B(S^2, k) = \left\{\frac{1}{k}\right\} \times_{\mathfrak{S}_k} F(S^2, k) \to \dot{\Delta}_{k-1} \times_{\mathfrak{S}_k} F(S^2, k)$$

is an equivalence. To summarize, S can be deformed onto the subspace

$$W_k = \{([x_1, \ldots, x_k], x) \in B(S^2, k) \times S^2 \mid x = x_i \text{ for some } i\}.$$

By projecting W_k onto $B(S^2, k)$ we get a covering. This implies that the homological dimension hd of W_k is that of $B(S^2, k)$, which is also the homological dimension of its covering space $F(S^2, k)$. We claim that this dimension is at most k. The projection onto the first coordinate $F(S^2, k) \to S^2$ is a bundle map with fiber $F(\mathbb{R}^2, k-1)$, so $hd(F(S^2, k)) \leq 2 + hd(F(\mathbb{R}^2, k-1))$. Since we also have a fibration $F(\mathbb{R}^2, k-1) \to F(\mathbb{R}^2, k-2)$ given by projecting onto the first $(k-2)$-entries, with fiber a copy of $\mathbb{R}^2 \setminus \{x_1, \ldots, x_{k-2}\}$ which is a bouquet of circles, the claim follows immediately by induction, knowing that $F(\mathbb{R}^2, 2) \simeq S^1$.

Note that we can identify W_k with the quotient $F(S^2, k)/\mathfrak{S}_{k-1}$ where the symmetric group acts on the first $(k-1)$-coordinates. In particular in the case $k = 2$, $S \simeq W_2 = F(S^2, 2) \simeq S^2$.

By projecting W_k onto S^2 via the last coordinate, we get a bundle with fiber $B(\mathbb{R}^2, k-1)$. Let us look at the inclusion of the fiber over $\{\infty\} \in S^2 = \mathbb{R}^2 \cup \{\infty\}$ in this bundle

$$B(\mathbb{R}^2, k-1) \hookrightarrow W_k = F(S^2, k)/\mathfrak{S}_{k-1},$$

$$[x_1, \ldots, x_{k-1}] \mapsto ([x_1, \ldots, x_{k-1}], \infty).$$

Let S^∞ be the direct union of the S^n's under inclusion: this is a contractible space. Now S^2 embeds in S^∞ and we have a map of quotients

$$F(S^2, k)/\mathfrak{S}_{k-1} \to F(S^\infty, k)/\mathfrak{S}_{k-1}.$$

The space on the right-hand side projects onto S^∞ with fiber $B(\mathbb{R}^\infty, k-1)$. Since the base space is contractible, there is a homotopy equivalence $F(S^\infty, k)/\mathfrak{S}_{k-1} \simeq B(\mathbb{R}^\infty, k-1)$. Let us consider the composition

$$B(\mathbb{R}^2, k-1) \xrightarrow{\iota} W_k = F(S^2, k)/\mathfrak{S}_{k-1} \to B(\mathbb{R}^\infty, k-1). \tag{3.86}$$

This composition is homotopic to the map induced on configuration spaces from the inclusion $\mathbb{R}^2 \subset \mathbb{R}^\infty$. It is a known useful fact that each embedding $B(\mathbb{R}^n, k) \hookrightarrow B(\mathbb{R}^{n+1}, k)$ induces a monomorphism in mod-2 homology[1]. In the case $k = 2$ for example, this is $B(\mathbb{R}^n, 2) \simeq \mathbb{R}P^{n-1} \to B(\mathbb{R}^{n+1}, 2) \simeq \mathbb{R}P^n$. This then implies that $B(\mathbb{R}^2, k-1) \hookrightarrow B(\mathbb{R}^\infty, k-1)$ induces in homology mod-2 a monomorphism as well, which then means that the first portion of the composition in (3.86), which is inclusion of the fiber, injects

[1]This follows from the work of F. Cohen [26] who first calculated $H_*(B(\mathbb{R}^n, k); \mathbb{F})$ for all n, k, and for $\mathbb{F} = \mathbb{Z}_2, \mathbb{Z}_p$, p odd.

in homology. Consider the Wang long exact sequence in homology associated to the bundle $W_k \to S^2$ (Theorem 2.5 in [74]):

$$H_{q+1}(W_k) \to H_{q-n+1}(B(\mathbb{R}^2, k-1)) \to H_q(B(\mathbb{R}^2, k-1)) \xrightarrow{\iota_*} H_q(W_k) \to H_{q-n}(B(\mathbb{R}^2, k-1))$$

with $n = 2$ in our case. Since ι_* is a monomorphism, the long exact sequence splits into short exact sequences and because we are working over a field, $H_q(W_k) \cong H_q(B(\mathbb{R}^2, k-1)) \oplus H_{q-2}(B(\mathbb{R}^2, k-1))$. Since $H_*(W_k) \cong H_*(S)$, the proof is complete. ■

Remark 3.4.2 The top mod-2 homology group $H_k(S)$ is trivial if $k-1$ is not a binary power and is a copy of $\mathbb{Z}_2$ if $k-1$ is a binary power. By Lemma 3.4.1, this is because $H_{k-2}(B(\mathbb{R}^2, k-1))$ satisfies the same condition ([41], p. 146).

Lemma 3.4.3 *Suppose $k \geq 3$. The manifold S defined in* (3.42) *is non-orientable.*

Proof. We first observe that the manifold $S_k^2 \setminus S_{k-1}^2$ is not orientable for any $k \geq 2$. From the proof of Lemma 3.4.1

$$S_k^2 \setminus S_{k-1}^2 = \dot{\Delta}_{k-1} \times_{\mathfrak{S}_k} F(S^2, k)$$

is a bundle over $B(S^2, k)$ with fiber the open simplex. Since $B(S^2, k)$ is orientable (because unordered configuration spaces of smooth manifolds are orientable if and only if the dimension of the manifold is even), the orientability of the total space is the same as the orientability of the bundle. But the braids generators of the fundamental group of $B(S^2, k)$ act (after restriction to the open simplex) by transpositions on the vertices of $\overline{\Delta}_{k-1}$ and this is orientation reversing, so the bundle is not orientable.

Now let V_k be the subset of $S_k^2 \setminus S_{k-1}^2$ of all sums $\sum t_i \delta_{x_i}$ with $x_i = \{\infty\}$ for some i. Again $\{\infty\}$ stands for the north pole of $S^2 = \mathbb{R}^2 \cup \{\infty\}$. Here $V_k \simeq B(\mathbb{R}^2, k-1)$. Note that $\pi_1(B(\mathbb{R}^2, k-1))$ embeds in $\pi_1(B(S^2, k))$ with similar braid generators. For the exact same reason as for $S_k^2 \setminus S_{k-1}^2$, V_k is not orientable.

Consider finally the manifold

$$\mathbf{S} = \left\{ \left(\nu, \delta_y, \frac{1}{2}\right) \in S_k^2 * S^2 \ : \ \nu \in S_k^2 \setminus S_{k-1}^2, \ y \in supp(\nu) \right\}.$$

Then S is a codimension 0 submanifold of $\mathbf{S}$ (with boundary) which is also a deformation retract. Both S and $\mathbf{S}$ have the same orientation. But there is a bundle map $\mathbf{S} \to S^2$ with fiber V_k. It is easy to see now that the orientation of $\mathbf{S}$ is that of V_k. Indeed the bundle over the open upper hemisphere D of S^2 is trivial homeomorphic to $V_k \times D$. This is an open subset of $\mathbf{S}$ which is non-orientable, thus $\mathbf{S}$ must be non-orientable. ■

Lemma 3.4.4 *Let $k \geq 3$. Then Y has the Euler characteristic of a contractible space, i.e. $\chi(Y) = 1$.*

Proof. By the previous lemma, S is up to homotopy a degree-k covering of $B(S^2, k)$. This gives that

$$\chi(S) = k\chi(B(S^2, k)) = k\frac{1}{k!}\chi(F(S^2, k)) \ = \ \frac{1}{(k-1)!}\chi(S^2)\chi(F(\mathbb{R}^2, k-1)) \ = \ 0.$$

Here what vanishes is $\chi(F(\mathbb{R}^2, k-1)) = 0$ since, letting $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$, there are homeomorphisms

$$F(\mathbb{R}^2, k-1) = \mathbb{R}^2 \times F\big(\mathbb{R}^2 \setminus \{(0,0)\}, k-2\big) = \mathbb{R}^2 \times \mathbb{C}^* \times F\big(\mathbb{C}^* \setminus \{1\}, k-3\big)$$

and $\chi(\mathbb{C}^*) = \chi(S^1) = 0$.

On the other hand, S is a smooth $(3k-1)$-dimensional manifold with boundary. A neighborhood of S in $S_k^2 * S^2$ is a $(3k+2)$-dimensional open manifold U_δ. This neighborhood is the union of two open subspaces A and B, where A is a fiberwise cone over the interior of S and B is a bundle over ∂S with fiber the cone over a hemisphere. The complement $\overline{U}_\delta \setminus S$ is the union of two subspaces $\tilde{A}$ and $\tilde{B}$, where $\tilde{A}$ retracts onto an S^2-bundle over the interior of S, while $\tilde{B}$ is up to homotopy ∂S. Clearly $\tilde{A} \cap \tilde{B}$ retracts onto an S^2-bundle over ∂S. We can then write

$$\begin{aligned} \chi(U_\delta \setminus S) \ = \ \chi(\tilde{A} \cup \tilde{B}) \ &= \ \chi(\tilde{A}) + \chi(\tilde{B}) - \chi(\tilde{A} \cap \tilde{B}) = 2\chi(S) + \chi(\partial S) - 2\chi(\partial S) \\ &= \ 2\chi(S) - \chi(\partial S). \end{aligned}$$

We know that for a manifold S of dimension m with boundary it holds

$$\chi(\partial S) = \chi(S) - (-1)^m \chi(S).$$

If $m = 3k - 1$ is odd, then $\chi(\partial S) = 2\chi(S)$ and so $\chi(U_\delta \setminus S) = 0$. If m is even, ∂S is odd dimensional closed and its Euler characteristic is null. But $\chi(S) = 0$ and here again $\chi(U_\delta \setminus S) = 0$.

Now cover $X = S^2_k * S^2$ by means of $U_\delta \simeq S$ and $Y = X \setminus S$. The universal property of the Euler characteristic gives that

$$\chi(X) = \chi(U_\delta) + \chi(Y) - \chi(U_\delta \setminus S) = \chi(S) + \chi(Y) = \chi(Y),$$

so that $\chi(Y) = \chi(X) = 1$ as claimed. The second equality follows from the fact that $\chi(X) = \chi(S^2_k * S^2) = \chi(S^2_k) + \chi(S^2) - \chi(S^2_k)\chi(S^2)$ and that

$$\chi(Z_k) = 1 - \frac{1}{k!}(1-\chi)(2-\chi)\cdots(k-\chi)$$

for any surface Z, see [67], and more generally for any simplicial complex Z, see [54], with $\chi = \chi(Z)$. ■

Lemma 3.4.5 *The set Y is simply connected.*

Proof. Using the same notation as in the proof of the previous lemma, we have the push-out

$$\begin{tikzcd} \tilde{A} \cap \tilde{B} \arrow[r] \arrow[d] & \tilde{A} \arrow[d] \\ \tilde{B} \arrow[r] & U_\delta \setminus S \end{tikzcd}$$

Recall that $\tilde{A}$ is up to homotopy an S^2-bundle over S, $\tilde{B} \simeq \partial S$ and that $\tilde{A} \cap \tilde{B}$ is an S^2-bundle over ∂S. This means that $\pi_1(\tilde{A} \cap \tilde{B}) = \pi_1(\partial S)$ and $\pi_1(\tilde{A}) \cong \pi_1(S)$. We therefore have the following push-out in the category of groups (by the Van-Kampen theorem):

$$\begin{tikzcd} \pi_1(\partial S) \arrow[r] \arrow[d, "\cong"] & \pi_1(S) \arrow[d] \\ \pi_1(\partial S) \arrow[r] & \pi_1(U_\delta \setminus S) \end{tikzcd}$$

which shows that $\pi_1(U_\delta \setminus S) \cong \pi_1(S)$. On the other hand we can use the same open covering of $X = S^2_k * S^2$ by U_δ and $Y = X \setminus S$. Since X is a join of connected spaces, it is 1-connected. The push-out of groups

$$\begin{tikzcd} \pi_1(U_\delta \setminus S) \arrow[r] \arrow[d, "\cong"] & \pi_1(X \setminus S) \arrow[d] \\ \pi_1(U_\delta) \arrow[r] & 0 \end{tikzcd}$$

implies that because the left-hand vertical map is an isomorphism, then so is the right-hand vertical map and $\pi_1(X \setminus S) = \pi_1(Y) = 0$. ■

Despite the fact that Y is simply connected and has unit Euler characteristic, it is not contractible.

Proposition 3.4.6 *Suppose $k \geq 2$, $k \neq 4$. Then the set*

$$Y = (S^2_k * S^2) \setminus S$$

is not contractible.

Proof. We assume that Y is contractible and derive a contradiction. The main step is to prove that under this condition with mod-2 coefficients we must have

$$H_*(S) \cong H_{3k-1-*}(S^2_k), \qquad 0 \le * \le k. \tag{3.87}$$

This will then be shown to be impossible.

The closed subset S has a neighborhood U_δ which is $(3k+2)$-dimensional with $(3k+1)$-dimensional boundary $\partial\overline{U}_\delta$. Using Poincaré's duality with mod-2 coefficients for the closed manifold $\partial\overline{U}_\delta$ gives us

$$H^*(\partial\overline{U}_\delta) \cong H_{3k+1-*}(\partial\overline{U}_\delta).$$

Since $\overline{U}_\delta \setminus S$ retracts onto $\partial\overline{U}_\delta$, and homology is dual to cohomology for finite type spaces and field coefficients, we can conclude that

$$H_*(\overline{U}_\delta \setminus S) \cong H_{3k+1-*}(\overline{U}_\delta \setminus S), \qquad * \ge 0. \tag{3.88}$$

Next we turn to the open covering of $X = S^2_k * S^2$ by U_δ and $Y = X \setminus S$. Using that $Y \cap U_\delta = U_\delta \setminus S$ and $U_\delta \simeq S$, the Mayer-Vietoris sequence for this union takes the form

$$H_*(U_\delta \setminus S) \to H_*(S) \oplus H_*(Y) \to H_*(X) \to H_{*-1}(U_\delta \setminus S) \to H_{*-1}(S) \oplus H_{*-1}(Y) \to H_{*-1}(X) \to \cdots$$

Since Y has trivial reduced homology by assumption, the sequence becomes

$$H_*(U_\delta \setminus S) \to H_*(S) \to H_*(X) \to H_{*-1}(U_\delta \setminus S) \to H_{*-1}(S) \to H_{*-1}(X) \to \cdots \tag{3.89}$$

But S has homological dimension k (see Lemma 3.4.1), so for $* > k+1$ we have the isomorphism $H_{*-1}(U_\delta \setminus S) \cong H_*(X)$. Since X is the third suspension of S^2_k, $H_*(X) \cong H_{*-3}(S^2_k)$ and thus

$$H_*(U_\delta \setminus S) \cong H_{*-2}(S^2_k), \quad * > k. \tag{3.90}$$

It is known generally (see [54]) that the barycentric set Z_k is $(2k+r-2)$-connected whenever Z is r-connected, $r \ge 1$. If $Z = S^2$, which is 1-connected, S^2_k is $(2k-1)$-connected and so X is $(2k+2)$-connected. In the range $* \le 2k+2$, $\tilde{H}_*(X) = 0$. The Mayer-Vietoris sequence (3.89) leads in this case to

$$H_*(U_\delta \setminus S) \cong H_*(S), \quad * < 2k+2.$$

Since S has no homology beyond degree k, we can focus on the range below so that

$$H_*(U_\delta \setminus S) \cong H_*(S), \quad 0 \le * \le k. \tag{3.91}$$

We can now combine all previous isomorphisms into one for $0 \le * \le k$

$$H_*(S) \xrightarrow[(3.91)]{\cong} H_*(U_\delta \setminus S) \xrightarrow[(3.88)]{\cong} H_{3k+1-*}(U_\delta \setminus S) \xrightarrow[(3.90)]{\cong} H_{3k-1-*}(S^2_k).$$

This is the claim in (3.87). Note that S^2_k is $(3k-1)$-dimensional as a CW-complex and is $(2k-1)$-connected, so its homology is non-zero only in the range $2k \le * \le 3k-1$.

The isomorphism $H_*(S) \cong H_{3k-1-*}(S^2_k)$ cannot hold. First let us check the case $k=2$. In that case we pointed out in the proof of Lemma 3.4.1 that $S \simeq F(S^2, 2) \simeq S^2$. Since $S^2_2 \simeq \Sigma^3\mathbb{R}P^2$ (the 3-fold suspension of $\mathbb{R}P^2$: see [54], Corollary 1.6), the isomorphism obviously cannot hold: in fact $H_1(S^2) = 0$ but $H_4(\Sigma^3\mathbb{R}P^2) = H_1(\mathbb{R}P^2) = \mathbb{Z}_2$.

Suppose that $k \ge 3$. According to Theorem 1.3 in [54], S^2_k has the same homology as (one desuspension) of the symmetric smash product $\overline{\mathrm{SP}}^k(S^3) = (S^3)^{\wedge k}/\mathfrak{S}_k$; i.e. $H_*(S^2_k) \cong H_{*+1}(\overline{\mathrm{SP}}^k(S^3))$. Combining this with (3.87) we get

$$H_*(S) \cong H_{3k-*}(\overline{\mathrm{SP}}^k(S^3)), \qquad 0 \le * \le k. \tag{3.92}$$

We will show that this is impossible. To that end we need describe the groups on both sides of (3.92). We work again mod-2. From Lemma 3.4.1 we have that

$$H_*(S) \cong H_*(B(\mathbb{R}^2, k-1)) \oplus H_{*-2}(B(\mathbb{R}^2, k-1)), \quad * \ge 0.$$

(when $*-2<0$ the corresponding group is zero). The mod-2 homology of $B(\mathbb{R}^2,k-1)$ has been computed by D.B. Fuks in [41] and it is best described as a subspace of the polynomial algebra (viewed as an infinite vector space generated by powers of the indicated generators)

$$\mathbb{Z}_2[a_{(1,2)},a_{(3,4)},\cdots,a_{(2^i-1,2^i)},\cdots], \tag{3.93}$$

where the notation $a_{i,j}$ refers to a generator having homological degree i and a certain *filtration* degree j, both degrees being additive under multiplication of generators. Now the condition for an element $a^{k_1}_{(2^{i_1}-1,2^{i_1})}\cdots a^{k_r}_{(2^{i_r}-1,2^{i_r})}\in H_*(B(\mathbb{R}^2,k-1))$ is that its filtration degree is less or equal than $k-1$; that is if and only if $\sum_{i_s}k_{i_s}2^{i_s}\le k-1$.

For example $\tilde H_*(B(\mathbb{R}^2,2))=\mathbb{Z}_2\{a_{(1,2)}\}$ (one copy of $\mathbb{Z}_2$ generated by $a_{(1,2)}$ having homological degree one and filtration degree two). Similarly $\tilde H_*(B(\mathbb{R}^2,4))=\mathbb{Z}_2\{a_{(1,2)},a^2_{(1,2)},a_{(3,4)}\}$, so that

$$H_1(B(\mathbb{R}^2,4))=\mathbb{Z}_2\{a_{(1,2)}\},\quad H_2(B(\mathbb{R}^2,4))=\mathbb{Z}_2\{a^2_{(1,2)}\},\quad H_3(B(\mathbb{R}^2,4))=\mathbb{Z}_2\{a_{(3,4)}\}.$$

Now $H_*(B(\mathbb{R}^2,5))\cong H_*(B(\mathbb{R}^2,4))$ and this turns out to be a general fact that is explained in Lemma 3.4.9 in more geometric terms.

On the other hand, the reduced groups $\tilde H_*(\overline{\mathrm{SP}}^k(S^3))$ form a subvector space of the polynomial algebra

$$\mathbb{Z}_2[\iota_{(3,1)},f_{(5,2)},f_{(9,4)},\ldots,,f_{(2^{i+1}+1,2^i)},\ldots,] \tag{3.94}$$

consisting of those elements of second filtration degree precisely k (see the Appendix in [54] and references therein). Here again $f_{(2^{i+1}+1,2^i)}$ denotes an element of homological degree $2^{i+1}+1$ and filtration degree 2^i. For example (here $\iota=\iota_{(3,1)}$)

$$\tilde H_*(\overline{\mathrm{SP}}^4S^3)=\mathbb{Z}_2\{\iota^4,\iota^2f_{(5,2)},f^2_{(5,2)},f_{(9,4)}\},$$

which is better listed as follows:

$$H_{12}(\overline{\mathrm{SP}}^4S^3)=\mathbb{Z}_2\{\iota^4\},\quad H_{11}(\overline{\mathrm{SP}}^4S^3)=\mathbb{Z}_2\{\iota^2f_{(5,2)}\},$$

$$H_{10}(\overline{\mathrm{SP}}^4S^3)=\mathbb{Z}_2\{f^2_{(5,2)}\},\quad H_9(\overline{\mathrm{SP}}^4S^3)=\mathbb{Z}_2\{f_{(9,4)}\}.$$

This space $\overline{\mathrm{SP}}^4(S^3)$ is 8-connected, and more generally $\overline{\mathrm{SP}}^k(S^3)$ is $2k$-connected, see [54].

Let us now compare the groups in (3.92). When $*=0$, $H_0(S)=\mathbb{Z}_2$ but so is $H_{3k}(\overline{\mathrm{SP}}^k(S^3))$ generated by the class $\iota^k_{(3,1)}$. Also when $*=1$, $k\ge3$, $H_1(S)=H_1(B(\mathbb{R}^2,k-1))=\mathbb{Z}_2$ but so is $H_{3k-1}(\overline{\mathrm{SP}}^k(S^3))$ generated by $\{\iota^{k-2}f_{5,2}\}$. There is no contradiction yet. When $*=2$, we get the generator $a^2_{(1,2)}\in H_2(B(\mathbb{R}^2,k-1))\cong\mathbb{Z}_2$ as soon as $k\ge5$ ($a^2_{(1,2)}$ is in filtration 4). This gives that $H_2(S)=\mathbb{Z}_2\oplus\mathbb{Z}_2$. We claim however that $H_{3k-2}(\overline{\mathrm{SP}}^k(S^3))=\mathbb{Z}_2$, which will give a contradiction in that case. Indeed a generator in filtration degree k in (3.94) is written as a finite product

$$\iota^{k_0}f^{k_1}_{5,2}\cdots f^{k_i}_{(2^{i+1}+1,2^i)}\cdots,\qquad\sum_{i\ge0}k_i2^i=k.$$

The homological degree of this class is $\sum_{i\ge0}k_i(2^{i+1}+1)=2\sum_{i\ge0}k_i2^i+\sum_{i\ge0}k_i$. To obtain the rank of H_{3k-2} we need to find all the possible sequences of integers $(k_0,k_1,k_2,\ldots)$ such that $\sum_{i\ge0}k_i2^i=k$ and $2\sum_{i\ge0}k_i2^i+\sum_{i\ge0}k_i=3k-2$. We have to solve for

$$\sum_{i\ge0}k_i2^i=k=2+\sum_{i\ge0}k_i.$$

This immediately gives that $k_i=0,i\ge2$. There is one and only one solution: $k_0=k-4$ and $k_1=2$; and the group $H_{3k-2}(\overline{\mathrm{SP}}^k(S^3))\cong\mathbb{Z}_2$ is generated by $\iota^{k-4}f^2_{5,2}$.

