Scuola Internazionale Superiore di Studi Avanzati (SISSA)

INTEGRABILITY OF CONTINUOUS TANGENT SUB-BUNDLES

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A thesis presented for the degree of Doctor of Philosophy

Contents

Abstract

In this thesis, the main aim is to study the integrability properties of continuous tangent sub-bundles, especially those that arise in the study of dynamical systems.

After the introduction and examples part we start by studying integrability of such sub-bundles under different regularity and dynamical assumptions.

Then we formulate a continuous version of the classical Frobenius theorem and state some applications to such bundles, to ODE and PDE.

Finally we close of by stating some ongoing work related to interactions between integrability, sub-Riemannian geometry and contact geometry.

Dedication

This thesis is dedicated to my family and to Cansu.

Acknowledgments

The biggest portion of thanks goes to my advisor Stefano Luzzatto. I have not yet seen any advisor who devotes so much time for his students on academic and personal level, even at times when he has to deal with his own problems. I also would like to thank my co-advisor Andrei Agrachev for spending time with me to answer my questions. Some of the geometric ideas that I have encountered in his courses have also been very useful through out my thesis.

I would like to thank my family for their continuous support and Cansu for always being there whenever I needed.

I would like to thank all my friends in Trieste, Turkey and other corners of the world for making my life in general enjoyable. I had the option of either putting half a page list of names or to upset some by not putting their names here. None of the choices seemed particularly pleasant so I will ask my friends to excuse me for not putting any names here.

Many thanks also goes to people whom I have interracted mathematically with throughout my Ph.D.; to Charles C. Pugh for although our contact had been brief, some of his advices have been central to shaping my thesis, to Raul Ures and Janah Hertz for spending a lot of time with me discussing integrability rather than going out and swimming in Barcola and also to Rodrigues Hertz, Marcelo Viana, Mehmet Kıral, Christian Bonnatti, Rafael Potrie and Amie Wilkinson.

Finally I would like to thank SISSA for providing me the opportunity to come here and giving me funding on various occasions and to ICTP for providing me for an office and also funding me on various occasions.

CHAPTER 1

Overview of Main Results

In this section we give an overview of the main results in the thesis. The introduction (chapter 2) does not contain any novel results except maybe slightly different proofs of already known theorems. Chapter 3 contains some theorems related to integrability of certain classes of invariant bundles. Chapter 4 contains the main result of the thesis which is a continuous version of the classical Frobenius theorem in three dimensions and its applications. The last chapter contains works in progress, some of it almost finished some still in development. In particular it contains an almost finished generalization of the results stated in chapter 4 to higher dimensions. We will only give a detailed overview of chapter 3 and chapter 4 here. In these two chapters, M is always a compact, three dimensional smooth Riemannian manifold.

The number convention is as follows;

(1) Theorems, corollaries, lemmas etc are numbered within chapters

(2) Sections are numbered with in chapters, subsections within sections and subsubsections within subsections

(3) Equations are numbered within subsections rather than subsubsections, to prevent lengthy strings of numbers.

For the relevant definitions about integrability, we invite the reader to take a look at section 2.1.

1.1. Statement of Results for Chapter 3

Let $\phi : M \to M$ be a C^2 diffeomorphism and $E \oplus F$ a $D\phi$ invariant splitting; that is $D\phi_x(E_x) = E_{\phi(x)}$, $D\phi_x(F_x) = F_{\phi(x)}$ and $T_xM = E_x \oplus F_x$. The main aim of this chapter is to derive sufficient conditions for integrability of E under different regularity assumptions. We start with C^1 invariant sub-bundles E and gradually decrease regularity assumptions on the bundles to only having some transversal regularity properties. Subsection 1.1.2 gives an overview of the C^1 case, subsection 1.1.3 gives an overview of the Lipschitz case and subsection 1.1.4 gives an overview of what we call the sequentially transversally Lipschitz case. The main emphasis is on the effect of some volume domination conditions on Lie brackets. Another main idea in the case when E is not differentiable or Lipschitz is to use C^1 approximations E^k and try to come up with weaker regularity conditions on these approximations which allow us to transfer geometric information about the Lie brackets of E^k to the limit bundle. This will pave the way to continuous Frobenius type theorems later on in chapter 4. The results presented in this chapter are published in [**57, 58**]. The results are, **for C^1 case:** theorems 1.3(pg 2), 1.5(pg 3), 1.6(pg 5), **for Lipschitz case:** theorems 1.7(pg 6), 1.10(pg 6), **for sequentially transversally Lipschitz case:** theorem 1.13(pg 9).

1.1.1. Notations and Definitions. Let

$$m(D\phi_x|_V) = \min_{v\in V, v\neq 0} \frac{\|D\phi_x(v)\|}{\|v\|}$$

$$||D\phi_x|_V|| = \max_{v\in V, v\neq 0} \frac{\|D\phi_x(v)\|}{\|v\|}$$

Then the decomposition $E \oplus F$ is *dynamically dominated* if there exists a Riemannian metric such that

$$\frac{\|D\phi_x|_E\|}{m(D\phi_x|_F)} < 1 \tag{1.1.1.1}$$

for every $x \in M$. The decomposition is *volume dominated* if moreover, for all $x \in M$,

$$|det(D\varphi_x^k|_{E_z})| \leq 1 \qquad ||D\varphi_x^{-k}|_{F_x}|| < 1. \tag{1.1.1.2}$$

REMARK 1.1. In literature, dynamical domination condition is usually known as only as domination but we preferred to state it as such in order to stress the difference from volume domination. Moreover we note that volume hyperbolic implies volume domination.

REMARK 1.2. When we work with dominated splittings we always assume that we work with the metric that gives the estimates in the definition above.

1.1.2. C^1 Case. Let $E\oplus F$ be a $D\phi$-invariant C^1 decomposition with $dim(E) = 2$ and $dim(F) = 1$. Our first result is the following.

THEOREM 1.3. *Let M be a 3-dimensional Riemannian manifold, $\phi : M \to M$ a C^2 diffeomorphism, and $E \oplus F$ a $D\phi$-invariant C^1 volume dominated decomposition with $dim(E) = 2$. Then E is uniquely integrable.*

Notice that since $\|D\phi_x|_E\|^2 \geq |\det(D\phi_x|_E^2)|$, volume domination in the three-dimensional setting is a strictly weaker condition than the well-known "2-partially hyperbolic" assumption used in [**22, 45, 37**] which, in the three-dimensional setting as above, can be written as $\|D\phi_x|_E\|^2/|\det D\phi_x|_F| < 1$ (see subsection 1.1.3.3

for a non-trivial example that demonstrates this). In the view of many results such as robust transitivity implying volume hyperbolicity, we believe that this condition is more natural to work with. Later on in subsection 1.1.3 and subsubsection 3.2.1.2 we also remark on some dynamical assumptions such as transitivity and chain recurrence that imply volume domination.

As an almost immediate corollary we get integrability for standard C^1 dominated decompositions for volume-preserving diffeomorphisms on three-dimensional manifolds. We have

COROLLARY 1.4. *Let M be a 3-dimensional Riemannian manifold, $\phi : M \to M$ a volume preserving C^2 diffeomorphism, and $E \oplus F$ a $D\phi$-invariant C^1 dominated decomposition with $dim(E) = 2$. Then E is uniquely integrable.*

To see that Corollary 1.4 follows from Theorem 1.3 we just observe that in the volume-preserving setting, dynamical domination implies volume domination. Indeed, the volume preservation property implies $|detD\phi_x|_E| \cdot |detD\phi_x|_F| = 1$ and so (1.1.1.1) implies $|detD\phi_x|_F| > 1$ (arguing by contradiction, $|detD\phi_x|_F| \leq 1$ would imply $|detD\phi_x|_E| \geq 1$ by the volume preservation, which would imply $\|D\phi_x|_E\| \ /|detD\phi_x|_F| \geq 1$ which, using the fact that F is one-dimensional and therefore $|detD\phi_x|_F| = m(D\phi_x|_F)$, would contradict (1.1.1.1)). Dividing the equation $|detD\phi_x|_E| \cdot |detD\phi_x|_F| = 1$ through by $(|detD\phi_x|_F|)^2$ we get (1.1.1.2).

We remark also that the assumptions of Corollary 1.4 imply that there are non-zero Lyapunov exponents on any regular orbit and that the two exponents of E sum to the negative of the (positive) exponent of F, and therefore the conclusions follow also from the results of [**37**].

1.1.2.1. *Weak domination.* Theorem 1.3 is a special case of our main result which applies to decompositions in arbitrary dimension and arbitrary co-dimension. Let M be a compact Riemannian manifold of $dim(M) \geq 3$, $\phi : M \to M$ a C^2 diffeomorphism and $E \oplus F$ a $D\phi$-invariant decomposition with $dim(E) = d \geq 2$ and $dim(F) = \ell \geq 1$. For every $k \geq 1$ we let $s_1^k(x) \leq s_2^k(x) \leq ..s_d^k(x)$ and $s_1^{-k}(x) \geq s_2^{-k}(x) \geq ..s_d^{-k}(x)$ denote the singular values[1] of $D\phi_x^k|_E$ and $D\phi_x^{-k}|_E$ respectively, and let $r_1^k(x) \leq r_2^k(x) \leq ..r_\ell^k(x)$ and $r_1^{-k}(x) \geq r_2^{-k}(x) \geq ..r_\ell^{-k}(x)$ denote the singular values of $D\phi_x^k|_F$ and $D\phi_x^{-k}|_F$ respectively.

THEOREM 1.5. *Let M be a compact Riemannian manifold of $dim(M) \geq 3$, $\varphi : M \to M$ a C^2 diffeomorphism, and $E \oplus F$ a $D\phi$-invariant C^1 tangent bundle splitting with $dim(E) = d \geq 2$ and $dim(F) = \ell \geq 1$. Suppose there exists a dense*

[1]We recall that the singular values of $D\phi_x^{\pm k}|_E$ at $x \in M$ are the square roots of the eigenvalues of the self-adjoint map $(D\varphi_x^{\pm k}|_E)^\dagger \circ D\varphi_x^{\pm k}|_E : E(x) \to E(x)$ where $(D\varphi_x^{\pm k})^\dagger|_E : E(\varphi^{\pm k}(x)) \to E(x)$ is the conjugate of $D\varphi_x^{\pm k}|_E$ with respect to the metric g, i.e. the unique map which satisfies $g(D\varphi_x^{\pm k}|_E u, v) = g(u, (D\varphi_x^{\pm k}|_E)^\dagger v)$ for all $u \in E(x), v \in E(\varphi^{\pm k}(x))$.

subset $\mathcal{A} \subset M$ such that for every $x \in \mathcal{A}$ we have

$$(\star) \qquad \liminf_{k\to\infty} \frac{s^k_{d-1}(x)s^k_d(x)}{r^k_1(x)} = 0 \quad \textit{and/or} \quad \liminf_{k\to\infty} \frac{s^{-k}_1(x)s^{-k}_2(x)}{r^{-k}_\ell(x)} = 0.$$

Then E is uniquely integrable.

As in the 3-dimensional setting above, one of the key points here is to relax the standard "2-partially hyperbolic" condition $m(D\phi_x|_F) > |D\phi_x|_E|^2$ used to prove unique integrability in [**22**] for C^2 dynamically dominated decompositions, in [**45**] for C^1 decompositions, and in [**64**] for Lipschitz decompositions (in [**45, 64**] the result is not stated explicitly but can be deduced from the results and methods stated there). Integrability is also proved in [**37**] under the assumption that there exist constants $a, b, c, d > 0$ with $[a^2, b^2] \cap [c, d] = \emptyset$ such that $a|v| < |D\phi_x v| < b|v|$ for all $v \in E(x)$ and $c|v| < |D\phi_x v| < d|v|$ for all $v \in F(x)$, which are also more restrictive than $(\star)$. Indeed, we have either $b^2 < c$ or $a^2 > d$. The first case implies that $(b^2/c)^k$ goes to zero exponentially fast and so, letting $x^\ell = \phi^\ell(x)$ we have $(b^2/c)^k \geq (s^1_n(x^{k-1})...s^1_n(x))^2/(r^1_1(x^{k-1})...r^1_1(x)) \geq (s^k_n)^2(x)/r^k_1(x) \geq s^k_n(x)s^k_{n-1}(x)/r^k_1(x)$ and hence the first condition in $(\star)$ is satisfied. In exactly the same way if $a^2 > d$ holds then the second condition in $(\star)$ is satisfied.

Notice that $(\star)$ implies volume domination in the three-dimensional setting and therefore Theorem 1.5 implies Theorem 1.4. Indeed, the largest singular value gives the norm of the map and smallest one gives the co-norm, so one has $s^k_d(x) = |D\phi^k_x|_E|, s^{-k}_1 = |D\phi^{-k}_x|_E|, r^k_1 = m(D\phi^k_x|_F), r^{-k}_\ell = m(D\phi^{-k}_x|_F)$. So if $dim(E) = 2$ we have $s^k_d(x)s^k_{d-1}(x) = |\det(D\phi^k_x|_E)|$ and $s^{-k}_2(x)s^{-k}_1(x) = |\det D\phi^{-k}_x|_E|$, and if $dim(F) = 1$ then we have $r^k_1(x) = m(D\phi^k_x|_F) = |det(D\phi^k_x|_F)|$ and $r^{-k}_\ell(x) = m(D\phi^{-k}_x|_F) = |det(D\phi^{-k}_x|_F)|$.

1.1.2.2. *Lyapunov regularity.* In the pioneering paper [**37**] of Hammerlindl, conditions for integrability are given in terms of certain conditions on the Lyapunov exponents for volume preserving diffeomorphisms (though the volume preserving condition is not strictly necessary in his arguments which only use the existence of a dense set $\mathcal{A}$ of *regular* points, which is automatically true in the volume-preserving setting[2]). A main feature of interest in the results of [**37**] is the

[2]A point $x \in M$ is said to be regular for the map φ when there is a splitting of the tangent space $T_xM = E_1(x)\oplus E_2(x)\oplus ...\oplus E_s(x)$ invariant with respect to $D\varphi$ such that certain conditions are satisfied such as the existence of a specific asymptotic exponential growth rate (Lyapunov exponent) is well defined, see [**7**] for precise definitions. The multiplicative ergodic Theorem of Oseledets [**7**] says that the set of regular points has *full probability* with respect to *any* invariant probability measure. In particular, the assumptions of the Theorem are satisfied if every open set has positive measure for some invariant probability measure, in particular this holds if φ is volume preserving or has an invariant probability measure which is equivalent to the volume. It is easy to see that the invariant subbundles E and F can be separately split on the orbit of x

absence of an overall global domination condition of the C^1 splitting $E \oplus F$, replaced instead by "intermediate", "asymptotic" domination conditions associated to the Oseledets' "sub-decomposition" of E and F. More precisely he assumes that for all pairs of Lyapunov exponents μ_1, μ_2 of $D\phi|_E$ and λ of $D\phi|_F$, we have $\mu_1 + \mu_2 \neq \lambda$.

The following result is simply a restatement, albeit not a completely trivial one, of Theorem 1.2 of [**37**]. We include it here because it is can be stated very naturally in the language of singular values introduced above and possibly help towards a better understanding, or at least a different understanding, of the results of [**37**]. In Section 3.1.5 we show that the assumptions on the Lyapunov exponents in [**37**] are equivalent to condition $(\star\star)$ below and thus essentially reduce Theorem 1.6 to Theorem 1.2 of [**37**].

THEOREM 1.6. *Let M be a compact Riemannian manifold of $dim(M) \geq 3$, $\varphi : M \to M$ a C^2 diffeomorphism and $TM = E \oplus F$ a $D\phi$-invariant C^1 tangent bundle splitting with $dim(E) = d \geq 2$ and $dim(F) = \ell \geq 1$. Suppose that there is a dense subset $\mathcal{A} \subset M$ of regular points such that for each $x \in \mathcal{A}$ and each set of indices $1 \leq i, j \leq d, 1 \leq m \leq \ell$ there exists a constant $\lambda > 0$ such that*

$$(\star\star) \qquad \frac{s_i^k(x) s_j^k(x)}{r_m^k(x)} \leq e^{-\lambda k} \quad \forall\, k \geq 1 \quad \textit{or} \quad \frac{s_i^{-k}(x) s_j^{-k}(x)}{r_m^{-k}(x)} \leq e^{-\lambda k} \quad \forall\, k \geq 1.$$

Then E is uniquely integrable.

We will make a few remarks comparing conditions $(\star)$ and $(\star\star)$.

A first observation is that the decay rate of the ratios in $(\star\star)$ is assumed to be exponential, whereas this is not required in $(\star)$, where decay at an a priori arbitrarily slow rate is required, and even then only for a subsequence. We do not know if this is a real weakening of exponential decay or if condition $(\star)$ effectively implies exponential decay, but in the absence of any assumption about x being a regular point, it seems likely that it is indeed strictly weaker.

A perhaps more relevant observation is that in both cases the decay of the ratios is allowed to occur either along the forward orbit of a point or along the backward orbit of a point, but in condition $(\star\star)$ the choice of which of these two estimates are satisfied *is allowed to depend on the choice of indices.* More precisely, notice that

$$(1.1.2.1) \qquad s_{d-1}^k(x) s_d^k(x) / r_1^k(x) \leq e^{-\lambda k} \quad \Longrightarrow \quad s_i^k(x) s_j^k(x) / r_m^k(x) \leq e^{-\lambda k}$$

for *every* set of indices $1 \leq i, j \leq d, 1 \leq m \leq \ell$, and similarly

$$(1.1.2.2) \quad s_1^{-k}(x) s_2^{-k}(x) / r_n^{-l}(x) \leq e^{-\lambda k} \quad \Longrightarrow \quad s_i^{-k}(x) s_j^{-k}(x) / r_m^{-k}(x) \leq e^{-\lambda k}$$

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using these subbundles, that is $E(x^k) = E_{i_1}(x^k) \oplus ... \oplus E_{i_d}(x^k)$, $F(x^k) = E_{j_1}(x^k) \oplus ... \oplus E_{j_\ell}(x^k)$ where the indice sets $\{i_1, ..., i_d\}$ and $\{j_1, ..., j_\ell\}$ do not intersect.

for *every* set of indices $1 \leq i, j \leq d, 1 \leq m \leq \ell$. Thus, (an exponential version of) condition ($\star$) requires that one of the equations (1.1.2.1) and (1.1.2.2) be satisfied and therefore forces ($\star\star$) to hold either *always* in forward time for all choices of indices or *always* in backward time for all choices of indices. The crucial point of condition ($\star\star$) is to weaken this requirement and to allow either the backward time condition or the forward time condition to be satisfied *depending on the choice of indices* (in fact we will see in Section 3.1.5 that under the assumption of Lyapunov regularity these two choices are mutually exclusive for a given point x but of course may depend on the choice of x). In particular this means that the sub-bundles E and F do not satisfy an overall domination condition but rather some sort of "non-resonance" conditions related to the further intrinsic Oseledets splittings of E and F.

1.1.3. Lipschitz Case. Let $E \oplus F$ a Lipschitz continuous $D\varphi$-invariant tangent bundle decomposition, with $dim(E) = 2$ and $dim(F) = 1$.

1.1.3.1. *Dynamical domination and robust transitivity.* We have the following integrability result related to robust transitivity and dynamical domination:

THEOREM 1.7. *Let M be a Riemannian 3-manifold, $\varphi : M \to M$ a volume-preserving or transitive C^2 diffeomorphism and $E \oplus F$ a $D\varphi$-invariant, Lipschitz, dynamically dominated, decomposition. Then E is uniquely integrable.*

REMARK 1.8. Under the stronger assumption that φ is *robustly transitive* (i.e. φ is transitive and any C^1 sufficiently close diffeomorphism is also transitive) instead of just transitive, the dynamically dominated condition is automatically satisfied [**33**] and so integrability follows under the additional assumption that the decomposition is Lipschitz.

REMARK 1.9. We could replace the transitivity assumption in Theorem 1.7 by chain recurrence or even just the absence of sources, see Section 3.2.1.2.

1.1.3.2. *Volume domination.* We will obtain Theorem 1.7 as a special case of the following more general result which replaces the volume preservation/transitivity and dynamical domination conditions with a single volume domination condition. We have the following result.

THEOREM 1.10. *Let M be a Riemannian 3-manifold, $\varphi : M \to M$ a C^2 diffeomorphism and $E \oplus F$ a $D\varphi$-invariant, Lipschitz continuous, volume dominated decomposition. Then E is uniquely integrable.*

Theorems 1.7 and 1.10 extend analogous statements in [**57**] obtained using different arguments, in arbitrary dimension but under the assumption that the decomposition is C^1. They also extend previous results of Burns and Wilkinson [**22**], Hammerlindl and Hertz-Hertz-Ures [**37, 45**] and Parwani [**64**] who prove analogous results[3] for respectively C^2, C^1 and Lipschitz distributions under the assumption of *center-bunching* or *2-partial hyperbolicity*:

$$\frac{\|D\varphi_x|_{E_x}\|^2}{\|D\varphi_x|_{F_x}\|} < 1 \tag{1.1.3.1}$$

for every $x \in M$. In the 3-dimensional setting condition (1.1.3.1) clearly implies volume domination and is therefore more restrictive. In Section 1.1.3.3 we sketch an example of a diffeomorphism and an invariant distribution E which does not satisfy condition (1.1.3.1) but does satisfy the dynamical domination and volume domination assumptions we require in Theorem 1.10. This particular example is uniquely integrable by construction and so is not a "new" example, but helps to justify the observation that our conditions are indeed less restrictive than center-bunching (1.1.3.1).

The techniques we employ here are similar to those of Parwani but to relax the center-bunching condition one needs a more careful analysis of the behaviour of certain Lie brackets, this is carried out in Section 3.2.1.4.

Remark 1.11. Volume hyperbolicity may also be obtained from other topological properties of the map φ in some generic setting, for example C^1 generically non-wandering maps are volume hyperbolic, see [**10, 68**] for this and other related results. It is not clear to what extent the results in the cited papers may improve this paper's results since we require our maps to be C^2, however we mention them as it seems interesting that weaker forms of partial hyperbolicity seem to be relevant in different settings.

Remark 1.12. The assumption that the diffeomorphisms in the Theorems above are C^2 is necessary for the arguments we use in the proofs. In the proof of Theorem 1.10, we need to be able to compute the Lie brackets of iterates of certain sections from E by $D\varphi$. For this reason $D\varphi$ needs to be C^1 to keep the regularity of a section along the orbit of a initial point p.

1.1.3.3. *Volume Domination versus 2-Partially Hyperbolic.* In this section we are going to sketch the construction of some non-trivial examples which satisfy the volume domination condition (1.1.1.2) but not the center-bunching condition (1.1.3.1). This is a variation of the "derived from Anosov" construction due to

[3] In some of the references mentioned, the relevant results are not always stated in the same form as given here but may be derived from related statements and the technical arguments. In some cases the setting considered is that of partially hyperbolic diffeomorphisms with a tangent bundle decomposition of the form $E^s \oplus E^c \oplus E^u$ where E^s is uniformly contracting and E^u uniformly expanding. In this setting, one considers the integrability of the sub-bundles $E^{sc} = E^s \oplus E^c$ and $E^{cu} = E^c \oplus E^u$ and it is not always completely clear to what extent the existence of a uniformly expanding sub-bundle is relevant to the arguments. We emphasize that the setting we consider here does not require the invariant distribution E to contain any further invariant sub-bundle.

Mañé [**60**] (see [**12**] for the volume preserving case, which is what we use here). We are very grateful to Raúl Ures for suggesting and explaining this construction to us. Consider the matrix

$$\begin{pmatrix} -3 & 0 & 2 \\ 1 & 2 & -3 \\ 0 & -1 & 1 \end{pmatrix}$$

This matrix has determinant 1 and has integer coefficients therefore induces a volume preserving toral automorphism on $\mathbb{T}^3$. It is Anosov since its eigenvalues are $r_1 \sim -0.11, r_2 \sim 3.11, r_3 \sim -3.21$. Note that $r_1 r_2 / r_3 < 1$ but $r_2^2 / r_3 > 1$. Hence (1.1.1.2) is satisfied but (1.1.3.1) is not. Now take a fixed point p and a periodic point q and a neighbourhood U of p so that forward iterates of q never intersect U. One can apply Mané's construction to perturb the map on U as to obtain a new partially hyperbolic automorphism of $\mathbb{T}^3$ which is still volume preserving. Such a perturbation is not a small one and therefore one can not claim integrability of the new system trivially by using standard theorems as in [**50**]. Since the perturbation is performed on U and orbit of q never intersects U, the perturbation does not change the splitting and the contraction and expansion rates around q and in particular (1.1.3.1) is still not satisfied on the orbit of q. Yet the new example is volume preserving therefore it is necessarily the case that (1.1.1.2) is satisfied.

1.1.4. Sequential transversal regularity. The kind of techniques we use lead naturally to the formulation of a somewhat unorthodox regularity condition, which we call "sequentially transversal Lipschitz regularity". The main reason that we choose to present this result is that the techniques used in the proof, especially those in Section 3.3.2, generalize naturally to yield continuous Frobenius-type theorems, such as those given in forthcoming papers [**59, 76**]; we also believe that there are some interesting questions to be pursued regarding the relation between Lipschitz regularity and sequential transversal Lipschitz regularity, we discuss these in subsection 1.1.4.1 below. We will later give a "detailed sketch" of the arguments concentrating mostly on techniques which are novel, the full arguments can be found in a previous version of our paper [**58**].

Let $E \oplus F$ be a continuous $D\varphi$-invariant tangent bundle decomposition with $dim(E) = 2$. We say that E is *sequentially transversally Lipschitz* if there exists a C^1 line bundle Z, everywhere transverse to E, and a C^1 distribution $E^{(0)}$ such that the sequence of C^1 distributions $\{E^{(k)}\}_{k>1}$ given by

$$E_x^{(k)} = D\varphi_{\varphi^k x}^{-k} E_{\varphi^k x}^{(0)}, \forall x \in M, k > 1 \tag{1.1.4.1}$$

are equi-Lipschitz along Z, i.e. there exists $K > 0$ such that for every $x, y \in M$ close enough and belonging to the same integral curve of Z, and every $k \geq 0$, we have $\measuredangle(E_x^{(k)}, E_y^{(k)}) \leq K d(x, y)$.

THEOREM 1.13. *Let M be a Riemannian 3-manifold, $\varphi : M \to M$ a C^2 diffeomorphism and $E \oplus F$ a $D\varphi$-invariant, sequentially transversally Lipschitz, dynamically and volume dominated, decomposition. Then E is uniquely integrable.*

1.1.4.1. *Relation Between Lipschitzness and Transversal Lipschitzness.* Here we discuss some general questions concerning the relationships between various forms of Lipschitz regularity. We say that a sub-bundle E is *transversally Lipschitz* if there exists a C^1 line bundle Z, everywhere transverse to E, along which E is Lipschitz. The relations between Lipschitz, transversally Lipschitz, and sequentially transversally Lipschitz are not clear in general. For example it is easy to see that sequentially transversally Lipschitz implies transversally Lipschitz but we have not been able to show that transversally Lipschitz, or even Lipschitz, implies sequentially transversally Lipschitz. Nevertheless certain equivalence may exist under certain forms of dominations for bundles which occur as invariant bundles for diffeomorphisms. We formulate the following question:

QUESTION 1.14. *Suppose $E \oplus F$ is a $D\varphi$-invariant decomposition satisfying* (1.1.3.1). *Then is E transversally Lipschitz if and only if it is Lipschitz ?*

One reason why we believe this question is interesting is that that transversal Lipschitz regularity is a-priori strictly weaker than full Lipschitz regularity. Thus a positive answer to this question would imply that transversal Lipschitzness of center-bunched dominated systems becomes in particular, by Theorem 1.10, a criterion for their unique integrability. More generally, a positive answer to this question would somehow be saying that one only needs some domination condition and transversal regularity to prevent E from demonstrating pathological behaviours such as non-integrability or non-Lipschitzness.

The notion of sequential transversal regularity and the result of Theorem 1.13 may play a role in a potential solution to the question above. Indeed, if E is sequentially transversally Lipschitz and volume dominated, then by Theorem 1.13 it is uniquely integrable. Then, under the additional assumption of centre-bunching, by arguments derived from theory of normal hyperbolicity (see [**50**]) it is possible to deduce that E is Lipschitz along its foliation $\mathcal{F}$. We also know that there is a complementary transversal foliation given by integral curves of Z along which E is sequentially Lipschitz and therefore Lipschitz. This implies that E is Lipschitz.

Thus center-bunching and sequential transverse regularity implies Lipschitz. The missing link would just be to show that if E is transversally Lipschitz along a direction then it is also sequentially transversally Lipschitz along that direction. This would yield a positive answer to the question.

1.2. Overview of Chapter 4

In this chapter we formulate a continuous version of Frobenius theorem by using certain notions of asymptotic involutivity. In this chapter we denote the continuous bundle by Δ. We do not work with a specific dynamical system as in the previous chapters and the manifold can be non-compact in this case.

We will formulate these conditions in two stages: the first one more natural and the second one more general and more technical and more useful for applications. Since integrability is a local property we will work in some fixed local chart of the manifold; in particular the distribution in this local chart can always be written as the kernel of some 1-form η. If η is sufficiently regular (e.g. C^1), it admits an exterior derivative which we will denote by $d\eta$. All norms to be used below will be the norms induced by the Riemannian volume. Unless specified otherwise all norms and converging sequences refer to the C^0 topology.

The main theorems are; theorem 1.16 in page 10 and theorem 1.18 in page 12.

1.2.1. Asymptotic involutivity. In the following, given a two dimensional distribution Δ, we will assume that it is given as the kernel of a non-vanishing differential 1-form. This is strictly true when the distribution is coorientable however integrability and unique integrability are local questions therefore everything will be done in local coordinates so we do not lose generality by assuming that $\Delta = ker(\eta)$ for some non-vanishing differential 1-form η.

Definition 1.15. A continuous distribution $\Delta = ker(\eta)$ is *asymptotically involutive* if there exists a sequence of C^1 differential 1-forms η_k with $\eta_k \to \eta$ such that

$$\|\eta_k \wedge d\eta_k\| e^{\|d\eta_k\|} \to 0$$

as $k \to \infty$. Δ is *uniformly asymptotically involutive* if moreover we have

$$\|\eta_k - \eta\| e^{\|d\eta_k\|} \to 0.$$

Theorem 1.16. *Let Δ be a 2-dimensional distribution on a 3-dimensional manifold. If Δ is asymptotically involutive then it is integrable. If Δ is uniformly asymptotically involutive then it is uniquely integrable.*

We recall that the classical Frobenius Theorem [**35**] yields unique integrability for C^1 distributions (in arbitrary dimension) under the assumption that Δ is *involutive*:

$$\eta \wedge d\eta = 0. \qquad (1.2.1.1)$$

This can be seen as a special case of Theorem 1.16 by choosing $\eta_k \equiv \eta$. Other generalizations and extensions of Frobenius' Theorem exist in the literature, related both to the regularity of the distribution and to the setting of the problem, see

[**24, 25, 26, 27, 41, 42, 48, 49, 39, 40, 53, 54, 56, 62, 63, 67, 69, 73, 74, 75**], including generalizations to Lipschitz distributions, for which condition (1.2.1.1) can be formulated almost everywhere, and the interesting, though apparently not very well known, generalization of Hartman [**41, 42**] to *weakly differentiable* distributions, i.e. distributions defined by a 1-form η which may not be differentiable or even Lipschitz but still admits continuous exterior derivative[4] $d\eta$ and for which, therefore, condition (1.2.1.1) can also still be formulated.

Our definition of asymptotic involutivity allows for a significant relaxation of the assumptions on the regularity of Δ and in particular *does not require* that η admit a continuous exterior derivative. Indeed, Hartman [**41**] showed that the existence of $d\eta$ is equivalent to the existence of a sequence of C^1 differential 1-forms η_k such that $\eta_k \to \eta$ and $d\eta_k \to d\eta$. Replacing, as we do here, the conditions $d\eta_k \to d\eta$ and $\eta \wedge d\eta = 0$ with $\|\eta_k \wedge d\eta_k\| e^{\|d\eta_k\|} \to 0$ and $\|\eta_k - \eta\| e^{\|d\eta_k\|} \to 0$ relaxes the assumption on the existence of $d\eta$.

1.2.2. Asymptotic involutivity on average. A key feature of the asymptotic involutivity condition is that the regularity condition is described in terms of $\|d\eta_k\|$ rather than C^1 norm of η^k. If η is non-Lipschitz than the C^1 norms of η^k necessarily blow up while $\|d\eta_k\|$ may not. We later give some examples of this kind. Moreover we actually do allow even $\|d\eta_k\|$ to blow up, albeit at some controlled rate. Our argument however yields some rather technical, but significantly more general, conditions which relax to some extent the requirement on the rate at which $\|d\eta_k\|$ is allowed to blow up, and instead only require some control on the rate at which the "average" value of $\|d\eta_k\|$ blows up. It seems that these weaker conditions are significantly easier to verify in applications and therefore we give here a precise formulation and statement of results in terms of these conditions.

We fix some arbitrary point $x_0 \in M$ and a local coordinate system $(x^1, x^2, x^3, \mathcal{U})$ around x_0. We suppose we are given a continuous form η defined in $\mathcal{U}$ and the corresponding distribution $\Delta = ker(\eta)$, and assume without loss of generality that Δ is everywhere transversal to the coordinate axis $\partial/\partial x^3$. For any sequence of C^1 forms η_k defined in $\mathcal{U}$ we write the corresponding exterior derivative $d\eta_k$ in coordinates as

$$d\eta_k = d\eta_{k,1} dx^1 \wedge dx^3 + d\eta_{k,2} dx^2 \wedge dx^3 + d\eta_{k,3} dx^1 \wedge dx^2$$

where $d\eta_{k,1}, d\eta_{k,2}, d\eta_{k,3}$ are C^1 functions defined in $\mathcal{U}$. If $\eta_k \to \eta$ then, for all k sufficiently large, the corresponding distributions $\Delta_k = ker(\eta_k)$ are also transversal

[4]More precisely we say that η is "weakly differentiable" if there exists a differential 2-form $d\eta$ that satisfies Stokes' Formula: $\int_J \eta = \int\int_S d\eta$ for every piece of C^1 surface $\mathcal{S}$ bounded by a C^1 piecewise Jordan curve J. Note that this condition holds for example under the assumption that η is Lipschitz (and therefore differentiable almost everywhere) and is therefore strictly weaker than assuming that η is C^1.

to the coordinate axis $\partial/\partial x^3$ and therefore there exist C^1 frames $\{X_k, Y_k\}$ for Δ_k in $\mathcal{U}$ where X_k, Y_k are C^1 vector fields of the form

$$X_k = \frac{\partial}{\partial x^1} + a_k \frac{\partial}{\partial x^3}, \qquad Y_k = \frac{\partial}{\partial x^2} + b_k \frac{\partial}{\partial x^3} \tag{1.2.2.1}$$

for some C^1 functions a_k, b_k. We let $e^{\tau X_k}, e^{\tau Y_k}$ denote the flows induced by the vector fields X_k, Y_k respectively. Fixing some smaller neighbourhood $\mathcal{U}' \subset \mathcal{U}$ we can choose $t_0 > 0$ such that the flow is well defined and $e^{\tau X_k}(x), e^{\tau Y_k}(x) \in \mathcal{U}$ for all $x \in \mathcal{U}'$ and $|\tau| \leq t_0$. Then, for every $x \in \mathcal{U}'$, $|t| \leq t_0$ we define

$$\widetilde{d\eta}_{k,1}(x,t) := \int_0^t d\eta_{k,1} \circ e^{\tau X^{(k)}}(x) d\tau,$$

$$\widetilde{d\eta}_{k,2}(x,t) := \int_0^t d\eta_{k,2} \circ e^{\tau Y^{(k)}}(x) d\tau$$

and

$$\widetilde{d\eta}_k(x,t) := \max\{\widetilde{d\eta}_{k,1}(x,t), \widetilde{d\eta}_{k,2}(x,t)\}.$$

Definition 1.17. A continuous distribution $\Delta = ker(\eta)$ is *asymptotically involutive on average* if, for every $x_0 \in M$, there exist local coordinates around x_0 and a sequence of C^1 differential 1-forms η_k with $\eta_k \to \eta$ and corresponding C^1 distributions $\Delta_k = ker(\eta_k)$ and C^1 local frames $\{X_k, Y_k\}$, and a neighbourhood $\mathcal{U}' \subset \mathcal{U}$ such that for every $x \in \mathcal{U}'$ and every $|t| \leq t_0$

$$\|\eta_k \wedge d\eta_k\|_x e^{\widetilde{d\eta}_k(x,t)} \to 0$$

as $k \to \infty$. Δ is *uniformly asymptotically involutive on average* if, moreover, for every $x \in \mathcal{U}'$ and every $|t| \leq t_0$,

$$\|\eta_k - \eta\|_x e^{\widetilde{d\eta}_k(x,t)} \to 0$$

as $k \to \infty$.

Theorem 1.18. *Let Δ be a 2-dimensional distribution on a 3-dimensional manifold. If Δ is asymptotically involutive on average then it is integrable. If Δ is uniformly asymptotically involutive on average then it is uniquely integrable.*

Notice that $\widetilde{d\eta}_k(x,t) \leq \|d\eta_k\|$ and therefore Theorem 1.16 follows immediately from Theorem 1.18.

We conclude this section with a question motivated by the observation that in the C^1 setting the involutivity condition (1.2.1.1) is both necessary and sufficient for unique integrability. It seems natural to ask whether the same is true for uniform asymptotic involutivity on average.

Question 1.19. *Let Δ be a 2-dimensional continuous uniquely integrable distribution on a 3-dimensional manifold. Is Δ uniformly asymptotically involutive on average?*

1.2.3. Applications. We discuss here three applications of our results: to the problem of the uniqueness of solutions of ODE's, of existence and uniqueness of solutions of PDE's, and to the problem of integrability of invariant bundles in Dynamical Systems. While none of these applications perhaps has the status of a major result in itself, we believe they are good "examples" and indicate the potential applicability of our main integrability results to a wide range of problems in different areas of mathematics.

