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Thesis

On C^*-algebras associated to Horocycle Flows

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Abstract

This thesis is mainly concerned with the study of the crossed product C^*-algebras associated to the horocycle flow on compact quotients of $\mathrm{SL}(2,\mathbb{R})$. Looking at the Cuntz semigroup, we retrieve some information about the structure of hereditary C^*-subalgebras and Hilbert modules for a class of C^*-algebras which contain the C^*-algebras we want to study. After translating these results in our context, we study the functoriality of the construction both for the case of discrete subgroups of $\mathrm{SL}(2,\mathbb{R})$ and for the case of hyperbolic Riemann surfaces. Also properties of another crossed product C^*-algebra that is Morita equivalent to the C^*-algebra of the horocycle flow are explored and from considerations about the associated dynamical system we can prove that the multiplier algebra of the crossed product C^*-algebra associated to the horocyce flow contains a Kirchberg algebra in the UCT class as a unital C^*-subalgebra in some cases.

A side chapter is devoted to a project that the author started during his PhD, concerning the construction of spectral triples on the Jiang-Su algebra. The construction we give is performed by means of a particular AF-embedding.

Contents

Chapter 0

Introduction

Transformation group C^*-algebras have been intensively studied during the last decades, mostly in order to give examples for classification (and structure) theorems and to interpret properties of dynamical systems on the C^*-side. In particular, if the crossed product C^*-algebra is in the class of unital and simple C^*-algebras with finite nuclear dimension which satisfy the UCT ([66]), then the C^*-algebra can be recovered by K-theoretical information and traces (up to $*$-isomorphism) (see [30]); this is the case for the transformation group C^*-algebras associated to irrational rotations on the circle ([58]), in which case the classification result tells us that two such C^*-algebras A_{θ_1} and A_{θ_2} are $*$-isomorphic if and only if the corresponding irrational angles differ by an integer, $\theta_1 = \pm\theta_2 \mod 1$, which turns out to be equivalent to the dynamical systems (S^1, θ_1) and (S^1, θ_2) being topologically conjugated. Another cornerstone example comes from minimal homeomorphisms on a Cantor set X ([56], [34]); in this case if ϕ_1 and ϕ_2 are two minimal homeomorphisms on X, the resulting crossed product C^*-algebras are $*$-isomorphic if and only if the corresponding dynamical systems (X, ϕ_1) and (X, ϕ_2) are strongly orbit equivalent (or equivalently, the K-theoretic data coincide). In this example the K_0-group of the crossed product C^*-algebra can be recovered from the one of a particular AF-subalgebra ([14]) that nowaday would be called a large C^*-subalgebra ([55]). The idea underlying this construction was inspiring for more general transformation group C^*-algebras. For example, if X is a compact metrizable space with finite covering dimension and h is a minimal homeomorphism, Toms and Winter proved in [73] that one can produce large subalgebras of $C(X) \rtimes_h \mathbb{Z}$ that are recursive subhomogeneus, hence $\mathcal{Z}$-absorbing, and deduce from this that also the transformation group C^*-algebra absorbs the Jiang-Su algebra $\mathcal{Z}$ tensorially, leading to classification in the case the projections separate traces. In the case of minimal actions of a more general class of discrete groups, including finite groups (see [39]) and residually finite groups with finite asymptotic dimension ([69]), such classification results can still be obtained if we restrict to actions with finite Rokhlin dimension and the resulting crossed product C^*-algebras turn out to be ASH-algebras ([18] 8.5).

The goal of this thesis is to study a class of simple $\mathcal{Z}$-stable transformation group C^*-algebras associated to actions of the group of real numbers on compact metric spaces of finite covering dimension. The C^*-algebras belonging to this class will no longer be unital and so in principle we cannot expect the classification and structure results mentioned above to hold true in this situation. The study of such crossed products was started in [38], where, using a suitable notion of Rokhlin dimension, it was proved that for a free flow on a locally compact metrizable space the corresponding crossed product C^*-algebra has finite nuclear dimension (Corollary 9.2) and that under the same hypothesis it is stable (Theorem 6.6); hence, in the case this C^*-algebra contains non-zero projections, it is classifiable and is the stabilization of a unital ASH-algebra. Thus, if we want to produce simple $\mathcal{Z}$-stable transformation group C^*-algebras arising from flows whose structure does not reflect the structure of a unital simple $\mathcal{Z}$-stable C^*-algebra, we should focus on the projectionless case. The absence of projections, in the case of simple $\mathcal{Z}$-stable C^*-algebras, turns out to be an interesting feature and despite the impossibility of applying the aforementioned results, by [59] it gives to the Cuntz semigroup a more tractable structure, since Cuntz (sub)equivalence in this case is the same as Blackadar (sub)equivalence (see [51] Definition 2.1). The point is that the C^*-algebras in this class have almost stable rank 1. Nevertheless, it is natural to expect, in view of the comparison results of [59], that in the case the crossed product C^*-algebra admits a unique trace, some of its properties can already be established by looking at the structure of its Cuntz semigroup. Hence, it would be desirable that our C^*-algebras share this property.
Sticking to this setting, we give, following an idea by Elliott, a characterization of the Pedersen ideal for C^*-algebras whose stabilization has almost stable rank 1 in Proposition 2.4.1. Furthermore, in the case of C^*-algebras admitting a unique lower semicontinuous 2-quasitrace, we prove in Proposition 2.4.11 that the Pedersen ideal coincides with the ideal associated to this quasitrace. This implies by Theorem 2.4.12 that the σ-unital C^*-algebras in this class are either algebraically simple or stable, depending on wether the dimension function associated to the quasitrace takes a finite or an infinite value when evaluated on a strictly positive element. An analogue statement holds at the level of countably generated Hilbert modules (Theorem 2.4.19).
Every free and minimal flow on a locally compact metrizable space produces, by the above considerations, a simple $\mathcal{Z}$-stable C^*-algebra. In order to ensure that it does not contain projections, some considerations have to be made. First of all note