The isomorphism (3.92) cannot hold for $k\ge5$. We are left to consider the cases $k=3$: here $H_3(S)=\mathbb{Z}_2$ but $H_6(\overline{\mathrm{SP}}^3(S^3))=0$ giving a contradiction.

In conclusion since the isomorphism (3.92) (equivalently (3.87)) cannot hold, Y must have non trivial mod-2 homology and thus cannot be contractible as we had asserted. ■

The next proposition treats the case $k = 4$: in preparation we need the following lemma. Recall that S is a manifold with boundary embedded in $\overline{U}_\delta \subset S^2_k * S^2$. We can write $\overline{U}_\delta$ as the union of two sets $\overline{A}$ and $\overline{B}$, where $\overline{A}$ is a three-dimensional-disk-bundle over S and $\overline{A} \cap \overline{B}$ its restriction over ∂S. We refer to this bundle as the *normal disk bundle* and its boundary as the *sphere normal bundle*. Note that in the proof of Lemma 3.4.4, we have used $\tilde{A} = \overline{A} \setminus S$ and $\tilde{B} = \overline{B} \setminus S$.

Lemma 3.4.7 *The sphere normal bundle over ∂S is orientable.*

PROOF. We will view this bundle as an extension of a normal sphere bundle over the interior $\dot{S}$:=int(S) which is orientable (in so doing we give more details on the construction of $\overline{A}$ and $\overline{A} \cap \overline{B}$).

We recall that the join is given by the equivalence relation $X * Y = X \times Y \times I/_\sim$, where $\sim$ are identifications at the endpoints of $I = [0, 1]$, see (1.17). The join contains the open dense subset $X \times Y \times (0, 1)$ (let us call it the *big cell*). This subset is a manifold of dimension $n + m + 1$ if X, Y are manifolds of dimensions n and m, respectively. In our case S is a subset of the big cell

$$(S^2_k \setminus (S^2_{k-1})^\delta) \times S^2 \times (0, 1) \subset (S^2_k \setminus (S^2_{k-1})^\delta) * S^2$$

and int(S) is regularly embedded as a differentiable submanifold. It has therefore a unit normal disk bundle (of dimension 3) in there. This is homeomorphic to a tubular neighborhood V^δ of int(S). Let us use the same name for the neighborhood and the normal bundle. The normal bundle of $\dot{S}$ in $(S^2_k \setminus (S^2_{k-1})^\delta) \times S^2 \times (0, 1)$ is the normal bundle of $\dot{S}$ in $(S^2_k \setminus (S^2_{k-1})^\delta) \times S^2 \times \{\frac{1}{2}\}$ to which we add a trivial line bundle. We can then consider directly $\dot{S}$ as a subset of $(S^2_k \setminus (S^2_{k-1})^\delta) \times S^2$ and show that it has an orientable rank 2 normal bundle there. Write $D_k := S^2_k \setminus (S^2_{k-1})^\delta$ and

$$S = \left\{ \left(\sum_{i=1}^k t_i \delta_{x_i}, x \right) \in D_k \times S^2, x = x_i \text{ for some } i \right\}.$$

Define V^δ the neighborhood of S in $D_k \times S^2$ as follows:

$$V^\delta = \left\{ \left(\sum_{i=1}^k t_i \delta_{x_i}, x \right) \in D_k \times S^2, |x - x_i| < \frac{\delta}{2} \text{ for some and hence unique } x_i \right\}.$$

The choice of x_i is unique as x cannot be strictly within $\delta/2$ from two distinct x_i, x_j since $d(x_i, x_j) \geq \delta$ according to the definition of S. The neighborhood retracts back to S via the map

$$\left(\sum_{i=1}^k t_i \delta_{x_i}, x \right) \mapsto \left(\sum_{i=1}^k t_i \delta_{x_i}, x_i \right),$$

where $d(x, x_i) < \delta/2$. Consider the projection map $\pi : \dot{S} \to S^2$ sending $\left(\sum_{i=1}^k t_i \delta_{x_i}, x \right) \mapsto x$. We claim that the normal bundle of $\dot{S}$ in $D_k \times S^2$ is isomorphic to the pullback via π of the tangent bundle TS^2 over S^2. We assume δ to be less than the injectivity radius of S^2. Define a homeomorphism between the tubular neighborhood V^δ of $\dot{S}$ and a normal disk bundle of the pullback of TS^2 over $\dot{S}$ by sending $\left(\sum_{i=1}^k t_i \delta_{x_i}, x \right)$ with $|x - x_i| < \delta$ for some i to the element in the pullback

$$\left(\left(\sum_{i=1}^k t_i \delta_{x_i}, x \right), v_i \right),$$

where $v_i = exp^{-1}_{x_i}(x)$ and exp_{x_i} is the exponential map at $x_i \in S^2$. This map is a homeomorphism onto its image and the normal bundle to $\dot{S}$ in $D_k \times S^2$ is isomorphic to TS^2. Since TS^2 is orientable (although non trivial), the normal bundle over $\dot{S}$ is orientable. This bundle can be extended to S by taking the closure of V^δ in $D_k \times S^2 := (S^2 \setminus (S^2_{k-1})^\delta) \times S^2 \times \{\frac{1}{2}\}$. This extension is orientable over all of S since it is orientable over the interior. By adding a line bundle we get the disk bundle over S in the big cell (which we have labeled $\overline{A}$). This bundle is orientable over all of S and in particular over ∂S. This is our claim. ■

Proposition 3.4.8 *The set $Y = (S^2_4 * S^2) \setminus S$ is not contractible.*

Proof. As before we assume Y is contractible and derive a contradiction. We first show that for any field coefficients $\mathbb{F}$ and $* > k$

$$H_{*+3}(U_\delta \setminus S) \cong H_*(\partial S). \tag{3.95}$$

Write as before $\overline{U}_\delta \setminus S$ as the union $\tilde{A} \cup \tilde{B}$, with $\tilde{A} \cap \tilde{B}$ retracting onto the S^2-bundle over ∂S discussed earlier. The Mayer-Vietoris sequence for the union $\tilde{A} \cup \tilde{B}$ is given by

$$H_{n+1}(\tilde{A} \cap \tilde{B}) \to H_{n+1}(\tilde{A}) \oplus H_{n+1}(\tilde{B}) \to H_{n+1}(U_\delta \setminus S) \to H_n(\tilde{A} \cap \tilde{B}) \to H_n(\tilde{A}) \oplus H_n(\tilde{B}) \to H_n(U_\delta \setminus S).$$

As S has homological dimension at most k and $\tilde{A}$ is an S^2-bundle over it, $H_n(\tilde{A})$ vanishes for $n > k+2$. On the other hand, the S^2-bundle over ∂S is orientable (Lemma 3.4.7) and has a global section given by the variation in the s-parameter (defining the join). By the Gysin sequence ([43],§4.D) one has a splitting

$$H_n(\tilde{A} \cap \tilde{B}) \simeq H_n(\partial S) \oplus H_{n-2}(\partial S).$$

Replacing in the Mayer-Vietoris sequence gives for $n > k+2$

$$\cdots \longrightarrow \begin{matrix} H_{n+1}(\partial S) \\ \oplus \\ H_{n-1}(\partial S) \end{matrix} \xrightarrow{\phi_{n+1}} H_{n+1}(\partial S) \longrightarrow H_{n+1}(U_\delta \setminus S) \longrightarrow \begin{matrix} H_n(\partial S) \\ \oplus \\ H_{n-2}(\partial S) \end{matrix} \xrightarrow{\phi_n} H_n(\partial S) \longrightarrow \cdots$$

Now, in every inclusion of $\tilde{A} \cap \tilde{B}$ into $\tilde{B}$, the fibers (i.e. S^2) contract to a point. Therefore ϕ_n is trivial on the bottom group, while restricted to the top group it is a bijection. This map is an epimorphism and the long exact sequence for $n > k+2$ splits into short exact sequences

$$0 \to H_{n+1}(U_\delta \setminus S) \to H_n(\partial S) \oplus H_{n-2}(\partial S) \to H_n(\partial S) \to 0.$$

As vector spaces we get $H_{n+1}(U_\delta \setminus S) \cong H_{n-2}(\partial S)$ which is our claim. Combined with (3.90) this yields

$$H_*(\partial S) \cong H_{*+1}(S^2_k), \quad * > k. \tag{3.96}$$

Next we look at the Mayer-Vietoris sequence for the union $S^2_k = (S^2_k \setminus S^2_{k-1}) \cup (S^2_{k-1})^\delta$. It is shown in [67] that $(S^2_{k-1})^\delta \setminus S^2_{k-1}$ retracts onto $\partial (S^2_{k-1})^\delta$ so that the long exact sequence becomes

$$\cdots \to H_{n+1}(\partial (S^2_{k-1})^\delta) \to H_{n+1}(S^2_{k-1}) \oplus H_{n+1}(S^2_k \setminus S^2_{k-1}) \to H_{n+1}(S^2_k) \to H_n(\partial (S^2_{k-1})^\delta) \to \cdots$$

Since the inclusion of S^2_{k-1} in S^2_k is contractible, and since $S^2_k \setminus S^2_{k-1} \simeq B(S^2, k)$ has homological dimension k (see Lemma 3.4.1), for $n > k$ the following short sequence is exact

$$0 \to H_{n+1}(S^2_k) \to H_n(\partial (S^2_{k-1})^\delta) \to H_n(S^2_{k-1}) \to 0$$

and we have the splitting

$$H_*(\partial (S^2_{k-1})^\delta) \cong H_*(S^2_{k-1}) \oplus H_{*+1}(S^2_k), \quad * > k. \tag{3.97}$$

Both isomorphisms (3.96) and (3.97) cannot hold simultaneously as we now explain.

A key point is to observe that ∂S is a degree-k regular covering of $\partial (S^2_{k-1})^\delta$. A property of a covering $\pi : X \to Y$ is the existence of a transfer morphism $tr : H_*(Y) \to H_*(X)$ so that $\pi_* \circ tr$ is multiplication in $H_*(Y)$ by the degree of the covering i.e. by k, see [43], Section 3.G. If the characteristic of the field of coefficients is prime to k, then this composite is not trivial and $H_*(Y)$ injects into $H_*(X)$.

When $k = 4$, we have a degree-4 covering $\partial S \to \partial (S^2_3)^\delta$ so that with $\mathbb{F} = \mathbb{F}_3$-coefficients (the finite field with 3 elements) we must have a monomorphism $H_*(\partial (S^2_3)^\delta; \mathbb{F}_3) \hookrightarrow H_*(\partial S; \mathbb{F}_3)$. When $* > 4$, upon combining (3.96) and (3.97) we get a monomorphism

$$H_*(S^2_3; \mathbb{F}_3) \oplus H_{*+1}(S^2_4; \mathbb{F}_3) \to H_{*+1}(S^2_4; \mathbb{F}_3).$$

This leads immediately to a contradiction if $H_*(S^2_3; \mathbb{F}_3) \neq 0$ in that range of dimensions.

We know that $H_*(S^2_3) \cong H_{*+1}(\overline{\mathrm{SP}}^3(S^3))$. We therefore wish to show that $H_*((\overline{\mathrm{SP}}^3(S^3); \mathbb{F}_3) \neq 0$ for some $* \geq 6$. It turns out that old calculations of Nakaoka give us precisely the answer [76]. Nakaoka's Theorem 15.5 states that

$$H^r(\mathrm{SP}^3(S^n); \mathbb{F}_3) \cong \mathbb{F}_3$$

for $r = 0, n, n+4k$ with $1 \leq k \leq [n/2]$ and $k \neq [n/4]$, $r = n + 4k + 1$ with $1 \leq k \leq [(2n-1)/4]$ and $k \neq [(n-1)/4]$, and $r = 2n$ with $n \equiv -2$ or $1 \pmod 4$. In our case $n = 3$, so $H^r(\mathrm{SP}^3(S^3); \mathbb{F}_3) \cong \mathbb{F}_3$ for $r = 0, 3, 7, 8$. Dually we obtain the same groups for $H_r(\mathrm{SP}^3(S^3); \mathbb{F}_3)$ (since working over a field). But $H_r(\mathrm{SP}^3(S^3); \mathbb{F}_3) \cong H_r(\overline{\mathrm{SP}}^3(S^3); \mathbb{F}_3)$ for $r > 3$ for the following three reasons:

- By construction $H_r(\overline{\mathrm{SP}}^3(S^3);\mathbb{F}_3) = H_r\big(SP^3(S^3), \mathrm{SP}^2(S^3);\mathbb{F}_3\big), \quad r \geq 1$.
- There is a splitting due originally to Steenrod (any coefficients, see [54]):

$$H_r(\mathrm{SP}^3(S^3)) \cong H_r\big(\mathrm{SP}^3(S^3), \mathrm{SP}^2(S^3)\big) \oplus H_r(\mathrm{SP}^2(S^3)).$$

- $H_r(\mathrm{SP}^2(S^3);\mathbb{F}_3) = 0$ if $r > 3$. In fact, from the covering $(S^3)^2 \to \mathrm{SP}^2(S^3)$, by a consequence of the transfer construction, $H_*(\mathrm{SP}^2(S^3);\mathbb{F}_3)$ is the subvector space of invariant cohomology classes in $H_*(S^3 \times S^3)$ under the induced permutation action interchanging the two spheres. Since S^3 is an odd sphere, the involution acts via $\tau_*([S^3]\otimes[S^3]) = -[S^3]\otimes[S^3]$ and the class $[S^3]\otimes[S^3]$ is not invariant so maps to zero in $H_*(\mathrm{SP}^2(S^3);\mathbb{F}_3)$.

As a consequence $H_r(\overline{\mathrm{SP}}^3(S^3);\mathbb{F}_3) \cong \mathbb{F}_3$ for $r = 7, 8$ which gives a contradiction as we had asserted. The proof is complete. ■

Using the above transfer property but with $\mathbb{F}_2$ coefficients, one can find an alternative proof of Proposition 3.4.6 for k odd. To conclude this topological discussion, it is worthwhile noting that Lemma 3.4.1 can be used to give a novel proof of the following result on the mod-2 homology of unordered configurations of points in $\mathbb{R}^n$.

Proposition 3.4.9 *For k odd and $n \geq 2$ one has*

$$H_*(B(\mathbb{R}^n, k);\mathbb{Z}_2) \cong H_*(B(\mathbb{R}^n, k-1);\mathbb{Z}_2).$$

Proof. All homology is with mod-2 coefficients. A starting point is the homology splitting

$$H_q(B(S^n, k)) \cong H_q(B(\mathbb{R}^n, k)) \oplus H_{q-n}(B(\mathbb{R}^n, k-1)). \tag{3.98}$$

One reference to this result is Theorem 18 (1) of [84]. It is also a special case of a similar result of the second author where one can replace the sphere by any closed manifold M and $\mathbb{R}^n$ by $M \setminus \{p\}$ its punctured version. Let $W_{n,k} := F(S^n, k)/\mathfrak{S}_{k-1}$ where $\mathfrak{S}_{k-1}$ acts by permutations on the first $(k-1)$-coordinates. By projecting onto the last coordinate we obtain a bundle over S^n with fiber $B(\mathbb{R}^n, k-1)$. Precisely as in the proof of Lemma 3.4.1, we see that

$$H_*(W_{n,k}) \cong H_*(B(\mathbb{R}^n, k-1)) \oplus H_{*-n}(B(\mathbb{R}^n, k-1)). \tag{3.99}$$

Consider next the degree-k regular covering $\pi : W_{n,k} \to B(S^n, k) := F(S^n, k)/\mathfrak{S}_k$. There is a transfer morphism $tr : H_*(B(S^n, k)) \to H_*(W_{n,k})$ so that the composite $\pi_* \circ tr$ is multiplication by k. Since k is odd and thus prime to the characteristic of the field $\mathbb{Z}_2$, multiplication by k is injective and necessarily $H_*(B(S^n, k))$ embeds in $H_*(W_{n,k})$; that is (3.98) embeds into (3.99). But $H_*(B(\mathbb{R}^n, k-1))$ always embeds into $H_*(B(\mathbb{R}^n, k))$ (in fact for any coefficients as it is relatively easy to see). This means that $H_*(B(\mathbb{R}^n, k);\mathbb{Z}_2) \cong H_*(B(\mathbb{R}^n, k-1);\mathbb{Z}_2)$ if k is odd as claimed. It also means that $H_*(B(S^n, k)) \cong H_*(W_{n,k})$. ■

3.4.2 Min-max scheme

To prove Theorem 1.1.8 we will run a min-max scheme based on (a retraction of) the set Y in (3.41). More precisely, we will consider the set $Y_{\mathcal{R}}$ introduced in (3.84) on which the test functions Φ_λ are modelled. Some parts are quite standard and follow the ideas of [35] (see [67] for a Morse theoretical point of view): for the specific problem (1.18) the crucial step is Proposition 3.4.10, giving information on the topology of the low sub-levels of J_ρ: see also the comments after the proof.

Given any $L > 0$, Proposition 3.3.17 guarantees us the existence of $\lambda > 1$ sufficiently large such that $J_\rho\big(\Phi_\lambda(\nu, p, s)\big) < -L$ for any $(\nu, p, s) \in Y_{\mathcal{R}}$. Recalling $\widetilde{\Psi}$ in (3.19), we take L so large that Corollary 3.2.8 applies, i.e. such that $\widetilde{\Psi}(J_\rho^{-L}) \subseteq Y$. The crucial step in describing the topology of the low sub-levels of J_ρ is the following result.

Proposition 3.4.10 *Let L, λ be as above and let $\mathcal{F}$ be the retraction given before* (3.84). *Then the composition*

$$Y_{\mathcal{R}} \xrightarrow{\Phi_\lambda} J_\rho^{-L} \xrightarrow{\mathcal{F}\circ\widetilde{\Psi}} Y_{\mathcal{R}}$$

is homotopically equivalent to the identity map on $Y_{\mathcal{R}}$.

PROOF. We divide the proof in three cases, depending on the values of the join parameter s.

Case 1. Let $s \in \left[\frac{3}{4}, 1\right]$. In this case the test functions we are considering have the form $(\varphi_1^t, \varphi_2^t)$, $t = t(s)$, as defined in Subsection 3.3.2. Notice that, as discussed at the beginning of the proof of Proposition 3.3.7, most of the integral of $e^{\varphi_2^t}$ is localized near p and $\sigma_2(\varphi_2^t) \ll \sigma_1(\varphi_1^t)$ for these values of s, which again implies $s(\varphi_1^t, \varphi_2^t) = 1$, see (3.16). It turns out that, by the construction in Subsection 3.2.1, one has

$$\widetilde{\Psi}\big(\Phi_\lambda(\nu, p, s)\big) = \widetilde{\Psi}(\varphi_1^t, \varphi_2^t) = (*, \tilde{p}, 1),$$

where $*$ is an irrelevant element of Σ_k (recall that they are all identified when the join parameter equals 1, see (1.17)) and where $\tilde{p} \in \Sigma$ is a point close to p. If $p(\mathfrak{t}) : [0,1] \to \Sigma$ is a geodesic joining p to $\tilde{p}$, one can realize the desired homotopy as

$$\big((\nu, p, s); \mathfrak{t}\big) \mapsto \big(\nu, p(\mathfrak{t}), (1-\mathfrak{t})s + \mathfrak{t}\big), \qquad \mathfrak{t} \in [0,1].$$

Case 2. Let $s \in \left[\frac{1}{4}, \frac{3}{4}\right]$. The test functions we are considering here are given in Subsection 3.3.2. For this range of s the exponential of the first component φ_1 (see (3.72)) is well concentrated around the points $\widetilde{x}_i$, see (3.67). The exponential of the second component φ_2, depending on the value of s, will be instead either concentrated near p or will be spread over Σ in the sense that $\sigma_2(\varphi_2)$ might not be small. Recall the maps $\widetilde{\psi}_l$ given in Proposition 2.4.1 and the definition of $\hat{\nu}$ involved in the construction of the test functions given in (3.65): $\hat{\nu} = \mathcal{R}_p(\nu) = \sum_{i=i}^k t_i \delta_{x_i}$. We then have

$$\widetilde{\Psi}\big(\Phi_\lambda(\nu, p, s)\big) = \widetilde{\Psi}(\varphi_1, \varphi_2) = \begin{cases} \Big(\widetilde{\psi}_k(\varphi_1), \widetilde{\psi}_1(\varphi_2), s(\varphi_1, \varphi_2)\Big) & \text{if } \sigma_2(\varphi_2) \text{ small,} \\ \Big(\widetilde{\psi}_k(\varphi_1), *, 0\Big) & \text{otherwise,} \end{cases}$$

with $\widetilde{\psi}_1(\varphi_2)$ close to p (whenever defined, i.e. for $\sigma_2(\varphi_2)$ small) and $\widetilde{\psi}_k(\varphi_1)$ close to $\sum_{i=1}^k t_i \delta_{\widetilde{x}_i}$ in the distributional sense. Furthermore, writing $\varphi_1 = \varphi_{1,\lambda}$ to emphasize the dependence on λ, it turns out that

$$\widetilde{\psi}_k(\varphi_{1,\lambda}) \to \sum_{i=1}^k t_i \delta_{\widetilde{x}_i} \qquad \text{as } \lambda \to +\infty,$$

which gives us the following homotopy:

$$(\nu\,; \mathfrak{t}) \mapsto \widetilde{\psi}_k\left(\varphi_{1,\frac{\lambda}{\mathfrak{t}}}\right), \qquad \mathfrak{t} \in [0,1].$$

Reasoning as in Step 3 of Subsection 3.3.2 we get a homotopy which deforms the points $\widetilde{x}_i$ to the original one x_i. Letting $\tilde{\gamma}_i$ be the geodesic joining $\widetilde{x}_i$ and x_i in unit time we consider

$$(\nu\,; \mathfrak{t}) \mapsto \sum_{i=1}^k t_i \delta_{\tilde{\gamma}_i(1-\mathfrak{t})}, \qquad \mathfrak{t} \in [0,1].$$

Notice that for $\mathfrak{t} = 0$ we get in the above homotopy $(\nu\,; 0) = \mathcal{R}_p(\nu)$. Observe now that $\mathcal{R}_p$ is homotopic to the identity map, see Remark 3.3.5, and let $\mathcal{H}_{\mathcal{R}_p}$ be the map introduced in Step 4 of Subsection 3.3.2 which realizes this homotopy. We then consider

$$(\nu\,; \mathfrak{t}) \mapsto \mathcal{H}_{\mathcal{R}_p}(\nu, 1-\mathfrak{t}), \qquad \mathfrak{t} \in [0,1].$$

Finally, letting $\mathcal{H}$ be the concatenation of the above homotopies (rescaling the respective domains of definition) and letting $p(\mathfrak{t}) : [0,1] \to \Sigma$ be again a geodesic joining p to $\widetilde{\psi}_1(\varphi_2)$ (whenever defined) we get the desired homotopy:

$$\big((\nu, p, s); \mathfrak{t}\big) \mapsto \begin{cases} \big(\mathcal{H}(\nu\,; \mathfrak{t}), p(\mathfrak{t}), (1-\mathfrak{t})s + \mathfrak{t}s(\varphi_1, \varphi_2)\big),\ \mathfrak{t} \in [0,1] & \text{if } \sigma_2(\varphi_2) \text{ small,} \\ \big(\mathcal{H}(\nu\,; \mathfrak{t}), p, (1-\mathfrak{t})s\big),\ \mathfrak{t} \in [0,1] & \text{otherwise.} \end{cases} \tag{3.100}$$

Case 3. Let $s \in \left[0, \frac{1}{4}\right]$. In this case the test functions we are considering are as in Subsection 3.3.2. Notice that for this range of s we always get $\sigma_2(\hat{\varphi}_2^t) \ll \sigma_1(\hat{\varphi}_1^t)$, see the beginning of the proof of Proposition 3.3.7, and therefore $s(\hat{\varphi}_1^t, \hat{\varphi}_2^t) = 0$. We have further to subdivide this case depending on the values of s due to the construction of the test functions in the Steps 1-4 of Subsection 3.3.2.