1.2.3.1. *Uniqueness of solutions for ODE's.* We consider a vector field

$$X = f(x) \tag{1.2.3.1}$$

defined in some local chart $\mathcal{U}$ of a two-dimensional Riemannian manifold M by a non-vanishing continuous function f. By a classical result of Peano, X admits locally defined integral curves at every point in $\mathcal{U}$ but uniqueness is not guaranteed as there exist simple counterexamples even if f is Hölder continuous. A natural question concerns the "weakest" form of continuity which guarantees uniqueness. We recall that the *modulus of continuity* of a continuous function f defined on $\mathcal{U}$ is a continuous function $w : [0,\infty) \to [0,\infty)$ such that $w(t) \to 0$ as $t \to 0$ and, for all $x, y \in \mathcal{U}$,

$$|f(x) - f(y)| \leq w(|x - y|).$$

As a Corollary of our arguments we obtain the following result which we will prove in Section 4.2.1.

THEOREM 1.20. *Suppose the modulus of continuity of f satisfies*

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_0^\epsilon \omega(t)dt \cdot \exp\left(\frac{1}{\epsilon^2}\int_0^\epsilon \omega(t)dt\right) = 0. \tag{1.2.3.2}$$

Then X has a unique local integral curve through every point in $\mathcal{U}$.

It would be interesting to compare our condition (1.2.3.2) with other conditions such as the classical and the well-known *Osgood* condition

$$\int_0^\epsilon \frac{1}{w(t)} dt = \infty \tag{1.2.3.3}$$

which also implies unique integrability [**66**] . For the moment however we have not been able to establish a relationship between the two conditions.

QUESTION 1.21. *Does either of* (1.2.3.2) *or* (1.2.3.3) *imply the other? Are there examples of functions which satisfy one and not the other?*

We remark that any function which admits a modulus of continuity also admits an *increasing* modulus of continuity $\hat{w}(t) \geq \omega(t)$. Then if (1.2.3.2) holds for the modulus of continuity $\omega(t)$ it clearly holds also for the increasing modulus of continuity $\hat{\omega}(t)$ (but notice that the converse is not true). If the modulus of continuity is increasing, then (1.2.3.2) is equivalent to the following more natural-looking condition

$$\lim_{t\to 0} \omega(t) e^{\frac{\omega(t)}{t}} = 0. \tag{1.2.3.4}$$

It is easy to check that Lipschitz functions satisfy (1.2.3.4) (and therefore also (1.2.3.2)) as well as (1.2.3.3), as do some standard non-Lipschitz functions such as $w(t) = t \ln t, w(t) = t \ln ... \ln t$ and more "exotic" examples such as $w(t) = t \ln^{1+t} t$, whereas functions such as $w(t) = t^{\alpha}, \alpha \in (0,1)$ and $w(t) = t \ln^{\alpha} t, \alpha > 1$ satisfy neither our condition (1.2.3.2) nor (1.2.3.3). It seems likely that if there exists any example of a modulus of continuity which satisfies (1.2.3.2) but not (1.2.3.3) it would have to have some significant amount of oscillation which could be controlled by the integrals in (1.2.3.2) but not by the simpler condition (1.2.3.4).

1.2.3.2. *Pfaff Equations.* Besides the intrinsic interest of the question of integrability from a purely geometric point of view, the issue of integrability classically arises in the context of the problem of existence and uniqueness of solutions of PDE's. Indeed, this seems to have been the main motivation of Frobenius [**35**], who applied previous results of Clebsch [**28**] and Deahna [**32**], see discussion in [**55**], to Pfaff equations, i.e. equations of the form

$$\begin{cases} \dfrac{\partial f}{\partial x}(x,y) = a(x,y,f(x,y)) \\ \dfrac{\partial f}{\partial y}(x,y) = b(x,y,f(x,y)) \end{cases} \tag{$\mathcal{P}$}$$

where $a(x,y,z), b(x,y,z)$ are scalar functions defined on $\mathcal{U} = \mathcal{V} \times I \subset \mathbb{R}^3$. When $f = f(x,y)$ exists (and is unique), with the initial condition $f(x_0,y_0) = z_0$, the system $(\mathcal{P})$ is said to be *(uniquely) integrable* at (x_0,y_0,z_0).

The existence and uniqueness of the Pfaff system of equations clearly depends on the properties of the functions a and b. The classical Theorem of Frobenius gives some *involutivity* conditions which imply integrability if the functions a, b are C^1. As a relatively straightforward application of our more general result, we can consider the situation where a, b are just continuous but have a particular, though not very restrictive, form. More specifically suppose that a, b have the form

$$a(x,y,z) := A(x,y)F(z) \quad \text{and} \quad b(x,y,z) := B(x,y)F(z) \tag{$\tilde{\mathcal{P}}$}$$

for continuous functions $A(x,y), B(x,y), F(z)$ satisfying:

($\tilde{\mathcal{P}}_1$) F is Lipschitz continuous

($\tilde{\mathcal{P}}_2$) There exist sequences $A^{(k)}, B^{(k)}$ of C^1 functions such that

i) $A^{(k)} \to A$ and $B^{(k)} \to B$,

ii) $A_y^{(k)} - B_x^{(k)} \to 0$.

Note that $A_y^{(k)}, B_x^{(k)}$ denote the partial derivatives of $A^{(k)}, B^{(k)}$ with respect to x and y respectively, and that the convergence in *i*) and *ii*) of $(\tilde{\mathcal{P}}_2)$ are intended in the C^0 topology.

THEOREM 1.22. *The Pfaff system* $(\mathcal{P})$ *defined by functions of the form* $(\tilde{\mathcal{P}})$ *satisfying* $(\tilde{\mathcal{P}}_1), (\tilde{\mathcal{P}}_2)$ *is uniquely integrable.*

We will prove Theorem 1.22 in Section 4.2.2.

REMARK 1.23. We remark that condition $(\tilde{\mathcal{P}}_2)$ seems relatively abstract but it is quite easy to construct examples of continuous functions A, B which satisfy it. Suppose for example that $\tilde{A}(x), \tilde{B}(y)$ are continuous functions and that $\varphi(x, y)$ is a C^2 function, and let $A(x, y) = \tilde{A}(x) + \varphi_x(x, y)$ and $B(x, y) = \tilde{B}(y) + \varphi_y(x, y)$. Let $\tilde{A}^{(k)}, \tilde{B}^{(k)}$ be sequences of C^1 functions with $\tilde{A}^{(k)} \to \tilde{A}, \tilde{B}^{(k)} \to B$. Then $A^{(k)}(x, y) = \tilde{A}^{(k)}(x) + \varphi_x(x, y)$ and $B^{(k)}(x, y) = \tilde{B}^{(k)}(y) + \varphi_x(x, y)$ are C^1 functions and it follows that $A^{(k)} \to A$, $B^{(k)} \to B$. Moreover the partial derivatives are $A_y^{(k)} = \varphi_{xy}$, $B_x^{(k)} = \varphi_{yx}$ and therefore $A_y^{(k)} - B_x^{(k)} = 0$.

1.2.3.3. *Dominated decompositions with linear growth.* Continuous distributions arise naturally in Dynamical Systems as $D\varphi$-invariant distributions for some diffeomorphism $\varphi : M \to M$. The integrability (or not) of such distributions can have significant implications for ergodic and topological properties of the dynamics generated by φ. The classical Frobenius Theorem and its various extensions have generally not been suitable for studying the integrability of such "dynamically defined" distributions which are usually given implicitly by asymptotic properties of the dynamics and therefore have low regularity. The conditions we give here, on the other hand, are naturally suited to treat these kind of distributions because they allow distributions with low regularity and also because they formulate the notion of involutivity in an asymptotic way which lends itself to be verified by sequences of dynamically defined approximations to the invariant distributions.

A first non-trivial application of the results is given in [**76**] for a class of C^2 diffeomorphisms $\varphi : M \to M$ of a 3-dimensional manifold which admit a *dominated splitting*: there exists a continuous $D\varphi$-invariant tangent bundle decomposition $TM = E \oplus F$ and a Riemannian metric for which derivative restricted to the 1-dimensional distribution F is *uniformly expanding*, i.e. $\|D\varphi_x\| > 1$ for all $x \in M$, and the derivative restricted to the 2-dimensional distribution E may have a mixture of contracting, neutral, or expanding behaviour but is in any case *dominated* by the derivative restricted to F, i.e. $\|D\varphi_x(v)\| < \|D\varphi_x(w)\|$ for all $x \in M$ and all unit vectors $v \in E_x, w \in F_x$.

Dominated splittings, even on 3-dimensional manifolds, are not generally uniquely integrable [**47**] but the main result of [**76**] is the unique integrability of dominated splittings on 3-dimensional manifolds under the additional assumption that the derivative restricted to E admits *at most linear* growth, i.e. there exists a constant $C > 0$ such that $\|D\varphi_x^k v\| \leq Ck$ for all $x \in M$, all unit vectors $v \in E_x$, and all $k \in \mathbb{N}$. This result is obtained by a non-trivial argument which leads to the verification that the distribution E is *uniformly asymptotically involutive on average* and therefore Theorem 1.18 can be applied, giving unique integrability.

Previous related results include unique integrability for splittings on the torus $\mathbb{T}^3$ which admit a strong form of domination [**20**] and other results which assume various, rather restrictive, geometric and topological conditions [**15, 38, 64, 45**]. The assumption on linear growth is a natural extension of the most classical of all integrability results in the dynamical systems setting, that of *Anosov diffeomorphisms*, where E is uniformly contracting, i.e. $\|D\varphi_x(v)\| < 1 < \|D\varphi_x(w)\|$ for all $x \in M$ and for every unit vector $v \in E_x, w \in F_x$ and for center-stable bundles of time 1 maps of Anosov flows for which < 1 is replaced by ≤ 1. For Anosov diffeomorphisms the integrability of the invariant distributions can be obtained by a very powerful set of techniques which yield so-called "Stable Manifold" Theorems, which go back to Hadamard and Perron, see [**50**]. These techniques however generally break down in settings where the domination is weaker. The application of our Theorem 1.18, as implemented in [**76**], includes the setting of Anosov diffeomorphisms on 3-dimensional manifolds and thus represents a perhaps more flexible, and maybe even more powerful in some respects, alternative to the standard/classical techniques.

CHAPTER 2

Integrability

2.1. Integrability

This section is devoted to intricacies of how to define a desired notion of integrability in the continuous setting. The main references for this section are [**1, 19, 22, 54**]. We will also have a few words to say about integrability in the differentiable setting, especially about properties that are still valid in the continuous setting.

Let E be a subbundle of TM where M is a smooth manifold. An immersed manifold will always mean injectively immersed for us. We also remind that if N is an immersed submanifold of M then given any open subset $U \subset M$, $U \cap N$ is also open in N since subspace topology of N is coarser than the manifold topology but an open set in the manifold topology need not be an open set in the subspace topology. If $i : N \hookrightarrow M$ is an immersion or embedding, by abuse of notation we identify $i(N)$ with N. Finally given an integrable vector field X, for obvious reasons, we will denote its flow starting at the point p by $e^{tX}(p)$ and the differential of this flow with respect to manifold coordinates as De^{tX}_p (whenever it is possible to differentiate it). We will occasionally use the pushforward notation. More precisely, let X, Y be vector fields defined on $U \subset M$ and let $V \subset U$ and ϵ be such that $e^{tX}(V) \subset U$ for all $t \leq \epsilon$. Then $t \to e^{tX}_* Y$ is a family of vector fields defined on $e^{tX}(V)$ by the formula $(e^{tX}_* Y)_{e^{tX}(p)} = De^{tX}_p Y_p$ for $p \in V$.

Definition 2.1. E is said to be **integrable at p** if there exists a $rank(E)$ dimensional immersed, connected submanifold N such that $p \in N$ and for all $q \in N$, $T_qN = E_q$. N will be called an integral manifold of E passing through p. E is said to be **integrable** if it has integral manifolds through every point in M. Two integral manifolds N_1, N_2 with non-empty intersection are said to satisfy **uniqueness of solutions** if whenever they intersect, their intersection is relatively open in both submanifolds. E is said to be **uniquely integrable** if it is integrable and every pair of integral manifolds N_1, N_2 satisfies uniqueness of solutions. Finally in this case **the maximal integral manifold** of E passing through p is the integral manifold that contains all the other integral manifolds passing through p .

We will later show in lemma 2.4 that if uniqueness of solutions are satisfied then arbitrary union of integral manifolds which all intersect is again an integral manifold. Therefore a unique maximal element always exist. This is constructed by taking union of all the connected integral manifolds passing through p. Existence of two maximal distinct integral manifolds is a contradiction since their union is an integral manifold that contains both.

We refer the reader to subsection 2.1.3 to understand better why we call the property above uniqueness of solutions. In particular, we show that the geometric question of integrability is locally equivalent to solving a homogenous system of linear, first order partial differential equations, hence the terminology "solutions".

DEFINITION 2.2. A rank n, C^r foliation on a smooth manifold M, is a family $\mathcal{F}$ of connected, disjoint, immersed C^r submanifolds of dimension n such that every point in M belongs to one of the submanifolds and the intersection of each submanifold in $\mathcal{F}$ with a coordinate neighbourhood of M consists of countably many connected components. The elements of $\mathcal{F}$ will be called leaves of the foliation.

The last condition is a technicality, which although is quite important for the topological properties of a foliation, will not be of interest to us very much. The most critical part is the decomposition into disjoint submanifolds part. We note that from a foliation one can build a subbundle of TM of rank n by taking the tangent space of the element of $\mathcal{F}$ at every point. We denote this bundle by $T\mathcal{F}$. Some authors prefer to call $\mathcal{F}$ a C^r foliation if $T\mathcal{F}$ is C^r and some others when leaves of $\mathcal{F}$ form a C^r family of C^r submanifolds [1]. It is possible to show that if $T\mathcal{F}$ is C^r for $r > 0$ then integral manifolds of $\mathcal{F}$ form a C^r family of C^r submanifolds: in every locality choose a basis of sections of $T\mathcal{F}$ and integrate them in order. Theorems from theory of ODE will tell you that you will get pieces of C^r integral manifolds which form a C^r family of integral manifolds due to differentiable dependence on initial conditions. The reverse only holds true if integral manifolds of $\mathcal{F}$ form a C^{r+1} family; having C^{r+1} leaves or a C^r family of C^r leaves is not enough. Thus the definition given in 2.2 is the one which makes the least assumptions on regularity. Nevertheless in the next chapters, we will almost always be dealing with continuous sub-bundles and will not be concerned by these regularity questions very much.

PROPOSITION 2.3. *If E is uniquely integrable then there exists a foliation $\mathcal{F}$ such that $E_p = T\mathcal{F}_p$.*

[1]The notion of C^r family of integral manifolds can be formulated in various ways. One way is just to choose some particular local coordinates and write the integral manifolds of a subbundle as family of graphs in the local variables. If these graphs are C^r in every variable in some neighbourhood of every point, then the integral manifolds can be called a C^r family. This is stronger than just the leaves being C^r as in definition 2.2

Proof. This will be a direct result of the following lemma:

Lemma 2.4. *Let $\{N_i\}_i$ be an arbitrary collection of C^r integral manifolds of E all of which have a point in common and all of which satisfy uniqueness of solutions. Then $N = \cup_i N_i$ is a C^r immersed manifold.*

Proof. We will start defining a topology on N. Define the topology by setting: $U \subset N$ to be open if $U_i = U \cap N_i$ is open in N_i for each i. By uniqueness of solutions, this topology contains all N_i (since uniqueness implies all $N_i \cap N_j$ are open), N (since $N \cap N_i = N_i$) and $\emptyset$. If U and V are open then U_i, V_i are open in N_i and so is $U_i \cap V_i$. But $(U \cap V) \cap N_i = U_i \cap V_i$ so $U \cap V$ is an element of this topology. Similar statement holds for $U_i \cup V_i$, so this is indeed a topology. It remains to prove that as a topological space N is Hausdorff, second-countable and can be given a smooth structure.

To show that N is Hausdorff, it is enough to show that inclusion $i : N \hookrightarrow M$ is continuous so that N becomes a subspace of M and so it is inherited from Hausdorff property of M. But given any $U \subset M$ open, since N_i are immersed submanifolds, $U \cap N_i$ are open in N_i and since $i^{-1}(U) = \cup_i U \cap N_i$, it is also open proving that i is continuous.

Now we show it is second countable. Cover M with countable open sets $\{V_i\}$ so that $V_i \cap N$ (when non-empty) are embedded submanifolds of some N_k and of M (and in particular are open in N), let $\{W\}_j$ be a countable basis of topology of M and select a subset of this $\{W_i\}_{i\in I}$ so that every V_i is a union of some $\{W_{i_k}\}$ and $W_{i_k} \subset V_i$ for some i. Then for $k \in I$, $\{W_k \cap N\}$ is a countable collection of open submanifolds which are second countable, whose union is N. So N itself is second countable.

It only remains to show that it has smooth structure. To do this we use $\{V_i\}$ defined previously. Since by assumption $V_i \cap N$ are embedded in M, they have a unique smooth structure that makes them smoothly embedded submanifolds with an atlas of the form $\{V_i^k, \phi_i^k = \psi_i^k|_{V_i^k}\}_k$ where ψ_i^k are coordinate maps on M and so that V_i^k are mapped to C^r submanifolds $U_i^k \subset \mathbb{R}^{rank(E)} \subset \mathbb{R}^{rank(M)}$. If $V_i^k \cap V_j^\ell$ is not empty then it is open by assumption. Therefore smoothness of the transition functions for $\{\psi_i^k\}_{i,k}$ easily imply the same for $\{\phi_i^k\}_{i,k}$ when restricted to images of these intersections. □

To finish the proof, given $p \in M$, let L_p be the unique maximal integral manifold which by lemma 2.4 is a connected, immersed manifold. The collection of such maximal integral manifolds cover all M and are either disjoint or coincide by construction. More over since N is an immersed manifold given any $U \subset M$ open, $N \cap U$ is open in N and so can be written as a countable union of open sets by second countability. In particular then the intersection $N \cap U$ has countable number of connected components. □

2.1.1. The differentiable case. In the case of a differentiable subbundle, question of integrability and unique integrability coincides.

LEMMA 2.5. *Let E be an integrable and differentiable subbundle of TM. Then E is uniquely integrable.*

PROOF. This is not so hard consequence of uniqueness of ODE with differentiable vector fields. Indeed assume there exists two integral surfaces W_1, W_2 which intersect. Take any $p \in W_1 \cap W_2$. Take a basis of sections $\{X_k\}_{k=1}^{rank(E)}$ of E defined in some neighbourhood of U, all of which are differentiable vector fields. Then their restrictions to tangent spaces of W_i are also differentiable vectors fields. By uniqueness of ODE, their integral curves around each point, for short enough times, also belongs to these surfaces and must also locally coincide whenever they pass through a common point. Define the map $(t_1, ..., t_{rank(E)}) \to e^{t_{rank(E)}X_{rank(E)}} \circ ... \circ e^{t_1 X_1}(p)$. For t_i small enough the image of this map belongs to both surfaces W_i and contains a relatively open set (since the flow of a differentiable vector fields has differentiable dependence on base points). Therefore $W_1 \cap W_2$ is relatively open. □

This statement can be improved by something called transversal Lipschitzness.

DEFINITION 2.6. Let E be a subbundle so that $dim(M) = rank(E) + 1$ (which will be called codimension 1 subbundle from now on). E is called transversally Lipschitz of for all $p \in M$, if there exists a coordinate neighbourhood, a vector field Y defined on U which is transverse to E and a local basis of sections $\{X_i\}$ of E such that if we write in coordinates

$$X_i = \sum_{k=1}^{rank(E)} a_i^k(x) \frac{\partial}{\partial x_k}$$

then $a_i^k(x)$ are differentiable along Y.

The following weaker proposition can be proven:

PROPOSITION 2.7. *If E is a transversally Lipschitz and integrable subbundle then it is uniquely integrable.*

We will prove a more general theorem; 3.25 in section 3.3. It can be shown that transversal Lipschitzness implies conditions in this theorem. The upshot of proposition 2.7 however is that one can find a basis of E and their restriction to every transverse plane can be written, up to rescaling, as a Lipschitz vector-field inside the plane which becomes uniquely integrable. This lemma can also be found in another form in [**69**] where it is required that E admits a foliation which is transversally Lipschitz (transversal Lipschitzness for a foliation can be defined by requiring that locally the leaves of a foliation form a Lipschitz family of integral manifolds). This lemma can be seen as an Analogue of Van Kampen's uniqueness theorem (and could quite possibly be derived from there directly), see [**41**]. It is possibly true, by use of Van Kampen's theorem, that if E is integrable with the leaves forming a Lipschitz family, then E is uniquely integrable.

Although the main focus of the thesis is to study merely continuous sub-bundles, weaker forms of differentiability such as transversal Lipschitzness or existence of continuous exterior derivatives for defining 1-forms seems to have interesting geometric consequences for sub-bundles such as analogues of sub-Riemannian distance estimates. These will not be covered in this thesis but will be subjects of future works. We invited the interested reader to check chapter 4.

2.1.1.1. *Frobenius Theorem.* In the case a distribution is differentiable there is a necessary and sufficient condition for integrability known as the Frobenius theorem. We will now state and prove the theorem using some ideas which will also be useful for us when we study the continuous case.

THEOREM 2.8. *Let E be subbundle on a smooth manifold M. M is uniquely integrable if and only if for every point $p \in M$, there exists a neighbourhood $U(p)$ and a basis $\{X_i\}_{i=1}^{rank(E)}$ of sections such that*

$$[X_i, X_j]_q \in E_q$$

for all i, j and $q \in U(p)$

PROOF. For necessity, given any p choose a neighbourhood with a local basis of sections $\{X_i\}$. Since E is uniquely integrable through every point q, there passes an integral manifold W of E. Note that since $T_sW = E_s$ for all $s \in W$, X_i restricts to a vector field on W. In particular then for all $s \in W$, $[X_i, X_j]_s \in T_sW = E_s$.

For sufficiency again given any p choose a neighbourhood with a local basis of sections $\{X_i\}$. Let $rank(E) = m$. Define

$$W(t_1, ..., t_m) = e^{t_m X_m} \circ ... \circ e^{t_1 X_1}(p)$$

For t_i small enough this is differentiable surface since $DW_{t=0}$ has columns as $X_i(p)$ which are linearly independent. Our aim will be to show that this surface is tangent to E. Denote $p(t_i) = e^{t_i X_i} \circ ... \circ e^{t_1 X_1}(p)$ and $t = (t_1, ..., t_m)$. We have that

$$\tilde{X}_i(t) = \frac{\partial W}{\partial t_i}(t) = De^{t_m X_m} \circ ...De^{t_{i+1} X_{i+1}} X_i(p(t_i))$$

We must show that $\tilde{X}_i(t) \in E_{p(t_m)}$. For this the following lemma suffices:

LEMMA 2.9. *Let X, Y be everywhere linearly independent, non-vanishing vector fields with $E_p = span < X_p, Y_p >$ defined in some coordinate chart U. Then if $[X, Y]_q \in E_q$, $(De^{tX}Y)_q \in E_q$ for all $q \in U$.*

PROOF. Consider the family of vector fields $Y(t, q) = (De^{tX}Y)_q$. Given q, let p be such that $q = e^{tX}(p)$ The derivative of this family with respect to t is given by (see for instance [**1**]):

$$\frac{\partial}{\partial t} Y(t, q) = De_p^{tX}[X, Y]_p$$

By assumption $[X, Y]_p = a(p)X_p + b(p)Y_p$. Therefore

$$\frac{\partial}{\partial t} Y(t, q) = a(p)De_p^{tX} X_p + b(p)De_p^{tX} Y_p$$

But $De_p^{tX} Y_p = Y(t, q)$ and $De_p^{tX} X_p = X_q$. So we obtain the following differential equation:

$$\frac{\partial}{\partial t} Y(t, q) = a(p)X_q + b(p)Y(t, q)$$

with the initial condition $Y(0, q) = Y_q$. Writing $a(t) = a(e^{-tX}(q))$ and $b(t) = b(e^{-tX}(q))$, the solution to this differential equation is given by:

$$Y(t, q) = Y_q + (\int_0^t ds\ a(s)X_q + b(s)Y_q)\ e^{\int_s^t dr\ \ b(r)}$$

Since $X_q, Y_q \in E_q$ we obtain directly that $Y(t, q) \in E_q$ for all t. □

Then a successive application of this lemma gives that $\tilde{X}_i(t) \in E_{p(t_i)}$ for all t, which finishes the proof using lemma 2.5. □

One can give also a differential ideal criterion to this theorem which will be central to most of the work in this thesis.

DEFINITION 2.10. Given a subbundle E of TM of $rank(E)$, the local defining differential 1-forms of E on the neighbourhood U are a collection of local differential 1-forms $\eta_1, ..., \eta_k$ (where $k = dim(M) - dim(E)$), such that for all sections X of E and all $i = 1, ..., k$, $\eta_i(X) = 0$ everywhere. Such a collection of differential 1-forms is called involutive if for all $i = 1, ..., k$,

$$\eta_1 \wedge ... \wedge \eta_k \wedge d\eta_i = 0$$

identically.

LEMMA 2.11. *E is integrable if and only if every set of local defining differential 1-forms is involutive.*

PROOF. Take any local basis of sections $\{X_i\}$ and any set of local defining differential 1-forms $\{\eta_i\}$. Complete $\{X_i\}$ to a basis by adding vector fields $\{Z_i\}$. Denote $\eta_1 \wedge .... \wedge \eta_k = \eta$.

$$\eta \wedge d\eta_\ell = 0 \quad \text{iff} \quad \eta \wedge d\eta_\ell(Z_1, ..., Z_k, X_i, X_j) = 0$$

for all i, j (note that putting more than two X_ℓ already nullifies the differential 1-form). Then

$$\eta \wedge d\eta_\ell(Z_1, ..., Z_k, X_i, X_j) = 0 \quad \text{iff} \quad \eta(Z_1, ..., Z_k)d\eta_\ell(X_i, X_j) = 0$$

Since $\eta(Z_1, ..., Z_k) \neq 0$ (by the assumption that $\{X_i, Z_j\}_{i,j}$ form a basis), using Cartan's formula

$$d\eta_n(X_i, X_j) = 0 \quad \text{iff} \quad \eta_\ell([X_i, X_j]) = 0 \quad \text{iff} \quad [X_i, X_j] \in E$$

□

On the passing we note that in the case E is Lipschitz, Frobenius theorem can still be formulated in terms of Lebesgue almost every point conditions on the Lie brackets but becomes more technical. Since uniqueness and existence of flows for Lipschitz vector fields always holds true, the proof in theorem 2.8, with some work, can be adapted to the case when E is Lipschitz. In this case the flow curves give a Lipschitz family of curves and so the differential is also defined almost everywhere. Since curves of the form $e^{t_i X_i}(p)$ are measure 0, the differential of another Lipschitz vector field may not be defined every where on this curve. Yet the curves of the form $e^{t_i X_i}(p)$ for all p form a regular foliation of the neighbourhood and almost surely these curves intersect the zero measure set of non-differentiability in relatively zero measure sets. So going like this one can form uniform sized Lipschitz integral manifolds passing through almost everywhere which satisfy uniqueness of solutions. This then can be extended to the whole manifold by taking limits, thanks to continuity of E.

2.1.2. Non-differentiable case. In the case E is not differentiable one can come up with many weaker forms of unique integrability all of which we review for completeness and for the interest of the reader. The terminology we have chosen here might be different from the literature.

Definition 2.12. A distribution E is said to admit a foliation if there exists a foliation $\mathcal{F}$ such that $T\mathcal{F} = E$. It said to admit a unique foliation if $\mathcal{F}$ is the unique foliation satisfying this property.

A distribution which admits a unique foliation is integrable but not necessarily uniquely integrable. Indeed one simply can consider the flow lines of the ODE $\dot{x} = x^{1/3}, \dot{y} = 1$. If one takes out the integral manifold $\gamma(t) = (0, t)$, then the remaining ones form a foliation. Yet $\gamma(t)$ crosses every other integral manifold transversally violating uniqueness. A more advanced example for a two dimensional distribution on $\mathbb{T}^3$ will be given later on in section 2.2.2. It is also possible that a bundle admits more than one foliation, see for instance [**11**] for a vector field that admits more than one foliation. Thus in the continuous world, many things are up to choice of integral leaves. If one collects all the integral leaves, that might not be a foliation, but if then some of them are thrown out one might obtain a foliation.

Obviously in the non-differentiable case it not possible to make directly sense of Frobenius theorem (although one could try to come up with curve version of it such as in [**50**]). It is the main contribution of this thesis to come up with "Frobenius like" conditions for continuous distributions and apply it integrability questions that arise for instance in dynamical systems, along with some other minor applications to ODE and PDE.

2.1.3. The Connection Between Integrability and PDE.

In this subsection we show how to interpret integrability question as a system of homogenous, linear, first order PDE.

Proposition 2.13. *Let E be a C^r subbundle with $rank(E) = d$ on a $m+d$ dimensional manifold M. Then E is integrable (with C^r family of leaves) iff for every $p \in M$, there exists a neighbourhood U, a collection of real valued C^r functions $f_1, f_2, ..., f_m$ and a local basis of sections $\{X_i\}_{i=1}^d$ of E such that*

- *$X_i(f_j) = 0$ for all $i = 1, ..., d$ and $j = 1, ..., m$*
- *$df_1, ..., df_m$ is a set of everywhere linearly independent differential 1-forms*
- *Setting $f = (f_1, ..., f_m)$, $f^{-1}(c)$ are integral manifolds of E.*

Proof. We first prove necessity. The intuition about the existence of a foliation is that for any point p, there exists a coordinate neighbourhood U and a diffeomorphism $\phi : U \to \mathbb{R}^d \oplus \mathbb{R}^m = (x, y)$ such that the images of the leaves of the foliation are flat i.e they are intersection of d dimensional planes with the image of U. This means that the diffeomorphism maps leaves to sets of the form $(x, y = cnst)$. Therefore if $\mathcal{L}$ is such a leaf, then dy_i restricted to tangent space of $\phi^{-1}L$, which spans $\phi_*^{-1}E$ at the given point, is zero. But then $\phi^* dy_i = d(y_i \circ \phi)$ restricted to tangent space of $\mathcal{L}$ is 0 and so $f_i = y_i \circ \phi$ are the required functions.

Linear independence follows from the fact that dy_i are linearly independent and ϕ is a diffeomorphism. To be done, lets prove the existence of ϕ.

One can prove that for any p, there exists some local coordinates $U = (x_1, ..., x_d, y_1, . .., y_m)$ and a basis $\{X_i\}_{i=1}^d$ of E of the following form:

$$X_i = \frac{\partial}{\partial x_i} + \sum_{j=1}^{m} a_i^j(x,y)\frac{\partial}{\partial y_j}$$

for some C^r functions $a_i^j(x,y)$. Consider now for some $V \subset U$, the subset $P = (0, ...0, y_1, ..., y_m) \cap V$. Consider the functions:

$$H(t_1, ..., t_d, y_1, ..., y_m) = e^{t_n X_n} \circ ... \circ e^{t_1 X_1}(0, ..., 0, y_1, ..., y_m)$$

As shown in theorem 2.8, for a fixed y, the image of this map is an integral manifold. Also the maps are defined for all $|t_i| \leq \epsilon$ for some $\epsilon > 0$. Then each $(x, y = cnst)$ is mapped to a leaf of the foliation. If we can invert this map, we obtain the required diffeomorphism ϕ. It is not hard to see that due to form of the vector fields X_i,

$$H(t_1, ..., t_d, y_1, ..., y_m) = (t_1, ..., t_d, y_1 + g_1(t,y), ..., y_m + g_m(t,y))$$

That this is a diffeomorphism for ϵ small enough can be seen in the following way: The differential of H is of the form $DH = I + M(t,y)$ where $M(t,y) \to 0$ as $t \to 0$. Then apply inverse function theorem to get the inverse (regularity of the functions is also an automatic consequence of; a) the inverse function theorem and b) the fact that H is C^r, since E is).

For sufficiency, the assumption that df_i are linearly independent means that $f^{-1}(c)$ are locally embedded d dimensional submanifolds. Moreover tangent space of these manifolds coincide with the intersection of the kernel of df_i. Indeed given any vector Y tangent to one of these submanifolds $\mathcal{M}$, extend it to a vector field Z on the manifold. Since f_i are constant on this manifold, their value does not change along the flow of Z. So for all $p \in \mathcal{M}$, $Z(f_i)(p) = df_i(Y_p) = 0$. Since the dimension of the tangent space and the dimension of the intersection of the kernels coincide the tangent space of these manifolds are the intersection of the kernels of df_i. And since $df_i(X_j) = 0$ for all i, j for basis $\{X_i\}_{i=1}^d$ of E, again by dimension tangent space of the manifolds coincide with E □

In the language of PDE's, $f^{-1}(c)$ is called a solution of the system of PDE's $X_i(f_j) = 0$ with the initial condition c. Uniqueness of intersecting solutions defined on two different neighbourhoods more or less means that the solutions that intersect coincide up to a translation of the initial condition. This can be used to give a different definition of a foliation which generalizes the notion of an immersion defined on a manifold, but we shall not be bothered by that.

2.1.4. Questions of Integrability in Dynamical Systems. A dynamical system is generally a self mapping of some space: $\phi : M \to M$. Differentiable dynamics is the case when M is a smooth manifold and ϕ is C^r for some $r \geq 1$. In this context there are important examples of dynamical systems where the tangent space of M splits into subspaces which are invariant under $D\phi$. That is there exists a decomposition $E_x \oplus F_x = T_xM$ such that $D\phi_x(E_x) = E_{\phi(x)}$ and $D\phi_x(F_x) = F_{\phi(x)}$. This can be seen as a natural generalization of the concept of an eigenspace. It turns out however that generally these sub-bundles are not differentiable (see for instance [**43**]). Our main motivation for trying to integrate continuous distributions will be to apply it to dynamical systems. We will defer more detailed examples of such invariant decompositions to next section.

2.1.5. Differentiable Approximations of Bundles. In studying integrability of continuous distributions, one can either proceed using topological ideas such as in [**18, 19, 38**] or try to use analysis and geometry via differentiable approximations of the continuous distribution. We will employ the second idea. For now we will try to give some insight into what one should expect from such approximations. Given any distribution E, it is always possible to approximate it by various methods. The most common such method is of course through mollifications as done in [**70**]. A method which is also particular to dynamical systems is to "pull back" a fixed C^1 bundle by the dynamics which we shall explore later.

We have seen that unique integrability of E on M is equivalent to unique integrability of E on every local chart. Therefore assume we replace M with a coordinate neighbourhood $U \subset \mathbb{R}^{n+m}$, equipped with the usual Euclidean inner product and metric which we respectively denote as $(\cdot, \cdot)$, $|\cdot|$. We use the same notation for the inner product and norm induced on differential k-forms and tangent vectors. Seeing vector fields X as maps from $\mathbb{R}^{n+m}$ to $\mathbb{R}^{n+m}$, DX simply denotes a matrix whose entries are partial derivatives of components of X. We use also the induced Euclidean norm on these matrices for now. In the following $|.|_\infty$ denotes sup-norm over some compact domain.

To start with, assume that we have a C^1 distribution locally spanned on some neighbourhood U by $\{X_i\} = \sum_k a_k \frac{\partial}{\partial x_k}$ and annihilated by $\{\eta_j\}$. Now if one further gets a C^1 approximation of this bundle (bear with this pointless exercise for a while please), it means that locally we have a basis of sections $\{X_i^k\}$ and annihilators $\{\eta_j^k\}$ such that as $k \to 0$:

$$|X_i^k - X_i|_\infty \to 0 \quad , \quad |DX_i^k - DX_i|_\infty \to 0 \quad , \quad |\eta_j^k - \eta_j|_\infty \to 0$$

for all i, j. This in particular means that

$$\eta_\ell^k([X_i^k, X_j^k](p)) \to \eta_\ell([X_i, X_j](p)) = 0$$

for all ℓ, i, j and $p \in U$.

Now if one wants to approximate a C^0 bundle by C^1 ones, we still get

$$|X_i^k - X_i|_\infty \to 0 \quad , \quad |\eta_j^k - \eta_j|_\infty \to 0$$

but the convergence of differentials DX_i^k are not possible. Indeed even if there exists a constant such that $|DX_i^k|_\infty < C$ for all k, i, this would mean that E is Lipschitz. Yet an observation one should make is the following:

$$[X, Y] = (X^i \frac{\partial Y^j}{\partial x_i} - Y^i \frac{\partial X^j}{\partial x_i}) \frac{\partial}{\partial x_j}$$

So if one is extremely lucky, even though derivatives of X_i^k blow up the differences of them might behave better. In fact later we show that for a wide range of dynamical systems with continuous invariant bundles $E \oplus F$, it is possible to approximate E by C^1 bundles so that

$$\eta_\ell^k([X_i^k, X_j^k]) \to 0 \tag{2.1.5.1}$$

Then a possible question is

Question 2.14. *Is the existence of such a basis as in equation 2.1.5.1 enough to guarantee integrability of the limit bundle?*

This is a naive question and should not be expected to hold true. One can even have C^1 approximations of C^1 functions, whose derivatives have nothing to do with the limit function's derivative. And indeed in section 2.2.3 we demonstrate an example of [**47**] which gives a negative answer to this question. Thus the approximations must be, in some way, "more" related to the limit distribution so that the behaviour of the Lie brackets of the approximations are reflected in the behaviour of the limit bundle in some way. So the correct question is:

Question 2.15. *What are some regularity conditions on $\{X_i^k\}$ that are weaker than requiring equi-Lipschitzness or C^1 convergence but which yet together with the existence of such a basis as in equation 2.1.5.1 is enough to guarantee integrability of the limit bundle?*

The only such condition that we know of is given in [**42**] (called continuous exterior differentiability), which we had briefly mentioned in section 1.2. However in our opinion this condition is quite difficult to check, especially for implicitly defined distributions as in dynamical systems. It also implies certain regularity properties on the foliation one gets by using this theorem which limits the scope of applicability in dynamical systems where pathological regularity behaviours may occur along directions transverse to leaves of a foliation. Thus one of the main motivations of this thesis is to come up with other weak regularity assumptions on the sequence $\{X_i^k\}$ which do not imply that the limit bundle is Lipschitz but still helps to transfer information about $\eta_\ell^k([X_i^k, X_j^k])$ to the limiting object.