that if the flow admits a compact transversal, it is shown in [38] Remark 9.5 that it is possible to explicitly construct a projection on the resulting crossed product C^*-algebra. Even if the flow does not admit a compact transversal, we should look closer at the specific flows in order to exclude the existence of projections.
If the flow is the *horocycle flow* on a compact quotient of $\mathrm{SL}(2,\mathbb{R})$, as already observed by Connes on page 129 of [21], the interplay with the associated geodesic flow denies the existence of projections on the resulting crossed product C^*-algebra; also, the unique ergodicity of this flow ([33]) and the correspondence between invariant probability measures and traces on the crossed product C^*-algebra ([49] Theorem 6.30) guarantee the existence of a unique trace. Hence this class of examples constitute a good candidate for an application of the above considerations.
In view of this, we begin in this thesis the study of the crossed product C^*-algebra associated to this flow.
After translating the general results concerning σ-unital simple $\mathcal{Z}$-stable C^*-algebras with almost stable rank 1 to this example, we proceed in Chapter 4 exploring the functoriality of this construction. Namely, using basic results about the possibility to lift equivariant $*$-homomorphisms to crossed products, we show that there are both a covariant and a contravariant functors from a category of discrete cocompact subgroups of $\mathrm{SL}(2,\mathbb{R})$ in which the morphisms are inclusions modulo conjugation in $\mathrm{SL}(2,\mathbb{R})$, to the category of C^*-algebras (in which the morphisms are $*$-homomorphisms), which are induced by the crossed product construction. Furthermore, specializing to cocompact groups that are the symmetrization of the fundamental groups of compact hyperbolic Riemann surfaces, we see that the crossed product construction induces a contravariant functor from a category of compact hyperbolic Riemann surfaces in which the morphisms are holomorphic covering maps, to the category of C^*-algebras.
In particular, two biholomorphic compact hyperbolic Riemann surfaces give rise to $*$-isomorphic C^*-algebras. It is interesting to note that the question wether two compact hyperbolic Riemann surfaces that are just homeomorphic produce $*$-isomorphic C^*-algebras, is equivalent to the question wether the C^*-algebras in the range of the functor are classified by their Elliott invariant (see Remark 4.3.6).
Using Mackey-Rieffell machinery it is readily seen that the C^*-algebra of the horocycle flow corresponding to a cocompact subgroup of $\mathrm{SL}(2,\mathbb{R})$ is the stabilization of the crossed product C^*-algebra obtained by considering the action of this cocompact group on the euclidean plane with the origin removed. Properties of the flow reflect into properties of the crossed product C^*-algebra we consider, whose structure is studied; in particular, it follows by comparing the strictly positive elements that the C^*-algebra associated to the horocycle flow and the one associated to the discrete

action we just mentioned must be $*$-isomorphic. This suggests that the properties of the C^*-algebra of the flow can be equivalently obtained by looking at the corresponding discrete dynamical system; in particular the C^*-algebras associated to the latter dynamical system must be stable. Now, any stable C^*-algebra has weak stable rank 1, is such that its multiplier algebra is properly infinite and in the σ-unital case has the property that any strictly positive element for it is properly infinite. Thus one should be able to recover these properties from the dynamics.
In this direction, we see that the discrete dynamical system is paradoxical; observe that by the Banach-Tarski paradox a paradoxical action on a compact Hausdorff space does not allow the existence of a full supported invariant measure, but this is no more true if the space is locally compact non-compact and the paradoxical sets have infinite measure. The study of crossed products associated to paradoxical actions constitutes an active area in the field of C^*-algebras and most of the results obtained in this direction are aimed to prove pure infiniteness of the transformation group C^*-algebra from dynamical considerations (see [65], [46] and [3]). Our aim in the present context would be to weaken the hypothesis that ensure pure infiniteness, namely the existence of a basis of clopen paradoxical sets ([65]), or strong proximality of the action ([46]), in order to obtain proper infiniteness for just the strictly positive elements in the C^*-algebra or for its multiplier algebra. These conditions would be enough to guarantee stability under some restriction on the stable rank.
We show in Proposition 5.1.3 that the existence of a contractive open set in a locally compact connected normal Hausdorff space guarantees infiniteness of the multiplier algebra of the transformation group C^*-algebra. Contractiveness of the action for discrete subgroups of $\mathrm{SL}(2,\mathbb{R})$ can be retrieved from the presence of hyperbolic elements in the subgroup. In the case of discrete groups containing two hyperbolic elements with different axes, the action turns out to be paradoxical and by Proposition 5.2.2 this is reflected in the proper infiniteness of the multiplier algebra of the associated crossed product C^*-algebra. In particular this gives a dynamical interpretation of this property in the case of cocompact discrete subgroups of $\mathrm{SL}(2,\mathbb{R})$.
Furthermore, if the cocompact subgroup of $\mathrm{SL}(2,\mathbb{R})$ belongs to a certain class, it is shown, using results from [45], [3] and [46] that the multiplier algebra of the corresponding crossed product C^*-algebra contains a Kirchberg algebra in the UCT class as a unital C^*-subalgebra.

This version of the present thesis is different from the original one in the following points:
The sentence "The author of this thesis could prove the following weaker version" on page 32 is replaced with the sentence "Fruitful discussions with the colleague and

friend André Schemaitat lead to the following weaker version";
A reference in the Bibliography has been added, namely [7]. Because of this, the enumeration of other references has changed.
Chapter 6 begins now with the sentence "This chapter is based on a joint work with Professor Ludwik Dąbrowski ([7]).";

0.1 Overview of the chapters

Chapter 1 The first Chapter contains notation and preliminaries that will be used in this thesis.

