THERMODYNAMICAL ASPECTS OF GRAVITY:
FROM EVENT HORIZON TO EMERGENT GRAVITY

by

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Abstract

In this thesis we explore various aspects of horizon thermodynamics and its relation with gravitational dynamics. We start with addressing the issue about the region of origin of the Hawking quanta, using both a heuristic argument and a detailed study of the renormalized stress energy tensor (RSET). We present compelling evidence that the Hawking quanta originate from what might be called a quantum atmosphere around the black hole with energy density and fluxes of particles peaked at about 4M, running contrary to the popular belief that these originate from the ultra high energy excitations very close to the horizon. We then study the behavior of the *effective Hawking temperature* as perceived by a free falling observer. We compute the energy density using this temperature and compare it with the energy density obtained from RSET measured by the same observer and notice a discrepancy. We further compute the adiabaticity of this temperature and try to explain the reason for this discrepancy. Next we move on to thermodynamics of local causal horizon (LCH) and in particular focus on derivation of equations of motion for theories beyond general relativity as an equation of state. Jacobson showed that the Einstein equation is implied by the Clausius relation imposed on a small patch of locally constructed causal horizon. The extension of this thermodynamic derivation of the field equation to more general theories of gravity has been attempted many times in the last two decades. In particular, equations of motion for minimally coupled higher-curvature theories of gravity, but without the derivatives of curvature, have previously been derived using a thermodynamic reasoning. In that derivation the horizon slices were endowed with an entropy density whose form resembles that of the Noether charge for diffeomorphisms, and was dubbed the Noetheresque entropy. Here we derive a new entropy density, closely related to the Noetheresque form, such that the field equation of any diffeomorphism-invariant metric theory of gravity can be obtained by imposing the Clausius relation on a small patch of a local causal horizon. Finally, we shall demonstrate how the equation of state derivation can be carried on to theories having torsion as an independent degree of freedom, such as Einstein–Cartan gravity, by using the irreversible Clausius equation.

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Contents

CHAPTER 1

Introduction

> It is the facts that matter, not the proofs. Physics can progress without the proofs, but we can't go on without the facts...if the facts are right, then the proofs are a matter of playing around with the algebra correctly.
>
> Richard P. Feynman (1918-88)

Ever since the General theory of relativity was published more than a century ago, it has been well verified by numerous experimental tests at astrophysical, cosmological and up to the millimeter scale[1]. While one can say it is one of the most successful theory that has been formulated and tested, it has puzzled scientists for decades. While we do know that some better theory has to supersede general relativity for dealing with the black holes or the big bang singularities, we still miss a full fledge alternative. One of the most significant, long standing, open problem in physics is trying to harmonize general relativity with quantum mechanics, which can consistently explain the other three fundamental forces of nature. So one can say despite all its correctness and verification, General relativity is *incomplete* given that it cannot be unified with quantum mechanics.

General relativity is a highly non-linear and complicated theory, so the challenges of dealing with it can become enormously difficult. In this regard one can easily attribute the failure of quantization of gravity on the technical difficulties involved in the program. On the other hand the question becomes even more puzzling on the philosophical ground as one may ask what does quantization of gravity would mean, as we know the foundation of general relativity relies on spacetime dynamics and general covariance (there is no preferred reference frame) which clashes with quantum theory, whose formulation requires a preferred foliation and splitting of space and time.

At this point it is quite tempting to ask, why would one need a quantized theory of gravity, also considering the fact that the two theories we are talking about belongs to two vividly different length and energy scales. Departing from the usual trend in theoretical physics about unification, like Maxwell's unification of electricity and magnetism or Weinberg–Salam's electroweak unification, one might think that gravity is fundamentally different and there is no need for quantization.

Thinking along the line where the gravitational field is not quantized one immediately faces the problem that such a theory would lead to the violation of the uncertainty principle, in such a scenario gravity could be used to determine the position and momentum of any particle up to an arbitrary precision [2]. As an alternative Moller and Rosenfeld [3] proposed a semiclassical model of gravity, in this approach one keeps the gravitational sector classical while the source is quantized matter fields. The Einsteins equation takes the form,

$$G_{ab} = 8\pi G \langle \psi | T_{ab} | \psi \rangle$$

here the coupling between quantized matter and classical gravity is made possible by taking the expectation value of the stress-tensor and not using it in its operator form, and hence one can equate it to the c–number Einstein tensor. The problems with such an approach and the inconsistency of the semiclassical Einstein equation was shown in [4].

Other than this there are the aforementioned problems with the singularities inside black hole and with the predicted singularity at the start of our cosmological history. These issues have no resolution in the classical or the semiclassical framework of gravity. In summary, all these evidences point towards the need for a quantized theory of gravity.

The discovery that black holes radiate as black bodies [5, 6] and have entropy [7, 8] has radically changed our understanding of general relativity and strongly hints towards a quantum nature of gravity. Indeed shortly after it was shown that black holes behave as thermodynamics systems [9], it was shown via a quantum field theory calculation on a classical black hole spacetime that the black hole has a characteristic temperature and entropy given as,

$$kT_H = \frac{\hbar\kappa}{2\pi}, \qquad S_{BH} = \frac{A_{hor}}{4\hbar G}$$

where κ is the surface gravity and A_{hor} is the area of the horizon. It is easy to see that both of the above quantities are *quantum gravitational* in nature because of their dependence on $\hbar$ and Newton's constant, G.

In ordinary thermodynamics the temperature and entropy give us information about the underlying microstates of the system, in particular the entropy is the measure of the number of possible microscopic configurations that the system can have. There is no reason to assume something different for the black holes and hence one can assume that the black hole entropy and temperature gives us a hint about the underlying gravitational microstates. This idea has been used and as a consistency check the black hole entropy has been computed in various proposed quantum gravity models successfully [10, 11, 12, 13, 14, 15].

One peculiar behavior of the black hole entropy is its holographic scaling, being proportionality to the area of the horizon rather than being extensive and being proportional to the volume. The black hole entropy brings many more interesting puzzles such as the *problem of universality*. This is related to the fact that Bekenstein–Hawking entropy can be computed using different models of quantum gravity as noted above. These models are in principle characterized by very different gravitational and matter microstates. If the black hole entropy is supposed to give information about the underlying degrees of freedom for a theory of quantum gravity then one might expect the result to be theory dependent. However, this is not the case, all the proposed theories like string theory, loop quantum gravity, induced gravity, gives the exact same result despite having very different microstates.

Another puzzling thing about black hole radiation is the *information loss problem* [16, 17, 18, 19]. If we consider a scenario where the black hole is initially formed by collapse of matter that is in a pure state, then if the black hole evaporates completely by Hawking radiation(into a thermal mixed state), it would require a non unitary transition from a pure state to mixed state which is not allowed by the known laws of quantum mechanics. On the other hand if Hawking radiation has to be a pure state then there must be some correlation between the "early" and "late" modes, which were not in any causal contact previously. This problem has been analyzed in several approaches and recently it was proposed in [20] that one must either give up on the equivalence principle near the horizon or low energy effective field theory beyond some microscopic distance away from the horizon or the other option is to end up with a highly entropic "remnant" at the end of the evaporation process. All these demands a deeper investigation of the radiation process and how the Hawking quanta are created.

The proportionality of the horizon area to the black hole entropy provides us a fundamental connection between thermodynamics and spacetime geometry. The next obvious question was, if such a relation can be formulated beyond black hole dynamics. The discovery of the Unruh effect [21] [22] [23], which shows that an accelerated observer would perceive the Minkowski vacuum as a thermal state and can associate a temperature to the Rindler horizon, further provides evidence towards the thermodynamic nature of any causal horizon rather than thermodynamic nature of black holes. At this level one might suspect that this thermodynamic behavior of the horizon has nothing to do with the gravitational dynamics as there is no gravitational field equations involved in these derivations. However, to clear out such doubts one can prove the equivalence between the gravitational field equation evaluated on a horizon and the first law of thermodynamics, $TdS = dE + PdV$ [24, 25, 26].

Similarly, a holographic relation between the surface term and the bulk term of gravitational action (even valid for some theories beyond general relativity) was shown [27], further it was also shown that the surface terms when evaluated on the horizon, for a given solution are equal to the horizon entropy [28]. Now because of the holographic relationship between the surface term and the bulk term and the link between the horizon entropy to the surface term, one can again think of an indirect connection between gravitational dynamics and horizon thermodynamics [1].

One of the most significant step to bridge the gap between gravitational dynamics and horizon thermodynamics was taken by Jacobson [29]. He derived the Einstein equation implementing the Clausius equation, starting from the entanglement entropy of a local Rindler horizon, assuming the Unruh temperature and a matter flux (interpreted as the heat flux) crossing the horizon. This established the Einstein equation as an equation of state, where one can interpret the gravitational dynamics as a manifestation of local microscopic degrees of freedom. Based on the Einstein equivalence principle [30] the local neighborhood of any arbitrary spacetime point can be approximated as a flat Minkowski patch. One can argue the existence of a local boost Killing vector within this flat patch. Now if there is a local accelerated observer within this local patch the observer will see a local causal surface which is called the *local Rindler horizon* in this context. Based on this geometric setup for the derivation in [29] one can easily infer that the derivation of the Einstein equation from the thermodynamic variable is completely based on the local physics and is not dependent on any global feature of the manifold and that is where the strength of this result lies.

Of course, one might view gravity as an effective field theory [31], within which the Einstein–Hilbert action is the first term in the derivative expansion of the effective action. The higher derivative corrections are not important in the solar system limit but when we are talking about local underlying microstructure of spacetime these terms must play a significant

[1]The only problem with this approach is that the entropy is derived "on shell", as one needs to know the gravitational action to get the surface term and obtain the horizon entropy. In this sense one cannot say that you can derive the equations determining the gravitational dynamics from the thermodynamics of the horizon.

role. In this regard it seems extremely important to push the spacetime thermodynamics construction beyond Einstein equation. If this will work, it would point towards a generic thermodynamical/emergent nature of gravitational theories. Otherwise, it might point out that such features are just a byproduct of some special simplicity of general relativity and very few other metric theories of gravity. As such it would not imply that this can be used as a guiding principle towards the understanding of the fundamental nature of spacetime. Some interesting generalization of the equation of state derivation has been performed so far for various higher derivative theories of gravity [32, 33, 34, 35, 36] and often using a more precise geometric construction rather than just using Einstein equivalence principle [37].

While moving beyond general relativity one major criticism for the equation of state derivation lies in the choice of the entropy. As discussed above, in [29] the variation of the entropy that enters the Clausius equation was chosen to be the entanglement entropy, which is proportional to the area of the horizon of the Local causal horizon. However for modifications to general relativity the black hole entropy is no more proportional to the area of the horizon but can have curvature corrections [38, 39]. For this reason the entropy endowed to the local causal horizon for such theories are often taken to be that of the Noethersque form. While it was shown in [40], that the renormalized entanglement entropy can get corrections which are reproducing the Wald entropy terms [38], something that can be used to somewhat justify the choice of entropy made in [33, 35, 36], we surely need a more concrete quantum mechanical calculation of the entropy for the local causal horizon.

A further improvement was made to the spacetime thermodynamics approach in [41], where the semi-classical Einstein equation was derived implementing the Clausius equation and using a causal diamond. The derivation is based on a hypothesis which states that, upon simultaneous variation of the geometry and quantum fields from maximal symmetry, the entanglement entropy in a small geodesic ball is maximal for a fixed volume of the ball. In this new approach as well, the Einstein equation is not just a dynamical constrain within a single solution but a relationship between infinitesimally separated spacetime histories and geometry. In the context of AdS/CFT [42] the Einstein equation was derived using the Ryu-Takayanagi formula [43] following the same philosophy [44, 45]. The advantage of the derivation in [41] is that it allows for a larger applicability as one does not need to rely on AdS/CFT. It can be applied to any general spacetime where a causal diamond can be constructed giving a better idea about which entropy to use [46] and making the use of the modular Hamiltonian for computing the infrared part of the entropy [47].

Surely the next step would be to generalize this derivation for higher derivative theories and see if this idea of gravity emerging from entanglement holds [48]. It requires some exact computation of the entanglement entropy for local null hypersurface (without neglecting the curvature of the causal diamond) and it opens a wide possibility for upcoming research in this field.

1.1. Plan of the Thesis

The content of the thesis is organized as follows: Chapter 2 introduces the basic mathematical tools that we would need in order to study horizon thermodynamics and understand the link between gravitational dynamics and the underlying local microscopic degrees of freedom. We start with the very basics of black hole physics and talk about various global and local coordinate systems that are important for our further studies. Next we setup the construction of the local causal horizon that becomes a very important ingredient for deriving the gravitational field equations from thermodynamical variables. We introduce quantum field theory in curved spacetime and specifically outline the derivation of Hawking radiation and finish by giving a brief synopsis of black hole thermodynamics.

Chapter 3 deals with the origin of Hawking radiation and is mainly based on two original works. First we considered the effect of tidal force on the spontaneously created particle anti-particle pair near the black hole horizon and we showed that the most likely region for the origin of the Hawking quanta is around $r = 4M$, where M is the mass of the black hole. The next section of the chapter deals with computation of the energy density and flux as perceived by an observer having zero acceleration and radial velocity at the horizon and infer the same result about the origin of the Hawking quanta. Using the Renormalized stress energy tensor we also computed the energy density as measured by a free falling observer and compared this with the energy density that can be computed using the *effective* temperature that this observer would perceive. Doing so we found a discrepancy and we speculate that the breaking down of the adiabatic approximation near the horizon for the *effective* temperature function can be a reason for this.

In chapter 4 we move on to thermodynamics of local causal horizons and at first review the derivation of field equations as an equation of state in the case of general relativity and for F(R) gravity. In the next section we show the extension of this derivation for higher curvature theories based on a Noethersque entropy functional and a better geometric construction of the local causal horizon as discussed in Chapter 2. Using this approach one cannot derive the equations of motion of theories having a derivative of Riemann term in the Lagrangian and also it is not easy to see how one can deal with theories having some non-minimal coupling between the matter and gravity sector. The next part of the chapter is based on an original work where we showed that one can modify the Noether charge entropy based on the ambiguities it has and one needs to consider a *test* stress energy tensor as the *heat* flux in the Clausius equation to make the extension possible.

In chapter 5 first we review some important aspects of Riemann–Cartan spacetime, which would be needed for the derivation of the Einstein–Cartan equation as an equation of state. We derive the Zeroth law for a local causal in Einstein–Cartan spacetime and then the Raychaudhuri equation for a null congruence in the presence of torsion. Finally, we use these tools to derive the Einstein–Cartan equation as an equation of state implementing an irreversible Clausius equation.

In chapter 6 we give a brief outline of the primary content of the other chapters and focus mainly on a discussion about the original results obtained. We finish with proposing the future scope that the work done for this thesis can lead to.

The content of the thesis is based on:

- **Higher derivative gravity: field equation as the equation of state**
  Ramit Dey, Stefano Liberati, Arif Mohd
  Phys. Rev. D 94, 044013 (2016) [arXiv:1605.04789 [gr-qc]]

- **The black hole quantum atmosphere**
  Ramit Dey, Stefano Liberati, Daniele Pranzetti
  submitted in PRD [arXiv:1701.06161 [gr-qc]]

- **Spacetime thermodynamics in presence of torsion**
  Ramit Dey, Stefano Liberati, Daniele Pranzetti
  in preparation

- **Effective Hawking temperature for free falling observer**
  Ramit Dey, Stefano Liberati, Zahra Mirzaiyan,Daniele Pranzetti
  in preparation

- **AdS and dS black hole solutions in analogue gravity: The relativistic and non-relativistic cases**
  Ramit Dey, Stefano Liberati, Rodrigo Turcati
  Phys. Rev. D 94, 104068 (2016) [arXiv:1609.00824 [gr-qc]]
  Not included in thesis

CHAPTER 2

Quantum meets gravity: The basic toolbox

The first and the simplest solution of the Einstein equations was given by Karl Schwarzschild in 1916 for a static and spherically symmetric spacetime having no external energy source. Solving the vacuum Einstein equation the Schwarzschild metric is given as

$$ds^2 = \left(1 - \frac{2M}{r}\right) dt^2 - \left(1 - \frac{2M}{r}\right)^{-1} dr^2 - r^2(d\theta^2 + \sin^2\theta \mathrm{d}\phi^2), \tag{2.1}$$

where M is the mass of the gravitating test body.

It can be seen directly from the solution (2.1) that this metric becomes singular at two points corresponding to $r = 0$ and $r = 2M$. The singularity related to the $r = 0$ point is a true singularity of the spacetime while the singularity at $r = 2M$ is just a coordinate singularity corresponding to the particular choice of the coordinate system we are working in. The strength of this solution lies in the fact that this solution is unique for the given symmetries of the spacetime and the predictions made by the Schwarzschild metric are among the experimentally verified tests of general relativity.

2.1. Kruskal–Szekeres coordinate

As mentioned earlier the singularity at $r = 2M$ is just the artifact of not using the correct coordinate system and indeed by looking at radial null trajectories it can be shown that this singularity is just a coordinate singularity. Furthermore it can be verified that the $r = 2M$ surface is not a physical singularity by extending the spacetime so that it is geodesically complete and is maximal. The maximal analytic extension of the Schwarzschild spacetime was done by Kruskal and we illustrate it in this section.

We define a new coordinate as

$$r^* = \int \frac{rdr}{r - 2M} = r + 2M\ln\left(\frac{r}{2M} - 1\right). \tag{2.2}$$

As we can see from the above equation that r^* changes logarithmically and thus slower than r near the horizon, this is known by the name of tortoise coordinate. We can define a new set of null coordinates as

$$u = (t - r^*) \qquad \text{and} \qquad v = (t + r^*). \tag{2.3}$$

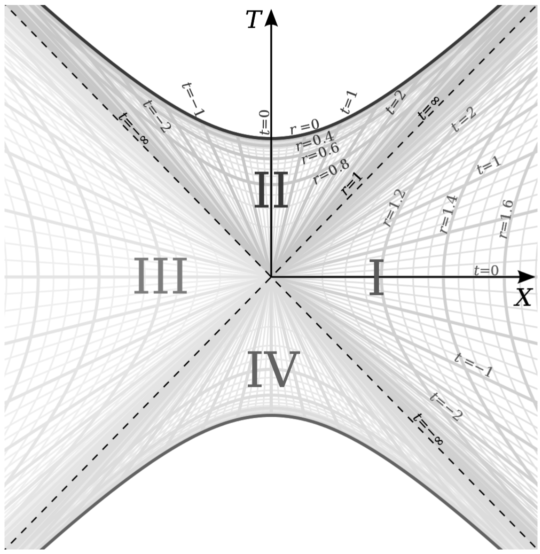

Figure 2.1: The figure shows the four quadrants of the Kruskal extension of Schwarzschild spacetime

Using these the Schwarzschild metric can be written as

$$ds^2 = (1 - 2M/r)\, du\, dv. \tag{2.4}$$

Using (2.2) and (2.3) we can write

$$r - 2M = 2M \exp\left[\frac{v-u}{4M}\right] e^{-r/2M}. \tag{2.5}$$

We can rewrite (2.4) using (2.5) as

$$ds^2 = 2M\, \frac{\exp\left[\frac{v-u}{4M}\right] e^{-r/2M}}{r}\, du\, dv \tag{2.6}$$

Near the horizon, $r \sim 2M$, this line element is regular but still this coordinate is defined only for $r > 0$. To achieve extension of the coordinate beyond the $r = 0$ point we need to re-parametrize the null geodesics using the coordinate transformation $U = U(u)$ and $V = V(v)$. For finding out the exact form of the transformation we can calculate the affine parameter along the null geodesic by using the equation

$$E = g_{ab} k^a (\xi_t)^b \tag{2.7}$$

Using this we can define the coordinate transformation as

$$U = p(u) = -e^{-u/4M} \tag{2.8}$$

$$V = q(v) = e^{v/4M} \tag{2.9}$$

which transforms the line element into the form

$$ds^2 = \frac{32M^3}{r} e^{-r/2M}\, dU\, dV. \tag{2.10}$$

We can further define two new coordinates as

$$T = (U + V)/2 \qquad \text{and} \qquad X = (V - U)/2 \tag{2.11}$$

to cast the line element defined in (2.10) as

$$ds^2 = \frac{32M^3}{r} e^{-r/2M} (dT^2 - dX^2) \tag{2.12}$$

2.2. Riemann Normal Coordinates

In the previous section we saw the maximal extension of the Schwarzschild spacetime, now it is possible to build a coordinate system by using geodesics through any given point, p, to determine the coordinates of a local neighborhood around p. Such a coordinate system is known as the Riemann normal coordinate (RNC). If we consider a point, q, sufficiently close to p, then there would be unique geodesic passing through p and q. Let the component of the unit tangent vector to this geodesic be k^a and s be the arc length measured from p to q. By doing so one can show trivially that all geodesic through p has the form

$$x^a(s) = sk^a, \tag{2.13}$$

also k^a is constant along each geodesic. Substituting this in the geodesic equation:

$$0 = \frac{d^2x^a}{ds^2} + \Gamma^a_{bc}(x) \frac{dx^b}{ds} \frac{dx^b}{ds}, \tag{2.14}$$

one obtains at the origin, p, the relations,

$$\Gamma^a_{bc}|_p = 0 \tag{2.15}$$

$$\Gamma^a_{bc,d}|_p + \Gamma^a_{c,db}|_p + \Gamma^a_{dc,b}|_p = 0 \tag{2.16}$$

Now it is possible to do an expansion of the metric in powers of x^a around p namely,

$$g_{ab}(x) = \eta_{ab} + g_{ab,cd}|_p \frac{x^c x^d}{2} + \mathcal{O}(\epsilon^3), \tag{2.17}$$

since $g_{ab,c}|_p = 0$ at p, there is no linear order term in the above expansion. Using 2.15 one can show

$$g_{cd,ab}|_p = -\frac{1}{3}(R_{cadb} + R_{cbda}). \tag{2.18}$$

Using this relation in 2.17 we finally get

$$g_{ab}(x) = \eta_{ab} + R_{acbd}|_p \frac{x^c x^d}{3} + \mathcal{O}(\epsilon^3). \tag{2.19}$$

Working in RNC we raise and lower indices using η_{ab}.

2.3. Local Inertial Null Normal Coordinates

In the context of black hole horizon we introduce a set of coordinate commonly known as Null Normal coordinate (NNC). These coordinates are defined as follows: let Σ be a codimension two spatial slice of the horizon cross section, at each point on Σ we can use a pair of null vectors (k^a, l^a) to span the tangent space orthogonal to Σ. There is an unique geodesic intersecting Σ orthogonally at point q on the surface and passing through a neighborhood point r. At point q the tangent to the geodesic can be expressed in terms of the linear combination of the null vectors as $Vk^a + Ul^a$. Now if point r lies at unit affine parameter then NNCs of r is given by the coefficients of this linear combination of the null vectors and the cordinates of point q on Σ, where we have used a single coordinate chart to cover Σ.

For a generic spacetime we can also define NNCs provided we cover Σ with a single coordinate chart. If Σ is sufficiently small, the null surface to the past of Σ are part of a local causal horizon(LCH), more precisely the horizon is the surface $U = 0$ restricted to $V \leq 0$. LCH are used to show how gravitational field equations can be derived from the Clausius equation of thermodynamics. In this section we would define inertial NNCs that are adapted to these kind of local surfaces.

2.3.1. Construction of the local inertial NNCs

We introduce a basis $\{l^a, k^a, e^a_A\}, A = 2, ..., n-1$, at some point p of an n-dimensional spacetime. Here l^a and k^a are null vectors orthogonal to e^a_A, where $\{e^a_A\}$ is an set of orthogonal spatial vectors. The basis vectors satisfies the relation

$$k_a k^a = l_a l^a = k_a e^a_A = l_a e^a_A = 0, k_a l^a = -1 \tag{2.20}$$

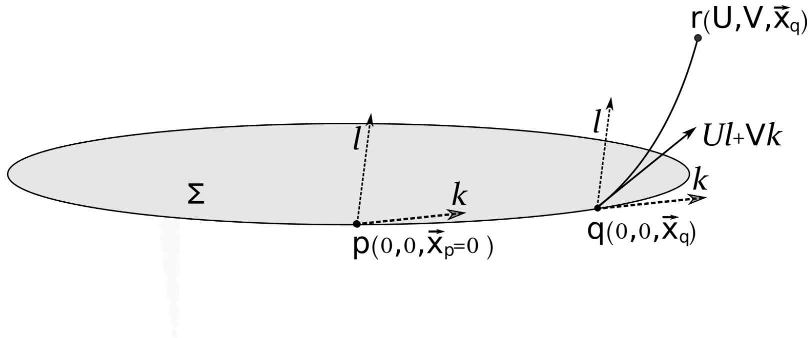

Figure 2.2: The point p lies on a $n-2$ dimensional surface Σ which is coordinatized by Riemann Normal Coordinates based upon p. Two null vectors $l^a = (\partial/\partial U)^a$ and $k^a = (\partial/\partial V)^a$ form the basis of the plane normal to Σ. The points off the surface Σ, say r, is coordinatized in terms of the geodesic from r to the surface Σ that meets Σ orthogonally at the point q. If q has coordinates $\{0, 0, \vec{x}_q\}$ and the tangent to the geodesic at q is $Vk^a + Ul^a$, then the coordinates of r are $\{U, V, \vec{x}_q\}$.

The coordinates of a point in the local neighborhood of p can be written in terms of RNC as $\{y^\alpha\} \equiv \{u, v, y^A\}$ provided it lies at unit affine parameter and the geodesic connecting p and that arbitrary point has a tangent $ul^a + vk^a + y^A e^a_A$, at p.

We can define a (n-2) dimensional spatial surface, Σ, by the equation $u = v = 0$. Now for defining NNCs we need a pair of null vector which will be orthogonal to Σ. The null vectors (k^a, l^a) defined in the beginning are orthogonal to Σ at point p but there is no guarantee that they will remain orthogonal when they are parallel transported along the generators of Σ. Also one must note that, if k^a is parallel transported along some generators of Σ and then its orthogonal projection is taken, there is no guarantee that the output would be a null vector. We have to take into consideration that the divergences of u and v are orthogonal to Σ everywhere but they are null just at point p. We can overcome this difficulty by choosing a linear combination of $\nabla^a u$ and $\nabla^a v$ and demand that they are null everywhere on Σ. After some algebra we get a set of orthogonal null vectors

$$K^a = -\nabla^a u + K\nabla^a v, \qquad L^a = -\nabla^a v + L\nabla^a u, \tag{2.21}$$

provided K and L has a very specific form which can be obtained easily. Also we must rescale K^a and L^a using the condition $K^a L_a = -1$.

From what we have established so far we can now introduce NNCs $\{U, V, x^A\}$ for a point r in the neighbourhood of p. Here point r lies on a unique geodesic that intersects Σ orthogonally at some point q and the tangent to the geodesic connecting r and q is given as UK^a+VL^a. By doing this we have established a locally inertial coordinate system for point p as it can be verified by looking at the coordinate transformation between the RNCs and NNCs and calculating the components of the Riemann tensor in terms of both coordinate systems, wich turns out to be the same upto first order.

2.3.2. Local Killing vector

A generic curved spacetime does not admit any global Killing vectors. However, one can still use the fact that there is a local inertial coordinate system defined at point p which can be used to define a Killing vector field ξ^a as a power series, writing its components in terms of the inertial NNCs. The covariant components of ξ^a are given as

$$\xi_a = U\delta_a^V - V\delta_a^U + \frac{1}{2}C_{cda}x^c x^d + \frac{1}{6}D_{cdea}x^c x^d x^e \\ +O(x^4). \tag{2.22}$$

Unlike a true Killing vector the approximate Killing vector that we defined above does not exactly satisfy the Killing identity in the entire local patch

$$\nabla_r \nabla_s \xi_a = R^m{}_{rsa}\xi_m \tag{2.23}$$

For this approximate Killing vector we can tune the coefficients of expansion given in (2.22) so that the Killing identity holds atleast upto second order (expanding in terms of NNCs) when restricted to a very specific curve, Γ. We can take this curve to be a geodesic passing through point p with tangent K^a and the points on the curve are given in terms of the NNCs as $(0, V, 0, ...)$. By doing so the power series expansion of the Killing vector becomes

$$\xi_a = U\delta_a^V - V\delta_a^U + O(x^3) \tag{2.24}$$

Also as a result of this the local Killing vector satisfies the following relations

$$\begin{aligned} \xi^a|_\Gamma &= (V - V_0)\, \mathrm{d}_V^a, & (2.25)\\ \nabla_{(a}\xi_{b)} &= O(x^2), & (2.26)\\ \nabla_a \nabla_b\, \xi_c|_\Gamma &= (R_{cba}{}^e\, \xi_e)|_\Gamma\,. & (2.27) \end{aligned}$$

The local causal horizon can be defined as the boundary of the past of Σ in the neighbourhood of point p. For making the description of space time thermodynamics more natural and analogous to first law of black hole thermodynamics we define the the bifurcation surface to the past of point p at some point p_o. The central generator, Γ, passes through p_0 having NNCs $\{0, V_0, 0, ...\}$ at p_0. By doing this all the identities of the aproximate Killing vector shown above holds provided we expand the covariant form of ξ^a as

$$\xi_a = U\delta_a^V - \tilde{V}\delta_a^U + \frac{1}{2}C_{cda}\tilde{x}^c\tilde{x}^d + \frac{1}{6}D_{cdea}\tilde{x}^c\tilde{x}^d\tilde{x}^e + O(x^4) \tag{2.28}$$

where $\tilde{x}^a = x^a - V_0\delta_V^a$ and $\tilde{V} = V - V_0$. Note this expansion does not have $\mathcal{O}(x^2)$ term as well because $C \sim \mathcal{O}(x)$.

As the Killing equation and Killing identity holds on the central geodesic Γ another consequence as shown in [35] is, the Killing vector ξ^a becomes tangent to Γ. So we get

$$\xi^a = \tilde{V}k^a \tag{2.29}$$

from this it is also clearly seen that the killing vector vanishes at point p_0 having NNCs $\{0, V_0, 0, ...\}$.

The hypersurface Σ_0 is deformed to the future only in a small neighbourhood of the point p_0, where the slice Σ_0 is given by the $V = V_0$ hypersurface. We can assume Σ to be the deformation of the horizon slice to the future and also make sure that the two slices coincide everywhere other than this small region. For this $\Sigma_0 \cup \Sigma$ forms a closed boundary of a local patch of the horizon enabling us to use Stokes theorem.

2.4. Equation of motion for a general theory of gravity

After reviewing some crucial tools we shall need later on for describing the kinematics of spacetime, we move on and discuss its dynamics in more general frameworks beyond general relativity. In particular, in this section we review the equation of motion of a general diffeomorphism invariant metric theory of gravity following ref. [39]. Lagrangian n-form is denoted in bold as $\mathbf{L} = \epsilon L$. The most general diffeomorphism invariant Lagrangian is of the form

$$\mathbf{L} = \mathbf{L}\left[g_{ab}, R_{abcd}, \nabla_{a_1}R_{abcd}, ..., \nabla_{(a_1}...\nabla_{a_m)}R_{abcd}, \psi, \nabla_{a_1}\psi, ..., \nabla_{(a_1}...\nabla_{a_l)}\psi\right], \tag{2.30}$$

where ψ denote the matter fields.

The equation of motion for g_{ab} following from the above Lagrangian is given by,

$$A^{ab} + E^{pqra} R_{pqr}{}^{b} + 2\nabla_p \nabla_q E^{pabq} = 0, \tag{2.31}$$

where E^{abcd} would be the equation of motion for R_{abcd} if we were to treat it as an independent field,

$$E^{abcd} = \frac{\partial L}{\partial R_{abcd}} - \nabla_{a_1} \frac{\partial L}{\partial \nabla_{a_1} R_{abcd}} + ... + (-1)^m \nabla_{(a_1} ... \nabla_{a_m)} \frac{\partial L}{\partial \nabla_{(a_1} ... \nabla_{a_m)} R_{abcd}}, \tag{2.32}$$

and A^{ab} is

$$A^{ab} = \frac{\partial L}{\partial g_{ab}} + \frac{1}{2} g^{ab} L + B^{ab}. \tag{2.33}$$

The origin of the last term B^{ab} is as follows: a typical term in the variation of the Lagrangian due to the derivatives of Riemann is of the form

$$\epsilon \frac{\partial L}{\partial \nabla_{(a_1} ... \nabla_{a_i)} R_{abcd}} \mathrm{d}\nabla_{(a_1} ... \nabla_{a_i)} R_{abcd}, \tag{2.34}$$

and this can be calculated as

$$\begin{aligned} &= \epsilon \ \frac{\partial L}{\partial \nabla_{(a_1} ... \nabla_{a_i)} R_{abcd}} \nabla_{a_1} \mathrm{d}\nabla_{(a_2} ... \nabla_{a_i)} R_{abcd} \\ &+ \epsilon \ \cdot (\text{terms proportional to } \nabla \delta g) \\ &= \text{exact differential} \\ &+ \text{terms contributing to } E_{abcd} \\ &+ \epsilon \ \cdot (\text{terms proportional to } \delta g), \end{aligned} \tag{2.35}$$

where integration by parts was used in both the terms in going from the first equality to the second equality. It is the last term of eq. (2.35), which is proportional to δg_{ab}, that we denoted as B^{ab} appearing as the last term in eq. (2.33). We direct the reader to ref. [39] for the details.