2.2. Examples

In this chapter we will give examples of integrable and non-integrable continuous sub-bundles, most of which are left invariant by some diffeomorphism acting on the manifold.

2.2.1. Examples of Integrable Invariant Distributions. We start with some well known examples of invariant bundles associated to diffeomorphisms on manifold which are known to be integrable. We concentrate on mostly the well known examples of Anosov and partially hyperbolic splittings and towards the end we also discuss dominated splittings.

2.2.1.1. *Partially Hyperbolic Linear Diffeomorphisms of the Torus.* In the case of $M = \mathbb{T}^n$ one can build diffeomorphisms with invariant sub-bundles easily using linear automorphisms of the torus. Modelling the torus as the quotient of $\mathbb{R}^n$ by the integer lattice, it is not hard to see that any matrix $A : \mathbb{R}^n \to \mathbb{R}^n$ with integer entries and determinant 1 determines a linear automorphism of the torus. Indeed the fact that it has integer entries means that it preserves the integer lattice and the fact that it has determinant 1 means its inverse has also integer entries and so the inverse also preserves it. And therefore this linear map of $\mathbb{R}^n$ descends to a map of the torus. In this case for the diffeomorphism we write $\phi = A$ and since A is linear its derivative on $\mathbb{R}^n$ and therefore the torus is the same. One can then find the generalized eigenspaces $E_1, E_2, ..., E_k$of A (corresponding to its real Jordan block form such that $\sum_{i=1}^{k}(dim(E_i)) = n$ and so that $A(E_i) = E_i$. The eigenvalues $\lambda_1, \lambda_2, ..., \lambda_k$ associated to each space maybe complex but their norm determines whether A expands E_i, contracts it or is neutral. Some examples of such maps are:

$$A_1 = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

and

$$A_2 = \begin{pmatrix} -3 & 0 & 2 \\ 1 & 2 & -3 \\ 0 & -1 & 1 \end{pmatrix}$$

The first map is very well known by the name Arnold's cat map and has two eigenspaces corresponding to eigenvalues $\frac{3}{2}+\frac{\sqrt{5}}{2}, \frac{3}{2}-\frac{\sqrt{5}}{2}$. Note that one is less than 1 in norm and the other greater. For the second map the eigenvalues are $\lambda_1 \sim -0.11, \lambda_2 \sim 3.11, \lambda_3 \sim -3.21$. Both are examples of what are called uniformly hyperbolic maps:

Definition 2.16. Given a compact Riemannian manifold M and a diffeomorphism $\phi : M \to M$, M is said to be uniformly hyperbolic if there exists a continuous splitting of the tangent space as $T_xM = E^u_x \oplus E^s_x$ for all x and a Riemannian metric so that denoting $\lambda_u(x) = \inf_{v\in E^u} \frac{||D\phi^u_x v||}{||v||}$ and $\mu_s(x) = \inf_{v\in E^s} \frac{||D\phi^s_x v||}{||v||}$ one has

$$\mu_s(x) < 1 < \lambda_u(x).$$

The bundle E^s is called the stable bundle and the bundle E^u is called the unstable bundle.

Although in this example these bundles are necessarily smooth, in general they might even fail to be Lipschitz (see for instance [**43**]). Using Toral automorphisms one can also build an example of a non uniformly hyperbolic diffeomorphism (less trivial than the identity):

$$A_3 = \begin{pmatrix} 2 & 2 & 1 \\ 2 & -3 & 2 \\ -3 & -1 & -2 \end{pmatrix} \tag{2.2.1.1}$$

with eigenvalues $\lambda_1 \sim -3.73, \lambda_2 = 1, \lambda_3 \sim -0.26$. Therefore it has an eigenvalue which is neither strictly less than 1 nor bigger as in definition 2.16. Such class of examples is called partially hyperbolic:

Definition 2.17. Given a compact Riemannian manifold M and a diffeomorphism $\phi : M \to M$, M is said to be pointwise partially hyperbolic if there exists a continuous splitting of the tangent space as $T_xM = E^u_x \oplus E^c_x \oplus E^s_x$ and a Riemannian metric so that denoting $\lambda_\sigma(x) = \inf_{v\in E^\sigma} \frac{||D\phi_x v||}{||v||}$ and $\mu_\sigma(x) = \inf_{v\in E^\sigma} \frac{||D\phi_x v||}{||v||}$ one has

$$\mu_s(x) < 1 < \lambda_u(x) \qquad \mu_s(x) < \lambda_c(x) \leq \mu_c(x) < \lambda_u(x)$$

It is called absolutely partially hyperbolic if

$$\sup_{x\in M}\mu_s(x) < \inf_{x\in M}\lambda_c(x) \qquad \sup_{x\in M}\mu_c(x) < \inf_{x\in M}\lambda_u(x)$$

The bundle E^c is called the central bundle.

We remark that in this case all the invariant bundles are smooth and integrable and moreover even the joint bundles such as E^{cu} and so on are integrable and integrate to trivial planes. Some results in [**50**] even state that small enough C^1 perturbations of these maps also have integrable bundles. However this is far from being the general case. First of all it is only known that these types of bundles are Hölder continuous and only stable and unstable bundles are uniquely integrable which give rise to a foliation (see [**50**]). In the next subsections we will give relatively more interesting examples of partially hyperbolic diffeomorphisms whose bundles may fail to be smooth. Secondly the center bundle (with dimension greater than 1) or the joint bundles may fail to be integrable even when the bundles are smooth, see for instance [**22**] which is a modification of an example in [**72**] attributed to Borel or [**47**]. Finally the distinction between point-wise and absolute partial hyperbolicity actually has topological consequences and is not a mere generalization for the generalization's sake. Indeed all absolute partially hyperbolic splittings on $\mathbb{T}^3$ are integrable ([**20**]) while point-wise are not ([**46**]). It will the topic of the next section to give some examples of partially hyperbolic diffeomorphisms in which the center fails to be integrable.

2.2.1.2. *Derived from Anosov Systems.* A DA system is obtained from modifying an Anosov (or uniformly hyperbolic) diffeomorphism A of the Torus. Some of the original papers on this account are [**60**],[**72**]. See [**29**] for a nice computer illustrations of the effects of such a modification. We will consider the three dimensional example $A = A_3$ given in the previous subsection in equation (2.2.1.1). The modification is performed around the fixed point p_0 of A which corresponds to 0 in the covering space $\mathbb{R}^3$. Note that A_3 is uniformly hyperbolic with $dim(E^u) = 2$, $dim(E^s) = 1$. The modification will be done in such a way so that for the new system, there will be a one dimensional central E^c which contracts at certain places and expands in others. Lets first explain the philosophy of the construction. One performs a modification (which will not be a small perturbation in C^1 topology) at small neighbourhood V of the fixed point p so that for the new system A_δ one has that at this point $dim(E^s(p)) = 2$, $dim(E^u(p)) = 1$ while at other certain places which are not affected by the modification one still will have $dim(E^u) = 2$, $dim(E^s) = 1$. One should note that a local modification of the map at V will also modify the invariant manifolds on the forward and backward images of V that is at $\{A_\delta^n(V)\}_{n=-\infty}^{\infty}$. But in the end one can simply find a periodic point $q \neq p$ of the original diffeomorphism of A and take V small enough so that the orbit of q does not intersect V. This means that none of the sets $A^n_\delta(U)$ intersect q and therefore we see that the structure of the hyperbolic splitting at q remains undisturbed. After having proven that this system is partially hyperbolic, this will necessitate the existence of a central bundle which expands at certain parts and contracts in others. We now pass to a detailed explanation of the technique.

Let $A : \mathbb{R}^3 \to \mathbb{R}^3$ be a matrix with integer entries and determinant 1 and has eigenvalues $\lambda_1 < 1 < \lambda_2 < \lambda_3$. As discussed before it gives rise to an automorphism ϕ of $\mathbb{T}^3$ with three bundles $E^1 \oplus E^2 \oplus E^3$ such that for any $v \in E^i$ one has that $D\phi v = \lambda_i v$. Let $\pi : \mathbb{R}^3 \to \mathbb{T}^3$ be the quotient map and $p = \pi(0)$. This point is fixed under ϕ i.e $\phi(p) = p$. It is known that the set of periodic points are dense for such a hyperbolic toral automorphism. Therefore let q be another point different from p which is periodic with orbit $p, \phi(p), ...\phi^{n+1}(p) = p$. Take a small enough neighbourhood V of q so that $\phi^i(p)$ never intersect V so that one has that both the backward images and the forward images of V are disjoint from p and its orbit. We are going to modify the map here. Take some small disk D_0 around 0 so that it is mapped diffeomorphically by π to its image V_0 around p. One can then find a map $\psi : V_0 \to D_0$ so that $\psi(p) = 0$ and

$$\psi \circ \phi \circ \psi^{-1}(x_1, x_2, x_3) = (\lambda_1 x_1, \lambda_2 x_2, \lambda_3 x_3)$$

This can simply be obtained by composing $\pi|_{D_0}$ with a change of basis that takes the standard basis to the eigenvectors of A. Then one also has that for $z \in V_0$

$$\psi \circ \phi(z_1, z_2, z_3) = (\lambda_1 \psi^1(z), \lambda_2 \psi^2(z), \lambda_3 \psi^3(z))$$

where $\psi^i(x)$ are components of the map ψ. Now define a new map $\tilde{\phi}$ by

$$\begin{aligned} \tilde{\phi}(z) =& \phi(z) \quad \text{if } z \notin V_0 \\ \psi \circ \tilde{\phi}(z) =& (\lambda_1 \psi^1(z), \mu(\psi(z)), \lambda_3 \psi^3(z)) \end{aligned}$$

where $\mu : \mathbb{R}^3 \to \mathbb{R}$ is a smooth map which has the properties $|D_2\mu(p)| < 1$, $\lambda_3 > |D_2\mu(z)| > \lambda_1$ and $\mu(z) = \lambda_2 z_2$ if $z \notin V_1 \subset V_0$. Such a map can be with some work obtained using bump functions or flows. In the case one uses volume preserving flows to get this modification, the new maps will also be volume preserving since ϕ was as well, see [**12**] for details. Now then one has that

$$D(\psi \circ \tilde{\phi}) = \begin{pmatrix} \lambda_1 & 0 & 0 \\ D_1\mu & D_2\mu & D_3\mu \\ 0 & 0 & \lambda_3 \end{pmatrix} \circ D\psi$$

where D_i denotes derivative with respect to x_i. Therefore $det(D\tilde{\phi}) = \lambda_1 D_2\mu\lambda_3 > 0$ hence this is a diffeomorphism since it glues smoothly to ϕ out of V_0. It is clear that the image of E_2 under $D\psi$ which is spanned by $(0,1,0)$ is invariant under this map such that for any $v \in E_2(0)$, $D\phi v = D_2\mu(0)v$ and for $v \in E_2(q)$, $D\phi v = \lambda_2 v$. Thus E_2 is an invariant bundle which contracts near 0 and which expands out of V_0. Now we will construct the remaining invariant bundles. This is done by first requiring $\sup_{x\in\mathbb{R}^3} |D_i\mu(x)|$ to be small for $i = 1, 3$. Under this assumption define the family of cones

$$C^u_\epsilon(x) = \{v \in T_x\mathbb{T}^3 \mid \epsilon|P_3 v| \geq |(P_2 + P_1)v|\}$$

$$C^s_\epsilon(x) = \{v \in T_x\mathbb{T}^3 \mid \epsilon|P_1 v| \geq |(P_3 + P_2)v|\}$$

where $P_i : T\mathbb{T}^3 \to E_i$ are the projections associated to the splitting $E_1 \oplus E_2 \oplus E_3$ and $|\cdot|$ is a norm coming from a metric which orthonormalizes the splitting E_1, E_2, E_3. Then for instance if $v \in C^u_\epsilon(x)$ one has that

$$\begin{aligned}
\frac{|(P_2 + P_1)D\tilde{\phi}v|^2}{|P_3 D\tilde{\phi}v|^2} &= \frac{|D_2\mu P_2 v + (\lambda_1 + D_1\mu)P_1 v|^2}{|\lambda_3 P_3 v|^2} \\
&\leq \frac{(D_2\mu)^2|P_2 v|^2 + (\lambda_1 + D_1\mu)^2|P_1 v|^2}{(\lambda_3 + D_3\mu)^2|P_3 v|^2} \\
&< \frac{|P_2 v|^2 + |P_1 v|^2}{|P_3 v|^2} \\
&= \frac{|P_2 v + P_1 v|^2}{|P_3 v|^2}
\end{aligned}$$

Therefore $D\phi_x(C^u_\epsilon(x)) \subset C^u_\epsilon(\phi(x))$. Similarly one can show that $D\phi^{-1}_x(C^u_\epsilon(x)) \subset C^u_\epsilon(\phi^{-1}(x))$. By iterating this procedure and taking the intersection of inter-nested sequence of cones one obtains possibly new invariant distributions $\tilde{E}_1, \tilde{E}_2$ which by the well known cone criterion (see [**50**].) become the stable and unstable distributions of the new systems, completing the construction. It can be shown that still the center, center-stable and center-unstable bundles are uniquely integrable.

We have infact used this construction in subsubsection 1.1.3.3. In particular we had chosen the modification as in [**12**] so that the new system is also volume preserving and which had the property that $\frac{|D\phi|_{E^c(x)}||D\phi|_{E^s(x)}|}{|D\phi|_{E^u(x)}|} < 1$ and that for some points $q \in \mathbb{T}^2$, $\frac{|D\phi|_{E^c(q)}|^2}{|D\phi|_{E^u(q)}|} > 1$. The first is due to the fact that the system is volume preserving and so $\det(D\phi_x) = 1$ and that $|D\phi|_{E^u(x)}| > 1$. For the second note that the modification does not effect the structure of the splitting or the expansivity properties at the periodic point q out of the neighbourhood V we have chosen. But since the original diffeomorphism is induced by the matrix given in (2.2.1.1) which satisfies $\frac{\lambda_2^2}{\lambda_3} > 1$ so does the splitting at q (with eigenvalues representing center and and unstable expansions).

2.2.1.3. *Time 1 Maps of Anosov Flows.* Let $\phi_t : M \to M$ be a C^2 flow on a compact Riemannian manifold M. Such a flow is called Anosov if there exists a continuous splitting

$$T_xM = E^s(x) \oplus E^c \oplus E^u(x)$$

such that for some constants K and $0 < \lambda < 1$ one has

- E^c is the flow direction
- $||D\phi_t|_{E^s(x)}|| \leq K\lambda^t$ for all $t > 0$
- $||D\phi_{-t}|_{E^s(x)}|| \geq K\lambda^{-t}$ for all $t > 0$
- E^σ for $\sigma \in \{s, c, u\}$ are invariant under $D\phi_t$ for all $t > 0$

Then it is easy to see that the time one map of this flow which is ϕ_1 gives rise to a partially hyperbolic splitting which again $E^s \oplus E^c \oplus E^u$ such that it acts isometrically on the center bundle. In this case the center bundle is trivially uniquely integrable and it is also true that also the bundles E^{cu} and E^{cs} are integrable. There are two main examples for Anosov flows which are described below.

For the first one let $\phi : M \to M$ be an Anosov diffeomorphism. Consider $\mathcal{M} = [0, 1] \times M$ and the equivalence relation $(0, x) \sim (1, \phi(x))$. Let $\pi : M \to \mathcal{M} \backslash \sim$ be the projection to the quotient space obtained by the equivalence relation. Let X be the vector field whose integral curves are the image, under π, of the lines $[0, 1] \times \{p\}$ for all $p \in M$. By abuse of notation we will occasionally denote by X the flow generated on M

Proposition 2.18. *Flow of X is Anosov*

Proof. First of all one needs to build the stable and unstable bundles. We set E^c to be the span of the vector field X as usual with the Anosov flows. Then at $t = 0$, we set $E^\sigma_{t=0} = E^\sigma$ for $\sigma = u$ or s, where E^u, E^s are the bundles associated to the map ϕ. Then we denote $E^\sigma_t = (e^{tX})_*(E^\sigma_{t=0})$ where e^{tX} is the pushforward of the flow of X. Since flow of X, e^{tX} satisfies $e^{1X}(0, p) = (0, \phi(p))$ (as a flow on M) we see that $(e^{1X})_*(E^\sigma_0) = \phi^* E^\sigma_0 = E^\sigma_0$ and invariance is satisfied. Now we need to get the hyperbolicity estimates. First lets look at $t = 0$. Given any t, we will let $\lfloor t \rfloor$ denote its lower integer part and $r \in [0, 1)$ the remainder. Let K be a constant such that

$$\frac{1}{C} \leq \sup_{|r|<1, p \in M} ||De^{rX}_{(0,p)}|| \leq C$$

We also denote $\mu_s = \sup_{p \in M} ||D\phi|_{E^s(p)}||$.

Then for instance

$$||De^{tX}_{(0,p)}||_{E^s(0,p)}| \leq C||De^{\lfloor t \rfloor X}_{(0,p)}|_{E^s(0,p)}|| \leq C\mu_s^{\lfloor t \rfloor} \leq \frac{C}{\mu_s}\mu_s^t$$

since $e^{\lfloor t \rfloor X}_{(0,p)} = (0, \phi^{\lfloor t \rfloor}(p))$. Thus we get the required condition for E^s at the points $(0,p)$ If we are at a point of the form (t,p) since $e^{(1-t)X}(t,p) = (0,q)$ for some q and $De^{(1-t)X}(E^\sigma(t,p)) = E^\sigma(0,q)$ hence again hyperbolicity estimates can be obtained in the same manner as above. The estimates for E^u follow exactly the same pattern.

□

The more famous example is the geodesic flow on a compact Riemannian manifold with constant negative curvature. On such a manifold M let SM be the unit tangent bundle which is compact. Let $\gamma_t(x,v)$ be the unique geodesic flow in TM passing through (x,v). Then it is a fundamental result due to Anosov ([**3**]) that the map $\gamma_t : SV \to SV$ is an Anosov flow. Note that each manifold with constant negative curvature has as its universal covering space the hyperbolic space $\mathbb{H}$. The flow in the unit tangent bundle can be lifted to $\mathbb{H}$ which gives rise to what is called a horocycle flow. In this setup it becomes geometrically more clear, without much calculations, that the flow should be Anosov. See [**65**] for a short explanation of this.

2.2.1.4. *Frame Flows.* An extension of the notion of a geodesic flow is the frame flow. In contrast to the previous cases, this is not an Anosov flow because the center bundle is larger than then only the flow direction. Assume either that M is oriented or move to an oriented cover. Let $n = dim(M)$. We let $\hat{SM}$ denote the space of positively oriented $n-$frames in SM. Defining $\pi : \hat{SM} \to SM$ be the projection to the first vector. This is fiber bundle with structure group isomorphic to $SO(n-1)$ where elements fix the first vector and rotate the remaining. Then the frame flow $\hat{\gamma_t}$ which acts on SM be the flow which acts like γ_t on the first vector and parallel transports the remaining vectors along the geodesic defined by the first vector. Note that parallel transport preserves orthogonality and thus is well defined and moreover it preserves norms, therefore it acts isometrically on the tangent space of fibers. One then has that $\pi \circ \hat{\gamma_t} = \gamma_t \circ \pi$ and so with the remarks above the flow given by this map has partially hyperbolic time-t maps (for $t > 0$)

has as the center direction an $n+1$ dimensional space which is the flow direction plus the tangent space to the fibers

2.2.1.5. *Direct Products, Skew Products and Algebraic Extensions.* Direct products, skew products and algebraic extensions are more of methods to produce partially hyperbolic diffeomorphism out of existing ones. Let $f : M \to M$ denote an Anosov diffeomorphism and $g : N \to N$ be a diffeomorphism of a compact manifold M satisfying

$$||Df|_{E^s}|| \leq m(Dg) \leq ||Dg|| \leq ||Df_{E^u}||$$

Then it is easy to see that the map $f \times g : M \times N \to M \times N$ given by $f \times g(x, y) = (f(x), g(y))$ is partially hyperbolic. The case $g = Id$ is an indispensable tool for creating counter examples, breaking conjectures and forcing people to use notions such as "Generically" and "C^r" dense.

For skew products, the map g is replaced by $g : M \times N \to N$, a family of maps modelled on M. In this case the requirements are the same (where infimum and supremums will now be over $M \times N$ rather than N) and then again the map $f \times_g g(x, y) = (f(x), g(x, y))$ is partially hyperbolic.

Finally let G be a Lie group equipped with a left-invariant metric and $g : M \to G$ be a smooth map. Then define the map $f \times_g g : M \times G \to M \times G$ by $f \times_g g(x, y) = (f(x), g(x)y)$ where $g(x)$ acts on G by left translations. Since G acts on its self by isometries, this is a partially hyperbolic map. We have already seen an example of Algebraic extension which are the frame flows where $\hat{\gamma}_t$ becomes a group extension of the geodesic flow γ_t for which $G = SO(n-1)$.

2.2.2. Non-Integrable Continuous Bundles. In this section we comment on some examples:

(1) Of a continuous invariant decomposition on three torus which is integrable but is not uniquely integrable and does not admit a foliation
(2) Of a continuous invariant decomposition on three torus which admits a foliation but is not uniquely integrable

The first two examples are due to a nice construction by [**47**]. We will study in depth these examples which arise as consequences of the same construction. We will present a slightly modified proof of the example in [**47**] which is more direct and in the Author's opinion more natural in the sense that it follows from basic definitions. We note that there are in literature other interesting examples of non-integrability. One is due to [**11**] which is a Hölder continuous vector field with more than one foliations and the other is due to [**22**] which a partially hyperbolic diffeomorphism on a Nilmanifold which has smooth but non-integrable two dimensional center bundle.

2.2.3. The Hertz-Hertz-Ures Construction. [**47**] has given the first known example of a three dimensional, point-wise partially hyperbolic system where the center bundle is non-integrable. The following is a slight modification of the proof which makes it more direct, transparent and less technical.

The germ of the idea in the original proof is their conjecture that all three dimensional, non-dynamically coherent examples contain an attracting or repelling torus tangent to center-unstable or center-stable.

Thus to get a three dimensional non-integrable system, the aim will be to come up with a three dimensional system which necessarily has such a leaf as described above. We start by a simple attempt. Let $A : \mathbb{T}^2 \to \mathbb{T}^2$ be a Anosov diffeomorphism of the torus with eigenvalues $\frac{1}{\lambda} < 1 < \lambda$ and eigendirections e_s, e_u. Let $\phi : S^1 \to S^1$ be a Morse-Smale diffeomorphism of the circle that is it has north pole as a fixed point and south pole as a fixed point and the dynamics is like the time one map of the flow from the north pole to the south pole. We identify these points respectively with 0 and $\frac{1}{2}$ and the "first half" of the circle with $(0, \frac{1}{2})$ and "second half" with $(\frac{1}{2}, 1)$. We denote the tangent direction to the circle by $\frac{\partial}{\partial\theta}$. Let $\sigma = \frac{d\phi}{d\theta}|_{\frac{1}{2}}$,$\alpha = \frac{d\phi}{d\theta}|_0$. We impose the following condition on this map:

$$\sigma < \lambda < \alpha < \frac{1}{\lambda} \tag{2.2.3.1}$$

Now consider the product map $F_0 : \mathbb{T}^2 \times S^1 \to \mathbb{T}^2 \times S^1$ given by $F(x,t) = (Ax, \phi(\theta))$. This system has two fixed torii, one repeller at 0 and one attractor at $\frac{1}{2}$ which we denote as $\mathbb{T}_0$ and $\mathbb{T}_{\nVdash/\nvDash}$. This system actually looks like what we want. Due to condition (2.2.3.1) one observes that $\mathbb{T}_{1/2}$ locally looks like a center unstable torus (while $\mathbb{T}_0$ looks locally like a E^{us} torus. However it is easy to see that no matter what the derivative of ϕ at 0 is, $e_s \oplus e_u \oplus \frac{\partial}{\partial\theta}$ is not a global partially hyperbolic splitting (it is dominated splitting of the form $E^{cs} \oplus E^u$ where $E^{cs} = \text{span} < e_s, \frac{\partial}{\partial\theta} >$). Therefore we have to modify this map to obtain a partially hyperbolic system whose center and stable bundles are mixes of e_s and $\frac{\partial}{\partial\theta}$ while the unstable bundle remains unchanged e_u. Moreover we must make sure that $\mathbb{T}_{1/2}$ remains as a fixed center unstable torus to be able to make use of the idea in the conjecture mentioned above. Hence we define a modification which perturbs along the stable direction with the perturbation depending on the S^1 parameter θ. The modification $F : \mathbb{T}^2 \times S^1 \to \mathbb{T}^2 \times S^1$ will be of the form

$$F(x,\theta) = (Ax + v(\theta)e_s, \phi(\theta)) \tag{2.2.3.2}$$

for some C^2 map $v : S^1 \to S^1$. We will later on add certain assumptions on $v(\theta)$. Note at this stage however since ϕ is not altered, $\mathbb{T}_0$ and $\mathbb{T}_{1/2}$ remain as fixed repeller and attractor torii.

To start with, we want to have, apart from e_u, two other distinct 1-dimensional distributions left invariant by the differential of this map (and these distributions will be candidates for the stable and center bundles). We will try obtaining this for two distributions which look like $X_i = (f_i(\theta)e_s, g_i(\theta))$ for $i = 1, 2$ which is a reasonable assumption since the perturbation function depends only on θ and is in the direction of e_s. The differential of the map F is

$$DF|_{x,\theta} = \begin{pmatrix} \lambda & 0 & v'(\theta) \\ 0 & \frac{1}{\lambda} & 0 \\ 0 & 0 & \phi'(\theta) \end{pmatrix}$$

The invariance condition is $DF_{x,\theta}(X_1(x,\theta)) = X_1(F(x,\theta))$ which is equivalent to requiring

$$\frac{f_i(\theta)\lambda + g_i(\theta)\mu'(\theta)}{\phi'(\theta)g_i(\theta)} = \frac{f_i(\phi(\theta))}{g_i(\phi(\theta))} \tag{2.2.3.3}$$

Now we make certain assumptions on the form of $X_i(x,\theta)$ which renders this equation solvable. Since $\mathbb{T}_0$ and $\mathbb{T}_{1/2}$ are already fixed and everything except $\mathbb{T}_0$ converge to $\mathbb{T}_{1/2}$ we will mainly be mainly interested in what happens for $\theta \neq 0$ and $\theta \neq \frac{1}{2}$. First of all we shall assume that $g_i(\theta)$ are non-zero out of 0 and $\frac{1}{2}$ which means that only torii tangent to E^{cu} and E^{su} can appear at 0 or $\frac{1}{2}$. Therefore rewriting $X_i(x,\theta) = (h_i(\theta)e_s, 1)$ equation (2.2.3.3) takes the form

$$h_i(\theta)\lambda + \mu'(\theta) = \phi'(\theta)h_i(\phi(\theta)) \tag{2.2.3.4}$$

Moreover since X_i are supposed to be tangent vector fields to invariant one dimensional foliation, it is reasonable that $h_i(\theta)$ must be obtainable as the derivative of a function $g_i : S^1 \to S^1$ that is $h_i(\theta) = \frac{dg_i}{d\theta}$. With this second assumption bu integrating the equation (2.2.3.4) one obtains

$$g_i(\theta)\lambda + \mu(\theta) = g_i(\phi(\theta)) \tag{2.2.3.5}$$

which is a much more tractable equation than the previous ones. Under certain assumptions these equations can be solved for two independent solutions which are continuous and C^1 for $\theta \neq 0$ and $\theta \neq \frac{1}{2}$ which is the content of the next lemma:

PROPOSITION 2.19. *There exists two solutions $g_1(\theta)$ and $g_2(\theta)$ to equation (2.2.3.4) which satisfy the following properties:*

i. $g_1(\theta)$ is defined and continuous everywhere on S^1 while $g_1'(\theta)$ is defined and continuous everywhere except $\frac{1}{2}$ and as $\theta \to \frac{1}{2}$, $g_1'(\theta) \to \infty$ and $g_1'(0) = 0$

ii. $g_2(\theta)$ is defined and continuous everywhere except 0 on S^1 while $g_2'(\theta)$ is defined and continuous everywhere except 0 and as $\theta \to 0$, $g_2'(\theta) \to \infty$ and $g_2'(\frac{1}{2}) = 0$

iii. $g_2'(\theta) - g_1'(\theta) \neq 0$ where ever defined

Before moving on two proving this technical lemma, we will first demonstrate that this is enough to get our result which is the content of the next lemma:

PROPOSITION 2.20. *Assume two two solutions as described in proposition 2.19 exists. Denote*

$$X_1(x,\theta) = span < g_1'(\theta)e_s, \frac{\partial}{\partial\theta} > \quad if \quad \theta \neq \frac{1}{2}$$

$$X_1(x,\theta) = e_s \quad if \quad \theta = \frac{1}{2}$$

$$X_2(x,\theta) = span < g_2'(\theta)e_s, \frac{\partial}{\partial\theta} > \quad if \quad \theta \neq 0$$

$$X_2(x,\theta) = e_s \quad if \quad \theta = 0$$

Then the following decomposition

$$TM = X_1 \oplus X_2 \oplus e_u$$

is a (continuous) partially hyperbolic decomposition for F where $X_1 = E^c$, $X_2 = E^s$, $X_3 = E^u$ and whose center bundle is not integrable at $\theta = \frac{1}{2}$

PROOF. The bundle X_1 is as smooth as g_1' outside $\frac{1}{2}$ and since $g_1'(\theta)$ goes to ∞ as θ goes to $\frac{1}{2}$ it means $X_1(x,\theta)$ also goes to e_s, which establishes continuity. Same argument also works for X_2. The invariance of the bundles is by construction the result of equation (2.2.3.4) (which was the derivative of equation (2.2.3.3)) and the invariance equation $DF|_{(x,\theta)}X_i(x,\theta) = X_i(F(x,\theta))$.

As for partial hyperbolicity note that

$$X_1(x,0) = \frac{\partial}{\partial\theta} \qquad X_1(x,\frac{1}{2}) = e_s$$
$$X_2(x,0) = e_s \qquad X_2(x,\frac{1}{2}) = \frac{\partial}{\partial\theta}$$

The assumption that $\sigma < \lambda < \alpha < \frac{1}{\lambda}$ means that X_1 dominates X_2 and is dominated by e_u at 0 and $\frac{1}{2}$ and hence the partial hyperbolicity requirements are satisfied in the neighbourhoods of 0 and $\frac{1}{2}$. Since given any $\theta \neq 0$ or $\theta \neq \frac{1}{2}$, $\phi^n(\theta) \to \frac{1}{2}$, $X_1(F^n(x,\theta))$, $X_2(F^n(x,\theta))$ are eventually in the neighbourhood (and stay there further on) where the partial hyperbolicity conditions are satisfied, so this decomposition is globally partially hyperbolic.

It remains to show that the center bundle is non-integrable around $\frac{1}{2}$. First of all notice that the curves tangent to e_s on $\mathbb{T}_{1/2}$ are center curves. Moreover one can see (by direct differentiation) that the following curves:

$$\gamma(\theta) = (x,\frac{1}{2}) + ((g_1(\theta + \frac{1}{2}) - g_1(\frac{1}{2}))e_s, \theta)$$

are integral curves of X_1 which satisfy $\gamma(0) = (x, \frac{1}{2})$. Since there is also the e_s curve (which is a central curve) passing through this point which is distinct from the curve $\gamma(\theta)$ one sees that the local center manifolds are not unique. □

Remark 2.21. One might wonder whether why the non-uniqueness argument for E^c at $\frac{1}{2}$ does not work to give a non-uniqueness for E^s at 0. This is simply because $g_2(\theta)$ is not continuous and defined at 0 and therefore one can not write a solution curve of E^s of the form

$$\alpha(\theta) = (x,0) + ((g_2(\theta) - g_2(0))e_s, \theta)$$

Instead the solution curves along the direction E^s wind up and accumulate on the e_s direction of $\mathbb{T}_0$ instead of intersecting it. This will become more evident when the solutions $h_1(\theta)$ and $h_2(\theta)$ are explicitly calculated and $v(\theta)$ is explicitly defined.

Now we move to proving proposition 2.19. We will divide it into several lemmas.

Lemma 2.22. *If the following series converge and are continuous,*

$$g_1(\theta) = \frac{1}{\lambda}\sum_{k=1}^{\infty} \lambda^k v(\phi^{-k}(\theta)) \tag{2.2.3.6}$$

$$g_2(\theta) = \frac{1}{\lambda}\sum_{k=0}^{\infty} \lambda^{-k} v(\phi^k(\theta)) \qquad (2.2.3.7)$$

then they are the solutions to the equation (2.2.3.5)*.*

PROOF. Both solutions are obtained by iteratively solving 2.2.3.5. We will do the second one. Indeed one has directly that

$$g_2(\theta) = \frac{1}{\lambda}(g_2(\phi(\theta)) - v(\theta)$$

$$g_2(\phi(\theta)) = \frac{1}{\lambda}(g_2(\phi^2(\theta)) - v(\phi(\theta))$$

$$\vdots$$

$$g_2(\phi^n(\theta)) = \frac{1}{\lambda}(g_2(\phi^{n+1}(\theta)) - v(\phi^n(\theta))$$

Therefore iteratively replacing the terms on the righthand side containing u_2, one obtains

$$g_2(\theta) = \frac{1}{\lambda}\sum_{i=0}^{\infty} \lambda^k v(\phi^k(\theta))$$

The first solution $g_1(\theta)$ is similarly obtained but by iterating backwards with ϕ. □

Now we show that these solutions indeed converge.

LEMMA 2.23. *Let $v : S^1 \to S^1$ be a C^2 function satisfying the following properties:*

$$v(0) = v(\frac{1}{2}) = 0$$

Then the series given in (2.2.3.6) *and* (2.2.3.7) *converges.*

PROOF. The convergence of first is easy. Indeed since v is continuous on a compact space $|\lambda^k v(\phi^{-k}(\theta)| < K\lambda^k$ and $\lambda < 1$ therefore the series converges absolutely and hence converges.

The second one is the one that requires a more careful analysis. The analysis that will be carried out will be a tool which we will repetitively use in the later sections to understand more properties. For $\theta = 0$ and $\theta = \frac{1}{2}$ convergence is automatic since $\mu(0) = \mu(\frac{1}{2}) = 0$. For $\theta \in (0, \frac{1}{2})$ or $(\frac{1}{2}, 1)$ note that $\phi^k(\theta) \to \frac{1}{2}$ as $k \to \infty$. For k large enough one has that (since $\psi'(\frac{1}{2}) = \sigma$)

$$|\psi^k(\theta) - \frac{1}{2}| < K(\sigma + \delta)^k$$

for δ as small as required. Therefore since $v(\frac{1}{2}) = 0$

$$|v(\psi^k(\theta))| < K|\frac{dv}{d\theta}|_\infty(\sigma + \delta)^k$$

But by (2.2.3.1) and for δ small enough one has $\frac{\sigma+\delta}{\lambda} < 1$ and therefore for k large enough, the terms $|\lambda^{-k}v(\psi^k(\theta))| \leq K(\lambda^{-1}(\sigma + \delta))^k$ are geometric so the series converges. □

The following lemma gives the continuity properties of the solutions:

LEMMA 2.24. *The series given by* $g_1(\theta)$ *is everywhere continuous while the series given by* $g_2(\theta)$ *is continuous outside* 0.

PROOF. Writing the terms of the partial sums in $g_1(\theta)$,

$$g_1^n(\theta) = \frac{1}{\lambda}\sum_{k=1}^{n} \lambda^k v(\phi^{-k}(\theta))$$

one sees that for all n, $g_1(\theta) - g_1^n(\theta)$ is the tail of a geometric series and therefore partial sums converge uniformly.

For the second case we will do a similar analysis as in lemma 2.23. Note that as remarked in that lemma the terms in the series behave as geometric after $\phi^k(\theta)$ is in some neighbourhood of $\frac{1}{2}$. Let this neighbourhood be U and let $k(\theta)$ denote the integer such that $\phi^{k(\theta)}(\theta) \in U$. Note that as $\theta \to 0$, $k(\theta) \to \infty$. Thus it is easy to see that "convergence speed" depends on θ and gets arbitrarily slow as $\theta \to 0$. Indeed for all $k > k(\theta)$

$$|g_1^k(\theta) - g_1(\theta)| < K\frac{\sigma}{\lambda}^k \qquad \text{for} k > k(\theta) \tag{2.2.3.8}$$

However outside any neighbourhood of 0, $\theta(k)$ is uniformly bounded above by some constant K, therefore the convergence speed is uniform since equation (2.2.3.8) holds for all θ and $k > K$ for K independent of θ. Therefore outside 0 the limit is continuous. □

Next the differentiability properties:

Lemma 2.25. *$g_1(\theta)$ is C^1 out of $\frac{1}{2}$ and $g_2(\theta)$ is C^1 out of 0. Moreover as $\theta \to \frac{1}{2}$ $|g_1'(\theta)| \to \infty$ and as $\theta \to 0$ $|g_2'(\theta)| \to \infty$. Moreover imposing that $v'(\theta) < 0 in (0, \frac{1}{2})$ and $v'(\theta) > 0 in (\frac{1}{2}, 0)$, derivatives $|g_1'(\theta)|$ and $|g_2'(\theta)|$ are non-zero for $\theta \neq 0$ and $\theta \neq 1$*

Proof. Differentiating the partial sums for the series $g_1(\theta)$ we obtain

$$\frac{dg_1^n}{d\theta} = \frac{1}{\lambda} \sum_{k=1}^{n} \lambda^k \frac{dv}{d\theta}|_{(\phi^{-k}(\theta))} \frac{d\phi^{-k}}{d\theta}(\theta))$$

Note now that $(\phi^{-k}(\theta)) \to 0$ and in a close enough neighbourhood of 0 we have that $|\frac{d\phi^{-k}}{d\theta}(\theta))| < K\alpha^{-k}$ where α is as in (2.2.3.1). But then since $\frac{\lambda}{\alpha} < 1$, for k large enough the terms behave like geometric and converges. The fact that the limit is continuous out of $\frac{1}{2}$ is exactly as in lemma 2.24. Indeed there is a $k(\theta)$ such that $(\phi^{-k(\theta)}(\theta))$ is in some prescribed neighbourhood of 0 and $k(\theta)$ goes to ∞ as θ goes to $\frac{1}{2}$. Therefore the convergence is uniform only outside any neighbourhood of $\frac{1}{2}$ and therefore the limit is continuous everywhere except $\frac{1}{2}$.