Chapter 2 In the first section of Chapter 2 we recall a definition of largeness for hereditary C^*-subalgebras of a given C^*-algebra, that first appeared in [63]; we give equivalent conditions for largeness in terms of Cuntz equivalence in Proposition 2.1.9 and give a necessary and sufficient condition for largeness to pass to the stabilization in Proposition 2.1.13. In particular, by Example 2.1.15, largeness does not pass in general to the stabilization.
In the second section we prove, using Brown's results from [15], that largeness passes to the stabilization for singly generated hereditary C^*-subalgebras of stable σ-unital C^*-algebras and give an expression for the approximants appearing in the definition of largeness in this case (Proposition 2.2.2).
In the third section we recall some results from [22] and [51] in order to investigate the relationship between stability, largeness in the stabilization and infiniteness of the multiplier algebra for a σ-unital C^*-algebra, under some assumptions on the stable rank.
In the last section, using the stability result of Section 3 and a characterization of the Pedersen ideal (Proposition 2.4.11), we are able to prove, using comparison, a dichotomy between algebraic simplicity and stability for a certain class of C^*-algebras in Theorem 2.4.12. The counterpart at the level of Hilbert modules is Theorem 2.4.19.

Chapter 3 This chapter is an introduction to the geodesic and horocycle flow on compact quotients of $\mathrm{SL}(2,\mathbb{R})$. In the first section we recall some basic facts about homogeneous spaces and give the definition of geodesic and horocycle flow on quotients of $\mathrm{SL}(2,\mathbb{R})$ by discrete subgroups; we also include a proof of the freeness of the horocycle flow in the cocompact case.
In the second section we introduce hyperbolic Riemann surfaces together with

their connection to Fuchsian groups.
In the third section we see the geometric interpretation of the geodesic and horocycle flow in the case the quotient is the unit tangent bundle of a hyperbolic Riemann surface and give the statement of the results by Hedlund and Furstenberg about the minimality and unique ergodicity of the horocycle flow on compact spaces.

Chapter 4 After a brief digression on the crossed product C^*-algebras associated to smooth flows on compact manifolds, we see that, as a consequence of freeness, minimality (Hedlund's Theorem) and unique ergodicity (Furstenberg's Theorem), the crossed product C^*-algebra associated to the horocycle flow on a quotient of $\mathrm{SL}(2,\mathbb{R})$ by a discrete cocompact subgroup is simple, stable, $\mathcal{Z}$-stable and admits a unique trace. Moreover, from the interplay between the geodesic and the horocycle flow, it follows that this C^*-algebra has almost stable rank 1 and thus the considerations from the last section of Chapter 2 apply. This is the content of Theorem 4.1.2, Theorem 4.1.4 and Theorem 4.1.5.
In the second section we study functoriality of the crossed product construction in this situation, defining functors from certain categories of discrete subgroups of $\mathrm{SL}(2,\mathbb{R})$ to C^*-algebras. In the third section we specialize to the case in which the domain category is a category of hyperbolic Riemann surfaces.

Chapter 5 This chapter is an attempt to derive stability of the C^*-algebra $C_0(\mathbb{E}) \rtimes \Gamma$ associated to a cocompact subgroup Γ of $\mathrm{SL}(2,\mathbb{R})$, which is the content of Theorem 4.1.4, from dynamical considerations. In view of Proposition 2.3.5 and Proposition 2.3.11 it would be enough to prove that the multiplier algebra is infinite, or that a strictly positive element is properly infinite, once we have information about the stable rank.
In the first section we study paradoxical and contractive actions on locally compact (normal) Hausdorff spaces and derive the infiniteness conditions just mentioned in these situations.
In the second section we just observe that the action of any discrete subgroup of $\mathrm{SL}(2,\mathbb{R})$ containing hyperbolic elements is contractive and if it contains two hyperbolic elements with different axes, it is paradoxical. In particular this is true in the cocompact case and so we can apply the results of the previous section.
In the last section we combine results from [46] (or [3]) and [45] to conclude that in the case a discrete subgroup of $\mathrm{SL}(2,\mathbb{R})$ is the lift of a Fuchsian group of the first kind not containing cyclic elements of order 2, the multiplier algebra of the crossed product C^*-algebra associated to the horocycle flow contains a

Kirchberg algebra in the UCT class as a unital C^*-subalgebra.

Chapter 1

Notation and Preliminaries

1.1 Notation

We will denote by $\mathbb{R}$ the topological group of real numbers, where the topology is the one generated by open intervals. We will denote by $\mathbb{C}$ the field of complex numbers, endowed with the topology generated by open balls and by $\mathbb{C}^\times$ the subset $\mathbb{C} - \{0\}$. Every C^*-algebra is assumed to be defined over $\mathbb{C}$. $\mathbb{N}$ will denote the set of natural numbers and $\mathbb{Z}$ the group of integers.
If A is a C^*-algebra we will denote by $\tilde{A}$ its minimal unitization, by $M(A)$ its multiplier algebra, by A_+ its positive cone and by $\mathrm{Aut}(A)$ the group of $*$-isomorphisms from A to itself; we will call the elements of $\mathrm{Aut}(A)$ automorphisms.
If $\mathfrak{H}$ is a Hilbert space, we will denote by $\mathbb{B}(\mathfrak{H})$ the C^*-algebra of bounded operators on it and by $U(\mathfrak{H})$ the group of unitaries in $\mathbb{B}(\mathfrak{H})$.
If S is a subset of a C^*-algebra A, we denote by $\overline{S}$ its closure in the C^*-norm of A. If x is a positive element in a C^*-algebra A, we will denote by $\mathrm{her}(x) = \overline{xAx}$ the hereditary C^*-subalgebra generated by x. We will denote by $\mathbb{K}$ the C^*-algebra of compact operators on a separable infinite-dimensional Hilbert space $\mathfrak{H}$. For every $n \in \mathbb{N}$, M_n is the simple finite-dimensional C^*-algebra of $n \times n$ complex-valued matrices. For x a positive element in a C^*-algebra A and $\epsilon > 0$, we will denote by $(x-\epsilon)_+$ the positive element of $C^*(x)_+ \subset A_+$ given by application of the functional calculus with respect to the function $f \in C_0((0,\infty))$ defined as

$$f(t) = \begin{cases} t - \epsilon & \text{for } t \geqslant \epsilon \\ 0 & \text{for } t < \epsilon \end{cases}.$$