2.5. Quantum field theory in flat spacetime

Now that we have all the classical tools, let us turn our attention toward the interface between gravity and the quantum world. As a first step let us consider the case when one wants to keep the spacetime description classical while allowing matter to be quantized. To begin with we review the case of quantum field theory in flat spacetime. Let us consider a $(1+1)$ dimensional system consisting of a real massless scalar field, $\Phi(x,t)$, satisfying the field equation

$$\partial^a\partial_a\Phi = 0. \tag{2.36}$$

This field equation can be derived from the action of the system which is given as

$$S = \frac{1}{2}\int d^2x\sqrt{-g}g_{ab}\partial^a\Phi\partial^b\Phi. \tag{2.37}$$

In the process of canonical field quantization [49] we define Φ and its conjugate momentum, $\pi(x,t)$, as an operator satisfying the equal time commutation relation given as

$$\begin{aligned}
\left[\Phi(x,t),\Phi(x',t)\right] &= 0, \\
\left[\pi(x,t),\pi(x',t)\right] &= 0, \\
\left[\Phi(x,t),\pi(x',t)\right] &= i\delta(x-x^{'}).
\end{aligned} \tag{2.38}$$

Using (2.36) we can define the Klein–Gordon inner product calculated on a spacelike hypersurface as

$$(\Phi_1,\phi_2) = -i\int d\Sigma^a\sqrt{-g_\Sigma}(\Phi_1\overleftrightarrow{\partial_a}\phi_2^*) \tag{2.39}$$

2.5.1. Quantization in Minkowski coordinates

We take the Minkowski line element in $(1+1)$ dimensions which is given by

$$ds^2 = dt^2 - dx^2. \tag{2.40}$$

For this line element, equation (2.39) can be written as

$$(\partial_t^2 - \partial_x^2)\Phi = 0. \tag{2.41}$$

The solution of this equation after proper normalization is give by

$$u_k = \frac{1}{\sqrt{4\pi\omega}} e^{-i(\omega t - kx)}. \tag{2.42}$$

These modes are defined to be positive frequency modes since they are eigenfunctions of the operator $(\partial/\partial t)$ with positive eigen values. Using (2.36) and calculating the scalar product on $t =$ constant hypersurface we define a set of orthogonality relation between the modes u_k and their complex conjugates u_k^* as

$$(u_k, u_{k'}) = \delta(k - k^{'}); \qquad (u_k^*, u_{k'}^*) = -\delta(k - k^{'}); \qquad (u_k, u_{k'}^*) = 0 \tag{2.43}$$

From the above relations it is clear that the normal modes defined in (2.42) and their complex conjugates form a complete orthonormal basis which can be used for expanding the scalar field as

$$\Phi(x, t) = \int dk \left(\hat{a}_k u_k(t, x) + \hat{a}_k^\dagger u_k^*(t, x) \right), \tag{2.44}$$

where $\hat{a}_k$ and $\hat{a}_k^\dagger$ are the annihilation and the creation operators which satisfy the standard commutation relations. Using the annihilation operator the Minkowski vacuum state can be defined as

$$\hat{a}_k |0_M\rangle = 0 \tag{2.45}$$

From the Minkowski vacuum state defined above we can obtain the multi particle states by repeated application of the creation operator, $\hat{a}_k^\dagger$.

2.5.2. Quantization in accelerated frame

In a similar way as in the previous section, we can write the field equation in an accelerated frame of reference (Rindler coordinates) and quantize the system. But before doing so let us set up the Rindler coordinates which describes an accelerated observer.

For an observer traveling with uniform acceleration, a, the equation of motion is given as

$$\frac{d}{dt}(\gamma u) = a, \tag{2.46}$$

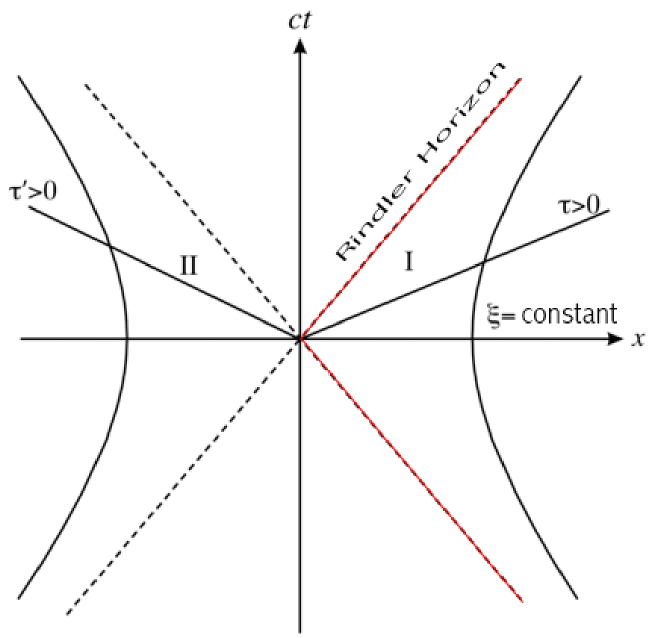

Figure 2.3: The figure shows the Rindler wedge (region bounded by red line) and the Rindler horizon.

where u is the velocity of the observer and $\gamma = 1/\sqrt{1-u^2}$. Integrating Eq. (2.46) and using the boundary condition, $u = 0$ at $t = 0$ gives

$$u = at\sqrt{1 + a^2t^2}. \tag{2.47}$$

We integrate Eq. (2.47) to get x in terms of t as

$$x = \frac{1}{a}\sqrt{1 + a^2t^2}, \tag{2.48}$$

where we have used the condition that $x = 1/a$ at $t = 0$. The proper time τ of an observer set in the accelerated frame is related to the Minkowski time t as follows

$$d\tau = dt\sqrt{1 - v^2}. \tag{2.49}$$

We integrate this equation and get

$$\tau = \frac{1}{a}\sinh^{-1}(at). \tag{2.50}$$

Using Eqs. (2.48) and (5.97), we can write

$$x = \frac{1}{a}\cosh(a\tau); \quad \text{and} \quad \mathrm{t} = \frac{1}{\mathrm{a}}\mathrm{sinh}(\mathrm{a}\tau) \tag{2.51}$$

Using the fact that any two dimensional coordinate system is conformally flat we can write the relation between the coordinates of an accelerated frame (τ, ξ) and Minkowski coordinate (x, t) as

$$t = \frac{1}{a}e^{\xi a}\sinh(a\tau), \quad \text{and} \quad x = \frac{1}{a}e^{\xi a}\cosh(a\tau). \tag{2.52}$$

In these newly defined coordinate the flat space line element takes the form

$$ds^2 = e^{2g\xi}(d\tau^2 - d\xi^2). \tag{2.53}$$

From the transformation defined in (2.52) we see that in the range $-\infty < \tau < \infty$ and $-\infty < \xi < \infty$ the coordinates only cover the right wedge of the two dimensional Minkowski spacetime. Thus the Rindler coordinates are incomplete and we can infer this by saying that the accelerated observer cannot observe more than $1/a$ in the direction opposite to its direction of motion. Since events beyond the right wedge cannot be observed we can think of it as a horizon.

In the Rindler coordinates the field equation (2.36) takes the form

$$(\partial_\tau^2 - \partial_\xi^2)\Phi(\tau, \xi) = 0. \tag{2.54}$$

Solving this equation we can write the wave modes after proper normalization as

$$v_{\tilde{k}}(\tau, \xi) = \frac{1}{\sqrt{4\pi\tilde{\omega}}}e^{-i(\tilde{\omega}\tau - \tilde{k}\xi)} \tag{2.55}$$

Using (2.39) the orthogonality relation between these modes and their complex conjuate $v^*_{\tilde{k}}$ can be defined as

$$(v_{\tilde{k}}, v_{\tilde{k}'}) = \delta(\tilde{k} - \tilde{k}'); \qquad (v^*_{\tilde{k}}, v^*_{\tilde{k}'}) = -\delta(\tilde{k} - \tilde{k}'); \qquad (v_{\tilde{k}}, v^*_{\tilde{k}'}) = 0. \tag{2.56}$$

As these modes and their complex conjugate form a complete basis, the expansion of the scalar field in terms of these modes are given by

$$\Phi(\tau, \xi) = \int d\tilde{\omega}(\hat{b}_{\tilde{k}} v_{\tilde{k}}(\tau, \xi) + \hat{b}^{\dagger}_{\tilde{k}} v^{*}_{\tilde{k}}(\tau, \xi)) \tag{2.57}$$

Here $\hat{b}_{\tilde{k}}$ and $\hat{b}_{\tilde{k}^{\dagger}}$ are the annihilation and the creation operators defined in the Rindler coordinated, which satisfy the standard commutation relation. The vacuum state in this new coordinate system can be defined as

$$\hat{b}_{\tilde{k}}|0_R\rangle = 0 \tag{2.58}$$

where $|0_R\rangle$ is refered to as the Rindler vacuum.

2.5.3. Unruh effect

The fact that the vacuum in Minkowski space $|0_M\rangle$ appears to be a thermal state when viewed by an accelerated observer is known as Unruh effect. For derivation of the Unruh effect we need to express the Minkowski modes in terms of the Rindler modes by means of the so called Bogolubov transformations.

As both sets of normal modes, u_k and $v_{\tilde{k}}$ are complete we can express one of them in terms of the other as

$$v_{\tilde{k}}(\tau, \xi) = \int dk \left(\alpha(k, \tilde{k}) u_k(t, x) + \beta^*(k, \tilde{\kappa}) u^*_{k}(t, x) \right) \tag{2.59}$$

$$u_k(t, x) = \int d\tilde{k} \left(\alpha^*(k, \tilde{k}) v_{\tilde{k}}(\tau, \xi) - \beta(k, \tilde{\kappa}) v^*_{\tilde{k}}(\tau, \xi) \right) \tag{2.60}$$

The quantities $\alpha(k, \tilde{k})$ and $\beta(k, \tilde{k})$ are known as the Bogolubov coefficients. We can use the inner product defined in (2.39) and the orthonormality conditions of the mode to express the Bogolubov coefficients as follows

$$\alpha(k, \tilde{k}) = (v_{\tilde{k}}, u_k) \qquad\qquad \beta(k, \tilde{k}) = -(v_{\tilde{k}}, u^*_k) \tag{2.61}$$

The annihilation operators $\hat{a}_k, \hat{a}^{\dagger}_k$ and $\hat{b}_{\tilde{k}}, \hat{b}^{\dagger}_{\tilde{k}}$ can be related using the Bogolubov coefficients as

$$\hat{a}_k = (u_k, \Phi(\tau, \xi)) = \int d\tilde{\omega}(\alpha(k, \tilde{k})\hat{b}_{\tilde{k}} + \beta^*(k, \tilde{k})\hat{b}^{\dagger}_{\tilde{k}}) \tag{2.62}$$

and

$$\hat{b}_{\tilde{k}} = (v_{\tilde{k}}, \Phi(t,x)) = \int d\omega(\alpha^*(k,\tilde{k})\hat{a}_k - \beta^*(k,\tilde{k})\hat{a}_k^\dagger). \tag{2.63}$$

Using the commutation relations

$$\left[\hat{a}_k, \hat{a}_{k'}^\dagger\right] = \delta(k - k'), \qquad \left[\hat{b}_{\tilde{k}}, \hat{b}_{\tilde{k}'}^\dagger\right] = \delta(\tilde{k} - \tilde{k}') \tag{2.64}$$

and from the expression for the creation and annihilation operator in Eqs. (2.62) and (2.63), we can arrive at the relations

$$\int dk \left(\alpha^*(k,\tilde{k}')\alpha(k,\tilde{k}) - \beta^*(k,\tilde{k}')\beta(k,\tilde{k})\right) = \delta(\tilde{k} - \tilde{k}'), \tag{2.65}$$

$$\int dk\big(\alpha(k,\tilde{k})\beta(k,\tilde{k}') - \beta(k,\tilde{k})\alpha(k,\tilde{k}')\big) = 0. \tag{2.66}$$

Next we calculate the Bogolubov coefficients between the Minkowski modes and the Rindler modes using Eq. (2.61). Computing the inner product on the $\tau = 0$ hypersurface we get

$$\alpha(k,\tilde{k}) = \frac{1}{4\pi\sqrt{\omega\tilde{\omega}}}\int d\xi(\omega e^{a\xi} + \tilde{\omega})e^{i\tilde{\omega}\xi}\exp\left[-i\left(ka^{-1}e^{a\xi}\right)\right], \tag{2.67}$$

$$\beta(k,\tilde{k}) = \frac{1}{4\pi\sqrt{\omega\tilde{\omega}}}\int d\xi(\omega e^{a\xi} - \tilde{\omega})e^{i\tilde{\omega}\xi}\exp\left[i\left(ka^{-1}e^{a\xi}\right)\right]. \tag{2.68}$$

When the Minkowski modes are expressed in terms of the Rindler modes, if the coefficient $\beta(k,\tilde{k})$ is non-zero then we can see from (2.63) the Minkowski vacuum will not be annihilated by the annihilation operator defined in the Rindler coordinate. The Bogolubov coefficients can be calculated by performing the integrals given in (2.67) and (2.68). At first we substitute $z = e^{a\xi}$ and the integrals reduce to

$$\alpha(k,\tilde{k}) = \frac{a^{-1}}{4\pi\sqrt{\omega\tilde{\omega}}}\int \mathrm{d}z(\omega z + \tilde{\omega})z^{ilg^{-1}-1}e^{-ikzg^{-1}}, \tag{2.69}$$

$$\beta(k,\tilde{k}) = \frac{a^{-1}}{4\pi\sqrt{\omega\tilde{\omega}}}\int \mathrm{d}z(\omega z - \tilde{\omega})z^{ilg^{-1}-1}e^{ikzg^{-1}}. \tag{2.70}$$

Using the known identity given by

$$\int_0^\infty x^{s-1}e^{-bx}dx = \exp(-s\ln b)\Gamma(s) \tag{2.71}$$

and using proper cut-off which is set to zero finally, the integrals can be evaluated to give

$$\alpha(k,\tilde{k}) = \frac{1}{4\pi a}\sqrt{\frac{\omega}{\tilde{\omega}}}(\omega\tilde{k} + k\tilde{\omega})\left(\frac{k}{a}\right)^{-i\tilde{k}a^{-1}} \Gamma(-i\tilde{k}a^{-1})e^{\pi\tilde{k}/2a} \tag{2.72}$$

and

$$\beta(k,\tilde{k}) = -\frac{1}{4\pi a}\sqrt{\frac{\omega}{\tilde{\omega}}}(\omega\tilde{k} + k\tilde{\omega})\left(\frac{k}{a}\right)^{-i\tilde{k}a^{-1}} \Gamma(-i\tilde{k}a^{-1})e^{-\pi\tilde{k}/2a}. \tag{2.73}$$

From the above expressions we get the relation

$$\beta(k,\tilde{k}) = -\alpha(k,\tilde{k})e^{-\pi\tilde{k}/a}. \tag{2.74}$$

The number operator can be defined in Rindler coordinates to be $\hat{b}_{\tilde{k}}\hat{b}^{\dagger}_{\tilde{k}}$. The expectation value of this Rindler number operator in Minkowski vacuum is given as

$$\langle 0_M|N_R|0_M\rangle = \langle 0_M|\hat{b}_{\tilde{k}}\hat{b}^{\dagger}_{\tilde{k}}|0_M\rangle = \int dk|\beta(k,\tilde{k})|^2, \tag{2.75}$$

where we use Eq. (2.63) for arriving at the final expression. Using equation (5.41) we get

$$\langle 0_M|N_R|0_M\rangle = \frac{1}{2\pi k}\int\left[\frac{a^{-1}dk}{\exp(2\pi\tilde{\omega}a^{-1}) - 1}\right] \tag{2.76}$$

Thus it is clear that the Rindler number operator in $|0_M\rangle$ state gives a thermal spectrum at temperature $a/2\pi$. This also shows that the quantization in Minkowski and the Rindler coordinates are inequivalent which we shall further discuss in the next section.

2.5.4. Inequivalent quantization and correlation in vacuum

As mentioned previously, the Rindler coordinates cover only a part of the Minkowski space and an observer in Rindler spacetime cannot obtain any information from the region beyond the Rindler horizon. Defining a new set of coordinates as

$$t + x = \tilde{v} \;=\; g^{-1}e^{gv}, \tag{2.77}$$

$$t - x = \tilde{v} \;=\; -g^{-1}e^{gu}, \tag{2.78}$$

where $v = \xi + \tau$ and $u = \xi - \tau$. In these coordinates, the Rindler line element (2.53) takes the from

$$ds^2 = e^{2g\xi} du dv \tag{2.79}$$

and the general solution of the wave equation in these coordinates can be written as $P(u) + F(v)$. The outgoing mode i.e. the mode dependent on u when expressed in terms of the Minkowski null coordinates defined in (2.78) is given as

$$p = \exp\left[i \frac{\omega}{g} \ln(-\tilde{u}) \right]. \tag{2.80}$$

Clearly this field mode cannot be defined for all values of $\tilde{u}$ as $p = 0$ for $\tilde{u} > 0$.

The Rindler spacetime exhibits the boost Killing vector given by $\chi = \partial/\partial\tau$. In terms of the null coordinates defined for Minkowski space χ takes the form

$$\chi = g(\tilde{v}\partial_{\tilde{v}} - \tilde{u}\partial_{\tilde{u}}) \tag{2.81}$$

It is easily seen that the field mode defined in (2.80) is a positive frequency mode with respect to ξ but it is not a purely positive frequency mode with respect to the timelike Killing vector $(\partial/\partial t)$ defined for Minkowski space. The field mode p is not purely positive frequency with respect to u coordinate is also evident from the fact that p vanishes for $u > 0$ and a purely positive frequency mode cannot vanish on any open interval.

For expressing the Minkowski wave modes in terms of the Rindler modes we can define p for $\tilde{u} > 0$ by analytic continuation. If we take the branch cut of ln$(\tilde{u})$ on the upper half plane we can define a function ln$(\tilde{u} + i\pi)$ which is analytic for positive values of $\tilde{u}$ and it agrees with ln$(-\tilde{u})$ on the negative real axis. Using this we can define a new mode which will have purely positive $\tilde{u}$ frequency. This mode is defined as

$$h = p(\tilde{u}) + e^{\frac{-\pi\omega}{g}} p(-\tilde{u}), \tag{2.82}$$

and the annihilation operator defined by $a(h) = (h, \phi)$ gives $a(h)|0\rangle_M = 0$. By using linearity of the Klein–Gordon product we can express the annihilation operator defined above in terms of the Rindler annihilation and creation operator as

$$a(h) = a(p) + e^{-\pi\omega/g} a(\tilde{p}), \tag{2.83}$$

where $\tilde{p}(\tilde{u}) = p(-\tilde{u})$ and $a(\tilde{p}) = -a^{\dagger}(\tilde{p}^{*})$ Using eq. (2.83) the expectation value of the number of particle in Minkowski space as detected by an Rindler observer can be calculated after normalizing the modes properly and a Planckian spectrum can be obtained with the temperature $a/2\pi$.

2.6. Quantum field theory in curved spacetimes

Let us now move to the more general case of curved spacetimes. In particular we shall consider in detail the black hole case. Even classically rotating black holes (Kerr black holes) exhibits the phenomenon of super radiant scattering [50, 51] which involves stimulated emission when a scalar field is incident on a rotating black hole. This suggests that when quantum fields are studied around a rotating black hole it should also exhibit spontaneous emission. Remarkably it was found by Hawking that even static black holes exhibit this phenomenon of spontaneous radiation in the presence of quantum fields. This phenomenon of spontaneous emission of particle from a black hole is known as Hawking radiation.

When a free quantized scalar field passes through the interior of a collapsing star its modes gets red-shifted. As this field crawls out of the surface of the star undergoing collapse the extent of red-shift increases. On performing the Bogolubov transformation between the standard outgoing Minkowski field modes and the red-shifted modes emerging from the star we get a Planckian spectrum of particle. Thus it implies that the initial "in vacuum" state contains a thermal flux of outgoing particles at late times. We work in $(1+1)$ dimension by choosing a two dimensional metric corresponding to a spherical collapsing object. This is done to avoid the difficulty of dealing with mathematical complexity while we get the same result when extended to $(3+1)$ dimensions. The way in which the star is collapsing is also kept arbitrary as it does not effect the final result as long as it asymptotically settles down to a Schwarzschild black hole.

2.6.1. Particle production by black hole

When a star start collapsing, initially spacetime is nearly flat and thus the Minkowski vacuum is a good approximation describing the vacuum state for such a configuration. When the star has collapsed sufficiently to form a black hole, the exterior spacetime of the star is described by the Schwarzschild metric and in this region we also need to define a new "out vacuum" state based on the late time annihilation operator. In this section we will calculate the Bogolubov transformation between the "in" and "out" vacuum state to obtain the thermal spectrum of particle at late times far away from the black hole.

Now we look at mode solutions of the standard Klein-Gordon equation for Schwarzschild spacetime. The Klein–Gordon equation can be written as

$$\frac{1}{\sqrt{-g}}\partial_a\left[\sqrt{-g}g^{ab}\partial_b\phi\right]=0 \tag{2.84}$$

In the asymptotic region defined by $r\to\infty$ the solution of radial part of Eq. (2.84) is simply given as $e^{\pm i\omega r}$. We change variable r to r^*, where r^* is defined as the tortoise coordinates. In terms of these redefined coordinates the solution of Eq. (2.84) is given as

$$\frac{1}{\sqrt{2\pi\omega}}e^{-i\omega(t-r^*)/rY_{lm}(\theta\phi)} = \frac{1}{\sqrt{2\pi\omega}}e^{i\omega u/rY_{lm}(\theta\phi)}, \tag{2.85}$$

$$\frac{1}{\sqrt{2\pi\omega}}e^{-i\omega(t+r^*)/rY_{lm}(\theta\phi)} = \frac{1}{\sqrt{2\pi\omega}}e^{i\omega v/rY_{lm}(\theta\phi)}. \tag{2.86}$$

When working in (1+1) dimensions and neglecting the effect of back scattering of field modes these mode solutions reduces to the standard flat space form for large distances. We can define a vacuum with respect to these modes as $a|0_M\rangle = 0$ where a is the annihilation operator defined with respect to the modes given in Eqs.(2.85)and (2.86). This suggests that there is no incoming radiation from $\mathcal{I}^-$. Due to the presence of the collapsing star these modes will get red-shifted which otherwise would have propagated in the same initial form.

We now compute the red-shifted modes reaching $\mathcal{I}^+$ after passing through the collapsing star. In $(1+1)$ dimensions the spacetime in the exterior region of the collapsing star can be best described by the Schwarzschild metric. We take an arbitrary form of the metric defined in terms of the null coordinates as

$$ds^2 = C(r)dudv, \tag{2.87}$$

where

$$u = t-(r^*-R_0^*), \tag{2.88}$$

$$v = t+(r^*-R_0^*), \tag{2.89}$$

where R_0^* is a constant and r^* is defined as

$$r^* = \int C(r)^{-1} dr. \tag{2.90}$$

This arbitrary metric is assumed to be asymptotically flat and this is given by the condition $C(r) \to 1$ in the limit $r \to \infty$. The interior spacetime of the collapsing star is defined by a metric which in a arbitrary form is given as

$$ds^2 = A(U, V) dU dV, \tag{2.91}$$

and

$$U = \tau - (r - R_0), \tag{2.92}$$

$$V = \tau + (r - R_0), \tag{2.93}$$

R_0 and R_0^* are related in the same way as r and r^*. It is assumed that at $\tau = 0$ the star is at rest and the surface of the star is given by $r = R_0$. To depict the scenario of a wave entering the collapsing star and emerging out we assume that the wave gets reflected at $r = 0$, which is the center of the star, and restrict the treatment to only positive values of r. To achieve this we need to impose the boundary condition $\phi = 0$ at $r = 0$.

We denote the relation between the interior and the exterior coordinates of the star, ignoring any reflection at the surface of the star, in an arbitrary functional form as

$$U = \alpha(u), \tag{2.94}$$

$$v = \beta(V). \tag{2.95}$$

Using Eqs. (2.92) and (2.93), we can define the centre $(r = 0)$ of the radial coordinate by the line

$$V = U - 2R_0. \tag{2.96}$$

For $\tau > 0$ the star starts collapsing along the worldline $r = R(\tau)$. Matching the interior and the exterior metric along this collapsing surface we get

$$\frac{dU}{du} = \frac{C(1 - \dot{R})}{[AC(1 - \dot{R}^2) + \dot{R}^2] - \dot{R}}, \tag{2.97}$$

$$\frac{dv}{dV} = \frac{[AC(1 - \dot{R}^2) + \dot{R}^2] + \dot{R}}{C(1 + \dot{R})}, \tag{2.98}$$

where C is evaluated at $r = R(\tau)$ and the overdot represents differentiation with respect to τ. When the star collapses to a sufficiently small region in spacetime, it forms a black hole. At this point the surface of the collapsing star coincides with the event horizon of the black hole which is given by $C = 0$. When this condition is used Eqs.(2.97) and (2.98) reduce to

$$\frac{dU}{du} \sim (\dot{R} - 1)C(R)/2\dot{R}, \tag{2.99}$$

$$\frac{dv}{dV} \sim A(1 - \dot{R})/2\dot{R}, \tag{2.100}$$

and calculating the second limit using the standard L Hospital's rule. Now, near the event horizon, $R(\tau)$ can be expanded as

$$R(\tau) = R_h - \dot{R}(\tau_h)(\tau_h - \tau) + O([\tau_h - \tau]^2), \tag{2.101}$$

where we have defined $R = R_h$ at the horizon.Using this and integrating Eq. (2.99) we get

$$U = De^{-\kappa u} + constant. \tag{2.102}$$

we define κ as the surface gravity of the black hole and it is given as

$$\kappa = \frac{1}{2}\frac{\partial C}{\partial r}|_{R_h}. \tag{2.103}$$

Integrating Eq.(2.100) we see that the relation between v and V is linear.

The mode functions are given by the solution of the equation (2.84) with the boundary condition that $\phi = 0$ at $r = 0$.We mentioned previously (2.96) how the center of the radial coordinate system is defined. Using these facts we can write the solution to the field mode as

$$\frac{1}{\sqrt{4\pi\omega}}(e^{-i\omega v} - e^{-i\omega\beta[\alpha(u)-2R_0]}), \tag{2.104}$$

where we defined v at $r = 0$ using (2.96) to be

$$v = \beta[V] = \beta[\alpha(u) - 2R_0]. \tag{2.105}$$

This solution shows how the outgoing modes get complicated due to the red-shifting. Using (2.102) we can write this outgoing mode as

$$f_\omega = \frac{1}{\sqrt{4\pi\omega}} e^{-i\omega(a\exp[-\kappa u]+b)} \,. \tag{2.106}$$

This mode can be expressed in terms of the standard outgoing modes(i.e the modes which have not suffered the red-shifting) as

$$f_\omega = \frac{1}{2\pi} \int \left[\alpha_{\omega\Omega} e^{-i\Omega u} + \beta_{\omega\Omega} e^{i\Omega u} \right] d\Omega. \tag{2.107}$$

The particle content of this outgoing modes can be calculated in a similar way as in section (2.5.3) by calculating the Bogoliubov coefficients we get

$$\langle N_\Omega \rangle = |\beta_{\omega\Omega}|^2 = \frac{2M}{\exp[8\pi M\Omega] - 1}, \tag{2.108}$$

where we have made use of the fact that the surface gravity, $\kappa = 1/4M$ and M as the mass of the black hole. Thus equation (2.108) gives a thermal spectrum of particles at the temperature of

$$T = \frac{\kappa}{2\pi}. \tag{2.109}$$

Here we neglected any back-scattering of the mode due the black hole. This assumption is not valid in $(3+1)$ dimensions as the angular part of the Klein-Gordon equation written in the Schwarzschild coordinates will act as an effective potential which effectively scatter off the incoming waves partially. Taking this fact into account we can modify the spectrum of particle by introducing an absorption factor as

$$N_p = \frac{\Gamma_p}{e^{\omega/T_H} - 1}, \tag{2.110}$$

Where T_H is the Hawking temperature and Γ_p is the factor that indicates the emissivity of the black hole and it is known as *greybody factor* [49, 52]. The presence of the *greybody factor* also shows that a black hole does not behave as a perfect black body.

One plausible way of explaining Hawking radiation is by the tunneling mechanism [53]. Another heuristic way of looking at black hole evaporation is considering the fact that due to quantum fluctuations virtual particles and anti-particles are continuously created. When the the separation between these virtual particles are of the order of the size of the black hole, strong tidal forces prevents re-annihilation of these pairs. The particle having positive energy escapes out to infinity and contributes to the flux of radiation obtained at $\mathcal{I}^+$ and the other particle having negative energy falls into the black hole singularity. As the black hole absorbs the negative energy particle the energy and the mass of the black hole reduces. We will look at a more precise mathematical derivation using this heuristic argument in section (3.2).

2.7. Hawking radiation – some essential aspects

In this section we discuss two of the most disturbing and unsolved aspects of Hawking radiation, the trans Planckian issue and the information loss paradox. Both these issues hints at the fact that the semi-classical setup we are working in is probably not the best framework for studying black hole thermodynamics and at some point the quantum effects of gravity must be considered.

2.7.1. The trans-Planckian issue

It was stated earlier that the model with which we are working, for investigating the Hawking radiation, is asymptoticly flat. For a static observer to detect particle at $\mathcal{I}^+$ the state near the horizon must be vacuum as described by a free-fall observer, i.e. an observe who is falling across the event horizon freely. The generic state which we defined at the past null infinity is Minkowski vacuum and thus one must show that the free-fall vacuum must result from the initial vacuum which we choose. This was done by tracing the v modes backward in time and through the collapsing star to the past null infinity. By doing this the free-fall frequency matches with the Killing frequency at $\mathcal{I}^-$ as we demanded. But tracing the mode backward in time has a subtle problem involved with it [54, 55] . As these modes are propagated backwards they get exponentially blue-shifted with respect to the Killing time. For a outgoing quanta of radiation at a time t after the black hole is formed the amount of blue-shifting of the propagated mode to $\mathcal{I}^-$ is determined by a factor of $e^{\kappa t}$ where κ is the surface gravity of the black hole. This factor can be found from the relation between the frequency of a mode at the past null infinity and the future null infinity. A mode of frequency Ω on $\mathcal{I}^-$ is related to a mode of frequency ω on $\mathcal{I}^+$ as

$$\omega(u, \Omega) = \alpha'(u)\Omega, \tag{2.111}$$

where $\alpha'(u)$ is defined by (2.102). For a Hawking mode of frequency $\sim \kappa$ the typical frequency of the field mode at $\mathcal{I}^+$ is given as

$$\Omega \sim \kappa/\alpha'(u) \propto e^{\kappa u}. \tag{2.112}$$

From this relation we see that for $u \to \infty$ the blue shifting is enormously large. This issue was tackled in several ways though it needs a more concrete resolution [56, 57, 58].

2.7.2. Information loss

By studying Hawking radiation within the framework of semi-classical gravity one can show that the final state resulting from the evaporation process should be a mixed state. This results from the fact that the *outgoing* Hawking modes are maximally entangled with the *ingoing* modes and eventually when the black hole has completely evaporated we are just left with thermal radiation in a mixed state. This evolution from a pure state to a mixed state is not allowed within the known laws of quantum mechanics, giving rise to the black hole information loss problem.

Before going into the details of the information loss problem we briefly review the concept of entanglement first. Let us assume we have two quantum mechanical system which are represented by the Hilber space $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively. The joint state of the two systems would be represented in a tensor product of the individual Hilbert space as $\mathcal{H}_1 \otimes \mathcal{H}_2$. A general state in this tensor product Hilbert space is given as

$$|\psi\rangle = \sum_i c_i |\psi_{1i}\rangle \otimes |\psi_{2i}\rangle. \tag{2.113}$$

It is possible to encode the information just relavant for the first observer as a part of the general state by a reduced density matrix defined as

$$\rho_1 = \sum_{ij} c_i c_j^* \langle\psi_{2j}|\psi_{2i}\rangle |\psi_{1i}\rangle\langle\psi_{1j}|. \tag{2.114}$$

A normalized state $|\psi\rangle$ can be written as a product state $|\psi_1\rangle \otimes |\psi_2\rangle$ only if $\rho_1^2 = \rho_1$, and in this case the system is said to be in *pure state*. This implies that the product state just behaves as the state on which the measurement is made and the other state is irrelevant for this measurement. However, a state as given in (2.113) cannot be written as a product state as mentioned above. In such a scenario both the states are said to be entangled and on its own each state would be a *mixed state*. For a mixed state the result of measurement done on one state would be dependent on the other state as well due to nontrivial correlation between the states. In particular, for two entangled states if there are two observables, $\mathcal{O}_1$ and $\mathcal{O}_2$ for two systems respectively, then

$$\langle\psi|\mathcal{O}_1 \otimes \mathcal{O}_2|\psi\rangle \neq \langle\psi|\mathcal{O}_1|\psi\rangle\langle\psi|\mathcal{O}_2|\psi\rangle\,. \tag{2.115}$$

The density matrix for a mixed state is essentially a probability density of an ensemble, thus one can naturally define the Von Neumann entropy as

$$S = -\mathrm{Tr}(\rho \mathrm{log}\rho). \tag{2.116}$$

For the two entangled systems we considered, the entanglement entropy of a subsystem would be just given as the Von Neumann entropy of the reduced density matrix for the subsystem under consideration,

$$S_1 = -Tr(\rho_1 \mathrm{log}\rho_1), \quad S_2 = -Tr(\rho_2 \mathrm{log}\rho_2). \tag{2.117}$$

Now we try to understand the information loss problem. Let us assume a pure state outside the black hole consisting of n numbers of EPR pairs, and if we throw one of each pair into the balck hole, we end up with

$$S_{inside} = S_{outside} = S_{entanglement} = n\mathrm{log}2\,. \tag{2.118}$$

When the black hole evaporates completely we are left with half of each pair in a highly mixed state with a total entanglement entropy given as,

$$S_{entanglement,final} = n\mathrm{log}2. \tag{2.119}$$

Now we could have started with a pure state having $S_{entanglement} = 0$, then we would be still inevitably left with a highly mixed state. We cannot describe this transition from a pure state to a mixed state by any unitary transformation [16] and this lies at the root of the information loss problem.