Now we will show that it blows up as $\theta \to \frac{1}{2}$. Let V be a neighbourhood of $\frac{1}{2}$ in which $\frac{d\phi}{d\theta} > \sigma - \delta$ for δ s.t $\frac{\lambda}{\sigma-\delta} > 1$. Let θ^k be a sequence of points in V s.t $\theta^k \to \frac{1}{2}$. Let N_k be the first time s.t $\phi^{-N_k}(\theta^k) \notin V$. Then $N_k \to \infty$. Then write

$$\frac{dg_1^n}{d\theta}| \geq \frac{1}{\lambda}|\sum_{k=1}^{N_k} \lambda^k \frac{dv}{d\theta}|_{(\phi^{-k}(\theta))} \frac{d\phi^{-k}}{d\theta}(\theta))| - K$$

where K is a term which represents the remaining terms of the summation and is uniformly bounded above since $\phi^{-k}(\theta)$ converges 0. Since $\frac{\lambda}{\sigma} > 1$, the first part of the summation blows up as $N_k \to \infty$ that is as $\theta \to \frac{1}{2}$.

The fact that the derivatives are non-zero where ever defined follows from the fact that the sign of the each of the term in the summation is the same. ☐

"Independence" of the conjugacies will also be of importance to us.

Lemma 2.26. *$g_1'(\theta) - g_2'(\theta) \neq 0$ for all $\theta \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$*

Proof. Note that u_1' and u_2' both satisfy the equation (2.2.3.4). Then $\ell(\theta) = u_1'(\theta) - u_2'(\theta)$ satisfies the equation, that is

$$\ell(\theta) = \frac{\phi'(\theta)}{\lambda}\ell(\phi(\theta))$$

$$= \prod_{i=0}^{n-1} \frac{\phi'(\phi^i(\theta))}{\lambda^{i+1}} \ell(\phi^n(\theta))$$

Since ϕ' is never zero, $\ell(\theta)$ is not zero if $\ell(\phi^n(\theta))$ is not zero for some n.

Now note that $g_2'(\theta)$ is continuous at $\theta = \frac{1}{2}$ in particular bounded while $g_1'(\theta) \to \infty$ as $\theta \to \frac{1}{2}$. Therefore there exists a neighbourhood V of $\frac{1}{2}$ s.t on $V - \{\frac{1}{2}\}$ $\ell(\theta) \neq = 0$. In particular it is non-zero on a fundamental domain for ϕ of the form $(t, \phi(t))$. But given any $\theta \neq 0$ and $\theta \neq \frac{1}{2}$ there exists an n s.t $\phi^n(\theta)$ is in this neighbourhood on which $\ell(\phi^n(\theta)) \neq 0$. Therefore by the observation above $\ell(\theta) \neq 0$ everywhere. □

This concludes the proof of proposition 2.19 and the construction of the example. We leave it as an exercise to check that depending on the signs of $v(\theta)$ at the fixed tori, one either obtains a system which does not admit a foliation (but admits a branching one infact) and another one for which if the fixed torus at $\frac{1}{2}$ is removed then one obtains a foliation.

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CHAPTER 3

Integrability of Certain Classes of Invariant Splittings

3.1. C^1 Case

3.1.1. Strategy of the proof. Theorem 1.6 follows from Proposition 3.6 which says that the rates of growth of the singular values are exactly the Lyapunov exponents and from [**37**, Theorem 1.2]. The core of the paper is the proof of Theorem 1.5 in section 3.1.4. We will use a classical result of Frobenius [**1, 54**] which gives necessary and sufficient conditions for integrability of C^1 distributions in terms of Lie brackets of certain vector fields spanning the distribution[1]. More precisely we recall that a distribution is *involutive* at a point x if any two C^1 vector fields $X, Y \in E$ defined in a neighbourhood of x satisfy $[X, Y]_x \in E_x$ where $[X, Y]_x$ denotes the Lie bracket of X and Y at x. Frobenius' Theorem says that a C^1 distribution is locally uniquely integrable at x if and only if it is involutive at x. Moreover it follows from the definition that non-involutivity is an open condition and thus it is sufficient to check involutivity on an a dense subset of M, in particular the subset $\mathcal{A}$, to imply unique integrability of E. We fix once and for all a point

$$x_0 \in \mathcal{A}$$

satisfying condition $(\star)$, and let W, Z be two arbitrary C^1 vector fields in E defined in a neighbourhood of x_0. Letting $\Pi : TM \to F$ denote the projection onto F along E we will prove that

$$|\Pi[W, Z]_{x_0}| = 0, \tag{3.1.1.1}$$

i.e. the component of $[W, Z]_{x_0}$ in the direction of F_{x_0} is zero, and so $[W, Z]_{x_0} \in E_{x_0}$. This implies involutivity, and thus unique integrability, of E. In the next section we will define fairly explicitly two families of local frames for E (recall that a local frame for E at x is a set of linearly independent vector fields which span E_z for all z in some neighbourhood of x) and show that the norm $|\Pi[W, Z]_{x_0}|$ is bounded above by the Lie brackets of the basis vectors of these local frames. We then show that the Lie brackets of these local frames are themselves bounded above by terms of the form $(\star)$ thus implying that they go to zero at least for some subsequence and thus proving (3.1.1.1).

[1]There exist also some partial generalisations of the Frobenius Theorem to distributions with less regularity, see for example, [**69, 67**]

3.1.2. Further questions. Condition $(\star\star)$ is more restrictive than $(\star)$ in the sense that it requires the decay to be exponential, but is less restrictive than $(\star)$ in the sense that there is no overall second order domination since it is sufficient that for each choice of indices i, j, m one of the two conditions in $(\star\star)$ be satisfied, and which one is satisfied may depend on the choice of indices. In this sense the domination behaviour of E and F are interlaced, that is there might exist some invariant subspaces of $F(x)$ that dominate (in a second order sense) some invariant subspaces of $E(x)$ and vice versa. With this perspective the comparison becomes more clear. An overall domination allows one to remove the condition of Lyapunov regularity and weaken exponential convergence to 0 to convergence at any rate. While if one wants to remove over all second order domination then, within the available results and techniques, one needs to impose Lyapunov regularity and exponential convergence to 0. This brings about several plausible questions:

QUESTION 1. Can condition $(\star)$ be replaced by condition $(\star\star)$ (in the sense of removing overall domination) with an assumption which is weaker than Lyapunov regularity?

QUESTION 2. Can the exponential convergence condition in $(\star\star)$ be replaced by a slower or general convergence to 0 under additional assumptions therefore generalizing theorem 1.2 [**37**]

3.1.3. Acknowledgements. We would like to thank Raúl Ures for many useful discussions which have benefited the paper.

3.1.4. Involutivity. In this Section we prove Theorem 1.5.

3.1.4.1. *Orienting the brackets.* The idea which allows us to improve on existing results and work with a condition as weak as $(\star)$, depends in a crucial way on the choice of the sequence of local frames which we use to bound $|\Pi[W, Z]_{x_0}|$. We will rely on a relatively standard general construction which is essentially the core of the proof of Frobenius's Theorem, see [**54**], and which is stated and proved in [**37**] exactly in the form which we need here, we therefore omit the proof.

LEMMA 3.1. *Let $E \subset TM$ be a C^1 d-dimensional distribution defined in a neighbourhood of x_0. Let $e_1, .., e_n$ be any choice of basis for E_{x_0}, and F_{x_0} a subspace complementary to E_{x_0}. Then there exists a C^1 local frame $\{E_i\}_{i=1}^d$ for E around x_0 s.t $E_i(x_0) = e_i$ and $[E_i, E_j]_{x_0} \in F_{x_0}$ for any $1 \leq i, j \leq d$.*

In what follows we evaluate all objects and quantities at x_0 so the reader should keep this in mind when we omit this index, when there is no risk of confusion. For each $k \geq 1$ we let $v_1^{(k)}, .., v_d^{(k)}$ be an orthonormal choice of eigenvectors at E_{x_0} which span the eigenspaces of $((D\phi^k)|_E)^\dagger (D\phi^k)|_E$ and which satisfy $|(D\phi^k)|_E v_i^k| = s_i^k$, where $s_1^k \leq s_2^k \leq ..$ are the singular values of the map $D\phi^k|_E$. We then let $\{Y_i^{(k)}\}$ be the C^1 local frame given by Lemma 3.1, i.e. such that

$$Y_i^k(x_0) = v_i^k \quad \text{ and } \quad [Y_i^k, Y_j^k]_{x_0} \in F_{x_0}.$$

Now for each $k \geq 1$, let $(i(k), j(k))$ denote a (not necessarily unique) pair of indices that maximise the norm of the bracket, i.e.

$$|[Y_{i(k)}^k, Y_{j(k)}^k]_{x_0}| \geq |[Y_\ell^k, Y_m^k]_{x_0}|$$

for all $1 \leq \ell, m \leq d$. The proof for the first condition in $(\star)$ will use these vector fields. The proof of the second condition is exactly the same by considering $D\phi^{-k}$ and using vector fields $\{X_i^{(k)}\}$ which satisfy the similar conditions as above, that is $X_i^k(x_0) = w_i^k$ and $[X_i^k, X_j^k]_{x_0} \in F_{x_0}$ where w_i^k are a choice of orthonormal eigenvectors associated to the singular values $s_1^{-k} \geq s_2^{-k} \geq ..$

3.1.4.2. *A priori bounds.* The following Lemma gives some upper bounds on $|\Pi[Z, W]_{x_0}|$ in terms of the Lie brackets $|[Y_{i(k)}^k, Y_{j(k)}^k]_{x_0}|$. These bounds are *a priori* in the sense that they do not depend on the specific form of the local frame $\{Y_i^k\}$, but simply on the fact that they are orthonormal. The statement we give here is thus a special case of a somewhat more general setting.

Lemma 3.2. *For every $k \geq 1$ we have*

$$|\Pi[Z, W]_{x_0}| \leq d(d-1)|[Y_{i(k)}^k, Y_{j(k)}^k]_{x_0}|$$

Proof. Write

$$Z = \sum_{\ell=1}^{d} \alpha_\ell^{(k)} Y_\ell^k \quad \text{ and } \quad W = \sum_{m=1}^{d} \alpha_m^{(k)} Y_m^k$$

for some functions $\alpha_l^{(k)}, \alpha_l^{(k)}$ which, by orthogonality of the frames at x_0, satisfy $|\alpha_m^{(k)}(x_0)|, |\alpha_m^{(k)}(x_0)| \leq 1$. We have by bilinearity of $[\cdot, \cdot]$:

$$[Z, W] = \sum_{l,m=1}^{d} \alpha_\ell^{(k)} \alpha_m^{(k)} [Y_\ell^k, Y_m^k] + \alpha_\ell^{(k)} Y_\ell^k(\alpha_m^{(k)}) Y_m^k - \alpha_m^{(k)} Y_m^k(\alpha_\ell^{(k)}) Y_\ell^k$$

Applying the projection Π to both sides and using the fact that $\Pi(Y_\ell^k) = \Pi(Y_m^k) = 0$, $[Y_\ell^k, Y_m^k]_{x_0} \in F(x_0)$, and taking norms, we have

$$|\Pi[Z, W]_{x_0}| \leq \sum_{\ell,m=1}^{d} |\alpha_\ell^{(k)} \alpha_m^{(k)}(x_0)| |[Y_\ell^k, Y_m^k]_{x_0}|$$

This clearly implies the statement. □

3.1.4.3. *Dynamical bounds.* By Lemma 3.2 it is sufficient to find a subsequence $k_m \to \infty$ such that

$$|[Y^{k_m}_{i(k_m)}, Y^{k_m}_{j(k_m)}]_{x_0}| \to 0$$

as $m \to \infty$, as this would imply (3.1.1.1) and thus our result. This is the key step in the argument. Notice first that by $(\star)$, there exists a subsequence $k_m \to \infty$ such that

$$\frac{s^{k_m}_d(x_0)s^{k_m}_{d-1}(x_0)}{m(D\phi^{k_m}|_{F_{x_0}})} \to 0 \tag{3.1.4.1}$$

as $m \to \infty$. Our result then follows immediately from the following estimate which, together with Lemma 3.2 and (3.1.4.1) implies that $|\Pi[Z,W]_{x_0}| = 0$. This is of course where we use in a crucial way the specific choice of local frames.

LEMMA 3.3. *There is a constant $K > 0$ s.t up to passing to a subsequence of k_m one has*

$$|[Y^{k_m}_{i(k_m)}, Y^{k_m}_{j(k_m)}]_{x_0}| \leq K\frac{s^{k_m}_d(x_0)s^{k_m}_{d-1}(x_0)}{m(D\phi^{k_m}|_{F_{x_0}})}$$

PROOF. We divide the proof into two parts. First we explain how to choose the required subsequence $k_m \to \infty$. Then we show that for such a subsequence we have the upper bound given in the statement.

Let $k_m \to 0$ be the subsequence such that (3.1.4.1) holds and consider the sequence of images $\phi^{k_m}(x_0)$ of the point x_0. By compactness of M this sequence has a converging subsequence and so, up to taking a further subsequence (which we still denote by k_m) if necessary, we can assume that there exists a point $y \in M$ such that $\lim_{m\to\infty} \phi^{k_m}(x) = y$. Fix m_0 large enough s.t for all $m > m_0$, $\phi^{k_m}(x_0)$ lies in a coordinate chart around y and let $A = \{\phi^{k_m}(x_0)\}_{m>m_0} \cup y$. Notice that A is a compact set. Now for each k_m, let $(i(k_m), j(k_m))$ denote the "maximizing" pairs of indices defined above, and let

$$\hat{Y}_{k_m} := \frac{D\phi^{k_m}_{x_0}[Y^{k_m}_{i(k_m)}, Y^{k_m}_{j(k_m)}]_{x_0}}{|D\phi^{k_m}[Y^{k_m}_{i(k_m)}, Y^{k_m}_{j(k_m)}]_{x_0}|}.$$

Then the sequence of vectors $\hat{Y}_{k_m}$ lies in the compact space $A \times S$, where S is the unit ball in $\mathbb{R}^m$ and therefore there exists a subsequence of k_m (which we still denote as k_m) and a vector $\hat{Y} \in T_yM$ such that

$$\hat{Y}_{k_m} \to \hat{Y}$$

as $m \to \infty$. Now recall that by our choice of local frames we have

$$[Y^{k_m}_{i(k_m)}, Y^{k_m}_{j(k_m)}]_{x_0} \in F_{x_0}$$

and since F is a $D\phi$ invariant and closed subset of TM this implies also

$$\hat{Y}_{k_m} \in F_{\phi^{k_m}(x_0)} \quad \text{and} \quad \hat{Y} \in F_y.$$

Now let $E^\perp$ be the complementary subbundle of TM orthonormal to E and $\vartheta \subset T^*M$ be the subbundle defined by $g(E^\perp, \cdot)$ which is the subbundle of T^*M that defines E by the orthogonality relation. For any η, a section of ϑ, one can write $\eta(\cdot) = g(V, \cdot)$ where V is a section of $E^\perp$. Therefore since the bundle F is uniformly bounded away from E there exists a section η of ϑ around y s.t $\eta(\hat{Y}_{k_m}), \eta(\hat{Y}) > c$ for all m large enough and for some constant $c > 0$. In the following we assume that everything is evaluated at x_0, so we generally omit x_0 unless needed for clarity.

Lemma 3.4. *For every m large enough we have*

$$\eta(D\phi^{k_m}[Y^{k_m}_{i(k_m)}, Y^{k_m}_{j(k_m)}]) \leq |d\eta| s^{k_m}_{n-1} s^{k_m}_n.$$

Proof. By the naturality of the Lie bracket we have

$$\eta(D\phi^{k_m}[Y^{k_m}_{i(k_m)}, Y^{k_m}_{j(k_m)}]) = \eta([D\phi^{k_m} Y^{k_m}_{i(k_m)}, D\phi^{k_m} Y^{k_m}_{j(k_m)}]) \tag{3.1.4.2}$$

We recall a formula in differential geometry (see [**54**, Page 475]), for any two vector fields Z, W and a 1-form η we have

$$\eta([Z, W]) = Z(\eta(W)) - Z(\eta(W)) + d\eta(Z, W)$$

Applying this formula to the right hand side of (3.1.4.2) and using the fact that η is a bilinear form, and that $\eta(Y^{k_m}_{i(k_m)}) = \eta(Y^{k_m}_{j(k_m)}) = 0$ by construction, and the choice of the vectors $v^{k_m}_{i(k_m)}, v^{k_m}_{j(k_m)}$ (see Section 3.1.4.1) we get

$$\begin{aligned} |\eta([D\phi^{k_m} Y^{k_m}_{i(k_m)}, D\phi^{k_m} Y^{k_m}_{j(k_m)}]_{x_0})| &= |d\eta(D\phi^{k_m} Y^{k_m}_{i(k_m)}, D\phi^{k_m} Y^{k_m}_{j(k_m)})_{x_0}| \\ &\leq |d\eta||D\phi^{k_m}_{x_0} v^{k_m}_{i(k_m)}||| D\phi^{k_m}_{x_0} v^{k_m}_{j(k_m)}| \\ &\leq |d\eta| s^{k_m}_{n-1}(x_0) s^{k_m}_n(x_0) \end{aligned}$$

Substituting into (3.1.4.2) we get the result. ☐

Lemma 3.5. *For every m large enough we have*

$$\eta(D\phi^{k_m}_{x_0}[Y^{k_m}_{i(k_m)}, Y^{k_m}_{j(k_m)}]_{x_0}) \geq |\eta(\hat{Y}_{k_m})| m(D\phi^{k_m}_{x_0}|_F) |[Y^{k_m}_{i(k_m)}, Y^{k_m}_{j(k_m)}]_{x_0}|$$

Proof. Notice first that by the definition of $\hat{Y}_{k_m}$ we have

$$|\eta(D\phi^{k_m}[Y^{k_m}_{i(k_m)}, Y^{k_m}_{j(k_m)}])| = |D\phi^{k_m}[Y^{k_m}_{i(k_m)}, Y^{k_m}_{j(k_m)}]||\eta(\hat{Y}_{k_m})|$$

Then, using the fact that $[Y^{k_m}_{i(k_m)}, Y^{k_m}_{j(k_m)}]_{x_0} \in F(x_0)$ we get

$$|D\phi^{k_m}[Y^{k_m}_{i(k_m)}, Y^{k_m}_{j(k_m)}]| \geq m(D\phi^{k_m}|_F)|[Y^{k_m}_{i(k_m)}, Y^{k_m}_{j(k_m)}]|$$

Substituting this bound into the previous inequality we get the result. ☐

Returning to the proof of Lemma 3.3, combining Lemmas 3.4 and 3.5 we get

$$|d\eta| s_{n-1}^{k_m} s_n^{k_m} \geq |\eta(\hat{Y}_{k_m})| m(D\phi^{k_m}|_F) |[Y_{i(k_m)}^{k_m}, Y_{j(k_m)}^{k_m}]|.$$

Using the fact that $|d\eta|$ is bounded and $|\eta(\hat{Y}_{k_m})| > c > 0$ for all m large enough there exists a constant $K > 0$ such that

$$|[Y_{i(k_m)}^{k_m}, Y_{j(k_m)}^{k_m}]| < K \frac{s_n^{k_m} s_{n-1}^{k_m}}{m(D\phi^{k_m}|_F)}.$$

This completes the proof. □

3.1.5. Lyapunov Regular Case. In this Section we prove Theorem 1.6 by showing that the assumptions of the Theorem are equivalent to the assumptions of [**37**, Theorem 1.2], which has the same conclusions. To state this equivalence recall that by definition of regular point, every $x \in \mathcal{A}$ admits an Oseledets splitting $E = E_1 \oplus E_2 \oplus ... \oplus E_n$ for some $n \leq m$, with associated Lyapunov exponents $\lambda_1 < \lambda_2 < ... < \lambda_n$. Let $d_0 = 0$, $d_n = m$ and, for each $i = 1, .., n-1$, let $d_i = \sum_{j=1}^{i} dim(E_j)$.

PROPOSITION 3.6. *For every $i = 1, ..., n$ and every $d_{i-1} < \ell \leq d_i$ we have*

$$\lim_{k \to \pm\infty} \frac{1}{k} \ln s_\ell^k = \lambda_\ell$$

Proposition 3.6 says that the rates of growth of the singular values are exactly the Lyapunov exponents. This implies that conditions $(\star\star)$ is equivalent to

$$\lim_{k \to \infty} \frac{1}{k} \ln \frac{s_i^k(x) s_j^k(x)}{r_m^k(x)} \neq 0. \tag{3.1.5.1}$$

Letting

$$\mu_i(x) = \lim_{k \to \infty} \frac{1}{k} \ln s_i^k(x) \quad \mu_j(x) = \lim_{k \to \infty} \frac{1}{k} \ln s_j^k(x) \quad \lambda_m(x) = \lim_{k \to \infty} \frac{1}{k} \ln r_m^k(x)$$

equation (3.1.5.1) holds true if and only if $\mu_i(x) + \mu_j(x) \neq \lambda_m(x)$ for all indices i, j, m which is exactly the condition given in [**37**, Theorem 1.2].

The proof of Proposition 3.6 is based on the following two statements. The first is the so-called Courant-Fischer Min-Max Theorem.

THEOREM 3.7 ([**30**]). *Let $\phi : X \to Y$ be a linear operator between n dimensional spaces. Let $s_1 \leq s_2 \leq ... \leq s_n$ be its singular values. Then*

$$s_\ell = \sup_{dim(V) = n-\ell+1} m(\phi|_V) = \inf_{dim(W)=\ell} |\phi|_V| \tag{3.1.5.2}$$

where sup/inf above are taken over all subspaces V, W of the given dimensions.

We refer the reader to [**30**] for the proof. The second statement we need is

LEMMA 3.8. *Let $E_{\ell m}(x) = E_\ell(x) \oplus ... \oplus E_m(x)$ where $\ell < m$ and the associated Lyapunov exponents are ordered as $\lambda_\ell(x) < \lambda_{\ell+1}(x) < ... < \lambda_m(x)$. Then*

$$\lim_{k\to\infty} \frac{1}{k} log(||D\phi_x^k|_{E_{\ell m}}||) = \lambda_m(x) \quad \text{and} \quad \lim_{k\to\infty} \frac{1}{k} log(m(D\phi_x^k|_{E_{\ell m}})) = \lambda_\ell(x)$$

Lemma 3.8 is part of the Multiplicative Ergodic Theorem of Oseledets, and is stated in the notes [**8**] and is also stated and proved, though with a somewhat different notation than that used here, in [**5**, Theorem 3.3.10].

PROOF OF PROPOSITION 3.6. For every $1 \le i \le d$ and any subspaces $V', W' \subset E_i$ with $dim(W') = \ell - d_{i-1}$ and $dim(V') = d_i - \ell + 1$ we write $W = E_1 \oplus E_2 \oplus ... \oplus E_{i-1} \oplus W'$ and $V = V' \oplus E_{i+1} \oplus ... \oplus E_n$ Then, by (3.1.5.2) we have $|D\phi^k|_W| \ge s_\ell^k \ge m(D\phi^k|_V)$. Denote $E_{1\ell} = E_1 \oplus ... \oplus E_\ell$ and $E_{\ell n} = E_\ell \oplus ... \oplus E_n$. Since $E_{1\ell} \supset W$ and $E_{\ell n} \supset V$ one has that $||D\phi^k|_{E_{1\ell}}|| \ge ||D\phi^k|_W||$ and $m(D\phi^k|_V) \ge m(D\phi^k|_{E_{\ell n}})$. Therefore $||D\phi^k|_{E_{1\ell}}|| \ge s_\ell^k \ge m(D\phi^k|_{E_{\ell n}})$ and the result follows by Lemma 3.8. □

The rest of this section is dedicated to the proof of Lemma 3.8.

PROOF. (3.8) Denote $E' = E_\ell(x) \oplus E_{\ell+1}(x) ... \oplus E_m(x)$ which is invariant, that is $D\phi^k(E'(x)) = E'(\phi^k(x))$. Let $|\cdot|_x$ denote the restriction of the ambient norm on T_xM to $E'(x)$.

LEMMA 3.9. $\lim_{n\to\infty} \frac{1}{n} log(||D\phi^n|_{E_i}||) = \lim_{n\to\infty} \frac{1}{n} log(m(D\phi^n|_{E_i})) = \lambda_i$

PROOF. Let $S_i^n(x)$ be the unit sphere in $E_i(x)$. Define the sequence of functions $f_n : S_i^n \to \mathbb{R}$ given by

$$f_n(v) = \frac{1}{n} log(|D\phi_x^n v|)$$

Then $f_n \to \lambda_i(x)$ point-wise but since S_i^n is compact it converges uniformly. That is for all ϵ there exists an n_0 s.t for all $n > n_0$ and for all $v \in S_i^n(x)$

$$|\frac{1}{n} log(|D\phi_x^n v|) - \lambda_i(x)| \le \epsilon \tag{3.1.5.3}$$

Now one has that $||D\phi^n|_{E_i}|| = |D\phi^n v_n|$, $m(D\phi^n|_{E_i}) = |D\phi^n w_n|$ for a sequence of vectors v_n, w_n in $S_i^n(x)$. By uniform convergence for all n large enough

$$|\frac{1}{n} log(||D\phi^n|_{E_i}||) - \lambda_i(x)| = |\frac{1}{n} log(|D\phi^n v_n|) - \lambda_i(x)| \le \epsilon$$

$$|\frac{1}{n}log(m(D\phi^n|_{E_i})) - \lambda_i(x)| = |\frac{1}{n}log(|D\phi^n w_n|) - \lambda_i(x)| \leq \epsilon$$

which gives the result.

□

COROLLARY 3.10. *For n large enough and for all i*

$$m(D\phi^n|_{E_\ell}) \leq ||D\phi^n|_{E_\ell}|| < m(D\phi^n|_{E_{\ell+1}}) \leq ||D\phi^n|_{E_{\ell+1}}|| <$$
$$... < m(D\phi^n|_{E_m}) \leq ||D\phi^n|_{E_m}||$$

PROOF. Fix an $\epsilon > 0$ s.t $\lambda_\ell + \epsilon < \lambda_{\ell+1} - \epsilon < ... < \lambda_m - \epsilon$. Now by equation 3.1.5.3 there exists n_0 s.t for all $n > n_0$ for all i and for all $v \in S_i^n$

$$e^{n(\lambda_i - \epsilon)} < |D\phi_x^n v| < e^{n(\lambda_i + \epsilon)}$$

$$\Rightarrow e^{n(\lambda_i - \epsilon)} < m(D\phi_x^n|_{E_i}) \leq ||D\phi_x^n|_{E_i}|| < e^{n(\lambda_i + \epsilon)}$$

and the result follows.

□

LEMMA 3.11. *Let $\Pi_i^n : E'(x^n) = E_\ell(x^n) \oplus ... \oplus E_m(x^n) \to E_i(x^n)$ be the projection to $E_i(x^n)$ with respect to this direct sum. For any $v \in E'(x^n)$ let $v_i^n = \Pi_i^n(v)$ (we denote $v_i^0 = v_i$). Let $|\cdot|_n$ be the norm defined on $E'(x^n)$ by*

$$|v|_n = \sqrt{\sum_{j=\ell}^{m} |v_j^n|^2}$$

which "orthonormalizes" $E_\ell(x^n) \oplus ... \oplus E_m(x^n)$. Then for all n large enough $||D\phi_x^n|_{E'}||_n = ||D\phi_x^n|_{E_m}||$ and $m_n(D\phi_x^n|_{E'}) = m(D\phi_x^n|_{E_\ell})$

PROOF. Let n be large enough such that the condition in corollary 3.10 is satisfied. Now for $v \in E'(x^n)$, by invariance of the splitting, one has $(D\phi^n v)_i^n = D\phi^n v_i$ therefore

$$|D\phi^n v|_n^2 = \sum_{l=\ell}^{m} |D\phi^n v_i|^2$$

But by corollary 3.10 $|D\phi^n v_\ell| < ... < |D\phi^n v_m|$ for all v and large enough n. Therefore the maximum is achieved when $v \in E_m(x^n)$ and minimum is achieved when $v \in E_\ell(x^n)$. Taking the supremum and infimum of $|D\phi^n v|^2$ for $v \in E_m(x^n)$ and for $v \in E_\ell(x^n)$ one obtains

$$||D\phi^n_x|_{E'}||^2_n = ||D\phi^n_x|_{E_m}||^2$$

$$m_n(D\phi^n_x|_{E'})^2 = m(D\phi^n_x|_{E_\ell})^2$$

□

Now we will conclude the proof of the lemma, that is, $\lim_{n\to\infty} \frac{1}{n}||D\phi^n|_{E'}|| = \lambda_m(x)$ and $\lim_{n\to\infty} \frac{1}{n} m(D\phi^n|_{E'}) = \lambda_\ell(x)$.

Note that there exists constants $c(n), d(n), e(n), f(n) > 0$ s.t

$$c(n)||\cdot||_n < ||\cdot|| < d(n)||\cdot||_n$$

$$e(n)m_n(\cdot) < m(\cdot) < f(n)m_n(\cdot)$$

so that

$$c(n)||D\phi^n_{E'}||_n < ||D\phi^n_{E'}|| < d(n)||D\phi^n_{E'}||_n$$

$$e(n)m_n(D\phi^n_{E'}) < m(D\phi^n_{E'}) < f(n)m_n(D\phi^n_{E'})$$

But by lemma 3.11 one has

$$||D\phi^n_{E'}||_n = ||D\phi^n|_{E_m}||$$

and

$$m_n(D\phi^n_{E'}) = m(D\phi^n|_{E_\ell})$$

Now since the angles between the splitting can go to zero at most sub-exponentially one has that the constants $c(n), d(n), e(n), f(n) > 0$ are sub-exponential with respect to n therefore by lemma 3.9

$$\lim_{n\to\infty} \frac{1}{n} log(c(n)||D\phi^n|_{E_m}||) = \lim_{n\to\infty} \frac{1}{n} log(d(n)||D\phi^n|_{E_m}||) = \lambda_m(x)$$

$$\lim_{n\to\infty}\frac{1}{n}log(e(n)m(D\phi^n|_{E_\ell})) = \lim_{n\to\infty}\frac{1}{n}log(f(n)m(D\phi^n|_{E_\ell})) = \lambda_\ell(x)$$

The result for s_l^{-k} similarly follows considering $D\phi_x^{-k}$ and Lyapunov exponents $-\lambda_n \leq ... \leq -\lambda_1$.

□

3.2. Lipschitz Case

3.2.1. Strategy and overview of the proof. We will first show that Theorem 1.7 is a special case of Theorem 1.10. We will consider the volume preserving setting and the transitive setting separately. We then discuss the proof of Theorem 1.10.

3.2.1.1. *Volume preserving implies volume domination.* We show that when φ is volume preserving, dynamical domination implies volume domination. Indeed, notice that $|det(D\varphi_x|_{F_x})| = \|D\varphi_x|_{F_x}\|$ since F is one-dimensional, so the difference between dynamical domination and volume domination consists of the difference between $\|D\varphi_x|_{E_x}\|$ and $|det(D\varphi_x|_{E_x})|$. These two quantities are in general essentially independent of each other; indeed considering the singular value decomposition of $D\varphi_x|_{E_x}$ and letting $s_1 \leq s_2$ denote the two singular values (since we assume E is 2-dimensional), we have that $\|D\varphi_x|_{E_x}\| = s_2$ and $|det(D\varphi_x|_{E_x})| = s_1s_2$. If $\|D\varphi_x|_{E_x}\| = s_2 < 1$ then we have a straightforward inequality $|det(D\varphi_x|_{E_x})| = s_1s_2 < s_2 = \|D\varphi_x|_{E_x}\|$ but this is of course not necessarily the case in general. However there is a relation in the volume preserving setting as this implies $|detD\varphi_x|_E|\cdot|detD\varphi_x|_F| = 1$ and so (1.1.1.1) implies $|detD\varphi_x|_F| > 1$ (arguing by contradiction, $|detD\varphi_x|_F| = \|D\varphi_x|_F\| \leq 1$ would imply $|detD\varphi_x|_E| \geq 1$ by the volume preservation, and this would imply $\|D\varphi_x|_E\|/\|D\varphi_x|_F\| \geq 1$ which would contradict (1.1.1.1)). Dividing the equation $|detD\varphi_x|_E| \cdot |detD\varphi_x|_F| = 1$ through by $(|detD\varphi_x|_F)^2|$ we get (1.1.1.2).

3.2.1.2. *Transitivity implies volume domination.* We show that when φ is transitive (or, as mentioned in Remark 1.9 when φ is chain-recurrent or just φ has no sources), dynamical domination implies volume domination. We are grateful to the referee for pointing out this fact and explaining the proof. The argument is based on the following Lemma whose proof we sketch below and which follows closely arguments in [**9, 31**].

LEMMA 3.12. *Let Λ be a compact invariant set with a continuous splitting $E \oplus F$ with $\dim E = 2$ and $\dim F = 1$. Then the splitting is volume dominated if and only if the Lyapunov exponents $\lambda_1 \leq \lambda_2 \leq \lambda_3$ of any invariant ergodic measure μ satisfy $\lambda_1 + \lambda_2 \leq \lambda_3 - a$ for some uniform value of a (depending on the constants of domination).*

Sketch of proof. One direction is trivial: if the splitting is volume dominated clearly the condition on the Lyapunov exponents satisfies the stated bounds. The other direction is non-trivial and we argue by contradiction. Suppose that Λ is not volume dominated, this means that there exists a sequence of points x_n and a sequence $k_n \to \infty$ such that for every $0 \le j \le k_n$ one has

$$|det(D\varphi^j|_E(x_n))| \ge \frac{1}{2}\|D\varphi^j|_F(x_n)\|. \tag{3.2.1.1}$$

Now consider the sequence of probability measures

$$\mu_n = \frac{1}{k_n}\sum_{i=0}^{k_n-1} \delta_{\varphi^i(x_n)}.$$

Up to passing to a subsequence if necessary, we can assume that μ_n is convergent to an invariant probability measure μ. Since the splitting is continuous we have

$$\int \log(det(D\varphi|_E)d\mu_n \to \int \log(det(D\varphi|_E))d\mu$$

and,

$$\int log(\|D\varphi|_F\|)d\mu_n \to \int log(\|D\varphi|_F\|)d\mu.$$

Since the determinant is multiplicative and F is one dimensional, the integrals with respect to μ_n are Birkhoff sums and therefore converge exactly to the sum of Lyapunov exponents (in the E case) and the Lyapunov exponent (in the F case). Using the ergodic decomposition and (3.2.1.1) it follows that there is an ergodic invariant measure with $\lambda_1 + \lambda_2 \ge \lambda_3$. This contradicts the assumption and does proves that Λ is volume dominated. □

To complete the proof of volume domination, notice that dynamical domination implies $\lambda_2 < \lambda_3 - a$ for all invariant ergodic probability measures for some a independent of μ. Assume by contradiction that φ is transitive and dynamically dominated but not volume dominated. Then, by Lemma 3.12, it admits a measure μ such that $\lambda_1+\lambda_2 \ge \lambda_3 - a$. But dynamical domination implies $\lambda_2 < \lambda_3 - a$ and so we get $\lambda_1 > 0$. Thus all Lyapunov exponents of μ are strictly positive and therefore μ is supported on a source, contradicting transitivity (or chain-recurrence, or that f does not have sources).

3.2.1.3. *Volume domination implies integrability.* From now on we concentrate on Theorem 1.10 and reduce it to a key technical Proposition. Therefore the assumptions are that $E \oplus F$ is Lipschitz and volume dominated. We don't require or use dynamical domination for any of the propositions that we prove here. We fix an arbitrary point $x_0 \in M$ and a local chart $(\mathcal{U}, x^1, x^2, x^3)$ centered at x_0. We can assume (up to change of coordinates) that $\partial/\partial x^i, i = 1, 2, 3$ are transverse to

E and thus we can define linearly independent vector fields X and Y, which span E and are of the form

$$X = \frac{\partial}{\partial x^1} + a\frac{\partial}{\partial x^3} \qquad Y = \frac{\partial}{\partial x^2} + b\frac{\partial}{\partial x^3}.$$

where a and b are Lipschitz functions. Notice that it follows from the form of the vector fields X, Y that at every point of differentiability the Lie bracket is well defined and lies in the x^3 direction, i.e.

$$[X, Y] = c\frac{\partial}{\partial x^3}$$

for some L^∞ function c. In Section 3.2.1.4 we will prove the following

PROPOSITION 3.13. *There exists $C > 0$ such that for every $k > 1$ and $x \in \mathcal{U}$, if the distribution E is differentiable at x then we have*

$$\|[X, Y]_x\| \leq C\frac{|det(D\varphi_x^k|_{E_x})|}{|det(D\varphi_x^k|_{F_x})|}.$$

Substituting the volume domination condition (1.1.1.2) into the estimate in Proposition 3.13, we get that the right hand side converges to 0 as $k \to \infty$, and therefore $\|[X, Y]_x\| = 0$ and so the distribution E is involutive at every point x at which it is differentiable. Theorem 1.10 is then an immediate consequence of the following general result of Simić [**69**] which holds in arbitrary dimension and a generalization of a well-known classical result of Frobenius proving unique integrability for involutive C^1 distributions.