We will denote by f_n the function on $\mathbb{R}$ defined by

$$f_n(t) = \begin{cases} 0 \text{ for } t \leqslant 1/(2n) \\ 2nt - 1 \text{ for } 1/(2n) \leqslant t < 1/n \\ 1 \text{ for } 1/n \leqslant t \end{cases} .$$

If X is a locally compact space, we will denote by $\mathcal{B}(X)$ the σ-algebra of Borel subsets of X. If X is locally compact and non-compact, we will denote by $C_0(X)$ the C^*-algebra of continuous functions on X vanishing at infinity; if X is compact we will denote by $C(X)$ the C^*-algebra of continuous functions on X. If X is a locally compact space and f a continuous function on it, we will denote by $\operatorname{supp}(f) \coloneqq \overline{\{x \in X \ : \ f(x) \neq 0\}}$ its support. If C is a closed subset of X, we denote by $\mathring{C}$ its interior.

1.2 The Cuntz semigroup

Let A be a C^*-algebra and let a, b be positive elements in A. As in [70] Definition 2.1 we say that a is *Cuntz subequivalent* to b and write $a \precsim b$ if there is a sequence of elements x_n in A such that

$$a = \lim_n x_n b x_n^*.$$

Furthermore we say that a and b are *Cuntz equivalent* and write $a \sim b$ if $a \precsim b$ and $b \precsim a$.

Cuntz comparison passes to hereditary C^*-subalgebras.

Proposition 1.2.1 ([70] Proposition 2.18)**.** *Let A be a C^*-algebra and B a hereditary C^*-subalgebra. If a and b are positive elements in B such that $a \precsim b$ in A, then $a \precsim b$ in B.*

Let A be a C^*-algebra and consider the set $(A \otimes \mathbb{K})_+/\sim$; for a positive element a in $A \otimes \mathbb{K}$ we will denote its class by $[a]$.
We can endow $(A \otimes \mathbb{K})_+/\sim$ with the structure of a semigroup in the following way. Choose a $*$-isomorphism $\psi : M_2 \otimes \mathbb{K} \to \mathbb{K}$ and for a, $b \in (A \otimes \mathbb{K})_+$ define $[a] + [b] \coloneqq [(\mathrm{id}_A \otimes \psi)(a \oplus b)]$. Since every $*$-automorphism of $\mathbb{K}$ is implemented by a unitary in $\mathbb{B}(\mathbb{H})$, this definition does not depend on the choice of ψ and for every $a \in (A \otimes \mathbb{K})_+$ we have $[a] = [(\mathrm{id}_A \otimes \psi)(a \oplus 0)]$, where we used the fact that if two positive elements in $A \otimes \mathbb{K}$ are Cuntz equivalent in $M(A \otimes \mathbb{K})$, then they are Cuntz equivalent in $A \otimes \mathbb{K}$ by Proposition 1.2.1.

Definition 1.2.2. *Let A be a C^*-algebra. The Cuntz semigroup of A $Cu(A)$ is the partially ordered semigroup defined as*

$$Cu(A) := (A \otimes \mathbb{K})/\sim,$$

where the semigroup structure is the one just defined and the partial order is the one induced by Cuntz subequivalence.

For every C^*-algebra A, $Cu(A)$ contains the zero element $[0]$. If $[a]$ and $[b]$ are such that $a \precsim b$, then we write $[a] \leqslant [b]$; if $[a] \leqslant [b]$ and $[a] \neq [b]$ we write $[a] < [b]$. We will say that $[a]$ is *way below* $[b]$, or that $[a]$ is *compactly contained* in $[b]$ and write $[a] \ll [b]$ if for every upward directed sequence $\{[b_n]\}_{n\in\mathbb{N}} \subset Cu(A)$ for which a supremum $\sup_n[b_n]$ exists and $[b] \leqslant \sup_n[b_n]$, there exists $m \in \mathbb{N}$ such that $[a] \leqslant [b_m]$.
We will list some properties of the Cuntz semigroup of a C^*-algebra.

Theorem 1.2.3 ([22] Theorem 1)**.** *Let A be a C^*-algebra. Then $Cu(A)$ satisfies the following*

- *Every nondecreasing sequence in $Cu(A)$ has a supremum.*
- *For every $[a]$ in $Cu(A)$ there is a sequence $[a_1] \ll [a_2] \ll \ldots$ with supremum $[a]$.*
- *If $[a]$, $[a']$, $[b]$ and $[b']$ are elements in $Cu(A)$ such that $[a] \ll [a']$ and $[b] \ll [b']$, then $[a] + [a'] \ll [b] + [b']$.*
- *If $[a_n]$, $[b_n]$ are increasing sequences in $Cu(A)$, then $\sup_n([a_n] + [b_n]) = \sup_n[a_n] + \sup_n[b_n]$.*

In the next chapter the following two results Theorem 1.2.4 and Proposition 1.2.5 will be used several times. A proof of Theorem 1.2.4 can be found in [70] Theorem 2.30.