One can follow the analysis of the information loss by Page as given in [59, 60] to have a better understanding of the problem in a nice quantitative way. The Bekenstein–Hawking entropy, $S_{BH} = A/4\hbar G$, follows the first and second law of thermodynamics, so one can assume that the maximum number of internal states available to a black hole of area A would be given by $e^{S_{BH}}$ [61]. In this sense S_{BH} should set the upper bound on the entanglement entropy (Von Neumann entropy) of any black hole.

Now evaporation of the black hole causes its surface to shrink and thus S_{BH} must decrease steadily due to Hawking radiation. On the other hand the entanglement entropy of the black hole must increase steadily as explained earlier. So in the lifetime of a black hole (within a finite time), a contradiction is reached when $S_{BH} = S_{entanglement}$, as after this point the entanglement entropy must exceed the Bekenstein–Hawking entropy. For a Black hole that begins in a pure state, this happens when S_{BH} has been reduced to half of its initial value due to the evaporation process. This is often referred to as the Page time, t_{page}. For a 4D Schwarzschild black hole $t_{page} = R_0^3/l_{Planck}^2$. If Hawking radiation is to be a pure state then $S_{entanglement}$ must start to decrease and eventually become zero when the black hole disappears. If this must happen it should begin to decline at the midpoint of the lifetime of a black hole (around t_{page}), this behavior of the entropy is shown in the *Page curve*.

In a ordinary thermodynamic system, such as burning a piece of coal, the entanglement entropy follows a similar behavior as depicted by the *Page curve*. The earlier radiated photons would be entangled to the remaining coal but the ones emitted in the late time can carry information from the burning coal and have an imprint of the quantum states due to the excitations inside the burning system. The main difference with such a system and Hawking radiation is the presence of the horizon. Due to the black hole horizon the internal excitations cannot imprint anything on the photons emitted at late time thus $S_{entanglement}$ cannot decrease by any means. This was also shown in a more rigorous way by using the *strong subadditivity* of the Von Neumann entropy in [18].

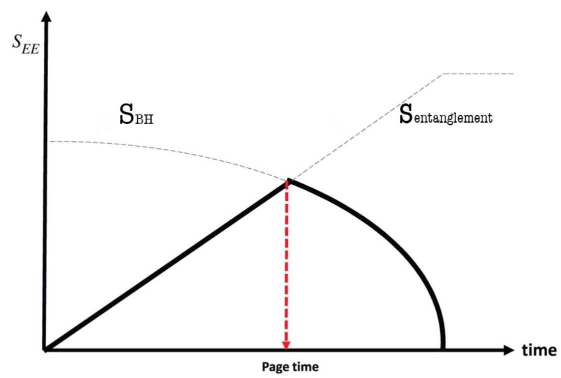

Figure 2.4: The plot shows how the different entropies would evolve as the black hole evaporates.

Examining the assumptions made for the formulation of the information loss problem and speculating that some of them can be wrong, it is possible to conjecture ways to avoid the problem. Some well known alternative scenarios are :

- **Firewall**: A hypothetical phenomenon known as black hole firewall was proposed in [20, 62], where an in-falling observer would encounter high energy quanta very close to the event horizon. As we know the *early* and *late* time modes cannot be entangled, due to no-cloning theorem of entanglement, as the *early* mode is already maximally entangled with modes inside the horizon. So it was proposed that the late time radiation can purify the early time one only when black hole complementarity (as proposed by Susskind in [63]) is violated in some way. It can be assumed that a black hole forms in the expected manner but the evaporation process would not follow semi-classical theory strictly. In this manner the entanglement between the state of quantum field observables inside and outside the black hole can be diminished.

  The firewall would imply some deviation from semi-classical physics at a low curvature region near the horizon even before Page time. Thus the diminishing or vanishing entanglement would be possible but eventually one cannot assume a Unruh state, which is the unique non-singular state at the horizon, as a result of a semi-classical collapse [64]. From this it is clear that the vacuum state at the horizon would be singular, converting it to a firewall (Unless some *effective* notion of complementarity is used). One can also say that the law governing the presence of a firewall has to be non-local/acausal in nature.

- **Fuzzball**: Based within the framework of string theory, it was proposed in [65], instead of a classic picture of a black hole with an event horizon, due to quantum tunneling phenomenon, a fuzzball is formed when the star collapses. More precisely according to the proposal of a fuzzball, for a black hole with entropy S, there are $\exp S$ *horizon-free non-singular* solutions that asymptotically look like the black hole but generically differ from the black hole up to the horizon scale. These solutions were called the fuzzballs and are considered to be the black hole microstates while the original black hole is represented by the average description of the system.

  In this scenario the interior singularity of the black hole is not present and the entire interior region of the black hole can be described as a ball of *strings*. Even though it is a radical proposal, considering that massive black holes can form at sufficiently low energy density and curvature, where semi-classical general relativity can provide a good description of spacetime dynamics, there is no information loss in this geometry as there is no event horizon present.

- **Remnant**: In this proposal the evaporation process stops when the black hole has reached Planck scale and quantum gravity effects starts dominating [66]. One way to realize such a remnant at the end of the evaporation is based on a modified Heisenberg uncertainty principle [67]. The remnant would contain all the information that went into the black hole hence one can say that the joint state of the remnant and its outside is pure. One major issue with such a remnant is it will violate the Bekenstein entropy bound as the remant would need to have arbitrarily high number of states to be able to entangle with all the emitted Hawking quanta.

2.8. Black hole thermodynamics

The event horizon of a black hole acts as a causal boundary which does not allow any propagation of information from the interior region of the black hole to the outside. This led Bekenstein to propose to associate an entropy with the area of the horizon of a black hole [7]. Indeed, soon after this proposal, Jim Bardeen, Brandon Carter, and Stephen Hawking derived four laws of black hole mechanics [9] which are analogous to the four laws of classical thermodynamics. In the case of Penrose process and super-radiance we can write a relation between the mass of the black hole, the area of the event horizon and its angular momentum which looks very similar to the first law of thermodynamics, upon interpreting the area of the event horizon as an entropy, the surface gravity as a temperature associated with the horizon.

The major break through came in this field after Hawking showed that a black hole can emit thermal flux of particle and the temperature is precisely related to the surface gravity of the black hole in the same way as it was demanded for the laws of black hole thermodynamics to hold and also the entropy is given as $A/4$, where A is the area of the black hole. It was thus evident that a black hole acts as a thermodynamic system which could be in thermal equilibrium with its surrounding. The importance of black hole thermodynamics lies in the fact that we can get an idea about the microscopic degrees of freedom of the space time by investigating and analyzing macroscopic quantities such as entropy. It was also shown later that the field equations governing the dynamics of gravity can be derived by extremising the entropy defined for such a system.

2.8.1. Zeroth law

The zeroth law of black hole thermodynamics states that the surface gravity of a stationary black hole remains uniform and unchanged over the entire event horizon. There are two ways in which the zeroth law can be proved, each having its own advantages and drawbacks. Firstly we can assume that a spacetime exhibits a bifurcation surface and it can be shown that the zeroth law holds. In the second approach we need to assume that the dominant energy condition holds and using a specific field equation of gravity zeroth law can be proved.

For a stationary axisymmetric black hole the Killing vector generating the horizon can be written as

$$\chi_a = \xi_t + \Omega_H \xi_\phi. \tag{2.120}$$

As this Killing field becomes null on the horizon $(\chi_a \chi^a = 0)$ we can write

$$\chi^b \chi_{a;b} = \kappa \chi_a, \tag{2.121}$$

where κ is a constant, defined as the surface gravity of the black hole. By taking Lie derivative of this equation with respect to the Killing vector field ξ_a we get

$$\kappa, a\xi^a = 0, \tag{2.122}$$

which shows that the surface gravity is constant along the generator of the horizon. Now we need to prove that the surface gravity is also constant along the event horizon (i.e. from one generator to the other).

Using the Killing equation, $\xi_{a;b} = -\xi_{b;a}$, and the Frobenius' theorem, which is given as

$$\xi_{[a} \nabla_b \xi_{c]} = 0, \tag{2.123}$$

we can write (2.121) as

$$\kappa^2 = -\frac{1}{2} \chi_{a;b} \chi^{a;b}. \tag{2.124}$$

To show that κ is constant on a bifurcate killing horizon we take the derivative of (2.124) along the tangent , k^a, to the event horizon. This gives

$$\kappa k^a \nabla_a \kappa = -\frac{1}{2} k^a \nabla_a \nabla_b \chi_c \nabla^b \chi^c. \tag{2.125}$$

Using the known identity

$$\nabla_a \nabla_b \xi_c = -R^d_{bca} \xi_a \tag{2.126}$$

we get

$$\kappa k^a \nabla_a \kappa = \frac{1}{2} k^a R^d_{abc} \xi_d \nabla^a \xi^b. \tag{2.127}$$

$$= 0 \tag{2.128}$$

This shows that the surface gravity defined on a bifurcate Killing horizon is constant. However, our spacetime might not posses a bifurcation surface and so one has to derive the zeroth law starting from Eq. (2.124) and using Einstein equation with the dominant energy condition, which states that matter should flow along timelike or null world lines. An alternative derivation to the zeroth law which does not incorporate the use of any field equation can be seen in [68]

2.8.2. First law

We now consider a stationary black hole being perturbed by some influx of matter across the horizon and ΔT_{ab} represents the variation of the energy momentum tensor. We assume that once the perturbation is removed the black hole settles down to a stationary state.

If we define a Killing parameter, τ, for the generators of the horizon then from (2.121) we see that, τ, is not an affine parameter along the null geodesic generators of the horizon. We can define an affine parameter, λ along these generators and the relation between both these parameter is

$$\lambda \propto e^{\kappa \tau}. \tag{2.129}$$

For small perturbation of the black hole we can write

$$\triangle M = \int d\tau \int d\sigma^2 \triangle T_{ab} (\xi_t)^a \chi^b, \tag{2.130}$$

$$\triangle J = -\int d\tau \int d\sigma^2 \triangle T_{ab} (\xi_\phi)^a \chi^b, \tag{2.131}$$

where $\triangle M$ and $\triangle J$ are the change in mass and angular momentum of the black hole and $d\sigma^2$ is the differential area element of the horizon. Using Eq. (2.130) and (2.131) we can write

$$\begin{aligned}\triangle M - \Omega\triangle J &= \int T_{ab}((\xi_t)^a + \Omega(\xi_\phi)^a)\chi^b d\sigma^2 d\tau \\ &= \int T_{ab}\chi^a\chi^b d\sigma^2 d\tau \end{aligned} \tag{2.132}$$

Using the Raychaudhuri's equation [69, 70] defined for the change of expansion along a geodesic which is non-affinely parametrized and retaining terms up to first order we get

$$\frac{d\theta}{d\tau} = \kappa\theta - 8\pi T_{ab}\chi^a\chi^b. \tag{2.133}$$

Using this equation from (2.132) we get

$$\begin{aligned}\triangle M - \Omega\triangle J &= -\frac{1}{8\pi}\int \tau d\sigma^2\left(\frac{d\theta}{d\tau} - \kappa\theta\right) \\ &= \frac{\kappa}{8\pi}\int \theta d\tau d\sigma^2 \\ &= \frac{\kappa}{8\pi}\int \frac{1}{\delta\sigma^2}\frac{d(\delta\sigma^2)}{d\tau}d\tau d\sigma^2 \\ &= \frac{\kappa}{8\pi}\delta\sigma^2. \end{aligned} \tag{2.134}$$

Thus we get an expression which is analogous to the first law of thermodynamics. Noticeably, there is a generalized version of the first law which does not takes into account any specific theory of gravity, and is valid for any classical theory of gravity arising from a diffeomorphism invariant Lagrangian [38].

2.8.3. Second law

This law states that the entropy of the black hole cannot decrease during any physical process if the null energy condition holds. As we saw that the entropy of a black hole, in general relativity, is related to the surface area of the event horizon, the second law states that the surface area of the black hole cannot decrease during any physical process. We saw this analogy in the case of Penrose process and super-radiance where the surface area of the black hole increased. The second law can be formulated in a mathematical way by looking at the evolution of the black hole surface area using the Raychaudhuri's equation and assuming null energy condition. From Raychaudhuri's equation we get

$$\frac{d\theta}{d\lambda} = -\frac{1}{2}\theta^2 - \sigma^{ab}\sigma_{ab} - R_{ab}k^a k^b. \tag{2.135}$$

where there is no twist in the above equation as the congruence of the null geodesics is hypersurface orthogonal. Using Einstein equation we can replace $R_{ab}k^a k^b$ by $T_{ab}k^a k^b$ and assuming null energy condition, $T_{ab}k^a k^b > 0$, we see that all the quantities on the right hand side of Eq. (2.135) are positive. Thus we arrive at the condition

$$\frac{d\theta}{d\lambda} < \frac{1}{2}\theta^2. \tag{2.136}$$

Integrating this equation we get

$$\frac{1}{\theta} > \frac{1}{\theta_0} + \frac{1}{2}\lambda. \tag{2.137}$$

Now suppose the expansion, θ_0, is negative at some point of time then from Eq. (2.137) we get that $\theta \sim -\infty$ within some finite value of the affine parameter τ. Now the expansion is given as

$$\theta = \frac{1}{A}\frac{dA}{d\tau}, \tag{2.138}$$

where A is the area of the black hole. For $\theta \sim -\infty$ we get $A = 0$, which implies that there will be caustics formed in the future null direction of the geodesic. Now according to Cosmic censorship conjecture, which states that there can be no naked singularities in spacetime or the generators of the event horizon can have no future end points, this is not allowed. Thus the expansion of the event horizon must be always positive which implies that the area of the event horizon always increases.

2.8.4. Third law

The *strong* version of the third law states that, entropy of any system tends to an universal constant independent of the macroscopic properties of the system, which can be taken to be zero, as the temperature of the system approaches absolute zero. This *strong* version of the third law does not hold for black hole thermodynamics [71, 72] and it can be shown very easily that a Kerr–Newman black hole can violate this. However, it can be shown that the weak form of the third law, in the sense of unattainability of the zero temperature state, holds. According to the third law of black hole mechanics, if the stress-energy tensor satisfies the weak energy condition, then the surface gravity of a black hole cannot be reduced to zero within a finite amount of time [71].

The actual proof of the third law for black holes can be quite complicated for this discussion so we outline a simpler version of the proof which shows, given that the weak energy condition holds that the surface gravity cannot be zero. To show this we start with a charged Vaidya metric (we need a dynamical solution that can potentially become extremal at a finite advanced time v), which is given as

$$ds^2 = -f dv^2 + 2dvdr + r^2 d\Omega^2 \tag{2.139}$$

if m is the mass and q is the charge of the black hole

$$f = 1 - \frac{2m(v)}{r} + \frac{q^2(v)}{r^2}\,. \tag{2.140}$$

The above spacetime violates the third law when $m(v) = q(v)$ i.e the black hole becomes extremal for some advanced time $v_0 < \infty$.

The weak energy condition states that the energy density measure by an observer with velocity k^a is always positive,or

$$T_{ab}k^a k^b > 0. \tag{2.141}$$

Here T_{ab} is the stress-energy tensor which gives the solution (2.139), when used as a source in the Einstein equation. For the Vaidya solution

$$T^{ab} = \rho l^a l^b + P\mathrm{diag}(-1,-1,1,1) \tag{2.142}$$

where l^a is a null vector, ρ and P are the density and pressure of the collapsing matter.

For a radial observer, given the weak energy condition one must have a positive ρ at the apparent horizon located at

$$r = r_+ = m + \sqrt{m^2 - q^2}. \tag{2.143}$$

From this one can arrive at the condition

$$4\pi r_+^3 \rho(r_+) = m\dot{m} - q\dot{q} + \sqrt{m^2 - q^2} > 0 \tag{2.144}$$

where dot represents differentiation with respect to the affine null parameter v. Now, let us assume that the black hole becomes extremal at a finite advanced time v_0, this means $\Delta(v_0) = 0$, where $\Delta(v) = m(v) - q(v)$. As the black hole was not extremal at $v < v_0$, we must have $\Delta(v) > 0$. Now from (2.144), using $m(v_0) = q(v_0)$, one gets

$$m(v_0)\dot{\Delta}(v_0) > 0, \tag{2.145}$$

which is clearly in contradiction to what we said earlier. This shows that if the weak energy condition holds then the black hole cannot become extremal in a finite advance time.

CHAPTER 3

Black hole evaporation

3.1. Introduction

The discovery of Hawking radiation [5] changed our perspective towards black holes, giving us a deeper insight about the microscopic nature of gravity. At the same time, within the semi-classical framework, the current understanding of such process still leaves open several issues. A well known unresolved problem of black hole physics as discussed before is the information loss paradox [73, 59, 74], i.e. the apparent incompatibility between the complete thermal evaporation of a black hole endowed with an event horizon and unitary evolution as prescribed by quantum mechanics.

For restoring unitarity of Hawking radiation and addressing the information loss problem correctly, it is important (among other things) to know from where the Hawking quanta originate. For example, if one assumes a near horizon origin of the Hawking radiation, then one way to restore unitarity is by conjecturing some sort of UV-dependent entanglement between partner Hawking quanta which would enable the late time Hawking flux to retrieve the information in the early stages of the evaporation process. Such scenario seems to lead to the so called "firewall" argument as the conjectured lack of maximal entanglement between the Hawking pairs makes the near horizon state singular and eventually demands some drastic modification of the near horizon geometry [20]. On the other hand, if one believes in a longer distance origin of the Hawking quanta, some effect must be operational at a larger scale for restoring unitarity rather than near the horizon, avoiding the "firewall".

A similar open issue is the transplanckian origin of Hawking quanta. Hawking's original calculation indicates that the quanta originate near the black hole horizon in a highly blue-shifted state requiring an assumption on the UV completion of the effective field theory used for the computation and on the lack of back-reaction on the underlying geometry [1]. While it was debated for a while if Hawking quanta could originate initially, during the star collapse, and later released over a very long time, it was convincingly argued in [77] that this cannot be the case if an event horizon indeed forms. This leads to the conclusion that the Hawking quanta are generated in a region outside the horizon. A conclusion corroborated by studies of the Hawking modes correlation structure where it was shown

[1] See, for instance, [75, 76] for a black hole evaporation analysis where these issues can be addressed in a quantum gravity context.

that mode conversion happens over a long distance from the horizon [78]. A more recent claim in this direction, based on calculating the size of the radiating body via the Stefan–Boltzmann law, showed that the Hawking quanta originate in a near horizon quantum region, a sort of black hole "*atmosphere*" [79]. It is a well known fact that the typical wavelength of the radiated quanta is comparable to the size of the black hole, so one might think that the point particle description is not very accurate. However, as measured by a local observer near the horizon, the wavelength is highly blue-shifted when traced back from infinity to the horizon, thus validating the point particle description.

The Hawking process can be explained heuristically as-well, for example via a tunneling mechanism where the particle tunnels out of the horizon or the anti particle (propagating backwards in time) tunnels into the horizon and as a result of this we get the constant Hawking flux at infinity [53]. Alternatively, one popular picture is to imagine that the strong tidal force near the black hole horizon stops the annihilation of the particle and anti-particle pairs that are formed spontaneously from the vacuum. Once the antiparticle is "hidden" within the black hole horizon, having a negative energy effectively, the other particle can materialise and escape to infinity [80, 81].

In the first two section of this chapter based on [82], we will explicitly make use of this latter heuristic picture as well as of a full calculation of the stress energy tensor in 1+1 dimensions to investigate where the Hawking quanta might originate. We shall see that both methods seem to agree in suggesting that the Hawking quanta originate from the black hole *atmosphere* and not from a region very close to the horizon. In section II, based on the heuristic picture of Hawking radiation described above and invoking the uncertainty principle and tidal forces, we show that most of the contribution to the radiation spectrum comes from a region far away from the horizon. In section III we further strengthen our claim by a detailed calculation of the renormalized stress energy tensor, which indicates a similar result, we also calculate the energy density and flux as a free falling observer would measure. In section IV we will see another view point of looking at hawking radiation based on effective temperature as proposed in [83, 84]. In section VI we will compare the energy densities as seen by a free falling observer using the RSET and the effective temperature function. We also investigate where the adiabatic condition breaks down indicating another hint about the region of particle creation and a plausible explanation for the discrepancy in the two energy densities.

One ingredient of our heuristic argument to identify a quantum atmosphere outside the black hole horizon, where particle creation takes place, is the uncertainty principle. However, the use of the uncertainty principle alone, as originally suggested by Parker [85], does not contain any physically relevant information about the location of particle production and why smaller black holes should be hotter. Indeed, the uncertainty principle in this case provides a rough estimate of the region of particle production as inversely proportional to the energy of the Hawking quanta when they are produced, but it does not take into account any dynamical mechanism to estimate the probability of spontaneous emission.

Thus one can improve this argument by invoking a physical process of creation of the Hawking quanta and using the uncertainty principle as a complementary tool to estimate the region of origin of the quanta. In this section, we try to achieve this goal by relying on tidal forces.

Let us then consider a situation where a virtual pair, consisting of a particle and anti-particle, pops out of the vacuum spontaneously for a very short time interval and then annihilates itself. In the Schwinger effect [86] a static electric field is assumed to act on a virtual electron-positron pair until the two partners are torn apart once the threshold energy necessary to become a real electron-positron pair is provided by the field. Energy is conserved due to the fact that the electric potential energy has opposite sign for partners with opposite charge. However, in its gravitational counterpart a priori only vacuum polarisation can be induced by a static field in the absence of an horizon.

In fact, only in the presence of the latter one has both the characteristic peeling structure of geodesics (diverging away from the horizon on both its sides) as well as the presence of an ergoregion behind it. [2] The presence of an ergoregion is crucial for energy conservation as it allows for negative energy states given that in it the norm of the timelike Killing vector, with respect to which we compute energy, changes sign.

Indeed, if a Schwinger-like process takes place near the black hole horizon, due to the tidal force of the black hole and the peeling of geodesics, the pair can get spatially separated and one partner can enter the black hole horizon following a timelike or null curve with negative energy while the other particle can escape to infinity and contribute to the Hawking flux. In this picture, we are implicitly assuming that virtual particles in

[2] This is strictly true only for non-rotating black holes, for rotating ones the ergoregion lies outside of the horizon allowing for the classical phenomenon of superradiance. However, the quantum emission still requires the peculiar peeling structure of geodesics typical of the horizon.

the vicinity of a black hole horizon move along geodesics when they are just about to go on-shell.

Therefore, the physical scenario we want to envisage is that of a particle-antiparticle pair pulled apart by the black hole tidal force outside the horizon until they go on-shell as one of them reaches the horizon located at $r_s = 2GM/c^2$ (actually an infinitesimal distance inside it so that the geodesic motion will drag it further inside) while the other particle is at a radial coordinate distance $r = r_*$. One could also consider the case where the ingoing particle tunnels through the horizon and goes on-shell well inside the horizon (as e.g. suggested by the results of [78]); however, since in our analysis below we are interested in the tidal force as computed in the outgoing particle rest frame, this should not affect the final expression for the force. Thus, from the point of view of an outside static observer, the work done by the gravitational field on the pair (in our heuristic derivation) is insensitive to the exact location where the ingoing particle becomes real.

Once on-shell, the outgoing particle eventually reaches infinity and contributes to the Hawking spectrum. In order to do so though, it has to be created with an energy corresponding to the energy of the Hawking quanta at a distance $r_* > r_s$ from the center of the black hole as measured by a local static observer; this can be reconstructed by noticing that

$$\omega_r = \frac{\omega_\infty}{\sqrt{g_{00}}} , \tag{3.1}$$

where ω_∞ is the energy at infinity and we are using the $(+, -, -, -)$ signature. At infinity, the thermal spectrum of Hawking radiation gives

$$\omega_\infty = \gamma \frac{k_B T_H}{\hbar} , \tag{3.2}$$

where the Hawking temperature for a black hole of mass M reads $k_B T_H = \frac{\hbar c^3}{8\pi GM}$. Thus, we get

$$\omega_\infty = \gamma \frac{c^3}{8\pi GM} \tag{3.3}$$

and

$$\omega_r = \gamma \frac{c}{4\pi r_s} \frac{1}{\sqrt{1 - \frac{r_s}{r}}} , \tag{3.4}$$

where γ is a numerical factor spanning the energy range of the quanta giving rise to the radiation thermal spectrum. At the peak of the spectrum $\gamma \approx 2.82$.

This energy is provided by the work done by the gravitational field to pull the two partners apart. We can compute this work in the static frame outside a black hole and compare it with $\omega(r_*)$. Using this relation, we can determine the region from which the Hawking quanta originate. This is the process we now want to implement.

Let us clarify that, in a general relativistic framework, the geodesic deviation equation does not describe the force acting on a particle moving along a geodesic. Rather, it expresses how the spacetime curvature influences two nearby geodesics, making them either diverge or converge, i.e. it effectively measures tidal effects. Therefore, we can interpret these effects as the pull of the gravitational force on particles and talk about the work done by the gravitational field only in an heuristic sense. Nevertheless, in the case considered here where the test particles have a mass much smaller than the black hole and we can neglect back-reaction effects, we expect this interpretation of the gravitational field effects to capture some relevant aspects of black hole physics. With these assumptions spelled out, let us proceed.

In the rest frame of the outgoing particle, one would see the antiparticle accelerating towards the horizon due to the tidal force. This radial acceleration in the rest frame of the particle can be computed using the geodesic deviation equation, namely

$$a^r|_{r_*} \equiv \left.\frac{Dn^r}{D\tau^2}\right|_{r_*} = R^r{}_{\mu\nu\rho}u^\mu u^\nu n^\rho|_{r_*} \ , \tag{3.5}$$

where the r.h.s. is expressed in terms of the Riemann tensor components, n^r denotes the separation between the two radially infalling geodesics followed by the pair of particles and $u^\mu = [1, 0, 0, 0]$ in the rest frame of the particle.

The separation between the particle and the anti-particle when the pair forms spontaneously (i.e. they go "on-shell") is given by their Compton wavelength, namely $n^\rho = [0, n^r, 0, 0]$ where $n^r \sim \lambda_C = \hbar/mc$, and $m \ll M$ is the particles rest mass (from now on we shall work in units where $\hbar = c = 1$). So in the end, Eq. (3.5) implies that the radial component of the tidal acceleration (as computed in the rest frame of the particle at coordinate r_*) is given by

$$a^r|_{r_*} = \frac{2M}{r_*^3}\lambda_C \tag{3.6}$$

For computation of the acceleration in the rest frame of the particle we need the Riemann tensor in the inertial frame of the particle. One can compute the Riemann tensor in the static Schwarzschild coordinates and then boost it using the free-fall velocity of the particle as measured in the static frame. A feature of the Schwarzschild geometry is that the components of the Riemann tensor remains invariant under such a boost [87]. Thus, in (3.5) we have $R_{rttr} = -2M/r^3$.

Our aim is to determine the work done on the spontaneously created particle pair by the tidal force in the static frame outside the black hole. For this we need to compute the tidal force as measured by a static observer outside the black hole at the instant when the outgoing partner goes on shell. This can be achieved by considering the particle rest frame and the static observer frame as locally two inertial frames: The latter sees the particle as moving with outward velocity given by the radial component of the geodesic tangent vector $u^r = dr/d\tau$. Once this is known, we can derive the radial acceleration observed by the static observer by performing a boost with rapidity $\zeta = \tanh^{-1}(u^r)$.

We thus need to determine the instantaneous radial component of the free fall velocity of the outgoing particle when it goes on-shell. This can be computed from the geodesic equation and it is given by

$$u^r = \frac{dr}{d\tau} = \sqrt{\frac{2M}{r}\left(1 - \frac{r}{r_0}\right)}\,, \tag{3.7}$$

where r_0 comes as an integration constant corresponding to the coordinate distance at which the particle velocity goes to zero. Since we are interested in the value of the radial component of the geodesic tangent vector at the instant when the outgoing particle goes on-shell and becomes an Hawking quantum which eventually reaches infinity, we can take the integration constant $r_0 \to \infty$, i.e. Hawking quanta can be created with zero velocity only at infinity. Hence, we get

$$u^r|_{r_*} = \sqrt{\frac{2M}{r_*}}\,. \tag{3.8}$$

We can now boost the acceleration vector $a^\mu = (0, a^r, 0, 0)$, where a^r given by (3.6), with a velocity parameter given by (3.8), in order to determine the tidal force in the static frame a^r_{st}. We get $a^r_{\text{st}} = a^r \cosh(\zeta)$ so that the radial component of the force under this transformation is given by

$$F^r_{\text{tidal-st}}\big|_{r_*} = \left.\frac{m a^r_{\text{st}}}{(1 - 2M/r)}\right|_{r_*} = \frac{m\lambda_C}{(1 - 2M/r_*)^2}\frac{2M}{r_*^3}\,, \tag{3.9}$$

where we have rescaled the mass in the rest frame by the appropriate Lorentz factor, $(1 - 2M/r_*)^{-1}$. Finally, using the fact that $\lambda_C \sim 1/m$, the magnitude of the force is given by

$$||F^r_{\mathrm{tidal-st}}|| = \frac{2M}{r_*^3}\left(1 - \frac{r_s}{r_*}\right)^{-\frac{3}{2}} . \tag{3.10}$$

In analogy with the Schwinger effect, we shall now assume that the work done by the tidal force to split the virtual pair can be approximated by the product of the force computed above with the distance over which it appears to have acted, i.e. the separation of the two Hawking quanta as they go on-shell as measured by a static observer at r_*. Given that we have assumed that the ingoing Hawking quantum goes on shell as soon as it can do so, i.e. at horizon crossing, this distance will coincide with the static observer's proper distance to the horizon $d(r_*)$.

Therefore, the work required by the tidal force to split the pair apart is given by [3]

$$W_{\mathrm{tidal}} \sim ||F^r_{\mathrm{tidal-st}}||\, d(r_*) = \frac{2M}{r_*^3}\left(1 - \frac{r_s}{r_*}\right)^{-\frac{3}{2}} d(r_*)\,, \tag{3.11}$$

where $d(r_*)$ is given by

$$\begin{aligned} d(r_*) &= \int_{r_s}^{r_*} \sqrt{g_{rr}}dr' \\ &= r_s\left(\sqrt{\alpha(\alpha-1)} + \frac{1}{2}\log\left[\alpha\left(1+\sqrt{1-\frac{1}{\alpha}}\right)^2\right]\right)\,, \end{aligned} \tag{3.12}$$

and we have defined $\alpha \equiv r_*/r_s$.

We can then equate this work to the total energy of the two Hawking quanta being created, namely $W_{\mathrm{tidal}} = 2\omega_r$. This gives us

$$\frac{2M}{r_*^3}\left(1 - \frac{2M}{r_*}\right)^{-\frac{3}{2}} d(r_*) = \frac{\gamma}{2\pi r_s}\left(1 - \frac{2M}{r_*}\right)^{-\frac{1}{2}} . \tag{3.13}$$

Finally, from eq. (3.13) we get

$$\gamma = \frac{2\pi}{\alpha^2}\left(1-\frac{1}{\alpha}\right)^{-\frac{1}{2}} \cdot \left(1 + \frac{1}{2\sqrt{\alpha^2-\alpha}}\log\left[\alpha\left(1+\sqrt{1-\frac{1}{\alpha}}\right)^2\right]\right) . \tag{3.14}$$

[3] Alternatively, we could introduce a 4-vector $\ell^\mu = (0, \ell^r, 0, 0)$, with $||\ell|| = \sqrt{g_{\mu\nu}\ell^\mu\ell^\nu} = d(r_*)$, and compute the work as $W_{\mathrm{tidal}} \sim g_{rr}F^r_{\mathrm{tidal-st}}\ell^r\big|_{r_*}$. This would give the same result.

The relation between γ and α, i.e the radial distance scaled as r_*/r_s, is better illustrated in Fig. 3.1. It is clear from the plot that the part of the Hawking thermal spectrum around the peak ($\gamma \sim 2.82$), where most of the radiation is concentrated, corresponds to a region which extends far outside the horizon, up to around $2r_s$ (at the peak $r_* \approx 4.38\, M$).

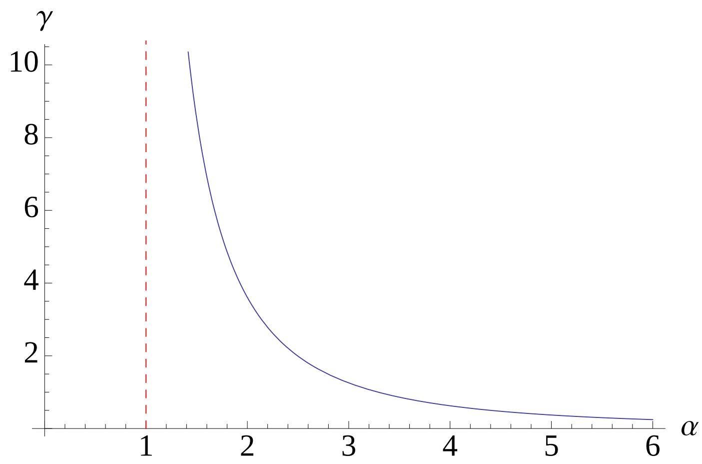

Figure 3.1: This plot shows the variation of γ with respect to the radial distance from the center of the black hole. The red dashed line corresponds to the horizon location at $\alpha = 1$ where the expression for the tidal force work diverges, indicating that the quanta in the far UV tail of the Hawking spectrum originate from very near the horizon.