THEOREM 3.14 ([**69**]). *Let E be an m dimensional Lipschitz distribution on a smooth manifold M. If for every point $x_0 \in M$, there exists a local neighbourhood $\mathcal{U}$ and a local Lipschitz frame $\{X_i\}_{i=1}^m$ of E in $\mathcal{U}$ such that for almost every point $x \in \mathcal{U}$, $[X_i, X_j]_x \in E_x$, then E is uniquely integrable.*

REMARK 3.15. We mention that there are some versions of Proposition 3.13 in the literature for C^1 distributions and giving an estimate of the $\|[X, Y]_x\| \leq |D\varphi_x^k|_{E_x}|^2/m(D\varphi_x|_{F_x})$, , see e.g. [**45, 64**]. For this quantity to go to zero, one needs the center bunching assumption (1.1.3.1). In our proposition, through more careful analysis, we relax the condition of center bunching (1.1.3.1) to volume domination (1.1.1.2).

3.2.1.4. *Lie bracket bounds.* This section is devoted to the proof of Proposition 3.13, which is now the only missing component in the proof of Theorems 1.7 and 1.10. As a first step in the proof, we reduce the problem to that of estimating the norm of a certain projection of the bracket of an orthonormal frame. More specifically, let π denote the orthogonal projection (with respect to the Lyapunov metric which orthogonalizes the bundles E and F) onto F.

LEMMA 3.16. *There exists a constant* $C_1 > 0$ *such that if* $\{Z, W\}$ *is an orthonormal Lipschitz frame for* E *and differentiable at* $x \in \mathcal{U}$ *then we have*

$$\|[X,Y]_x\| \leq C_1 \|\pi[Z,W]_x\|.$$

PROOF. Notice that since F and $\frac{\partial}{\partial x_3}$ are transverse to E, then one has that $K_1 \leq ||\pi \frac{\partial}{\partial x^3}|| \leq K_2$ for some constants $K_1, K_2 > 0$. Moreover since $\|\pi[X,Y]\| = |c|.\|\pi\partial/\partial x^3\|$ and $\|[X,Y]\| = |c|$ then it is sufficient to get an upper bound for $\|\pi[X,Y]\|$. Writing X, Y in the local orthonormal frame $\{Z, W\}$ we have

$$X = \alpha_1 Z + \alpha_2 W \quad \text{ and } \quad Y = \beta_1 Z + \beta_2 W.$$

By bilinearity of the Lie bracket and the fact that $\pi(Z) = \pi(W) = 0$ since π is a projection along E, straightforward calculation gives

$$\|\pi[X,Y]\| = |\alpha_1\beta_2 - \alpha_2\beta_1|.\|\pi[Z,W]\|$$

By orthonormality of $\{Z, W\}$, we have $|\alpha_i| \leq \|X\|, |\beta_i| \leq \|Y\|$ and since these are uniformly bounded, the same is true for $|\alpha_1\beta_2 - \alpha_2\beta_1|$ and so we get the result. □

By Lemma 3.16 it is sufficient to obtain an upper bound for the quantity $\|\pi[Z,W]\|$ for some Lipschitz orthonormal frame. In particular we can (and do) choose Lipschitz orthonormal frames $\{Z, W\}$ of E such that for every $x \in \mathcal{U}$ and every $k \geq 1$ we have

$$\|D\varphi_x^k Z\| \|D\varphi_x^k W\| = |det(D\varphi^k|_E)|.$$

For these frames will we prove the following.

LEMMA 3.17. *There exists* $C_2 > 0$ *such that for every* $k \geq 1$ *and* $x \in \mathcal{U}$*, if the distribution* E *is differentiable at* x *we have*

$$\|\pi[Z,W]_x\| \leq C_2 \frac{|det(D\varphi^k|_{E_x})|}{||D\varphi^k|_{F_x}||}.$$

Combining Lemma 3.17 and Lemma 3.16 and letting $C = C_1C_2$ we get:

$$\|[X,Y]_x\| \leq C_1\|\pi[Z,W]_x\| \leq C_1C_2 \frac{|det(D\varphi_x^k|_{E_x})|}{\|D\varphi_x^k|_{F_x}\|} = C \frac{|det(D\varphi_x^k|_{E_x})|}{|det(D\varphi_x^k|_{F_x})|}$$

which is the desired bound in Proposition 3.13 and therefore completes its proof.

To prove Lemma 3.17, observe first that for every $y \in M$ there exist 2 orthonormal Lipschitz vector fields A_y, B_y that span E in a neighborhood of y and by compactness we can suppose that we have finitely many pairs, say $(A_1, B_1), ..., (A_\ell, B_\ell)$ of such vector fields which together cover the whole manifold. We denote by $\mathcal{U}_i$ the domain where the vector fields A_i, B_i are defined and let

$$C_2 := \sup\{|\pi[A_i, B_i](x)| : 1 \leq i \leq l \text{ and almost every } x \in \mathcal{U}_i\}.$$

Note this constant C_2 is finite. In fact, by the standard fact that Lipschitz functions have weak differential which is essentially bounded (or L^∞), then for every $i \in \{1, ..., l\}$ the function $|[A_i, B_i]|$ is bounded. To complete the proof we will use the following observation.

LEMMA 3.18. *For any Lipschitz orthonormal local frame $\{Z, W\}$ for E which is differentiable at $x \in M$, we have*

$$|\pi[Z, W]| \leq C_2$$

PROOF. Write $Z = \alpha_1 A_i + \alpha_2 B_i$ and $W = \beta_1 A_i + \beta_2 B_i$ for some $1 \leq i \leq \ell$. Using the bilinearity of the Lie bracket and the fact that $\pi(A_i) = \pi(B_i) = 0$ we get $|\pi[Z, W]| = |\alpha_1\beta_2 - \alpha_2\beta_1||\pi[A_i, B_i]|$. Since $\{A_i, B_i\}$ and $\{Z, W\}$ are both orthonormal frames, we have $|\alpha_1\beta_2 - \alpha_2\beta_1| = 1$, and so we get result. □

PROOF OF LEMMA 3.17. For $k > k_0$ and $x \in \mathcal{U}$ such that E is differentiable at x, Let

$$\tilde{Z}(\varphi^k x) = \frac{D\varphi_x^k Z}{\|D\varphi_x^k Z\|} \quad \text{and} \quad \tilde{W}(\varphi^k x) = \frac{D\varphi_x^k W}{\|D\varphi_x^k W\|}$$

Recall that $D\varphi_x^k(E_x) = E_{\varphi^k(x)}$. Therefore, since Z, W span E in a neighborhood of x, then $\tilde{Z}, \tilde{W}$ span E in a neighbourhood of $\varphi^k(x)$ and in particular $\pi(\tilde{Z}) = \pi(\tilde{W}) = 0$. Therefore we get

$$\|\pi[D\varphi^k Z, D\varphi^k W]\| = |det(D\varphi^k|_{E^{(k)}})|\|\pi[\tilde{Z}, \tilde{W}]\| \tag{3.2.1.2}$$

Note that $\|\pi[D\varphi^k Z, D\varphi^k W]\| = \|\pi D\varphi^k[Z, W]\|$. Then by the invariance of the bundles we have

$$\|\pi D\varphi^k[Z, W]\| = \|D\varphi^k \pi[Z, W]\|. \tag{3.2.1.3}$$

Since F is one dimensional,

$$\|D\varphi^k \pi[Z, W]\| = \|D\varphi^k|_F\|\|\pi[Z, W]\| \tag{3.2.1.4}$$

Combining (3.2.1.3) and (3.2.1.4) we get

$$\|D\varphi^k|_F\|\|\pi[Z, W]\| = \|\pi D\varphi^k[Z, W]\|.$$

Putting this into equation (3.2.1.2) and using the fact that $\|\pi[\tilde{Z}, \tilde{W}]\|$ is uniformly bounded by lemma 3.18 one gets

$$\|\pi[Z, W]\| \leq C_2 \frac{|det(D\varphi^k|_E)|}{\|D\varphi^k|_F\|}$$

This concludes the proof of Lemma 3.17. □

3.3. Sequentially Transversally Lipschitz Case

3.3.1. General philosophy and strategy of proof. Since our distribution is no longer Lipschitz we are not able to apply any existing general involutivity/integrability result, such as that of Simić quoted above[2]. Instead we will have to essentially construct the required integral manifolds more or less explicitly "by hand".

The standard approach for this kind of construction is the so-called *graph transform* method, see [**50**], which takes full advantage of certain hyperbolicity conditions and consists of "pulling back" a sequence of manifolds and showing that the sequence of pull-backs converges to a geometric object which can be shown to be a unique integral manifold of the distribution. This method goes back to Hadamard and has been used in many different settings but, generally, cannot be applied in the partially hyperbolic or dominated decomposition setting where the dynamics is allowed to have a wide range of dynamical behaviour and it is therefore impossible to apply any graph transform arguments to E^{sc} under our assumptions. This is perhaps one of the main reasons why this setting has proved so difficult to deal with.

The strategy we use here can be seen as a combination of the Frobenius/Simić involutivity approach and the Hadamard graph transform method. Rather than approximating the desired integral manifold by a sequence of manifolds we approximate the continuous distribution E by a sequence $\{E^{(k)}\}$ of C^1 distributions obtained dynamically by "pulling back" a suitably chosen initial distribution. Since these approximate distributions are C^1, the Lie brackets of C^1 vector fields in $E^{(k)}$ can be defined. *If the $E^{(k)}$ were involutive*, then each one would admit an integral manifold $\mathcal{E}^{(k)}$ and it is fairly easy to see that these converge to an integral manifold of the original distribution E. However this is generally not the case and we need a more sophisticated argument to show that the distributions $E^{(k)}$ are "asymptotically involutive" in a particular sense which will be defined formally below. For each k we will construct an "approximate" local center-stable manifold $\mathcal{W}^{(k)}$ which is not an integral manifold of $E^{(k)}$ (because the $E^{(k)}$ are not necessarily involutive) but is "close" to being integral manifolds. Further estimates, using also the asymptotic involutivity of the distributions $E^{(k)}$, then allow us to show that these manifolds converge to an integral manifold of the distribution E. We will then use a separate argument to obtain uniqueness, taking advantage of a result of Hartman.

[2]Some notion of Lie bracket can be formulated in lower regularity, see for example [**19**, Proposition 3.1], but it is not clear to us how to obtain a full unique integrability result using these ideas.

3.3.2. Almost involutive approximations. In this section we state and prove a generalization of Proposition 3.13 which formalizes the meaning of "almost" involutive. We consider the sequence of C^1 distributions $\{E^{(k)}\}_{k>1}$ as in the definition of sequential transverse regularity in (1.1.4.1). We fix a coordinate system $(x^1, x^2, x^3, \mathcal{U})$ so that $\partial/\partial x^i$ are all transverse to E and therefore to $E^{(k)}$ for k large enough since $E^{(k)} \to E$ uniformly in angle. Then thanks to this transversality assumption we can find vector fields defined on $\mathcal{U}$ of the form

$$X^{(k)} = \frac{\partial}{\partial x^1} + a^{(k)}\frac{\partial}{\partial x^3} \quad \text{and} \quad Y^{(k)} = \frac{\partial}{\partial x^2} + b^{(k)}\frac{\partial}{\partial x^3}. \tag{3.3.2.1}$$

that span $E^{(k)}$ and converge to vector fields of the form

$$X = \frac{\partial}{\partial x^1} + a\frac{\partial}{\partial x^3} \quad \text{and} \quad Y = \frac{\partial}{\partial x^2} + b\frac{\partial}{\partial x^3}. \tag{3.3.2.2}$$

that span E for $a^{(k)}$, a, $b^{(k)}$, b everywhere non-vanishing functions. Moreover we can choose $\partial/\partial x^3$ to be the direction where sequential transversal Lipschitzness holds true (since it is already transversal to E) so we have the property that there exists $C > 0$

$$\left|\frac{\partial a^{(k)}}{\partial x^3}\right| < C \quad \text{and} \quad \left|\frac{\partial b^{(k)}}{\partial x^3}\right| < C \tag{3.3.2.3}$$

for all k. It is easy to check that $[X^{(k)}, Y^{(k)}]$ lies in the $\partial/\partial x^3$ direction. As before we will have some estimates about how fast the Lie brackets of these vector fields decay to 0. For the following let $F^{(k)}$ be the continuous bundle which is orthogonal to $E^{(k)}$ with respect to Lyapunov metric on E so that $F^{(k)}$ goes to F in angle (since F is orthogonal to E with respect to the Lyapunov metric). We have the following analogue of Proposition 3.13

PROPOSITION 3.19. *There exists $C > 0$ such that for every $k > 1$ and $x \in \mathcal{U}$, we have*

$$\|[X^{(k)}, Y^{(k)}](x)\| \leq C \frac{|det(D\varphi_x^k|_{E_x^{(k)}})|}{\|D\varphi_x^k|_{F_x^{(k)}}\|}.$$

SKETCH OF PROOF. The proof of Proposition 3.19 is very similar to that of Proposition 3.13 and it is not hard to get the result with the difference that E and F in the right hand side of the estimate in Proposition 3.13 are replaced by $E^{(k)}$ and $F^{(k)}$. In this case we choose our collection of C^1 orthonormal collection of frames $\{Z^{(k)}, W^{(k)}\}$ of $E^{(k)}$ so that $||D\varphi^k Z^{(k)}||||D\varphi^k W^{(k)}|| = det(D\varphi^k|_{E^{(k)}})$ Then exactly as in lemma 3.18, to get an upper bound on $|[X^{(k)}, Y^{(k)}]|$, it is enough to bound $[Z^{(k)}, W^{(k)}]$. The proof of the inequality $\|[Z^{(k)}, W^{(k)}]\| \leq det(D\varphi^k|_{E^{(k)}})/\|D\varphi^k|_{F^{(k)}}\|$ follows quite closely the proof of lemma 3.17 where the projection π is replaced by $\pi^{(k)}$ which the projection to $F^{(k)}$ along $E^{(k)}$ at relevant places. □

The next, fairly intuitive but in fact quite technical, step is to replace the estimates on the approximations with estimates on the limit bundle.

PROPOSITION 3.20. *There exists $C > 0$ such that for every $k > 1$ and $x \in \mathcal{U}$, we have*

$$|det(D\varphi_x^k|_{E_x^{(k)}})| \leq C|det(D\varphi_x^k|_{E_x})| \tag{3.3.2.4}$$

and

$$\|D\varphi_x^k|_{F_x^{(k)}}\| \geq C\|D\varphi_x^k|_{F_x}\| \tag{3.3.2.5}$$

SKETCH OF PROOF. (3.3.2.5) is fairly easy since for any vector $v \notin E$, $|D\varphi^k v| \geq CD\varphi_x^k|_{F_x}|v|$ (since F has dimension 1). The real technical estimate is (3.3.2.4). One first needs to make the observation that there exists a cone $C(\alpha)$ of angle α around E such that $D\varphi^k E^{(k)} \subset C(\alpha)$. This is the main observation that allows to relate two determinants independent of k. Indeed given a basis v_1^k, u_1^k of $E^{(k)}$, then $K|det(D\varphi_x^k|_{E_x^{(k)}})| = |D\varphi_x^k v_1^k \wedge D\varphi_x^k u_1^k| = |\Lambda(D\varphi_x^k)v_1^k \wedge u_1^k|$ where $\Lambda(D\varphi_x^k)$ is the induced action of $D\varphi_x^k$ on $TM \wedge TM$. But then $\Lambda(D\varphi_x^k)$ allows a dominated splitting of $TM \wedge TM$ whose invariant spaces are $E_1 = E \wedge E$, $E_2 = E \wedge F$ where E_1 is dominated by E_2. We have that for all k, $E_1^{(k)} = \text{span}(v_1^k \wedge u_1^k)$ is a space which is inside a cone $C(\alpha)$ around E_1. Therefore usual dominated splitting estimates give that $|\Lambda(D\varphi_x^k)|_{E_1^{(k)}(x)}| \leq K|\Lambda(D\varphi_x^k)|_{E_1(x)}|$ which proves the claim about determinants since $|\Lambda(D\varphi_x^k)|_{E_1(x)}| = |det(D\varphi_x^k|_{E_x})|$. □

3.3.3. Almost Integral Manifolds. We will use the local frames $\{X^{(k)}, Y^{(k)}\}$ to define a family of local manifolds which we will then show converge to the required integral manifold of E. We emphasize that these are *not* in general integral manifolds of the approximating distribution $E^{(k)}$. We will use the relatively standard notation $e^{tX^{(k)}}$ to denote the flow at time $t \in \mathbb{R}$ of the vector field $X^{(k)}$. Then we let

$$\mathcal{W}_{x_0}^{(k)}(t,s) := e^{tX^{(k)}} \circ e^{sY^{(k)}}(x_0).$$

This map is well defined for all sufficiently small s, t so that the composition of the corresponding flows remains in the local chart $\mathcal{U}$ in which the vector fields $X^{(k)}, Y^{(k)}$ are defined. Since the vector fields $X^{(k)}, Y^{(k)}$ are uniformly bounded in norm, choosing ϵ sufficiently small the functions $\mathcal{W}_{x_0}^{(k)}$ can be defined in a fixed domain $U_\epsilon = (-\epsilon, \epsilon) \times (-\epsilon, \epsilon)$ independent of k such that $\mathcal{W}_{x_0}^{(k)}(U_\epsilon) \subset \mathcal{U}$. By a direct application of the chain rule and the definition of $\mathcal{W}_{x_0}^{(k)}$, for every $(t, s) \in U_\epsilon$ we have

$$\tilde{X}^{(k)}(t,s) = \frac{\partial \mathcal{W}_{x_0}^{(k)}}{\partial t}(t,s) = X^{(k)}(\mathcal{W}_{x_0}^{(k)}(t,s))$$

and

$$\tilde{Y}^{(k)}(t,s) = \frac{\partial \mathcal{W}_{x_0}^{(k)}}{\partial s}(t,s) = (e^{tX^{(k)}})_* Y^{(k)}(\mathcal{W}_{x_0}^{(k)}(t,s)).$$

where for two vector fields V, Z and $t \in \mathbb{R}$, $(e^{tV})_* Z$ denotes that pushforward of Z by the flow of V defined by

$$[(e^{tV})_* Z]_p = De^{tV}_{e^{-tV}(p)} Z_{e^{-tV}(p)}.$$

The following lemma gives a condition for this family of maps to have a convergent subsequence whose limits becomes a surface tangent to E:

LEMMA 3.21. *If $\tilde{X}^{(k)} \to X$ and $\tilde{Y}^{(k)} \to Y$ then the images of $\mathcal{W}_{x_0}^{(k)}$ are embedded submanifolds and this sequence of submanifolds has a convergent subsequence whose limit is an integral manifold of E.*

PROOF. Since X and Y are linearly independent by assumption of convergence, the differential of the map $\mathcal{W}_{x_0}^{(k)}$ is invertible at every point $(t,s) \in U_\epsilon$, i.e. the partial derivatives $\partial\mathcal{W}_{x_0}^{(k)}/\partial s$ and $\partial\mathcal{W}_{x_0}^{(k)}/\partial t$ are linearly independent for every $(t,s) \in U_\epsilon$. Thus the maps $\mathcal{W}_{x_0}^{(k)}$ are embeddings and define submanifolds through x_0 (which are not in general integral manifolds of $E^{(k)}$). Moreover, since $X^{(k)}, Y^{(k)}$ have uniformly bounded norms, it follows by Proposition 3.22 that $D\mathcal{W}_{x_0}^{(k)}$ has bounded norm uniformly in k and therefore the family $\{\mathcal{W}_{x_0}^{(k)}\}$ is a compact family in the C^1 topology. By the Arzela-Ascoli Theorem this family has a subsequence converging to some limit

$$\mathcal{W}_{x_0} : U_\epsilon \to \mathcal{U}. \tag{3.3.3.1}$$

We claim that $\mathcal{W}_{x_0}(U_\epsilon)$ is an integral manifold of E. Indeed, as $k \to \infty$, $X^{(k)} \to X$, $Y^{(k)} \to Y$ and $\{X, Y\}$ is a local frame of continuous vector fields for E, in particular X, Y are linearly independent and span the distribution E. Moreover, by Proposition 3.22, the partial derivatives $\partial\mathcal{W}_{x_0}^{(k)}/\partial t$ and $\partial\mathcal{W}_{x_0}^{(k)}/\partial s$ are converging uniformly to X and Y and therefore

$$\frac{\partial \mathcal{W}_{x_0}}{\partial t} = X \quad \text{and} \quad \frac{\partial \mathcal{W}_{x_0}}{\partial s} = Y.$$

This shows that $\mathcal{W}_{x_0}(U_\epsilon)$ is a C^1 submanifold and its tangent space coincides with E and thus $\mathcal{W}_{x_0}(U_\epsilon)$ is an integral manifold of E, thus proving integrability of E under these assumptions. □

It therefore just remains to verify the assumptions of Lemma 3.21, i.e. to show that the vectors $X^{(k)}$ and $(e^{tX^{(k)}})_* Y^{(k)}$ converge to X and Y. The first convergence is obviously true. Thus it remains to show the latter which we show in the next result, thus completing the proof of the existence of integral manifolds.

PROPOSITION 3.22. *For all $t \in (-\epsilon, \epsilon)$ we have*

$$\lim_{k\to\infty} \|(e^{tX^{(k)}})_* Y^{(k)} - Y^{(k)}\| = 0.$$

PROOF. To obtain this proposition one uses the following standard property of the pushforward (see proof of Proposition 2.6 in [**1**] for instance):

$$\frac{d}{dt}[(e^{tX^{(k)}})_* Y^{(k)} - Y^{(k)}] = (e^{tX^{(k)}})_*[X^{(k)}, Y^{(k)}] \tag{3.3.3.2}$$

together with the following proposition:

PROPOSITION 3.23. *There exists $C > 0$ such that for every $k \geq 1, x \in \mathcal{U}$ and $|t| \leq \epsilon$, we have*

$$||(e^{tX^{(k)}})_* \frac{\partial}{\partial x^3}|_x\| = \exp \int_0^t \frac{\partial a^{(k)}}{\partial x^3} \circ e^{-\tau X^{(k)}}(x) d\tau$$

This latter proposition follows by integrating the equality

$$\frac{d}{dt}((e^{tX^{(k)}})_* \frac{\partial}{\partial x^3}|_x) = \left(e^{tX^{(k)}}\right)_* [X^{(k)}, \frac{\partial}{\partial x^3}]|_x$$

Once this is established since we know that $\partial a^{(k)}/\partial x^3$ is uniformly bounded (3.3.2.3), we obtain that the effect of $(e^{tX^{(k)}})_*$ on $\partial/\partial x^3$ is bounded. Since $[X^{(k)}, Y^{(k)}]$ is a vector in this direction whose norm goes to 0 we directly obtain by equation (3.3.3.2) that $\frac{d}{dt}[(e^{tX^{(k)}})_* Y^{(k)} - Y^{(k)}]$ goes to 0 uniformly and hence by mean value theorem that $|(e^{tX^{(k)}})_* Y^{(k)} - Y^{(k)}|$ goes to 0 which is the proposition. □

REMARK 3.24. From the proof, it is seen that the most crucial ingredient is for the approximations to satisfy the pushforward bound in Proposition 3.22. One can generalize this observation to get geometric theorems about integrability of continuous sub-bundles, not just those arising in dynamical systems, with some additional assumptions such as the Lie brackets going to 0. This idea, which originated in this paper, is employed in forthcoming works [**59, 76**].

3.3.4. Uniqueness. To get uniqueness of the integral manifolds we will take advantage of a general result of Hartman which we state in a simplified version which is sufficient for our purposes.

THEOREM 3.25 ([**42**], Chapter 5, Theorem 8.1). *Let $X = \sum_{i=1}^n X^i(t,p)\frac{\partial}{\partial x^i}$ be a continuous vector field defined on $I \times U$ where $U \subset \mathbb{R}^n$ and $I \subset \mathbb{R}$. Let $\eta_i = X^i(t,p)dt - dx^i$. If there exists a sequence of C^1 differential forms η_i^k such that $|\eta_i^k - \eta_i|_\infty \to 0$ and $d\eta_i^k$ are uniformly bounded then X is uniquely integrable on $I \times U$. Moreover on compact subset of $U \times I$ the integral curves are uniformly Lipschitz continuous with respect to the initial conditions.*

We recall that a two form being uniformly bounded is equivalent to each of its component is being uniformly bounded.

Corollary 3.26. *Vector fields X and Y defined in* (3.3.2.2) *are uniquely integrable.*

Proof. We will give the proof for X, that for Y is exactly the same. Since X has the form

$$X = \frac{\partial}{\partial x^1} + a\frac{\partial}{\partial x^3},$$

its solutions always lie in the $\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^3}$ plane. Therefore given a point (x_0^1, x_0^2, x_0^3), it is sufficient to consider the restriction to such a plane. Then the C^1 differential 1-forms defined in Theorem 3.25 are

$$\eta_1 = dt - dx^1 \qquad \eta_2 = a(x)dt - dx^3$$

where $x = (x^1, x_0^2, x^3)$, and for the approximations we can write

$$\eta_1^k = dt - dx^1 \qquad \eta_2^k = a^{(k)}(x)dt - dx^3$$

where $a^{(k)}(x)$ are functions given in equation (3.3.2.1), again for some fixed x_0^2. But then by sequential transversal Lipschitz assumption and choice of coordinates we have that $|\frac{\partial a^{(k)}}{\partial x^3}| < C$ for all k and

$$d\eta_1^k = 0 \qquad d\eta_2^k = \frac{\partial a^{(k)}}{\partial x^3} dx^3 \wedge dt$$

(since we restrict to $x^2 = \text{const}$ planes) and the requirements of Theorem 3.25 are satisfied which proves that X is uniquely integrable. $\square$

Now we have that X and Y are uniquely integrable at every point. Assume there exist a point $p \in \mathcal{U}$ such that through p there exist two integral surfaces $\mathcal{W}_1, \mathcal{W}_2$. This means both surfaces are integral manifolds of E and in particular the restriction of X and Y to their tangent space are uniquely integrable vector fields. Therefore there exists ϵ_1 such that the integral curve $e^{tX}(p)$ for $|t| \leq \epsilon_1$ belongs to both surfaces. Now consider an integral curve of Y starting at the points of $e^{tX}(p)$, that is $e^{sY} \circ e^{tX}(p)$. For ϵ_1 small enough, there exists ϵ_2 small enough such that for every $|t| < \epsilon_1$ and $|s| < \epsilon_2$ this set is inside both surfaces since Y is also uniquely integrable (ϵ_i only depend on norms $|X|$, $|Y|$ and size of $\mathcal{W}_i$ and therefore can be chosen uniformly independent of point). This set is a C^1 disk and therefore $\mathcal{W}_1$ and $\mathcal{W}_2$ coincide on an open domain. Applying this to every point $p \in U$ we obtain that through every point in U there passes a single local integral manifold. This concludes the proof of the uniqueness.

CHAPTER 4

A Continuous Frobenius Theorem in Three Dimensions

4.1. Proof of the Theorem

4.1.1. Strategy and main technical steps. Our approach is quite geometrical and implements the simple idea that if Δ is a distribution which is "almost involutive", it should be possible to apply a small perturbation to obtain a new distribution $\widetilde{\Delta}$ which is involutive. It turns out that finding *some* perturbation to make the distribution involutive is easy, but making sure this perturbation is small is non-trivial and essentially constitutes the key estimate in our argument. We will obtain an estimate on the size of the perturbation which allows us o conclude that the asymptotic involutivity on average condition implies that Δ can be approximated by involutive distributions. This can then be shown to imply (not necessarily unique) integrability of Δ. Finally, we will use an additional argument to show that we have unique integrability with the additional assumption of *uniform* asymptotic involutivity on average.

In this section we reduce the proof of Theorem 1.18 to the proof of some more technical statements, albeit also of independent interest. Before we proceed, however, we remark that we can assume without loss of generality that the forms η_k approximating η in the definition of asymptotic involutivity are actually C^2. Indeed, by a standard "mollification" procedure we can replace the original sequence with smoother ones which still satisfy the (uniform) asymptotic involutivity on average conditions. Thus, from now on and for the rest of the paper we assume that the approximating forms η_k are C^2.

We will prove the following general perturbation result which does not require any involutivity or asymptotic involutivity assumptions. For two distributions Δ, Δ' defined in some local chart $\mathcal{U}$, we will use the notation $\measuredangle(\Delta, \Delta')$ to denote the maximum angle between subspaces of Δ and Δ' at all points of $\mathcal{U}$.

THEOREM 4.1. *Let Δ be a continuous 2-dimensional distribution on a 3-dimensional manifold M. Then, for every $x_0 \in M$, there exist neighborhoods $\mathcal{U}' \subset \mathcal{U}$ of x_0 and $\epsilon > 0$ such that if Δ_ϵ is a C^2 distribution with $\measuredangle(\Delta_\epsilon, \Delta) \leq \epsilon$ then there exists a local frame $\{X, Y\}$ of Δ_ϵ and a C^1 vector field W such that the distribution*

$$\widetilde{\Delta}_\epsilon = span\{X + W, Y\}$$

is involutive. Moreover, X, Y and W can be chosen so that for every C^2 form η with $\Delta_\epsilon = \ker(\eta)$ and $\|\eta\|_x \geq 1$ for every $x \in \mathcal{U}$, we have

$$\|W\| \leq \sup_{x\in\mathcal{U},|t|\leq t_0} \{\|\eta\|_x \|\eta \wedge d\eta\|_x e^{\widetilde{d\eta}(x,t))}\}.$$

Theorem 4.1 will be proved in Sections 4.1.2-4.1.5. We remark that the condition $\|\eta\|_x \geq 1$ is not a restriction since the condition $\Delta_\epsilon = \ker(\eta)$ is preserved under multiplication of η by a scalar and therefore we can always assume without loss of generality that this lower bound holds. Its purpose is just to simplify the form of the upper bound on $\|W\|$ (where, as mentioned in the introduction, the norm $\|\cdot\|$ refers to the C^0 topology). Notice that this bound is perfectly adapted to work with the asymptotic involutivity assumption of our main theorem. Indeed, by this assumption, for sufficiently large k we have that $\Delta^{(k)}$ is close to Δ and we can apply Theorem 4.1 to get a corresponding involutive distribution $\widetilde{\Delta}^{(k)}$ after a perturbation whose norm is bounded by $\|\eta\|\|\eta \wedge d\eta\|e^{\widetilde{d\eta}(x,t))}$. Since $\eta_k \to \eta$ we have that $\|\eta_k\|$ is uniformly bounded, hence $\|\eta\|\|\eta \wedge d\eta\|e^{\widetilde{d\eta}(x,t))} \to 0$ and therefore the sequence of perturbed involutive distributions $\widetilde{\Delta}^{(k)}$ approximates the original distribution Δ. In Section 4.1.6 we will show that this implies that Δ is (weakly) integrable in the sense that it admits (not necessarily unique) local integral surfaces through every point. We formalize this statement in the following

PROPOSITION 4.2. *Suppose there exists a sequence of involutive distributions $\widetilde{\Delta}^{(k)}$ which converges to a continuous distribution Δ uniformly on some open set $\mathcal{U}$. Then there exists an open subset $\mathcal{V} \subset \mathcal{U}$ such that Δ is (not necessarily uniquely) integrable at every $x \in \mathcal{V}$.*

We note that the (local) convergence of a family of hyperplanes and line bundles is to be understood here in terms of the maximal angle going to zero uniformly in a given neighbourhood. To get uniqueness of these integral manifolds we will prove the following statement.

PROPOSITION 4.3. *Let Δ be a continuous 2-dimensional distribution on a 3-dimensional manifold M. Suppose that Δ is uniformly asymptotically involutive on average. Then Δ is locally spanned by two uniquely integrable vector fields X, Y.*

The unique integrability of Δ follows from Proposition 4.3 by a simple contradiction argument: if there are two integral manifolds of Δ through a point then at least one of the vector fields X and Y does not satisfy uniqueness of solutions, thus contradicting the statement of Proposition 4.3.

We have thus reduced the proof of Theorem 1.18 to the proofs of Theorem 4.1 which will be given in Sections 4.1.2-4.1.5, Proposition 4.2 which will be given in Section 4.1.6, and Proposition 4.3, which will be given in Section 4.1.7.

4.1.2. The perturbation. We now fix once and for all an arbitrary point $x_0 \in M$. Our aim in this section is to define a neighbourhood $\mathcal{U}$ of x_0 and a perturbation of a C^2 distribution Δ_ϵ sufficiently close to our original distribution Δ which yields a new C^1 distribution $\widetilde{\Delta}_\epsilon$. In the following sections we will show that $\widetilde{\Delta}_\epsilon$ satisfies the required properties for the conclusions of Theorem 1.18, in particular that it is involutive and that it is a *small* perturbation of Δ_ϵ.

First of all we fix a local chart $(x^1, x^2, x^3, \mathcal{U}_0)$ centered at x_0. Notice that we can (and do) assume without loss of generality that Δ is everywhere transversal to the coordinate axes in $\mathcal{U}_0$ and that therefore this transversality also holds for Δ_ϵ if ϵ is sufficiently small. In particular this implies that we can define a local frame $\{X, Y\}$ for Δ_ϵ in $\mathcal{U}_0$ where X, Y are C^2 vector fields of the form

$$X = \frac{\partial}{\partial x^1} + a\frac{\partial}{\partial x^3} \quad \text{and} \quad Y = \frac{\partial}{\partial x^2} + b\frac{\partial}{\partial x^3}. \tag{4.1.2.1}$$

for suitable C^2 functions $a(x), b(x)$. Notice that the transversality condition implies that the C^0 norms of a and b are uniformly bounded below for all Δ_ϵ with ϵ sufficiently small.

Remark 4.4. The vector fields X, Y are C^2 and thus define local flows, which we will denote by e^{tX}, e^{tY} respectively, and admit unique integral curves through every point $x \in \mathcal{U}_0$, which we denote by $\mathcal{X}_x, \mathcal{Y}_x$ respectively. These integral curves will play an important role in the following construction and it will be sometimes convenient to mix the notation a little bit. For example we will refer to the "natural" parametrization of an integral curve $\mathcal{Y}_x$ to intend parametrization by the flow so that once we specify some point $y = \mathcal{Y}_x(0)$ (which may be different from the point x which we use to specify the curve) we then have $\mathcal{Y}_x(t) := e^{tY}(y)$.

We are now ready to fix the neighbourhood $\mathcal{U}$ in which we define the perturbation. At this stage we make certain choices motivated by the fact that the involutive distribution we are constructing is given by the span of two vector fields of the form $\{X + W, Y\}$. We could similarly obtain a pair of vector fields of the form $\{X, Y + Z\}$ since the situation is completely symmetric. We let $\mathcal{S}$ denote the integral manifold through x_0 of the coordinate planes given by $< \partial/\partial x^1, \partial/\partial x^3 >$ in the local chart $\mathcal{U}_0$. Then the vector field Y and its unique integral curves are everywhere transversal to $\mathcal{S}$ and indeed, by the uniform bounds on $|b(x)|$, this transversality is uniform in Δ_ϵ as long as ϵ is sufficiently small. In particular this means that we can choose a smaller neighbourhood $\mathcal{U} \subset \mathcal{U}_0$ which is "saturated" by the integral curves of Y in the sense that every point $x \in \mathcal{U}$ lies on an integral curve of Y through some point of $\mathcal{S} \cap \mathcal{U}$. Moreover, this saturation condition can be guaranteed for a fixed neighbourhood $\mathcal{U}$ for any Δ_ϵ sufficiently close to Δ. For every $x \in \mathcal{U}$ we let $\mathcal{Y}_x$ denote the integral curve through x of the vector field Y. We consider the natural parametrization of each integral curve $\mathcal{Y}_x$ by fixing the initial condition $\mathcal{Y}_x(0) \in \mathcal{S}$ and then let t_x be the time such that $\mathcal{Y}_x(t_x) = x$. Notice that by choosing our neighbourhood $\mathcal{U}$ sufficiently small, we can also assume that the integration time t_x is bounded by any a priori given arbitrarily small constant. To simplify the final expression it will be convenient to have

$$|t_x| \leq \frac{1}{\|X\| \cdot \|Y\|} = \frac{1}{\sqrt{(1+a^2)(1+b^2)}} \leq 1. \tag{4.1.2.2}$$

This upper bound is uniform for all distributions Δ_ϵ is ϵ if sufficiently small.

We are now ready to define our perturbation. Notice first that the explicit forms of the vector fields X and Y implies that the Lie bracket $[X, Y]$ always lies in the $\partial/\partial x^3$ direction. Indeed, we can compute explicitly the Lie bracket and use it to define a function $h : \mathcal{U} \to \mathbb{R}$ by

$$[X, Y] = \left(\frac{\partial b}{\partial x^1} - \frac{\partial a}{\partial x^2} + a\frac{\partial b}{\partial x^3} - b\frac{\partial a}{\partial x^3} \right) \frac{\partial}{\partial x^3} =: h\frac{\partial}{\partial x^3}. \tag{4.1.2.3}$$

Thus h is the (signed) magnitude of the Lie bracket $[X, Y]$ (which happens to be always in the $\partial/\partial x^3$ direction). We define the function $\alpha : \mathcal{U} \to \mathbb{R}$ by

$$\alpha(x) := \int_0^{t_x} h(\mathcal{Y}_x(\tau)) \exp\left(\int_\tau^{t_x} \frac{\partial b}{\partial x^3}(\mathcal{Y}_x(s))ds \right) d\tau. \tag{4.1.2.4}$$

At the moment the function α is just defined "out of the blue" with no immediately obvious motivation, but we will show below that it is exactly the right form for the perturbation we seek. Using this function we define the perturbed distribution by

$$\widetilde{\Delta}_\epsilon := span\left\{ X + \alpha\frac{\partial}{\partial x^3}, Y \right\} \tag{4.1.2.5}$$

In Section 4.1.3 we will show that $\widetilde{\Delta}_\epsilon$ is C^1, in Section 4.1.4 that it is involutive, and in Section 4.1.5 we will show that the perturbation α satisfies the required upper bounds.

4.1.3. Differentiability. In this section we prove the following

PROPOSITION 4.5. *The function α is C^1.*

Since X and Y are C^2, it follows immediately from Proposition 4.5 and (4.1.2.5) that $\widetilde{\Delta}_\epsilon$ is C^1 as required.