Emphasizing in the test functions the dependence on λ and recalling that $t = t(s)$, for $s \in \left[\frac{3}{16}, \frac{1}{4}\right]$ we get the following property: $\widetilde{\psi}_k(\check{\varphi}_{1,\lambda}^t) \xrightarrow{\lambda\to\infty} \sum_{i=1}^k t_i \delta_{\widetilde{x}_i}$ (see Step 1). When $s \in \left[\frac{1}{8}, \frac{3}{16}\right]$ one has by construction that $\widetilde{\psi}_k(\tilde{\varphi}_{1,\lambda}^t) \xrightarrow{\lambda\to\infty} \sum_{i=1}^k t_i \delta_{\widetilde{x}_i}$ (see Step 2). For $s \in \left[\frac{1}{8}, \frac{3}{16}\right]$ we get instead $\hat{\psi}_k(\tilde{\varphi}_{1,\lambda}^t) \xrightarrow{\lambda\to\infty} \sum_{i=1}^k t_i \delta_{\tilde{\gamma}_i}$ (see Step 3). Finally, when $s \in \left[\frac{1}{8}, \frac{3}{16}\right]$ we obtain $\bar{\psi}_k(\tilde{\varphi}_{1,\lambda}^t) \xrightarrow{\lambda\to\infty} \mathcal{H}_{\mathcal{R}_p}(\nu, t)$ (see Step 4).

In any case we then proceed analogously as in Step 2 and the desired homotopy is given as in the second part of (3.100). ■

Recall now that Y is not contractible, see Proposition 3.4.6; being $Y_{\mathcal{R}}$ a deformation retract of Y, see Remark 3.3.16, we get that $Y_{\mathcal{R}}$ is not contractible too. Therefore, by the latter result we deduce that

$$\Phi_\lambda(Y_{\mathcal{R}}) \text{ is not contractible in } J_\rho^{-L}.$$

Moreover, one can take λ large enough so that $\Phi_\lambda(Y_{\mathcal{R}}) \subset J_\rho^{-2L}$. Similarly as in Section 2.3 we next define the topological cone over $Y_{\mathcal{R}}$ by the equivalence relation

$$\mathcal{C} = \frac{Y_{\mathcal{R}} \times [0,1]}{Y_{\mathcal{R}} \times \{0\}},$$

where $Y_{\mathcal{R}} \times \{0\}$ is identified to a single point and consider the min-max value:

$$m = \inf_{h\in\Gamma} \max_{\xi\in\mathcal{C}} J_\rho(h(\xi)),$$

where

$$\Gamma = \Big\{ h : \mathcal{C} \to H^1(\Sigma) \times H^1(\Sigma) : \ h(\nu, p, s) = \Phi_\lambda(\nu, p, s) \ \ \forall (\nu, p, s) \in \partial\mathcal{C} \simeq Y_{\mathcal{R}} \Big\}. \tag{3.101}$$

First, we observe that the map from $\mathcal{C}$ to $H^1(\Sigma) \times H^1(\Sigma)$ defined by $(\cdot, t) \mapsto t\,\Phi_\lambda(\cdot)$ belongs to Γ, hence this is a non-empty set. Moreover, by the choice of Φ_λ we have

$$\sup_{(\nu,p,s)\in\partial\mathcal{C}} J_\rho\big(h(\nu,p,s)\big) = \sup_{(\nu,p,s)\in Y_{\mathcal{R}}} J_\rho\big(\Phi_\lambda(\nu,p,s)\big) \le -2L.$$

The crucial point is to show that $m \ge -L$. This is done exactly as in Section 2.3. We repeat here the argument for the reader's convenience. It holds that $\partial\mathcal{C}$ is contractible in $\mathcal{C}$, and hence in $h(\mathcal{C})$ for any $h \in \Gamma$. On the other hand by the fact that $Y_{\mathcal{R}}$ is not contractible and by Proposition 3.4.10 $\partial\mathcal{C}$ is not contractible in J_ρ^{-L}, so we deduce that $h(\mathcal{C})$ is not contained in J_ρ^{-L}. Being this valid for any $h \in \Gamma$, we conclude that necessarily $m \ge -L$.

It follows from standard variational arguments (see [90]) that the functional J_ρ admits a Palais-Smale sequence at level m. However, this does not guarantee the existence of a critical point, since it is not known whether the Palais-Smale condition holds or not. To bypass this problem one needs a *monotonicity trick* introduced by Struwe in [88], see Lemma 2.3.1, jointly with the compactness result given in Theorem 1.1.4, see Section 2.3 for full details.

3.5 Appendix: proof of Proposition 3.3.7

The energy estimates of Proposition 3.3.7 will follow from the next three Lemmas.

Lemma 3.5.1 *If φ_1, φ_2 are defined as in (3.72), we have that*

$$\fint_\Sigma \varphi_1 \, dV_g = O(1), \qquad \fint_\Sigma \varphi_2 \, dV_g = O(1).$$

PROOF. From elementary inequalities (see also Figure 2) it is easy to show that there exists a constant C so that

$$|\varphi_1| + |\varphi_2| \leq C \left(1 + \log \frac{1}{d(\cdot, p)} + \sum_i \frac{1}{d(\cdot, \tilde{x}_i)} \right).$$

As the logarithm of the distance from a fixed point is integrable, the conclusion easily follows. ■

In the following, for positive numbers a, b we will use the notation

$$a \simeq_{\mathrm{C}} b \quad \Leftrightarrow \quad \exists C > 1 \text{ such that } \frac{b}{C} \leq a \leq Cb. \tag{3.102}$$

Lemma 3.5.2 *Under the above assumptions one has*

$$\int_\Sigma e^{\varphi_1}\, dV_g \simeq_C \hat{\mathfrak{s}}^4 \tau_\lambda^2 \check{\lambda}^2, \qquad \int_\Sigma e^{\varphi_2}\, dV_g \simeq_C \max\left\{ \frac{\widetilde{\tau}^2}{\hat{\mathfrak{s}}^2 \mu^4}, 1 \right\}.$$

PROOF. Let $\tau \in (0, +\infty]$ be fixed and let $\hat{\nu} \in \Sigma_{k,p,\bar{\tau}}$ be as in (3.65). For simplicity we may assume that there is only one point in the support of $\hat{\nu}$, i.e. $\hat{\nu} = \delta_{x_j}$. The case of a general $\hat{\nu}$ is then treated in analogous way. It is not difficult to show that the terms $-\frac{1}{2}v_2, -\frac{1}{2}v_{1,1}$ do not affect the integrals of e^{φ_1} and e^{φ_2}, respectively, and that

$$\int_\Sigma e^{\varphi_1}\, dV_g \simeq_{\mathrm{C}} \int_\Sigma e^{v_1}\, dV_g, \qquad \int_\Sigma e^{\varphi_2}\, dV_g \simeq_{\mathrm{C}} \int_\Sigma e^{v_2}\, dV_g.$$

Therefore, it is enough to prove the following:

$$\int_\Sigma e^{v_1}\, dV_g \simeq_{\mathrm{C}} \hat{\mathfrak{s}}^4 \tau_\lambda^2 \check{\lambda}^2, \qquad \int_\Sigma e^{v_2}\, dV_g \simeq_{\mathrm{C}} \max\left\{ \frac{\widetilde{\tau}^2}{\hat{\mathfrak{s}}^2 \mu^4}, 1 \right\}. \tag{3.103}$$

We start by observing that, by definition, for $d(x_j, p) \leq \frac{4}{\lambda_j}$ one has

$$v_1(x) = \log \frac{1}{\left((\hat{\mathfrak{s}}\tau_\lambda)^{-2} + d(x,p)^2\right)^3}.$$

By an elementary change of variables we find

$$\int_\Sigma e^{v_1}\, dV_g = \int_\Sigma \frac{1}{\left((\hat{\mathfrak{s}}\tau_\lambda)^{-2} + d(x,p)^2\right)^3}\, dV_g \simeq_{\mathrm{C}} \hat{\mathfrak{s}}^4 \tau_\lambda^4. \tag{3.104}$$

By the definition of τ and $\hat{\nu} \in \Sigma_{k,p,\bar{\tau}}$ (see in particular (3.62) and (3.63)), recalling that $d(x_j, p) \leq \frac{4}{\lambda_j}$ and that $\lambda_j \geq \lambda$ by construction, we get

$$\frac{1}{\tau} \leq d(x_j, p) \leq \frac{4}{\lambda_j} \leq \frac{C}{\lambda}. \tag{3.105}$$

By taking λ sufficiently large we deduce $\tau \gg 1$. It follows that $\check{s} = 1$ and $\check{\lambda} = \lambda$, see (3.69). Moreover, by (3.105) we have

$$\frac{C}{\lambda} \leq \tau_\lambda \leq \lambda.$$

Therefore, we can rewrite (3.104) as

$$\int_\Sigma e^{v_1}\, dV_g = \int_\Sigma \frac{1}{\left((\hat{\mathfrak{s}}\tau)^{-2} + d(x,p)^2\right)^3}\, dV_g \simeq_{\mathrm{C}} \hat{\mathfrak{s}}^4 \tau_\lambda^2 \check{\lambda}^2$$

and the proof of the first part of (3.103) is concluded. Suppose now $d(x_j, p) > \frac{4}{\lambda_j}$ and divide Σ into three subsets:

$$\mathcal{A} = A_{\widetilde{x}_j}\left(\frac{1}{s_j \lambda_j}, \frac{d(\widetilde{x}_j, p)}{4} \right), \qquad \mathcal{B} = B_{\frac{1}{s_j \lambda_j}}(\widetilde{x}_j), \qquad \mathcal{C} = \Sigma \setminus (\mathcal{A} \cup \mathcal{B}).$$

We start by estimating

$$\int_{\mathcal{B}} e^{v_1}\, dV_g = \int_{B_{\frac{1}{s_j\lambda_j}}(\widetilde{x}_j)} \frac{s_j^4\lambda_j^4 d(\widetilde{x}_j,p)^4}{\big((\hat{\mathfrak{s}}\tau_\lambda)^{-2} + d(x,p)^2\big)^3}\, dV_g.$$

Observe that if in the latter formula we substitute $d(x,p)$ with $d(\widetilde{x}_j,p)$ we get negligible errors which will be omitted. Therefore, we can rewrite it as

$$\begin{aligned}\int_{\mathcal{B}} e^{v_1}\, dV_g &= \int_{B_{\frac{1}{s_j\lambda_j}}(\widetilde{x}_j)} \frac{s_j^4\lambda_j^4}{d(\widetilde{x}_j,p)^2} \frac{1}{\big((\hat{\mathfrak{s}}\tau_\lambda d(\widetilde{x}_j,p))^{-2} + 1\big)^3}\, dV_g \\ &= \frac{s_j^2\lambda_j^2}{d(\widetilde{x}_j,p)^2} \frac{C}{\big((\hat{\mathfrak{s}}\tau_\lambda d(\widetilde{x}_j,p))^{-2} + 1\big)^3} \;=\; s_j^2\widetilde{s}_j^2 \frac{\lambda_j^2}{d(x_j,p)^2} \frac{C}{\big((\hat{\mathfrak{s}}\tau_\lambda d(\widetilde{x}_j,p))^{-2} + 1\big)^3},\end{aligned}$$

where in the last equality we have used (3.67). Exploiting now the conditions (3.70) and (3.71), the assumption $d(x_j,p) > \frac{4}{\lambda_j}$ and recalling that $d(x_j,p) \geq \frac{1}{\tau}$ by definition (3.63), we conclude that

$$\int_{\mathcal{B}} e^{v_1}\, dV_g = \hat{\mathfrak{s}}^4\tau_\lambda^2\check{\lambda}^2 \frac{C}{\big((\hat{\mathfrak{s}}\tau_\lambda d(\widetilde{x}_j,p))^{-2} + 1\big)^3} \simeq_{\mathrm{C}} \hat{\mathfrak{s}}^4\tau_\lambda^2\check{\lambda}^2.$$

It is then not difficult to show that

$$\int_{\mathcal{A}} e^{v_1}\, dV_g \leq \hat{\mathfrak{s}}^4\tau_\lambda^2\check{\lambda}^2 C, \qquad \int_{\mathcal{C}} e^{v_1}\, dV_g \leq \hat{\mathfrak{s}}^4\tau_\lambda^2\check{\lambda}^2 C,$$

for some $C > 0$. This concludes the proof of the first part of (3.103).

For the second part of (3.103), similarly as before, we divide Σ into

$$\widetilde{\mathcal{A}} = A_p\left(\frac{1}{\hat{\mathfrak{s}}\widetilde{\tau}}, \frac{1}{\hat{\mathfrak{s}}\mu}\right), \quad \widetilde{\mathcal{B}} = B_{\frac{1}{\hat{\mathfrak{s}}\widetilde{\tau}}}(p), \quad \widetilde{\mathcal{C}} = \Sigma \setminus (\widetilde{\mathcal{A}} \cup \widetilde{\mathcal{B}}).$$

For $x \in \widetilde{\mathcal{B}}$ we have $v_2(x) = \log\left(\frac{\mu}{\widetilde{\tau}}\right)^{-4}$, hence

$$\int_{\widetilde{\mathcal{B}}} e^{v_2}\, dV_g = \int_{B_{\frac{1}{\hat{\mathfrak{s}}\widetilde{\tau}}}(p)} \left(\frac{\mu}{\widetilde{\tau}}\right)^{-4} dV_g = \frac{\widetilde{\tau}^2}{\hat{\mathfrak{s}}^2\mu^4} C. \tag{3.106}$$

Moreover, working in normal coordinates around p one gets

$$\int_{\widetilde{\mathcal{A}}} e^{v_2}\, dV_g \leq \frac{\widetilde{\tau}^2}{\hat{\mathfrak{s}}^2\mu^4} C, \tag{3.107}$$

for some $C > 0$. On the other hand, we have

$$\int_{\widetilde{\mathcal{C}}} e^{v_2}\, dV_g \simeq_{\mathrm{C}} 1. \tag{3.108}$$

From (3.106), (3.107) and (3.108) it follows that

$$\int_{\Sigma} e^{v_2}\, dV_g \simeq_{\mathrm{C}} \max\left\{\frac{\widetilde{\tau}^2}{\hat{\mathfrak{s}}^2\mu^4}, 1\right\},$$

which concludes the proof of the second part of (3.103). ■

Recalling the definition of $\hat{\nu} \in \Sigma_{k,p,\widetilde{\tau}}$ in (3.65) we introduce now the following sets of indices: let $I \subseteq \{1,\dots,k\}$ be given by

$$I = \left\{ i : d(x_i,p) > \frac{4}{\lambda_i} \right\}.$$

We then subdivide I into two subsets $I_1, I_2 \subseteq I$:

$$I_1 = \left\{ i : d(x_i,p) \leq \frac{1}{\tau_\lambda} \right\}, \qquad I_2 = \left\{ i : d(x_i,p) > \frac{1}{\tau_\lambda} \right\}. \tag{3.109}$$

Lemma 3.5.3 *Under the above assumptions one has*

$$\int_\Sigma Q(\varphi_1,\varphi_2)\,dV_g \;\leq\; 8\pi\big(\log\widetilde{\tau}-\log\mu\big)+8|I_1|\pi\big(\log\check{\lambda}-\log\tau_\lambda\big)+\sum_{i\in I_2}8\pi\big(\log s_i+\log\lambda_i-\log d(\widetilde{x}_i,p)\big)+$$
$$+\;16\pi\sum_{i\in I_2}\log d(\widetilde{x}_i,p)+\big(24\pi\log\tau_\lambda+24\pi\log\hat{\mathfrak{s}}\big)+C,$$

for some $C=C(\Sigma)$.

Proof. We start by observing that, by definition, $\nabla v_{1,1}=0$ in $\Sigma\setminus\bigcup_{i\in I}A_{\widetilde{x}_i}\left(\frac{1}{s_i\lambda_i},\frac{d(\widetilde{x}_i,p)}{4}\right)$, while $\nabla v_2=0$ in $\Sigma\setminus A_p\left(\frac{1}{\hat{\mathfrak{s}}\widetilde{\tau}},\frac{1}{\hat{\mathfrak{s}}\mu}\right)$. We next prove the following estimates on the gradients of $v_{1,1},v_{1,2}$ and v_2:

$$|\nabla v_{1,1}(x)|\leq\frac{4}{d_{min}(x)}\qquad\text{in }\bigcup_{i\in I}A_{\widetilde{x}_i}\left(\frac{1}{s_i\lambda_i},\frac{d(\widetilde{x}_i,p)}{4}\right),\tag{3.110}$$
$$|\nabla v_2(x)|\leq\frac{4}{d(x,p)}\qquad\text{in }A_p\left(\frac{1}{\hat{\mathfrak{s}}\widetilde{\tau}},\frac{1}{\hat{\mathfrak{s}}\mu}\right),\tag{3.111}$$
$$|\nabla v_{1,2}(x)|\leq\frac{6}{d(x,p)}\qquad\text{for every }x\in\Sigma,\tag{3.112}$$

where $d_{min}(x)=\min\limits_{i\in I}d(x,\widetilde{x}_i)$ and

$$|\nabla v_{1,2}(x)|\leq C\hat{\mathfrak{s}}\tau_\lambda\qquad\text{for every }x\in\Sigma,\tag{3.113}$$

where C is a constant independent of τ_λ and $\hat{\mathfrak{s}}$.