The plot above also shows how, in this tidal force derivation, the quanta with higher velocity (kinetic energy) are produced closer to the horizon. This is consistent with our analysis since the higher the initial radial velocity the stronger the Lorentz contraction of the outgoing particles distance from the horizon in their rest frame, given by λ_C, resulting in a shorter proper distance $d(r_*)$ at which they are detected.

Also, by using Eq. (3.12) and expressing the rest of Eq. (3.11) in terms of α, we can see that the work doable at fixed α by the tidal forces scales as the inverse of the mass of the black hole so making evident that smaller holes can produce hotter particles at the same relative distance from the horizon.

Let us stress again the heuristic nature of our argument. We are considering the instantaneous value of the tidal force observed by the outgoing partner at a given coordinate distance r_* where it goes on-shell. However, we then use this instantaneous value to compute the work done by the gravitational field over a distance $d(r_*)$, as if the force was actually at work with the same constant value throughout the whole splitting process. A similar approach was also used in [88] to give an estimate of the wavelength of the Hawking quanta as produced by the gravitational tidal force.

So, although the analogy with the Schwinger effect for the electron-positron pair production by an electric field may be advocated to lend support to our description of Hawking quanta production from a quantum atmosphere that extends well beyond the horizon, we now want to present a more sound analysis based on the renormalized stress energy tensor in order to confirm this picture.

3.3. Stress-energy tensor

By analyzing the renormalized stress energy tensor (RSET) in the 1+1 dimensional case, one can understand Hawking radiation in a better way as this is a local object which can help to probe the physics in the vicinity of the black hole. Derivation of the RSET components has been considered in many places in the literature [89, 90, 49, 91, 92], here we build on these previous results and compute the energy density and flux as seen by an observer which has no zero radial velocity(thus giving rise to no kinematical effects) and zero acceleration at the horizon.

3.3.1. Computation of RSET

Following [64], let us introduce a set of globally defined affine coordinates U, V on $\mathscr{I}^-_{\text{left}}, \mathscr{I}^-_{\text{right}}$ respectively. Restricting to the radial and time dimensions, the metric reads

$$ds^2 = C(U,V)dUdV\ . \tag{3.15}$$

In $(1+1)$ dimensions the renormalised stress energy tensor for any massless scalar field in terms of these affine null coordinates can be easily computed using the conformal anomaly [89, 90, 93, 49, 94]. The components of the RSET computed in some arbitrary vacuum state are given as:

$$\langle T_{UU}\rangle = -\frac{1}{12\pi}C^{1/2}\partial_U^2 C^{-1/2} = \frac{1}{24\pi}\left[\frac{C_{,UU}}{C} - \frac{3}{2}\frac{(C_{,U})^2}{C^2}\right], \tag{3.16}$$

$$\langle T_{VV}\rangle = -\frac{1}{12\pi}C^{1/2}\partial_V^2 C^{-1/2} = \frac{1}{24\pi}\left[\frac{C_{,VV}}{C} - \frac{3}{2}\frac{(C_{,V})^2}{C^2}\right], \tag{3.17}$$

$$\langle T_{UV}\rangle = \frac{RC}{96\pi} = \frac{1}{24\pi}\partial_U\partial_V \ln C\,, \tag{3.18}$$

where C is the conformal factor introduced in the above metric and R is the scalar curvature.

Now let us also introduce a null coordinate u affine on $\mathscr{I}^+_{\text{right}}$ such that

$$U = p(u)\,; \tag{3.19}$$

from this we get

$$\partial_U = \dot{p}^{-1}\partial_u\,. \tag{3.20}$$

In terms of the set (u, V), the metric reads

$$ds^2 = \bar{C}(u,V)dudV\,, \tag{3.21}$$

with

$$\bar{C}(u,V) = \dot{p}(u)C(U,V)\,. \tag{3.22}$$

Assuming that the observer is always outside the collapsing star, $\bar{C}(u,V)$ would be the metric component of a static spacetime. In terms of this newly defined null coordinate, a simple computation shows that T_{UU} is given as

$$\langle T_{UU}\rangle = -\frac{\dot{p}^{-2}}{12\pi}\left[\bar{C}^{1/2}\partial_u^2\bar{C}^{-1/2} - \dot{p}^{1/2}\partial_u^2\dot{p}^{-1/2}\right]\,. \tag{3.23}$$

Now T_{VV} will have only a static contribution if $V = v$ but if the affine null coordinate on $\mathscr{I}^+_{\text{left}}$ is defined as

$$V = q(v) \tag{3.24}$$

and we define $C'(U,v) = \dot{q}(v)C(U,V)$, T_{VV} is given as

$$\langle T_{VV}\rangle = -\frac{\dot{q}^{-2}}{12\pi}\left[C'^{1/2}\partial_v^2 C'^{-1/2} - \dot{q}^{1/2}\partial_v^2\dot{q}^{-1/2}\right]\,. \tag{3.25}$$

As mentioned earlier $\bar{C}(u,V)$ is the metric component of a static spacetime, so all the dynamics of the collapsing geometry is captured in the $\dot{p}$ term of (3.23). In the above analysis, by using another affine null coordinate, we can differentiate between the static contribution to the RSET and that due to the the dynamics associated with the collapse [64].

3.3.2. RSET for different vacuum states.

Capturing the dependence at different radii of the RSET components would require a knowledge of the full $p(u)$ at any value of u, i.e. specify a collapse history. However, this would lead to the inclusion of transient effects which are not relevant for the present discussion. For this reason, we shall here rely on the fact that, well after the collapse has settle down, the black hole geometry is formally indistinguishable from that of an eternal configuration [68, 95] (where the form of $p(u)$ is simply fixed by the geometry, see (2.8)).

So, in order to extract physical information from the RSET, we shall compute the energy density and the flux experienced by an observer at constant Kruskal position long after the collapse has taken place in the two physically relevant states for Hawking radiation in the eternal black hole case, namely the Hartle–Hawking and the Unruh states. We shall start in this section by explicitly evaluating the general expressions for the RSET components expectation values. Using (2.8) we get the relations

$$\begin{aligned} \dot{p}(u) &\equiv \partial_u p(u) = -\frac{p(u)}{2r_s}\,, && (3.26)\\ \ddot{p}(u) &= \frac{p(u)}{4r_s^2} = -\frac{\dot{p}(u)}{2r_s}\,. && (3.27)\end{aligned}$$

For computing the first term of (3.23) we can write

$$\bar{C}^{1/2}\partial_u^2\bar{C}^{-1/2} = \frac{3}{4}\bar{C}^{-2}\left(\partial_u\bar{C}\right)^2 - \frac{1}{2}\bar{C}^{-1}\partial_u^2\bar{C}\,. \qquad (3.28)$$

Using the metric conformal factor C from (2.12) we get

$$\begin{aligned} \partial_u\bar{C} &= \partial_u[\dot{p}(u)C] = \ddot{p}C + \dot{p}\partial_u C\\ &= \dot{p}(u)\left(-\frac{1}{2r_s} + \frac{r^2 - r_s^2}{2r^2 r_s}\right)C && (3.29)\\ &= -\frac{r_s}{2r^2}\bar{C}\,, && (3.30)\end{aligned}$$

and

$$\partial_u^2\bar{C} = -\frac{1}{2}r_s\partial_u\left(\frac{\bar{C}}{r^2}\right) = \frac{r_s^2}{4r^4}\bar{C} - \frac{1}{2}\frac{r_s f(r)\bar{C}}{r^3}\,. \qquad (3.31)$$

Using the above relation in (3.28) we have

$$\begin{aligned} \bar{C}^{1/2}\partial_u^2\bar{C}^{-1/2} &= \frac{3}{4}\bar{C}^{-2}\left[\frac{r_s^2}{4r^4}\bar{C}^2\right] - \frac{1}{2}\bar{C}^{-1}\left[\frac{r_s^2}{4r^4}\bar{C} - \frac{1}{2}\frac{r_s f(r)\bar{C}}{r^3}\right]\\ &= -\frac{3}{16}\frac{r_s^2}{r^4} + \frac{r_s}{4r^3} - \frac{3}{4}\frac{M^2}{r^4} + \frac{M}{2r^3}, && (3.32)\end{aligned}$$

where $f(r)$ is given in (2.1) and we used $r_s = 2M$ in the last step. For the second term on the r.h.s. of (3.23), we have

$$\dot{p}^{1/2}\partial_u^2\dot{p}^{-1/2} = -\frac{\dot{p}^{1/2}}{2}\partial_u\left(\frac{\ddot{p}}{\dot{p}^{3/2}}\right) = \frac{1}{(8M)^2}\,. \qquad (3.33)$$

We are now ready to compute explicitly the expectation value of the different RSET components for the Hartle–Hawking ($|H\rangle$) and Unruh ($|U\rangle$) states.

We can start by observing that for the T_{UU} and T_{UV} components, the expectation values are the same in the two vacuum states [49]. Therefore, in the following we simply denote

$$\langle T_{UU}\rangle \equiv \langle H|T_{UU}|H\rangle = \langle U|T_{UU}|U\rangle\,, \tag{3.34}$$

$$\langle T_{UV}\rangle \equiv \langle H|T_{UV}|H\rangle = \langle U|T_{UV}|U\rangle\,. \tag{3.35}$$

By means of (3.32), (3.33), $\langle T_{UU}\rangle$ is given by

$$\begin{aligned}\langle T_{UU}\rangle &= \frac{\dot{p}^{-2}}{24\pi}\left[\frac{3}{2}\frac{M^2}{r^4}-\frac{M}{r^3}+\frac{1}{32M^2}\right] &(3.36)\\ &= (768\pi M^2)^{-1}\frac{V^2}{4r^2}e^{-r/M}\left[1+\frac{4M}{r}+\frac{12M^2}{r^2}\right]. &(3.37)\end{aligned}$$

To compute $\langle T_{UV}\rangle$ we use (3.18), from which

$$\begin{aligned}\langle T_{UV}\rangle &= \frac{1}{24\pi}\partial_U\partial_V\ln C = \frac{1}{24\pi}(\dot{p}\dot{q})^{-1}\partial_u\partial_v\ln C\\ &= -\frac{1}{96\pi}(\dot{p}\dot{q})^{-1}C\partial_r^2 C. \end{aligned}\tag{3.38}$$

Using $C(t,r)$ from (2.12) and the exact values of $q(u)$ and $p(v)$, we get

$$\langle T_{UV}\rangle = -\frac{M^2}{12\pi r^4}e^{-r/2M}\,. \tag{3.39}$$

On the other hand, the dependence of $\langle T_{VV}\rangle$ on the state in which we are computing the expectation value is important. For the Hartle–Hawking state (eternal black hole scenario, non-singular vacuum state in both past and future horizons) in Kruskal coordinates the modes are given by $e^{-i\omega U}, e^{-i\omega V}$, where we defined V as

$$V \equiv q(v) = 2r_s e^{v/2r_s}\,. \tag{3.40}$$

Using this definition of V we can proceed in a similar way as for the computation of $\langle T_{UU}\rangle$. From (3.25), we obtain

$$\begin{aligned}\langle H|T_{VV}|H\rangle &= \frac{\dot{q}^{-2}}{24\pi}\left[\frac{3}{2}\frac{M^2}{r^4}-\frac{M}{r^3}+\frac{1}{32M^2}\right] &(3.41)\\ &= (768\pi M^2)^{-1}\frac{U^2}{4r^2}e^{-\frac{r}{M}}\left[1+\frac{4M}{r}+\frac{12M^2}{r^2}\right]. &(3.42)\end{aligned}$$

For the Unruh state in Kruskal coordinates, the modes are given by $e^{-i\omega U}, e^{-i\omega v}$ and there is no regularization condition imposed in the past horizon. The expectation value of the T_{VV} component can be obtained from the relation

$$\langle U|T_{VV}|U\rangle = 16M^2\dot{q}^{-2}\langle U|T_{vv}|U\rangle\,, \tag{3.43}$$

where $\langle U|T_{vv}|U\rangle$ can be computed from

$$\langle U|T_{vv}|U\rangle = -\frac{1}{12\pi}f(r)^{1/2}\partial_v^2 f(r)^{-1/2} \tag{3.44}$$

using $f(r) = \left(1 - \frac{2M}{r}\right)$, as follows from the metric of a black hole in static Schwarzschild coordinates. We have

$$\langle U|T_{vv}|U\rangle = \frac{1}{24\pi}\left[\frac{3M^2}{2r^4} - \frac{M}{r^3}\right], \tag{3.45}$$

and from (3.43) we get

$$\langle U|T_{VV}|U\rangle = \frac{1}{6\pi}\frac{M^2}{V^2}\left[\frac{3M^2}{2r^4} - \frac{M}{r^3}\right]. \tag{3.46}$$

3.3.3. Energy density

We now have all the ingredients to extract physical information from the RSET. Let us first analyze the energy density as measured in the frame of an observer moving along fixed position in Kruskal coordinates.

Let us consider an observer at a given Kruskal position with 2-velocity $v^\mu = C^{-1/2}(1,0)$ (in $[T,X]$ coordinates). This choice of trajectory is not geodesic; however the acceleration that the observer experiences is irrelevant compared to the Hawking temperature and the acceleration is zero at the horizon. If one considers a free falling trajectory it will have a non zero, non constant radial velocity near the horizon as well as at the horizon. This would imply that the observer is accelerating with respect to the black hole and this would lead to additional contribution to the energy density and flux [96]. The observer we considered has zero radial velocity thus giving rise to no kinematical effects. The energy density, ρ, measured by this observer for the Unruh state is given by

$$\begin{aligned}\rho &= \langle U|T_{\mu\nu}|U\rangle v^\mu v^\nu = C^{-1}\langle U|T_{TT}|U\rangle \\ &= C^{-1}\langle U|T_{VV} + T_{UU} + 2T_{UV}|U\rangle\,.\end{aligned} \tag{3.47}$$

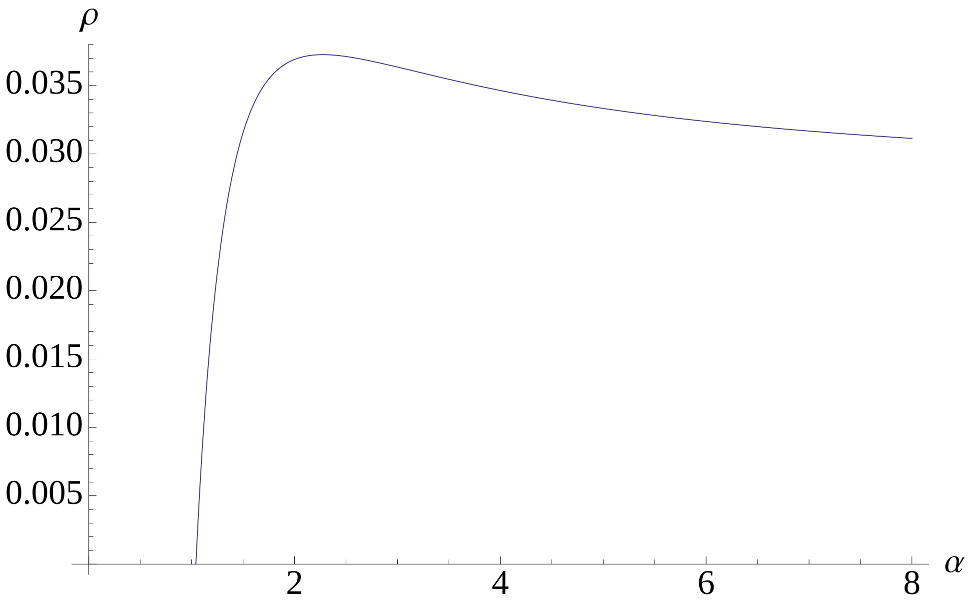

Figure 3.2: Plot of the energy density at a given time as a function of the the radial distance from the centre of the black hole in Unruh state at a given instant of time. We removed the divergence ar $\alpha = 1$ which arises due to the divergent nature of the Unruh vacuum in the past horizon.

Using (3.36), (3.39), (3.46) we can compute the energy density exactly and we plot it in FIG. 3.2 (where $\alpha \equiv r/r_s$).

The energy density (3.47) blows up at the horizon $(r = 2M)$ since we are computing the energy density as observed by a free falling observer in the Unruh state which is well known to be ill defined on the past horizon. In fact, such divergence arise from the $1/V^2$ term in the component (3.46) when $V = 0$, i.e. at the past horizon. The horizon location condition in Schwarzschild radial coordinate, $\alpha = 1$, cannot distinguish between past and future horizons and thus the divergent contribution would enter in the plot above of the energy density expression (3.47) when evaluated at $\alpha = 1$. However, a free falling observer at the future horizon would not see this divergence, which is just an artifact of Kruskal coordinates [4]. This is a well known fact already pointed in [90]. For this reason, we have removed

[4] Let us stress that also the calculation in [64] of the RSET components in the collapse scenario shows that at the white hole horizon the Unruh state will necessarily be singular. This can be easily realised by applying time reversal to the subdominant terms in the dynamical contribution (3.33) derived in [64] (see Eq. (52) there), which then shows an exponentially growing flux at the white horizon which very rapidly would create a divergence in the T_{UU} component of the RSET soon after horizon formation.

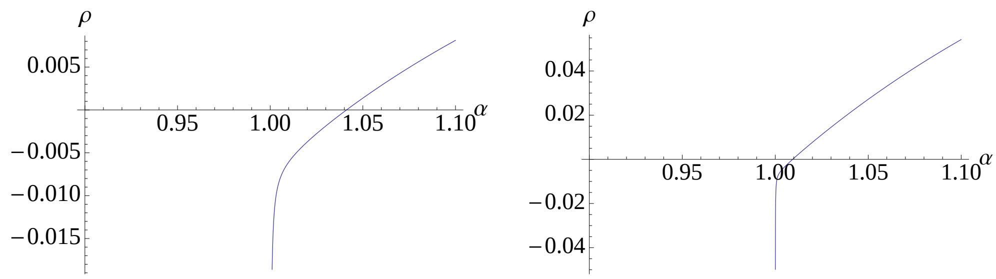

Figure 3.3: Near horizon behavior of the energy density in the Unruh state at different times. The first plot corresponds to the same instant of time as the plot in FIG. 3.2; the second one to a close instant after.

the point $\alpha = 1$ in the plot shown in FIG. 3.2.

Near the horizon the energy density becomes negative; these negative values are attained closer to the horizon as the energy density is measured at later times. We show this near horizon behavior in the two plots in FIG. 3.3, where the first is evaluated at the same time as the plot in FIG. 3.2 and the second one at a close instant after (a similar behavior was found also in [97]); the negative divergent behavior of the energy density at the horizon is clear from the plots.

The significant aspect of the plot in FIG. 3.2 for us is the peak in the distribution of ρ that is obtained outside the horizon which is at $r \approx 4.32$ M. Quite in agreement with our heuristic prediction based on the gravitational analogue of the Schwinger effect. Let us point out that, although we have shown the plot at a given instant of time, the behavior of the energy density remains the same at any time, in particular the presence of the peak at the same location persists; the only difference is that the value of the energy density increases since it accumulates, given that we are not taking into account the effect of back-reaction.

To get a non-singular energy density plot for the free falling observer we should consider the Hartle–Hawking state. This is given by

$$\begin{aligned} \rho &= \langle H|T_{\mu\nu}|H\rangle v^\mu v^\nu = C^{-1}\langle H|T_{TT}|U\rangle \\ &= C^{-1}\langle H|T_{VV} + T_{UU} + 2T_{UV}|H\rangle\,. \end{aligned} \tag{3.48}$$

Using the expectation values given in (3.36), (3.39), (3.41), we can plot the energy density (3.48) with respect to radial distance parametrized by α. This is shown in FIG. 3.4, where we see a similar nature of the distribution with a peak outside the horizon; however, as expected, in this case the energy density is regular everywhere. Remarkably, the peak is located at $r \approx 4.37M$, in close agreement with our heuristic findings.

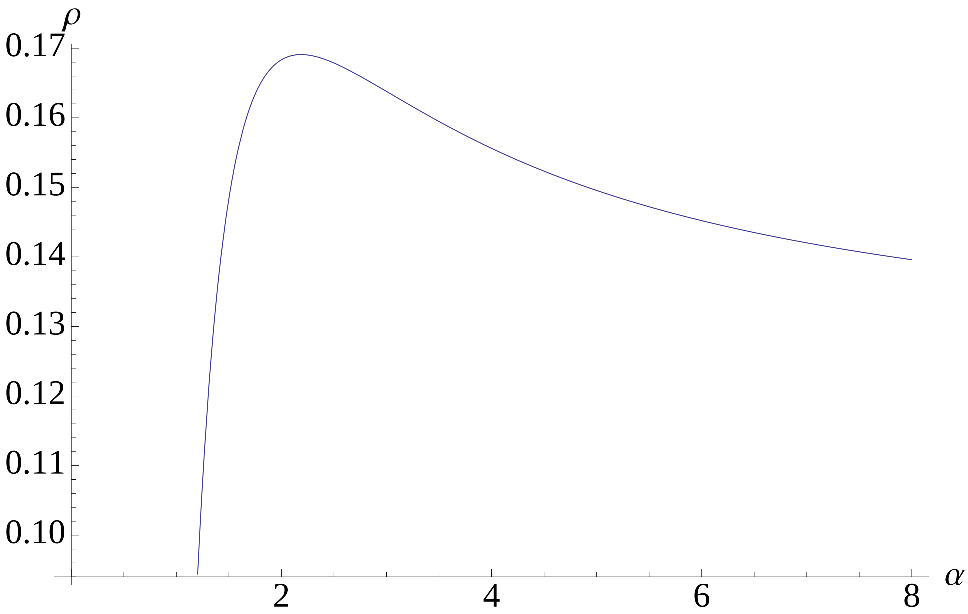

Figure 3.4: Plot of the variation of energy density computed in Hartle–Hawking state with respect to the radial distance from the centre of the black hole at fixed time measured in the static frame. Notice that close to the horizon the energy density is negative also in this case, but it remains finite at the horizon due to the non-divergent behavior of the T_{VV} component (3.41) in the Hartle–Hawking vacuum.

These results strongly support our previous claim that the radiation density is maximized in a region outside the horizon. We now show that a similar behavior with a peak away from the horizon is exhibited also by the flux part of the RSET.

3.3.4. Flux

The flux of the Hawking radiation in the Unruh vacuum is given by [98] [5]

$$F = -\langle U|T_{\mu\nu}|U\rangle v^{\mu} z^{\nu} \,, \tag{3.49}$$

where v^{μ} is the velocity of the observer and z^{ν} is the contravariant component of the normal to the observer. Let us consider a static observer at fixed distance in a Kruskal frame with $v^{\mu} = C^{-1/2}[1, 0]$ and indicate the normal

[5] In the Hartle–Hawking vacuum the flux vanishes due to the thermal equilibrium of the state.

vector as $z^\nu = [A, B]$. The latter has to satisfy the following conditions

$$g_{\mu\nu} z^\mu z^\nu = -1, \; z^\mu v_\mu = 0 \,. \tag{3.50}$$

Using the second relation we get $A = 0$ and from the first relation we get $B = C^{-1/2}$. Therefore, $z^\nu = C^{-1/2}[0, 1]$.

Using these expressions for v^μ, z^ν, we get

$$F = -C^{-1}\langle U|T_{TX}|U\rangle = C^{-1}\langle U|[-T_{VV} + T_{UU}]|U\rangle \,. \tag{3.51}$$

Plugging in the expectation values (3.36), (3.46) found above, we can plot the flux as a function of α. This is shown in FIG. 3.5. Also in this case the plot of the flux would receive a fictitious (for a free falling observer at the future horizon) divergent contribution from the component (3.46), and we have thus removed the point $\alpha = 1$ from the plot, thus avoiding the divergence at the past horizon $V = 0$. We see that the flux has a maximum

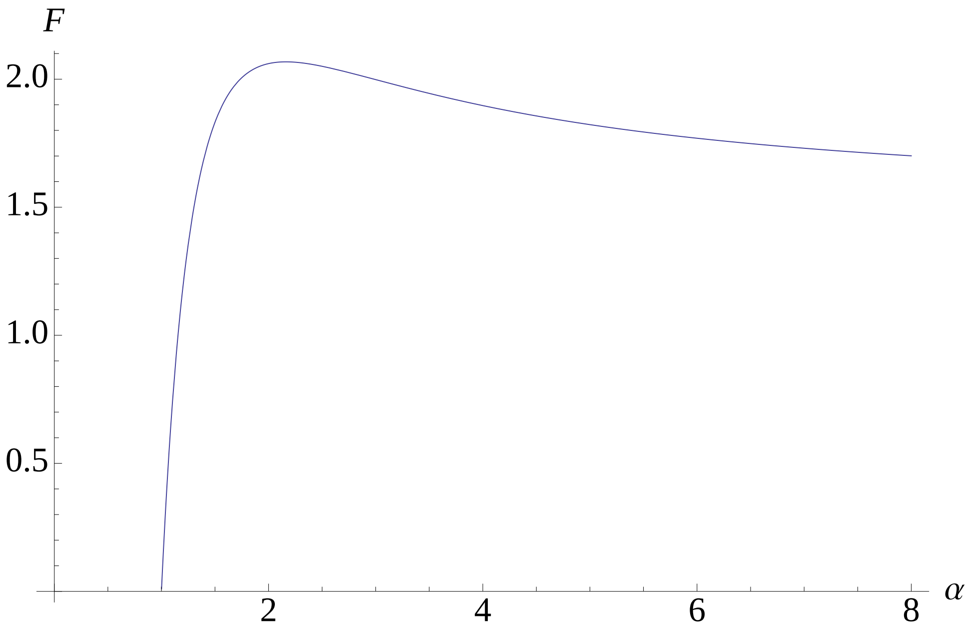

Figure 3.5: This plot shows the variation of the flux of Hawking radiation with respect to the radial distance as measured by an observer in the Unruh state at a given instant of time.

at $r = 4.32M$ and most of the contribution to the Hawking radiation comes from a region between the horizon and $r \approx 6M$.

We have seen before the energy density as obtained from the RSET for a Kruskal observer. In this section we compute the energy density as measured by a free falling observer. For this we first review Painleve coordinate and then we obtain RSET components using a coordinate transformation. The derivation of Painleve coordinate requires the definition of a new time coordinate as

$$t_p = t - f(r), \tag{3.52}$$

for some arbitrary function $f(r)$ such that

$$f'(r) = -\frac{1}{1-\frac{2M}{r}}\sqrt{\frac{2M}{r}}. \tag{3.53}$$

Substituting Eq.(3.53) in Schwarzschild metric Eq.(2.1) one gets

$$ds^2 = (1-\frac{2M}{r})dt^2 - 2\sqrt{\frac{2M}{r}}dt_p dr - dr^2 - r^2 d{\Omega_2}^2. \tag{3.54}$$

As we see the spatial slices of the metric (3.54) is the flat metric in spherical polar coordinates. Also there is no coordinate singularity at the Schwarzschild radius ($r = 2M$). The time coordinate of the Painleve metric follows the proper time of a free falling observer who starts from infinity at zero velocity.

Now let's consider the Painleve coordinate as (t_p, x) and Schwarzschild coordinate as (t, r) and find the Jacobian for the coordinate transformation. According the definition of time translation Eq.(3.52) we have

$$\frac{\partial(t_p, x)}{\partial(t, r)} = \begin{pmatrix} \frac{\partial t_p}{\partial t} & \frac{\partial t_p}{\partial r} \\ \frac{\partial x}{\partial t} & \frac{\partial x}{\partial r} \end{pmatrix} = \begin{pmatrix} 1 & -f' \\ 0 & 1 \end{pmatrix}$$

The inverse of the transformation matrix (what we need to find the RSET components in Paileve coordinate from what we have for Schwartzchild ones)

$$\frac{\partial(t, r)}{\partial(t_p, x)} = \begin{pmatrix} \frac{\partial t}{\partial t_p} & \frac{\partial t}{\partial x} \\ \frac{\partial r}{\partial t_p} & \frac{\partial r}{\partial x} \end{pmatrix} = \begin{pmatrix} 1 & f' \\ 0 & 1 \end{pmatrix}$$

Here we compute the components of the RSET for a free falling observer in Unruh vacuum. Choosing the Unruh vacuum state for a free falling observer we already know the RSET component in Schwarzschild spacetime [77] and we can find the RSET components in Painleve coordinate by a simple coordinate transformation as

$$\begin{aligned}
T_{t_p t_p} &= \frac{\partial t}{\partial t_p}\frac{\partial t}{\partial t_p}T_{tt} + 2\frac{\partial t}{\partial t_p}\frac{\partial r}{\partial t_p}T_{tr} + \frac{\partial r}{\partial t_p}\frac{\partial r}{\partial t_p}T_{rr} \\
&= T_{tt}
\end{aligned} \tag{3.55}$$

$$\begin{aligned}
T_{t_p x} &= \frac{\partial t}{\partial t_p}\frac{\partial t}{\partial x}T_{tt} + \frac{\partial t}{\partial t_p}\frac{\partial r}{\partial x}T_{tr} + \frac{\partial r}{\partial t_p}\frac{\partial t}{\partial x}T_{rt} + \frac{\partial r}{\partial t_p}\frac{\partial r}{\partial x}T_{rr} \\
&= f'T_{tt} + T_{rr} = T_{x t_p}
\end{aligned} \tag{3.56}$$

$$\begin{aligned}
T_{xx} &= \frac{\partial t}{\partial x}\frac{\partial t}{\partial x}T_{tt} + 2\frac{\partial t}{\partial x}\frac{\partial r}{\partial x}T_{tr} + \frac{\partial r}{\partial x}\frac{\partial r}{\partial x}T_{rr} \\
&= f'^2 T_{tt} + 2f'T_{tr} + T_{rr}
\end{aligned} \tag{3.57}$$

With the components of RSET in Unruh vacuum being as

$$\begin{aligned}
T_{tt} &= \frac{1}{24\pi}\left(\frac{7M^2}{r^4} - \frac{4M}{r^3} + \frac{1}{32M^2}\right) \\
T_{tr} &= -\frac{1}{24\pi}\frac{1}{(1-\frac{2M}{r})}\frac{1}{32M} \\
T_{rr} &= -\frac{1}{24\pi}\frac{1}{(1-\frac{2M}{r})^2}\left(\frac{M^2}{r^4} - \frac{1}{32M}\right)
\end{aligned} \tag{3.58}$$

The velocity for the free falling observer in Painleve coordinate is

$$v = (1, -\sqrt{\frac{2M}{r}}) \tag{3.59}$$

The energy density for the free falling in Painleve coordinate can be found as

$$\varepsilon = T_{ab}v^a v^b = T_{t_p t_p} - 2\sqrt{\frac{2M}{r}}T_{t_p x} + \frac{2M}{r}T_{xx} \tag{3.60}$$

Now one can easily obtain the energy density for the free falling observer in Painleve coordinate using Eqs.(3.55),(3.56),(3.57) and (3.58). In Fig.3.6 we plot the energy density as a function of the distance r. As we see the energy density increases as the observer gets closer to the black hole. This can be accounted to the non zero radial velocity as the observer is crossing the horizon.

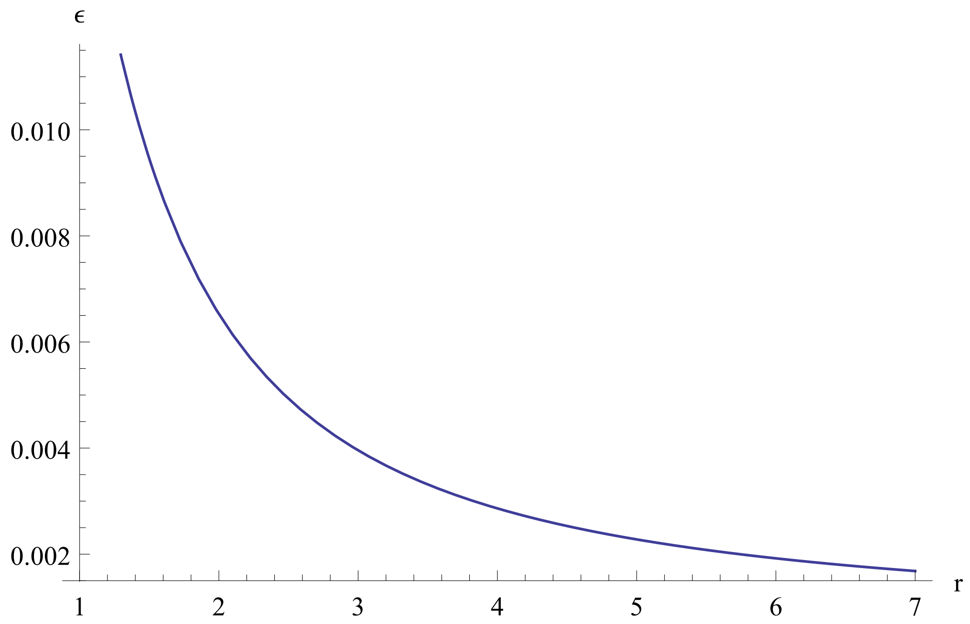

Figure 3.6: Energy density as a function of r. As the observer get closer to the horizon the energy density increases.