Theorem 1.2.4 (Rørdam Lemma)**.** *Let A be a C^*-algebra and a, b two positive elements in A. The following are equivalent*

- *$a \precsim b$.*
- *For every $\epsilon > 0$, there is $\delta > 0$ such that $(a - \epsilon)_+ \precsim (b - \delta)_+$.*
- *For every $\epsilon > 0$, there are $x \in A$ and $\delta > 0$ such that $(a - \epsilon)_+ = x^*x$ and $xx^* \in \mathrm{her}((b - \delta)_+)$.*

- *For every* $\epsilon > 0$*, there are* $\delta > 0$ *and* $r \in A$ *such that* $(a-\epsilon)_+ = r(b-\delta)_+ r^*$.

Proposition 1.2.5. *Let* A *be a* C^**-algebra and* a *be a positive element in* $A \otimes \mathbb{K}$*. Then the following hold:*

(i) $[a] = \sup_{\epsilon>0}[(a-\epsilon)_+] = \sup_n[(a-1/n)_+]$,

(ii) for any $\epsilon > 0$, $[(a-\epsilon)_+] \ll [a]$.

Proof. (i) follows directly from the definition of supremum in a partially ordered set and Rørdam Lemma Theorem 1.2.4.
(ii) is Remark 2.62 of [70]. □

One important class of C^*-algebras is the one for which the Cuntz semigroup has the following property

Definition 1.2.6 ([70] Definition 6.33)**.** *Let* $(S, +, \leqslant) = S$ *be a partially ordered semigroup. Then* S *is almost unperforated if whenever* s *and* t *are elements in* S *for which there is an* $n \in \mathbb{N}$ *satisfying* $(n+1)s \leqslant nt$*, then it follows* $s \leqslant t$.

One of the most striking signs of the importance of the Cuntz semigroup is the Toms-Winter conjecture. As is customary we say that a C^*-algebra A is $\mathcal{Z}$-stable if $A \otimes \mathcal{Z} \simeq A$, where $\mathcal{Z}$ is the Jiang-Su algebra (see [41]) and we refer to [78] for the definition of nuclear dimension.

Conjecture 1.2.7 (Toms-Winter)**.** *Let* A *be a separable, simple, nonelementary (i.e. not* $*$*-isomorphic to* $\mathbb{C}$, $\mathbb{K}$ *or* $M_n(\mathbb{C})$ *for some* $n \in \mathbb{N}$*) nuclear* C^**-algebra. Then the following are equivalent*

(i) A *has finite nuclear dimension,*

(ii) A *is* $\mathcal{Z}$*-stable,*

(iii) $Cu(A)$ *is almost unperforated.*

Partial confirmations of this conjecture have been obtained by many authors. For example Winter showed in [77] that (i) implies (ii) and Rørdam showed in [64] that (ii) implies (iii). Furthermore the conjecture is true for purely infinite C^*-algebras.

There is a natural notion of functional on the Cuntz semigroup of a C^*-algebra and in the next section we will see how this concept is related to quasitraces on the C^*-algebra

Definition 1.2.8 ([70] Definition 6.12)**.** *Let A be a C^*-algebra and $Cu(A)$ its Cuntz semigroup. A functional on $Cu(A)$ is a map $\lambda : Cu(A) \to [0, \infty]$ that preserves the order, addition, the zero element and suprema of increasing sequences. Denote by $F(Cu(A))$ the set of functionals on $Cu(A)$.*

1.3 Traces and functionals

In this section we recall some definitions and results concerning quasitraces on C^*-algebras and functionals on Cuntz semigroups. All the material of this section is taken from [70] Chapter 6.

Definition 1.3.1. *Let A be a C^*-algebra. A 1-quasitrace on A is a map $\tau : A_+ \to [0, \infty]$ that is homogeneus, additive on commuting elements and that satisfies the trace property $\tau(xx^*) = \tau(x^*x)$ for every $x \in A$.*
For $n > 1$, an n-quasitrace on A is a 1-quasitrace $\tau : A_+ \to [0, \infty]$ that extends to a 1-quasitrace on $M_n \otimes A$.
A trace on A is a 1-quasitrace that is additive (not only on commuting elements).
If τ is a trace or an n-quasitrace with $n \geqslant 1$, we will say that τ is lower semicontinuous if for every $t \in \mathbb{R}_+$, the set $\tau^{-1}([0, t]) \subset A_+$ is closed.

In [12] it was shown that any 2-quasitrace τ on a C^*-algebra A is an n-quasitrace for every $n > 1$. Moreover the extension of τ to $M_n \otimes A$ is unique and is of the form $\tau \otimes \mathrm{tr}_n$, where tr_n denotes the standard trace on M_n. It follows that every 2-quasitrace τ on A extends to a 1-quasitrace on $A \otimes \mathbb{K}$, which we will still denote by τ.
It is not true that every 1-quasitrace on a C^*-algebra A is a 2-quasitrace; a counterexample with A unital was echibited in [37].
Furthermore, in the case A is exact, every 2-quasitrace on A is a trace by [43].

For a C^*-algebra A we will denote by 2-$QT(A)$ the set of lower semicontinuous 2-quasitraces on A. and by $T(A)$ the set of lower semicontinuous traces on A.

Theorem 1.3.2 ([70] Theorem 6.17)**.** *Let A be a C^*-algebra. Then there is a bijection*

$$\kappa : \begin{array}{ccc} 2\text{-}QT(A) & \to & F(Cu(A)) \\ \tau & \mapsto & d_\tau, \end{array}$$

where

$$d_\tau([a]) = \sup_n \tau(f_n(a))$$

for $a \in (A \otimes \mathbb{K})_+$*. Its inverse is*

$$\kappa^{-1}: \begin{array}{ccc} F(Cu(A)) & \to & 2\text{-}QT(A) \\ \lambda & \mapsto & \tau_\lambda, \end{array},$$

where

$$\tau_\lambda(a) = \int_0^\infty \lambda([(a-t)_+])dt$$

for $a \in (A \otimes \mathbb{K})_+$*.*

In the case A is a commutative C^*-algebra, every 2-quasitrace is a trace and the Riesz Theorem gives a way to compute the range of κ.