Concerning (3.110) and (3.111) we show the inequalities just for $v_{1,1}$, as for v_2 the proof is similar. We have that

$$\begin{aligned}\nabla v_{1,1}(x)\;&=\;-4\frac{\sum_{i=1}^k t_i\left(\frac{d(x,\widetilde{x}_i)}{d(\widetilde{x}_i,p)}\right)^{-5}\nabla_x\left(\frac{d(x,\widetilde{x}_i)}{d(\widetilde{x}_i,p)}\right)}{\sum_{j=1}^k t_j\left(\frac{d(x,\widetilde{x}_j)}{d(\widetilde{x}_j,p)}\right)^{-4}}=-4\frac{\sum_{i=1}^k t_i\left(\frac{d(x,\widetilde{x}_i)}{d(\widetilde{x}_i,p)}\right)^{-4}\frac{\nabla_x d(x,\widetilde{x}_i)}{d(x,\widetilde{x}_i)}}{\sum_{j=1}^k t_j\left(\frac{d(x,\widetilde{x}_j)}{d(\widetilde{x}_j,p)}\right)^{-4}}\\&=\;-4\frac{\sum_{i=1}^k t_i\left(\frac{d(x,\widetilde{x}_i)}{d(\widetilde{x}_i,p)}\right)^{-4}\frac{\nabla_x d(x,\widetilde{x}_i)}{d_{min}(x)}}{\sum_{j=1}^k t_j\left(\frac{d(x,\widetilde{x}_j)}{d(\widetilde{x}_j,p)}\right)^{-4}}.\end{aligned}$$

Exploiting the fact that $|\nabla_x d(x,\widetilde{x}_i)|\leq 1$ we obtain (3.110). Moreover, by direct computations one gets (3.111). We consider now

$$\nabla v_{1,2}(x)=-3\frac{\hat{\mathfrak{s}}^2\tau_\lambda^2\nabla_x(d^2(x,p))}{1+\hat{\mathfrak{s}}^2\tau_\lambda^2 d^2(x,p)}.$$

Using the estimate $|\nabla_x(d^2(x,p))|\leq 2d(x,p)$ the properties (3.112) and (3.113) easily follow by the inequalities

$$\frac{\hat{\mathfrak{s}}^2\tau_\lambda^2 d^2(x,p)}{1+\hat{\mathfrak{s}}^2\tau_\lambda^2 d^2(x,p)}\leq 1,\qquad\frac{\hat{\mathfrak{s}}\tau_\lambda d(x,p)}{1+\hat{\mathfrak{s}}^2\tau_\lambda^2 d^2(x,p)}\leq 1;\qquad\text{for every }x\in\Sigma,$$

respectively. Recalling the definitions of φ_1,φ_2 in (3.72) and that $v_1=v_{1,1}+v_{1,2}$, we obtain

$$\int_\Sigma Q(\varphi_1,\varphi_2)\,dV_g\;=\;\frac13\int_\Sigma\big(|\nabla\varphi_1|^2+|\nabla\varphi_2|^2+\nabla\varphi_1\cdot\nabla\varphi_2\big)\,dV_g\tag{3.114}$$

$$\begin{aligned}&=\;\frac13\int_\Sigma\left(|\nabla v_1|^2+\frac14|\nabla v_2|^2-\nabla v_1\cdot\nabla v_2\right)dV_g+\frac13\int_\Sigma\left(|\nabla v_2|^2+\frac14|\nabla v_{1,1}|^2-\nabla v_2\cdot\nabla v_{1,1}\right)dV_g+\\&+\;\frac13\int_\Sigma\left(\nabla v_1-\frac12\nabla v_2\right)\cdot\left(\nabla v_2-\frac12\nabla v_{1,1}\right)\,dV_g\\&=\;\frac14\int_\Sigma|\nabla v_{1,1}|^2\,dV_g+\frac14\int_\Sigma|\nabla v_2|^2\,dV_g+\frac13\int_\Sigma|\nabla v_{1,2}|^2\,dV_g+\int_\Sigma\left(\frac16\nabla v_{1,1}\cdot\nabla v_{1,2}-\frac{7}{12}\nabla v_{1,1}\cdot\nabla v_2\right)dV_g.\end{aligned}$$

We start by observing that the integral of the mixed terms is uniformly bounded. Indeed, we claim that

$$\nabla v_{1,1} \cdot \nabla v_2 = 0. \tag{3.115}$$

By the remark before (3.110), (3.115) will follow by proving that $A_{\widetilde{x}_i}\left(\frac{1}{s_i \lambda_i}, \frac{d(\widetilde{x}_i, p)}{4}\right) \cap A_p\left(\frac{1}{\hat{\mathfrak{s}}\widetilde{\tau}}, \frac{1}{\hat{\mathfrak{s}}\mu}\right) = \emptyset$ for all $i \in I$. Recall the constant $\bar{\delta}$ in (3.67). Clearly, when all the points of the support of $\hat{\nu}$ are bounded away from p, i.e. $d(x_i, p) > \bar{\delta}$ for all i, we get the conclusion. Consider now the case $d(x_i, p) \leq \bar{\delta}$ for some i and observe that in this case $\widetilde{s}_i = \hat{\mathfrak{s}}$, see (3.67). Moreover, by taking $\bar{\delta}$ sufficiently small, one has also $\check{s} \leq C$ by the definition (3.69) (see also (3.105) and the motivation above it). To prove that the above two subsets are disjoint, one has just to ensure that $d(\widetilde{x}_i, p) \gg \frac{1}{\hat{\mathfrak{s}}\mu}$. We distinguish between two cases. Suppose first that $d(x_i, p) > \frac{1}{\tau_\lambda}$. By the assumptions we have made and by (3.70), one gets

$$d(\widetilde{x}_i, p) = \frac{1}{\widetilde{s}_i} d(x_i, p) = \frac{1}{\hat{\mathfrak{s}}} d(x_i, p) \geq \frac{1}{\hat{\mathfrak{s}} \lambda_i} = \frac{1}{\hat{\mathfrak{s}}\, d(x_i, p) \tau_\lambda \check{\lambda}} \geq \frac{1}{C\, \hat{\mathfrak{s}}\, \tau_\lambda \check{\lambda}} = \frac{1}{C\, \hat{\mathfrak{s}}\, \tau_\lambda \check{s} \lambda} \geq \frac{1}{C \hat{\mathfrak{s}}\, \tau_\lambda \lambda} \gg \frac{1}{\hat{\mathfrak{s}} \mu}$$

by the choice of the parameters μ and λ. The case $d(x_i, p) \leq \frac{1}{\tau_\lambda}$ is treated in the same way with minor modifications. This conclude the proof of (3.115).

We claim now that

$$\int_\Sigma \nabla v_{1,1} \cdot \nabla v_{1,2} \, dV_g \leq C. \tag{3.116}$$

We introduce the sets

$$A_i = \left\{ x \in \Sigma : d(x, \widetilde{x}_i) = \min_{j \in I} d(x, x_j) \right\}. \tag{3.117}$$

By (3.110) and (3.113) we get

$$\begin{aligned}
\int_\Sigma \nabla v_{1,1} \cdot \nabla v_{1,2} \, dV_g &\leq \int_\Sigma \frac{C}{d_{min}(x)\, d(x,p)} \, dV_g \leq \sum_{i \in I} \int_{A_i} \frac{C}{d(x, \widetilde{x}_i)\, d(x,p)} \, dV_g \\
&\leq \sum_{i \in I} \int_{A_{\widetilde{x}_i}\left(\frac{1}{s_i \lambda_i}, \frac{d(\widetilde{x}_i, p)}{4}\right)} \frac{C}{d(x, \widetilde{x}_i)\, d(\widetilde{x}_i, p)} \, dV_g \leq C,
\end{aligned}$$

which proves the claim (3.116).

Using the estimate (3.110) one has

$$\begin{aligned}
\frac{1}{4} \int_\Sigma |\nabla v_{1,1}|^2 \, dV_g &\leq 4 \int_\Sigma \frac{1}{d^2_{min}(x)} \, dV_g \leq 4 \sum_{i \in I} \int_{A_i} \frac{1}{d^2(x, \widetilde{x}_i)} \, dV_g \\
&\leq 4 \sum_{i \in I} \int_{A_{\widetilde{x}_i}\left(\frac{1}{s_i \lambda_i}, \frac{d(\widetilde{x}_i, p)}{4}\right)} \frac{1}{d^2(x, \widetilde{x}_i)} \, dV_g \\
&\leq \sum_{i \in I} 8\pi \big(\log s_i + \log \lambda_i + \log d(\widetilde{x}_i, p) \big) + C.
\end{aligned} \tag{3.118}$$

Recalling the definition of $I_1, I_2 \subseteq I$ given in (3.109) we observe the following: for $i \in I_1$ we get $\lambda_i = \check{\lambda}$ and $\widetilde{s}_i = \hat{\mathfrak{s}}$, see (3.70) and (3.67), respectively. Moreover, taking into account (3.71) we deduce

$$\begin{aligned}
\frac{1}{4} \int_\Sigma |\nabla v_{1,1}|^2 \, dV_g &\leq 8|I_1| \pi \big(\log \check{\lambda} - \log \tau_\lambda \big) + \sum_{i \in I_2} 8\pi \big(\log s_i + \log \lambda_i + \log d(\widetilde{x}_i, p) \big) + C \\
&= 8|I_1| \pi \big(\log \check{\lambda} - \log \tau_\lambda \big) + \sum_{i \in I_2} 8\pi \big(\log s_i + \log \lambda_i - \log d(\widetilde{x}_i, p) \big) + \\
&+ 16\pi \sum_{i \in I_2} \log d(\widetilde{x}_i, p) + C.
\end{aligned} \tag{3.119}$$

Similarly as for (3.118), by (3.111) we get

$$\frac{1}{4} \int_\Sigma |\nabla v_2|^2 \, dV_g = 4 \int_{A_p\left(\frac{1}{\hat{\mathfrak{s}}\widetilde{\tau}}, \frac{1}{\hat{\mathfrak{s}}\mu}\right)} \frac{1}{d^2(x,p)} \, dV_g \leq 8\pi \big(\log \widetilde{\tau} - \log \mu \big) + C. \tag{3.120}$$

To estimate the term $|\nabla v_{1,2}|^2$ we consider $\Sigma = B_{\frac{1}{\hat{\mathfrak{s}}\tau_\lambda}}(p) \cup \big(\Sigma \setminus B_{\frac{1}{\hat{\mathfrak{s}}\tau_\lambda}}(p)\big)$. From (3.112) we deduce that

$$\int_{B_{\frac{1}{\hat{\mathfrak{s}}\tau_\lambda}}(p)} |\nabla v_{1,2}|^2 \, dV_g \le C.$$

Using then (3.112) one finds

$$\frac{1}{3}\int_{\Sigma\setminus B_{\frac{1}{\hat{\mathfrak{s}}\tau_\lambda}}(p)} |\nabla v_{1,2}|^2 \, dV_g \le 12 \int_{\Sigma\setminus B_{\frac{1}{\hat{\mathfrak{s}}\tau_\lambda}}(p)} \frac{1}{d^2(x,p)} \, dV_g \le 24\pi\big(\log\tau_\lambda + \log\hat{\mathfrak{s}}\big) + C. \tag{3.121}$$

Finally, by (3.115), (3.116) and inserting (3.119), (3.120) and (3.121) into (3.114) we get the conclusion. ■

Proof of Proposition 3.3.7. Using Lemmas 3.5.1, 3.5.2 and 3.5.3, the energy estimate we get is

$$\begin{aligned} J_\rho(\varphi_1,\varphi_2) &\le 8\pi\big(\log\widetilde{\tau} - \log\mu\big) + 8|I_1|\pi\big(\log\check{\lambda} - \log\tau_\lambda\big) + \sum_{i\in I_2} 8\pi\big(\log s_i + \log\lambda_i - \log d(\widetilde{x}_i,p)\big) + 16\pi\sum_{i\in I_2}\log d(\widetilde{x}_i,p) + \\ &+ \big(24\pi\log\tau_\lambda + 24\pi\log\hat{\mathfrak{s}}\big) - \rho_1\big(4\log\hat{\mathfrak{s}} + 2\log\tau_\lambda + 2\log\check{\lambda}\big) - \rho_2\log\max\left\{\frac{\widetilde{\tau}^2}{\hat{\mathfrak{s}}^2\mu^4},1\right\} + C \\ &\le 8\pi\big(\log\widetilde{\tau} - \log\mu\big) + 8|I_1|\pi\big(\log\check{\lambda} - \log\tau_\lambda\big) + \sum_{i\in I_2} 8\pi\big(\log s_i + \log\widetilde{s}_i + \log\lambda_i - \log d(x_i,p)\big) + \\ &+ 16\pi\sum_{i\in I_2}\log d(\widetilde{x}_i,p) + \big(24\pi\log\tau_\lambda + 24\pi\log\hat{\mathfrak{s}}\big) - \rho_1\big(4\log\hat{\mathfrak{s}} + 2\log\tau_\lambda + 2\log\check{\lambda}\big) + \\ &- \rho_2\log\max\left\{\frac{\widetilde{\tau}^2}{\hat{\mathfrak{s}}^2\mu^4},1\right\} + C, \end{aligned}$$

for some constant $C > 0$. Exploiting the conditions (3.70) and (3.71) we obtain

$$\begin{aligned} J_\rho(\varphi_1,\varphi_2) &\le 8\pi\big(\log\widetilde{\tau} - \log\mu\big) + 8|I_1|\pi\big(\log\check{\lambda} - \log\tau_\lambda\big) + \sum_{i\in I_2} 8\pi\big(2\log\hat{\mathfrak{s}} + \log\check{\lambda} + \log\tau_\lambda\big) + \\ &+ 16\pi\sum_{i\in I_2}\log d(\widetilde{x}_i,p) + \big(24\pi\log\tau_\lambda + 24\pi\log\hat{\mathfrak{s}}\big) - \rho_1\big(4\log\hat{\mathfrak{s}} + 2\log\tau_\lambda + 2\log\check{\lambda}\big) + \\ &- \rho_2\log\max\left\{\frac{\widetilde{\tau}^2}{\hat{\mathfrak{s}}^2\mu^4},1\right\} + C. \end{aligned} \tag{3.122}$$

Recalling the definition of I_1, I_2 in (3.109), we distinguish between two cases.

Case 1. Suppose first that $I_1 \neq \emptyset$. By construction it follows that $\tau \gg 1$, see (3.62) and (3.63). Therefore, by (3.68) we get $\hat{\mathfrak{s}} = \mathfrak{s}$. On the other hand, using (3.69) and the definition of $\check{\lambda}$ under it, we deduce $\check{\lambda} \le C\lambda$.

For $\hat{\mathfrak{s}} \ll \frac{\widetilde{\tau}}{\mu^2}$ we get in (3.122) the following:

$$\max\left\{\frac{\widetilde{\tau}^2}{\hat{\mathfrak{s}}^2\mu^4},1\right\} = \frac{\widetilde{\tau}^2}{\hat{\mathfrak{s}}^2\mu^4}. \tag{3.123}$$

In this case (3.122) can be rewritten as

$$\begin{aligned} J_\rho(\varphi_1,\varphi_2) &\le \log\widetilde{\tau}\,\big(8\pi - 2\rho_2\big) + \log\lambda\,\big(8(|I_1|+|I_2|)\pi - 2\rho_1\big) + \log\hat{\mathfrak{s}}\,\big(24\pi + 16|I_2|\pi - 4\rho_1 + 2\rho_2\big) + \\ &+ \log\tau_\lambda\big(8|I_2|\pi - 8|I_1|\pi + 24\pi - 2\rho_1\big) + \log\mu\big(4\rho_2 - 8\pi\big) + C. \end{aligned} \tag{3.124}$$

Recalling that $\hat{\mathfrak{s}} \ll \frac{\widetilde{\tau}}{\mu^2}$, the latter estimate is negative by the choice of the parameters $\widetilde{\tau} \gg \mu \gg \lambda$ and $\rho_2 > 4\pi$.

When instead $\hat{\mathfrak{s}} = \frac{\widetilde{\tau}}{\mu^2} + O(1)$ we have

$$\max\left\{\frac{\widetilde{\tau}^2}{\hat{\mathfrak{s}}^2\mu^4},1\right\} = 1. \tag{3.125}$$

Considering now (3.122) and observing that $\log \hat{\mathfrak{s}} = \log \widetilde{\tau} - 2\log\mu + C$, we end up with

$$\begin{aligned} J_\rho(\varphi_1,\varphi_2) &\leq \log\widetilde{\tau}\left(32\pi + 16|I_2|\pi - 4\rho_1\right) + \log\lambda\left(8(|I_1|+|I_2|)\pi - 2\rho_1\right) \\ &+ \log\tau_\lambda\big(8|I_2|\pi - 8|I_1|\pi + 24\pi - 2\rho_1\big) + \log\mu\big(8\rho_1 - 56\pi - 32|I_2|\pi\big) + C. \end{aligned}$$

The crucial fact is that by construction of $\Sigma_{k,p,\bar{\tau}}$, see (3.60), it holds $|I_2| \leq k-2$ whenever $|I_1| \neq \emptyset$. Hence, we conclude that

$$\begin{aligned} J_\rho(\varphi_1,\varphi_2) &\leq \log\widetilde{\tau}\left(16k\pi - 4\rho_1\right) + \log\lambda\left(8(|I_1|+|I_2|)\pi - 2\rho_1\right) + \log\tau_\lambda\big(8|I_2|\pi - 8|I_1|\pi + 24\pi - 2\rho_1\big) + \\ &+ \log\mu\big(8\rho_1 - 56\pi - 32|I_2|\pi\big) + C. \end{aligned}$$

which is large negative since $\rho_1 > 4k\pi$ and by the choice of the parameters.

Case 2. Suppose now $I_1 = \emptyset$. By construction we deduce that $\tau \leq C$, see (3.62) and (3.63). Therefore, using (3.68) we obtain $\hat{\mathfrak{s}} \leq C$. In this case the equality in (3.123) always holds true. Moreover, by (3.69) we have $\check{\lambda} = \mathfrak{s}\lambda$. Hence, (3.122) can be rewritten as

$$\begin{aligned} J_\rho(\varphi_1,\varphi_2) &\leq \log\mathfrak{s}\left(8|I_2|\pi - 2\rho_1\right) + \log\widetilde{\tau}\left(8\pi - 2\rho_2\right) + \log\lambda\left(8|I_2|\pi - 2\rho_1\right) + \\ &+ \log\tau_\lambda\big(8|I_2|\pi + 24\pi - 2\rho_1\big) + \log\mu\big(4\rho_2 - 8\pi\big) + C. \end{aligned}$$

Observing that $|I_2| \leq k$ we conclude that the latter estimate is large negative since $\rho_1 > 4k\pi$, $\rho_2 > 4\pi$ and by the choice of the parameters. ■

Chapter 4

A mean field equation: a first existence result in a doubly supercritical case

We start here to discuss the second topic of the thesis, namely a class of a mean field equations with two parameters defined on a compact surface Σ of the following type:

$$-\Delta u = \rho_1 \left(\frac{h_1\, e^{u}}{\int_\Sigma h_1\, e^{u}\, dV_g} - 1 \right) - \rho_2 \left(\frac{h_2\, e^{-u}}{\int_\Sigma h_2\, e^{-u}\, dV_g} - 1 \right), \tag{4.1}$$

where ρ_1, ρ_2 are real parameters and h_1, h_2 are two smooth positive functions. For an introduction to the above equation see Section 1.2.

In this chapter we will give the first existence result in a doubly supercritical case, namely when $\rho_i > 8\pi\ i = 1, 2$, see Subsection 1.2.1. The argument presented here is stated in the paper [45] and the main result is the following:

Theorem 4.0.1 *Let h_1, h_2 be two smooth positive functions. Assume that $\rho_1, \rho_2 \in (8\pi, 16\pi)$. Then there exists a solution to the equation* (4.1)*.*

The plan of the chapter is the following: in Section 4.1 we state some preliminary results such as variants of the Moser-Trudinger inequality and a compactness property, in Section 4.2 we introduce the *rate of concentration* and the *center of mass* of a function and we provide a new improved Moser-Trudinger inequality and finally in Section 4.3 we prove the main result using min-max theory.

4.1 Preliminaries

In this section we collect some useful preliminary facts. We begin with a compactness result which is deduced from the blow-up theorem in [82].

Theorem 4.1.1 *Suppose that u_n satisfies*

$$-\Delta u_n = \rho_{1,n} \left(\frac{h_1\, e^{u_n}}{\int_\Sigma h_1\, e^{u_n}\, dV_g} - 1 \right) - \rho_{2,n} \left(\frac{h_2\, e^{-u_n}}{\int_\Sigma h_2\, e^{-u_n}\, dV_g} - 1 \right).$$

Assume that $\rho_{1,n}, \rho_{2,n} \in (8\pi, 16\pi)$ for any $n \in \mathbb{N}$ and that $\rho_{1,n} \to \rho_1 \in (8\pi, 16\pi)$ and $\rho_{2,n} \to \rho_2 \in (8\pi, 16\pi)$. Then the solution sequence $(u_n)_n$ (up to adding suitable constants) is uniformly bounded in $L^\infty(\Sigma)$ and there exist u and a subsequence $(u_{n_k})_k$ such that

$$u_{n_k} \to u,$$

where this u is a solution to (4.1) *for these ρ_1 and ρ_2.*

Proof. Since I_ρ is invariant under translation by constants in the argument, we can restrict ourselves to considering the subspace of $H^1(\Sigma)$ of functions with zero average.

Consider the blow-up sets of the sequence $(u_n)_n$ given by

$$S_1 =\Big\{x \in \Sigma : \exists x_n \to x \text{ such that } u_n(x_n) \to +\infty\Big\},$$

$$S_2 =\Big\{x \in \Sigma : \exists x_n \to x \text{ such that } u_n(x_n) \to -\infty\Big\}.$$

From the blow-up theorem in [82], it is sufficient to show that $S_1 \cap S_2 = \emptyset$. We argue by contradiction. Assume that $x_0 \in S_1 \cap S_2$. Define the blow-up values at x_0 by

$$m_1(x_0) = \lim_{r\to 0}\lim_{n\to+\infty} \rho_{1,n} \frac{\int_{B_r(x_0)} h_1(x)\, e^{u_n}\, dV_g}{\int_\Sigma h_1(x)\, e^{u_n}\, dV_g},$$

$$m_2(x_0) = \lim_{r\to 0}\lim_{n\to+\infty} \rho_{2,n} \frac{\int_{B_r(x_0)} h_2(x)\, e^{-u_n}\, dV_g}{\int_\Sigma h_2(x)\, e^{-u_n}\, dV_g}.$$

Since $\rho_{1,n}, \rho_{2,n} \in (8\pi, 16\pi)$, from the blow-up theorem in [82], we have

$$4\pi \le m_1(x_0) < 16\pi, \qquad 4\pi \le m_2(x_0) < 16\pi, \tag{4.2}$$

and

$$\big(m_1(x_0) - m_2(x_0)\big)^2 = 8\pi\big(m_1(x_0) + m_2(x_0)\big). \tag{4.3}$$

By the last equality we derive

$$m_1(x_0) = m_2(x_0) + 4\pi \pm 4\sqrt{\pi m_2(x_0) + \pi^2}.$$

First, let us consider the case $m_1(x_0) = m_2(x_0)+4\pi+4\sqrt{\pi m_2(x_0) + \pi^2}$. Using the fact that $4\pi \le m_2(x_0)$, we derive that $m_1(x_0) \ge 16\pi$, which is a contradiction to the first estimate in (4.2).

If instead we consider the case $m_1(x_0) = m_2(x_0)+4\pi-4\sqrt{\pi m_2(x_0) + \pi^2}$, the estimate $4\pi \le m_2(x_0) < 16\pi$ implies that $m_1(x_0) < 12\pi$. By interchanging the roles of $m_1(x_0)$ and $m_2(x_0)$, we obtain the same inequality for $m_2(x_0)$. Therefore we have

$$4\pi \le m_1(x_0) < 12\pi, \qquad 4\pi \le m_2(x_0) < 12\pi. \tag{4.4}$$

On the other hand, using (4.3) jointly with the fact that $m_i(x_0) \ge 4\pi$, $i = 1, 2$, we deduce that

$$|m_1(x_0) - m_2(x_0)| \ge 8\pi,$$

which is a contradiction to (4.4). ■

We collect now some versions of Moser-Trudinger inequalities. It is well known that an improved inequality will hold if e^u has integral bounded from below on different regions of Σ of positive mutual distance. Notice that for the Toda system (1.18) an analogous property was proved in [69], while a more general result in this direction was given in [9] (see Lemma 2.2.3 in Chapter 2).

Proposition 4.1.2 *([100]) For a fixed integer l, let $\Omega_1, \dots, \Omega_l$ be subsets of Σ satisfying $d(\Omega_i, \Omega_j) \ge \delta_0$ for $i \ne j$, where δ_0 is a positive real number, and let $\gamma_0 \in \big(0, \frac{1}{l}\big)$. Then, for any $\varepsilon > 0$ there exists a constant $C = C(\Sigma, l, \varepsilon, \delta_0, \gamma_0)$ such that*

$$l \log \int_\Sigma e^{u-\bar u}\, dV_g + \log \int_\Sigma e^{-u+\bar u}\, dV_g \le \frac{1}{16\pi - \varepsilon} \int_\Sigma |\nabla u|^2\, dV_g + C$$

for all the functions $u \in H^1(\Sigma)$ satisfying

$$\frac{\int_{\Omega_i} e^u\, dV_g}{\int_\Sigma e^u\, dV_g} \ge \gamma_0, \qquad \forall\, i \in \{1, \dots, l\}.$$

We next state a result which is a local version of the inequality (1.21), that will be of use later on.