To prove Proposition 4.5, notice first that from the definition of α in (4.1.2.4) it follows immediately that α is C^1 in the direction of Y. It is therefore sufficient to prove that α is also C^1 along two other vector fields that together with Y form a coordinate system in $\mathcal{U}$. The existence of such a coordinate system in $\mathcal{U}$ is guaranteed by classical results on the representation of vector fields near a regular point, see e.g. [**54**], however we will need here a particular choice of coordinate system, in particular one defined by Y and two additional vector fields Z, V which span the tangent space of $\mathcal{S}$. Therefore we give a self contained proof.

PROPOSITION 4.6. *There are two C^1 vector fields Z and V that span the tangent space of $\mathcal{S}$, such that the system $\{Y, V, Z\}$ is a trivialization of the tangent bundle TM by commuting vector fields in the neighborhood $\mathcal{U}$, which is to say that*

$$[Y, Z] = [Y, W] = [Z, W] = 0.$$

PROOF. We first recall that the neighborhood $\mathcal{U}$ is parameterized in such a way that any point can be joined to a point of $\mathcal{S}$ by an integral curve of Y, and so we can choose $\epsilon > 0$, and modify $\mathcal{U}$ slightly, such that the map

$$\phi(t_1, t_2, t_3) = e^{t_1 Y} \circ e^{t_2 \frac{\partial}{\partial x^1}} \circ e^{t_3 \frac{\partial}{\partial x^3}}(x_0)$$

is a diffeomorphism from $(-\epsilon, \epsilon)^3$ to $\mathcal{U}$. We define

$$Z := \phi_* \frac{\partial}{\partial t_2} \quad \text{and} \quad V := \phi_* \frac{\partial}{\partial t_3}$$

where the subscript $*$, here and below, denotes the standard push-forward of vector fields. Observe that by the chain rule, for every $\bar{t} = (t_1, t_2, t_3) \in (-\epsilon, \epsilon)^3$ we have

$$\phi_* \frac{\partial}{\partial t_2}(\phi(\bar{t})) = \frac{\partial \phi}{\partial t_2}(\bar{t}) = e_*^{t_1 Y} \frac{\partial}{\partial x^1}(\phi(\bar{t}))$$

and

$$\phi_* \frac{\partial}{\partial t_3}(\phi(\bar{t})) = \frac{\partial \phi}{\partial t_3}(\bar{t}) = e_*^{t_1 Y} \frac{\partial}{\partial x^3}(\phi(\bar{t})).$$

Since the vector field Y is C^2 it follows that e_*^{tY} is C^1 which implies that the vector fields Z and V are C^1. By the naturality of the Lie bracket and observing that $Y = \phi_* \frac{\partial}{\partial t_1}$ we have

$$[V, Y] = \left[\phi_* \frac{\partial}{\partial t_3}, \phi_* \frac{\partial}{\partial t_1}\right] = \phi_* \left[\frac{\partial}{\partial t_3}, \frac{\partial}{\partial t_1}\right] = 0$$

and similarly $[Z, Y] = [Z, V] = 0$. This shows that the vector fields commute. Now we are only left to prove that $\mathcal{S}$ is spanned by V and Z. Since $\mathcal{S}$ is by definition the integral surface of the local coordinates $\partial/\partial x^1$ and $\partial/\partial x^3$, it is sufficient to show that V and Z span this plane. We will show this by computing explicit formulas for the vector fields. By standard calculus for vector fields on manifolds [**1, 36**], for any $x \in \mathcal{U}$, we have that

$$\frac{d}{dt}\left(e_*^{tY} \frac{\partial}{\partial x^3}|_x\right) = e_*^{tY} \left[\frac{\partial}{\partial x^3}, Y\right]|_x = \frac{\partial b}{\partial x^3} \circ e^{-tY}(x) \cdot e_*^{tY} \frac{\partial}{\partial x^3}|_x.$$

Integrating both side gives

$$e_*^{tY} \frac{\partial}{\partial x^3}|_x = \exp\left(\int_0^t \frac{\partial b}{\partial x^3} \circ e^{-\tau Y}(x) d\tau\right) \frac{\partial}{\partial x^3}|_x$$

which shows that V always lies in the direction $\partial/\partial x^3$. By the same calculations we also get

$$e_*^{tY}\frac{\partial}{\partial x^1}|_x = \frac{\partial}{\partial x^1}|_x + \int_0^t \frac{\partial b}{\partial x^1}\circ e^{-sY}(x)\exp\left(\int_s^t \frac{\partial b}{\partial x^3}\circ e^{-\tau Y}(x)d\tau\right)ds\frac{\partial}{\partial x^3}|_x$$

which shows that Z always lies in the span of $\partial/\partial x^1$ and $\partial/\partial x^3$. Therefore we have that V and Z span the tangent space of $\mathcal{S}$. □

To complete the proof of Proposition 4.5 it is sufficient to show that α is C^1 along the vector fields Z and V defined above. We will need the following simple fact which constitutes the main motivation for our specific choice of the coordinate system.

LEMMA 4.7. *If x and y belong to the same integral curve of V or to the same integral curve of Z, then we have*

$$t_x = t_y.$$

PROOF. Let x and y be in the same integral curve of V. Then, since $[Y, V] = 0$, it follows that $e^{t_x Y}(x)$ and $e^{t_x Y}(y)$ are in the same integral curve of V. Since $e^{t_x Y}(x) \in \mathcal{S}$ and $V \in T\mathcal{S}$ then $e^{t_x Y}(y) \in \mathcal{S}$ and it follows that $t_x = t_y$. The proof for Z is exactly the same. □

PROOF OF PROPOSITION 4.5. To show that α is differentiable along the vector fields Z and V we will show directly from first principle that for every $x \in \mathcal{U}$, the limits

$$\lim_{\delta\to 0}\frac{\alpha(x) - \alpha(e^{\delta V}(x))}{\delta} \quad \text{and} \quad \lim_{\delta\to 0}\frac{\alpha(x) - \alpha(e^{\delta Z}(x))}{\delta}$$

exist. We will prove the statement for the first limit, the second follows by exactly the same arguments. We fix some $x \in \mathcal{U}$ and for $\delta \neq 0$, by Lemma 4.7 we have $t_x = t_{e^{\delta V}(x)} =: t$. To simplify the notation in the calculations below, we shall write

$$\mathcal{B}(\delta,\tau) := \exp\left(\int_\tau^t \frac{\partial b}{\partial x^3}(\mathcal{Y}_{e^{\delta V}(x)}(s))ds\right).$$

Notice that for $\delta = 0$ we have $e^{\delta V}(x) = x$ and so we have

$$\alpha(x) = \int_0^t h(\mathcal{Y}_x(\tau))\mathcal{B}(0,\tau)d\tau.$$

Then

$$\alpha(x) - \alpha(e^{\delta V}(x)) = \int_0^t h(\mathcal{Y}_x(\tau))\mathcal{B}(0,\tau)d\tau - \int_0^t h(\mathcal{Y}_{e^{\delta V}(x)}(\tau))\mathcal{B}(\delta,\tau)d\tau.$$

By adding and subtracting the term

$$\int_0^t h(\mathcal{Y}_x(\tau))\mathcal{B}(\delta,\tau)d\tau$$

to the right hand side we get

$$\begin{aligned}\alpha(x) - \alpha(e^{\delta V}(x)) &= \int_0^t h(\mathcal{Y}_x(\tau))\mathcal{B}(0,\tau)d\tau - \int_0^t h(\mathcal{Y}_x(\tau))\mathcal{B}(\delta,\tau)d\tau \\ &\quad + \int_0^t h(\mathcal{Y}_x(\tau))\mathcal{B}(\delta,\tau)d\tau - \int_0^t h(\mathcal{Y}_{e^{\delta V}(x)}(\tau))\mathcal{B}(\delta,\tau)d\tau \\ &= \int_0^t h(\mathcal{Y}_x(\tau))[\mathcal{B}(0,\tau) - \mathcal{B}(\delta,\tau)]d\tau \\ &\quad + \int_0^t [h(\mathcal{Y}_x(\tau)) - h(\mathcal{Y}_{e^{\delta V}(x)}(\tau))]\mathcal{B}(\delta,\tau)d\tau.\end{aligned}$$

Dividing both sides by δ it is therefore sufficient to show that the limit exists for each integral on the last two lines above. For the first integral notice that $h(\mathcal{Y}_x(\tau))$ does not depend on δ and therefore it is sufficient to show that

$$\lim_{\delta\to 0} \frac{\mathcal{B}(0,\tau) - \mathcal{B}(\delta,\tau)}{\delta}$$

exists. To see this, notice first that it is equal to

$$\lim_{\delta\to 0} \frac{1}{\delta}\left[\exp\left(\int_\tau^t \frac{\partial b}{\partial x^3}(\mathcal{Y}_x(s))ds\right) - \exp\left(\int_\tau^t \frac{\partial b}{\partial x^3}(\mathcal{Y}_{e^{\delta V}(x)}(s))ds\right)\right],$$

This is by definition the directional derivative of the function

$$\exp \int_\tau^t \frac{\partial b}{\partial x^3}(\mathcal{Y}_x(s))ds \tag{4.1.3.1}$$

in the direction of V. Since $\partial b/\partial x^3$ is C^1 it follows that (4.1.3.1) is also C^1 and therefore this directional derivative exists. Similarly, for the second integral above, the limit of $\mathcal{B}(\delta,\tau)$ as $\delta \to 0$ is just $\mathcal{B}(0,\tau)$ (and thus exists), and so it is sufficient to show that the limit

$$\lim_{\delta\to 0} \frac{h(\mathcal{Y}_x(\tau)) - h(\mathcal{Y}_{e^{\delta V}(x)}(\tau))}{\delta}.$$

exists. Again, this is exactly the directional derivative of h in the direction of V. Since h is C^1 this derivative exists. This proves that α is C^1. □

4.1.4. Involutivity. In this Section we prove

$$\left[X + \alpha\frac{\partial}{\partial x^3}, Y\right] = 0.$$

This implies involutivity as required, since the vanishing of the Lie bracket for a C^1 local frame of a C^1 distribution is well known to be equivalent to the involutivity condition $\eta \wedge d\eta = 0$ given above; this follows for example from Cartan's formula given in (4.1.5.4) below, or see any standard reference such as [**54**]. By the linearity of the Lie bracket we have

$$\left[X + \alpha \frac{\partial}{\partial x^3}, Y\right] = [X, Y] + \left[\alpha \frac{\partial}{\partial x^3}, Y\right]$$

and, applying the general formula $[\varphi X, \psi Y] = \varphi X(\psi)Y - \psi Y(\varphi)X + \varphi\psi[X, Y]$ for C^1 functions φ, ψ, where $X(\psi), Y(\varphi)$ denote the "directional derivatives" of the functions ψ, φ in the directions of the vector fields X, Y respectively, we get

$$\left[\alpha \frac{\partial}{\partial x^3}, Y\right] = -Y(\alpha)\frac{\partial}{\partial x^3} + \alpha \frac{\partial b}{\partial x^3}\frac{\partial}{\partial x^3}$$

Notice that this bracket lies in the x^3 direction. Substituting above and using the fact that $[X, Y] = h\partial/\partial x^3$ also lies in the x^3 direction we get

$$\left[X + \alpha \frac{\partial}{\partial x^3}, Y\right] = \left(h + \alpha \frac{\partial b}{\partial x^3} - Y(\alpha)\right)\frac{\partial}{\partial x^3}. \tag{4.1.4.1}$$

Thus the involutivity of Δ_ϵ is equivalent to the condition that the bracket on the left hand side of (4.1.4.1) is equal to 0, or equivalently that $Y(\alpha) = h + \alpha\partial b/\partial x^3$, i.e. that α is a solution to the partial differential equation

$$Y(u) = h + u\frac{\partial b}{\partial x^3}. \tag{4.1.4.2}$$

To see that α is a solution of (4.1.4.2) note that by the definition of α in (4.1.2.4), for any integral curve $\mathcal{Y}$ of Y in $\mathcal{U}$ parameterized so that $\mathcal{Y}(0) \in \mathcal{S}$ and for any $|t| \le t_0$ we have

$$\alpha(\mathcal{Y}(t)) = \int_0^t h(\mathcal{Y}(\tau)) \exp\left(\int_\tau^t \frac{\partial b}{\partial x^3}(\mathcal{Y}(s))ds\right) d\tau.$$

Differentiating α along $\mathcal{Y}$ we get

$$\begin{aligned}
Y(\alpha)(\mathcal{Y}(t)) =& \frac{d}{dt}\alpha(Y(t)) \\
&= h(\mathcal{Y}(t)) \exp\left(\int_t^t \frac{\partial b}{\partial x^3}(\mathcal{Y}(s))ds\right) \\
&+ \frac{\partial b}{\partial x^3}(\mathcal{Y}(t)) \int_0^t h(\mathcal{Y}(\tau)) \exp\left(\int_\tau^t \frac{\partial b}{\partial x^3}(\mathcal{Y}(s))ds\right) d\tau \\
&= h(\mathcal{Y}(t)) + \frac{\partial b}{\partial x^3}(\mathcal{Y}(t))\alpha(\mathcal{Y}(t))
\end{aligned}$$

which proves that α is the required solution of (4.1.4.2).

4.1.5. Perturbation bounds. In this Section we prove the upper bound on the norm of α which gives the upper bound required in the statement of Theorem 4.1. Notice first of all that by the definition of the function h in (4.1.2.3) we have

$$\begin{aligned}\alpha(x) &= \int_0^{t_x} h(\mathcal{Y}_x(\tau)) \exp\left(\int_\tau^{t_x} \frac{\partial b}{\partial x^3}(\mathcal{Y}_x(s))ds\right) d\tau \\ &\leq \int_0^{t_x} \|[X,Y](\mathcal{Y}_x(\tau))\| \exp\left(\int_\tau^{t_x} \frac{\partial b}{\partial x^3}(\mathcal{Y}_x(s))ds\right) d\tau\end{aligned} \tag{4.1.5.1}$$

We will estimate the two terms in two Lemmas.

LEMMA 4.8. $\|[X,Y](\mathcal{Y}_x(\tau))\| \leq \{\|X\|_{\mathcal{Y}_x(\tau)}\|Y\|_{\mathcal{Y}_x(\tau)}\|\eta\wedge d\eta\|_{\mathcal{Y}_x(\tau)}\}$

LEMMA 4.9. $\exp\left(\int_\tau^{t_x} \frac{\partial b}{\partial x^3}(\mathcal{Y}_x(s))ds\right) \leq \|\eta\|_{\mathcal{Y}_x(\tau)} \exp(\widetilde{d\eta}(\mathcal{Y}_x(\tau), t_x-\tau)$

Combining these two estimates, substituting into (4.1.5.1), and using the bound on t given by (4.1.2.2), we obtain

$$\begin{aligned}|\alpha(x)| &\leq t\sup_{x\in\mathcal{U}}\{\|\eta\|_x \exp(\widetilde{d\eta}(t,x))\|X\|_x\|Y\|_x\|\eta\wedge d\eta\|_x\} \\ &\leq \sup_{x\in\mathcal{U}}\{\|\eta\|_x\|\eta\wedge d\eta\|_x \exp(\widetilde{d\eta}(t,x))\}\end{aligned}$$

which is the required bound and thus completes the proof of Theorem 4.1 modulo the proof of the two Lemmas. For the proof of both Lemmas, notice first that, since the vector fields X, Y defined in (4.1.2.1) lie in $\ker(\eta)$, any C^1 form η such that $\Delta = ker(\eta)$ is of the form

$$\eta = c(dx^3 - adx^1 - bdx^2) \tag{4.1.5.2}$$

for some non-vanishing C^1 function $c(x)$ defined in $\mathcal{U}$. Notice that $\eta(\partial/\partial x^3) = c$ and therefore $c \leq \|\eta\|$ everywhere and we can even assume, up to multiplying η by a (possibly negative) scalar if necessary, that

$$1 \leq c \leq \|\eta\| \tag{4.1.5.3}$$

We can now prove the two Lemmas.

PROOF OF LEMMA 4.8. All the estimates below are made for a given fixed point in $\mathcal{U}$ and so for simplicity we omit this from the notation. By the definition of h in (4.1.2.3) we have $\eta([X,Y]) = h\eta(\partial/\partial x^3) = ch$ and therefore $h = \eta([X,Y])/c$ and in particular

$$\|[X,Y]\| = \frac{|\eta([X,Y])|}{c}$$

Since $X, Y \in \ker(\eta)$, we have $\eta(X) = \eta(Y) = 0$ and the "Cartan formula" gives

$$d\eta(X,Y) = X(\eta(Y)) - Y(\eta(X)) - \eta([X,Y]) = -\eta([X,Y]). \tag{4.1.5.4}$$

On the other hand, we have

$$\eta \wedge d\eta \left(\frac{\partial}{\partial x^3}, X, Y \right) = \eta \left(\frac{\partial}{\partial x^3} \right) d\eta(X, Y) = c d\eta(X, Y).$$

Substituting into the equations above we then get

$$\|[X, Y]\| = \frac{|\eta([X, Y])|}{c} = \frac{|d\eta(X, Y)|}{c} = \frac{1}{c^2} \left| \eta \wedge d\eta \left(\frac{\partial}{\partial x^3}, X, Y \right) \right|$$

Using that $c > 1$ and the multilinearity of $\eta \wedge d\eta$ this gives the bound $\|[X, Y]\| \leq \|X\|\|Y\|\|\eta \wedge d\eta\|$ as required. $\square$

Proof of Lemma 4.9. By direct calculation we have

$$\begin{aligned} d\eta &= (\frac{\partial c}{\partial x^1} + a\frac{\partial c}{\partial x^3} + c\frac{\partial a}{\partial x^3})dx^1 \wedge dx^3 + (\frac{\partial c}{\partial x^2} + b\frac{\partial c}{\partial x^3} + c\frac{\partial b}{\partial x^3})dx^2 \wedge dx^3 \\ &+ (a\frac{\partial c}{\partial x^2} - b\frac{\partial c}{\partial x^1} + c\frac{\partial a}{\partial x^2} - c\frac{\partial b}{\partial x^1})dx^1 \wedge dx^2 \\ &= (X(c) + c\frac{\partial a}{\partial x^3})dx^1 \wedge dx^3 + (Y(c) + c\frac{\partial b}{\partial x^3})dx^2 \wedge dx^3 \\ &+ (a\frac{\partial c}{\partial x^2} - b\frac{\partial c}{\partial x^1} + c\frac{\partial a}{\partial x^2} - c\frac{\partial b}{\partial x^1})dx^1 \wedge dx^2 \end{aligned}$$

On the other hand we can write

$$d\eta = d\eta_1 dx^1 \wedge dx^3 + d\eta_2 dx^2 \wedge dx^3 + d\eta_3 dx^1 \wedge dx^2$$

and so, by comparing the terms of the two formulae for $d\eta$, we have

$$Y(c) = -c\frac{\partial b}{\partial x^3} + d\eta_2.$$

Dividing both sides by c gives

$$\frac{Y(c)}{c} = -\frac{\partial b}{\partial x^3} + \frac{d\eta_2}{c}.$$

Since $Y(c)$ is exactly the derivative of c along integral curves of Y, integrating along these integral curves we get

$$\log \left| \frac{c(\mathcal{Y}_x(t_x))}{c(\mathcal{Y}_x(\tau))} \right| = -\int_\tau^{t_x} \frac{\partial b}{\partial x^3}(\mathcal{Y}_x(s))ds + \int_\tau^{t_x} \frac{d\eta_2(\mathcal{Y}_x(s))}{c(\mathcal{Y}_x(s))}ds$$

which implies

$$\int_\tau^{t_x} \frac{\partial b}{\partial x^3}(\mathcal{Y}_x(s))ds = \int_\tau^{t_x} \frac{d\eta_2(\mathcal{Y}_x(s))}{c(\mathcal{Y}_x(s))}ds - \log \left| \frac{c(\mathcal{Y}_x(t_x))}{c(\mathcal{Y}_x(\tau))} \right|$$

hence we have

$$\exp\left(\int_\tau^{t_x} \frac{\partial b}{\partial x^3}(\mathcal{Y}_x(s))d\tau \right) = \left| \frac{c(\mathcal{Y}_x(\tau))}{c(\mathcal{Y}_x(t_x))} \right| \exp\left(\int_\tau^{t_x} \frac{d\eta_2(\mathcal{Y}_x(s))}{c(\mathcal{Y}_x(s))}ds \right).$$

Using that $1 \leq c \leq \|\eta\|$ by (4.1.5.3) we then get

$$\exp\left(\int_{\tau}^{t_x} \frac{\partial b}{\partial x^3}(\mathcal{Y}_x(s))d\tau\right) \leq \|\eta\|_{\mathcal{Y}_x(\tau)} \exp\left(\int_{\tau}^{t_x} d\eta_2(\mathcal{Y}_x(s))ds\right).$$

Notice that the integral on the right hand side is not exactly in the form used in the definition of $\widetilde{d\eta_2}$ in Section 1.2.2, where the limits in the integral go from 0 to t_x. Recalling that $\mathcal{Y}_x(s) = e^{(s-t_x)Y}(x)$, we have

$$\int_{\tau}^{t_x} d\eta_2(\mathcal{Y}_x(s))ds = \int_{\tau}^{t_x} d\eta_2 \circ e^{(s-t_x)Y}(x)ds = \int_{0}^{t_x-\tau} d\eta_2 \circ e^{(s-t_x)Y}(e^{\tau Y}(x)))ds$$

This last integral is by definition equal to $\widetilde{d\eta_2}(\mathcal{Y}_x(\tau), t_x - \tau)$ and so substituting into the expression above this completes the proof. □

4.1.6. Convergence. In this section we prove Proposition 4.2. We suppose throughout that we have a sequence $\tilde{\Delta}^{(k)}$ of C^1 involutive distributions converging uniformly to a continuous distribution Δ in some open set $\mathcal{U}$. The involutivity of the distributions $\tilde{\Delta}^{(k)}$ implies that they are uniquely integrable by the classical Frobenius Theorem, but to prove the required convergence we will need to construct these integral manifolds rather explicitly.

We assume without loss of generality that the open set $\mathcal{U}$ is contained inside some local chart and that Δ is everywhere transversal to the coordinate axes in this local chart. By the convergence of the sequence of distributions $\tilde{\Delta}^{(k)}$ to Δ, the same transversality property holds for $\tilde{\Delta}^{(k)}$ for all sufficiently large k. This implies that each $\tilde{\Delta}^{(k)}$ admits a local frame $\{X_k, Y_k\}$ formed by C^1 vector fields of the form

$$X_k = \frac{\partial}{\partial x^1} + a_k\frac{\partial}{\partial x^3} \quad \text{and} \quad Y_k = \frac{\partial}{\partial x^2} + b_k\frac{\partial}{\partial x^3}. \tag{4.1.6.1}$$

for C^1 functions a_k, b_k. By the convergence of the sequence of distributions $\tilde{\Delta}^{(k)}$ to Δ it follows that the sequences of vector fields X_k, Y_k converge to continuous vector fields X, Y which form a continuous local frame of Δ and have the form

$$X = \frac{\partial}{\partial x^1} + a\frac{\partial}{\partial x^3} \quad \text{and} \quad Y = \frac{\partial}{\partial x^2} + b\frac{\partial}{\partial x^3}. \tag{4.1.6.2}$$

for continuous functions a, b (cf. (4.1.2.1)). Since the approximating vector fields X_k, Y_k are C^1, their Lie bracket is well defined and their specific form implies it lies in the $\partial/\partial x^3$ direction, see (4.1.2.3), and in particular is transversal to $\tilde{\Delta}^{(k)}$. Therefore by the involutivity of $\tilde{\Delta}^{(k)}$ it follows that the vector fields commute, i.e.

$$[X_k, Y_k] = 0. \tag{4.1.6.3}$$

Notice that of course we cannot draw the same conclusion for the vector fields X, Y since they are only continuous and the Lie bracket is not defined.

We now fix an open subset $\mathcal{V} \subset \mathcal{U}$ and some sufficiently small $\epsilon > 0$ such that for any $x \in \mathcal{V}$, any $|s_1|, |s_2| \leq \epsilon$ and all sufficiently large k, we have

$$W_x^{(k)}(s_1, s_2) := e^{s_1 X_k} \circ e^{s_2 Y_k}(x) \in \mathcal{U}.$$

Notice that by (4.1.6.3) we have that

$$\begin{aligned} \frac{\partial W_x^{(k)}}{\partial s_1}(s_1, s_2) &= X_k(W_x^{(k)}(s_1, s_2)) \\ \frac{\partial W_x^{(k)}}{\partial s_2}(s_1, s_2) &= Y_k(W_x^{(k)}(s_1, s_2)). \end{aligned} \tag{4.1.6.4}$$

In particular the Jacobian of $W_x^{(k)}$ is non-zero everywhere on $[-\epsilon, \epsilon]^2$ and therefore we have a sequence of embeddings

$$W_x^{(k)} : [-\epsilon, \epsilon]^2 \to \mathcal{U}.$$

By (4.1.6.4), the tangent spaces of $W_x^{(k)}$ are exactly the hyperplanes of the distributions $\widetilde{\Delta}^{(k)}$ and therefore the images of the maps $W_x^{(k)}$ are exactly the local integral manifolds of the distributions through the point x.

LEMMA 4.10. *For every $x \in \mathcal{V}$, sequence $\{W_x^{(k)}\}$ is equicontinuous and equibounded*

PROOF. We have that $W_x^{(k)}(0) = x$ so $W_x^{(k)}(s_1, s_2) \in U$ and therefore equiboundedness is easy. For equicontinuity note that $DW^{(k)}(s_1, s_2)$ is a matrix whose columns are $X_k(W^{(k)}(s_1, s_2)$ and $Y_k(W^{(k)}(s_1, s_2)$. Therefore the differential is equibounded and so $W_x^{(k)}(s_1, s_2)$ is equicontinuous. $\square$

By the Arzela-Ascoli Theorem there exists a continuous function

$$W_x : [-\epsilon, \epsilon]^2 \to \mathcal{U}.$$

which is the uniform limit of some subsequence of $\{W_x^{(k)}\}_{k=1}^{\infty}$(which we assume, without loss of generality, to be the full sequence from now on). To complete the proof of Proposition 4.2 it is therefore sufficient to show that $W_x([-\epsilon, \epsilon]^2)$ is an integral manifold of the limiting distribution Δ, i.e. that W_x is actually differentiable and that its tangent spaces coincide with the hyperplanes of Δ. Thus, letting $DW_x = DW_x(s_1, s_2)$ denote the matrix whose columns are $X(W_x(s_1, s_2))$ and $Y(W_x(s_1, s_2))$ it is sufficient to prove the following

LEMMA 4.11. *W_x is a differentiable function whose derivative is DW*

PROOF. Notice first of all that from (4.1.6.4) and the fact that $X_k \to X$, $Y_k \to Y$ and $W_x^{(k)}(s_1, s_2) \to W_x(s_1, s_2)$ it follows that the partial derivatives $\partial W_x^{(k)}/\partial s_i(s_1, s_2)$ converge to $X(W_x(s_1, s_2))$ and $Y(W_x(s_1, s_2))$ respectively, and therefore, the derivative $DW_x^{(k)}$ converges uniformly to DW_x. Now, for any two points $p, q \in W_x([-\epsilon, \epsilon]^2)$ and a smooth curve $\gamma = \gamma(t)$ connecting p to q with $\gamma(0) = p, \gamma(\tau) = q$, the fundamental theorem of calculus implies that

$$W_x^{(k)}(q) = W_x^{(k)}(p) + \int_0^\tau DW_x^k(\gamma(t)) \circ \frac{d\gamma(t)}{dt} dt$$

Thus, taking limits and exchanging the limit and the integral (which can be done due to uniform convergence) we get

$$W_x(q) = W_x(p) + \int_0^\tau DW_x(\gamma(t)) \circ \frac{d\gamma(t)}{dt} dt$$

This implies that DW_x is the derivative of W_x and thus proves in particular that W_x is differentiable as required. □

4.1.7. Uniqueness. In this section we will prove Proposition 4.3 and thus complete the proof of our main result, Theorem 1.18. We will first state a general uniqueness result for continuous vector fields on surfaces, Proposition 4.12 below, and then show that there exist vector fields X, Y which span Δ and satisfy the assumptions of Proposition 4.12 and are therefore uniquely integrable. The conditions for uniqueness are given here in the same spirit as the uniqueness conditions of Theorem 1.18 but for completeness and clarity we give the full details in this simpler setting.

We consider a Riemannian surface $\mathcal{S}$ and a vector field X on $\mathcal{S}$ which we can suppose to be given as $X = ker(w)$ for some continuous 1-form w defined on $\mathcal{S}$. We can restrict our attention to a local chart $\mathcal{U}$ with local coordinates $(z^1, z^2, \mathcal{U})$ and suppose without loss of generality that X is everywhere transversal to both coordinate axes $\partial/\partial z^1, \partial/\partial z^2$ in $\mathcal{U}$ and that therefore in particular it can be written in the form

$$X = \frac{\partial}{\partial z^1} + a\frac{\partial}{\partial z^2}$$

for some non-zero continuous function $a(z)$. If w_k is a sequence of C^1 1-forms on $\mathcal{U}$ with $w_k \to w$ then, for all sufficiently large k, the corresponding vector fields X_k will also be transversal to both axes and the corresponding vector fields $X_k = ker(w_k)$ can also be written in the form

$$X_k = \frac{\partial}{\partial z^1} + a_k\frac{\partial}{\partial z^2}$$

for C^1 functions a_k. Choosing some smaller domain $\mathcal{U}' \subset \mathcal{U}$ there exists some $t_0 > 0$ such that the flow $e^{\tau X_k}$ is well defined for all $x \in \mathcal{U}'$ and $|\tau| \leq t_0$ and $e^{\tau X_k}(x) \in \mathcal{U}$. Notice that in this case the external derivatives dw_k of the 1-forms w_k have only one component and so, by some slight abuse of notation we can simply write $dw_k = dw_k dz^1 \wedge dz^2$. Then, for every $z \in \mathcal{U}'$ and every $|t| \leq t_0$ define

$$\widetilde{dw}_k(z,t) := \int_0^t dw_k \circ e^{\tau X_k}(z) d\tau.$$

With this notation we then have the analogue of Theorem 1.18 as follows.

Proposition 4.12. *Let $X = ker(w)$ be a continuous vector field defined on a surface $\mathcal{S}$. Suppose that for every point there is a local chart $\mathcal{U}$ and a sequence of C^1 differential 1-forms w_k on $\mathcal{U}$ such that $w_k \to w$, the corresponding C^1 vector fields X_k, and a neighbourhood $\mathcal{U}' \subset \mathcal{U}$ such that for every $z \in \mathcal{U}'$ and every $|t| \leq t_0$*

$$\|w_k - w\|_z e^{\widetilde{dw}_k(z,t)} \to 0 \tag{4.1.7.1}$$

as $k \to \infty$. Then X is uniquely integrable.

It might be interesting to know if this result also admits an analogue for vector fields on $\mathbb{R}$ giving conditions for uniqueness in this most simple setting, and thus allowing comparison with existing results such as [**14**]. The argument for the proof that we give below does not admit an immediate "restriction" to the one-dimensional setting. We first show how it implies Proposition 4.3.

Proof of Proposition 4.3 assuming Proposition 4.12. Let $\Delta = \ker(\eta)$ be a uniformly asymptotically involutive on average continuous distributions as per the hypotheses of Proposition 4.3. Then, by definition, for every $x_0 \in M$ there exists local coordinates (x^1, x^2, x^3) in some local chart $\mathcal{U}$ in which we have vector fields X_k, Y_k, X, Y as in (4.1.6.1) and (4.1.6.2). We just need to show that X and Y are uniquely integrable. We will prove unique integrability for X using Proposition 4.12, the argument for Y is completely analogous.

Notice first of all that X is contained in the surface tangent to the coordinate axes $< \partial/\partial x^1, \partial/\partial x^3 >$. Moreover, as in (4.1.5.2) above, the explicit form of the vector fields X_k and X mean that the forms η_k and η can be written as

$$\eta_k = c_k(dx^3 - a_k dx^1 - b_k dx^2) \quad \text{and} \quad \eta = c(dx^3 - a dx^1 - b dx^2)$$

for some non-vanishing C^1 function $c_k(x)$ and continuous function $c(x)$ respectively, and that their restriction to the surface $\mathcal{S}$ locally tangent to $< \partial/\partial x^1, \partial/\partial x^3 >$ yields the forms

$$w_k = c_k(dx^3 - a_k dx^1) \quad \text{and} \quad w = c(dx^3 - a dx^1)$$

with the property that $X_k = \ker(w_k)$ and $X = \ker(w)$. We therefore just need to show that the forms w_k, w satisfy the assumptions of Proposition 4.12 to get unique integrability of the vector field X. The convergence is immediate since the assumption that $\eta_k \to \eta$ implies that $a_k \to a$, $b_k \to b$ and $c_k \to c$ and therefore in particular that $w_k \to w$ as $k \to \infty$. To show that $\|w_k - w\|_x e^{\widetilde{dw_k(x,t)}} \to 0$ we have, by direct calculation,

$$d\eta_k = \left(X_k(c_k) + c_k \frac{\partial a_k}{\partial x^3} \right) dx^1 \wedge dx^3 + \left(Y_k(c_k) + c_k \frac{\partial b_k}{\partial x^3} \right) dx^2 \wedge dx^3$$
$$+ \left(a \frac{\partial c_k}{\partial x^2} - b_k \frac{\partial c_k}{\partial x^1} + c_k \frac{\partial a_k}{\partial x^2} - c \frac{\partial b_k}{\partial x^1} \right) dx^1 \wedge dx^2$$

and

$$dw_k = \left(X_k(c_k) + c_k \frac{\partial a_k}{\partial x^3} \right) dx^1 \wedge dx^3.$$

Therefore $dw_k = d\eta_{k,1}$ and thus (4.1.7.1) follows immediately from the assumption that η is uniformly asymptotically involutive on average. $\square$

The rest of this section is devoted to the proof Proposition 4.12 which is a sort of one-dimensional version of Theorem 4.1. Indeed, in Theorem 4.1 we showed that each distribution could be perturbed to yield an involutive distribution and that the size of this perturbation could be controlled. Here we show that any 1-form η defining a vector field on a surface can be "rescaled" to a *closed* 1-form defining *the same vector field* and that this rescaling has controlled norm.

LEMMA 4.13. *Let w be C^2 differential 1-form on a surface $\mathcal{S}$ and $(z^1, z^2, \mathcal{U})$ be a coordinate systems whose axes are transverse to $ker(w)$ and let $\mathcal{L}$ be an integral curve of $\partial/\partial z^2$. Then there is a C^1 differential 1-form $\hat{w}$ with* $\ker(\hat{w}) = \ker(w)$ *and $\hat{w}(\partial/\partial z^2) = 1$ along $\mathcal{L}$ such that for every $z \in \mathcal{U}$ we have*

$$d\hat{w} = 0 \quad \textit{and} \quad \|\hat{w}\|_z \leq \sup_{|t| \leq t_0} \{e^{\widetilde{dw(z,t))}}\} \|w\|_z.$$

PROOF. The proof also proceeds along quite similar lines to the proof of Theorem 4.1, though the situation here is considerably simpler. By the transversality of $ker(w)$ to the axes we have that $w = c(dz^2 - bdz^1)$ for some C^2 functions b, c and, without loss of generality, we assume that $1 \leq |c| \leq \|\eta\|$. We assume that the neighborhood $\mathcal{U}$ is parameterized such that every point $z \in \mathcal{U}$ corresponds to a time t_z such that the integral curve $\mathcal{X}$ of $X := \partial/\partial x^1 + b\partial/\partial x^2$ is so that $e^{-t_z X}(z) \in \mathcal{L}$. For every $z \in \mathcal{U}$ let

$$\beta(z) := \exp \left(- \int_0^{t_z} \frac{\partial b}{\partial x^2} (e^{(\tau - t_z)X}(z)) d\tau \right).$$

We can now define the form

$$\hat{w} := \beta(dz^2 - bdz^1).$$

Then, by definition of $\hat{w}$ we have $\ker(\hat{w}) = \ker(w)$ and for $z \in \mathcal{L}$ we have $t_z = 0$ which implies that the integral above vanishes and so $\beta(z) = 1$ and so $\hat{w}(\partial/\partial z^2) =$

1. Therefore we just need to show that $\hat{w}$ is C^1 and satisfies the required bounds. From the form of w and $\hat{w}$ we have

$$\|\hat{w}\|_z \leq |\beta(z)|\|dz^2 - bdz^1\|_z = |\beta(z)|\frac{\|w\|_z}{c(z)} = \frac{\beta(z)}{c(z)}\|w\|_z. \tag{4.1.7.2}$$

It is enough therefore to bound $\beta(z)/c(z)$. Notice first that, as can be verified in a straightforward way, the function β is the unique solution of the partial differential equation

$$X(u) = -u\frac{\partial b}{\partial z^2}$$

with boundary conditions $u = 1$ on $\mathcal{L}$. By the same arguments as in the proof of Proposition 4.5, we have that β is C^1 and, by direct calculation,

$$d\hat{w} = \left(X(\beta) + \beta\frac{\partial b}{\partial z^2}\right) dz^1 \wedge dz^2 = 0.$$

Thus $\hat{w}$ is closed. Finally, to estimate the norm of $\hat{w}$, again by direct calculation, we have

$$dw = \left(X(c) + c\frac{\partial b}{\partial z^2}\right) dz^1 \wedge dz^2.$$

By a slight abuse of notation again we write $dw = X(c) + c\partial b/\partial z^2$. Then, dividing through by c, we have

$$\frac{X(c)}{c} + \frac{\partial b}{\partial z^2} = \frac{dw}{c}.$$

Hence for every $z \in \mathcal{U}$ and $\tau \in [0, t_z]$ we have

$$\frac{X(c)}{c} \circ e^{(\tau - t_z)X}(z) + \frac{\partial b}{\partial z^2} \circ e^{(\tau - t_z)X}(z) = \frac{dw}{c} \circ \circ e^{(\tau - t_z)X}(z).$$

Integrating this equality along an integral curve of X we get

$$\log\left|\frac{c(z)}{c \circ e^{-t_z X}(z))}\right| - \int_0^{t_z} \frac{dw}{c} \circ e^{(\tau - t_z)X}(z)d\tau = -\int_0^{t_z} \frac{\partial b}{\partial x^2} \circ e^{(\tau - t_z)X}(z)d\tau$$

and hence, taking exponentials,

$$\begin{aligned}\exp\left(-\int_0^{t_z} \frac{\partial b}{\partial x^2} \circ e^{(\tau - t_z)X}(z)d\tau\right) &= \left|\frac{c(z)}{c \circ e^{-t_z X}(z))}\right| \exp\left(-\int_0^{t_z} \frac{dw}{c} \circ e^{(\tau - t_z)X}(z)d\tau\right) \\ &= \left|\frac{c(z)}{c \circ e^{-t_z X}(z))}\right| \exp\left(\int_0^{-t_z} \frac{dw}{c} \circ e^{\tau X}(z)d\tau\right)\end{aligned}$$

Finally, using that $1 \leq |c| \leq \|\eta\|$, this gives

$$\beta(z) \leq c(z)\exp\left(-\int_0^{t_z} \frac{dw}{c} \circ e^{(\tau - t_z)X}(z)d\tau\right) \leq c(z)\exp(\widetilde{dw}(z, -t_z)).$$

Substituting into (4.1.7.2) gives the result. ∎

Proof of Proposition 4.12. We suppose by contradiction that the vector field X admits two integral curves $\mathcal{X}^1$ and $\mathcal{X}^2$ through a given point $z_0 \in \mathcal{S}$ parameterized so that $\mathcal{X}^1(0) = \mathcal{X}^2(0) = z_0$. Now let $(z^1, z^2, \mathcal{U})$ be the local coordinate system around z_0 given by the assumptions of the Theorem. In particular X is transverse to both coordinate axes, and in particular to $\partial/\partial z^2$ and therefore there exists an integral curve $\mathcal{L}$ of $\partial/\partial x^2$ which joins two points $z_1 = \mathcal{X}^1(s_1)$ and $z_2 = \mathcal{X}^2(s_2)$ for some s_1, s_2. We can suppose that $\mathcal{L}$ is parameterized such that $\mathcal{L}(0) = z_1$ and $\mathcal{L}(t_2) = z_2$. Let Γ be the closed curve given by union of $\mathcal{L}$ and the two integral curves of X through z_0, let D be the region bounded by Γ, and let $\hat{w}_k$ be the sequence of 1-forms given by Lemma 4.13. By Stokes' formula we have

$$\int_\Gamma \hat{w}_k = \int_D d\hat{w}_k = 0 \tag{4.1.7.3}$$

and therefore

$$\int_\Gamma \hat{w}_k = \int_{\mathcal{X}^1} \hat{w}_k + \int_{\mathcal{X}^2} \hat{w}_k + \int_{\mathcal{L}} \hat{w}_k = 0$$

and so

$$\left| \int_{\mathcal{X}^1} \hat{w}_k + \int_{\mathcal{X}^2} \hat{w}_k \right| = \left| \int_{\mathcal{L}} \hat{w}_k \right|. \tag{4.1.7.4}$$

By Lemma 4.13 we have $\hat{w}_k(\partial/\partial z^2) = 1$ along $\mathcal{L}$ and therefore the right hand side of (4.1.7.4) is equal to $|t_2|$ where t_2 is the "distance" between z_1 and z_2 along $\mathcal{L}$. By assumption $t_2 \neq 0$ (and is independent of k) but we will show that the left hand side of (4.1.7.4) tends to 0 as $k \to \infty$, thus giving rise to a contradiction as required.