Definition 1.3.3 ([70] page 42)**.** *Let X be a locally compact Hausdorff space. An extended Radon measure on X is a positive Borel measure $\mu : \mathcal{B}(X) \to [0,\infty]$ that is inner regular on all Borel sets (i.e. the measure of any Borel set can be approximated by the measure of its compact subsets) and outer regular on compact sets (i.e. the measure of any compact set can be approximated by the measure of the open sets containing it). We denote by $M_+(X)$ the set of extended Radon measures on X.*

Proposition 1.3.4 ([70] Proposition 6.4)**.** *Let X be a locally compact Hausdorff space. There is a bijection*

$$\delta : M_+(X) \to T(C_0(X))$$

where $\delta(\mu)$ is integration with respect to μ for $\mu \in M_+(X)$.
For any $f \in C_0(X)$ and any extended Radon measure μ, the corresponding functional on $Cu(C_0(X))$ gives

$$d_{\delta(\mu)}([f]) = \mu(\overset{\circ}{\operatorname{supp}}(f)).$$

1.4 Crossed products

Let G be a topological group and A a C^*-algebra. An *action* of G on A is a group homomorphism $\alpha : G \to \operatorname{Aut}(A)$ such that for every $a \in A$ the map $g \mapsto \alpha_g(a)$ is continuous.

A natural choice of morphisms in the category of C^*-algebras admitting an action of a fixed topological group G is given by the following

Definition 1.4.1. *Let G be a topological group and A, B be C^*-algebras on which G acts by means of actions $\alpha : G \to \operatorname{Aut}(A)$ and $\beta : G \to \operatorname{Aut}(B)$. A $*$-homomorphism $\phi : A \to B$ is called equivariant if $\phi \circ \alpha_g = \beta_g \circ \phi$ for every $g \in G$.*

For the construction of the crossed product, we will need to restrict our attention to locally compact groups. In this case, if $\alpha : G \to \mathrm{Aut}(A)$ is an action of a group G on a C^*-algebra A, we will refer to the triple (A, G, α) as to a *C^*-dynamical system.*

First of all we define what will be a dense $*$-subalgebra of the crossed product C^*-algebra we will define.

Let G be a locally compact group and $\alpha : G \to \mathrm{Aut}(A)$ an action on a C^*-algebra A. Let $C_c(G, A)$ be the set of continuous compactly supported functions from G to A; we define the "twisted" convolution product

$$(x * y)(g) \coloneqq \int_G x(h)\alpha_h(y(h^{-1}g))d\mu(h),$$

where μ is a fixed left Haar measure on G, x, y are elements in $C_c(G, A)$ and g is an element in G. Let Δ be the modular function of G. Define

$$x^*(g) \coloneqq \Delta(g)^{-1}\alpha_g(x(g^{-1})^*).$$

With these operations $C_c(G, A)$ becomes a $*$-algebra. Define $L^1(G, A)$ to be the completion of $C_c(G, A)$ with respect to the norm

$$\|x\|_1 \coloneqq \int_G \|x(g)\|d\mu(g).$$

Then for every x and y in $L^1(G, A)$ we have $\|x * y\|_1 \leqslant \|x\|_1\|y\|_1$ and $\|x\|_1 = \|x^*\|_1$, that is, $L^1(G, A)$ is an involutive Banach algebra.

Definition 1.4.2 ([53] 7.4.8)**.** *Let G be a locally compact group. A unitary representation of G on a Hilbert space $\mathfrak{H}$ is a group homomorphism $v : G \to U(\mathfrak{H})$ that is continuous for the strong operator topology on $U(\mathfrak{H})$.*
If $\alpha : G \to \mathrm{Aut}(A)$ is an action on a C^*-algebra A, a covariant representation of (A, G, α) on a Hilbert space $\mathfrak{H}$ is a pair (π, v), where $\pi : A \to \mathbb{B}(\mathfrak{H})$ is a $*$-representation of A and $v : G \to U(\mathfrak{H})$ is a unitary representation, satisfying the covariance condition

$$v(g)\pi(a)v(g)^* = \pi(\alpha_g(a))$$

for all $g \in G$ and $a \in A$. (π, v) is called nondegenerate if π is nondegenerate.

Let (π, v) be a covariant representation of a C^*-dynamical system (A, G, α) on a Hilbert space $\mathfrak{H}$. For $x \in C_c(G, A)$ define the operator $\int_G \pi(x(g))v(g)d\mu(g) \in \mathbb{B}(\mathfrak{H})$

by $(\int_G \pi(x(g))v(g)d\mu(g))\xi = \int_G \pi(x(g))v(g)\xi d\mu(g)$ for every $\xi \in \mathfrak{H}$. The map

$$\pi \rtimes v : \begin{array}{ccc} C_c(G, A) & \to & \mathbb{B}(\mathfrak{H}) \\ x & \mapsto & \int_G \pi(x(g))v(g)d\mu(g) \end{array}$$

is continuous in the norm $\|\cdot\|_1$ and thus extends to a representation of $L^1(G, A)$, which we still denote by $\pi \rtimes v$ and call the *integrated form* of (π, v).