Proposition 4.1.3 *Fix $\delta > 0$, and let $\Omega_1 \subset \Omega_2 \subset \Sigma$ be such that $d(\Omega_1, \partial\Omega_2) \geq \delta$. Then, for any $\varepsilon > 0$ there exists a constant $C = C(\varepsilon, \delta)$ such that for all $u \in H^1(\Sigma)$*

$$\log \int_{\Omega_1} e^u \, dV_g + \log \int_{\Omega_1} e^{-u} \, dV_g \leq \frac{1}{16\pi - \varepsilon} \int_{\Omega_2} |\nabla u|^2 \, dV_g + C.$$

Proof. The proof is developed exactly as in Proposition 2.3 of [71], with obvious modifications. Here we just sketch the proof for the reader's convenience. First, we consider a spectral decomposition of the Laplacian on Ω_2 (with Neumann boundary conditions), in order to write u as $u = v + w$ with $v \in L^\infty(\Omega_2)$ and $w \in H^1(\Omega_2)$. We next consider a smooth cutoff function χ with values into $[0,1]$ satisfying

$$\begin{cases} \chi(x) = 1 & \text{for } x \in \Omega_1, \\ \chi(x) = 0 & \text{if } d(x, \Omega) > \delta/2, \end{cases}$$

and then define $\tilde{w}(x) = \chi(x) w(x)$. We now apply the Moser-Trudinger inequality (1.21) to $\tilde{w}$ to deduce the desired inequality. ■

We give now a criterion which is a first step in studying the properties of the low sub-levels of I_ρ. We first state a lemma concerning a covering argument, which is a particular case of a more general setting in [71], Lemma 2.5.

Lemma 4.1.4 *([71]) Let $\delta_0 > 0$, $\gamma_0 > 0$ be fixed, and let $\Omega_{i,j} \subseteq \Sigma$, $i, j = 1, 2$, satisfy $d(\Omega_{i,j}, \Omega_{i,k}) \geq \delta_0$ for $j \neq k$. Suppose that $u \in H^1(\Sigma)$ is a function verifying*

$$\frac{\int_{\Omega_{1,j}} e^u \, dV_g}{\int_\Sigma e^u \, dV_g} \geq \gamma_0, \quad \frac{\int_{\Omega_{2,j}} e^{-u} \, dV_g}{\int_\Sigma e^{-u} \, dV_g} \geq \gamma_0, \quad j = 1, 2.$$

Then there exist positive constants $\tilde{\gamma}_0$, $\tilde{\delta}_0$, depending only on γ_0, δ_0, and two sets $\tilde{\Omega}_1, \tilde{\Omega}_2 \subseteq \Sigma$, depending also on u such that

$$d(\tilde{\Omega}_1, \tilde{\Omega}_2) \geq \tilde{\delta}_0; \qquad \frac{\int_{\tilde{\Omega}_i} e^u \, dV_g}{\int_\Sigma e^u \, dV_g} \geq \tilde{\gamma}_0, \quad \frac{\int_{\tilde{\Omega}_i} e^{-u} \, dV_g}{\int_\Sigma e^{-u} \, dV_g} \geq \tilde{\gamma}_0; \quad i = 1, 2.$$

Using this result it is indeed possible to obtain an improvement of the constant in the Moser-Trudinger inequality (1.21).

Proposition 4.1.5 *Let $u \in H^1(\Sigma)$ be a function satisfying the assumptions of Lemma 4.1.4 for some positive constants δ_0, γ_0. Then for any $\varepsilon > 0$ there exists $C = C(\varepsilon) > 0$, depending on ε, δ_0, and γ_0 such that*

$$\log \int_\Sigma e^{u - \bar{u}} \, dV_g + \log \int_\Sigma e^{-u + \bar{u}} \, dV_g \leq \frac{1}{32\pi - \varepsilon} \int_\Sigma |\nabla u|^2 \, dV_g + C.$$

Proof. To obtain the thesis we can argue exactly as in Proposition 2.6 of [71]. First we set $\tilde{\delta}_0, \tilde{\gamma}_0$ and $\tilde{\Omega}_1, \tilde{\Omega}_2$ as in Lemma 4.1.4. Then we apply Proposition 4.1.3 with $\tilde{\Omega}_i$ and $U_i = \{x \in \Omega : \; d(x, \tilde{\Omega}_i) < \tilde{\delta}_0/2\}$ for $i = 1, 2$. Observing that

$$\log \int_{\tilde{\Omega}_i} e^u dV_g \geq \log \left(\int_\Sigma e^u dV_g \right) + \log \tilde{\gamma}_0,$$

$$\log \int_{\tilde{\Omega}_i} e^{-u} dV_g \geq \log \left(\int_\Sigma e^{-u} dV_g \right) + \log \tilde{\gamma}_0$$

for $i = 1, 2$, and that $U_1 \cap U_2 = \emptyset$, we deduce the thesis. ■

Proposition 4.1.5 implies that on low sub-levels of the functional I_ρ, at least one of the components of the couple (e^u, e^{-u}) must be very concentrated around a certain point. We will present in the sequel a more detailed description of the topology of low sub-levels.

4.2 Improved inequality

Following the ideas presented by Malchiodi and Ruiz in [71], in this section we exhibit an improved Moser-Trudinger inequality under suitable conditions of concentration of the involved function.

First, we give continuous definitions of *center of mass* and *scale of concentration* of positive functions normalized in L^1. Let us consider the set

$$A = \left\{ f \in L^1(\Sigma) \ : \ f > 0 \ \text{ a. e. and } \int_\Sigma f dV_g = 1 \right\},$$

endowed with the topology inherited from $L^1(\Sigma)$. Then we have the following result.

Proposition 4.2.1 *([71]) Let us fix a constant $R > 1$. Then there exist $\delta = \delta(R) > 0$ and a continuous map:*

$$\psi : A \to \overline{\Sigma}_\delta, \qquad \psi(f) = (\beta, \sigma),$$

satisfying the following property: for any $f \in A$ there exists $p \in \Sigma$ such that

a) *$d(p, \beta) \leq C'\sigma$ for $C' = \max\{3R + 1, \delta^{-1} diam(\Sigma)\}$.*

b) *There holds:*

$$\int_{B_\sigma(p)} f \, dV_g > \tau, \qquad \int_{B_{R\sigma}(p)^c} f \, dV_g > \tau,$$

where $\tau > 0$ depends only on R and Σ.

This result is obtained in several steps, which we summarize in the sequel. The explicit definition of the map $\psi(f) = (\beta, \sigma)$ is given below.

First, take $R_0 = 3R$, and define $\sigma : A \times \Sigma \to (0, +\infty)$ such that:

$$\int_{B_{\sigma(x,f)}(x)} f \, dV_g = \int_{B_{R_0 \sigma(x,f)}(x)^c} f \, dV_g. \tag{4.5}$$

The map $\sigma(x, f)$ is clearly uniquely determined and continuous. Moreover we have the following lemma.

Lemma 4.2.2 *([71]) The map σ satisfies:*

$$d(x, y) \leq R_0 \max\{\sigma(x, f), \sigma(y, f)\} + \min\{\sigma(x, f), \sigma(y, f)\}. \tag{4.6}$$

We now define

$$T : A \times \Sigma \to \mathbb{R}, \qquad T(x, f) = \int_{B_{\sigma(x,f)}(x)} f \, dV_g.$$

Lemma 4.2.3 *([71]) If $x_0 \in \Sigma$ is such that $T(x_0, f) = \max_{y \in \Sigma} T(y, f)$, then we have $\sigma(x_0, f) < 3\,\sigma(x, f)$ for any other $x \in \Sigma$.*

As a consequence of the previous lemma, one can obtain the following:

Lemma 4.2.4 *([71]) There exists a fixed $\tau > 0$ such that*

$$\max_{x \in \Sigma} T(x, f) > \tau > 0 \qquad \text{for all } f \in A.$$

Let us define

$$\sigma : A \to \mathbb{R}, \qquad \sigma(f) = 3 \min\{\sigma(x, f) : \ x \in \Sigma\},$$

which is obviously a continuous function. Given τ as in Lemma 4.2.4, consider the set

$$S(f) = \{x \in \Sigma \ : \ T(x, f) > \tau, \ \sigma(x, f) < \sigma(f)\}, \tag{4.7}$$

which is a nonempty open set for any $f \in A$, by Lemmas 4.2.3 and 4.2.4. Moreover, from (4.6), we have that

$$diam\big(S(f)\big) \leq (R_0 + 1)\sigma(f). \tag{4.8}$$

As in Subsection 3.1.2 we can assume that $\Sigma \subset \mathbb{R}^N$ isometrically, $N \in \mathbb{N}$ and take an open tubular neighborhood $\Sigma \subset U \subset \mathbb{R}^N$ of Σ, and $\delta > 0$ small enough so that

$$co\big[B_{(R_0+1)\delta}(x) \cap \Sigma\big] \subset U \quad \forall\, x \in \Sigma, \tag{4.9}$$

where co denotes the convex hull in $\mathbb{R}^N$.

We define now

$$\eta(f) = \frac{\displaystyle\int_\Sigma \big(T(x,f)-\tau\big)^+ \big(\sigma(f)-\sigma(x,f)\big)^+ x \, dV_g}{\displaystyle\int_\Sigma \big(T(x,f)-\tau\big)^+ \big(\sigma(f)-\sigma(x,f)\big)^+ \, dV_g} \in \mathbb{R}^N,$$

which can be interpreted as a center of mass in $\mathbb{R}^N$. As observed in Subsection 3.1.2, the integrands become nonzero only on the set $S(f)$. Moreover, whenever $\sigma(f) \leq \delta$, (4.8) and (4.9) imply that $\eta(f) \in U$, and so we can define

$$\beta : \big\{ f \in A : \; \sigma(f) \leq \delta \big\} \to \Sigma, \quad \beta(f) = P \circ \eta(f),$$

where $P : U \to \Sigma$ is the orthogonal projection.

Then the map $\psi(f) = \big(\beta(f), \sigma(f)\big)$ satisfies the conditions given by Proposition 4.2.1. If $\sigma(f) \geq \delta$, β is not defined. Observe that *a)* is then satisfied for any $\beta \in \Sigma$.

Remark 4.2.5 *The above map $\psi(f) = (\beta, \sigma)$ gives us a center of mass of f and its scale of concentration around that point. The identification in $\overline{\Sigma}_\delta$ is somehow natural, indeed, if σ exceeds a certain positive constant, we do not have concentration at a point and so β could not be defined.*

We next state an improved Moser-Trudinger inequality for functions $u \in H^1(\Sigma)$ such that both e^u and e^{-u} are concentrated at the same point with the same rate of concentration. In terms of Proposition 4.2.1, we have the following result. Notice that for the Toda system (1.18) an analogous improved inequality was given in [71], Proposition 3.2.

Proposition 4.2.6 *Given any $\varepsilon > 0$, there exist $R = R(\varepsilon) > 1$ and ψ as given in Proposition 4.2.1, such that for any $u \in H^1(\Sigma)$ with:*

$$\psi\left(\frac{e^u}{\int_\Sigma e^u dV_g}\right) = \psi\left(\frac{e^{-u}}{\int_\Sigma e^{-u} dV_g}\right),$$

the following inequality holds:

$$\log \int_\Sigma e^{u-\bar{u}} \, dV_g + \log \int_\Sigma e^{-u+\bar{u}} \, dV_g \leq \frac{1}{32\pi - \varepsilon} \int_\Sigma |\nabla u|^2 \, dV_g + C,$$

for some $C = C(\varepsilon)$.

Before proving the proposition, we need some preliminary lemmas concerning Moser-Trudinger type inequality for small balls, and also for annuli with small internal radius. The first one is obtained just by using a dilation argument.

Lemma 4.2.7 *For any $\varepsilon > 0$ there exists $C = C(\varepsilon) > 0$ such that*

$$\log \int_{B_{s/2}(p)} e^u \, dV_g + \log \int_{B_{s/2}(p)} e^{-u} \, dV_g \;\leq\; \frac{1}{16\pi - \varepsilon} \int_{B_s(p)} |\nabla u|^2 \, dV_g + 4 \log s + C$$

for any $u \in H^1(\Sigma)$, $p \in \Sigma$, $s > 0$ small.

Proof. Notice that, as $s \to 0$ we consider quantities defined on smaller and smaller geodesic balls centered at p. By considering normal geodesic coordinates at p, gradients, averages and the volume element will almost correspond to the Euclidean ones. If we assume that near p the metric of Σ is flat, we will get negligible error terms which will be omitted.

We just perform a convenient dilation of u given by

$$v(x) = u(sx + p).$$

We have the following equalities:

$$\int_{B_s(p)} |\nabla_g u|^2 \, dV_g = \int_{B_1(0)} |\nabla v|^2 \, dV_g,$$

$$\int_{B_{s/2}(p)} e^u \, dV_g = s^2 \int_{B_{1/2}(0)} e^v \, dV_g.$$

We apply then Proposition 4.1.3 to the function v to deduce the desired inequality. ■

Remark 4.2.8 *Observe that in Lemma 4.2.7 and in the results that will be present in the sequel there is no explicit dependence of the average of u, due to the fact that the average of u is cancelled by the average of $-u$.*

We next deduce a Moser-Trudinger type inequality on thick annuli (recall the notation in Section 1.3). In order to do this, we use the Kelvin transform to exploit the geometric properties of the problem.

Lemma 4.2.9 *Given $\varepsilon > 0$, there exists a fixed $r_0 > 0$ (depending only on Σ and ε) satisfying the following property: for any $r \in (0, r_0)$ fixed, there exists $C = C(r, \varepsilon) > 0$ such that, for any $u \in H^1(\Sigma)$ with $u = c \in \mathbb{R}$ in $\partial B_{2r}(p)$,*

$$\log \int_{A_p(s,r)} e^u \, dV_g + \log \int_{A_p(s,r)} e^{-u} \, dV_g \leq \frac{1}{16\pi - \varepsilon} \int_{A_p(s/2,2r)} |\nabla u|^2 \, dV_g - 4 \log s + C,$$

with $p \in \Sigma$, $s \in (0, r)$.

Proof. As in the proof of Lemma 4.2.7, by taking r_0 small enough, also here the metric becomes close to the Euclidean one. We can then assume that the metric is flat around the point p.

We consider the Kelvin transform $K : A_p(s/2, 2r) \to A_p(s/2, 2r)$ given by

$$K(x) = p + rs \frac{x - p}{|x - p|^2}.$$

Observe that K maps the interior boundary of $A_p(s/2, 2r)$ onto the exterior one and viceversa. We next define the function $\tilde{u} \in H^1(B_{2r}(p))$ as:

$$\tilde{u}(x) = \begin{cases} u\big(K(x)\big) & \text{if } |x - p| \geq s/2, \\ c & \text{if } |x - p| \leq s/2. \end{cases}$$

Our goal is to apply the local Moser-Trudinger inequality given by Proposition 4.1.3 to $\tilde{u}$. First of all, observe that

$$\int_{A_p(s,r)} e^{\tilde{u}} \, dV_g = \int_{A_p(s,r)} e^{u(K(x))} \, dV_g = \int_{A_p(s,r)} e^{u(x)} \frac{s^2 r^2}{|x - p|^4} \, dV_g, \tag{4.10}$$

since the Jacobian of K is $J\big(K(x)\big) = -r^2 s^2 |x - p|^{-4}$. Moreover, for $|x - p| \geq s/2$, we have

$$|\nabla \tilde{u}(x)|^2 = |\nabla u(K(x))|^2 \frac{s^2 r^2}{|x - p|^4} \tag{4.11}$$

Therefore,

$$\begin{aligned} \log \int_{A_p(s,r)} e^u \, dV_g + \log \int_{A_p(s,r)} e^{-u} \, dV_g + 4 \log s \; &= \; \log \int_{A_p(s,r)} e^u s^2 \, dV_g + \log \int_{A_p(s,r)} e^{-u} s^2 \, dV_g \\ &\leq \; \log \int_{A_p(s,r)} e^u \, \frac{s^2}{r^2} \, dV_g + \log \int_{A_p(s,r)} e^{-u} \, \frac{s^2}{r^2} \, dV_g + C \\ &\leq \; \log \int_{A_p(s,r)} e^u \, \frac{s^2 r^2}{|x - p|^4} \, dV_g + \log \int_{A_p(s,r)} e^{-u} \, \frac{s^2 r^2}{|x - p|^4} \, dV_g + C, \end{aligned}$$

where we have used the trivial inequality $r \geq |x-p|$ for $x \in A_p(s,r)$. By using (4.10), applying Proposition 4.1.3 to $\tilde{u}$ and then using (4.11), we have

$$\begin{aligned}
&\log \int_{A_p(s,r)} e^u \frac{s^2 r^2}{|x-p|^4}\, dV_g + \log \int_{A_p(s,r)} e^{-u} \frac{s^2 r^2}{|x-p|^4}\, dV_g + C \;= \\
&= \log \int_{A_p(s,r)} e^{u(K(x))}\, dV_g + \log \int_{A_p(s,r)} e^{-u(K(x))}\, dV_g + C \\
&\leq \frac{1}{16\pi - \varepsilon} \int_{B_{2r}(p)} |\nabla \tilde{u}|^2\, dV_g + C = \frac{1}{16\pi - \varepsilon} \int_{A_p(s/2,2r)} |\nabla \tilde{u}|^2\, dV_g + C \\
&= \frac{1}{16\pi - \varepsilon} \int_{A_p(s/2,2r)} |\nabla u(K(x))|^2 \frac{r^2 s^2}{|x-p|^4}\, dV_g + C \\
&= \frac{1}{16\pi - \varepsilon} \int_{A_p(s/2,2r)} |\nabla u|^2\, dV_g + C.
\end{aligned}$$

This concludes the proof of the lemma. ■

Remark 4.2.10 *We are now able to prove the improved inequality given in Proposition 4.2.6. The spirit of the proof is to use jointly Lemmas 4.2.7 and 4.2.9. Indeed, assume that e^u and e^{-u} concentrate around the same point at the same rate (in the sense of Proposition 4.2.1). If we sum the inequalities given by Lemmas 4.2.7 and 4.2.9, the extra term $4 \log s$ cancels and we can deduce the improved inequality of Proposition 4.2.6.*

We have to manage the case that when $\psi\left(\frac{e^u}{\int_\Sigma e^u dV_g}\right) = \psi\left(\frac{e^{-u}}{\int_\Sigma e^{-u} dV_g}\right)$ we do not really have concentration around the same point. Moreover, the property in Lemma 4.2.9 of u being constant on the boundary of a ball need not be satisfied.

Proof of Proposition 4.2.6. Fixed $\varepsilon > 0$, take $R > 1$ (depending only on ε) and let ψ be the continuous map given by Proposition 4.2.1. Fix also $\delta > 0$ small.

Let $u \in H^1(\Sigma)$ be a function with $\int_\Sigma u\, dV_g = 0$, such that

$$\psi\left(\frac{e^u}{\int_\Sigma e^u\, dV_g}\right) = \psi\left(\frac{e^{-u}}{\int_\Sigma e^{-u}\, dV_g}\right) = (\beta, \sigma) \in \overline{\Sigma}_\delta.$$

If $\sigma \geq \frac{\delta}{R^2}$, then applying Proposition 4.1.5 we get the result. Therefore, assume $\sigma < \frac{\delta}{R^2}$. Proposition 4.2.1 implies the existence of $\tau > 0$, $p_1,\ p_2 \in \Sigma$ satisfying:

$$\int_{B_\sigma(p_1)} e^u\, dV_g \geq \tau \int_\Sigma e^u\, dV_g, \qquad \int_{B_\sigma(p_2)} e^{-u}\, dV_g \geq \tau \int_\Sigma e^{-u}\, dV_g \tag{4.12}$$

and

$$\int_{B_{R\sigma}(p_1)^c} e^u\, dV_g \geq \tau \int_\Sigma e^u\, dV_g \qquad \int_{B_{R\sigma}(p_2)^c} e^{-u}\, dV_g \geq \tau \int_\Sigma e^{-u}\, dV_g, \tag{4.13}$$

with $d(p_1, p_2) \leq (6R+2)\sigma$. We divide the proof into two cases:

CASE 1: Assume that

$$\int_{A_{p_1}(R\sigma,\delta)} e^u\, dV_g \geq \tau/2 \int_\Sigma e^u\, dV_g, \qquad \int_{A_{p_2}(R\sigma,\delta)} e^{-u}\, dV_g \geq \tau/2 \int_\Sigma e^{-u}\, dV_g. \tag{4.14}$$

In order to satisfy the hypothesis of Lemma 4.2.9, we need to modify our function outside a certain ball. Via a dyadic decomposition, choose $k \in \mathbb{N}$, $k \leq 2\varepsilon^{-1}$, such that

$$\int_{A_{p_1}(2^{k-1}\delta, 2^{k+1}\delta)} |\nabla u|^2\, dV_g \leq \varepsilon \int_\Sigma |\nabla u|^2\, dV_g.$$

We define $\tilde{u} \in H^1(\Sigma)$ by:

$$\begin{cases} \tilde{u}(x) = u(x) & x \in B_{2^k\delta}(p_1), \\ \Delta\tilde{u}(x) = 0 & x \in A_{p_1}(2^k\delta, 2^{k+1}\delta), \\ \tilde{u}(x) = c & x \notin B_{2^{k+1}\delta}(p_1), \end{cases}$$

where $c \in \mathbb{R}$. Moreover, since we want to apply Lemma 4.2.9 to $\tilde{u}$, we have to choose δ small enough so that $2^{3\varepsilon^{-1}}\delta < r_0$, where r_0 is given by that lemma.

We have that

$$\int_{A_{p_1}(2^{k-1}\delta, 2^{k+1}\delta)} |\nabla\tilde{u}|^2 \, dV_g \leq C \int_{A_{p_1}(2^{k-1}\delta, 2^{k+1}\delta)} |\nabla u|^2 \, dV_g \leq C\varepsilon \int_\Sigma |\nabla u|^2 \, dV_g, \tag{4.15}$$

for some universal constant $C > 0$.

Case 1.1: Suppose that $d(p_1, p_2) \leq R^{\frac{1}{2}}\sigma$.