To estimate the left hand side of (4.1.7.4), notice that since the curves $\mathcal{X}^i$ are tangent to X we have

$$\int_{\mathcal{X}^i} \hat{w}_k = \int_0^{s_i} \hat{w}_k(X)(\mathcal{X}^i(t))dt$$

Also since $\hat{w}_k(X_k) = 0$ (since $X_k = ker(\hat{w}_k)$) we can write

$$\int_{\mathcal{X}^i} \hat{w}_k = \int_0^{s_i} \hat{w}_k(X - X_k)(\mathcal{X}^i(t))dt$$

Now let $|t_i| \leq s_i$ be such that

$$\int_0^{s_i} \hat{w}_k(X - X_k)(\mathcal{X}^i(t))dt = s_i \hat{w}_k(X - X_k)(\mathcal{X}^i(t_i)).$$

Then, letting $y = \mathcal{X}^i(t_i)$ we have

$$\begin{aligned} \int_{\mathcal{X}^i} \hat{w}_k &= \int_0^{s_i} \hat{w}_k(X - X_k)(\mathcal{X}^i(t))dt \\ &\leq s_i \|\hat{w}_k\|_y \|X - X_k\|_y \leq s_i \|\hat{w}_k\|_y \|w - w_k\|_y \end{aligned} \tag{4.1.7.5}$$

By Lemma 4.13 we have

$$\|\hat{w}_k\|_y \leq \sup_{|t|\leq t_0} \{e^{\widetilde{dw(y,t))}}\}\|w\|_y$$

Substituting this into (4.1.7.5) and applying the assumptions of the Theorem we get that $\int_{\mathcal{X}^i} \hat{w}_k \to 0$ for $i = 1, 2$ and, as explained above, this leads to a contradiction and thus completes the proof. □

4.2. Applications

In this section we prove Theorem 1.20 on the uniqueness of solutions for ODE's, and Theorem 1.22 on the existence and uniqueness of solutions for a class of Pfaff PDE's.

4.2.1. Uniqueness of solutions for continuous ODE's.

To prove Theorem 1.20 we fix once and for all some reference point $x_0 \in \mathcal{U}$ and choose a sufficiently small ball $B(x_0, r) \subset \mathcal{U}$. We will show that the vector field X is uniquely integrable at each point $x \in B(x_0, r)$. Notice first of all that, since $f(x) = (f_1(x), f_2(x))$ is non-vanishing in $\mathcal{U}$, we can assume, without loss of generality and passing to a smaller radius r if necessary, that there exist local coordinates where $f_1(x) > 0$ for all $x \in B(x_0, r)$. Then we can rescale the vector field X by dividing through by f_1 and define

$$Y := \frac{\partial}{\partial x_1} + g(x)\frac{\partial}{\partial x_2} \quad \text{where} \quad g(x) = \frac{f_2}{f_1}(x)$$

Note that g has the same modulus of continuity $\omega(t)$ as f and unique integrability of Y is equivalent to that of X since the two vector fields are just rescalings one of the other (and thus define the same one-dimensional distributions). It is therefore sufficient to show that Y is uniquely integrable. To do this we first show that we can approximate the function g be a family of smooth functions g^ϵ satisfying certain approximation bounds, and thus approximate the continuous vector field Y by corresponding smooth vector fields Y^ϵ. We then apply Proposition 4.14 to get unique integrability of Y.

For any small $\epsilon > 0$ let $V_{x_0,\epsilon}$ denote the Riemannian volume of the ball $B(x_0, \epsilon)$. Then, letting $w(t)$ denote the modulus of continuity of the function g above, we have the following result.

PROPOSITION 4.14. *There exists a family of smooth functions $\{g^\epsilon\}_{\epsilon>0}$ defined on $B(x_0, r)$ and a constant $K > 0$ such that for every $\epsilon > 0$ we have*

$$|g^\epsilon - g|_\infty \leq \frac{K}{\epsilon}\int_0^\epsilon \omega(t)dt \quad \textit{and} \quad \left|\frac{\partial g^\epsilon}{\partial x}\right| \leq \frac{K}{\epsilon^2}\int_0^\epsilon \omega(t)dt$$

Proof of Theorem 1.20 assuming Proposition 4.14. Notice first of all that $Y = ker(w)$, where w is the continuous 1-form $w := dx - g(x)dt$. Then there exists a family of approximating vector fields Y^ϵ given by the smooth 1-forms $w^\epsilon := dx - g^\epsilon(x)dt$ where the smooth functions g^ϵ are given by Proposition 4.14. Notice that since w^ϵ are smooth they admit exterior derivatives dw^ϵ and $|dw^\epsilon|_\infty = |\partial g/\partial x|_\infty$. Then from Proposition 4.14 we get

$$|w^\epsilon - w|_\infty e^{\epsilon|dw^\epsilon|_\infty} \leq \frac{K}{\epsilon}\int_0^\epsilon \omega(t)dt \exp\left(\frac{K}{\epsilon}\int_0^\epsilon \omega(t)dt\right)$$

and therefore, from the assumptions of Theorem 1.20 we have that

$$|w^\epsilon - w|_\infty e^{\epsilon|dw^\epsilon|_\infty} \to 0$$

as $\epsilon \to 0$. Thus the assumptions of Proposition 4.12 are satisfied (note that in Proposition 4.12, for generality the bound is stated in terms of $\widetilde{dw_k}(z,t)$, but since $|\widetilde{dw_k}(z,t)| \leq t|dw_k|_\infty$ the above condition is sufficient) and we get the unique integrability of Y. □

Proof of Proposition 4.14. We will use some fairly standard approximations by mollifiers. In particular the calculations below follow closely the ones in [**70**] but are formulated in terms of modulus of continuity rather than the Hölder norm. Let $r \in (0,1)$ and ϕ be at the standard mollifier supported on $B(x_0, r/2)$ and, for every $\epsilon \in (0,1)$, let

$$\phi_\epsilon := \frac{1}{V_{x_0,\epsilon r/2}}\phi\left(\frac{x}{\epsilon}\right). \quad \text{and} \quad g^\epsilon(x) := \int_{B(x_0,\epsilon)} \phi_\epsilon(y)g(x-y)dy.$$

By the well-known properties of mollifiers we have that ϕ_ϵ is supported on $B(x_0,\epsilon)$, $\int \phi_\epsilon = 1$, $|\phi_\epsilon|_\infty \leq 1/V_{x_0,\epsilon r/2}$,

$$\int_{B(x_0,\epsilon)} \frac{\partial \phi_\epsilon}{\partial x} = 0, \quad \text{and} \quad \frac{\partial \phi_\epsilon}{\partial x}(x) = \frac{1}{\epsilon V_{x_0,\epsilon r/2}}\frac{\partial \phi}{\partial x}(\frac{x}{\epsilon}).$$

Therefore, using these properties

$$|g^\epsilon(x) - g(x)| \leq \int_{B(0,\epsilon)} |\phi_\epsilon(y)||g(x-y) - g(x)|dy \leq |\phi_\epsilon|_\infty \int_{B(x_0,\epsilon)} \omega(|y|)dy.$$

Passing to polar coordinates (r,θ) and noting that $|y| = r$ and that the volume form in polar coordinates has the form $dV = rdrd\theta$, we get

$$|g^\epsilon(x) - g(x)| \leq |\phi_\epsilon|_\infty \int_{B(x_0,\epsilon)} \omega(|y|)dy \leq \frac{2\pi}{V_{x_0,\epsilon r/2}}\int_0^\epsilon t\omega(t)dt$$
$$\lesssim \frac{1}{\epsilon^2}\int_0^\epsilon t\omega(t)dt \leq \frac{1}{\epsilon}\int_0^\epsilon \omega(t)dt$$

which proves the first claim. Similarly

$$\begin{aligned}
\left|\frac{\partial g^\epsilon}{\partial x}(x)\right| &= \left|\int_{B(x_0,\epsilon)} \frac{\partial \phi_\epsilon}{\partial x}(y) g(x-y)dy\right| \\
&= |\int_{B(x_0,\epsilon)} \frac{\partial \phi_\epsilon}{\partial x}(y)(g(x-y)-g(x))dy| \\
&\leq \int_{B(x_0,\epsilon)} \left|\frac{\partial \phi_\epsilon}{\partial x}(y)\right| \omega(|y|)dy \\
&= |d\phi_\epsilon|_\infty \int_{B(x_0,\epsilon)} |\omega(|y|)dy \\
&= |d\phi|_\infty \frac{2\pi}{\epsilon V_{x_0,\epsilon r/2}} \int_0^\epsilon t\omega(t)dt \\
&\lesssim \frac{1}{\epsilon^3}\int_0^\epsilon t\omega(t)dt \leq \frac{1}{\epsilon^2}\int_0^\epsilon \omega(t)dt
\end{aligned}$$

where last line is again achieved by passing to polar coordinates This completes the proof of the proposition. □

4.2.2. Pfaff's Problem. The proof of Theorem 1.22 is based on the observation that notion of integrability for the PDE $(\mathcal{P})$ is closely related to the geometric integrability of distributions in the following way. Let Δ be the two-dimensional distribution in $\mathcal{U}$ spanned by the local frame $\{X, Y\}$ with

$$X := \frac{\partial}{\partial x} + a\frac{\partial}{\partial z} \quad \text{and} \quad Y := \frac{\partial}{\partial y} + b\frac{\partial}{\partial z} \tag{4.2.2.1}$$

where $a(x,y,z), b(x,y,z)$ are the functions in $(\mathcal{P})$.

PROPOSITION 4.15. *$(\mathcal{P})$ is (uniquely) integrable iff Δ is (uniquely) integrable.*

PROOF. Suppose first that $(\mathcal{P})$ is integrable. Let $(x_0, y_0) \in \mathcal{V}$ and let f be a solution of $(\mathcal{P})$ with initial condition $f(x_0, y_0) = z_0$. Then the graph

$$\Gamma(f) = \{(x, y, f(x, y)), (x, y) \in \mathcal{V}\}$$

is an embedded surface in $\mathbb{R}^3$ and the tangent space at a point $(x, y, f(x, y))$ is spanned by

$$\frac{\partial}{\partial x} + \frac{\partial f}{\partial x}\frac{\partial}{\partial z} \quad \text{and} \quad \frac{\partial}{\partial y} + \frac{\partial f}{\partial y}\frac{\partial}{\partial z}. \tag{4.2.2.2}$$

Since f is a solution of $(\mathcal{P})$, then (4.2.2.2) is exactly of the form (4.2.2.1) and so the graph $\Gamma(f)$ is a (unique) integral manifold of Δ. Conversely, suppose that the distribution Δ with local frame (4.2.2.1) is (uniquely) integrable. Then the integral manifold of Δ through a point $(x_0, y_0, z_0) \in \mathcal{U}$ can be realized as a graph of a (unique) function $f = f(x, y)$. The tangent spaces of this graph are by definition given by the span of (4.2.2.1) but also of (4.2.2.2) and thus f is a solution of $(\mathcal{P})$ which is therefore (uniquely) integrable. □

PROOF OF THEOREM 1.22. We translate the problem into the geometric problem of the integrability of the corresponding distribution Δ spanned by vector fields of the form (4.2.2.1). Then Δ can be written as the kernel $\Delta = ker(\eta)$ of the 1-form

$$\eta = dz - adx - bdy.$$

We will construct a sequence η_k of C^1 with the property that $\eta_k \to \eta$, $\|d\eta_k\|$ uniformly bounded in k, and $\|\eta_k \wedge d\eta_k\| \to 0$. This implies that Δ is uniformly asymptotically involute and is thus uniquely integrable by Theorem 1.16. Let $F^{(k)}(z)$ be a sequence of C^1 functions such that $F^{(k)} \to F$ in the C^0 topology and such that the derivative $F_z^{(k)}$ is uniformly bounded in k (this is possible because F is Lipschitz continuous). Now let

$$\eta_k := dz - A^{(k)}F^{(k)}dx - B^{(k)}F^{(k)}dy.$$

Then η is a C^1 form and clearly $\eta_k \to \eta$. Moreover, by direct calculation, we have

$$\begin{aligned} d\eta_k &= -d(A^{(k)}F^{(k)}) \wedge dx - d(B^{(k)}F^{(k)}) \wedge dy \\ &= -A_y^{(k)}F^{(k)}dy \wedge dx - A^{(k)}F_z^{(k)}dz \wedge dx - B_x^{(k)}F^{(k)}dx \wedge dy - B^{(k)}F_z^{(k)}dz \wedge dy \\ &= (A_y^{(k)} - B_x^{(k)})F^{(k)}dx \wedge dy + A^{(k)}F_z^{(k)}dx \wedge dz + B^{(k)}F_z^{(k)}dy \wedge dz. \end{aligned}$$

The functions $A^{(k)}, B^{(k)}, F^{(k)}$ are converging to the corresponding functions A, B, F and are therefore uniformly bounded, the functions $F_z^{(k)}$ are uniformly bounded by construction, and $A_y^{(k)} - B_x^{(k)} \to 0$ by assumption. It follows that $\|d\eta_k\|$ is uniformly bounded in k. Finally, again by direct calculation we have

$$\begin{aligned} \eta_k \wedge d\eta_k =&[(A_y^{(k)} - B_x^{(k)})F^{(k)} - A^{(k)}F^{(k)}B^{(k)}F_z^{(k)} + B^{(k)}F^{(k)}A^{(k)}F_z^{(k)}]dx \wedge dy \wedge dz \\ =&(A_y^{(k)} - B_x^{(k)})F^{(k)}dx \wedge dy \wedge dz. \end{aligned}$$

Therefore $\|\eta_k \wedge d\eta_k\| \to 0$ and this completes the proof. □

CHAPTER 5

Future Directions

In this chapter we state two partial results of still ongoing work. The first one is an extension of results in chapter 4 to any dimension and codimension. The second one is related to interactions between sub-Riemannian geometry and integrability. It encompasses a generalization of the integrability theorem found in [**22**] in an attempt to answer a question of continuous sub-Riemannian geometry found in [**71**]. The third one is an attempt to give a geometric sufficient condition that implies the sufficient but analytic conditions given in the continuous Frobenius theorem 5.2 stated in previous chapters. In particular we show a connection to existence of contact structures whose Reeb vector-fields satisfy certain properties. The final part is about some relations between sub-Riemannian geometry and Hartman's continuous exterior differentiability.

The first part about the generalization of the Frobenius theorem is almost complete so we present it in full detail. For the remaining we prefer to present everything here without proofs as they are ongoing work at the moment of writing the thesis. The interested reader can consult the authors for details.

5.1. A Continuous Frobenius Theorem in Higher Dimensions

The work in this section is a work in progress and is a generalization of the main theorem in chapter 4. Where as the methods employed in chapter 4 were analytic and required PDE analysis, the methods employed here are much more geometric and the conditions derived are simpler to state. As in the previous chapter, we will present the formulation in two steps: We'll first formulate a more natural and easier to understand version of the theorem in section 5.1.2 and then a more technical version in section 5.1.3 for which it is much more straight forward to apply in dynamical systems.

5.1.1. The Setting. Unique integrability of E on M is equivalent to unique integrability of E on every local chart. Therefore from now on we replace M with a coordinate neighbourhood $U \subset \mathbb{R}^{n+m}$, equipped with the usual Euclidean inner product and metric which we respectively denote as $(\cdot,\cdot)$, $|\cdot|$. We use the same notation for the inner product and norm induced on differential k-forms and tangent vectors. Given a continuous scalar valued function f_p defined on U, we will denote by f_∞ the supremum over $p \in U$ and by $f_{\inf}$ infimum over $p \in U$ (main example for this is the norm $|\cdot|_p$ of objects defined on U). For product of two such functions we then will have $(f.g)_\infty = \sup_{p \in U} f(p)g(p)$.

Let $\eta(z) = \sum_{i=1}^{n+m} a_i(z)dz^i$ be a continuous differential 1-form on U which is a continuous map from U to T^*U. Then a sequence of differential 1-forms η^k are said to converge to η uniformly if $|\eta^k - \eta|_\infty \to 0$ as $k \to \infty$. If η is a C^1 differential form, then the exterior derivative $d\eta$ is the 2-form which in components is $d\eta = \sum_{i,j=1}^{n+m} \frac{1}{2}(\frac{\partial a_i}{\partial z^j} - \frac{\partial a_j}{\partial z^i})dz^i \wedge dz^j$. We denote the normalized differential 1-forms as $\hat{\eta} = \frac{\eta}{|\eta|}$.

The kernel of a differential 1-form η, denoted as $ker(\eta)$, is defined to be the collection of all vectors $v \in T_xU$ for all $x \in U$ such that $\eta(v) = 0$. By a common kernel E of a collection of differential 1-forms $\{\eta_i\}_{i=1}^n$ we mean $E = \cap_{i=1}^n ker(\eta_i)$. Such a collection is said to be linearly independent if at each point p, these 1-forms span a n dimensional space over reals (which in particular means they are everywhere non-vanishing). Note that if these differential 1-forms are linearly independent at each point, then their common kernel is a m dimensional bundle inside TU which has the same regularity as the differential forms themselves. We call a collection of differential 1-forms $\{\eta_i\}_{i=1}^n$ weakly involutive if $E = \cap_{i=1}^n ker(\eta_i)$ is integrable and involutive if it is uniquely integrable.

As mentioned before, we will state two main theorems. The theorem (5.2) has the advantage that it is much simpler and the analogy to the classical Frobenius theorem is very clear. Therefore for the first time reader it is much more convenient to focus on this theorem. The second theorem (5.5), which is the one we will actually prove and which implies the first theorem, is more technical yet is much convenient for applications in such areas as integrability of invariant splittings in dynamical systems. Theorem 5.5 will be stated in a subsection following some discussions about theorem 5.2. The proof that follows will be the proof of theorem 5.5 and after the proof we will comment on some applications to integrability of continuous sub-bundles that arise in dynamical systems, to PDE's and to ODE's (which are basically straightforward generalizations of the ones presented in chapter 4).

5.1.2. Type I Asymptotic Involutivity.

Definition 5.1. Let $\{\eta_i\}_{i=1}^n$ be a collection of linearly independent differential 1-forms. Let η_i^k be a sequence of approximations such that $\max_i |\eta_i^k - \eta_i|_\infty \to 0$ as $k \to \infty$. Then $\{\eta_i\}_{i=1}^n$ is said to be *type I asymptotically involutive* if there exists a constant ϵ_0 such that

$$e^{\epsilon_0 \max_i |d\eta_i^k|_\infty} \max_j |\eta_1^k \wedge ...\eta_n^k \wedge d\eta_j^k|_\infty \to 0 \quad \text{as } k \to \infty$$

and is said to be a *type I exterior regular* if there exists an ϵ_0 such that

$$e^{\epsilon_0 \max_i |d\eta_i^k|_\infty} \max_j |\eta_j^k - \eta_j|_\infty \to 0 \quad \text{as } k \to \infty$$

Then the main result is:

THEOREM 5.2. *Let $\{\eta_i\}_{i=1}^n$ be a collection of linearly independent differential 1-forms. If this collection is type I asymptotically involutive then it is integrable. If it is also type I exterior regular then it is uniquely integrable.*

One might wonder how the "exterior regularity" condition is related to being Lipschitz or differentiable. An easy example which shows that they are not directly related is given by setting $\eta = df$ where f is a function who is only differentiable once. Then as shown in [**42**] the mollifications η_ϵ all satisfy $d\eta_\epsilon = 0$ where as η can be as "bad" as required in terms of regularity. For less trivial examples of non-regular but integrable distributions see the section 5.1.6 about applications.

For comparison, we recall that the classical Frobenius Theorem [**35**] dating back to 1877 yields unique integrability for C^1 distributions under the assumption that E is *involutive*, i.e.

$$\eta_1 \wedge ... \wedge \eta_n \wedge d\eta_i = 0 \text{ for all i} \tag{5.1.2.1}$$

This can be seen as a special case of our main Theorem by choosing $\eta_i^k \equiv \eta_i$. One interesting generalization of this theorem is theorem of Hartman [**42**]. This theorem gives a necessary and sufficient condition for existence of solutions with certain regularity properties (i.e the integral manifolds locally form a C^1 family of C^1 submanifolds) . More precisely Hartman introduces a notion of continuous exterior differential. Then he assumes that the differential system $\{\eta_i\}_{i=1}^n$ possesses (up to taking linear combinations with continuous coefficients) *continuous exterior derivatives* [1] $d\eta_i$ for which again condition (5.1.2.1) is required to hold. However the regularity condition obtained on the solutions is restrictive in situations such as continuous bundles that arise in dynamical systems that are known to exhibit pathological behaviour in terms of regularity (see for instance [**43**]). Therefore in this perspective, our theorem not only introduces conditions which are in principle much more easier to check but also has increased range of applicability.

5.1.3. Type II Asymptotic Involutivity.

To state the more generalized version of theorem 5.2, we need some more definitions and shorthand conventions.

[1] More precisely we say that η admits a continuous exterior derivative if there exists a differential 2-form $d\eta$ that satisfies Stokes' Formula: $\int_J \eta = \int_S d\eta$ for every piece of C^1 surface S bounded by a C^1 piecewise Jordan curve J. Hartman shows that such a continuous exterior derivative exists if and only if there exists a sequence of differential 1-forms η^k such that η^k converges to η uniformly and $d\eta^k$ converges to some continuous differential 2-form.

As above let $\{\eta_i^k\}_{k=1}^{\infty}$, $i = 1, ..., n$, be a sequence of collection of everywhere non-vanishing linearly independent differential 1-forms, E^k be their common kernel. We remind that $\hat{\eta}_i^k$ indicates the normalized version of η_i^k.

DEFINITION 5.3. For $p \in U$ we denote

$$|\hat{\eta}^k \wedge d\hat{\eta}^k|_p = \max_i |\hat{\eta}_1^k \wedge ... \wedge \hat{\eta}_n^k \wedge d\hat{\eta}_i^k|_p \tag{5.1.3.1}$$

$$|\hat{\eta}^k - \eta|_p = \max_i |\hat{\eta}_i^k - \eta_i|_p \tag{5.1.3.2}$$

$$|\eta^k|_p = \max_i |\eta_i^k|_p \quad , \quad m(\eta^k)_p = \min_i |\eta_i^k|_p \tag{5.1.3.3}$$

$$|d\eta_i^k(E^k, \cdot)|_p = \max_{X \in E_p^k, Y \in T_pM, |X|=1, |Y|=1} |d\eta_i^k(X, Y)|_p \tag{5.1.3.4}$$

$$\overline{d\eta}^k = \max_{i, p \in M} \frac{|d\eta_i^k(E^k, \cdot)|_p}{m(\eta^k)_p} \tag{5.1.3.5}$$

DEFINITION 5.4. Let $\{\eta_i\}_{i=1}^n$ be a collection of everywhere linearly independent differential 1-forms defined on $U \subset \mathbb{R}^{n+m}$. Let η_i^k be a sequence of families of differential 1-forms such that for all i one has $|\hat{\eta}_i^k - \eta_i|_\infty \to 0$ for all i as $k \to \infty$. Then $\{\eta_i\}_{i=1}^n$ is said to be *type II asymptotically involutive* if there exists a constant ϵ_0 such that

$$\frac{(|\hat{\eta}^k \wedge d\hat{\eta}^k| \ |\eta^k|)_\infty}{m(\eta^k)_{\inf}} e^{\epsilon_0 \overline{d\eta}^k} \to 0 \quad \text{as } k \to \infty$$

and is said to be a *type II exterior regular* if there exists an ϵ_0 such that

$$\frac{(|\hat{\eta}^k - \eta| \ |\eta^k|)_\infty}{m(\eta^k)_{\inf}} e^{\epsilon_0 \overline{d\eta}^k} \to 0 \quad \text{as } k \to \infty$$

Then the general version of our theorem is

THEOREM 5.5. *Let $\{\eta_i\}_{i=1}^n$ be a collection of linearly independent differential forms. If this collection is type II asymptotically involutive then it is integrable. If it is also type II exterior regular then it is uniquely integrable.*

It is easy to check that respectively type I asymptotic involutivity and exterior regularity implies their type II versions. Since from now on we are going to concentrate on the latter case, we drop type II in the naming.

Remark 5.6. The main differences between conditions of theorem 5.2 and 5.5 are that in the former one the approximations η_i^k converge as differential 1-forms while in the latter η_i^k converge only in angle and in the latter one does not require a bound on $|d\eta^k|_p$ but rather on some of its components. We will see later that this distinction is important to be able to apply this theorem in a straightforward manner for dynamical systems.

5.1.4. Proof of the Theorem. The proof of the theorem can be divided, as usual, into two parts: 1- The existence of integral manifolds of E through every point x and then 2- the uniqueness property of the integral manifolds through every point. The proof of existence already carries much of the flavor of the approach and is complete. The proof of the second part is also complete however not written in a reader friendly way, yet. So in this partial presentation we will only show the existence part.

It can be shortly summarized as follows: The condition of asymptotic involutivity in definition 5.4 will be used in subsection 5.1.5 to prove the existence of integral manifolds (for distributions which have dimension greater than 1, since 1 dimensional distributions are always integrable) while condition of exterior regularity can be used to prove uniqueness.

The existence proof in section 5.1.5 is carried out by constructing certain convenient basis $\{X_i^k\}$ of $E^k = \cap_{i=1}^n ker(\eta_i^k)$, and then integrating them in order to get certain surfaces W^k. Asymptotic involutivity guarantees that these surfaces have a convergent subsequence which becomes increasingly more tangent to E as $k \to \infty$.

The uniqueness proof can be carried out by showing that E is spanned by vector fields X_i which are uniquely integrable. To obtain this exterior regularity is used to create a contradiction to existence of two distinct solution curves of any X_i that passes through the same point. This will then easily imply the uniqueness of local integral manifolds for E.

5.1.5. Existence of Integral Manifolds. Through out this section we assume that the dimension of the distribution is greater than 1 since 1 dimensional distributions can always be integrated due to Peano's theorem. Given a vector field X on U we denote by $e^{tX}(p)$ its flow starting at p and De_p^{tX} denotes the differential of the flow with respect to coordinates evaluated at the point p, which is a map from T_pU to $T_{e^{tX}(p)}U$. The main proposition of this section, which implies the existence of the integral manifolds, is the following:

PROPOSITION 5.7. *There exist some $V \subset U$, $\epsilon > 0$ and a basis $\{X_i^k\}_{i=1}^m$ of E^k such that defining $W_x^k : (-\epsilon, \epsilon)^m \to U$,*

$$W_x^k(t_1, ..., t_m) = e^{t_m X_m^k} \circ ... \circ e^{t_1 X_1^k}(x)$$

the images $\mathcal{W}_x^k$ of W_x^k for all $x \in V$ and $|t_i| \leq \epsilon$ are embedded, m dimensional, C^1 submanifolds which converge (up to subsequence) to an embedded, C^1 submanifold $\mathcal{W}_x$ which is an m dimensional integral manifold of E.

This proposition depends on two other important propositions. The first one is a general proposition that does not depend on the choice the basis $\{X_i^k\}_{i=1}^m$. Given such a basis we denote

$$L_p^k = \text{span}_{i,j}\{[X_i, X_j]_p\}$$

where $[\cdot, \cdot]$ denotes lie brackets of vector fields.

PROPOSITION 5.8. *Let $\{X_i^k\}_{i=1}^m$ be basis of sections for E^k. Let $\epsilon > 0$ and $V \subset U$ be an open neighbourhood such that the maps*

$$W_p^k(t) = e^{t_m X_m^k} \circ ... \circ e^{t_1 X_1^k}(p)$$

(which we shortly denote as $p(t)$) are defined and differentiable for all $|t_i| \leq \epsilon$ and $p \in V$ (where $t = (t_1, ..., t_m)$). Then defining

$$\tilde{X}_i^k(t) = \frac{\partial W_x^k}{\partial p_i}(t) \in T_{p(t)}U$$

there exists a constant K_1 such that

$$\begin{aligned} |\tilde{X}_i^k(t) - X_i^k(p(t))|_{p(t)} \leq \\ K_1 \sup_{q \in V,\ |t_j| \leq \epsilon} \{\ max_{i,j}|[X_i^k, X_j^k]|_q \ \|De^{t_m X_m} \circ ... \circ De_q^{t_1 X_1}|_{L_q^k}\|_{q(t)}\ \} \end{aligned} \tag{5.1.5.1}$$

Then the next proposition says that one is able to choose the basis $\{X_i^k\}_{i=1}^m$ so that the quantity on the right hand side of the equation (5.1.5.1) goes to 0 and X_i^k converge to a basis of sections for E:

PROPOSITION 5.9. *There exists constants $\epsilon, K_1, K_2 > 0$, a neighbourhood $V \subset U$ and a sequence of linearly independent family of vector fields $\{X_i^k\}_k$ for $i = 1, 2, ..., m$ defined on V such that span$\{X_i^k\} = E^k$ and X_i^k converge uniformly to a linearly independent basis X_i of E with the additional properties that for all $t_1, ..., t_m$ such that $|t_j| \leq \epsilon$, for all i, for all $p \in V$ and for all $Y \in L_p^X$ of unit norm:*

$$\|De^{t_m X_m^k} \circ \ldots \circ De_p^{t_1 X_1^k}\ Y\|_{p(t)} \tag{5.1.5.2}$$
$$\leq 1 + K_1 \frac{|\eta_i^k|_p}{m(\eta^k)_{p(t)}}\ e^{m\epsilon K_2 \overline{d\eta}^k}$$

and that

$$\max_{i,j} |[X_i^k, X_j^k]|_p\ (1 + K_1 \frac{|\eta^k|_p}{m(\eta^k)_{p(t)}}\ e^{m\epsilon K_2 \overline{d\eta}^k}) \to 0 \tag{5.1.5.3}$$

as $k \to \infty$.

We can now prove the main proposition assuming these two last ones:

Proof. (of proposition 5.7 assuming propositions 5.8 and 5.9)
By proposition 5.9 we have a local section $\{X_i^k\}_{i=1}^m$ of E^k such that $X_i^k \to X_i$ where $\{X_i\}_{i=1}^m$ is a basis of E. Moreover the estimates in equations (5.1.5.2), (5.1.5.3) of proposition 5.9 imply that the right hand side of equation (5.1.5.1) in proposition 5.8 goes to 0. This means that $|\tilde{X}_i^k(t) - X_i^k(p(t))|_{p(t)}$ goes to zero at every $p(t)$. Since $X_i^k \to X_i$, we get that $\tilde{X}_i^k(t) \to X_i(p(t))$. Therefore if one computes the derivative of the mappings $DW_x^k(t)$ then as a matrix (whose columns are $\tilde{X}_i^k$), it converges to a matrix M whose columns are X_i. This means that for k large enough $DW_x^k(t)$ is invertible for every x and t which in particular means that $\mathcal{W}_x^k$ are m dimensional manifolds (as an application of inverse function theorem). Since derivatives of $W_x^k(t)$ are uniformly bounded, $\{W_x^k\}_k$ form an equi-bounded and equi-Lipschitz family of functions. Therefore they possess a subsequence which converge to some function $W_x(t)$. Since derivatives $DW_x^k(t)$ also converge to M, we get that $W_x(t)$ is differentiable with derivative M which means that it is a m dimensional manifold whose tangent space at every point q coincides with E_q. □

Now it remains to prove propositions 5.8 and 5.9. In the first subsection 5.1.5.1 we construct the basis $\{X_i^k\}_{i=1}^m$ and prove proposition 5.9. In the second subsection 5.1.5.2 we prove proposition 5.8.

5.1.5.1. *Construction of the Vector Fields* X_i^k *and Proof of Proposition 5.9.* The main proposition that we are going to prove in this section is proposition 5.9. Although we need this proposition to prove theorem 5.5, if one settles for proving only the less general theorem 5.2, then one can work with the easier version of this proposition where the exponential terms in the inequalities are replaced by terms of the form $K_1 e^{\epsilon_0 K_2 |d\eta^k|_\infty}$. We will spread the proof of this proposition over the remaining of this subsection. To define the vector fields X_i^k, we will first bring $\hat{\eta}_i^k$ to a convenient form $\hat{\eta}_i^{0,k}$ by taking linear combinations.

LEMMA 5.10. *There exists coordinates $(x^1,...,x^m,y^1,...,y^n)$, a continuous family of invertible matrices $M(x,y)$ and a sequence of C^1 family of invertible matrices (for k large enough) $M^k(x,y)$ such that*

$$\eta_i^0 = \sum_j M_{ij}\eta_j = dy^i - a_i^1 dx^1 - ... - a_i^m dx^m$$

$$\hat{\eta}_i^{0,k} = \sum_j M_{ij}^k \hat{\eta}_j^k = dy^i - a_i^{k1} dx^1 - ... - a_i^{km} dx^m$$

$$\hat{\eta}_i^{0,k} \to \eta_i^0 \quad ask \to \infty \tag{5.1.5.4}$$

for some continuous functions $a^i(x,y)$ and sequence of C^1 functions $a_j^{ki}(x,y)$

PROOF. This is a not so hard consequence of the fact that the families η_i and $\hat{\eta}_i^k$ are linearly independent and that $\hat{\eta}_i^k \to \eta_i$. Indeed writing the components η_i as a $n\times(n+m)$, the linear independence of η^i means there will be an invertible $n\times n$ minor of this matrix (the coordinates corresponding to this minor will be the dy^i coordinates). The inverse of this matrix will be M. To find M^k one then repeats the same argument for η_i^k and since $\hat{\eta}_i^k \to \eta_i$, same minors will be invertible for k large enough which will give M^k. The fact that $\hat{\eta}_i^{0,k} \to \eta_i^0$ follows easily from the fact that $\hat{\eta}_i^k \to \eta_i$, since the entries of the matrices M and M^k are some fractional polynomials of components of η_i and $\hat{\eta}_i^k$. $\square$

Now we can define the vector fields as

$$X_i^k = \frac{\partial}{\partial x^i} + \sum_{j=1}^m a_j^{ki}\frac{\partial}{\partial y^j} \tag{5.1.5.5}$$

These vector fields are linearly independent and satisfy $X_i^k \to E$ since

$$\lim_{k\to\infty} X_i^k = X_i = \frac{\partial}{\partial x^i} + \sum_{j=1}^m a_j^i \frac{\partial}{\partial y^j} \tag{5.1.5.6}$$

so that X is in the common kernel of η_i's and therefore in E. Now we want to have some uniform bound on $|De_p^{tX_i^k}Y|$. Given a vector field X the usual well known theorems from theory of ODE state that

$$|De_p^{tX}|_\infty < K_1 e^{tK_2|X|_{C^1}}$$

for some constants K_1, K_2 which depends on $|X|$ and where $|\cdot|_{C^1}$ denotes the C^1 sup norm. But this estimate can be sharpened by use of differential forms, which is the main content of proposition 5.9. This proposition can be found in [**42**] in a different way. The proof given there however is very analytic in nature and depends heavily on ODE analysis. We will give a very geometric proof of this theorem using Stoke's theorem which will end up being more efficient for use in applications.