Theorem 1.4.3 ([53] Proposition 7.6.4)**.** *Let $\alpha : G \to \mathrm{Aut}(A)$ be an action of a locally compact group G on a C^*-algebra A. The integrated form construction defines a bijection between the nondegenerate covariant representations of (A, G, α) and the continuous nondegenerate representations of $L^1(G, A)$.*

Definition 1.4.4. *Let $\alpha : G \to A$ be an action of a locally compact group G on a C^*-algebra A. The (full) crossed product C^*-algebra $A \rtimes_\alpha G$ is the completion of $L^1(G, A)$ with respect to the norm*

$$\|x\| := \sup_{\tilde{\pi}} \|\tilde{\pi}(x)\| = \sup_{\pi \rtimes v} \|\pi \rtimes v(x)\|,$$

where the sup *is taken over all the nondegenerate continuous representations of $L^1(G, A)$ or equivalently over all the integrated forms of nondegenerate covariant representations of (A, G, α). Whenever the action is clear we may just write $A \rtimes G$.*

Let again $\alpha : G \to \mathrm{Aut}(A)$ be an action of a locally compact group G on a C^*-algebra A. Choose a faithful nondegenerate representation π of A on a Hilbert space $\mathfrak{H}_0$ and for $x \in A$, $g \in G$ define operators in $\mathbb{B}(\mathfrak{H}_0 \otimes L^2(G))$ by

$$(\pi_\alpha(x)\xi)(g) := \alpha_g(x)(\xi(g)),$$

$$(\lambda(h)\xi)(g) := \xi(h^{-1}g)$$

for every $\xi \in \mathfrak{H}_0 \otimes L^2(G)$. Then (π_α, λ) is a nondegenerate covariant representation of (A, G, α). By [53] Theorem 7.7.5 for any $x \in L^1(G, A)$, the value $\|\pi_\alpha(x)\|$ does not depend on the faithful representation π.

Definition 1.4.5. *Let $\alpha : G \to A$ be an action of a locally compact group G on a C^*-algebra A. The reduced crossed product C^*-algebra $G \rtimes_{r,\alpha} A$ is the competion of $L^1(G, A)$ with respect to the norm*

$$\|x\|_r := \|\pi_\alpha \rtimes \lambda(x)\|$$

Whenever the action is clear we may just write $A \rtimes_r G$.

By definition there always is a surjective $*$-homomorphism $A \rtimes G \to A \rtimes_r G$.

It is well known that in particular cases $A \rtimes_r G$ and $A \rtimes G$ are $*$-isomorphic. Suppose we are given a locally compact group G. Fix a left Haar measure μ on G; for every $f \in L^1(G, \mu) = L^1(G)$ and for every $g \in G$ denote by ${}_g f$ the function in $L^1(G)$ given by ${}_g f(h) = f(g^{-1}h)$ for every $h \in G$.

Definition 1.4.6. *Let G be a locally compact group. G is said to be amenable if for every $\epsilon > 0$ and every compact subset $K \subset G$ there is a $\phi \in L^1(G)$ with $\|\phi\|_1 = 1$ such that*

$$\|{}_g\phi - \phi\|_1 < \epsilon \qquad \forall g \in K.$$

Note that by the unicity of the Haar measure up to a scaling constant, the definition of amenability does not depend on the choice of μ. The following is well known

Theorem 1.4.7 ([2] Theorem 5.3)**.** *Let $\alpha : G \to A$ be an action of an amenable locally compact group on a C^*-algebra A. Then*

$$A \rtimes_r G \simeq A \rtimes G$$

and they are both nuclear.

Since we are interested in crossed product C^*-algebras arising from actions of the group of real numbers, we state the following

Theorem 1.4.8 ([36] Theorem 1.2.1)**.** *Every locally compact abelian group is amenable.*

We will also be interested in the existence of traces on crossed product C^*-algebras and in particular in which cases they are unique.

Theorem 1.4.9 ([49] Theorem 6.30)**.** *Let G be a simply connected Lie group acting freely on a connected compact manifold M. Then the set of lower semicontinuous traces on $C(M) \rtimes_r G$ is in bijection with the transverse measures of the associated foliation.*

1.5 Morita equivalence

Let A be a σ-unital C^*-algebra and p a projection in $M(A)$. Following [15] Lemma 2.5, there is an element $v \in M(A \otimes \mathbb{K})$ such that $v^*v = 1 \otimes e_{1,1}$ and $vv^* = p \otimes 1$.

Hence if $B \subset A$ is the full corner pAp, the map

$$\begin{array}{ccc} B \otimes \mathbb{K} & \to & A \otimes \mathbb{K} \\ x & \mapsto & v^* x v \end{array}$$

is a $*$-isomorphism. The passage from corners to full hereditary C^*-subalgebras is made by means of the *linking algebra*. If $B \subset A$ is a full hereditary σ-unital C^*-subalgebra, then the corresponding linking algebra C is given in matrix form by

$$C = \begin{pmatrix} B & \overline{BA} \\ \overline{AB} & A \end{pmatrix}.$$

Since B and A are both full corners of C, we obtain the followin

Theorem 1.5.1 ([15] Theorem 2.8)**.** *Let $B \subset A$ be a full hereditary C^*-subalgebra of a C^*-algebra A. Then*

$$B \otimes \mathbb{K} \simeq A \otimes \mathbb{K}.$$

The same procedure applies to the situation in which A and B are two C^*-algebras that are *Morita equivalent* (see [17]).

Definition 1.5.2. *[[57] Definition 2.8] Let A be a C^*-algebra. A right Hilbert module over A is a complex Banach space E endowed with the structure of a right (algebraic) A-module and with an A-valued map*

$$\langle \cdot, \cdot \rangle_A : E \times E \to A$$

that is linear in the second variable and such that

$$\begin{aligned} \langle \xi, \eta \rangle_A^* &= \langle \eta, \xi \rangle_A, \\ \langle \xi, \eta \rangle_A a &= \langle \xi, \eta \cdot a \rangle_A, \\ \|\xi\|^2 &= \|\langle \xi, \xi \rangle_A\| \end{aligned}$$

for every $\xi, \eta \in E$ and $a \in A$. The map $\langle \cdot, \cdot \rangle_A$ is called an A-valued inner product. In a similar way one defines a left Hilbert module over A and we denote the corresponding A-valued inner product with ${}_A\langle \cdot, \cdot \rangle$ in this case .
A right (left) Hilbert module over A is said to be full if the span of the range of the right (left) A-valued inner product is dense in A.