We first apply Lemma 4.2.7 to u for $p = p_1$ and $s = 2(R^{1/2} + 1)\sigma$, and take into account (4.12), to obtain:

$$\begin{aligned} \frac{1}{16\pi - \varepsilon} \int_{B_s(p)} |\nabla u|^2 \, dV_g &\geq \log \int_{B_{s/2}(p)} e^u \, dV_g + \log \int_{B_{s/2}(p)} e^{-u} \, dV_g - 4\log\sigma - C \\ &\geq \log \int_\Sigma e^u \, dV_g + \log \int_\Sigma e^{-u} \, dV_g - 4\log\sigma - C. \end{aligned} \tag{4.16}$$

We next apply Lemma 4.2.9 to $\tilde{u}$ for $p = p_1$, $s' = 4(R^{1/2} + 1)\sigma$ and $r = 2^{k+1}\delta$:

$$\frac{1}{16\pi - \varepsilon} \int_{A_p(s'/2, 2r)} |\nabla\tilde{u}|^2 \, dV_g \geq \log \int_{A_p(s',r)} e^{\tilde{u}} \, dV_g + \log \int_{A_p(s',r)} e^{-\tilde{u}} \, dV_g + 4\log\sigma - C. \tag{4.17}$$

Using the estimate (4.13), we get

$$\frac{1}{16\pi - \varepsilon} \int_{A_p(s'/2, 2r)} |\nabla\tilde{u}|^2 \, dV_g \geq \log \int_\Sigma e^u \, dV_g + \log \int_\Sigma e^{-u} \, dV_g + 4\log\sigma - C. \tag{4.18}$$

Finally, combining (4.16), (4.18) and (4.15) we obtain our thesis (after renaming ε conveniently).

Case 1.2: Suppose $d(p_1, p_2) \geq R^{\frac{1}{2}}\sigma$ and

$$\int_{B_{R^{1/3}\sigma}(p_1)} e^{-u} \, dV_g \geq \tau/4 \int_\Sigma e^{-u} \, dV_g.$$

Here we argue as in Case 1.1. First, we apply Lemma 4.2.7 to u for $p = p_1$ and $s = 2(R^{1/3} + 1)\sigma$. Then we use Lemma 4.2.9 with $\tilde{u}$ for $p = p_1$, $s' = 4(R^{1/3} + 1)\sigma$ and $r = 2^{k+1}\delta$.

Case 1.3: Suppose $d(p_1, p_2) \geq R^{\frac{1}{2}}\sigma$ and

$$\int_{B_{R^{1/3}\sigma}(p_2)} e^{u} \, dV_g \geq \tau/4 \int_\Sigma e^{u} \, dV_g.$$

This case can be treated as in Case 1.2, just by interchanging the indices.

Case 1.4: Suppose $d(p_1, p_2) \geq R^{\frac{1}{2}}\sigma$ and

$$\int_{B_{R^{1/3}\sigma}(p_2)} e^{u} \, dV_g \leq \tau/4 \int_\Sigma e^{u} \, dV_g, \qquad \int_{B_{R^{1/3}\sigma}(p_1)} e^{-u} \, dV_g \leq \tau/4 \int_\Sigma e^{-u} \, dV_g.$$

Take $n \in \mathbb{N}$, $n \leq 2\varepsilon^{-1}$ so that

$$\sum_{i=1}^{2} \int_{A_{p_i}(2^{n-1}\sigma, 2^{n+1}\sigma)} |\nabla u|^2 \, dV_g \leq \varepsilon \int_\Sigma |\nabla u|^2 \, dV_g,$$

where we have chosen R such that $2^{3\varepsilon^{-1}} < R^{1/3}$. We define now the function $v \in H^1(\Sigma)$ by:

$$\begin{cases} v(x) = u(x) & x \in B_{2^n\sigma}(p_1) \cup B_{2^n\sigma}(p_2), \\ \Delta v(x) = 0 & x \in A_{p_1}(2^n\sigma, 2^{n+1}\sigma) \cup A_{p_2}(2^n\sigma, 2^{n+1}\sigma), \\ v(x) = 0 & x \notin B_{2^{n+1}\sigma}(p_1) \cup B_{2^{n+1}\sigma}(p_2). \end{cases}$$

As before we have that

$$\sum_{i=1}^{2} \int_{A_{p_i}(2^n\sigma,2^{n+1}\sigma)} |\nabla v|^2 \, dV_g \leq C \sum_{i=1}^{2} \int_{A_{p_i}(2^{n-1}\sigma,2^{n+1}\sigma)} |\nabla u|^2 \, dV_g \leq C\varepsilon \int_\Sigma |\nabla u|^2 \, dV_g,$$

where $C > 0$ is a universal constant.

Taking into account (4.12), we now apply Lemma 4.2.7 to v with $p = p_1$ and $s = 4(6R+2)\sigma$:

$$\begin{aligned} \frac{1}{16\pi - \varepsilon} \int_{B_{p_1}(2^n\sigma) \cup B_{p_2}(2^n\sigma)} |\nabla u|^2 \, dV_g + C\varepsilon \int_\Sigma |\nabla u|^2 \, dV_g \;&\geq\; \frac{1}{16\pi - \varepsilon} \int_{B_p(s)} |\nabla v|^2 \, dV_g \\ &\geq\; \log \int_{B_p(s/2)} e^v \, dV_g + \log \int_{B_p(s/2)} e^{-v} \, dV_g - 4 \log \sigma - C \\ &\geq\; \log \int_\Sigma e^u \, dV_g + \log \int_\Sigma e^{-u} \, dV_g - 4 \log \sigma - C. \end{aligned} \tag{4.19}$$

Next, we define $w \in H^1(\Sigma)$ by:

$$\begin{cases} w(x) = 0 & x \in B_{p_1}(2^n\sigma) \cup B_{p_2}(2^n\sigma), \\ \Delta w(x) = 0 & x \in A_{p_1}(2^n\sigma, 2^{n+1}\sigma) \cup A_{p_2}(2^n\sigma, 2^{n+1}\sigma), \\ w(x) = \tilde{u}(x) & x \notin B_{p_1}(2^{n+1}\sigma) \cup B_{p_2}(2^{n+1}\sigma). \end{cases}$$

Again we have

$$\sum_{i=1}^{2} \int_{A_{p_i}(2^n\sigma,2^{n+1}\sigma)} |\nabla w|^2 \, dV_g \leq C \sum_{i=1}^{2} \int_{A_{p_i}(2^{n-1}\sigma,2^{n+1}\sigma)} |\nabla u|^2 \, dV_g \leq C\varepsilon \int_\Sigma |\nabla u|^2 \, dV_g,$$

where also here C is a universal constant.

We apply Lemma 4.2.9 to w for any point p' such that $d(p', p_1) = \frac{1}{2} R^{1/3}\sigma$, $s' = \sigma$ and $r = 2^{k+1}\delta$, to obtain:

$$\begin{aligned} \frac{1}{16\pi - \varepsilon} \int_{(B_{2^{n+1}\sigma}(p_1) \cup B_{2^{n+1}\sigma}(p_2))^c} |\nabla u|^2 \, dV_g + C\varepsilon \int_\Sigma |\nabla u|^2 \, dV_g \;&\geq\; \frac{1}{16\pi - \varepsilon} \int_{A_{p'}(s'/2,2r)} |\nabla w|^2 \, dV_g \\ &\geq\; \log \int_{A_{p'}(s',r)} e^w \, dV_g + \log \int_{A_{p'}(s',r)} e^{-w} \, dV_g + 4 \log \sigma - C. \end{aligned}$$

We now use (4.14) and the hypothesis of Case 1.4 to conclude that

$$\begin{aligned} \frac{1}{16\pi - \varepsilon} \int_{(B_{2^n\sigma}(p_1) \cup B_{2^n\sigma}(p_2))^c} |\nabla u|^2 \, dV_g + C\varepsilon \int_\Sigma |\nabla u|^2 \, dV_g \;&\geq \\ \geq\; \log \int_\Sigma e^u \, dV_g + \log \int_\Sigma e^{-u} \, dV_g + 4 \log \sigma - C.& \end{aligned} \tag{4.20}$$

The inequality (4.20) jointly with (4.19) implies our result (after properly renaming ε).

CASE 2: Assume that

$$\int_{B_\delta(p_1)^c} e^u \, dV_g \geq \tau/2 \int_\Sigma e^u \, dV_g \quad \text{or} \quad \int_{B_\delta(p_2)^c} e^{-u} \, dV_g \geq \tau/2 \int_\Sigma e^{-u} \, dV_g.$$

Without loss of generality, suppose that the first alternative holds true. Let now $\delta' = \frac{\delta}{2^{3/\varepsilon}}$. If moreover:

$$\int_{B_{\delta'}(p_2)^c} e^{-u}\, dV_g \geq \tau/2 \int_\Sigma e^{-u}\, dV_g,$$

then we can apply Proposition 4.1.5 to deduce the thesis. Therefore we can assume that

$$\int_{A_{p_2}(R\sigma,\delta')} e^{-u}\, dV_g \geq \tau/2 \int_\Sigma e^{-u}\, dV_g. \tag{4.21}$$

We can apply the whole procedure of Case 1 to u, just by replacing δ with δ'. In fact, as in Case 1.1, we would get the inequalities (4.16) and (4.17). However, in this case we have to manage the fact that we do not know whether holds

$$\int_{A_p(s',r)} e^{u}\, dV_g \geq \alpha \int_\Sigma e^{u}\, dV_g,$$

for some fixed $\alpha > 0$. This property is needed in (4.17) to get the estimate

$$\log \int_{A_p(s',r)} e^{\tilde{u}}\, dV_g \geq \log \int_\Sigma e^{u}\, dV_g - C,$$

which allows us to deduce (4.18). To do this, we first apply Jensen and Poincaré-Wirtinger inequalities, to get

$$\log \int_{A_p(s',r)} e^{\tilde{u}}\, dV_g \geq \log \int_{A_p(r/8,r/4)} e^{u}\, dV_g \geq$$

$$\log \fint_{A_{p_1}(r/8,r/4)} e^{u}\, dV_g - C \geq \fint_{A_{p_1}(r/8,r/4)} u\, dV_g - C \geq -\varepsilon \int_\Sigma |\nabla u|^2\, dV_g - C.$$

Therefore, taking into account (4.21) and the last inequality, from (4.17) we obtain (after properly renaming ε):

$$\frac{1}{16\pi - \varepsilon} \int_{A_p(s'/2,2r)} |\nabla \tilde{u}|^2\, dV_g \geq \log \int_\Sigma e^{u}\, dV_g + 4\log\sigma - C. \tag{4.22}$$

Next, we apply Proposition 4.1.3, to get

$$\frac{1}{16\pi - \varepsilon} \int_{B_{\delta/2}(p_1)^c} |\nabla u|^2\, dV_g \geq \log \int_{B_\delta(p_1)^c} e^{u}\, dV_g + \log \int_{B_\delta(p_1)^c} e^{-u}\, dV_g.$$

Reasoning as above and using the hypothesis of Case 2, we can deduce:

$$\frac{1}{16\pi - \varepsilon} \int_{B_\delta(p_1)^c} |\nabla u|^2\, dV_g \geq \log \int_\Sigma e^{u}\, dV_g + 4\log\sigma - C. \tag{4.23}$$

Finally we obtain our result by combining (4.23), (4.22) and (4.16).

If we are under the conditions of Cases 1.2, 1.3 and 1.4, the thesis follows arguing in the same way. ■

Remark 4.2.11 *Our goal is to use Proposition 4.2.6 to obtain a lower bound of the functional I_ρ under suitable conditions. The presence of the two functions h_1 and h_2 in I_ρ is not so relevant because of the following estimates:*

$$\begin{aligned}\log \int_\Sigma h_1(x)\, e^{u}\, dV_g &\leq \log \int_\Sigma e^{u}\, dV_g + \log \|h_1\|_\infty,\\ \log \int_\Sigma h_2(x)\, e^{-u}\, dV_g &\leq \log \int_\Sigma e^{-u}\, dV_g + \log \|h_2\|_\infty.\end{aligned}$$

4.3 Min-max scheme

Let $\overline{\Sigma}_\delta$ be the topological cone over Σ defined in (1.26), and let us set

$$\overline{D}_\delta = diag\big(\overline{\Sigma}_\delta \times \overline{\Sigma}_\delta\big) = \big\{(\vartheta_1, \vartheta_2) \in \overline{\Sigma}_\delta \times \overline{\Sigma}_\delta \ : \ \vartheta_1 = \vartheta_2\big\},$$

$$X = \big(\overline{\Sigma}_\delta \times \overline{\Sigma}_\delta\big) \setminus \overline{D}_\delta.$$

Let $\varepsilon > 0$ be sufficiently small and let R, δ, ψ be as in Proposition 4.2.1. Consider then the map Ψ defined by

$$\Psi(u) = \left(\psi\left(\frac{e^u}{\int_\Sigma e^u \, dV_g}\right), \psi\left(\frac{e^{-u}}{\int_\Sigma e^{-u} \, dV_g}\right)\right). \tag{4.24}$$

By Proposition 4.2.6 and Remark 4.2.11, we have a lower bound of the functional I_ρ on functions u such that $u \in \overline{D}_\delta$. Therefore, there exists a large $L > 0$ such that if $I_\rho(u) \leq -L$ then it follows that $\Psi(u) \in X$.

In [71] the authors proved that even though the set X is non compact, it retracts to some compact subset $\mathcal{X}_\nu$. Indeed, we have the following lemma.

Lemma 4.3.1 *([71]) For $\nu \ll \delta$, define*

$$\mathcal{X}_{\nu,1} = \Big\{\big((x_1, t_1), (x_2, t_2)\big) \in X : |t_1 - t_2|^2 + d(x_1, x_2)^2 \geq \delta^4, \\ \max\{t_1, t_2\} < \delta, \min\{t_1, t_2\} \in \big[\nu^2, \nu\big]\Big\};$$

$$\mathcal{X}_{\nu,2} = \Big\{\big((x_1, t_1), (x_2, t_2)\big) \in X \ : \ \max\{t_1, t_2\} = \delta, \min\{t_1, t_2\} \in \big[\nu^2, \nu\big]\Big\},$$

and set

$$\mathcal{X}_\nu = \big(\mathcal{X}_{\nu,1} \cup \mathcal{X}_{\nu,2}\big) \subseteq X.$$

Then there is a retraction R_ν of X onto $\mathcal{X}_\nu$.

Our next goal is to introduce a family of test functions labelled on the set $\mathcal{X}_\nu$ on which the functional I_ρ attains large negative values. For $(\vartheta_1, \vartheta_2) = \big((x_1, t_1), (x_2, t_2)\big) \in \mathcal{X}_\nu$ define

$$\varphi(y) = \varphi_{(\vartheta_1, \vartheta_2)}(y) = \log \frac{\big(1 + \tilde{t}_2^2 d(x_2, y)^2\big)^2}{\big(1 + \tilde{t}_1^2 d(x_1, y)^2\big)^2}, \tag{4.25}$$

where

$$\tilde{t}_i = \tilde{t}_i(t_i) = \begin{cases} \frac{1}{t_i} & \text{for } t_i \leq \frac{\delta}{2}, \\ -\frac{4}{\delta^2}(t_i - \delta) & \text{for } t_i \geq \frac{\delta}{2}, \end{cases}$$

for $i = 1, 2$.

We start by proving the following estimate.

Lemma 4.3.2 *For ν sufficiently small, and for $(\vartheta_1, \vartheta_2) \in \mathcal{X}_\nu$, there exists a constant $C = C(\delta, \Sigma) > 0$, depending only on Σ and δ, such that*

$$\frac{1}{C} \frac{t_1^2}{t_2^4} \leq \int_\Sigma e^\varphi \, dV_g \leq C \frac{t_1^2}{t_2^4}. \tag{4.26}$$

Proof. First, observe that the following equality holds true for some fixed positive constant C_0:

$$\int_{\mathbb{R}^2} \frac{1}{\big(1 + \lambda^2 |x|^2\big)^2} \, dx = \frac{C_0}{\lambda^2}; \qquad \lambda > 0. \tag{4.27}$$

To prove the lemma, we distinguish the two cases

$$|t_1 - t_2| \geq \delta^3 \qquad \text{and} \qquad |t_1 - t_2| < \delta^3,$$

in order to exploit the properties of $\mathcal{X}_\nu$. Starting with the first alternative, by the definition of $\mathcal{X}_\nu$ and by the fact that $\nu \ll \delta$, it turns out that one of the t_i's belongs to $[\nu^2, \nu]$, while the other is greater or equal to $\frac{\delta^3}{2}$.

If $t_1 \in [\nu^2, \nu]$ and if $t_2 \geq \frac{\delta^3}{2}$ then the function $1 + \tilde{t}_2^2 d(x_2, y)^2$ is bounded above and below by two positive constants depending only on Σ and δ. Therefore, using (4.27) we get

$$\frac{t_1^2}{C} = \frac{1}{C\tilde{t}_1^2} \leq \int_\Sigma e^{\varphi(y)}\, dV_g(y) \leq \frac{C}{\tilde{t}_1^2} = Ct_1^2.$$

On the other hand, if $t_2 \in [\nu^2, \nu]$ and if $t_1 \geq \frac{\delta^3}{2}$ then the function $1 + \tilde{t}_1^2 d(x_1, y)^2$ is bounded above and below by two positive constants depending only on Σ and δ, hence

$$\int_\Sigma e^{\varphi(y)}\, dV_g(y) \geq \frac{1}{C}\int_\Sigma \left(1 + \tilde{t}_2^2 d(x_2, y)^2\right)^2 dV_g(y) \geq \frac{\tilde{t}_2^4}{C} = \frac{1}{Ct_2^4},$$

and similarly

$$\int_\Sigma e^{\varphi(y)}\, dV_g(y) \leq C\int_\Sigma \left(1 + \tilde{t}_2^2 d(x_2, y)^2\right)^2 dV_g(y) \leq C\tilde{t}_2^4 = \frac{C}{t_2^4}.$$

In both the last two cases we then obtain the conclusion.

Suppose now that we are in the second alternative, i.e. $|t_1 - t_2| < \delta^3$. Then by the definition of $\mathcal{X}_\nu$ we have that $d(x_1, x_2) \geq \frac{\delta^2}{2}$ and that $t_1, t_2 \leq \nu + \delta^3$. Using (4.27) we obtain

$$\int_\Sigma e^{\varphi(y)}\, dV_g(y) \geq \int_{B_{\delta^3}(x_1)} e^{\varphi(y)}\, dV_g(y) \geq \frac{1}{C}\frac{\left(1 + \tilde{t}_2^2 d(x_1, x_2)^2\right)^2}{\tilde{t}_1^2} \geq \frac{1}{C}\frac{t_1^2}{t_2^4}.$$

In an analogous way we derive

$$\int_{B_{\delta^3}(x_1)} e^{\varphi(y)}\, dV_g(y) \leq C\frac{\left(1 + \tilde{t}_2^2 d(x_1, x_2)^2\right)^2}{\tilde{t}_1^2} \leq C\frac{t_1^2}{t_2^4}.$$

Finally, by the estimate

$$\int_{B_{\delta^3}(x_1)^c} e^{\varphi(y)}\, dV_g(y) \leq \frac{C}{\tilde{t}_1^4}\int_{B_{\delta^3}(x_1)^c} \left(1 + \tilde{t}_2^2 d(x_2, y)^2\right)^2 dV_g(y) \leq C\frac{t_1^4}{t_2^4},$$

we are done. ∎

Remark 4.3.3 *Notice that for $e^{-\varphi}$ the same result holds true just by exchanging the indices of t_1 and t_2.*

Proposition 4.3.4 *For $(\vartheta_1, \vartheta_2) \in \mathcal{X}_\nu$, let $\varphi_{(\vartheta_1,\vartheta_2)}$ be defined as in (4.25). Then*

$$I_\rho(\varphi_{(\vartheta_1,\vartheta_2)}) \to -\infty \qquad \text{as } \nu \to 0,$$

uniformly for $(\vartheta_1, \vartheta_2) \in \mathcal{X}_\nu$.

Proof. We start by showing the following estimates:

$$\int_\Sigma \varphi\, dV_g = 4\big(1 + o_\delta(1)\big)\log t_1 - 4\big(1 + o_\delta(1)\big)\log t_2; \tag{4.28}$$

$$\frac{1}{2}\int_\Sigma |\nabla\varphi|^2\, dV_g \leq 16\pi\big(1 + o_\delta(1)\big)\log\frac{1}{t_1} + 16\pi\big(1 + o_\delta(1)\big)\log\frac{1}{t_2}. \tag{4.29}$$

We begin by proving (4.28). It is convenient to divide Σ into the two subsets

$$A_1 = B_\delta(x_1) \cup B_\delta(x_2); \qquad A_2 = \Sigma \setminus \mathcal{A}_1.$$

Moreover, we write

$$\varphi(y) = 2\log\left(1 + \tilde{t}_2^2 d(x_2, y)^2\right) - 2\log\left(1 + \tilde{t}_1^2 d(x_1, y)^2\right).$$

For $y \in A_2$ we clearly have that

$$\frac{1}{C_{\delta,\Sigma}t_1^2} \leq 1 + \tilde{t}_1^2 d(x_1, y)^2 \leq \frac{C_{\delta,\Sigma}}{t_1^2}; \qquad \frac{1}{C_{\delta,\Sigma}t_2^2} \leq 1 + \tilde{t}_2^2 d(x_2, y)^2 \leq \frac{C_{\delta,\Sigma}}{t_2^2},$$

therefore we derive

$$\int_{A_2} \varphi \, dV_g = 4\big(1 + o_\delta(1)\big) \log t_1 - 4\big(1 + o_\delta(1)\big) \log t_2.$$

Moreover, working in normal geodesic coordinates at x_i one also finds

$$\int_{B_\delta(x_i)} \log\left(1 + \tilde{t}_i^2 d(x_i, y)^2\right) \, dV_g = o_\delta(1) \log t_i.$$

Using jointly the last two inequalities we obtain (4.28).

We prove now (4.29). We have that

$$\begin{aligned} \nabla\varphi(y) &= 2\nabla \log\left(1 + \tilde{t}_2^2 d(x_2, y)^2\right) - 2\nabla \log\left(1 + \tilde{t}_1^2 d(x_1, y)^2\right) \\ &= \frac{4\,\tilde{t}_2^2 d(x_2, y) \nabla d(x_2, y)}{1 + \tilde{t}_2^2 d(x_2, y)^2} - \frac{4\,\tilde{t}_1^2 d(x_1, y) \nabla d(x_1, y)}{1 + \tilde{t}_1^2 d(x_1, y)^2}. \end{aligned}$$

From now on we will assume, without loss of generality, that $t_1 \leq t_2$. We distinguish between the case $t_2 \geq \delta^3$ and $t_2 \leq \delta^3$.