3.4.1. Calculation of the flux

We now calculate the flux of energy which the free falling observer measures in Painleve coordinate along its trajectory. As always we should first have the velocity of free falling observer in $(1+1)$ dimensions. The velocity of a free falling observer in Painleve coordinate is

$$u = (1, -\sqrt{\frac{2M}{r}}) \tag{3.61}$$

As we see the first component shows that the time coordinate is the proper time of the free falling observer. For calculating the flux we need to have the normal vetor to the velocity. The following conditions leads us to find the normal vector

$$\begin{aligned} g_{ab}n^a n^b &= -1 \\ n_a u^a &= 0 \end{aligned} \tag{3.62}$$

The normal vector can be found as

$$n^a = (0, 1) \tag{3.63}$$

Now the flux in Painleve coordinate as measured by the given observer would be

$$F = T_{ab}n^a u^b = -Tt_p x + \sqrt{\frac{2M}{r}} T_{xx} \tag{3.64}$$

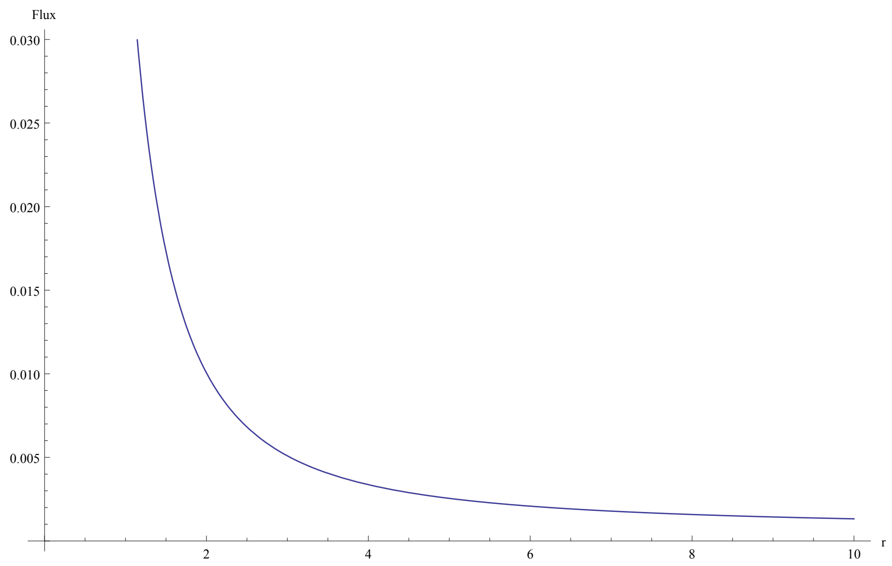

Figure 3.7: Fluxof energy measured by the free falling observer as a function of r.

As we see there is no divergence observed in energy density or flux by the free falling observer in crossing the horizon and for both, the observer measures a finite value at the horizon.

3.5. Effective temperature as percieved by an arbitrary observer

To study how different observers would perceive Hawking radiation, an effective temperature function was introduced in [83, 96, 84], which depends on the trajectory of the observer. We introduce and briefly review the effective temperature function here.

Considering a Schwarzschild black hole in $(t, r, \theta\phi)$ coordinate one can define an outgoing null coordinate as

$$\bar{u} = t - r^* \tag{3.65}$$

where r^* is the tortoise coordinate given as $r^* = r + 2M\log\left(\frac{\mathrm{r}}{2\mathrm{M}} - 1\right)$. For doing quantum field theory on a background containing such a black hole, one can define a new null coordinate as $U = U(\bar{u})$ which would determine the choice of the vacuum state for the observer. To analyze what an observer would perceive in a given vacuum state determined by U, we can introduce the proper time of the observer as τ and then the timelike trajectory of the observer would be defined as $(t(\tau), r(\tau))$. If one defines the proper time of the observer as another null coordinate given as

$$u = \tau \tag{3.66}$$

one can obtain the relation $U = U(u)$. Using this relation it is possible to compute the Bogoliubov coefficients which would give the particle content as perceived by the observer in a given vacuum state. One can define the perceived temperature function as

$$\kappa_{per}(u) = -\frac{d^2U}{du^2}/\frac{dU}{du}. \tag{3.67}$$

If u corresponds to the future null coordinate for Schwarzschild geometry then κ_{per} contains information about the peeling of null geodesics and this would be the relevant quantity for calculating the Hawking temperature. In this case the way κ_{per} is defined, it contains information about about the peeling as well as about the vacuum state and the observer.

After choosing an appropriate vacuum state by specifying $U(\bar{u})$, (3.67) can be written as

$$\begin{aligned}\kappa_{per} &= \left(-\frac{d^2U}{d\bar{u}^2}/\frac{dU}{d\bar{u}}\right)\frac{d\bar{u}}{du} - \frac{d^2\bar{u}}{du^2}/\frac{d\bar{u}}{du} \\ &= \frac{d\bar{u}}{du}\kappa(\bar{u}) - \frac{d^2\bar{u}}{du^2}/\frac{d\bar{u}}{du},\end{aligned} \tag{3.68}$$

where $\kappa(\bar{u})$ is defined as

$$\kappa(\bar{u}) = -\frac{d^2U}{d\bar{u}^2}/\frac{dU}{d\bar{u}}, \tag{3.69}$$

which is the 'state effective temperature' as it solely depends on the choice of the vacuum state and not on the trajectory of the observer.

One can check the value of $\kappa(\bar{u})$ for different vacuum states by considering $U(\bar{u})$ for the states. For Unruh vacuum $U(\bar{u}) = -4Me^{-\bar{u}/(4M)}$, which gives $\kappa(\bar{u}) = \frac{1}{4M}$, similarly for Boulware vacuum $U(\bar{u}) = \bar{u}$, which gives $\kappa(\bar{u}) = 0$.

As shown in [84] the effective temperature for an arbitrary observer following a trajectory $(t(u), r(u))$, where u acts as the proper time, is given as

$$\kappa_{per(u)} = \sqrt{\frac{1-v_l}{1+v_l}}1/\sqrt{1-\frac{2M}{r}}\left(\kappa(\bar{u}) - \frac{M}{r^2}\right) + a_p, \tag{3.70}$$

where v_l is the velocity of the observer with respect to the black hole as measured by a local inertial observer, a_p is the proper acceleration of the observer. One can express v_l as

$$v_l = \frac{v_r}{\sqrt{1 - \frac{2M}{r} + v_r^2}}, \tag{3.71}$$

where $v_r = \frac{dr}{d\tau}$, which can be calculated using the given metric describing the background geometry. For a generic observer a_p can be calculated from v_l but since we are just interested in free falling observer, $a_p = 0$ for our case.

For a free falling observer the velocity of the observer expressed in terms of Painleve coordinate is given as

$$v = \left[1, -\sqrt{\frac{2M}{r}}\right]. \tag{3.72}$$

Using the radial component of the velocity one can compute v_l and then the effective temperature function (3.70) as

$$\kappa_{eff} = \frac{1}{(1 - \sqrt{\frac{2M}{r}})}\left(\frac{1}{4M} - \frac{M}{r^2}\right), \tag{3.73}$$

where $\kappa(\bar{u}) = 1/4M$, which is the usual Hawking temperature.

3.6. Discrepancy in energy density

In the previous section we calculated the effective temperature function and now using κ_{eff} (3.73) as obtained for a free falling observer, we can assume a Planckian spectrum and compute the energy density as perceived by the free falling observer as

$$\rho_{per} = \frac{1}{48\pi}(\kappa_{eff})^2 \tag{3.74}$$

One can verify the energy density calculated by the above method matches with the known energy density for particles emitted at hawking temperature $\kappa = 1/4m$ which is the same as one that one would asymptotically obtain from (3.60). There would be a difference between the energy density which a free falling observer would observe as calculated in the Painleve coordinate (3.60) and the one which is calculated by the effective temperature method (3.74). One can see this clearly by plotting the difference between these two energy densities as shown in Fig3.8.

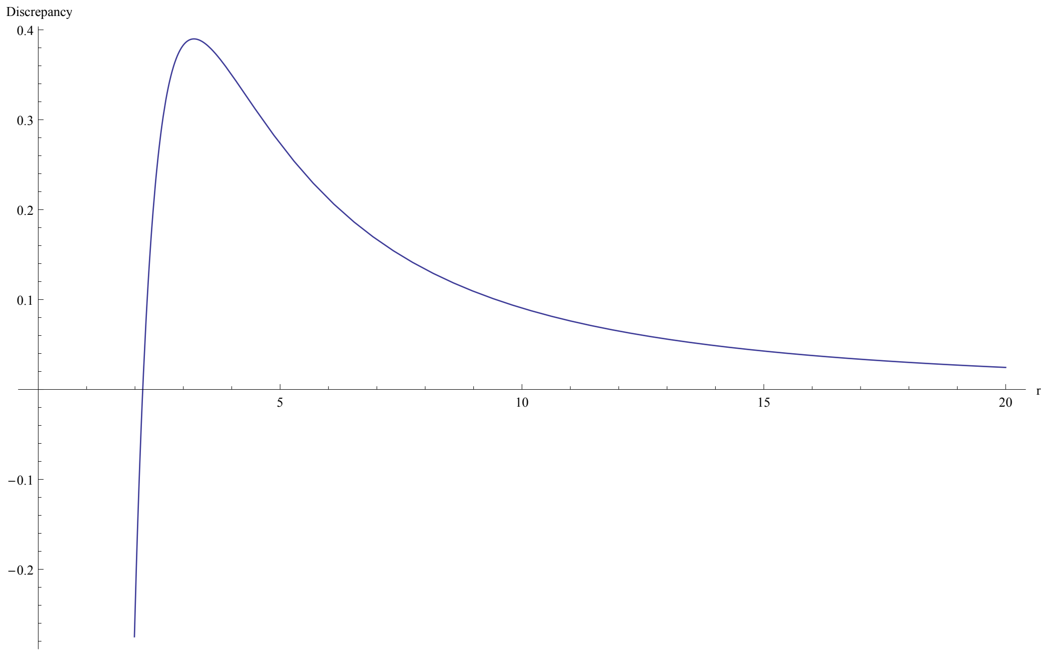

Figure 3.8: Discrepancy in energy density calculated by two methods

We can see a peak in the discrepancy near $r = 3M$ and as expected it vanishes for larger r. One can speculate that the difference in the energy densities comes from the assumption we made about a Plankian spectrum while calculating the energy density in (3.74). One way to check this would be to calculate the adiabatic condition for κ_{eff} and see if κ_{eff} changes adiabatically along the entire geodesic flow for the free falling observer.

3.7. Adiabatic condition

The adiabatic condition is given as

$$\epsilon = \frac{\dot{\kappa}_{eff}}{{\kappa_{eff}}^2} \ll 1 \tag{3.75}$$

where *dot* denotes differentiation with respect to the proper time of the observer. As seen before κ_{eff} is given as

$$\kappa_{eff} = \frac{1}{(1 - \sqrt{\frac{2M}{r}})} \left(\frac{1}{4} - \frac{1}{r^2} \right) \tag{3.76}$$

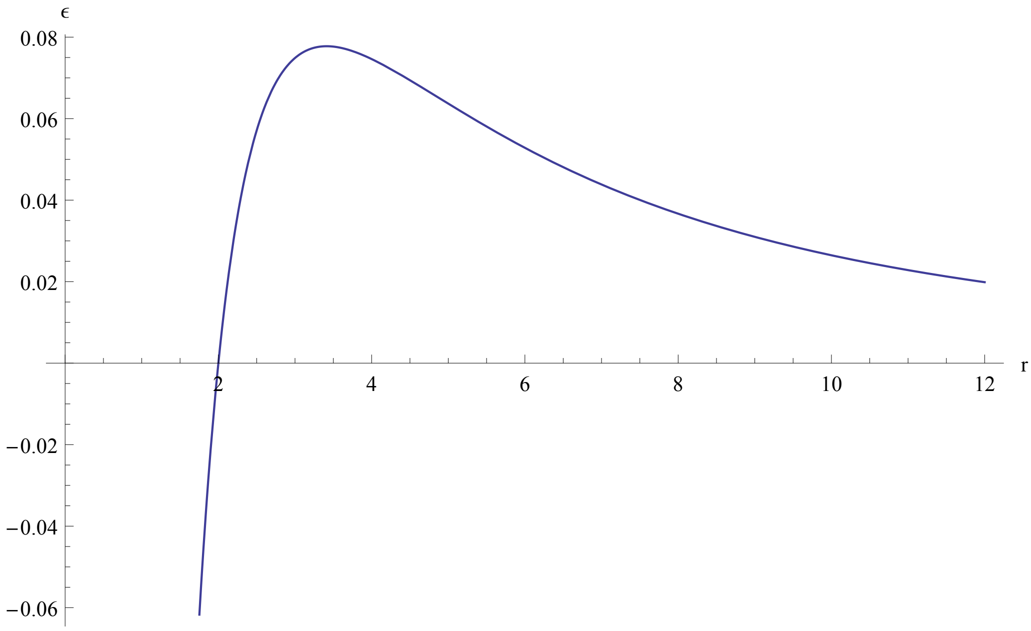

Figure 3.9: Adiabatic function as a function of distance (r). The peak in the diagram shows where there is the maximum violation of the adiabatic condition

To investigate better if there is a violation of the adiabatic condition or a fluctuation in its value along the trajectory of the observer we plot ϵ in FIG3.9

As we can see the adiabatic condition is not strictly violated anywhere but there is a deviation at around $3M$ where its value increases. So one can speculate the small difference between the energy densities around $3M$ in FIG3.8 due to this behavior of κ_{eff} around that region.

The energy density measured by a free falling observer was computed in [99] in terms of the *effective temperature* function. It was found that the energy density is not strictly given by (3.74) but has additional correction terms having derivatives of κ_{eff}. We found that even after including these terms the discrepancy is still there. This can be accounted to the fact that the quantity computed in [99] is not the same energy density for the free falling observer as we computed in (3.60). A "perceived" stress energy tensor was computed in [99] by subtracting a contribution, as measured by the observer in a local vacuum state defined in its local inertial frame (based on the coordinate (u,v)), from the energy density as measured in the vacuum state globally defined based on the null coordinates (U, V).

So one can say that from our analysis we learned that there are several ways to look at the origin of Hawking quanta and the ones that are considered here hints strongly at a long distance origin of the Hawking radiation. By studying the effective temperature we got a deviation in the adiabatic condition which also says that the WKB approximation does not hold exactly and this hints at particle creation in that region. Finally we propose that one can use the difference in the energy density as considered in [99] and the one we computed for a free falling observer in (3.74) to compute what this observer will perceive in its local inertial frame.

CHAPTER 4

Spacetime thermodynamics

4.1. Introduction

As a well known fact and as we saw before general relativity, and other diffeomorphism invariant theories of gravity, admit special states called black holes whose mechanics is governed by the laws that are in exact correspondence to the known laws of thermodynamics [100, 38, 39]. The energy and the entropy of these black holes depend upon the theory under consideration and their temperature is given by a geometric quantity, namely the surface gravity associated to the black hole horizon. As seen before the identification of the surface gravity of the black hole with the temperature comes by studying quantum field theory on the curved gravitational background of the black hole [101]. Remarkably analogy between the laws of black hole mechanics and the thermodynamic laws can be seen just using classical General Relativity without using any quantized matter field in the background. One may then raise the question how a classical theory of gravity can correctly predict the thermodynamic behavior of black holes in spite of the fact that their temperature can be derived only within a quantum mechanical framework.

As emphasized earlier, classical and quantum dynamics of black holes is widely believed to provide important lessons for understanding the underlying quantum theory of gravity. However, the underlying quantum theory should describe all gravitational macrostates and not merely the black holes (which is just a special state of the theory). Thus it seems plausible that if we restrict our attention to a region of spacetime small enough (with respect to the curvature scale) such that the spacetime curvature can be ignored and in that local region spacetime is “close to” Minkowski (by invoking the Einstein Equivalence Principle [30]), then locally the state should look like an equilibrium one and a coarse-grained/thermodynamic description of the degrees of freedom contained in that region of spacetime should be possible.

About twenty years ago, this chain of reasoning led Jacobson to derive the Einstein equation as the equation of state of the underlying microscopic degrees of freedom [29]. Assuming that the heat flow corresponds to the energy-momentum flux of matter across the Rindler horizon of a local observer, the entropy corresponds to the area of the horizon, and the temperature has the Unruh value ($= \hbar/2\pi$), Jacobson showed that the horizon must be dynamical in order for the Clausius relation $\mathrm{d}S = \mathrm{d}Q/T$ to hold true, and that its evolution is governed by the Einstein equation. The far reaching consequence of such a result is that it might be hinting that gravity is emergent or in other words dynamics of space-time is a manifestation of some fundamental dynamical degrees of freedom underlying the gravitational ones.

This naturally leads to the question whether equations of motion for more generalised theories of gravity, such as higher derivative theories, can be derived from local thermodynamical variables for a LCH by implementing the Clausius equation in the same way as it was done for General Relativity [102, 103, 33, 104, 105, 106, 107]. As we will see in Section 4.2, Jacobson's construction was applied to $f(R)$ theory [32, 34] after deforming the Clausius relation to account for the internal entropy production terms, $\mathrm{d}S = \mathrm{d}_i S + \mathrm{d}Q/T$. Further extension of the derivation of the F(R) field equations in this thermodynamic approach was presented in [34], where it was shown that one proceed through the derivation without the need of any internal entropy production term, but taking into account an additional scalar field flux to the heat flow across the horizon which can be reasoned by looking at the equivalence of F(R) theory with a scalar tensor theory of gravity [108].

In section III we will follow the work of ref. [35] where a careful construction of the geometry of local causal horizon (LCH) was used based on Riemann normal coordinates and the approximate Killing vector field constructed in ref. [37]. Horizon slices were then assumed to have an entropy density whose form resembles the form of Noether charge conjugate to diffeomorphisms. By imposing the Clausius relation on a small patch of the horizon enclosed between two slices sharing a common boundary, it was shown that the field equations for a wide class of higher curvature theories of gravity can be derived if a given consistency condition holds. Unfortunately, this consistency condition is not satisfied for general theories of gravity containing derivatives of Riemann tensor. Therefore the thermodynamic derivation of field equation is expected to fail in general higher derivative theories of gravity. Can it be salvaged?

One might wonder why should the entropy density be of the Noetheresque form at all. Could one come up with another definition of entropy of the local causal horizon such that the field equation can be derived from the Clausius relation? Or could one use the ambiguities in the construction of diffeomorphism Noether charge in order to get an entropy that does the job? Even in theories without derivatives of curvature there is a lingering question: how does one define the heat-flux when the matter is non-minimally coupled to the metric? For in that case, there is no canonical splitting of the total Lagrangian between the gravitational part and the matter part. Therefore there is no canonically defined stress tensor that can be used to define the energy flow across the horizon appearing on the right hand side of the Clausius relation.

To address these question in section IV we propose an entropy density that would lead to the derivation of the field equation as an equation of state for any diffeomorphism invariant metric theory of gravity. More precisely, we assume that we have a Lagrangian description of a diffeomorphism invariant theory, and we construct an entropy density associated to slices of the local causal horizon such that imposing the Clausius relation yields the equation of motion of the theory. We define the heat flux on the right hand side of the Clausius relation by using the stress tensor for a probe field minimally coupled to the metric that we put to zero at the end. This will allow us to work with the total Lagrangian of the theory irrespective of the minimal/non-minimal nature of the matter coupling thus evading the lingering question mentioned above.

4.2. Review of previous derivations

In this section we look at the derivation of the Einstein equation as an equation of state, stating all the necessary conditions [29]. We further review the thermodynamic derivation of the F(R) equations of motion using both, the reversible and irreversible Clausius equation [32, 34].

4.2.1. Einsteins equation of state

Using the Einsteins equivalence principle one can view the local neighborhood of any arbitrary space-time point p as flat space-time. Through p one can consider a spacelike 2-surface element Σ and take the expansion and shear of the past directed null congruence on one side to be zero. The past horizon of Σ is called the local causal horizon, which can be thought of as a localized Rindler horizon passing through the point p. This is a necessary condition for equilibrium thermodynamics of the horizon defined within this local patch around p, as a non-zero expansion or shear would imply the surface (cross section of the LCH) is shearing or expanding or contracting.

If one assumes the validity of the Clausius equation within this local patch (i.e $\delta Q = TdS$) one can interpret the heat as energy flow across the horizon and interpret the entropy as the entanglement entropy of some field degrees of freedom across the horizon, which results to be proportional to the area of the horizon. For an accelerated observer just outside the horizon, the vacuum fluctuations as perceived by the observer are thermal in nature, so the temperature of the system can be simply interpreted as the well known Unruh temperature which is given as $T = 1/2\pi$. For consistency, one should use the same accelerated observer to define the energy flux (or in other words the heat flow across the LCH). This can be better understood by using the Bisognano–Wigman theorem which tells us about the thermal nature of the Rindler patch based on the boost invariance of the Rindler Hamiltonian.

Assuming a local Rindler horizon through p one can assume an approximate local (boost) Killing vector, ξ^a, generating the horizon. To the past of Σ the heat flux can be defined as the boost energy across the horizon

$$\delta Q = \int_{\mathcal{H}} T_{ab}\xi^a dd^b \,, \tag{4.1}$$

where the integral is over the generators of the "inside" horizon $\mathcal{H}$ of Σ. We can assume a vector k^a to be tangent to the horizon and parametrized by an affine paprameter λ, then we have the relation $\xi^a = -\lambda k^a$, asuuming λ is negative on the past of Σ. We also have the relation $d\Sigma^a = k^a d\lambda dA$, where dA is the area element of the cross section of the LCH. Using these relations, the heat flux is given as

$$\delta Q = -\int_{\mathcal{H}} \lambda\, T_{ab}k^a k^b d\lambda dA \,. \tag{4.2}$$

Now, for what concerns the entropy term in the Clausius equation, one can say that the mere presence of fields in our spacetime allows to associate to our LCH an entanglement entropy. This is generically proportional to the area as well as divergent at the horizon and hence in need of a regularization UV scale, α. Further restricting the equivalence principle to its strong version assures that such regulator will be a constant so $dS = \alpha\delta A$, where the variation of the area, δA, of the LCH cross section given by

$$\delta A = \int_{\mathcal{H}} \theta d\lambda dA \,, \tag{4.3}$$

where θ is the expansion of the null congruence. As mentioned earlier, at p the expansion of the congruence must vanish along with the shear to have a thermodynamical system in equilibrium[1] Then, we can expand the variation of the the area about point p and use $\theta_p = 0$ to get

$$\delta A = \int_{\mathcal{H}} \left.\frac{d\theta}{d\lambda}\right|_p \lambda d\lambda dA = -\int_{\mathcal{H}} \lambda R_{ab}k^a k^b d\lambda dA \,, \tag{4.4}$$

[1] One has the freedom to assumes a non zero shear, then we would not have equilibrium thermodynamics because of gravitational dissipation [109] but that is not the case under consideration here.

where in the second step we have used the Raychaudhuri equation with shear and expansion assumed to be zero, as an equilibrium condition, and the twist is also set to zero since k^a is hypersurface orthogonal. By plugging (4.4) and the Unruh temperature in the Clausius equation one gets, for all null k^a, the equation

$$T_{ab}k^a k^b = (1/2\pi)R_{ab}k^a k^b\,. \tag{4.5}$$

After peeling off the k's and using the local conservation of the stress-energy tensor together with the Bianchi identity, one recovers the full Einstein equation from (4.5).

4.2.2. Spacetime thermodynamics for F(R) gravity: Non equilibrium case

As we have seen in the previous section, we can choose any arbitrary space-time point, p, and consider a locally flat patch around this point in order to exploit the existence of a local approximate boost Killing vector by means of Riemann normal coordinates [37]. For deriving the $F(R)$ equations of motion in the thermodynamics approach, the basic geometric construction remains the same as before, except now one needs to consider a non-zero expansion at p (still one can assume the shear to vanish) for validity of the Clausius equation at $\mathcal{O}(\lambda)$, thus implying non-equilibrium thermodynamics of the LCH as we will see below [32].

If we assume the entropy density to be a general function of the scalar curvature, then the entropy of the LCH reads

$$S = \alpha \int f(R)dA \tag{4.6}$$

and its variation along λ is given as

$$dS = \alpha \int_{\mathcal{H}} \left(f\theta + \frac{df}{d\lambda}\right) d\lambda dA\,. \tag{4.7}$$

As in the previos section, we expand this entropy about the point p, we plug it in the Clausius equation and we equate it to the heat flux (4.2), which is $\mathcal{O}(\lambda)$. Performing the series expansion in λ, we get

$$dS = \alpha \int_{\mathcal{H}} \left[\left(f\theta + \frac{df}{d\lambda}\right) + \lambda\left(\theta\frac{df}{d\lambda} + f\frac{d\theta}{d\lambda} + \frac{d^2 f}{d\lambda^2} + f\theta^2 + \theta\frac{df}{d\lambda}\right)\right]_p d\lambda dA\,. \tag{4.8}$$

From the above expression, we see that, in order to match this expression to the heat flux, we must retain only the $\mathcal{O}(\lambda)$ terms; thus, we must use the condition

$$\left[f\theta + \frac{df}{d\lambda}\right]_p = 0\,. \tag{4.9}$$

From the above equation one can see that the expansion will be non zero at p, hence even if we neglect shear tit is not possible to have equilibrium thermodynamics at the hypersurface p. For non-equilibrium thermodynamics, one needs to use the Clausius equation modified by an additional internal entropy term, $\delta Q = T\delta S + dS_i$. Using (4.8) and (4.2) in the Clausius equation and using the Raychaudhuri equation to replace $d\theta/d\lambda$, one gets

$$\alpha(-fR_{ab}k^a k^b + k^a k^b \nabla_a \nabla_b f - \frac{3}{2f}\theta^2) + dS_i = 2\pi(-T_{ab}k^a k^b - \xi_B \theta^2)\,. \tag{4.10}$$

From (4.9) one can write the expansion as a kinetic term for f. By doing so and peeling off the k's, we get

$$fR_{ab} - \nabla_a \nabla_b f + \frac{3}{2f}\nabla_a f \nabla_b f + \Phi g_{ab} + dS_i = \frac{2\pi}{e}T_{ab}\,. \tag{4.11}$$

At this point, when one tries to use the conservation of the stress-energy tensor to recover the full $F(R)$ equation of motion, one runs into troubles; namely, one cannot solve for Φ unless the kinetic term $\frac{3}{2f}\nabla_a f \nabla_b f$ is canceled by a similar term coming from the internal entropy production term, dS_i. Therefore, by making the identification

$$dS_i = -\frac{3}{2}\int_{\mathcal{H}} \alpha f \theta^2 \lambda d\lambda dA\,, \tag{4.12}$$

one can use the conservation of stress-energy tensor and recover the correct $F(R)$ equations of motion.

4.2.3. Spacetime thermodynamics for F(R) gravity: equilibrium approach

We review a further extension of this derivation as presented in [34], where one proceed through the derivation without the need of any internal entropy production term (as long as the shear at p is assumed to be zero), but taking into account an additional scalar field flux contribution to the heat flow across the horizon using the equivalence of F(R) gravity to scalar tensor theories.

If we start with the action

$$L = \frac{\alpha}{4\pi}\int (F(R) + L_{matter})\sqrt{-g}d^4x, \tag{4.13}$$

one can take $f = \frac{dF(R)}{dR}$ as an auxillary field φ and $V(\varphi)$ as the Legendre transform of $F(R)$ to get

$$L = \frac{\alpha}{4\pi} \int (\varphi R + V(\varphi) + L_{matter}) \sqrt{-g} d^4x \,. \tag{4.14}$$

The corresponding equations of motion for this Lagrangian are given as

$$\begin{aligned} R &= V'(\varphi)\,, \\ \varphi G_{ab} &= \nabla_a \nabla_b \varphi + 2\pi/e T_{ab} - g_{ab} - 1/2 g_{ab} V(\varphi)\,. \end{aligned} \tag{4.15}$$

To derive these field equations in a thermodynamical approach, one can start with an entropy density defined as

$$S = \alpha \int \varphi dA \,. \tag{4.16}$$

Now, promoting the entropy density to an independent field $\varphi(x)$, in order to be consistent with background independence, this field must be varied as well while performing variation of the total Lagrangian $(L_{gravity} + L_{matter})$ of the theory, which now reads

$$L_{total} = F(R) + L_{matter}(g_{ab}, \psi) + L_{scalar}(g_{ab}, \varphi)\,, \tag{4.17}$$

where ψ represents the ordinary matter fields. Variation with respect to the metric tensor gives the usual stress-energy tensor coming from L_{matter}; but now there would be an addition contribution from L_{scalar} as well. The heat flux across the LCH in such a setup is given by

$$\delta Q \sim k^a k^b (T^M_{ab} + T^{\varphi}_{ab}). \tag{4.18}$$

Only the kinetic part of the scalar field action will contribute to the heat flux, as the potential part contribution is always proportional to g_{ab}. Using dimensional analysis, one can write the most general contribution of the scalar field to the heat flux as

$$\delta Q_{scalar} \sim \Omega(\varphi)/\varphi k^a k^b \nabla_a \varphi \nabla_b \varphi. \tag{4.19}$$

Following the same steps as before and using the reversible Clausius equation one gets

$$\varphi R_{ab} - \nabla_a \nabla_b \varphi + \frac{(3/2 - \Omega)}{\varphi} \nabla_a \varphi \nabla_b \varphi + \Phi g_{ab} = 2\pi T^M_{ab}\,, \tag{4.20}$$

which would correspond to the fields equation of any Brans-Dickie theory if we set the Dickie constant as $\omega = \Omega - 3/2$.

As we saw in the previous derivations there are three essential ingredients involved in the construction of local spacetime thermodynamics. First, definition of the co-dimension three surface, called the local causal horizon (LCH), which plays the role of the local Rindler horizon. Second, specification of a special observer that measures the entropy and the energy flux. Since a general spacetime has no symmetries there is no Killing vector playing the role of the Rindler observer. Therefore one needs to construct a vector field ξ that is "approximately" Killing and plays the role of local observers in whose frame one formulates the local thermodynamics. The third and the final ingredient is the specification of the entropy functional associated with the slices of the LCH.

For the derivation of the higher curvature theories other than F(R) gravity, one cannot just invoke Einstein equivalence principle and assume the existence of a local Killing vector. As we will see in this section one needs to use the Killing identity as well as the Killing equation, so a better construction of the local Killing vector needs to be used. The geometric construction of the LCH and the approximate Killing vector used here is mainly based on refs. [35, 37] as we discussed it earlier in section 2.3.

Let us then come back again on the definition of LCH. Consider a spacelike codimension-two surface Σ_p passing through a spacetime point p. This surface has four congruences of null geodesics emanating orthogonally from it: future-pointing and outgoing, future-pointing and ingoing, past-pointing and outgoing, past-pointing and ingoing. The boundary of the past of Σ_p has two components generated by the latter two congruences. Pick one of those past boundary components, for concreteness, say, the ingoing one, then our LCH is defined as a small patch of this ingoing past boundary component centered at the point p.

As usual the equation of state derivation of the field equation of a theory of gravity proceeds by imposing the Clausius relation,

$$\mathrm{d}S = \frac{\mathrm{d}Q}{T}, \tag{4.21}$$

on a thin patch of the LCH, denoted as $\mathcal{H}$, centered on the central generator Γ (see fig. 4.1). The left hand side of eq. (4.21) is the change in entropy as one evolves the slice of the LCH from Σ_0 to Σ such that they have a common boundary. The right hand side of eq. (4.21) contains the temperature, which we choose to have the Unruh value $T = \hbar/2\pi$, and the heat flux across the patch as measured by the local Killing observer ξ^a,

$$\mathrm{d}Q = \int_{\mathcal{H}} (-T_a{}^b\,\xi^a)\,k_b \mathrm{d}V\,\mathrm{d}A, \tag{4.22}$$

where T_{ab} is matter energy-momentum tensor, and the integral is over the thin patch of LCH (see fig. 4.1) with the integration measure $k_a \mathrm{d}V \mathrm{d}A$, and $\mathrm{d}A$ being the volume element on the cut of $\mathcal{H}$. The integrand of the heat flux is of $O(x)$ since the approximate Killing vector ξ is of $O(x)$.

In the known literature related to the equation of state derivation of the field equation, the stress tensor above is taken to be that of the matter fields in the theory. This implicitly assumes that the matter is minimally coupled to the metric, for only then can one separate the total Lagrangian into a gravitational part and the matter part, and use the latter to define the canonical stress energy tensor. However, in general theories of gravity non-minimal couplings are allowed and there is no natural split between gravity and matter Lagrangian, and thus no natural stress tensor providing the heat flux. We will overcome this problem by deforming the theory with a probe action. We will introduce a probe field minimally coupled to the metric whose flow drives the evolution of LCH. In the end, we will put this probe field to zero. Therefore, in our derivation of the equation of state we will take T_{ab} above to be the stress energy tensor of this probe field, $T_{ab} = -2\frac{1}{\sqrt{-g}}\frac{\mathrm{d}S_{\text{probe}}}{\mathrm{d}g^{ab}}$, where S_{probe} is the action for the probe field minimally coupled to the metric.