PROOF. (of equation (5.1.5.2)) To start with, fix some ϵ and $V \subset U$ such that for all $|t_j| \leq \epsilon$ and $p \in V$, $e^{t_m X_m^k} \circ ... \circ e^{t_1 X_1^k}(p) \in U$. V and ϵ depend only on $\max_i |X_i^k|_\infty$ which are uniformly bounded above, so ϵ and V can be chosen so as not to depend on k. Note that by the form of the vector fields X_i^k, one has that $[X_i^k, X_j^k]_p \in \text{span}_i\{\frac{\partial}{\partial y_i}\}$ for all $p \in V$ and k, i, j, therefore $L_p^k \subset \text{span}_i\{\frac{\partial}{\partial y_i}\}$. To estimate the effect of the differential $De^{t_i X_i^k} \circ ... \circ De_p^{t_1 X_1^k}$ on $Y \in \text{span}\{\frac{\partial}{\partial y_i}\}|_p$, we will also first consider the case where there is the differential of only a single vector field, that is when the differential is $De^{t_j X_j^k}$. Then we will obtain the general case as successive applications of this.

Pick $p \in V$ and $|t_j| < \epsilon$. Let $\gamma \subset V$ be a curve defined on $[0, \tilde{t}]$ for some $\tilde{t} > 0$ such that $\gamma(0) = p$, $\gamma'(0) = Y$ and $e^{t_j X_j^k}(\gamma) \subset U$. We choose $\tilde{t}$ small enough so that X_j^k is always transverse to γ. Denote $q = \gamma(\tilde{t})$. Define the parameterized surface with corners Γ by

$$r(s_1, s_2) = e^{s_2 X_j^k} \circ \gamma(s_1) \qquad \Gamma = im(r)$$

for $0 < s_1 \leq \tilde{t}$ and $0 < s_2 \leq t_j$. Then $\partial\Gamma$, the boundary, is composed of the curve γ and the following piecewise smooth curves:

$$\xi_1(s) = e^{s X_j^k}(p) \qquad \xi_2(s) = e^{s X_j^k}(q) \qquad \beta(s) = e^{t_j X_j^k} \circ \gamma(s)$$

Since $\eta_i^k(X_j^k) = 0$ for all i, j we get using Stoke's theorem that for all i

$$\int_\beta \eta_i^k - \int_\gamma \eta_i^k = \int_\Gamma d\eta_i^k$$

which gives

$$\begin{aligned} \int_0^{\tilde{t}} ds_1 \eta_i^k(\beta'(s_1)) = \int_0^{\tilde{t}} ds_1 \eta_i^k(\gamma'(s_1)) + \\ \int_0^{\tilde{t}} ds_1 \int_0^{t_j} ds_2 d\eta_i^k(\frac{\partial r}{\partial s_1}, \frac{\partial r}{\partial s_2})(s_1, s_2) \end{aligned} \tag{5.1.5.7}$$

Differentiating this equality with respect to $\tilde{t}$ at $\tilde{t} = 0$ one gets:

$$\eta_i^k(\beta'(0)) = \eta_i^k(\gamma'(0)) + \int_0^{t_j} ds_2 d\eta_i^k(\frac{\partial r}{\partial s_1}, \frac{\partial r}{\partial s_2})(0, s_2)$$

Using chain rule one gets

$$\beta'(0) = De_p^{tX_j^k} Y \qquad \gamma'(0) = Y(p)$$

and

$$\frac{\partial r}{\partial s_1}(0, s_2) = De_p^{s_2 X_j^k} Y \qquad \frac{\partial r}{\partial s_2}(0, s_2) = X_j^k(e^{s_2 X_j^k}(p))$$

one can write the equality (5.1.5.7) as

$$\eta_i^k(De_p^{tX_j^k} Y) = \eta_i^k(Y_p) + \int_0^{t_j} ds_2 d\eta_i^k(De_p^{s_2 X_j^k} Y, X_j^k)(0, s_2)$$

To obtain the proposition we will use Grönwall lemma but we need some preliminary estimates.

Let $\alpha_i^k = dx^i$ for $i \leq m$ and $\alpha_{m+i}^k = \eta_i^k$ for $i \leq n$. Then α_i spans T^*V at every point and the angles between $\alpha_i(p)$ are uniformly bounded away from 0 with respect to k and p. In particular if $\hat{\alpha}_i^k$ are their normalized forms then $|\hat{\alpha}_1^k \wedge ... \wedge \hat{\alpha}_{m+n}^k|_p \geq c > 0$ for all k and $p \in V$. Now there exists a constant C_1 (independent of k) so that for any vector $v \in T_y V$, there exists an ℓ such that $|\alpha_\ell(v)| \geq C_1 |v||\alpha_\ell|$. Therefore there exists a ℓ such that $|\alpha_\ell(De_p^{tX_j^k} Y)| \geq C_1 |De_p^{tX_j^k} Y|_{p(t)} |\alpha_\ell|_{p(t)}$. In our particular case, it is easy to check that since $Y \in \text{span}\{\frac{\partial}{\partial y^i}\}_i$ and X_i^k have the particular form given in equation (5.1.5.5) one has that $De_p^{tX_j^k} Y \in \text{span}\{\frac{\partial}{\partial y^i}\}_i$ therefore in fact α_ℓ is η_ℓ^k for some ℓ. Also $|\eta_i^k(Y)| \leq |\eta_i^k|_p |Y|$. Then

$$(|De_p^{tX_j^k} Y| \min_i |\eta_i^k|)_{p(t)} \leq \frac{1}{C_1} |\eta_i^k|_p |Y|_p + \frac{1}{C_1} \int_0^{t_j} ds_2 |d\eta_i^k(De_p^{s_2 X_j^k} Y, X_j^k)|_{p(s_2)}$$

To complete the proof note that

$$|d\eta_i^k(De_p^{s_2 X_j^k} Y, X_j^k)|_{p(s_2)} \leq |d\eta_i^k(E^k, \cdot)|_{p(s_2)} |De_p^{s_2 X_j^k}|_{p(s_2)} \frac{\min_i |\eta_i^k|_{p(s_2)}}{\min_i |\eta_i^k|_{p(s_2)}}$$

Therefore using Grönwall lemma one obtains that there exists some constants K_1, K_2 such that for all i,

$$(|De_p^{tX_j^k} Y| \min_i |\eta_i^k|)_{p(t)} \leq K_1 |Y|_p |\eta_i^k|_p e^{\epsilon K_2 \overline{d\eta}^k}$$

Then one obtains the following upper bound for $Y \in \text{span}\{\frac{\partial}{\partial y_i}\}|_p$ and for all i

$$|De_p^{tX_j^k} Y|_{p(t)} \leq K_1 \frac{|\eta_i^k|_p |Y|_p}{m(\eta^k)_{p(t)}} e^{|t_j| K_2 \overline{d\eta}^k} \tag{5.1.5.8}$$

Now to get the theorem for successive applications first of all write $p_m = e^{t_m X_m^k} \circ ... \circ e^{t_1 X_1^k}(p)$. Note that for all m, $De^{t_m X_m^k} \circ ... \circ De^{t_1 X_1^k} Y$ is always in the span of $\text{span}\{\frac{\partial}{\partial y_i}\}|_{p(t_m)}$. Then applying the previous theorem repeatedly we get that for all choices of indices $i_m, ..., i_1$.

$$|De^{t_m X_m^k} \circ ... \circ De^{t_1 X_1^k} Y|_{p_m} \leq \frac{|\eta_{i_m}^k|_{p_{m-1}} ... |\eta_{i_1}^k|_p}{m(\eta^k)_{p_m} m(\eta^k)_{p_{m-1}} ... m(\eta^k)_{p_1}} e^{m\epsilon K_2 \overline{d\eta}^k}$$

In particular now choose i_ℓ for $\ell > 1$ so that $|\eta_{i_\ell}^k|_{p_{\ell-1}} = m(\eta^k)_{p_{\ell-1}}$ which results in cancellations in the above estimate and we get

$$|De^{t_m X_m^k} \circ ... \circ De^{t_1 X_1^k} Y|_{p_m} \leq \frac{|\eta_{i_1}^k|_p}{m(\eta^k)_{p_m}} e^{m\epsilon K_2 \overline{d\eta}^k}$$

which gives the estimate for Y in span of $\text{span}\{\frac{\partial}{\partial y_i}\}|_p$

□

Now it only remains to prove the claim about the Lie brackets (equation (5.1.5.3)).

PROOF. (of equation (5.1.5.3)) To prove this the following lemma will be sufficient (see equation (5.1.3.1) for the notations used below)

LEMMA 5.11. *There exists a constant $K_3 > 0$ such that,*

$$\max_{i,j} |[X_i^k, X_j^k]|_p \leq K_3 |\hat{\eta} \wedge d\hat{\eta}^k|_p$$

PROOF. Note that $\hat{\eta}_i^k(X_j^k) = 0$ for all i, j so $d\hat{\eta}_\ell^k(X_i^k, X_j^k) = \hat{\eta}_\ell^k([X_i^k, X_j^k])$. By the form of the vector fields X_i^k in equation (5.1.5.5) one has that $[X_i^k, X_j^k] \in \text{span}\{\frac{\partial}{\partial y^i}\}_i$. Since $\hat{\eta}_i^k$ converge to $\hat{\eta}_i$ which are transverse to $dx^1, ..., dx^m$ one has that there exists a constant K and ℓ such that $|\hat{\eta}_\ell^k([X_i^k, X_j^k])|_p \geq |[X_i^k, X_j^k]|_p$. Then

$$|[X_i^k, X_j^k]|_p \leq |d\hat{\eta}_\ell^k(X_i^k, X_j^k)|_p$$

Let Y_i^k be such that $\hat{\eta}_i^k(Y_j^k) = \delta_{ij}$ so that

$$|d\hat{\eta}_\ell^k(X_i^k, X_j^k)|_p = |\hat{\eta}_1^k \wedge ... \wedge \hat{\eta}_n^k \wedge d\hat{\eta}_\ell^k(Y_1^k, ..., Y_n^k, X_i^k, X_j^k)|_p$$
$$\leq K_3 |\hat{\eta}_1^k \wedge ... \wedge \hat{\eta}_n^k \wedge d\hat{\eta}_\ell^k|_p$$

where K_3 does not depend on k since all the vector fields have uniformly bounded norm. Therefore

$$\max_{i,j} |[X_i^k, X_j^k]|_p \leq K_3 \max_\ell |\hat{\eta}_1^k \wedge ... \wedge \hat{\eta}_n^k \wedge d\hat{\eta}_\ell^k|_p$$

$$= |\hat{\eta}^k \wedge d\hat{\eta}^k|_p$$

□

Combining this lemma with the weak asymptotic involutivity condition given in 5.1 one gets that

$$K_1 \max_{i,j} |[X_i^k, X_j^k]|_p \frac{|\eta^k|_p}{m(\eta^k)_{p(t)}} e^{m\epsilon c_\infty^k K_2 \overline{d\eta}^k} \leq$$
$$K_1 \max_i |\hat{\eta} \wedge d\hat{\eta}^k|_p \frac{|\eta^k|_p}{m(\eta^k)_{p(t)}} e^{m\epsilon c_\infty^k K_2 \overline{d\eta}^k}$$

goes to zero if ϵ is chosen so that $m\epsilon K_2 < \epsilon_0$. □

This finishes the proof of proposition 5.9.

5.1.5.2. *Construction of the Approximating Surfaces W^k and Proof of 5.8.* Now our aim will be to first construct through every point $x \in U(x_0)$ an integral manifold $W(x)$ of E (not necessarily unique). To build these integral manifolds, we will use a certain sequence of approximations using the flows of vector fields in proposition 5.9. Note that since X_m^k are uniformly bounded they have a uniform integration domain and time. Therefore there exists $U \subset \mathbb{R}^{n+m}$ and ϵ such that the following map is well defined for all $|t_i| \leq \epsilon$ and $x \in U$:

$$W^k(t_1, ..., t_m, x) = e^{t_m X_m^k} \circ ... \circ e^{t_1 X_1^k}(x)$$

The main proposition of this subsection is:

Proposition 5.12. *There exists an ϵ such that for all $x \in U$, images of $W^k(t_1, ..., t_m, x)$ are C^1 surfaces and this sequence of surfaces has a convergent subsequence whose limit is a surface tangent to E.*

Writing

$$\frac{\partial W^k}{\partial t_i}|_t = \tilde{X}_i^k$$

we need to show then that $\tilde{X}_i^k$ converge to linearly independent vector fields in E. The next proposition which is the second main proposition of this section, proves this.

PROPOSITION 5.13. *There exists a constant $C > 0$ and ϵ small enough, such that for all $t \in [-\epsilon, \epsilon]^m$ and $y = W^k(t)$, one has*

$$|\tilde{X}_i^k - X_i|_y \to 0$$

PROOF. To prove this it is sufficient to show

$$|\tilde{X}_i^k - X_i^k|_y \to 0$$

since X_i^k converge to X_i. We have that

$$\tilde{X}_i^k = (De^{t_m X_m^k} \circ ... \circ De^{t_{i+1} X_{i+1}^k}) X_i^k (e^{t_{i-1} X_{i-1}^k} \circ ... \circ e^{t_1 X_1^k}(x)) \tag{5.1.5.9}$$

Therefore in view of the Lie bracket and pushforward estimates given in proposition 5.9 and assumptions given in *iv* of definition 5.1, we only need to prove the following proposition:

PROPOSITION 5.14. *Let X_i and X_j^k be some C^1 vector fields. Then for all $t_i < \epsilon$*

$$|(De^{t_m X_m}) \circ ... \circ (De^{t_{j+1} X_{j+1}}) X_j^k - X_j^k|_\infty \leq K_1 \frac{(\max_{i,j} |[X_i^k, X_j^k]||\eta^k|)_\infty}{m(\eta^k)_{inf}} e^{m\epsilon c_\infty^k K_2 \overline{d\eta}^k}$$

which in particular goes to 0 if ϵ is small enough so that $m\epsilon K_2 \leq \epsilon_0$ where ϵ_0 is as in definition 5.4.

PROOF. This proposition will be a result of successive application of the following lemma:

LEMMA 5.15. *Let X, Y be vector fields. Then*

$$|De^{tX} Y - Y|_p \leq |t||De^{tX}[X, Y]|_p$$

PROOF. Note that

$$(\frac{d}{dt}(De^{tX}Y))_p = (De^{tX}[X,Y])_p \tag{5.1.5.10}$$

Then the lemma follows directly from mean value inequality. □

For simplicity of notation we will only consider $t_m = t_3$ and the general case directly follows with the same calculations. We will denote $p(t_2) = e^{t_2 X_2^k}(p), p(t_3) = e^{t_3 X_3^k}(p(t_2))$. Using equation (5.1.5.10) in lemma 5.15 one has that

$$De^{t_2 X_2^k} X_1^k = X_1^k + Z$$

for $Z_q = \int_0^{t_2}(De^{sX_2^k}[X_2^k, X_1^k])(q)ds \in P$ since $[X_2^k, X_1^k]$ is and $De^{t_i X_i^k}$ preserves P . Therefore

$$|(De^{t_3 X_3^k} \circ De^{t_2 X_2^k})_p X_1^k(p) - X_1^k(p)|_{p(t_3)}$$
$$< |(De^{t_3 X_3^k}_{p(t_2)} X_1^k(p_{t_2}))_{p(t_3)} - X_1^k(p(t_3))|_{p(t_3)}$$
$$+|\int_0^{t_2}(De^{t_3 X_3}_{p(t_2)}) De^{sX_2^k}[X_2^k, X_1^k])(p(t_2))ds|_\infty$$
$$< |t_3||De^{t_3 X_3}[X_3^k, X_1^k]|_p + |\int_0^{t_2}(De^{t_3 X_3} De^{sX_2^k}[X_2^k, X_1^k])(p)ds|_\infty$$

Then using equation (5.1.5.8) of proposition 5.9 specific to $Y \in \text{span}\frac{\partial}{\partial y_j}$ one has

$$|(De^{t_3 X_3^k} \circ De^{t_2 X_2^k}) X_1^k - X_1^k|_p \leq$$
$$4\epsilon K_1 \frac{(\max_{i,j} |[X_i^k, X_j^k]||\eta^k|)_\infty}{m(\eta^k)_{inf}} e^{m\epsilon c_\infty^k K_2 \overline{d\eta}^k}$$

which goes to zero using proposition 5.9 and definition 5.4. This was the claim to be proven.

□

□

5.1.6. Applications. All the applications presented in chapter 4 trivially generalize by simply changing the dimensions and assuming codimension 1 systems. In particular with this theorem one can show that any dominated splitting $E \oplus F$ where $dim(F) = 1$ and E is at most linearly growing and F is expanding is uniquely integrable.

5.2. Sub-Riemannian Geometry and Integrability

The aim of this section is two-folds. We first show that the question of integrability for center-stable and center-unstable bundles of a partially hyperbolic systems is related to a question about "continuous sub-Riemannian geometry" posed by Simic [**71**]. Second we use this relation and a recent pathological example of a partially hyperbolic system due to Hertz,Hertz and Ures in [**47**] with non-integrable center-unstable bundle to give a partial answer to this question.

5.2.1. Introduction. Given a differentiable dynamical system $\phi : M \to M$ on a compact Riemannian manifold M, it is quite common to encounter a situation where there exists a tangent subbundle (or distribution in short) $E \subset TM$, which is "dynamically defined" by the condition $D\phi_x(E(x)) = E(\phi(x))$ for all $x \in M$. Study of geometric properties of these "dynamical distributions" suffer from one serious drawback; the lack of differentiability. With this deficiency it becomes quite hard to understand how to use the tools offered by branches of geometry like sub-Riemannian geometry and Foliation theory. The former area studies properties of non-integrable distributions and virtually all of the result only hold for distributions which are at least C^1. One of the most indispensable tool here is the classical theory of ODE for which the most fundamental theorems are usually only stated for atleast Lipschitz vectorfields. The latter one studies the properties of integrable distributions and some of the fundamental theorems in this area allows generalizations to C^0 setting and these generalizations have been very useful in studying integrability properties of dynamical distributions (see for instance [**38**]). Thus it is evidently of interest to try to understand up to what extend sub-Riemannian geometry can be generalized to apply to Hölder continuous distributions so as to use these generalizations to answer questions in dynamical systems. Conversely to understand which are the right questions to ask about continuous sub-Riemannian geometry it is of importance to setup a bridge between possible questions of this area and dynamical systems. With such a connection one can harness the knowledge and constructions of dynamical systems to give insight into whether if it is meaningful to ask a certain question about sub-Riemannian geometry. Here we aim for this goal. As far as we are aware we setup the first bridge between two fundamental questions one of which is the integrability of certain dynamical distributions and the other one the sub-Riemannian distance estimates for accessible distributions. This is achieved by generalizing a previous integrability result in [**22**] about C^1 dynamical systems to the C^0 case by observing that sub-Riemannian distance estimates play a critical role in the proof given in [**22**]. We then harness this connection to give a partial answer and some insight into a question of "continuous sub-Riemannian geometry" posed by Simic in [**71**]. In giving this answer we rely on a very interesting example of a non-integrable dynamical distribution constructed in [**47**]

5.3. The Connection Between Dynamical Systems and Sub-Riemannian Geometry

Let M be a smooth, compact, Riemannian manifold of dimension m and $H_x \subset T_xM$ for all $x \in M$ a distribution of dimension n. A piecewise smooth path γ such that $\dot{\gamma}(t) \in H_{\gamma(t)}$ Lebesgue almost everywhere is called an H-admissible path. We denote the length of such a curve by $\ell(\gamma)$. The sub-Riemannian distance with respect to this subbundle is defined as

$$d_H(x,y) = \inf\{\ell(\gamma), \gamma \,\text{is H admissible}, \gamma(0) = x, \gamma(t) = y\} \tag{5.3.0.1}$$

where in the case no such curves exists, one sets $d_H(x,y) = \infty$. Let $\mathcal{O}_\eta(x)$ be the set of points connectable to x by H admissible curves of length less than η.

Definition 5.16. If for each η, $\mathcal{O}_\eta(x)$ contains an open neighbourhood of x then we say that E is locally accessible at x. If for each η, $\mathcal{O}_\eta(x)$ contains an open disk of dimension strictly greater than n, it is called non-integrable at x and integrable if the converse holds true. A distribution is called integrable if it is integrable at every point x.

Remark 5.17. A remark is in order here. In the case a distribution is C^1, the classical definition of non-integrable means that there does not exists an integral foliation. This is also equivalent to existence of a point x and a pair of vector fields X, Y inside this distribution so that $[X,Y]$ lies out of the distribution. This implies the existence of a disk inside some $\mathcal{O}_\eta(x)$ with dimension greater than n. Thus the converse of the integrability definition above implies integrability in the usual sense when the distribution is C^1. However the relation is much less clear in the case of continuous distributions. It is for sure true that if a C^0 foliation is uniquely integrable in the classical sense of existence of a unique integral foliation then the definition of integrability above holds true. If one has the situation that $\mathcal{O}_\eta(x)$ is a differentiable manifold (which is true in the case of C^1 distributions again) then the above definition would be again equivalent to classical integrability (since it is easy to show that if $\mathcal{O}_\eta(x)$ is a manifold then it must have dimension atleast n). Yet the converse is probably not true without further assumptions. Yet still due to the analogy to C^1 case we prefer to set the above definition as the definition of integrability that we use. The reader should keep this difference in mind however.

In this paper we first show that if one is able to compare the sub-Riemannian distance of a non-integrable θ-Hölder continuous codimension 1 distribution to the Riemannian distance as follows:

$$C_1 d(x,y)^{r(\theta)} \leq d_H(x,y) \leq C_2 d(x,y)^{r(\theta)} \quad \forall y \in \gamma \subset D \tag{5.3.0.2}$$

where D is a $n+1$ ball and $C_1, C_2 > 0$ are constants (which possibly depend on the distribution, the point x and the manifold but not on the points y) then one gets the following theorem of integrability:

THEOREM 5.18. *Let $\phi : M \to M$ be a C^2 diffeomorphism of a compact smooth Riemannian manifold (M, g) and $E^s \oplus E^c \oplus E^u$ be a point-wise partially hyperbolic splitting as in definition 2.17 with θ Hölder distributions E^σ for $\sigma \in \{s, c, u\}$. Assume that the upper bound given in equation* (5.3.0.2) *holds true and that there exist a constant $\epsilon_0 < 1$ and $x \in M$ s.t for all $k > k_0$ large enough,*

$$\frac{\mu_c(x^k)}{\lambda_u(x^k)^{r(\theta)}} < 1 - \epsilon_0 \qquad \frac{\mu_s(x^k)^{r(\theta)}}{\lambda_c(x^k)} < 1 - \epsilon_0$$

where $x^k = \phi^k(x)$. Then E^{cs} and E^{cu} are integrable in the sense of definition 5.16.

This theorem is a generalization of a theorem due to Wilkinson and Burns [**22**] in which they that C^1 partially hyperbolic splittings which satisfy

$$\frac{\sup_{x\in M} \mu_c(x)}{\inf_{x\in M} \lambda_u(x)^{\frac{1}{2}}} < 1 \qquad \frac{\sup_{x\in M} \mu_s(x)^{\frac{1}{2}}}{\inf_{x\in M} \lambda_c(x)} < 1$$

are E^{sc} and E^{cu} are shown to be integrable.

Our theorem is a generalization of this since it can be shown by the standard techniques in sub-Riemannian geometry that in the case a codimension 1 C^1 distribution is non-integrable at a point x, there exists a $n+1$ dimensional ball around x for which the condition in (5.3.0.2) is satisfied for $r(\theta) = \frac{1}{2}$ (see for instance [**2**],[**36**],[**61**]). This fact is also employed in the proof given in [**22**] although the authors of the paper do not point the connection of their proof to this important aspect of sub-Riemannian geometry. It is also interesting to note that the condition given in equation (5.3.0.2) can be seen as a non-integrable analogue of the quasi-isometry condition for foliations given in [**15**]. It does not seem likely however that there are connections between the two as quasi-isometry is a global property while sub-Riemannian length estimates are local. We also remark that the lower bound in 5.3.0.2 has been shown to hold true in [**71**] with $r(\theta) = \frac{1}{1+\theta}$. Therefore the upper bound still remains as an open question. In this direction, using theorem 5.18 and a famous example due to Hertz,Hertz,Ures in [**47**] we show that

THEOREM 5.19. *If $r(\theta)$ exists, it must satisfy $r(\theta) \leq 1 - \theta$.*

Therefore unlike the lower bound $\frac{1}{1+\theta}$, the upper bound cannot satisfy $f(1) \neq \frac{1}{2}$ which seems like a natural property in the case such a bound should exists (due to the fact that $\theta = 1$ corresponds to C^1 distributions for which both the upper and lower bound holds true with $r = \frac{1}{2}$). Not only that but as the regularity increases the estimate has to become less tight, that is it must change in an inversely correlated way. Even more one has that $f(1) = 0$ which says that the information one can obtain about the sub-Riemannian geometry of Hölder continuous distributions does not always necessarily improve as the regularity increases. These and several other reasons motivate us in the next section to conjecture actually that such a function can not exist.

5.3.1. Detailed Discussion and Motivation. Sub-Riemannian geometry is the study of metric properties of distributions on a Riemannian manifold M. Given a distribution (a subbundle inside the tangent bundle) $H \subset TM$, it is of natural interest to study the piecewise smooth paths $\gamma(t)$ in M that a.e satisfy $\dot{\gamma}(t) \in H_{\gamma(t)}$, which are called H admissible paths. Indeed in physical situations and control problems admissible curves describe the possible paths along which a system might evolve where the distribution represents the infinitesimal direction constraints [**1**] while in dynamical systems admissible paths along certain invariant distributions play a prominent role in studying the ergodic properties of the system [**23**],[**50**]. It is called sub-Riemannian because one can define a notion of distance between points x, y with respect to this distribution as in (5.3.0.1)

In the case where accessibility is satisfied around a point x, one might be interested in comparing the sub-Riemannian distance to the Riemannian distance. From an optimal control problem point of view, the sub-Riemannian distance might be related to the cost for a system to follow certain admissible paths[**1**]. From dynamical systems point of view, such comparisons gives an idea about the relationship between length of admissible invariant curves connecting two points and their actual distance from each other [**47**]. A wealth of literature on this comparison exists and a special version of this comparison for codimension 1 distributions are given in (5.3.0.2).However as far as we know the only result for distributions which are Hölder is given in [**71**], which is the equation (5.3.0.2). But non-smooth distributions are abundant in dynamical systems [**43**], [**47**] and such distance comparisons play important roles in studying these systems (in fact our theorem 5.18 precisely shows that they do). So one would like to know whether if such estimates somehow generalize to class of Hölder continuous distributions.

After this work Simic has raised the question whether if analogous upper bound holds true. To state it more precisely (in the way the lower bound is stated in [**71**])

QUESTION 5.20. *Does there exist a function defined in* $(0, 1]$ *s.t for all* $H \in \mathcal{H}(M, \theta)$ *(which is locally accessible at some* $p \in M$*) and for any* C^1 *path* γ *transverse to* H *starting at* p *and for every point* q *on* γ *with* $d(p, q) < \rho(M)$*, one has*

$$d_H(p,q) \leq C_2(H,M,\gamma)d(p,q)^{r(\theta)}$$

where $\rho(M)$ is a constant which depends on the manifold and $0 < C_2(H,M,\gamma)$ is a constant which only depends on the manifold, on the Hölder sup-norm of the distribution H and on the minimal angle between $\dot{\gamma}$ and H?

REMARK 5.21. To make the nature of the expected constant C_2 clear, we put here the exact formula for of the constant C_1 calculated in [**71**] for the lower bound. In there it is shown that the lower bound holds as

$$C_1(H,M,\gamma)d(p,q)^{\frac{1}{1+\theta}} \leq d_H(p,q) \text{ for all } q \in \gamma$$

where

$$C_1 = \{2c_M^{\frac{\theta}{1+\theta}} d_M^{\frac{1}{1+\theta}} sin(\theta_0)^{-\frac{1}{1+\theta}} |\alpha|_{C^\theta}^{\frac{1}{1+\theta}}\}^{-1}$$

Here c_m and d_m are constants depending on the manifold, θ_0 is the minimal angle between H and γ, α is a 1-form defining H as $H = ker(\alpha)$ and $|\cdot|_{C^\theta}$ is the Hölder sup-norm.

The proof in [**71**] is obtained by cleverly modifying a proof of this inequality for the smooth case given by Gromov [**36**]. The geometric nature of the Gromov's proof in the smooth case is quite important for the generalization. Unfortunately every proof of the upper bound in the smooth case depends heavily on notions such as existence and uniqueness of solutions to ODE making a modification of these proofs for the Hölder case very hard. As far as we are aware, this is still an open question.

Note that since every smooth distribution is Hölder for any θ, if one tries to use $r(\theta) = \frac{1}{1+\theta}$ as the exponent, then there is a problem. Indeed when applied to smooth distributions, this would require the fulfilment of both the lower bound and the upper bound in (5.3.0.2) for any θ which when combined together requires

$$C_1 d^{\frac{1}{2}}(x,y) \leq C_2 d^{\frac{1}{1+\theta}}(x,y)$$

for all y arbitrarily close to x in transversal direction where $1+\theta < 2$. This however can not be satisfied for small distances. Thus if the upper bound is to be satisfied for some function f then itself must satisfy $r(\theta) \leq \frac{1}{2}$. By the same reasoning it must also satisfy $r(\theta) \geq \frac{1}{1+\theta}$. Note moreover that when the exponent in (5.3.0.2) is evaluated at $\theta = 1$, that is the case of C^1 distributions, one recovers the original exponent for C^1 distributions given in (5.3.0.2). This somehow signifies that the exponent for the lower bound "behaves well" with respect to the Hölder exponent of the distribution. Therefore one might try to see what other properties the exponent for the upper bound satisfies. For instance is it true that as in the lower bound case in [**71**] one has that $r(\theta)$ is a continuous function such that $f(1) = \frac{1}{2}$.

Functions that satisfy all the properties above do exist, for instance $r(\theta) = \frac{\theta}{2}$ or $r(\theta) = \frac{\theta}{1+\theta}$ and many variations of these. However as we show in theorem 5.19, unfortunately $r(\theta)$ can not satisfy some of the expected properties above.

To obtain this result, we use a recent example due to Hertz, Hertz, Ures of a family of pathological partially hyperbolic systems where the center-unstable bundle fails to be integrable and non-accessible. In particular we show that if $r(\theta)$ exists and satisfies $r(\theta) > 1 - \theta$ then this class of a partially hyperbolic systems can not be non-accessible which leads to a contradiction. In proving 5.19 the example due to [**47**] is used without modification and therefore has only limited applicability. The author strongly believes that with the right modifications one can actually arrive at a much stronger conclusion:

CONJECTURE 5.22. *$r(\theta)$ does not exist.*

5.4. Contact Geometry and The Continuous Frobenius Theorem

In principle the conditions given in theorem 5.2 are much more easier to check then some of its predecessors such as those given in [**42**]. Although this has allowed us to state a variety of applications, it still leaves a lot to be desired. Therefore in this section we sketch a first attempt geometrize the conditions given in the previous chapter, mainly the conditions given in definition 5.1. We restrict to codimension 1 subbundle E given as the kernel of some 1-form η. As we are sketching an idea here, we will leave orientability issues aside. Note that the conditions there are satisfied if there exists a constant C such that for all k:

$$||d\eta^k||_\infty < C$$

$$||\eta^k \wedge d\eta^k||_\infty \to 0$$

The second condition given above is naturally expected of the approximation differential forms as it is like an asymptotic involutivity condition. And as the previous sections have demonstrated that condition is quite abundantly satisfied in many dynamical systems. It is therefore important to make more sense of the first condition. First of all note that the second condition implies that

$$||d\eta^k|_{E^k}||_\infty \to 0 \qquad \text{as } k \text{ goes to } \infty$$

That is the norm of $d\eta^k$ goes to 0 in the direction E^k. Thus one only needs transversal control to achieve the rest. A seemingly possible way to obtain this is to use a sequence of contact 1-forms for η^k. The reason is that each contact 1-form possesses what is called a Reeb vector field R^k such that R^k is transverse to E^k

and $d\eta^k(R^k,\cdot) = 0$. For this to give enough control over the norm $|d\eta^k|$ however we need R^k to not to converge to E too fast so that E^k, R^k forms a reasonable basis of the tangent space to estimate the norm of $|d\eta^k|$. More precise condition on the norm is the following:

LEMMA 5.23. *Assume η is a contact 1-form and R is its Reeb vector field. Let θ be the angle between R and $ker(\eta)$. Then there exists constant K_1, K_2 (depending only on $|\eta|$) such that*

$$||d\eta||_\infty \leq K_1 e^{\frac{K_2||\eta\wedge d\eta||_\infty}{\theta}}$$

Coupled with theorem 5.2, this gives that if

$$|\eta^k \wedge d\eta^k|_\infty e^{\frac{K_2|\eta^k\wedge d\eta^k|_\infty}{\theta^k}} \to 0$$

$$|\eta^k - \eta|_\infty e^{\frac{K_2|\eta^k\wedge d\eta^k|_\infty}{\theta^k}} \to 0$$

then the limit subbundle E is integrable. In the context of dynamical systems with a partially hyperbolic splitting $E^s \oplus E^c \oplus E^u$ with $E = E^s \oplus E^c$ and $F = E^u$ on a three manifold, it is just enough to have a single contact structure η^0 whose kernel E^0 is transverse to F with a Reeb vector field R^0. Then one can build a sequence of contact structures by

$$E^k = D\phi^{-k}E^0$$

with contact 1-forms $\eta^k = (\phi^k)^*\eta$ and Reeb vector fields $R^k = D\phi^{-k}R^0$. Both E^k and R^k converge to E. A condition for which lemma 5.23 is satisfied is the case where $R \in E^c \oplus E^u$. Note that contact 1-forms are defined up to multiplying by a function and upon such a rescaling, the Reeb vector field might change completely. This gives us some freedom and hope to achieve this. Here is the lemma:

LEMMA 5.24. *Let $E = ker(\eta)$ be a contact structure and P a plane distribution. Given a non-vanishing function f we denote $\eta_f = f\eta$ and its Reeb vector field R_f. Then there exists a function f such that $R_f \in P$ iff*

$$d\eta_f|_P = 0$$

It turns out that when P is differentiable, the condition above can be turned into a first order non-homogenous linear partial differential equation in terms of $ln(f)$ which locally admits solutions. However we need global solutions and at the moment it is not clear to us if it is possible to achieve this since theory of global solutions to PDE are quite under developed. In any case the full theorem is the following:

THEOREM 5.25. *Let $\phi : M \to M$ admit a partially hyperbolic splitting $E^s \oplus E^c \oplus E^u$. Assume there exists a contact structure $E^0 = \ker(\eta)$ transverse to E^u and a non-vanishing function f such that*

$$d\eta_f|_{E^{cu}} = 0$$

Then E^{cs} is uniquely integrable.

Since the non-existence of global solutions to first order linear PDE given by a vector field X, are quite related to existence of closed integral curves of X, we hope that this might lead to some connections between topology of E and its integrability.

5.5. Continuous Exterior Differentiability and Sub-Riemannian Geometry

We remind the reader that a continuous 1-form η is said to admit a continuous exterior differential if there exists a non-vanishing function f and a continuous 2-form β such that

$$\int_D \beta = \int_J f\eta$$

for every piecewise smooth Jordan curve J and surface S bounded by J. This condition is equivalent to existence of a sequence of C^1 differential 1-forms η^k which converge to η in C^0 norm and $d\eta^k$ converge to β in C^0 norm. Lets define $H = ker(\eta)$ and $H^k = ker(\eta^k)$. Then one can prove the following:

PROPOSITION 5.26. *There exists a local basis $\{X_i^k\}_{i=1}^d$ of H^k and $\{X_i\}_{i=1}^d$ of H such that*

- *X_i^k converge to X_i in C^0 topology*
- *X_i are uniquely integrable with C^1 family of flows*
- *$De_p^{tX_i^k}$ converge in C^0 topology to De_p^{tX}*
- *If H is integrable then it is uniquely integrable with a C^1 family of integral manifolds. Moreover in this case H^k can also be chosen integrable.*

This theorem basically means that the C^1 behaviour of the solutions of X_i^k can be transferred to solutions of X_i. In particular much of the techniques used in the C^1 case for sub-Riemannian estimates are valid. Thus we conjecture:

THEOREM 5.27. *Assume that $H = ker(\eta)$ where η admits a continuous exterior derivative. Moreover assume that H is non-integrable. Then the sub-Riemannian distance estimate*

$$C_1 d(x,y)^{\frac{1}{2}} \leq d_H(x,y) \leq C_2 d(x,y)^{\frac{1}{2}} \quad \forall y \in \gamma \subset D \tag{5.5.0.1}$$

is valid for H.

If this conjecture is true then by virtue of theorem 5.18 we get

THEOREM 5.28. *Assume $E^s \oplus E^c \oplus E^u$ is a point-wise partially hyperbolic splitting. Assume E^{sc} is codimension 1 whose local defining 1-forms have continuous exterior differentials. Then if*

$$\frac{\mu_c(x)^2}{\lambda_c(x)} < 1$$

for all x, E^{sc} is uniquely integrable in the classical sense. Similar statement holds true for E^{cu} under the assumption

$$\frac{\mu_s(x)}{\lambda_c(x)^2} < 1$$

for all x.

We remark that the center-stable foliation thus obtained will actually posses C^1 family of leaves and therefore this condition might be too stringent. We are not sure if the ideas here can be generalized to a less stringent setting such as in chapter 4. We actually also hope to show that theorem 5.27 holds true in the case E is transversally Lipschitz. If this is true then we will have answered the question 1.14 with the help of theorem 5.18.

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