In the first case the function $1 + \tilde{t}_2^2 d(x_2, y)^2$ is uniformly Lipschitz with bounds depending only on δ, and therefore we have

$$\nabla\varphi(y) = -\frac{4\tilde{t}_1^2 d(x_1, y) \nabla d(x_1, y)}{1 + \tilde{t}_1^2 d(x_1, y)^2} + O_\delta(1).$$

Let us fix a large constant $C_1 > 0$ and consider the subdivision of the surface Σ into the three domains

$$B_1 = B_{C_1 t_1}(x_1); \qquad B_2 = B_{C_1 t_2}(x_2); \qquad B_3 = \Sigma \setminus (B_1 \cup B_2).$$

In B_1 we have that $|\nabla\varphi| \leq C\tilde{t}_1$, while

$$\frac{\tilde{t}_1^2 d(x_1, y) \nabla d(x_1, y)}{1 + \tilde{t}_1^2 d(x_1, y)^2} = \big(1 + o_{C_1}(1)\big) \frac{\nabla d(x_1, y)}{d(x_1, y)} \qquad \text{in } \Sigma \setminus B_1. \tag{4.30}$$

These estimates imply that

$$\begin{aligned} \frac{1}{2} \int_\Sigma |\nabla\varphi|^2 \, dV_g &= \int_{\Sigma \setminus B_1} |\nabla\varphi|^2 \, dV_g + o_\delta(1) \log \frac{1}{t_1} + O_\delta(1) \\ &= 16\pi \int_{C_1 t_1}^1 \frac{dt}{t} + o_\delta(1) \log \frac{1}{t_1} + O_\delta(1) \\ &= 16\pi\big(1 + o_\delta(1)\big) \log \frac{1}{t_1} + 16\pi\big(1 + o_\delta(1)\big) \log \frac{1}{t_2} + O_\delta(1), \end{aligned}$$

recalling that $t_2 \geq \delta^3$.

If instead $t_2 \leq \delta_3$, by the definition of $\mathcal{X}_\nu$ we have that $d(x_1, x_2) \geq \frac{\delta^2}{2}$, and therefore $B_1 \cap B_2 = \emptyset$. Similarly to (4.30) we get

$$\begin{cases} \dfrac{\tilde{t}_1^2 d(x_1, y) \nabla d(x_1, y)}{1 + \tilde{t}_1^2 d(x_1, y)^2} = \big(1 + o_{C_1}(1)\big) \dfrac{\nabla d(x_1, y)}{d(x_1, y)} \\ \dfrac{\tilde{t}_2^2 d(x_2, y) \nabla d(x_2, y)}{1 + \tilde{t}_2^2 d(x_2, y)^2} = \big(1 + o_{C_1}(1)\big) \dfrac{\nabla d(x_2, y)}{d(x_2, y)} \end{cases} \qquad \text{in } B_3.$$

Moreover we have

$$|\nabla\varphi| \leq C\tilde{t}_i \quad \text{in } B_i, \ i = 1, 2.$$

Therefore we find

$$\begin{aligned} \frac{1}{2} \int_\Sigma |\nabla\varphi|^2 \, dV_g &= \int_{B_3} |\nabla\varphi|^2 \, dV_g + o_\delta(1) \log \frac{1}{t_1} + o_\delta(1) \log \frac{1}{t_2} + O_\delta(1) \\ &= 16\pi\big(1 + o_\delta(1)\big) \log \frac{1}{t_1} + 16\pi\big(1 + o_\delta(1)\big) \log \frac{1}{t_2} + O_\delta(1), \end{aligned}$$

for $t_2 \leq \delta^3$. This concludes the proof of (4.29).

Finally, the estimates (4.28) and (4.28), jointly with (4.26) and Remark 4.2.11 yield the inequality

$$I_\rho(\varphi) \leq \big(2\rho_1 - 16\pi + o_\delta(1)\big)\log t_1 + \big(2\rho_2 - 16\pi + o_\delta(1)\big)\log t_2 \to -\infty$$

as $\nu \to 0$, uniformly for $(\vartheta_1, \vartheta_2) \in \mathcal{X}_\nu$, since $\rho_1, \rho_2 > 8\pi$. ■

We next state a technical lemma, that will be of use later on.

Lemma 4.3.5 *Let $\varphi_{(\vartheta_1,\vartheta_2)}$ be as in (4.25): then, for some $C = C(\delta, \Sigma) > 0$, the following estimates hold uniformly in $(\vartheta_1, \vartheta_2) \in \mathcal{X}_\nu$:*

$$\sup_{x\in\Sigma} \int_{B_{rt_1}(x)} e^{\varphi}\, dV_g \leq C r^2 \frac{t_1^2}{t_2^4} \qquad \forall r > 0. \tag{4.31}$$

Moreover, given any $\varepsilon > 0$ there exists $C = C(\varepsilon, \delta, \Sigma)$, depending only on ε, δ and Σ (but not on ν), such that

$$\int_{B_{Ct_1}(x_1)} e^{\varphi}\, dV_g \geq (1-\varepsilon) \int_\Sigma e^{\varphi}\, dV_g, \tag{4.32}$$

uniformly in $(\vartheta_1, \vartheta_2) \in \mathcal{X}_\nu$.

Proof. By the elementary inequalities $\big(1 + \tilde t_2^2 d(x_2, y)^2\big)^2 \leq \frac{C}{t_2^4}$ and $1 + \tilde t_1^2 d(x_1, y)^2 \geq 1$ we have

$$\int_{B_{rt_1}(x)} e^{\varphi(y)}\, dV_g(y) \leq \frac{C}{t_2^4} \int_{B_{rt_1}(x)} \frac{1}{\big(1 + \tilde t_1^2 d(x_1, y)^2\big)^2}\, dV_g(y) \leq C r^2 \frac{t_1^2}{t_2^4} \qquad \text{for all } x \in \Sigma,$$

which gives the inequality (4.31).

We now prove (4.32). Using again that $\big(1 + \tilde t_2^2 d(x_2, y)^2\big)^2 \leq \frac{C}{t_2^4}$ we have that

$$\int_{\Sigma\setminus B_{Rt_1}(x_1)} e^{\varphi(y)}\, dV_g(y) \leq \frac{C}{t_2^4} \int_{\Sigma\setminus B_{Rt_1}(x_1)} \frac{1}{\big(1 + \tilde t_1^2 d(x_1, y)^2\big)^2}\, dV_g(y). \tag{4.33}$$

Finally, using normal geodesic coordinates centered at x_1 and (4.27) with a change of variable, we find

$$\lim_{t_1\to 0^+} t_1^{-2} \int_{\Sigma\setminus B_{Rt_1}(x_1)} \frac{1}{\big(1 + \tilde t_1^2 d(x_1, y)^2\big)^2}\, dV_g = o_R(1) \qquad \text{as } R \to +\infty.$$

This fact and (4.33), with the estimate (4.26), conclude the proof of the (4.32), by choosing R sufficiently large, depending on ε, δ and Σ. ■

Remark 4.3.6 *The same result holds if we consider $e^{-\varphi}$, interchanging the indices of t_1 and t_2.*

We next present a crucial step in describing the topology of low sub-levels, which will allow us to find a min-max scheme later on.

Proposition 4.3.7 *Let $L > 0$ be so large that $\Psi\big(\{I_\rho \leq -L\}\big) \in X$, and let ν be so small that $I_\rho(\varphi_{(\vartheta_1,\vartheta_2)}) < -L$ for $(\vartheta_1, \vartheta_2) \in \mathcal{X}_\nu$. Let R_ν be the retraction given in Lemma 4.3.1. Then the map $T_\nu : \mathcal{X}_\nu \to \mathcal{X}_\nu$ defined as*

$$T_\nu\big((\vartheta_1, \vartheta_2)\big) = R_\nu\big(\Psi(\varphi_{(\vartheta_1,\vartheta_2)})\big)$$

is homotopic to the identity on $\mathcal{X}_\nu$.

Proof. Let us denote $\vartheta_i = (x_i, t_i)$ and

$$f_1 = \frac{e^{\varphi_{(\vartheta_1,\vartheta_2)}}}{\int_\Sigma e^{\varphi_{(\vartheta_1,\vartheta_2)}}\, dV_g}, \quad \psi(f_1) = (\beta_1, \sigma_1),$$

$$f_2 = \frac{e^{-\varphi_{(\vartheta_1,\vartheta_2)}}}{\int_\Sigma e^{-\varphi_{(\vartheta_1,\vartheta_2)}}\, dV_g}, \quad \psi(f_2) = (\beta_2, \sigma_2),$$

where ψ is given in Proposition 4.2.1. First, observe that we have the following relations

$$\frac{1}{C} \leq \frac{\sigma_i}{t_i} \leq C, \qquad d\left(\beta_i, x_i\right) \leq C t_i, \tag{4.34}$$

for some constant $C = C(\delta, \Sigma) > 0$, depending only on Σ and δ. Indeed, by (4.32), we have that

$$\sigma\left(x_i, f_i\right) \leq C t_i,$$

where $\sigma(x, f)$ is the continuous map defined in (4.5). From that, we get that $\sigma_i \leq C t_i$. Moreover, by (4.31), we get the relation $t_i \leq C \sigma_i$.

Next, by (4.6) and using again the fact that $\sigma(x_i, f) \leq C t_i$, we obtain that

$$d\big(x_i, S\left(f_i\right)\big) \leq C t_i,$$

where $S(f)$ is the set defined in (3.5). But since we have the inequality

$$d\big(\beta_i, S\left(f_i\right)\big) \leq C \sigma_i,$$

we can conclude the proof of (4.34).

We are now able to prove the proposition. The proof will follow by taking into account a composition of three homotopies. The first deformation H_1 is defined in the following way:

$$\left(\begin{pmatrix} (\beta_1, \sigma_1) \\ (\beta_2, \sigma_2) \end{pmatrix}, s \right) \xmapsto{H_1} \begin{pmatrix} \big(\beta_1,\ (1-s)\sigma_1 + s\kappa_1\big) \\ \\ \big(\beta_2,\ (1-s)\sigma_2 + s\kappa_2\big) \end{pmatrix},$$

where $\kappa_i = \min\left\{\delta, \frac{\sigma_i}{\sqrt{\nu}}\right\}$.

We introduce now a second deformation H_2, given by

$$\left(\begin{pmatrix} (\beta_1, \kappa_1) \\ (\beta_2, \kappa_2) \end{pmatrix}, s \right) \xmapsto{H_2} \begin{pmatrix} \big((1-s)\beta_1 + s x_1,\ \kappa_1\big) \\ \\ \big((1-s)\beta_2 + s x_2,\ \kappa_2\big) \end{pmatrix},$$

where $(1-s)\beta_i + s x_i$ stands for the geodesic joining β_i and x_i in unit time. Observe that, if $\kappa_i < \delta$, then we have that $\sigma_i < \sqrt{\nu}\delta$. Therefore by choosing ν small enough, we have that β_i and x_i are close to each other, by (4.34). Instead, if $\kappa_i = \delta$, the equivalence relation in $\overline{\Sigma}_\delta$ makes the above deformation a trivial identification.

We perform a third deformation H_3 defined by

$$\left(\begin{pmatrix} (x_1, \kappa_1) \\ (x_2, \kappa_2) \end{pmatrix}, s \right) \xmapsto{H_3} \begin{pmatrix} \big(x_1,\ (1-s)\kappa_1 + s t_1\big) \\ \\ \big(x_2,\ (1-s)\kappa_2 + s t_2\big) \end{pmatrix}.$$

Finally, we define H as the composition of these three homotopies. Then,

$$\big((\vartheta_1, \vartheta_2), s\big) \mapsto R_\nu \circ H\big(\Psi(\varphi_{(\vartheta_1, \vartheta_2)}), s\big)$$

gives us the desired homotopy to the identity. Indeed, we observe that, since $\nu \ll \delta$, $H(\Psi(\varphi_{(\vartheta_1, \vartheta_2)}), s)$ always belongs to X, so that R_ν can be applied. ■

Remark 4.3.8 *In [71] the authors proved that the set $X = \overline{\Sigma}_\delta \times \overline{\Sigma}_\delta \setminus \overline{D}_\delta$ is not contractible. Indeed, if $\Sigma = S^2$, then $\overline{\Sigma}_\delta$ can be identified with $B_1(0) \subset \mathbb{R}^3$ and it turns out that $X \simeq S^2$, where $\simeq$ stands for homotopical equivalence. The case of positive genus is not so easy. However, the authors proved that X is not contractible by showing that its cohomology group $H^4(X)$ is not trivial.*

We now introduce the min-max scheme which provides existence of solutions for equation (4.1). The argument is developed exactly as in Section 2.3, so we will be sketchy.

Let $\overline{\mathcal{X}}_\nu$ be the topological cone over $\mathcal{X}_\nu$, which can be represented as

$$\overline{\mathcal{X}}_\nu = \frac{\mathcal{X}_\nu \times [0,1]}{\mathcal{X}_\nu \times \{1\}},$$

where the equivalence relation identifies all the points in $\mathcal{X}_\nu \times \{1\}$. We choose $L > 0$ so large that $I_\rho(u) \leq -L$ implies that $\Psi(u) \in X$ and then ν so small that

$$I_\rho(\varphi_{(\vartheta_1,\vartheta_2)}) \leq -4L \tag{4.35}$$

uniformly for $(\vartheta_1, \vartheta_2) \in \mathcal{X}_\nu$. The existence of such ν is guaranteed by Proposition 4.3.4. Fixing this value of ν, we define the following class:

$$\mathscr{H} = \Big\{ h : \overline{\mathcal{X}}_\nu \to H^1(\Sigma) \ : \ h \text{ is continuous and } h\big(\,\cdot\, \times \{0\}\big) = \varphi_{(\vartheta_1,\vartheta_2)} \text{ on } \mathcal{X}_\nu \Big\}. \tag{4.36}$$

To prove that $\mathscr{H} \neq \emptyset$, we just notice that the map

$$\bar{h}(\vartheta, s) = s\,\varphi_{(\vartheta_1,\vartheta_2)}, \qquad (\vartheta, s) \in \overline{\mathcal{X}}_\nu, \tag{4.37}$$

belongs to $\mathscr{H}$. Consider then the min-max value

$$m = \inf_{h \in \mathscr{H}} \sup_{\xi \in \overline{\mathcal{X}}_\nu} I_\rho\big(h(\xi)\big).$$

Letting $\Phi_\nu : \mathcal{X}_\nu \to H^1(\Sigma)$ be the map defined by

$$\Phi_\nu((\vartheta_1, \vartheta_2)) = \varphi_{(\vartheta_1,\vartheta_2)}, \qquad (\vartheta_1, \vartheta_2) \in \mathcal{X}_\nu,$$

from Proposition 4.3.7 and Remark 4.3.8 we deduce that

$$\Phi_\nu(\mathcal{X}_\nu) \text{ is not contractible in } J_\rho^{-L}.$$

Reasoning as in Section 2.3 we then obtain that $m > -2L$.

By classical arguments, the latter estimate and (4.35) yield a Palais-Smale sequence at level m. However, we cannot directly conclude the existence of a critical point, since it is not known whether the Palais-Smale condition holds or not. To avoid this problem and get the conclusion, we need a different argument, namely the *monotonicity trick* introduced by Struwe in [88], see Lemma 2.3.1, jointly with the compactness result in Theorem 4.1.1, see Section 2.3 for the description of the general strategy.

Chapter 5

A mean field equation: existence and multiplicity results

In this chapter we continue the analysis of the following mean field equation with two parameters on a compact surface Σ in a general non-coercive regime:

$$-\Delta u = \rho_1 \left(\frac{h\, e^u}{\int_\Sigma h\, e^u \, dV_g} - 1 \right) - \rho_2 \left(\frac{h\, e^{-u}}{\int_\Sigma h\, e^{-u} \, dV_g} - 1 \right), \tag{5.1}$$

where ρ_1, ρ_2 are real parameters and h is a smooth positive function, see Section 1.2 for an introduction to the topic. As noticed in Subsection 1.2.2, we consider here just one potential h, differently from equation (4.1) in Chapter 4, in order to apply the general compactness result in Theorem 1.2.1.

The chapter is divided into three parts. The first one (Section 5.1) concerns the existence problem to equation (5.1). Adapting the strategy presented for the Toda system in Chapter 2 (see also Subsection 1.1.1), we will give the following general existence result (still part of the paper [9]), see Section 5.1 for the proof.

Theorem 5.0.1 *Let h be a smooth positive function. Suppose Σ is not homeomorphic to S^2 nor $\mathbb{RP}^2$, and that $\rho_i \notin 8\pi\mathbb{N}$ for $i = 1, 2$. Then* (5.1) *has a solution.*

In the second part of the chapter (Section 5.2) we address instead the multiplicity aspect of the problem, see Subsection 1.2.2. Indeed, exploiting the analysis developed for the existence problem, we are able to get the following result, which is stated in the paper [47], see Section 5.2 for the proof.

Theorem 5.0.2 *Let $\rho_1 \in (8k\pi, 8(k+1)\pi)$ and $\rho_2 \in (8l\pi, 8(l+1)\pi)$, $k, l \in \mathbb{N}$ and let Σ be a compact surface with genus $g(\Sigma) > 0$. Then, for a generic choice of the metric g and of the function h it holds*

$$\#\big\{ \textit{solutions of } (5.1) \big\} \geq \binom{k + g(\Sigma) - 1}{g(\Sigma) - 1} \binom{l + g(\Sigma) - 1}{g(\Sigma) - 1}.$$

Here, by generic choice of (g, h) we mean that it can be taken in an open dense subset of $\mathcal{M}^2 \times C^2(\Sigma)^+$, where $\mathcal{M}^2$ stands for the space of Riemannian metrics on Σ equipped with the C^2 norm, see Proposition 5.2.4.

In the last part of the chapter (Section 5.3) we attack the problem with a different point of view (differently from Chapter 4 and the first two parts of the present chapter) and for the first time we analyze the associated Leray-Schauder degree, see Subsection 1.2.2. The argument presented here is stated in the note [46]. More precisely, by considering the parity of the Leray-Schauder degree we prove the following existence result, see Section 5.3 for the proof.

Theorem 5.0.3 *Let $h > 0$ be a smooth function and suppose $\rho_i \in (8k\pi, 8(k+1)\pi)$, $k \in \mathbb{N}$ for $i = 1, 2$. Then problem* (5.1) *has a solution.*

As observed in Subsection 1.2.2, the above theorem provides a new existence result in the case when the underlying manifold Σ is a sphere and gives a new proof for other known results.

The chapter is organized as follows: in Section 5.1 we adapt the strategy developed in Chapter 2 to prove the existence result of Theorem 5.0.1, in Section 5.2 we collect some results concerning Morse theory and we provide a proof for the multiplicity result of Theorem 5.0.2, in Section 5.3 we prove the new existence result of Theorem 5.0.3 by using the degree theory.

5.1 A general existence result

In this section we prove the main result concerning the existence problem, see Theorem 5.0.1. The proof is an adaptation of the argument introduced for the Toda system in Chapter 2 (see also Subsection 1.1.1). Therefore, we will present here just the main steps. Roughly speaking the role of the function u_2 is played by $-u$.

We start by taking two curves $\gamma_1, \gamma_2 \in \Sigma$ with the same properties as in Lemma 2.1.1 (see also Figure 2.1). We consider then the topological join $(\gamma_1)_k * (\gamma_2)_l$, see (1.17) and (1.15), on which we will base the min-max scheme. Let $\zeta \in (\gamma_1)_k * (\gamma_2)_l, \zeta = (1-s)\sigma_1 + s\sigma_2$, with

$$\sigma_1 := \sum_{i=1}^{k} t_i \delta_{x_i} \in (\gamma_1)_k \qquad \text{and} \qquad \sigma_2 := \sum_{j=1}^{l} s_j \delta_{y_j} \in (\gamma_2)_l.$$

We define now a test function labelled by $\zeta \in (\gamma_1)_k * (\gamma_2)_l$, namely for large L we will find a non-trivial map

$$\widetilde{\Phi}_\lambda : (\gamma_1)_k * (\gamma_2)_l \to I_\rho^{-L}.$$

We set $\widetilde{\Phi}_\lambda(\zeta) = \varphi_{\lambda,\zeta}$ given by

$$\varphi_{\lambda,\zeta}(x) = \log \sum_{i=1}^{k} t_i \left(\frac{1}{1+\lambda_{1,s}^2 d(x,x_i)^2} \right)^2 - \log \sum_{j=1}^{l} s_j \left(\frac{1}{1+\lambda_{2,s}^2 d(x,y_j)^2} \right)^2,$$

where $\lambda_{1,s} = (1-s)\lambda, \lambda_{2,s} = s\lambda$.

The following result holds true.

Proposition 5.1.1 *Suppose $\rho_1 \in (8k\pi, 8(k+1)\pi)$ and $\rho_2 \in (8l\pi, 8(l+1)\pi)$. Then one has*

$$I_\rho(\varphi_{\lambda,\zeta}) \to -\infty \quad \text{as } \lambda \to +\infty \qquad \text{uniformly in } \zeta \in (\gamma_1)_k * (\gamma_2)_l.$$

PROOF. The proof is developed exactly as in Proposition 2.1.3. Here we just sketch the main features.

We define $\tilde{v}_1, \tilde{v}_2 : \Sigma \to \mathbb{R}$ by

$$\tilde{v}_1(x) = \log \sum_{i=1}^{k} t_i \left(\frac{1}{1+\lambda_{1,s}^2 d(x,x_i)^2} \right)^2,$$
$$\tilde{v}_2(x) = \log \sum_{j=1}^{l} s_j \left(\frac{1}{1+\lambda_{2,s}^2 d(x,y_j)^2} \right)^2,$$

so that $\varphi = \tilde{v}_1 - \tilde{v}_2$.

The Dirichlet part of the functional I_ρ is given by

$$\frac{1}{2}\int_\Sigma |\nabla \varphi|^2 \, dV_g = \frac{1}{2}\int_\Sigma \big(|\nabla \tilde{v}_1|^2 + |\nabla \tilde{v}_2|^2 - 2\nabla \tilde{v}_1 \cdot \nabla \tilde{v}_2 \big) \, dV_g \le \frac{1}{2}\int_\Sigma |\nabla \tilde{v}_1|^2 \, dV_g + \frac{1}{2}\int_\Sigma |\nabla \tilde{v}_2|^2 \, dV_g + C,$$

where we have used

$$\left| \int_\Sigma \nabla \tilde{v}_1 \cdot \nabla \tilde{v}_2 \, dV_g \right| \le C.$$

We first study the cases $s = 0$ and $s = 1$, starting from $s = 0$. The case $s = 1$ can be treated in the same way and will be omitted. Observing that $\nabla \tilde{v}_2 = 0$ and taking into account the estimates (3.110), (3.111) on the gradient of $\tilde{v}_1$ we get

$$\frac{1}{2}\int_\Sigma |\nabla\varphi|^2\, dV_g \leq 16k\pi\big(1 + o_\lambda(1)\big)\log\lambda + C,$$

where $o_\lambda(1) \to 0$ as $\lambda \to +\infty$.