To proceed further one needs to specify the change in entropy on the right hand side of eq. (4.21). Intuition from the thermodynamics of black holes suggests that we associate entropy to the slices of LCH. Following ref. [35], let s^{ab} denote the entropy density (in the dualized form) associated to an arbitrary slice of LCH. Total entropy of a slice Σ is then given by the integral

$$S = \int_{\Sigma} s^{ab} n_{ab}\, \mathrm{d}A, \tag{4.23}$$

where n^{ab} is binormal to the cut Σ. Hence, the change in entropy between two slices Σ and Σ_0 of LCH is given by

$$\begin{aligned} \mathrm{d}S &= \int_{\Sigma \cup \Sigma_0} s^{ab} n_{ab} dA \\ &= -2 \int_{\mathcal{H}} \nabla_b s^{ab} k_a dV dA, \end{aligned} \tag{4.24}$$

where the Stokes' theorem was used in the second step. It is at this step that we used that Σ_0 and Σ have the same boundary. Now imposing the Clausius relation (4.21) in the limit $p_0 \to p$, we get from eqns. (4.22) and (4.24)

$$-(\hbar/\pi)\nabla_b s^{ab}\, k_a = T^{ab}\xi_b\, k_a + O(x^2). \tag{4.25}$$

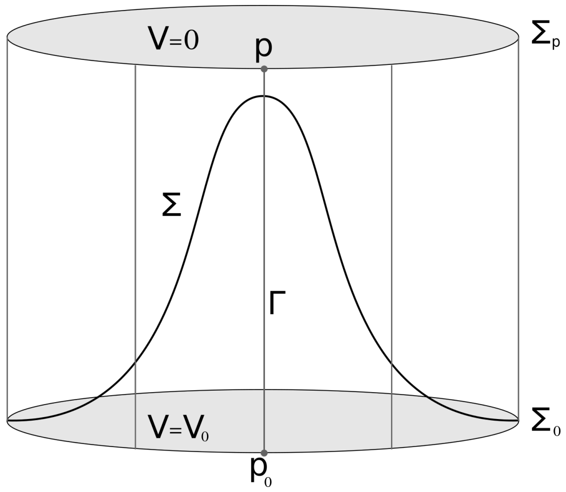

Figure 4.1: The thin narrow patch of LCH surrounding the central generator Γ on which the Clausius relation is imposed.

By equating the $O(x)$ terms on both sides of eq. (4.25) at all points p and for all null vectors k^a, if we recover the field equation of the theory of gravity under consideration (after putting $T^{ab} = 0$ because our T^{ab} is that of the probe field) then we deem the program to derive the equation of motion as the equation of state to be successful. Now it is clear that the last ingredient in this program is the specification of the entropy density s^{ab} such that eq. (4.25) gives the field equation of the theory.

Now as seen in (2.4), we know what is the correct field equation that should be recovered by using this thermodynamic argument. In the rest of this section we will put $\hbar/2\pi$ to be equal to 1, i.e., the Unruh temperature is scaled to unity, which is equivalent to choosing a convenient unit for the entropy density.

4.3.1. Field equations from Noetheresque entropy density

A specific proposal for the entropy density s^{ab} was made in ref. [35] (see also, refs. [33, 104]). Taking clue from the Noether charge entropy in black hole thermodynamics ref. [35] proposed a Noetheresque form for the entropy density,

$$s^{ab} \;=\; W^{abc}\xi_c + X^{abcd}\nabla_{[c}\xi_{d]}, \tag{4.26}$$

where the tensors W and X are theory dependent quantities and X is antisymmetric in the last two indices. One could also add a term proportional to the symmetric derivative of ξ but it can be shown using the properties (2.26) and (2.27) of the approximate Killing vector that such a term contributes at $O(x^A)$ (where x^A is transverse coordinate in NNC system) to the divergence of entropy density and hence does not contribute to $\mathrm{d}S$ when integrated over small and narrow horizon patches [35].

Calculating the divergence of the entropy density (4.26) we get,

$$\begin{aligned}\nabla_b s^{ab} &= \left(\nabla_p W^{aps} + X^{apqr} R_{rqp}{}^{s}\right)\xi_s \\ &+ X^{apqr}\left(\nabla_p \nabla_q \xi_r - R_{rqp}{}^{s}\,\xi_s\right) \\ &+ \left(W^{apq} + \nabla_r X^{arpq}\right)\nabla_p \xi_q. \end{aligned} \tag{4.27}$$

In this equation the first term is $O(x)$, the second term is $O(x^A)$ due to the Killing identity of eq. (2.27), and the third term has an $O(x^2)$ term due to the approximate Killing equation (2.26) and an $O(x^0)$ term due to the antisymmetric part of the derivative of ξ. Since the heat-flux in eq. (4.22) is of $O(x)$ the latter should vanish. Thus we are forced to impose

$$W^{a[pq]} + \nabla_r X^{ar[pq]} = 0. \tag{4.28}$$

Since W is antisymmetric in the first two indices, this equation can be solved for W in terms of X [35] as

$$W^{apq} = \nabla_r \left(X^{rapq} + X^{rqpa} + X^{rpqa}\right). \tag{4.29}$$

Putting this back in the eq. (4.27), then substituting $\nabla_b s^{ab}$ in eq. (4.25), and imposing Clausius relation for all k^a we get,

$$X^{pqr(a} R_{pqr}{}^{b)} - 2\nabla_p \nabla_q X^{p(ab)q} + \Phi g^{ab} = -\frac{1}{2} T^{ab}, \tag{4.30}$$

where Φ is a scalar that is a function of metric and curvature. The origin of the factor 1/2 on the right hand side is the convention we adopted at the end of sec. (4.3) that $\hbar/2\pi = 1$. Comparing eq. (4.30) with the equation of motion for a general diffeomorphism invariant theory eq. (2.31) we see that in general there is no choice of X that would make them identical.

We now recall that in refs. [35, 33, 104] matter was assumed to be minimally coupled, i.e., total Lagrangian L was the the sum of gravitational part and the minimally coupled matter part $L = L_{(gr)} + L_{(m)}$, and the gravitational part $L_{(gr)}$ was assumed to depend only on the metric and its curvature but not on the derivatives of curvature. Furthermore, the heat flux in the Clausius relation was sourced by the matter stress energy tensor,

$$\frac{1}{2} T^{ab}_{(m)} = \frac{\partial L_{(m)}}{\partial g_{ab}} + \frac{1}{2} L_{(m)} g^{ab}. \tag{4.31}$$

Now choosing $-X^{abcd} = \partial L_{(gr)}/\partial R_{abcd} \equiv P^{abcd}$, and $\Phi = 1/2L_{(gr)}$ in eq. (4.30) we get,

$$-P^{pqr(a}R_{pqr}{}^{b)} + 2\nabla_p\nabla_q P^{p(ab)q} + \frac{1}{2}L_{(gr)}g^{ab} = -\frac{1}{2}T^{ab}_{(m)}. \tag{4.32}$$

If the gravity Lagrangian $L_{(gr)}$ does not contain the derivatives of Riemann then there exists an interesting identity,

$$\frac{\partial L_{(gr)}}{\partial g_{ab}} = -2P^{pqr(a}R_{pqr}{}^{b)}. \tag{4.33}$$

This identity, first derived in ref. [106], is derived in (4.6) where we slightly generalize by considering the gravity Lagrangians containing upto one derivative of curvature. Substituting the identity (4.33) in eq. (4.32), plugging in the expression for $T^{ab}_{(m)}$ from eq. (4.31) and bringing it to the left hand side, we get

$$\frac{\partial L}{\partial g_{ab}} + P^{pqr(a}R_{pqr}{}^{b)} + 2\nabla_p\nabla_q P^{p(ab)q} + \frac{1}{2}Lg^{ab} = 0, \tag{4.34}$$

where we have combined the contributions of $L_{(m)}$ and $L_{(gr)}$ into that of the total Lagrangian L. Now noticing that $\partial L_{(gr)}/\partial R_{abcd} \equiv P^{abcd} = \partial L/\partial R_{abcd}$ since the matter is minimally coupled, we find that eq. (4.34) is identical to eq. (2.31) since for higher curvature theories without the derivatives of curvature we have that $B^{ab} = 0$, $A^{ab} = \partial L/\partial g_{ab} + 1/2Lg_{ab}$ and $E^{abcd} = P^{abcd}$. Therefore we see that for higher curvature gravity the Noetheresque entropy (4.26) of ref. [35] reproduces the equation of motion via the Clausius relation.

However, for the theories containing derivatives of curvature the equation of motion (4.30) obtained from the Clausius relation, assuming the Noetheresque entropy as in eq. (4.26), is not the same as the equation of motion of the theory (2.31). The difference can be traced back to the presence of two terms in A^{ab} (2.33) appearing in the equation of motion: first is $\partial L/\partial g_{ab}$, and the second is that arising from the variation $\mathrm{d}\nabla \ldots \nabla(\mathrm{Riem})$ of derivative(s) of curvature terms in the Lagrangian that we have collectively denoted as B^{ab}.

In the next section we propose a new definition of entropy that takes care of the uncompensated terms and yields the equation of motion via the Clausius relation. We will view the heat flux on the right hand side of the Clausius relation as due to the T^{ab} of a probe field that we will put to zero at the end of the calculation.

In this section we present the key finding of [36]. We modify the Noetheresque entropy of eq. (4.26) by adding a term quadratic in the approximate Killing vector. Let us introduce a symmetric tensor M^{ab}, which will be fixed later depending upon the theory, and consider the following entropy density,

$$s^{ab} \;=\; W^{abc}\xi_c + X^{abcd}\nabla_{[c}\xi_{d]} + 2M^{c[a}\xi^{b]}\xi_c. \tag{4.35}$$

The M term we have added is of $O(x^2)$ but it contributes at $O(x)$ to the left hand side in eq. (4.25). We recall that the effect of adding an exact form $\mathrm{d}\mu$ to the Lagrangian n-form is to shift the Noether charge from $\mathbf{Q}$ to $\mathbf{Q}+\xi\cdot\mu$ (see ref. [39]). It is easy to check that if we choose the $(n-1)$-form $\mu_{a_1\ldots a_{n-1}} = \boldsymbol{\epsilon}_{a_1\ldots a_{n-1}p}M^{pq}\xi_q$ then the Noether procedure will reproduce our proposed additional term in the entropy. Let us calculate the divergence of the M term,

$$\begin{aligned}&\nabla_b(2M^{c[a}\xi^{b]}\xi_c)\\ &= 2(\nabla_b M^{c[a})\xi^{b]}\xi_c + 2M^{c[a}(\nabla_b\xi^{b]})\xi_c + 2M^{c[a}\xi^{b]}\nabla_b\xi_c.\end{aligned} \tag{4.36}$$

Here, the first term on the right hand side is of $O(x^2)$. The second term, upon opening the antisymmetrization, has two sub-terms: the first containing $\nabla_b\xi^b$ is of $O(x^3)$, while the second containing $\nabla_b\xi^a$ will give zero when contracted with k_a. This is so because the approximate Killing vector ξ is proportional to k on the central generator Γ. The third term in eq. (4.36) again has two sub-terms: the first one with the free index a on M gives $M^{ca}\xi_c$ while the second with free index a on ξ will give zero after contracting with k. Therefore, the only contribution of the M term is to add the tensor M^{ab} to the first line of eq. (4.27). The relation between X and W as determined in eq. (4.29) remains the same. Thus the equation of motion obtained by imposing Clausius relation with the entropy (4.35) is

$$X^{pqr(a}R_{pqr}{}^{b)} - 2\nabla_p\nabla_q X^{p(ab)q} + \Phi g^{ab} + M^{ab} = -\frac{1}{2}T^{ab}, \tag{4.37}$$

where T^{ab} is the stress tensor of the probe field. Now, for a given theory of gravity we can simply choose M^{ab} such that eq. (4.37) is the equation of motion for the theory (after putting the probe stress tensor on the right hand side to zero). Comparing with the equation of motion of a general theory of gravity eq. (2.31) we see that we could choose

$$X^{abcd} = -E^{abcd}, \tag{4.38}$$

$$M^{ab} = \frac{\partial L}{\partial g_{ab}} + 2E^{pqr(a}R_{pqr}{}^{b)} + B^{ab}, \tag{4.39}$$

$$\Phi = \frac{1}{2}L. \tag{4.40}$$

Actually, the equation of motion only determines the combination $\Phi g^{ab} + M^{ab}$. Once we have specified M^{ab} then Φ can be determined by the Bianchi identity. For the choice of M^{ab} that we have made above, by comparing with the actual equation of motion we can immediately recognize that Φ should be equal to $1/2L$ up to a constant. Since M^{ab} is what appears in the expression of the horizon entropy we see that the entropy is not unique, for the terms proportional to g^{ab} in M^{ab} could equally well be lumped into Φ.

One important thing to note is if the new proposed entropy density alters the black hole entropy by any means. We see that the new term in the entropy that we have proposed does not alter the black hole entropy because the Killing vector vanishes on the bifurcation surface. Compatibility with black hole thermodynamics is a stringent requirement. Without it, we could have simply taken the whole entropy as given by the quadratic term and chosen M^{ab} to be the equation of motion. But then the black hole entropy in the theory would be zero. The X term in eq. (4.35) is thus dictated by the black hole entropy. The W term is necessary for the equation of state argument to go through for the higher curvature theories. For higher derivative theories the M term in eq. (4.35) is needed to get the equation of motion via the Clausius relation.

We must say that the non-uniqueness of the entropy density is beyond the non-uniqueness pointed out above. It is easy to write down higher order terms in ξ and its derivatives such that their contribution to the change in entropy of the patch of LCH is just $M^{ab}\xi_b$ for some effective M^{ab}. We think that the underlying problem is our completely classical treatment of the fields. We believe that the correct notion of entropy to be used in any thermodynamic derivation of field equation has to be quantum mechanical one. This is exemplified by a recent derivation of the semiclassical Einstein equation by Jacobson that involves an ansatz on the nature of entanglement entropy of the vacuum [41]. It has recently been pointed out in ref. [46] that relative entropy is not the right quantity to use on the left hand side of the Clausius relation in the geometric framework used here. It remains to be seen what quantum mechanical measure of entropy is rich enough to encode the dynamics of gravity.

Our approach so far has been very general. The use of probe field and the addition of a term quadratic in the local Killing vector to entropy density allowed us to give a thermodynamic derivation of the field equation for a general theory of gravity. In this section we illustrate our approach in several examples.

4.5.1. General relativity

As the simplest illustration of our approach let us consider the Einstein-Hilbert Lagrangian with the matter minimally coupled to the metric. The total Lagrangian is $L = L_{(EH)} + L_{(m)}$, where $L_{(EH)} = R$ and $L_{(m)}$ is the minimally coupled matter Lagrangian. The coefficients appearing in the entropy density (4.35), as defined in eqns. (4.38) (4.39) (4.40), can be calculated to be,

$$\begin{aligned} X^{abcd} &= -\frac{1}{2}(g^{ac}g^{bd} - g^{ad}g^{bc}), \\ M^{ab} &= \frac{\partial L}{\partial g_{ab}} + 2R^{ab} = \frac{\partial L_{(m)}}{\partial g_{ab}}, \\ \Phi &= \frac{1}{2}L = \frac{1}{2}R + \frac{1}{2}L_{(m)}, \end{aligned}$$

and $W^{abc} = 0$, and where in second equality of the M term we used that $\partial R/\partial g_{ab} = -2R^{ab}$. The equation implied by the Clausius relation (4.37) is then

$$-R^{ab} + \frac{1}{2}(R + L_{(m)})g^{ab} + \frac{\partial L_{(m)}}{\partial g_{ab}} = -\frac{1}{2}T^{ab}, \tag{4.41}$$

where T^{ab} on the right hand side is the stress tensor of the probe. For vanishing probe, recognizing that $\partial L_{(m)}/\partial g_{ab} + 1/2 L_{(m)} g^{ab} = 1/2\, T^{ab}_{(m)}$ is the matter stress energy tensor, we get the Einstein field equation (in the units such that $16\pi G = 1$),

$$R^{ab} - \frac{1}{2}Rg^{ab} = \frac{1}{2}T^{ab}_{(m)}.$$

This example illustrates explicitly that the matter Lagrangian, even if minimally coupled, makes a contribution to the entropy associated with the slices of LCH because of the M term. Therefore our entropy is different from that of ref. [35] even for the simplest possible case of general relativity.

The second example that we consider is a model in two dimensions: a non-minimally coupled dilaton φ with coupling constant λ and a Tachyon T, given by the action,

$$S = \int \mathrm{d}^2x\sqrt{-g}\, e^{\varphi}(R + (\nabla\varphi)^2 - (\nabla T)^2 + \mu^2T^2 + \lambda). \tag{4.42}$$

The black hole solutions in this model were studied in ref. [110] and it was shown that black hole physics in general relativity have counterparts in these two-dimensional models. In particular, black hole entropy of charged black holes in this theory was shown to be proporional to e^{φ_H}, where φ_H is the value of dilaton on the horizon. This result can also be obtained from the Noether charge method (see ref. [39]). The field equation obtained from the action (4.42) is

$$\begin{aligned}\nabla^a\nabla^b\varphi + \nabla^aT\nabla^bT + g^{ab}\Big(- \frac{1}{2}(\nabla\varphi)^2 - \Box\varphi - \frac{1}{2}(\nabla T)^2 \\ + \frac{\mu^2T^2}{2} + \frac{\lambda}{2}\Big) = 0.\end{aligned} \tag{4.43}$$

There does not seem to be a natural way to write this equation in terms of separate contributions from geometry and matter. That is, it is not clear how to decompose the action (4.42) into gravitation and matter piece. Therefore, we do not know what stress tensor should be used to calculate the heat flux. We could use the Tachyon stress tensor for this purpose but there does not seem to be a good justification for doing that.

According to the idea pursued in this chapter, we use the whole Lagrangian to contribute to the entropy while the heat flux is to be determined by a probe field that we put to zero at the end. Then the field equation (4.43) can be obtained by assigning entropy density (4.35) to LCHs with the coefficient tensors given by:

$$\begin{aligned} X^{abcd} &= -\frac{1}{2}e^{\varphi}(g^{ac}g^{bd} - g^{ad}g^{bc}), \\ M^{ab} &= e^{\varphi}\left(-\nabla^a\varphi\nabla^b\varphi + \nabla^aT\nabla^bT\right), \\ \Phi &= \frac{1}{2}L, \end{aligned} \tag{4.44}$$

where L is the total Lagrangian for the dilaton theory (4.42). Notice that X^{abcd} corresponds to the black hole entropy. In this example we have a non-zero M^{ab} not because of the higher derivative terms (there are none) but because of the non-minimal coupling of the matter. Even if we were to define the heat flux not by our probe field but by using the stress tensor of the Tachyon T, there would still be non-trivial contributions to M^{ab} and therefore this term is needed in the entropy density to get the field equation from local thermodynamics.

4.5.3. *Higher curvature gravity*

Let us now consider higher curvature gravity with minimally coupled matter field for which the Noetheresque entropy of ref. [35] also gives the field equation via the Clausius relation. For the total Lagrangian given by

$$L = L(g_{ab}, R_{abcd}, \psi, \nabla_a \psi), \tag{4.45}$$

the field equation is

$$\frac{\partial L}{\partial g_{ab}} + \frac{1}{2} g^{ab} L + P^{pqra} R_{pqr}{}^{b} + 2\nabla_p \nabla_q P^{pabq} = 0, \tag{4.46}$$

where $P^{abcd} = \partial L / \partial R_{abcd}$. The coefficient tensors in our entropy density (4.35) are given by

$$\begin{aligned} X^{abcd} &= -P^{abcd}, \\ M^{ab} &= \frac{\partial L}{\partial g_{ab}} + 2P^{pqr(a} R_{pqr}{}^{b)}, \\ \Phi &= \frac{1}{2} L, \end{aligned}$$

and W^{abc} is given by eq. (4.29). This should be contrasted with the entropy density of ref. [35] that we reviewed in sec. (4.3.1) where X and Φ were defined by only the gravitational part of the Lagrangian and there was no M term. If we allow for the non-minimal coupling in the higher derivative gravity then our approach of using the probe stress tensor to define the heat flux and the new entopy density will continue to yield the field equation via the Clausius relation.

4.5.4. $S = \int \sqrt{-g} f(\Box R) + S_{matter}$

As a final example we consider a higher derivative theory with matter minimally coupled to the metric. The gravitational part of the Lagrangian is a general function of $\Box R$ that we denote by $f(\Box R)$. Some special cases of these theories were studied in ref. [111] to show their equivalence to general relativity coupled to matter fields with exotic potentials. The equation of motion of this theory is

$$\nabla^a\nabla^b\Box f' - \Box f' R^{ab} + \nabla^a f'\nabla^b R - \frac{1}{2}g^{ab}\nabla_c f'\nabla^c R$$
$$-g^{ab}\Box^2 f' + 1/2 f g^{ab} = \frac{1}{2}T^{ab}_{(m)}, \tag{4.47}$$

where $f' = \partial f(\Box R)/\partial\Box R$ and $T^{ab}_{(m)}$ is the canonical stress energy tensor determined by the matter action, $\sqrt{-g}\,T^{ab}_{(m)} = 2\mathrm{d}S_{matter}/\mathrm{d}g_{ab}$. Since the matter is minimally coupled we could in principle use it to define the heat flux in the Clausius relation. From the point of view of this section though we will treat the whole action to contribute to the entropy while the heat flux would be given by the stress tensor of the probe field.

For this theory the coefficients appearing in the entropy density (4.35), as defined in eqns. (4.38 4.39 4.40), can be calculated to be,

$$\begin{aligned}
X^{abcd} &= -\frac{1}{2}(g^{ac}g^{bd} - g^{ad}g^{bc})\Box f', \\
M^{ab} &= -\frac{1}{2}T^{ab}_{(m)} + \nabla^a f'\nabla^b R - g^{ab}\Box^2 f' - \frac{1}{2}g^{ab}\nabla_c f'\nabla^c R, \\
\Phi &= \frac{1}{2}f,
\end{aligned}$$

and W^{abc} is given by eq. (4.29). Had we considered the heat flux to be sourced by the matter instead of the probe field then $T^{ab}_{(m)}$ would have appeared on the right hand side of the Clausius relation and would not have apppeared in M^{ab}. As we mentioned at the end of sec. 4.4 the terms proportional to g^{ab} in M^{ab} could be absorbed in Φ. But there would still be left the second term $\nabla^a f'\nabla^b R$ in M^{ab}. This is precisely the type of term whose origin lies in the derivatives of curvature in the action and could not be produced by the entropy density of ref. [35]. Therefore, even in the minimally coupled case and without the use of probe fields, the M term would be needed in the entropy density to yield the correct field equation.

4.6. Properties of Lagrangians for higher derivative gravity

This identity was first derived in ref. [106] whose treatment we follow here. A slight generalization here is that we consider the Lagrangians containing upto one derivative of curvature, $L = L(g_{ab}, R_{abcd}, \nabla_{a_1}R_{abcd})$. In this section L will stand for pure gravitational Lagrangian that we denoted as $L_{(gr)}$ above.

Let us consider an infinitesimal diffeomorphism $x^a \to x^a + \xi^a$ generated by a vector field ξ. The infinitesimal change in L is given by the Lie derivative of L that can be calculated in two different ways. In the first way, by considering the dependence of L on x^a through g_{ab}, R_{abcd} and $\nabla_{a_1} R_{abcd}$, we can write

$$\begin{aligned} \pounds_\xi L \quad &= \xi^m \nabla_m L = P^{abcd} \xi^m \nabla_m R_{abcd} \\ &+ Z^{a_1 abcd} \xi^m \nabla_m \nabla_{a_1} R_{abcd} + A^{ab} \xi^m \nabla_m g_{ab}, \end{aligned} \tag{4.48}$$

where $A^{ab} = \frac{\partial L}{\partial g_{ab}}$, $P^{abcd} = \frac{\partial L}{\partial R_{abcd}}$, and $Z^{a_1 abcd} = \frac{\partial L}{\partial \nabla_{a_1} R_{abcd}}$. The $abcd$ indices of P and Z are taken to have the symmetries of Riemann.

The second way is to consider the infinitesimal variation δL in L as due to the variation in g_{ab}, R_{abcd} and $\nabla_{a_1} R_{abcd}$ due to the diffeomorphism. The latter are given by the corresponding Lie derivatives. Thus we have,

$$\pounds_\xi L = A^{ab} \pounds_\xi g_{ab} + P^{abcd} \pounds_\xi R_{abcd} + Z^{a_1 abcd} \pounds_\xi \nabla_{a_1} R_{abcd}. \tag{4.49}$$

Taking into consideration the symmetries of R_{abcd} and P^{abcd}, the second term can be calculated as

$$P^{abcd} \pounds_\xi R_{abcd} = P^{abcd} \big[\xi^m \nabla_m R_{abcd} + 4 \nabla_a \xi^m R_{mbcd} \big]. \tag{4.50}$$

Similarly, the third term can be calculated as

$$\begin{aligned} & Z^{a_1 abcd} \pounds_\xi \nabla_{a_1} R_{abcd} = Z^{a_1 abcd} \big[\xi^m \nabla_m \nabla_{a_1} R_{abcd} \\ & + 4 \nabla_{a_1} R_{mbcd} \nabla_a \xi^m + \nabla_m R_{abcd} \nabla_{a_1} \xi^m \big]. \end{aligned} \tag{4.51}$$

Plugging these expressions in eq. (4.49), and using $\pounds_\xi g_{ab} = \nabla_a \xi_b + \nabla_b \xi_a$, we get

$$\begin{aligned} \pounds_\xi L \;&=\; P^{abcd} \xi^m \nabla_m R_{abcd} + Z^{a_1 abcd} \xi^m \nabla_m \nabla_{a_1} R_{abcd} \\ &+\; 2 \nabla_p \xi_q \Big[\frac{\partial L}{\partial g_{pq}} + 2 P^{pabc} R^q{}_{abc} \\ &+\; 2 Z^{a_1 pabc} \nabla_{a_1} R^q{}_{abc} + \frac{1}{2} Z^{pabcd} \nabla^q R_{abcd} \Big]. \end{aligned} \tag{4.52}$$

Now, we see from eq. (4.48) that the first two terms on the right hand side are already equal to $\pounds_\xi L$. This implies, since ξ is arbitrary, that the expression within the brackets must vanish. We thus get the identity

$$\frac{\partial L}{\partial g_{pq}} = -2P^{pabc}R^{q}{}_{abc} - 2Z^{a_1 pabc}\nabla_{a_1}R^{q}{}_{abc} - \frac{1}{2}Z^{pabcd}\nabla^{q}R_{abcd}. \tag{4.53}$$

For theories containing no derivatives of curvature, we have that $Z^{pabcd} = 0$ and eq. (4.53) reduces to the identity (4.33).

CHAPTER 5

Spacetime thermodynamics: Riemann–Cartan spacetime

5.1. Introduction

As we saw in the previous chapter, the equation of state derivation of the gravitational field equations gives us a direct hint towards gravity being emergent. In this context it becomes extremely interesting to push the derivation beyond general relativity as looking at gravity from an effective field theory point of view, the Einstein–Hilbert action is the first term in the derivative expansion of the action.

Previously we saw the higher derivative extension of spacetime thermodynamics, on the other hand, one can try to extend General Relativity by including the intrinsic spin of the particle in the geometric description of space-time itself [112]. For this one needs to introduce torsion as an additional degree of freedom for space-time, besides the metric, and use Riemann–Cartan geometry instead of Riemannian geometry. The most well known example of such a theory is the Einstein–Cartan (EC) theory [113] in which the Einstein equations are replaced by the Cartan–Sciama–Kibble field equations [114, 115] after inclusion of torsion.

In this chapter we address the question if field equations of such a theory, which includes torsion as well, can emerge out of the local thermodynamic variables of a LCH in the framework of Riemann–Cartan geometry.

5.2. Riemann-Cartan space-time

In this section we introduce some key features of a Riemann-Cartan spacetime as it will be further required for the equation of state derivation including torsion. A general affine connection is parametrized by its connection coefficients. Assuming metric compatibility, the non-Riemannian part of the connection is uniquely determined by the torsion tensor. In a generic coordinate basis, the connection coefficients read

$$\Gamma^{a}{}_{bc} = \gamma^{a}{}_{bc} + K^{a}{}_{bc}\,, \tag{5.1}$$

where, without loss of generality, we have separated the contribution of a Levi–Civita part γ and a contortion tensor K which contains the torsion properties. We will denote the Levi-Civita covariant derivative with ∇, while the general one will be barred $\overline{\nabla}$.

Definition 1 *Torsion tensor:*

$$T^a{}_{bc} := \Gamma^a{}_{bc} - \Gamma^a{}_{cb}\,. \tag{5.2}$$

The contortion tensor K can be rewritten in terms of the torsion one as

Definition 2 *Cotortion tensor:*

$$K^a{}_{bc} := \frac{1}{2}\left(T^a{}_{bc} - T_{bc}{}^a - T_{cb}{}^a\right)\,. \tag{5.3}$$

Indeed, we can always split the tensor K into the symmetric and anti-symmetric parts

$$K^a{}_{bc} = U^a{}_{bc} + \frac{1}{2}T^a{}_{bc}, \tag{5.4}$$

where U is another tensor, symmetric in the two lower indices. From the metricity condition

$$\overline{\nabla}_a g_{bc} = 0\,, \tag{5.5}$$

it is immediate to get

$$U^a{}_{bc} = -\frac{1}{2}\left(T_{bc}{}^a + T_{cb}{}^a\right)\,, \tag{5.6}$$

where space-time indices are raised, lowered and contracted with the metric g_{ab} (for instance, $T_{bc}{}^a = g_{be}g^{af}T^e{}_{cf}$).

Now the modified torsion tensor is given as:

Definition 3 *Modified torsion tensor:*

$$S^a{}_{bd} := T^a_{bd} + T_d\delta^a_b - T_b\delta^a_d\,, \tag{5.7}$$

where

$$T_d := T^a{}_{da} \tag{5.8}$$

is the trace of the torsion tensor. It is immediate to see that this satisfies $S^a{}_{bd} = -S^a{}_{db}$.

The commutator of the covariant derivatives in presence of torsion is given by

$$\begin{aligned}
[\overline{\nabla}_a, \overline{\nabla}_f]k^b &= \partial_a(\overline{\nabla}_f k^b) - \Gamma^c_{af}\overline{\nabla}_c k^b + \Gamma^b_{ac}\overline{\nabla}_f k^c - a \leftrightarrow f \\
&= (\partial_a\Gamma^b_{fd} - \partial_f\Gamma^b_{ac} + \Gamma^b_{ac}\Gamma^c_{fd} - \Gamma^b_{fc}\Gamma^c_{ad})k^d - (\Gamma^c_{af} - \Gamma^c_{fa})\overline{\nabla}_c k^b \\
&= -\overline{R}_{afd}{}^b k^d - T^c_{af}\overline{\nabla}_c k^b\,,
\end{aligned} \tag{5.9}$$

where we have used the standard definition of the Riemann tensor, although expressed in terms of the Riemann–Cartan connection (5.1). Notice that from this definition of the Riemann tensor the following symmetry properties still hold:

$$\overline{R}_{acd}{}^{b} = -\overline{R}_{cad}{}^{b} \tag{5.10}$$

$$\overline{R}_{acd}{}^{e} g_{eb} = \overline{R}_{acdb} = -\overline{R}_{acbd} = -\overline{R}_{acb}{}^{e} g_{ed}. \tag{5.11}$$

The first relation (5.10) is obvious from the definition (5.9) of $\overline{R}_{afd}{}^{b}$; the second (5.11) can be shown by applying the commutator (5.9) to the metric g_{ab} and using the metricity condition to eliminate the extra terms proportional to the torsion. However, the other symmetry property does not hold anymore, namely

$$\overline{R}_{acdb} \neq \overline{R}_{dbac}; \tag{5.12}$$

this is the case since the property $\overline{R}_{[acd]}{}^{b} = 0$ is no longer true in the presence of torsion. As a consequence, the Ricci tensor is not symmetric anymore. More precisely, it can be shown that

$$\overline{R}_{ab} = \overline{R}_{ba} - 3\overline{\nabla}_{[a} T^{d}{}_{db]} + T_{d} T^{d}{}_{ab}\,. \tag{5.13}$$

5.2.1. Killing equation

Let us now derive the Killing equation for a Riemann-Cartan space-time. The Killing equation gives a partial differential equation for vector fields generating isometries

$$\pounds_{\xi} g_{ab} = 0\,. \tag{5.14}$$

Explicitly the Lie derivative of the metric tensor is given as,

$$\pounds_{\xi} g_{ab} = \xi^{c}\partial_{c} g_{ab} + g_{cb}\partial_{a}\xi^{c} + g_{ca}\partial_{b}\xi^{c}\,. \tag{5.15}$$

If we convert partial derivatives into covariant derivatives, namely

$$\begin{aligned} \partial_{a}\xi^{b} &= \overline{\nabla}_{a}\xi^{b} - \Gamma^{b}{}_{ac}\xi^{c}\,, \\ \partial_{c} g_{ab} &= \overline{\nabla}_{c} g_{ab} + \Gamma_{bca} + \Gamma_{acb}\,, \end{aligned} \tag{5.16}$$

Then

$$\pounds_{\xi} g_{ab} = \overline{\nabla}_{a}\xi_{b} + \overline{\nabla}_{b}\xi_{a} - g_{bc}\Gamma^{c}_{af}\xi^{f} - g_{ac}\Gamma^{c}_{bf}\xi^{f} + \xi^{f}\overline{\nabla}_{f} g_{ab} + \xi^{f}\Gamma^{c}_{fa} g_{cb} + \xi^{f}\Gamma^{c}_{fb} g_{ca} \tag{5.17}$$

$$= g_{bc}\overline{\nabla}_{a}\xi^{c} + g_{ac}\overline{\nabla}_{b}\xi^{c} + \xi^{f} Q_{fab} + \xi^{f}\left(T^{c}_{fa} g_{cb} + T^{c}_{fb} g_{ca}\right) \tag{5.18}$$

it is straightforward to see that a Killing vector field ξ satisfies

$$\pounds_{\xi} g_{ab} = \overline{\nabla}_{a}\xi_{b} + \overline{\nabla}_{b}\xi_{a} - \xi^{c}\left(T_{abc} + T_{bac}\right) = 0\,. \tag{5.19}$$

In the case of the Levi-Civita connection, $T = 0$, we recover the standard $\nabla_{(a}\xi_{b)} = 0$ Killing equation.

5.2.2. Autoparallel curves

Autoparallel curves on a Riemann-Cartan space-time are defined through the equation

$$\xi^a \overline{\nabla}_a \xi^b = -\tilde{\kappa}\xi^b, \tag{5.20}$$

in a generic parametrization where $\tilde{\kappa}$ is a measure of non-affinity.

By means of the connection coefficient expression (5.1), the previous relation implies

$$\xi^a \nabla_a \xi^b - T_{ac}{}^{b} \xi^a \xi^c = -\tilde{\kappa}\xi^b\,, \tag{5.21}$$

where ∇ is the covariant derivative w.r.t. the Levi–Civita connection. Hence, autoparallel curves in a Riemann–Cartan space-time are not extremal curves, since the latter is a notion defined with respect to the metric of the manifold and it yields the standard geodesic equation in terms of the Levi–Civita connection only.

5.2.3. Hypersurface orthogonal congruence in presence of torsion

In this section we show that, in presence of torsion, hypersurface orthogonality does not imply a vanishing twist. If we consider a surface defined by the implicit equation $\phi(x) = 0$, the vector field normal to the surface is given by

$$\chi_a = h\partial_a\phi \tag{5.22}$$

where h is a proportionality constant. Now in Riemann–Cartan geometry, the modified commutator (5.9) implies

$$\overline{\nabla}_{[a}\overline{\nabla}_{b]} f = -T^c{}_{ab}\overline{\nabla}_c f, \tag{5.23}$$

where f is any arbitrary scalar and thus the Frobenius theorem takes the form

$$\chi_{[a}\overline{\nabla}_b\chi_{c]} = -\chi_a T^d{}_{bc}\chi_d - \chi_b T^d{}_{ca}\chi_d - \chi_c T^d{}_{ab}\chi_d\,. \tag{5.24}$$

In the Riemannian case ($T = 0$), the r.h.s. of (5.24) vanishes and the resulting relation can be used to prove that the twist of the congruence vanishes as well. If we now try to reproduce the standard proof in presence of torsion, we see that this is no longer necessarily the case (as demonstrated later).

For our derivation we are interested in a section of null surface, $\mathcal{H}$. We now look at what happens to the result that a null surface is described by the geodesic flow of its normal vector field.

A null horizon surface can be defined by the implicit equation

$$\phi(x) = 0\,, \tag{5.25}$$

with the condition that the normal vector field

$$\chi^a = h g^{ab}\partial_b\phi\,, \tag{5.26}$$

where h is any non-vanishing function, is null

$$\chi^a\chi_a = 0 \qquad \text{on } \mathcal{H}\,. \tag{5.27}$$

This last condition implies that the gradient of this norm is orthogonal to the surface $\mathcal{H}$, *i.e.* it is still proportional to the normal vector; namely the null normal to the horizon satisfies

$$\overline{\nabla}_b(\chi^a\chi_a) = 2\chi^a\overline{\nabla}_b\chi_a = -2\kappa\chi_b, \qquad \text{on } \mathcal{H}\,, \tag{5.28}$$

with κ a function corresponding to the "normal" surface gravity, which a priori represents a different notion of surface gravity than the inaffinity surface gravity defined through the geodesic equation (5.20). One should further note that, in Riemann–Cartan space-time, $\tilde{\kappa}$ is a priori different from the surface gravity defined from the condition of the Killing horizon generator to be a null vector field at the horizon.

For our horizon to be a Killing horizon we need to further require that the null vector field χ^a is a Killing vector field, we can then use (5.19) to write

$$\begin{aligned} -\kappa\chi_b &= \chi^a\overline{\nabla}_b\chi_a = -\chi^a\overline{\nabla}_a\chi_b - 2U^c{}_{ab}\chi^a\chi_c \\ &= -\chi^a\overline{\nabla}_a\chi_b + (T_{ba}{}^c + T_{ab}{}^c)\chi^a\chi_c = -\chi^a\overline{\nabla}_a\chi_b + T_{abc}\chi^a\chi^c\,, \end{aligned} \tag{5.29}$$

from which

$$\chi^a\overline{\nabla}_a\chi_b = \kappa\chi_b + T_{abc}\chi^a\chi^c\,; \tag{5.30}$$

in terms of the Levi-Civita connection, the previous equation implies

$$\chi^a\nabla_a\chi_b = \kappa\chi_b\,. \tag{5.31}$$

Hence, we see that the Killing vector generating the horizon is not geodesic with respect to the affine connection but rather with respect to the Levi-Civita connection associated to the metric (i.e the χ flows along extremal curves).

The non–geodesic flow of the Killing vector is quite a departure from the standard behaviour of Killing horizons in General Relativity and indeed it does have striking consequences. Let us first note that from (5.30) it is clear that the non-geodesic behaviour is due to the presence of the term $T_{\mu\nu\rho}\chi^\rho\chi^\mu$, i..e of a torsion current across the horizon. In theories with non-propagating torsion this could be only carried on by a flow of particles which has an associated spin current. What is the effect of such a current on the horizon? (apart from inducing a non geodesic flow of the Killing vector)

An important observation in this sense is that for a non–vanishing tensor current across the horizon the above introduced definitions of surface gravity, the inaffinity $\tilde{\kappa}$ (defined by the geodesic equation (5.20)) and the "normal surface gravity" defined via (5.28), do not coincide. This is also definitely at odd with what one has in General Relativity where this (and others) definitions of the surface gravity do coincide for a stationary Killing horizon [116].

In order to show this explicitly let us conveniently write the "normal surface gravity" as

$$\kappa = -n^b\chi^a\overline{\nabla}_b\chi_a\,, \tag{5.32}$$

where n^a is an auxiliary null vector defined at the horizon such that $\chi^a n_a = -1$. Now, by means of the Killing equation (5.19), one has

$$\begin{aligned}
\tilde{\kappa} &= n^b\chi^a\overline{\nabla}_a\chi_b\nabla & (5.33)\\
&= -n^b\chi^a\overline{\nabla}_b\chi_a + 2n^bT_{(ab)c}\chi^a\chi^c\nabla & (5.34)\\
&= \kappa + 2n^bT_{(ab)c}\chi^a\chi^c\nabla\,. & (5.35)
\end{aligned}$$

In GR the non coincidence of the inaffinity and normal definitions of the surface gravity for Killing horizon is generally associated to departure from equilibrium/stationarity, like for example in the case of evaporating/shrinking black holes (see e.g. the discussion in Section 2.2 of [116] keeping in mind that the normal surface gravity basically coincides with the surface gravity notion associated to the near horizon peeling structure of outgoing light rays). In analogy one might says that non-vanishing tensor currents across the horizon should not be allowed for a truly stationary description of the horizon and henceforth we shall ask them to be zero

$$T_{abc}\chi^c\chi^a = 0\,. \tag{5.36}$$

Remarkably, the above restriction on the torsion current across the horizon is crucial not only to remove ambiguities among otherwise inequivalent definitions of surface gravity, but also it is necessary in order to carry on the demonstration of the zeroth law of the black hole mechanics, i.e. the proof of the constancy of the temperature across the horizon. In fact, in order to do so we first need to introduce a local set of null tetrads at the horizon, playing the role of coordinate vector fields. This construction can be implemented by means of Gaussian null coordinates. Since we need to use the Killing vector field as one of the coordinate vector field, we then need to impose (5.36) for it to be geodesic along the horizon (see section 5.4).

Finally, we want to find the actual generator of the horizon, which is normally affinely parametrized on it. Let us then define on the horizon the vector field

$$k^a = e^{-\kappa v}\chi^a = \frac{1}{\kappa\lambda}\chi^a \,, \tag{5.37}$$

where v is the non-affine parameter (Killing time) defined by

$$\chi^a \overline{\nabla}_a v = 1 \,. \tag{5.38}$$

We then have

$$k^a \overline{\nabla}_a k_b = e^{-2\kappa v}(-\chi_b \chi^a \overline{\nabla}_a(\kappa v) + \kappa\chi^b + T^a_{bc}\chi^c\chi_a)\nabla \tag{5.39}$$

$$= T^a{}_{bc}k^c k_a = \frac{1}{\kappa^2\lambda^2}T_{abc}\chi^a\chi^c \tag{5.40}$$

As expected k^a is an affinely parametrised null horizon generator only once the geodesic condition (5.36) is imposed.

5.4. Zeroth law

For the derivation of the Einstein Cartan equation in a thermodynamic approach it is important to verify if the temperature of the LCH, which is proportional to the surface gravity, remains constant and if there is a consistent definition of the surface gravity. In this section we prove that the normal surface gravity defined by (5.28) provides a good notion of horizon temperature even in the case of non-vanishing torsion, namely it satisfies the zeroth law of horizon thermodynamics.

In order to do so we introduce a set of null tetrads made by the Killing vector (using the geodesic condition (5.36)) plus a second null geodesic vector n^μ and a complex null vector m^μ tangent to the horizon 2-sphere cross-section such that

$$\chi^a n_a = -1 = -m^a \bar{m}_a \quad \text{and} \quad n^a n_a = m^a m_a = \bar{m}^a \bar{m}_a = 0. \tag{5.41}$$

In this coordinate system adapted to the the null surface, these four null vector represent well-defined basis and the pull-back of the metric on the 2-sphere can be written as

$$h_{ab} = m^{(a}\bar{m}^{b)} = g_{ab} + \chi_a n_b + \chi_b n_a \tag{5.42}$$

and

$$\chi^a h_{ab} = n^a h_{ab} = 0\,; \tag{5.43}$$

moreover, being elements of a coordinate basis, the null vectors satisfy

$$[k,n]^a = [k,m]^a = [n,m]^a = 0\,. \tag{5.44}$$

We are now ready to compute the Lie derivative along the horizon Killing vector field generator of κ. From (5.28), we can write

$$\kappa = -n^c\chi^a\overline{\nabla}_c\chi_a\,. \tag{5.45}$$

We then have

$$\begin{aligned}
\pounds_\chi\kappa &= \chi^b\overline{\nabla}_b\kappa = -\chi^b\overline{\nabla}_b(n^c\chi^a\overline{\nabla}_c\chi_a) \\
&= -\chi^b n^c\chi^a\overline{\nabla}_b\overline{\nabla}_c\chi_a - \chi^b\chi^a\overline{\nabla}_b n^c\overline{\nabla}_c\chi_a - \chi^b n^c\overline{\nabla}_b\chi^a\overline{\nabla}_c\chi_a \\
&= -\chi^b n^c\chi^a\overline{\nabla}_b\overline{\nabla}_c\chi_a + \kappa\chi_c\chi^b\overline{\nabla}_b n^c - \chi^b n^c\overline{\nabla}_b\chi^a\overline{\nabla}_c\chi_a \\
&= -\chi^b n^c\chi^a\overline{\nabla}_c\overline{\nabla}_b\chi_a + \chi^b n^c\chi^a R_{bcda}\chi^d + \chi^b n^c\chi^a T^d_{bc}\overline{\nabla}_d\chi_a \\
&+ \kappa\chi_c\chi^b\overline{\nabla}_b n^c - \chi^b n^c\overline{\nabla}_b\chi^a\overline{\nabla}_c\chi_a \\
&= -\chi^b n^c\chi^a\overline{\nabla}_c\overline{\nabla}_b\chi_a - \kappa\chi^b n^c\chi_d T^d_{bc} \\
&+ \kappa\chi_c\chi^b\overline{\nabla}_b n^c - \chi^b n^c\overline{\nabla}_b\chi^a\overline{\nabla}_c\chi_a\,,
\end{aligned} \tag{5.46}$$

where in the last step we have used the symmetry properties of the Riemann tensor, the definition (5.28) and the commutator (5.9). We now compute

$$\begin{aligned}
-&\ \chi^b n^c\chi^a\overline{\nabla}_c\overline{\nabla}_b\chi_a = -\overline{\nabla}_c(\chi^b n^c\chi^a\overline{\nabla}_b\chi_a) \\
&+(n^c\overline{\nabla}_c\chi^b + \chi^b\overline{\nabla}_c n^c)\chi^a\overline{\nabla}_b\chi_a + \chi^b n^c\overline{\nabla}_b\chi^a\overline{\nabla}_c\chi_a \\
=&\ -2\overline{\nabla}_c(\chi^b n^c\chi^a\overline{\nabla}_{(b}\chi_{a)}) - \kappa n^c\chi_b\overline{\nabla}_c\chi^b + \chi^b n^c\overline{\nabla}_b\chi^a\overline{\nabla}_c\chi_a \\
=&\ \overline{\nabla}_c(\chi^b n^c\chi^a U_{dba}\chi^d) - \kappa n^c\chi_b\overline{\nabla}_c\chi^b + \chi^b n^c\overline{\nabla}_b\chi^a\overline{\nabla}_c\chi_a \\
=&\ -\overline{\nabla}_c\left(n^c\chi^b\chi^a\chi^d\left(T_{abd} + T_{bad}\right)\right) - \kappa n^c\chi_b\overline{\nabla}_c\chi^b + \chi^b n^c\overline{\nabla}_b\chi^a\overline{\nabla}_c\chi_a \\
=&\ -\kappa n^c\chi_b\overline{\nabla}_c\chi^b + \chi^b n^c\overline{\nabla}_b\chi^a\overline{\nabla}_c\chi_a\,,
\end{aligned} \tag{5.47}$$

where in the last passage we have used the Killing eq. (5.19).

Therefore,

$$\begin{aligned}
\pounds_\chi \kappa &= -\kappa \chi^b n^c \chi_d T^d_{bc} - \kappa n^c \chi_b \overline{\nabla}_c \chi^b + \kappa \chi_c \chi^b \overline{\nabla}_b n^c \\
&= \kappa \chi_c [\chi, n]^c \\
&= 0\,, \qquad (5.48)
\end{aligned}$$

where we have used the definition

$$[\chi, n]^a = \chi^b \overline{\nabla}_b n^a - n^b \overline{\nabla}_b \chi^a - \chi^d n^b T^a{}_{db} \qquad (5.49)$$

and the property (5.44).

To show the surface gravity is constant we need to further show $h_e^{\ b} \overline{\nabla}_b \kappa = 0$. If we assume the existence of bifurcation surface S_0 at which $\chi^a = 0$ we can show $h_e^{\ b} \overline{\nabla}_b \kappa = 0$ at the bifurcation surface as follows

$$\begin{aligned}
h_e^{\ b} \overline{\nabla}_b \kappa |_{S_0} &= h_e^{\ b} \overline{\nabla}_b (n^c \chi^a \overline{\nabla}_c \chi_a)|_{S_0} \\
&= -h_e^{\ b} n^c \chi^a \overline{\nabla}_b \overline{\nabla}_c \chi_a - h_e^{\ b} \chi^a \overline{\nabla}_b n^c \overline{\nabla}_c \chi_a - h_e^{\ b} n^c \overline{\nabla}_b \chi_a \overline{\nabla}_c \chi^a |_{S_0} \\
&= -h_e^{\ b} \overline{\nabla}_b \chi_a n^c \overline{\nabla}_c \chi^a |_{S_0} \\
&= -h_e^{\ b} \overline{\nabla}_b \chi_a \left(\chi^c \overline{\nabla}_c n^a - \chi^d n^c T^a{}_{dc} \right) |_{S_0} \\
&= 0\,, \qquad (5.50)
\end{aligned}$$

where we have used the fact that the horizon Killing vector field commutes with the auxiliary null vector n^a, as well as the vanishing of χ^a at the bifurcation surface.

Now one can show that

$$\mathcal{L}_\chi (h_e^{\ b} \overline{\nabla}_b \kappa) = h_e^{\ b} \mathcal{L}_\chi (\overline{\nabla}_b \kappa) + \mathcal{L}_\chi (h_e^{\ b}) \overline{\nabla}_b \kappa\,. \qquad (5.51)$$

In order to do so, let us notice first that the use of Gaussian null coordinates adapted to the horizon implies $\mathcal{L}_\chi (h_e^{\ b}) = 0$ (as follows from (5.44)); furthermore,

$$\begin{aligned}
h_e^{\ b} \mathcal{L}_\chi (\overline{\nabla}_b \kappa) &= h_e^{\ b} (\chi^a \overline{\nabla}_a \overline{\nabla}_b \kappa + \overline{\nabla}_a \kappa \overline{\nabla}_b \chi^a + \chi^a \overline{\nabla}_d \kappa\, T^d{}_{ab}) \\
&= h_e^{\ b} (\chi^a \overline{\nabla}_b \overline{\nabla}_a \kappa - \chi^a T^d{}_{ab} \overline{\nabla}_d \kappa + \overline{\nabla}_a \kappa \overline{\nabla}_b \chi^a + \chi^a \overline{\nabla}_d \kappa T^d{}_{ab}) \\
&= h_e^{\ b} (\overline{\nabla}_b (\chi^a \overline{\nabla}_a \kappa) - \overline{\nabla}_b (\chi^a) \overline{\nabla}_a \kappa + \overline{\nabla}_a \kappa \overline{\nabla}_b \chi^a) \\
&= 0\,, \qquad (5.52)
\end{aligned}$$

where we used the constancy of κ along the horizon Killing vector field, as we derived earlier. Therefore, $h_e^{\ b} \overline{\nabla}_b \kappa$ is constant over the horizon and thus if it is 0 at one point, it will be 0 everywhere.

The gravitational field strength is related to the components of a linear connection (which may be with or without torsion e.g. the Levi-Civita connection in Riemann space-times or the Cartan connection in Riemann-Cartan space-time) in a local inertial frame. Mathematically this means the existence of a unique local frame in which the connection components vanish at a point about which the local frame is described. The thermodynamical derivation strongly relies on the existence of such local inertial frame at each point of space-time, in order to define a LCH in terms of a local Rindler horizon. In this section we give a viable notion of Einstein Equivalence Principle (EP) for a non-Riemannian space-time having torsion.

In General Relativity one deals with a n-dimensional Riemannian manifold denoted by M_n. At each point p of M_n, the tangent vector space is denoted as $T_p(M_n)$. One can then introduce a local vector basis e_a. Given a local coordinate system $\{x^a\}$, the frame e_a is expanded in terms of the local coordinate basis $\partial_a = \partial/\partial x^a$

$$e_a = e_a{}^b \partial_b \,. \tag{5.53}$$

For a n-dimensional Riemannian manifold M_n, the EP states that for any point p a local inertial frame is introduced via a local coordinate transformation

$$dx^a \to dx^b(x(p)) = e_a{}^b(x(p))dx^a \tag{5.54}$$

relating the flat Minkowski metric η_{ab} to the induced metric, $g_{cd} = e^a{}_c e^b{}_d \eta_{ab}$, of the curved Riemannian space-time.

The coordinate basis is a set of n linearly independent vectors, defined in each point of the manifold, which are tangent to the n coordinate lines, which pass through that point and belong to a coordinate system (also called the global coordinate system or natural coordinate system) imposed on the manifold. For two different coordinate systems, the transformation between two coordinate bases can be defined as

$$e_a = e_a{}^b e_b \,, \quad \text{where} \quad e_a{}^b = \partial x^b / \partial x^a \,; \tag{5.55}$$

such transformations are integrable and called holonomic. They satisfy the condition

$$\partial_c e_a{}^b - \partial_a e_c{}^b = 0 \,, \tag{5.56}$$

which corresponds to some integrability conditions for the coordinate system, given by

$$(\partial_a \partial_b - \partial_b \partial_a) x^c = 0 \,. \tag{5.57}$$

In the Riemannian case, the $n^2(n-1)/2$ integrability conditions effectively reduce the number of unknowns $\partial_d e_j{}^a$ from n^3 to $n^2(n+1)/2$. This solves the system of $n^2(n+1)/2$ transformation equations

$$\Gamma^a_{pq} = e^a{}_d(e_p{}^b e_q{}^c \Gamma^d_{bc} - \partial_q e_p{}^d) = 0\,, \tag{5.58}$$

thus providing the coordinate basis in which all the components of the Levi-Civita connection are locally set to zero at a point [117].

Now we want to reproduce the argument for Einstein-Cartan space-time, that is to introduce a local frame in which the components of a more general affine connection vanish at some point p. The problem is, the connection $\Gamma(g,T)$ defined in (5.1) can be set to zero, under holonomic coordinate transformation (5.55), only if the torsion tensor vanishes identically. Now more generally, one can introduce a local inertial (Lorentz) frame in Einstein–Cartan space–time by means of local vector basis h_a defined by the non-integrable or anholonomic [118] transformations

$$h_a = h_a{}^b(p)\partial_b\,, \tag{5.59}$$

with

$$\partial_b h^c{}_a - \partial_a h^c{}_b \neq 0\,. \tag{5.60}$$

This basis is also called a non-coordinate basis. Although it is always possible to find a coordinate basis which will coincide with an anholonomic basis locally, in one point, there is no coordinate system which would correspond to the anholonomic basis globally.

The tensor which would encode the anholonomicity is defined by

$$\Omega_{bc}{}^a = h_d{}^a(\partial_b h^d{}_c - \partial_c h^a{}_b)\,. \tag{5.61}$$

With this basis, at every point, we define a local Lorentz frame by means of the set of coordinate differentials

$$dx^a = h^a{}_b(x)dx^b\,. \tag{5.62}$$

Local Lorentz frames are then obtained by requiring the induced metric in these coordinates to be Minkowskian,

$$\eta_{ab} = h_a{}^c h_b{}^d g_{cd}\,. \tag{5.63}$$

Referred to such an anholonomic system, the affine connection goes to

$$\Gamma^a_{bc} \to \Gamma^c_{ab} = h_a{}^a h_b{}^b h_c{}^c(T^a{}_{bc} - T_b{}^a{}_c + T_{bc}{}^a - \Omega^a{}_{bc} + \Omega_b{}^a{}_c - \Omega_{bc}{}^a) \tag{5.64}$$

At any arbitrary point p of a 4-D space-time, the orthonormality condition $g_{ab} = h_a{}^c h_b{}^d \eta_{cd}$ only determines 40 components of ∂h. The remaining 24 components may be locally fixed by $T(p) = \Omega(p)$. Then $\Gamma^c_{ab}(p) = 0$ and, accordingly, torsion does not violate the EP. The local coordinate system $\{x^a\}$ associated with the anholonomic frame at p can be extended to an infinitesimal neighborhood of the point p, in a similar way as a local normal coordinate basis is extended in the Riemannian case.

5.6. Raychaudhuri equation

The next important tool we need in order to show the thermodynamical origin of the Einstein-Cartan equation is the Raychaudhuri equation for a non-Riemannian space-time. Let us thus proceed in its derivation.

We consider a local horizon $\mathcal{H}$ generated by the affinely parametrized null vector k^a. Since the local Killing vector field is required to be geodesic on the horizon, in order for a local notion of temperature to be well defined, and $\chi^a = -\kappa\lambda k^a$, on $\mathcal{H}$ we demand the condition (5.36) to hold, which implies that k^a is also geodesic,

$$k^a \overline{\nabla}_a k_b = 0\,. \tag{5.65}$$

If we use this horizon null generator as an element of the horizon local coordinate basis and an auxiliary null vector field n^a such that $k^a n_a = -1$, the space-time metric can be decomposed as

$$h_{ab} = g_{ab} + k_a n_b + k_b n_a\,, \tag{5.66}$$

where h_{ab} is the transverse metric, namely

$$k^a h_{ab} = n^a h_{ab} = 0. \tag{5.67}$$

Let us denote η^a as the deviation vector between two neighboring flux lines of k^a. The Lie derivative of η^a along the tangent (to the horizon) vector k^a has to to vanish, namely

$$\pounds_k \eta^a = 0\,. \tag{5.68}$$

This implies

$$[k, \eta]^a = k^b \overline{\nabla}_b \eta^a - \eta^b \overline{\nabla}_b k^a - k^d \eta^b T^a_{db} = 0\,. \tag{5.69}$$

Thus, the failure of the deviation vector to be parallely transported along the horizon is measured by

$$k^b \overline{\nabla}_b \eta^a = \eta^b (\overline{\nabla}_b k^a + k^d T^a_{db}) = \eta^b B^a{}_b\,, \tag{5.70}$$

where we have defined the deviation tensor

$$B_{ab} := \overline{\nabla}_b k_a + k^d T_{adb} \,. \tag{5.71}$$

By means of the condition (5.36), it follows that the deviation tensor (5.71) is orthogonal to the generator vector field k^a, namely

$$k^a B_{ab} = 0 = k^b B_{ab} \,, \tag{5.72}$$

where the first equality holds due to the null condition $k^b \overline{\nabla}_a k_b = 0$. However, the deviation tensor above is not fully transversal since it is not orthogonal to n^a and has a component along n^a. To obtain a purely transverse deviation tensor we can define the projection of B_{ab} as

$$\begin{aligned} \tilde{B}_{ab} &= h_a^{\ e} h_b^{\ f} B_{ef} \\ &= B_{ab} + k_a n^e B_{eb} + k_b n^e B_{ae} + k_a k_b n^e n^f B_{ef} \,. \end{aligned} \tag{5.73}$$

We can define the expansion of the congruence as

$$\overline{\theta} = \frac{1}{2} h^{ab} \mathcal{L}_k h_{ab} = g^{ab} \tilde{B}_{ab} = g^{ab} B_{ab} \,, \tag{5.74}$$

where we have used the orthogonality condition (5.72) as well as (5.67). Therefore, the expansion reads

$$\overline{\theta} = g^{ab} B_{ab} = \overline{\nabla}_b k^b + k^b T_b \,. \tag{5.75}$$

The evolution of the expansion along k^a (i.e along the horizon in our case) is given by

$$\begin{aligned} \frac{d\overline{\theta}}{d\lambda} &= k^a \overline{\nabla}_a \overline{\theta} = k^a \overline{\nabla}_a \overline{\nabla}_b k^b + k^a \overline{\nabla}_a (k^b T_b) \\ &= k^a \overline{\nabla}_a \overline{\nabla}_b k^b + k^a k^b \overline{\nabla}_a T_b + T^b T_{abc} k^a k^c \,, \end{aligned} \tag{5.76}$$

where the last term can be set to zero by using (5.36), but we leave it for now since we want to give the Raychaudhuri equation in full generality for a null congruence.

We can expand the first term as

$$\begin{aligned} k^a \overline{\nabla}_a \overline{\nabla}_b k^b &= k^a \overline{\nabla}_b \overline{\nabla}_a k^b - k^a \overline{R}_{abd}{}^b k^d - k^a T^c{}_{ab} \overline{\nabla}_c k^b \\ &= \overline{\nabla}_b (k^a \overline{\nabla}_a k^b) - \overline{\nabla}_b k^a \overline{\nabla}_a k^b - \overline{R}_{ad} k^a k^d - k^a T^c{}_{ab} \overline{\nabla}_c k^b \\ &= \overline{\nabla}^b (T^c{}_{ba} k^a k_c) - \overline{\nabla}_b k^a \overline{\nabla}_a k^b - \overline{R}_{ad} k^a k^d - k^a T^c{}_{ab} \overline{\nabla}_c k^b \\ &= (\overline{\nabla}^b T^c{}_{ba}) k^a k_c + T^c{}_{ba} \overline{\nabla}^b (k^a k_c) - \overline{\nabla}_b k^a \overline{\nabla}_a k^b - \overline{R}_{ad} k^a k^d \\ &\quad - k^a T^c{}_{ab} \overline{\nabla}_c k^b \,. \end{aligned} \tag{5.77}$$

Therefore

$$\begin{aligned}\frac{d\overline{\theta}}{d\lambda} &= -k^a k^b \overline{R}_{ab} + k^a k_b \overline{\nabla}^d T^b_{da} + k^b k^a \overline{\nabla}_a T_b \\ &- \overline{\nabla}_b k^a \overline{\nabla}_a k^b + T^c_{ba}(\overline{\nabla}^b k_c + \overline{\nabla}_c k^b)k^a + T^c_{ba}(\overline{\nabla}^b k^a)k_c \\ &= -k^a k^b \overline{R}_{ab} + k^a k_b \overline{\nabla}^d T^b_{da} + k^b k^a \overline{\nabla}_a T_b + T^d T_{adb} k^a k^b \\ &- B^{ba} B_{ab} + k^c T_{bca} B_{ab} + \overline{\nabla}^b k^a (T_{abc} + T_{cba}) k^c \,, \end{aligned} \tag{5.78}$$

where in the last passage we have used the definition (5.71). One can show using (5.73) and (5.36) that

$$B^{ba} B_{ab} = \tilde{B}^{ba} \tilde{B}_{ab} \,. \tag{5.79}$$

Putting everything together, the Raychaudhuri equation reads

$$\begin{aligned}\frac{d\overline{\theta}}{d\lambda} &= -\overline{R}_{ab} k^a k^b + k^a k_b \overline{\nabla}^d T^b_{da} + k^b k^a \overline{\nabla}_a T_b + T^d T_{adb} k^a k^b \\ &- \tilde{B}^{ba} \tilde{B}_{ab} - \tilde{B}^{ab}(T_{acb} - T_{bca} + T_{cab})k^c + k^d k^c T^{a\,b}_{\;\,d}(T_{acb} + T_{cab}) \end{aligned} \tag{5.80}$$

Written in terms of expansion, shear and twist, respectively

$$\overline{\theta} := h^{ab} \tilde{B}_{ab} \,, \tag{5.81}$$

$$\overline{\sigma}_{ab} := \tilde{B}_{(ab)} - 1/2 \overline{\theta} h_{ab} \,, \tag{5.82}$$

$$\overline{\omega}_{ab} := \tilde{B}_{[ab]} \,, \tag{5.83}$$

the Raychaudhuri equation takes the form

$$\begin{aligned}\frac{d\overline{\theta}}{d\lambda} &= -\overline{R}_{ab} k^a k^b + k^a k_b \overline{\nabla}^d T^b_{da} + k^b k^a \overline{\nabla}_a T_b + T^d T_{adb} k^a k^b \\ &- \frac{1}{2}\overline{\theta}^2 - \overline{\sigma}^{ab}\overline{\sigma}_{ab} + \overline{\omega}^{ab}\overline{\omega}_{ab} + K^{ba}{}_c K_{abd} k^c k^d \end{aligned} \tag{5.84}$$

Contrary to the Riemannian case, we show if the 'twist' is zero the congruence is not necessarily hypersurface orthogonal. by means of (5.73), we have

$$k_{[a}\omega_{bc]} = k_{[a}\tilde{B}_{bc]} = k_{[a}B_{bc]} + k_{[a}B_{b|d|}k_{c]}n^d + k_{[a}B_{d|c}k_{b]}n^d \,; \tag{5.85}$$

it is straightforward to see that the last two terms vanish, so we can write

$$\begin{aligned} k_{[a}B_{bc]} &= k_{[a}\overline{\nabla}_c k_{b]} + k_{[a}T_{b|d|c]}k^d \\ &= k_{[a}\overline{\nabla}_c k_{b]} + k_a T_{bdc} k^d - k_b T_{adc} k^d + k_b T_{cda} k^d \\ &\quad -k_c T_{bda} k^d + k_c T_{adb} k^d - k_a T_{cdb} k^d \,. \end{aligned} \tag{5.86}$$

Therefore, the condition for hypersurface orthogonality expressed as eq. (5.24) no longer imply $k_{[a}B_{bc]} = 0$ and, hence, in general $\omega_{ab} \neq 0$.

As for the Riemannian case, we can still assume that the entropy of a local causal horizon in Riemann-Cartan space-time to be proportional to area. In fact, for a causal horizon at equilibrium the origin of its entropy is believed to be due to the vacuum entanglement across the horizon [119] (see, e.g. [120, 121], for a derivation of horizon entropy from the entanglement between quantum gravitational degrees of freedom) and, hence, the eventual presence of torsion due to some matter distribution away from the horizon is not expected to affect the proportionality of the entanglement entropy to the horizon area (at least not in a theory with non-propagating torsion, like Einstein-Cartan). Therefore, let us still assume an entropy-area law of the form

$$S = \alpha A = \alpha \int dA = \alpha \int d^2x\sqrt{h}\,, \tag{5.87}$$

where A is area of horizon cross section, h is determinant of the induced metric on the horizon cross section and e is a proportionality constant which is generally dependent on the UV cut-off; the variation of this entropy (due to some physical process changing the horizon area) is given by

$$dS = \alpha\delta A = \int d^2x\delta\sqrt{h} = \int d^2x\mathcal{L}_k\sqrt{h}\,. \tag{5.88}$$

By means of the relation

$$\frac{d\sqrt{h}}{d\lambda} = \frac{1}{2}\sqrt{h}h^{ab}\frac{dh_{ab}}{d\lambda}\,, \tag{5.89}$$

we obtain

$$\mathcal{L}_k\sqrt{h} = \xi^a\partial_a\sqrt{h} = \frac{1}{2}\sqrt{h}h^{ab}\frac{dh_{ab}}{d\lambda} = \frac{1}{2}\sqrt{h}h^{ab}\mathcal{L}_k h_{ab}\,, \tag{5.90}$$

where the last step is only true if we are working in Gaussian null coordinate.