Reasoning as in Proposition 2.1.3 we obtain

$$\int_\Sigma \varphi\, dV_g = -4\big(1+o_\lambda(1)\big)\log\lambda;\ \ \log\int_\Sigma e^{\varphi}\, dV_g = -2\big(1+o_\lambda(1)\big)\log\lambda;\ \ \log\int_\Sigma e^{-\varphi}\, dV_g = 4\big(1+o_\lambda(1)\big)\log\lambda.$$

Therefore we get

$$I_\rho(\varphi_{\lambda,\zeta}) \leq \big(16k\pi - 2\rho_1 + o_\lambda(1)\big)\log\lambda + C,$$

where C is independent of λ and σ_1, σ_2.

We consider now the case $s \in (0, 1)$. We can reason as before to estimate the Dirichlet part by

$$\frac{1}{2}\int_\Sigma |\nabla\varphi|^2\, dV_g \leq 16k\pi\big(1 + o_\lambda(1)\big)\log\big(\lambda_{1,s} + \delta_{1,s}\big) + 16l\pi\big(1 + o_\lambda(1)\big)\log\big(\lambda_{2,s} + \delta_{2,s}\big) + C,$$

where $\delta_{1,s} > \delta > 0$ as $s \to 1$ and $\delta_{2,s} > \delta > 0$ as $s \to 0$. Following the argument in Proposition 2.1.3 we obtain

$$\int_\Sigma \varphi\, dV_g = -4\big(1 + o_\lambda(1)\big)\log\big(\lambda_{1,s} + \delta_{1,s}\big) + 4\big(1 + o_\lambda(1)\big)\log\big(\lambda_{2,s} + \delta_{2,s})\big) + O(1),$$

$$\log\int_\Sigma e^{\varphi}\, dV_g = 4\log\big(\lambda_{2,s} + \delta_{2,s}\big) - 2\log\big(\lambda_{1,s} + \delta_{1,s}\big) + O(1),$$

$$\log\int_\Sigma e^{-\varphi}\, dV_g = 4\log\big(\lambda_{1,s} + \delta_{1,s}\big) - 2\log\big(\lambda_{2,s} + \delta_{2,s}\big) + O(1).$$

Using these estimates we get

$$I_\rho(\varphi_{\lambda,\zeta}) \leq \big(16k\pi - 2\rho_1 + o_\lambda(1)\big)\log\big(\lambda_{1,s} + \delta_{1,s}\big) + \big(16l\pi - 2\rho_2 + o_\lambda(1)\big)\log\big(\lambda_{2,s} + \delta_{2,s}\big) + O(1).$$

By assumption we have $\rho_1 > 8k\pi, \rho_2 > 8l\pi$ and exploiting the fact that $\max_{s\in[0,1]}\{\lambda_{1,s}, \lambda_{2,s}\} \to +\infty$ as $\lambda \to \infty$, we deduce the thesis. ■

Once we have this result we can proceed exactly as in Section 2.2. One gets indeed an analogous improved Moser-Trudinger inequality as in Lemma 2.2.3. We have just to observe that a local Moser-Trudinger inequality still holds in this case, as pointed out in Chapter 4, see Proposition 4.1.3

Therefore, considering $\rho_1 \in (8k\pi, 8(k+1)\pi)$ and $\rho_2 \in (8l\pi, 8(l+1)\pi)$, we deduce that on low sub-levels of the functional I_ρ at least one of the component of $\left(\frac{he^u}{\int_\Sigma he^u dV_g}, \frac{he^{-u}}{\int_\Sigma he^{-u} dV_g}\right)$ has to be very close to the sets of k- or l- barycenters over Σ, respectively, see Proposition 2.2.6 for details. It is then possible to construct a continuous map

$$\widetilde{\Psi} : I_\rho^{-L} \to (\gamma_1)_k * (\gamma_2)_l$$

with L sufficiently large, such that the composition

$$(\gamma_1)_k * (\gamma_2)_l \ \xrightarrow{\widetilde{\Phi}_\lambda}\ I_\rho^{-L} \ \xrightarrow{\widetilde{\Psi}}\ (\gamma_1)_k * (\gamma_2)_l$$

is homotopically equivalent to the identity map on $(\gamma_1)_k * (\gamma_2)_l$ provided that λ is large enough. $\widetilde{\Psi}$ is defined as in (3.2), where basically e^{u_2} is replaced by e^{-u}:

$$\widetilde{\Psi}(u) = (1 - \widetilde{s})(\Pi_1)_*\psi_k\left(\frac{he^u}{\int_\Sigma he^u dV_g}\right) + \widetilde{s}(\Pi_2)_*\psi_l\left(\frac{he^{-u}}{\int_\Sigma he^{-u} dV_g}\right).$$

With this at hand we argue as in Section 2.3 introducing a min-max scheme based on the set $(\gamma_1)_k * (\gamma_2)_l$. Allowing (ρ_1, ρ_2) to vary in a compact set of $(8k\pi, 8(k+1)\pi) \times (8l\pi, 8(l+1)\pi)$ we obtain a sequence of solutions $(u_n)_n$ corresponding to $(\rho_{1,n}, \rho_{2,n}) \to (\rho_1, \rho_2)$, see Lemma 2.3.1. We finally get a solution for (ρ_1, ρ_2) by applying the compactness result in Theorem 1.2.1.

5.2 A multiplicity result

In this section we present the proof concerning the multiplicity issue to problem (5.1), see Theorem 5.0.2 (see Subsection 1.2.2 for the strategy). We start by presenting a deformation lemma for equation (5.1) and by stating some useful results in Morse theory, see the next two subsections.

5.2.1 Compactness property and a Deformation Lemma

We recall here the compactness result concerning equation (5.1), see Theorem 1.2.1. It hold that if $\rho_i \neq 8\pi\mathbb{N}, i = 1, 2$, then the set of solutions to equation (5.1) is compact. We will need the latter compactness property to bypass the Palais-Smale condition, since it is not know whether it holds or not for this class of equations. More precisely, one can adapt the strategy in [65], where a deformation lemma for the Liouville equation (1.9) was presented, for our framework, see also [67], [89]. One has the next alternative: either there exists a critical point of the functional I_ρ inside some interval or there is a deformation retract between the relative sub-levels. Recall the notation for the sub-levels I_ρ^a given in Section 1.3.

Lemma 5.2.1 *If $\rho_i \neq 8\pi\mathbb{N}, i = 1, 2$ and if $a < b \in \mathbb{R}$ are such that I_ρ has no critical levels inside the interval $[a, b]$, then I_ρ^a is a deformation retract of I_ρ^b.*

Here, by deformation retract of a space X onto some subspace $A \subseteq X$ we mean a continuous map $R : [0, 1] \times X \to X$ such that $R(t, a) = a$ for all $(t, a) \in [0, 1] \times A$ and such that the final target of R is contained in A, i.e. $R(1, \cdot) \in A$.

Notice now that by the compactness result of Theorem 1.2.1 it follows that I_ρ has no critical points above some high level $b \gg 0$. Therefore, one can obtain a deformation retract of the whole Hilbert space $H^1(\Sigma)$ onto the sub-level I_ρ^b by following a suitable gradient flow, see for example Corollary 2.8 in [67] (with minor adaptations). Somehow, the absence of critical points of I_ρ above the level b prevents us from having *obstructions* while following the flow, see Figure 5.1.

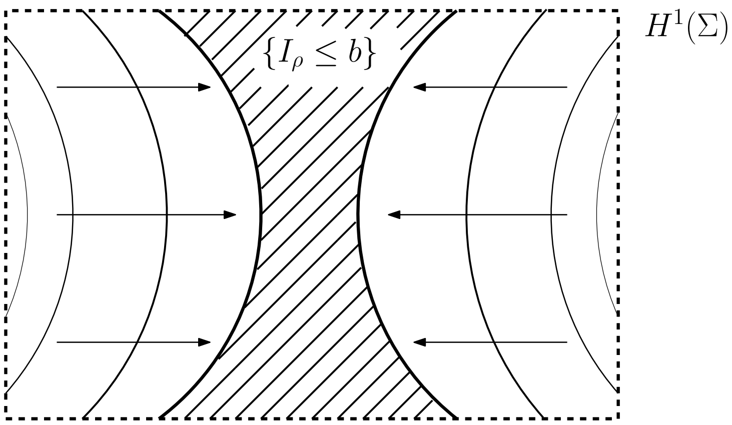

Figure 5.1: The deformation retract onto the sub-level I_ρ^b.

Proposition 5.2.2 *Suppose $\rho_i \neq 8\pi\mathbb{N}, i = 1, 2$. Then, there exists $b > 0$ sufficiently large such that the sub-level I_ρ^b is a deformation retract of $H^1(\Sigma)$. In particular, it is contractible.*

The aim will be then to show how rich is the topological structure of the very low sub-levels of I_ρ and apply the Morse inequalities of Theorem 5.2.3 to deduce the main result of Theorem 5.0.2.

5.2.2 Morse Theory

We recall here some classical results from Morse theory, which will be the main tool in proving Theorem 5.0.2.

Letting N be a Hilbert manifold, we recall first that a function $f \in C^2(N,\mathbb{R})$ is called a Morse function if all its critical points are non-degenerate. Moreover, the number of negative eigenvalues of the Hessian matrix at a critical point is called the index of the critical point. If $a < b$ are regular values of f we then define the following sets:

$$\begin{aligned} C_q(a,b) &= \#\big\{\text{critical points of } f \text{ in } \{a \le f \le b\} \text{ with index } q\big\}, \\ \beta_q(a,b) &= \operatorname{rank}\big(H_q\big(\{f \le b\},\{f \le a\}\big)\big). \end{aligned} \tag{5.2}$$

For the proof of the following result we refer for example to Theorem 4.3 in [25].

Theorem 5.2.3 *([25]) Let N be a Hilbert manifold and $f \in C^2(N,\mathbb{R})$ be a Morse function satisfying the Palais-Smale condition. Let $a < b$ be regular values of f and $C_q(a,b), \beta_q(a,b)$ be as in* (5.2)*. Then the (respectively) strong and weak Morse inequalities hold true:*

$$\sum_{q=0}^{n}(-1)^{n-q}C_q(a,b) \ge \sum_{q=0}^{n}(-1)^{n-q}\beta_q(a,b), \qquad n = 0,1,2,\dots$$

$$C_q(a,b) \ge \beta_q(a,b), \qquad n = 0,1,2,\dots$$

The strategy will be to apply this result in our framework, namely with $N = H^1(\Sigma)$ and $f = I_\rho$. We point out that the Palais-Smale condition is not necessarily needed for the Theorem 5.2.3 to hold, in fact it can be replaced by appropriate deformation lemmas for f, see Lemma 3.2 and Theorem 3.2 in [25]. The validity of such deformation lemmas can be obtained by following the ideas in [67], where a gradient flow for the scalar case (1.9) is defined.

For what concerns the assumption of f to be a Morse function, one can repeat (with minor adaptations) the argument in [30], which relies on a transversality result from [85], to obtain the following result:

Proposition 5.2.4 *([30]) Suppose $\rho_i \neq 8\pi\mathbb{N}, i = 1,2$. Then, for (g,h) in an open dense subset of $\mathcal{M}^2 \times C^2(\Sigma)^+$, I_ρ is a Morse function.*

By the above discussion it follows that we are in position to apply Theorem 5.2.3 in our setting.

5.2.3 Proof of Theorem 5.0.2

We have now all the tools in order to prove the main result of Theorem 5.0.2. Since the high sub-levels of I_ρ are contractible, see Proposition 5.2.2, the goal will be to describe the topology of the low sub-levels.

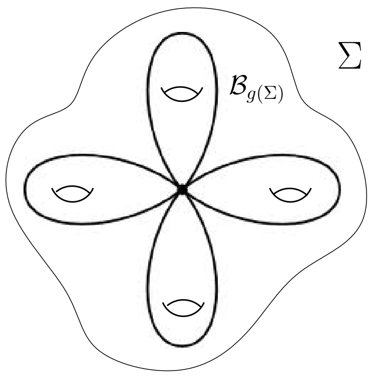

Figure 5.2: The bouquet $\mathcal{B}_{g(\Sigma)}$ of $g(\Sigma)$ circles.

This will be done by means of a bouquet of circles and its homology will give then a bound of the number of solutions to (5.1) by Theorem 5.2.3.

We recall that a bouquet $\mathcal{B}_N$ of N circles (see Figure 5.2) is defined as $\mathcal{B}_N = \cup_{i=1}^N \mathcal{S}_i$, where $\mathcal{S}_i$ is homeomorphic to S^1 and $\mathcal{S}_i \cap \mathcal{S}_j = \{c\}$, and c is called the center of the bouquet. The first simple result we need is the following, see the proof of Proposition 3.1 in [3]:

Lemma 5.2.5 *Let Σ be a surface with $g(\Sigma) > 0$. Then, there exist two curves $\gamma_1, \gamma_2 \subseteq \Sigma$ satisfying (see Figure 5.3)*

(1) γ_1 and γ_2 do not intersect each other;

(2) each of γ_1 and γ_2 are homeomorphic to respectively two disjoint bouquets of $g(\Sigma)$ circles, see Figure 5.2;

(3) there exist global retractions $\Pi_i : \Sigma \to \gamma_i$, $i = 1, 2$.

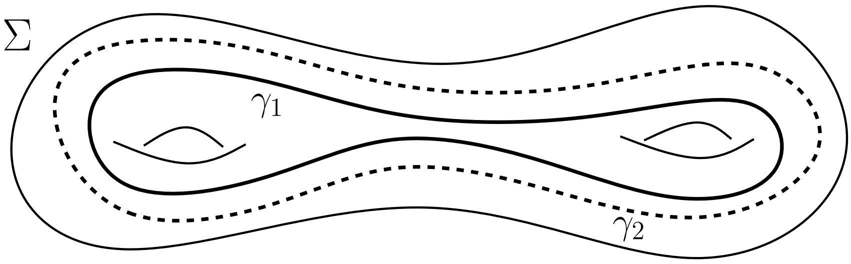

Figure 5.3: The curves γ_1 and γ_2.

We will now exploit the analysis developed in Section 5.1 to describe the topology of the low sub-levels of the functional I_ρ. As mentioned before, by means of improved Moser-Trudinger inequalities one can deduce that if $\rho_1 < 8(k+1)\pi$ and $\rho_2 < 8(l+1)\pi$, then either e^u is close to Σ_k or e^{-u} is close to Σ_l, see (1.15). This alternative is then expressed using the notion the topological join of Σ_k and Σ_l, see (1.17). Finally, applying the retractions Π_1, Π_2 introduced in the above lemma, low energy sub-levels may be described in terms of $(\gamma_1)_k * (\gamma_2)_l$ only.

In fact, one can project the low sub-levels of I_ρ onto the latter set, see the proof of Proposition 2.2.7 and Section 5.1: for $\rho_1 \in (8k\pi, 8(k+1)\pi)$, $\rho_2 \in (8l\pi, 8(l+1)\pi)$ and for L sufficiently large there exists a continuous map

$$\widetilde{\Psi} : I_\rho^{-L} \to (\gamma_1)_k * (\gamma_2)_l.$$

One the other hand, it is possible to do the converse, mapping $(\gamma_1)_k * (\gamma_2)_l$ into the low sub-levels using suitable test functions, see Proposition 5.1.1 and the notation before it:

$$\widetilde{\Phi} : (\gamma_1)_k * (\gamma_2)_l \to I_\rho^{-L}.$$

The above maps are somehow natural in the description of the low sub-levels as we have the following important result, see Proposition 2.2.7 and Section 5.1: the composition of the above maps $\widetilde{\Phi}$ and $\widetilde{\Psi}$ is homotopically equivalent to the identity map on $(\gamma_1)_k * (\gamma_2)_l$, i.e.

$$\widetilde{\Phi} \circ \widetilde{\Psi} \simeq Id_{(\gamma_1)_k * (\gamma_2)_l}.$$

By the latter homotopy equivalence we directly deduce that the homology groups of $(\gamma_1)_k * (\gamma_2)_l$ are mapped injectively into the homology groups of I_ρ^{-L} through the map induced by $\widetilde{\Phi}$.

Corollary 5.2.6 *Suppose $\rho_1 \in (8k\pi, 8(k+1)\pi)$, $\rho_2 \in (8l\pi, 8(l+1)\pi)$ and L sufficiently large. Then, for any $q \in \mathbb{N}$ we have*

$$H_q\big((\gamma_1)_k * (\gamma_2)_l\big) \hookrightarrow H_q\left(I_\rho^{-L}\right).$$

As a consequence of the above result we obtain a bound of the number of solutions to (5.1) by Theorem 5.2.3. One has just to observe that by Proposition 5.2.2, taking $L \geq b$, the sub-level I_ρ^L is contractible and therefore, by the long exact sequence of the relative homology, it follows that

$$H_{q+1}\left(I_\rho^L, I_\rho^{-L}\right) \cong \widetilde{H}_q\left(I_\rho^{-L}\right), \quad q \geq 0,$$

$$H_0\left(I_\rho^L, I_\rho^{-L}\right) = 0,$$

where $\widetilde{H}_q(X)$ of a topological set X is defined in the Section 1.3 and $\cong$ stands for homeomorphisms between topological spaces or isomorphisms between groups. Recalling the definition of $\beta_q(a,b)$ introduced in (5.2) and the notation of $\widetilde{\beta}_q$ given in the Section 1.3, the next result holds true by the above discussion and by taking $a = -L$ in Theorem 5.2.3.

Proposition 5.2.7 *Suppose $\rho_1 \in (8k\pi, 8(k+1)\pi)$, $\rho_2 \in (8l\pi, 8(l+1)\pi)$ and L sufficiently large. Then, for any $q \in \mathbb{N}$ it holds that*

$$\beta_{q+1}(L,-L) \geq \widetilde{\beta}_q\big((\gamma_1)_k * (\gamma_2)_l\big).$$

The next step is then to compute the homology groups of the topological join $(\gamma_1)_k * (\gamma_2)_l$. We recall that the two curves γ_1 and γ_2 were chosen such that there are homeomorphic to respectively two disjoint bouquets, see Lemma 5.2.5. The homology group of the barycenters over this object was computed in Proposition 3.2 of [3].

Proposition 5.2.8 *([3]) Let $\mathcal{B}_N$ be a bouquet of N circles. Then, we have that*

$$\widetilde{H}_q\big((\mathcal{B}_N)_j\big) \cong \begin{cases} \mathbb{Z}^{\binom{j+N-1}{N-1}} & if \quad q = 2N-1, \\ 0 & if \quad q \neq 2N-1. \end{cases}$$

Finally, it is well known that the homology groups of the topological join of two sets A and B are expressed in terms of the sum of the homology groups of each set, see [43].

Proposition 5.2.9 *([43]) Given two topological sets A and B we have*

$$\widetilde{H}_q(A * B) \cong \bigoplus_{i=0}^{q} \widetilde{H}_i(A) \otimes \widetilde{H}_{q-i-1}(B).$$

In particular it holds that

$$\widetilde{\beta}_q(A * B) = \sum_{i=0}^{q} \widetilde{\beta}_i(A)\, \widetilde{\beta}_{q-i-1}(B).$$

We are now in position to deduce the main Theorem 5.0.2. The proof will follow by applying the weak Morse inequality stated in Theorem 5.2.3 jointly with Proposition 5.2.7 and Propositions 5.2.8, 5.2.9. More precisely we get

$$\#\big\{\text{solutions of } (5.1)\big\} \geq C_{q+1}(L,-L) \overset{Thm\,5.2.3}{\geq} \beta_{q+1}(L,-L) \overset{Prop\,5.2.7}{\geq} \widetilde{\beta}_q\big((\gamma_1)_k * (\gamma_2)_l\big)$$

$$\overset{Prop\,5.2.8\,+\,Prop\,5.2.9}{\geq} \binom{k+g(\Sigma)-1}{g(\Sigma)-1}\binom{l+g(\Sigma)-1}{g(\Sigma)-1}$$

and the proof is concluded.

5.3 New existence results via degree theory

We give here the proof of the new existence result, see Theorem 5.0.3, which is based on the Leray-Schauder degree associated to the equation (5.1).

For some $\alpha \in (0,1)$ let $C_0^{2,\alpha}(\Sigma)$ be the class of $C^{2,\alpha}$ functions with zero average. Consider now the mapping $T : C_0^{2,\alpha}(\Sigma) \to C_0^{2,\alpha}(\Sigma)$ defined by

$$T(u) = (-\Delta)^{-1}\left(\rho_1\left(\frac{h\,e^u}{\int_\Sigma h\,e^u\,dV_g} - 1\right) - \rho_2\left(\frac{h\,e^{-u}}{\int_\Sigma h\,e^{-u}\,dV_g} - 1\right)\right), \tag{5.3}$$

where $(-\Delta)^{-1}f, f \in C^\alpha(\Sigma)$, is intended as the solution v, with zero average, of the problem $-\Delta v = f$, which is unique. We are concerned with the map $\Psi = Id - T$ and the solutions of equation (5.1) will correspond to zeros of Ψ.

Clearly, by elliptic regularity theory the operator T is compact. Moreover, the set of the solutions is compact for parameters $(\rho_1, \rho_2) \notin (8\pi\mathbb{N} \times \mathbb{R}) \cup (\mathbb{R} \times 8\pi\mathbb{N})$, see Theorem 1.2.1. Therefore, we can consider the associated degree $\deg\big(\Psi_{(\rho_1,\rho_2)}, B_r(0), 0\big)$ which is well-defined for r sufficiently large.

Consider now $\rho_i \in (8k\pi, 8(k+1)\pi)$, $k \in \mathbb{N}$ for $i = 1, 2$. Letting $\rho = \frac{1}{2}(\rho_1 + \rho_2)$, we perform the following homotopy which takes place in a connected component of $\mathbb{R}^2 \setminus ((8\pi\mathbb{N} \times \mathbb{R}) \cup (\mathbb{R} \times 8\pi\mathbb{N}))$:

$$\mathfrak{h}(t) = (1-t)(\rho_1, \rho_2) + t(\rho, \rho).$$

From the fact that the degree is constant along homotopies we obtain that

$$\deg\left(\Psi_{(\rho_1,\rho_2)}, B_r(0), 0\right) = \deg\left(\Psi_{(\rho,\rho)}, B_r(0), 0\right).$$

Observe now that by the structure of T we deduce

$$\Psi_{(\rho,\rho)}(-u) = -\Psi_{(\rho,\rho)}(u).$$

Therefore, we conclude that $\Psi_{(\rho,\rho)}$ is an odd operator. By the Borsuk theorem, see [57], it follows that the associated degree is odd an hence non zero. This guarantees us the existence of a solution to equation (5.1).

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