Using the metric decomposition (5.66) in terms of the two null vectors and the projected metric, we have

$$h^{ab}\mathcal{L}_k h_{ab} = h^{ab}\mathcal{L}_k[g_{ab} + k_a n_b + k_b n_a] = h^{ab}\mathcal{L}_k g_{ab}\,, \tag{5.91}$$

as $h^{ab}\mathcal{L}_k(k_a n_b) = 0$ due to the orthogonality of k_a, n_a with h^{ab}. Furthermore, one can show starting from (5.91) that

$$\begin{aligned}
h^{ab}\mathcal{L}_k g_{ab} &= 2h^{ab}(\overline{\nabla}_a k_b) + T^c{}_{da}g_{cb}k^d h^{ab} + T^c{}_{db}g_{ca}k^d h^{ab} \\
&= 2(\overline{\nabla}^a k_a - T_{adb}k^d n^b k^a + T_{bda}k^d h^{ab}) \\
&= 2(\overline{\nabla}^a k_a + T_{bda}k^d g^{ab}) \\
&= 2(\overline{\nabla}^a k_a + T_a k^a)\,.
\end{aligned} \tag{5.92}$$

Finally using (5.92) in (5.88) we get the variation of entropy as,

$$dS = \int d\lambda d^2x\sqrt{h}(\overline{\nabla}^a k_a + T_a k^a) = \int d\lambda d^2x\sqrt{h}\,\overline{\theta}\,. \tag{5.93}$$

where $\overline{\theta}$ is the expansion of the congruence in Rimeann-Cartan space-time, as defined in (5.75).

5.8. Einstein–Cartan field equations as an equation of state

At this point we have all the elements to take on our proposed task. Our objective is to start from thermodynamical variables, which ideally one can obtain as a result of coarse graining of space-time, and show how the Einstein–Cartan field equation emerges after using the Clausius equation to relate them. For doing so we will use the variation of entropy we derived in previous section and the Raychaudhuri equation (5.80).

Before proceeding with the thermodynamical derivation, let us first recall the form of the Einstein–Cartan equation that we want to recover.

5.8.1. Einstein–Cartan equation

The Einstein–Cartan field equations read

$$\overline{G}_{ab} = 8\pi G\left(T^{M}_{ab} + (\overline{\nabla}_d + T_d)(\tau^{d}{}_{ab} - \tau_{ab}{}^{d} - \tau_{ba}{}^{d})\right)\,, \tag{5.94}$$

$$S^{d}{}_{ab} = 16\pi G\,\tau^{d}{}_{ab}\,, \tag{5.95}$$

where T^M is the metric (hence symmetric) stress-energy tensor (SET) containing also non-Riemannian contributions and τ the spin angular momentum tensor. Given a matter Lagrangian L_M depending only on the matter field ψ, its first derivatives $\overline{\nabla}\psi$ and the metric g, these two quantities are defined as

$$T^{M}_{ab} := \frac{2}{\sqrt{-g}}\frac{\delta L_M}{\delta g^{ab}}\,, \tag{5.96}$$

$$\tau_{a}^{\;db} := \frac{1}{2\sqrt{-g}}\frac{\delta L_M}{\delta K^{a}{}_{db}}\,. \tag{5.97}$$

The total action function yielding the Einstein–Cartan field equations above reads

$$W = \int d^4x\left(L_M + \frac{1}{16\pi G}L_G\right)\,, \tag{5.98}$$

where the gravity Lagrangian density reads

$$L_G = \sqrt{-g} g^{ab} \overline{R}_{ab} \,. \tag{5.99}$$

By combining the two Einstein–Cartan equations (5.94), (5.95) we get

$$G_{ab} = 8\pi G\, T^{M}_{ab} + \frac{1}{2}(\overline{\nabla}_d + T_d)(S^{d}{}_{ab} - S_{ab}{}^{d} - S_{ba}{}^{d}) \,. \tag{5.100}$$

This is the equation we want to recover via the thermodynamical approach.

Let us start by splitting the equation (5.100) into its symmetric and antisymmetric parts and expand. The symmetric part of the Einstein–Cartan equation reads

$$\begin{aligned}\overline{R}_{(ab)} - \frac{1}{2} g_{ab}(\overline{R} - 2\lambda) &= 8\pi G\, T^{M}_{ab} - \frac{1}{2}(\overline{\nabla}_d + T_d)(S_{(ab)}{}^{d} + S_{(ba)}{}^{d}) \\ &= 8\pi G\, T^{M}_{ab} - \overline{\nabla}^{d} T_{(ab)d} + \overline{\nabla}_{(a} T_{b)} - T^{d} T_{(ab)d} + T_{(a} T_{b)} \\ .\end{aligned} \tag{5.101}$$

For later convenience, let us contract the field equation (5.101) with two k vectors; we find

$$\overline{R}_{ab} k^a k^b = 8\pi G\, T^{M}_{ab} k^a k^b + \overline{\nabla}^{d} T^{a}{}_{db} k_a k^b + \overline{\nabla}_a T_b k^a k^b + T^{d} T_{adb} k^a k^b + T_a T_b k^a k^b \,. \tag{5.102}$$

The antisymmetric part of (5.100) reads

$$\overline{R}_{[ab]} = \frac{1}{2}(\overline{\nabla}_d + T_d) S^{d}{}_{ab} \,. \tag{5.103}$$

However, this part of the Einstein-Cartan equations does not have a dynamical origin, but it follows simply from the definition of the Riemann tensor in terms of the connection (5.1); in fact, (5.103) is equivalent to the Ricci tensor property (5.13), once the modified torsion tensor definition (5.7) is applied. Therefore, it is enough to recover the symmetric part of the Einstein-Cartan equations through the thermodynamical argument in order to capture their dynamical content.

5.8.2. Einstein–Cartan equation of state:Torsion as a geometric degree of freedom

In section 5.2.3 we showed that, in presence of torsion, hypersurface orthogonality of the horizon generators does not imply the vanishing of the null congruence twist. Therefore, in the general case, we cannot set shear and twist to zero in the Raychaudhuri eq. (5.84) and we need to use the non-equilibrium thermodynamical approach in order to recover the Einstein–Cartan equation.

As we reviewed above, In presence of dissipative terms, the Clausius equation has to be generalized to take into account an internal entropy term; namely, one has to use the entropy balance law

$$dS = \frac{\delta Q}{T} + dS_i\,. \tag{5.104}$$

By means of the entropy formula (5.88), we then have

$$\alpha \delta A = \frac{2\pi}{\hbar}\delta Q + dS_i\,. \tag{5.105}$$

The heat flux across the LCH is given by the expression

$$\delta Q = \int_{\mathcal{H}} T^{M}_{ab}\chi^a d\Sigma^b = -\int_{\mathcal{H}} \sqrt{h} d\lambda d^2x\, \lambda\, T^{M}_{ab} k^a k^b\,, \tag{5.106}$$

where $d\Sigma^b = \sqrt{h} d\lambda d^2 x k^a$ is the horizon volume element.

From the result of the previous Section, we have

$$\begin{aligned} \alpha\delta A &= \alpha \int_{\mathcal{H}} \sqrt{h} d\lambda d^2x\, \overline{\theta} \\ &\approx \alpha \int_{\mathcal{H}} \sqrt{h} d\lambda d^2x \left(\overline{\theta}_p + \lambda \left.\frac{d\overline{\theta}}{d\lambda}\right|_p \right). \end{aligned} \tag{5.107}$$

Therefore, using (5.107) and (5.106) in (5.104) the Clausius equation reads

$$\alpha \int_{\mathcal{H}} \sqrt{h} d\lambda d^2x \left(\overline{\theta}_p + \lambda \left.\frac{d\overline{\theta}}{d\lambda}\right|_p \right) = -\frac{2\pi}{\hbar} \int_{\mathcal{H}} \sqrt{h} d\lambda d^2x\, \lambda\, T^{M}_{ab} k^a k^b + dS_i \tag{5.108}$$

In the above equation, the l.h.s. has a first term of zeroth order in λ and a second term of first order in λ , while the r.h.s. is entirely first order in λ. Therefore in order to match the two sides of (5.104), at the zeroth order in λ, we need the condition $\theta_p = 0$, which by (5.75) implies

$$\overline{\nabla}_a k^a = - \left. T_a k^a \right|_p\,. \tag{5.109}$$

Now, before plugging in the Raychaudhuri equation (5.84) in (5.108), let us note that in presence of torsion, hypersurface orthogonality of the horizon generators does not imply anymore the vanishing of the null congruence twist. This is explicitly shown in section 5.2.3. Therefore, in the general case, we can set neither the shear nor twist to zero in the Raychaudhuri eq. (5.84) and they will contribute to the internal entropy term.

In order to correctly identify all the contributions to this non-equilibrium term, we need to open up the non-Riemannian shear and twist in (5.84), which in general will contribute terms both in k^μ as well as in its covariant affine derivatives, and identify only the latter as non-equilibrium contributions. Hence, we use the explicit form of the horizon shear, twist and (5.109), so to rewrite the Raychaudhuri equation (5.84) as

$$\begin{aligned} \left.\frac{d\overline{\theta}}{d\lambda}\right|_p &= -\overline{R}_{ab}k^a k^b + k^a k_b \overline{\nabla}^d T^b{}_{da} + k^b k^a \overline{\nabla}_a T_b + T^d T_{adb} k^a k^b \\ &- \left.\overline{\nabla}_b k^a \overline{\nabla}_a k^b + 2\overline{\nabla}_a k^b K^a{}_{bc} k^c\right|_p \end{aligned} \tag{5.110}$$

and identify the terms in the second line on the r.h.s. of (5.110) as internal entropy terms. Therefore, combining (5.108) with (5.110), the generalized Clausius law (5.104) implies at first order in λ

$$-\frac{2\pi}{\hbar} T^M_{ab} k^a k^b = \alpha\left(-\overline{R}_{ab} + \overline{\nabla}^d T^b{}_{da} + \overline{\nabla}_a T_b + T^d T_{adb} + T_a T_b\right) k^a k^b \tag{5.111}$$

and

$$\begin{aligned} dS_i &= \alpha \int_{\mathcal{H}} \sqrt{h} d\lambda d^2x\, \lambda \left.\left(-\overline{\nabla}_b k^a \overline{\nabla}_a k^b + 2\overline{\nabla}_a k^b K^a{}_{bc} k^c - \overline{\nabla}_a k^a \overline{\nabla}_b k^b\right)\right|_p \\ &= \alpha \int_{\mathcal{H}} \sqrt{h} d\lambda d^2x\, \lambda \left.\left(-\sigma^{ab}\sigma_{ab} + \omega^{ab}\omega_{ab} + K^{ba}{}_c K_{abd} k^c k^d - T_a T_b k^a k^b\right)\right|_p \end{aligned} \tag{5.112}$$

where, in the last equation, we have first included all the dissipative, non-equilibrium terms inside the Raychaudhuri equation (all the ones containing a covariant derivative of the horizon generator), and then re-expressed them, by means of (5.84), (5.109), in terms of the Riemannian shear and twist plus torsion contributions[1].

The first relation (5.111) yields, for

$$\alpha = \frac{1}{4\hbar G}, \tag{5.113}$$

[1]Notice that the condition (5.109) induces an ambiguity in the identification of the equilibrium and the non-equilibrium parts of the Raychaudhuri, since the last term on the r.h.s. of (5.111) can always be compensated by a non-equilibrium one like the last one on the r.h.s. of the first line in (5.112). We have thus included them in order to consider the most general case.

the symmetric part (5.102) of the Einstein-Cartan equation.

The second one, eq. (5.112), provides a definition of the internal entropy contribution in presence of torsion. Notice that, for vanishing torsion (and hence twist), eq. (5.112) reproduces the dissipative term obtained in [32, 109], namely

$$dS_i = -\alpha \int_{\mathcal{H}} \sqrt{h} d\lambda d^2x \, \lambda \, ||\sigma||_p^2 \,, \tag{5.114}$$

where the shear $\sigma_{\mu\nu}$ is the one defined w.r.t. the Levi-Civita connection. In the presence of torsion, the internal entropy (5.112) contains contributions coming from both the shear squared term and the twist terms inside the Raychaudhuri equation (5.80). This could be considered the generalisation to the Riemann–Cartan geometries of the Hartle–Hawking term describing the dissipation of a distortion of the horizon and seems to imply a different output of gravitational waves w.r.t. to what expected in General Relativity (of course just in cases where at the horizons there are fluxes of matter generating torsion in non-propagating torsion theories like Einstein–Cartan).

As in the original argument of [29], we could now try to use the Bianchi identity for a Riemann–Cartan spacetime in order to recover the Ricci scalar part of the equation of motion. However, in presence of torsion, the modified Bianchi identity contains torsion dependent terms which are not total covariant derivatives. Therefore, in this case, the modified Bianchi identity is of little help.

However, one can split the symmetric part of the Einstein tensor into a Riemannian part and a non-Riemanninan part (see next Subsection where this approach is carried out explicitely); the Riemannian term will be the standard Einstein tensor written in terms of the Levi–Civita connection and it will satisfy the standard Riemannian spacetime Bianchi identity. The non-Riemanninan part of the symmetric Einstein tensor comprises terms involving the affine covariant derivative of the torsion and quadratic contractions of the torsion tensor. These terms can be moved to the r.h.s. of (5.101) in order to define an effective SET. The important point is that, on an Einstein–Cartan space–time such SET will be conserved w.r.t. the Levi–Civita connection, since the l.h.s. is conserved due to the Riemannian Bianchi identity. It follows that, on an Einstein–Cartan spacetime, the torsion tensor has to be such that the Levi–Civita covariant derivative of the non-Riemanninan part of the symmetric Einstein tensor has to be equal to the Levi–Civita covariant derivative of the r.h.s. of (5.101), namely

$$\nabla^b \overline{R}_{(ab)} - \frac{1}{2}\nabla_a(\overline{R} - 2\lambda) = \nabla^b(8\pi G\, T^{\scriptscriptstyle M}_{ab} - \overline{\nabla}^d T_{(ab)d} + \overline{\nabla}_{(a} T_{b)} - T^d T_{(ab)d} + T_{(a}T_{b)}) \tag{5.115}$$

once we split $\overline{R}_{(ab)}, \overline{R}$ into their Riemannian and non-Riemannian parts and use the Riemannian Bianchi identity. The explicit splitting is obtained from

$$\begin{aligned}
\overline{R}_{ab} &= R_{ab} + \overline{\nabla}_d K^d{}_{ab} - \overline{\nabla}_a K^d{}_{db} + K^d{}_{dc}K^c{}_{ab} - K^d{}_{ac}K^c{}_{db}\,, \\
\overline{R} &= g^{ab}\overline{R}_{ab} = R + 2g^{ab}\overline{\nabla}_d K^d{}_{ab} + g^{ab}(K^d{}_{dc}K^c{}_{ab} - K^d{}_{ac}K^c{}_{db}) \\
&= R + 2\overline{\nabla}^a T_a - T_a T^a - g^{ab}K^d{}_{ac}K^c{}_{db}\,,
\end{aligned}$$

where we have used the relations

$$g^{ab}\overline{\nabla}_d K^d{}_{ab} = -g^{ab}\overline{\nabla}_a K^d{}_{db} = \overline{\nabla}^a T_a \tag{5.116}$$

and

$$g^{ab}K^d{}_{dc}K^c{}_{ab} = -T_a T^a\,. \tag{5.117}$$

The Ricci scalar and cosmological constant parts in the symmetric Einstein-Cartan equation (5.101) can then be obtained from the condition (5.115), in analogy to the standard Riemannian case. In fact, let us now go back to the part of the symmetric Einstein-Cartan equation that we recovered so far through the Clausius law, namely eq. (5.111). By peeling off the two k's, we can rewrite this as

$$\overline{R}_{(ab)} + g_{ab}F(x) = 8\pi G\, T^{\scriptscriptstyle M}_{ab} - \overline{\nabla}^d T_{(ab)d} + \overline{\nabla}_{(a}T_{b)} - T^d T_{(ab)d} + T_{(a}T_{b)}\,, \tag{5.118}$$

where, as in the original thermodynamical derivation of [29], we have added a term proportional to the metric and depending on some function $F(x)$. By taking the Levi-Civita covariant derivative and enforcing the condition (5.115), it is then immediate to obtain

$$F(x) = -\frac{1}{2}\overline{R} + \lambda\,. \tag{5.119}$$

Plugging this last relation back into (5.118) we thus recover the full symmetric Einstein-Cartan equation (5.101).

5.8.3. Einstein–Cartan equation of state: Torsion as a background field

We now want to derive Einstein–Cartan equation from the non-equilibrium thermodynamical approach where we take the point of view of torsion as an external (or background) field, with the torsion terms defining an effective SET for a Riemannian space–time. In fact, if we write the symmetric part of the non–Riemannian Einstein tensor in terms of the Riemannian one plus torsion terms, namely

$$\overline{R}_{(ab)} - \frac{1}{2}g_{ab}(\overline{R} - 2\lambda) = R_{ab} - \frac{1}{2}g_{ab}(R - 2\lambda) - \overline{\nabla}^d T_{(ab)d} + \overline{\nabla}_{(a}T_{b)}$$
$$+T^d T_{(ab)d} + K^d{}_{c(a}K^c{}_{d|b)} - \frac{1}{2}g_{ab}\left(2\overline{\nabla}^d T_d - T_d T^d - g^{qp}K^d{}_{qc}K^c{}_{dp}\right) , \quad (5.120)$$

then the symmetric part of the Einstein–Cartan equation (5.101) can be written in terms of the Riemannian Einstein tensor and an effective SET, namely

$$\begin{aligned} R_{ab} - \frac{1}{2}g_{ab}(R - 2\lambda) &= 8\pi G\, T^{M}_{ab} - 2T^d T_{(ab)d} + T_{(a}T_{b)} - K^d{}_{c(a}K^c{}_{d|b)} \\ &+ \frac{1}{2}g_{ab}\left(2\overline{\nabla}^d T_d - T_d T^d - g^{qp}K^d{}_{qc}K^c{}_{dp}\right) . \quad (5.121) \end{aligned}$$

As the next step, we rewrite the Raychaudhuri equation (5.84) in terms of the Riemannian Ricci tensor; this yields

$$\frac{d\theta}{d\lambda} = -R_{ab}k^a k^b - \frac{1}{2}\theta^2 - \sigma^{ab}\sigma_{ab} + \omega^{ab}\omega_{ab} - 2T^d T_{(ab)d}k^a k^b . \quad (5.122)$$

We can now run the non-equilibrium thermodynamical argument, similarly to the previous Subsection. By means of the generalized Clausius relation (5.104), at first order in λ, it is immediate to see that the Einstein–Cartan equation written as in (5.121), modulo the terms proportional to the metric g_{ab}, is recovered once we use exactly the same definition of internal entropy production term as in the previous derivation (namely, the second line of (5.112)); explicitly, we recover

$$-\frac{2\pi}{\hbar}T^{M}_{ab}k^a k^b = \alpha\left(-R_{ab} - 2T^d T_{(ab)d} + T_{(a}T_{b)} - K^d{}_{c(a}K^c{}_{d|b)}\right)k^a k^b \quad (5.123)$$

for

$$dS_i = \alpha \int_{\mathcal{H}} \sqrt{h}\, d\lambda d^2x\, \lambda\left(-\sigma^{ab}\sigma_{ab} + \omega^{ab}\omega_{ab} + K^{ba}{}_c K_{abd}k^c k^d - T_a T_b k^a k^b\right)\Big|_p . \quad (5.124)$$

The Ricci scalar part of the equation of motion can be recovered similarly like in the previous derivation, by means of the Riemannian Bianchi identity. In the present case, this approach is even more well justified given that we have explicitly expressed the Einstein–Cartan equation in terms of the Riemannian Einstein tensor.

This second approach to the thermodynamical derivation of the Einstein–Cartan equation, with all the non–Riemannian torsion contributions reabsorbed in an effective SET, may seem more linear and clean at first. However, if we had proceeded along these lines from the beginning, the definition of the internal entropy production term (5.124) would have appeared as an ad hoc one, in order to recover the desired result. On the other hand, in the derivation presented in Subsection 5.8.2, where we work with the geometrical structures of the Riemann–Cartan spacetime, this form of dS_i follows naturally from the Raychaudhuri equation (5.84).

CHAPTER 6

Conclusion

6.1. Summary of results

The link between gravitational dynamics and thermodynamics of horizons has been studied intensively in the past few decades. Though a lot has been understood, many new paradoxes and problems also appeared, calling for better explanations and sometimes drastic changes to the known theory of gravity and quantum mechanics. One clear message that we can take from all these studies is that general relativity (GR) is not the complete theory of gravity and for the unification of gravity with quantum mechanics we have to go beyond GR. The other important thing that we have to check for consistency is how compatible these new theories are with the new innovative experimental results that are capable of giving us some information about possible characteristics (such as the validation of local Lorentz invariance, Locality, scale invariance etc) of a quantum theory of gravity.

In this thesis we explored two well studied directions of understanding the thermodynamical nature of gravity. In the first part we were concerned about thermodynamics of *event horizon* (black hole thermodynamics) and presented our studies about the region of origin of Hawking radiation. In the next part of the thesis we were dealing with thermodynamics of *local causal horizon* and how this can lead to the derivation of the equations of motion for a theories of gravity beyond GR (to be precise, higher derivative gravity and theories including torsion in this case) as an equation of state.

6.1.1. Thermodynamics of global horizons: Black hole thermodynamics

It has been widely believed that Hawking radiation originates from the excitations close to the horizon and this eventually suggested some drastic modification of the states in the near horizon regime as a resolution to the information loss paradox [20, 122, 123, 62]. One of the primary reasons for such an argument is based on the way Hawking did his original calculation, tracing back the modes all the way from future infinity to the past null infinity through the collapsing matter so that one has a vacuum state at the horizon for a free-falling observers.

The other disturbing feature about this argument is, when the modes are traced back they become highly blueshifted near the horizon and we are not well aware of the laws of physics in such high trans Planckian do-main. Some resolutions to the above problem has been proposed several times in the literature [124, 57, 125] but they all demand some challenging modification to our present knowledge of gravitation or quantum field theory.

Let us stress, however, that the UV departures from Lorentz invariance through the introduction of a fundamental cutoff postulated in [126, 127] are relevant only very close to the horizon for large black holes (in units of the Lorentz breaking scale). Hence, even contemplating such scenario, our analysis in section 3.3 of Chapter 3 would be basically unchanged and unaffected away from the horizon, as also stressed in the similar analysis carried out in [128].

In Chapter 3 we have further shown evidence that the Hawking quanta originate from a region which is far outside the horizon, which can be called a black hole *atmosphere* [82]. More precisely, from the plots of the energy density and the flux in the Unruh state, as presented in section 3.3, we get a maximum at $r \approx 4.32M$, for the energy density in the Hartle–Hawking state the peak is at $r \approx 4.37M$. This is strikingly close to our previous finding based on the heuristic argument using tidal forces, in section 3.2, for an origin at about $r \approx 4.38M$ for the peak of the thermal spectrum. By large this is also in agreement with some previous claims using various other methods, such as calculating the effective radius of a radiating body using the Stefan–Boltzmann law or computing the effective Tolman temperature [79, 129, 130, 131], as well as in close correspondence with the results of the study of the null component of the stress-energy tensor in the Unruh vacuum of [132].

If the radiation has a long distance origin then we might not need to worry about the trans Planckian issue at the horizon. Moreover, concerning the fundamental issue of unitarity of black hole evaporation, this result suggests to consider some effect operational at this new scale in order to eventually restore unitarity of Hawking radiation. A possible scenario is the one of non-violent nonlocality advocated in [133, 134]; see also the proposal of [135, 136].

In Chapter 3 we further investigated the behavior of an *effective temperature function* as perceived by an free falling observer starting with zero initial velocity at infinity. We found that the energy density computed from this temperature function does not match with the energy density computed by using RSET for the same observer. To confront the reason for this discrepancy we computed the adiabaticity of the temperature function and found that the deviation exactly tracks the deviation of the adiabaticity from its minimum value. From this one can infer that due to breaking down of the WKB approximation one cannot trust the validity of a plankian distribution at the region outside the horizon (extending well beyond 3M). We further proposed a way to study what an local inertial observer would perceive in its local inertial frame, getting rid of any kinematical effect that arises due to the acceleration of the observer with respect to the black hole.

6.1.2. Thermodynamics of local horizons: Spacetime thermodynamics

Ever since Jacobson's seminal work in [29], where he derived the Einstein equation starting from local thermodynamical variables, implementing the Clausius equation, a lot has been done in this field yet leaving some open issues and scopes for improvement. One of the main issue with the extension of the original derivation beyond GR is in the choice of the entropy that should go as an input in the Clausius equation. In [29] the proportionality of the entanglement entropy with the area of the local causal horizon was used but this cannot be the case if one needs to move beyond GR. As we know for black hole thermodynamics, for theories beyond GR the entropy is no more proportional to just the area of the horizon, it would often involve curvature terms as well.

Chapter 4 of this thesis begins with a brief review of spacetime thermodynamics and the rest of the chapter is mainly based on the work done in [36]. We have proposed a new expression for the entropy associated to the slices of local causal horizon, eq. (4.35), such that the Clausius relation imposed on a patch of the horizon implies the field equation of the theory under consideration. The theory in question could be any diffeomorphism invariant metric theory of gravity. In order to achieve this result we introduced two new ingredients: first, the heat flux in the Clausius relation is provided by a minimally coupled probe field that we put to zero in the end, and second, the entropy has a new term quadratic in the approximate Killing vector. Let us discuss these two inputs one by one.

The reason to introduce the probe matter providing the heat flux is to be able to work with the most general diffeomorphism invariant theory. This was done because a general diffeomorphism covariant Lagrangian does not admit a canonical split between a gravitational part and a matter part. For example, consider a scalar field ϕ in the Lagrangian whose coupling to metric is non-minimal of the form $R^{ab}\nabla_a\phi\nabla_b\phi$. If we consider this term as contributing to the matter stress tensor and use it to define the heat flux, then on the left hand side of the equation of motion (2.31) we will not include its contribution to E^{abcd}. The resulting entropy density will however not match with the black hole entropy in the theory which is determined by the E^{abcd} of the total Lagrangian by the Walds' formula.

Alternatively, we could count this term as "gravitational" and use only the canonical kinetic term of ϕ to define the heat flux. This would be a viable option, but it does not appear to be a very natural thing to do. On the other hand, the approach of characterising a system completely by perturbing it with probe fields and observing its response is ubiquitous in physics. In short, the need to work with complete generality, and the compatibility with black hole thermodynamics led us to define the heat flux in Clausius relation by a probe, minimally coupled matter field that we put to zero in the end. If we were not to use the probe fields to define the heat flux, then we would have to restrict to only the theories with minimally coupled matter field. But even then the new M term in the entropy density would still be needed to derive the field equation of higher derivative gravity.

The new term in the entropy that we have proposed does not alter the black hole entropy because the Killing vector vanishes on the bifurcation surface. Compatibility with black hole thermodynamics is a stringent requirement. Without it, we could have simply taken the whole entropy as given by the quadratic term and chosen M^{ab} to be the equation of motion. But then the black hole entropy in the theory would be zero. The X term in eq. (4.35) is thus dictated by the black hole entropy. The W term is necessary for the equation of state argument to go through for the higher curvature theories. For higher derivative theories the M term in eq. (4.35) is needed to get the equation of motion via the Clausius relation.

The generality of our approach seems to suggest that there is no obstacle for the equation of state derivation for any diffeo-invariant metric theory of gravity, irrespective of whether it is Lorentz invariant or not. In particular, one could then derive local thermodynamics in Lorentz violating theories, e.g., the Einstein-Æther theory. However, this expectation faces two challenges. First of all, local Lorentz invariance is crucial to associate the Unruh temperature with the local causal horizon. Second, the existence of black hole thermodynamics in such theories is not well-settled yet, and is under active investigation [137, 138, 139, 140, 141, 142, 143, 144, 145]. Finding a thermodynamic route to the equation of motion in such theories thus appears to be a premature enterprise at the moment.

We would like to mention in passing that some authors [146, 104] have taken the converse route to the one discussed here. That is, their goal is to understand if the field equation implies the Clausius relation for an appropriately defined entropy density. It is not too difficult to show that given our entropy one can follow this program of running the argument backwards to its completion for any diffeomorphism invariant theory of gravity.

In Chapter 5 we have extended the thermodynamics of space–time formalism to the Einstein–Cartan theory of gravity. In doing so we have had to reconsider several ingredients entering in the original derivation [29]. First by redefining the notion of the local inertial frame in Riemann–Cartan geometries as well as by reconsidering the notion of Killing horizon surface gravity and derive the Raychaudhuri equation in this framework. In doing so we have obtained several original results and we have understood that the notion of a stationary Killing horizon in this setting requires an additional condition on the torsion current through the horizon which enforces the geodesic flow of the Killing vector and the horizon generator.

Then we have applied this toolkit to the space–time thermodynamics approach and shown that using a generalised Clausius equation it is possible to recover the relevant part of the Einstein–Cartan–Sciama–Kibble equations. In doing so we have identified the relevant non-equilibrium terms, finding a novel dependence on the twist, which as such represent a generalisation of the usual Hartle–Hawking term. Let us stress that this term is calculated on the horizon and as such can present non-zero twist and torsion even in theories with non-propagating torsion as Einstein–Cartan, as long as the deformation of the local Rindler horizon is generated by matter fluxes endowed with spin currents. This seems to suggest that if these terms are as usual associated to the actual energy that can be observed at infinity as carried away by gravitational waves, then they can provide a signature of the actual presence of torsion in the case e.g. of black hole mergers in environments with suitable matter.

6.2. FUTURE SCOPE

The study of the thermodynamic nature of gravity, as presented in this thesis for both global and local causal horizons has been conceptually compelling. These studies gives us the best hints about what are the essential features for a quantum theory of gravity and surely will help us to get more insight about the fundamental structure of spacetime.

A very interesting and informative way to study various features of Hawking radiation is through analogue models of gravity [124] [147]. Analogue models of black holes can be build based on various condensed matter systems such as acoustics in fluid, superfluid Helium, Bose–Einstein condensate(BEC), propagation of electromagnetic waves in dielectric medium, and they can be realized in a laboratory setup to verify certain features of black hole thermodynamics that exists within the analogue models. In particular we have shown the existence of the analogue models for Anti de-Sitter and de-Sitter black holes in BEC [148]. This opens the scope to study various other tantalizing features of gravity, such as the fluid/gravity correspondence [149] associated to AdS geometry, within the analogue framework. We have also shown in [148] that the stress energy tensor computed on the boundary for Nordstrom gravity (which captures the dynamics of analogue models of gravity as shown in [150]) will have the same form as in general relativity [151], which can be that of a conformal fluid if computed for asymptotic AdS solutions.

The nature of Hawking radiation has intrigued people for decades now. Knowing whether the Hawking quanta are created few Planck distances away from the horizon or in some *quantum atmosphere*, extending well beyond the horizon, can give us a better understanding about open issues, such as the information loss and trans-Planckian problem as discussed here. It will be specially important to know this region of origin for the radiation if someone believes that some new effects must be considered to unitarize Hawking radiation at the same characteristic scale as that of radiation itself. This would immediately give us a clue about a "nonviolent" scenario [152] [134] operational at the *quantum atmosphere* or something dramatic like a "firewall" at the horizon [20] that will ultimately help us to solve the information loss problem. We have given some arguments to favor the origin of Hawking radiation in the *quantum atmosphere* [82] [79] but the debate is far from over and more concrete evidence is needed to get a definite answer.

Of course, an alternative possibility to escape conclusions like a firewall at the horizon is that new physics ensues in the far UV. Partially inspired by analogue models scenarios people have conjectured for example Lorentz breaking UV completions of GR and the standard model. In particular, theories of gravity entailing a violation of local Lorentz invariance at high energies have received much attention. Einstein–Aether theory [153] [154] [155] is one such theory for which it is possible to write a covariant Lagrangian but due to presence of a globally defined timelike vector field, Lorentz invariance is broken. A very surprising discovery was that this theory can still have black hole solutions even though superluminal propagation is allowed [156]. It was found that black holes in such theories would have spacelike causal boundaries called universal horizon, beyond which nothing can escape from the black hole. It is interesting to study the thermodynamics of these causal surface and investigate how it differs from the thermodynamics of event horizon. Several independent studies have been conducted to understand the thermodynamics of black holes in Einstein–Aether gravity [141] [143] [144] but not much is understood on the sort of regular vacuum state that could be accommodated on such spacetimes. Given that in the standard (GR) case the correlation structure of hawking pairs in the unique regular vacuum state (Unruh) is a crucial ingredient to achieve the conclusion that purification requires a firewall, it would be quite interesting to see what the correlation structure for the equivalent (if any) state would be in these Lorentz breaking black hole solutions.

Finally, in the spacetime thermodynamics derivation for higher derivative gravity we mentioned that one of the main drawback is in a quantum mechanical description for the entropy that is used in the Clausius equation. In the derivation of the semi-classical Einstein equation [41] a better framework was used where one can compute the UV and IR part of the entanglement entropy and then based on a conjecture of entanglement equilibrium, link these two parts to arrive at the Einstein equation. While the IR part of the entropy is computed quite robustly [47] there is still no definite computation about the UV part of the entanglement entropy (which is taken proportional to area) for a causal diamond, including the first order curvature corrections. These curvature corrections would be particularly important if one tries to move beyond general relativity. One surprising feature that we found was the presence of boost invariance even after including the first order curvature corrections in the metric for the causal diamond, written in terms of Riemann normal coordinates. This allows us to define positive frequency modes and a nice time slicing to do local quantum field theory within a causal diamond. There are some computations of entanglement entropy [157] for such a setup considering a S-J vacuum state [158] [159] but using the local boost invariance we believe an extension of these calculations are possible.

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