By a Hilbert module over A we will always mean a right Hilbert module over A. Note that if $B \subset A$ is a full corner such that $B = pAp$, $p \in M(A)$, then pA is a

Hilbert module over A under the action $\xi \cdot a = \xi a$ and the inner product $\langle \xi, \eta \rangle_A = \xi^* \eta$ for $\xi, \eta \in pA$ and $a \in A$. This is the prototype example of an imprimitivity bimodule.

Definition 1.5.3 ([57] Definition 3.1). *Let A and B be C^*-algebras. An $A - B$ imprimitivity bimodule X is a complex Banach space with the structure of a right Hilbert module over B and a left Hilbert module over A, that is full with respect to both inner products and such that*

$$_A\langle \xi, \eta \rangle \cdot \zeta = \xi \cdot \langle \eta, \zeta \rangle_B$$

for all ξ, η and $\zeta \in X$.
Two C^*-algebras A and B are said to be Morita equivalent ($A \simeq_{M.e.} B$) if there exists an $A - B$ imprimitivity bimodule.

Let A, B and C be C^*-algebras. As one would expect Morita equivalence is an equivalence relation ([57] Proposition 3.18):

- A is an $A - A$ imprimitivity bimodule with the inner products defined as

  $$_A\langle a, b \rangle = ab^*, \qquad \langle a, b \rangle_A = a^* b$$

  for $a, b \in A$;

- If X is an $A - B$ imprimitivity bimodule and Y is a $B - C$ imprimitivty bimodule, then one can define an $A - C$ imprimitivity bimodule $X \otimes_B Y$ (see [57] Proposition 3.16);

- If X is an $A - B$ imprimitivity bimodule, then one can construct a $B - A$ imprimitivity bimodule $X^* = \{\xi^*, : \xi \in X\}$ with

  $$_B\langle \xi^*, \eta^* \rangle := \langle \xi, \eta \rangle_B$$

  for all $\xi^*, \eta^* \in X^*$.

Hence if two C^*-algebras A and B are both full corners of another C^*-algebra C, they are Morita equivalent (and also stably isomorphic by Brown's Theorem). Let now X be an $A - B$ imprimitivity bimodule and set

$$L(X) := \begin{pmatrix} A & X \\ X^* & B \end{pmatrix}$$

with multiplication given by

$$\begin{pmatrix} a_1 & \xi_1 \\ \eta_1^* & b_1 \end{pmatrix} \begin{pmatrix} a_2 & \xi_2 \\ \eta_2^* & b_2 \end{pmatrix} = \begin{pmatrix} a_1a_2 +_A \langle \xi_1, \eta_2 \rangle & a_1\xi_2 + \xi_1 b_2 \\ \eta_1^* a_2 + b_1 \eta_2^* & \langle \eta_1, \xi_2 \rangle_B + b_1 b_2 \end{pmatrix}$$

and involution given by

$$\begin{pmatrix} a & \xi \\ \eta^* & b \end{pmatrix}^* = \begin{pmatrix} a^* & \eta \\ \xi^* & b^* \end{pmatrix}.$$

By [17] Theorem 1, $L(X)$ can be completed to a C^*-algebra and A and B are full corners of $L(X)$ in the case they are both σ-unital. Hence for σ-unital C^*-algebras we have the following

$$A \otimes \mathbb{K} \simeq B \otimes \mathbb{K} \Leftrightarrow A \text{ and } B \text{ are full corners in a } C^*\text{-algebra } C \ \Leftrightarrow A \simeq_{M.e} B.$$

A particular situation that will be important for us comes from actions of locally compact groups. Let G be a locally compact group and H, K two closed subgroups. Denote respectively by $\mathrm{lt} : K \to \mathrm{Aut}(C_0(G/H))$ the action induced by the left multiplication for elements of K and by $\mathrm{rt} : H \to \mathrm{Aut}(C_0(K\backslash G))$ the action induced by right multiplication for elements of H. The proof of the following can be found in [75] Corollary 4.10.

Proposition 1.5.4 (Green's Symmetric Imprimitivity Theorem)**.** *Let G be a locally compact group and H, K two closed subgroups. Then*

$$C_0(G/H) \rtimes_{\mathrm{lt}} K \simeq_{M.e.} C_0(K\backslash G) \rtimes_{rt} H.$$

Denote by Δ_H and Δ_K the modular functions of H and K respectively. Let $A = C_0(G/H) \rtimes_{\mathrm{lt}} K$ and $B = C_0(K\backslash G) \rtimes_{rt} H$. The linear space $C_c(G)$ can be completed to an A-B-imprimitivity bimodule. The actions and inner products are given by

$$(x \cdot \xi)(g) = \int_K x(k, gH) xi(k^{-1}g) \Delta_K(k)^{1/2} d\mu_K(k),$$

$$(\xi \cdot y)(g) = \int_H \xi(gh^{-1}) y(h, Kgh^{-1}) \Delta_H(h)^{-1/2} d\mu_H(h),$$

$${}_A\langle \xi, \eta \rangle(k, gH) = \Delta_K(k)^{-1/2} \int_H \xi(gh) \overline{\eta(k^{-1}gh)} d\mu_H(h),$$

$$\langle \xi, \eta \rangle_B(h, Kg) = \Delta_H(h)^{-1/2} \int_K \overline{\xi(k^{-1}g)} \eta(k^{-1}gh) d\mu_K(k)$$

for all $x \in C_c(K, C_0(G/H))$, $y \in C_c(H, C_0(K\backslash G))$ *and* ξ, $\eta \in C_c(G)$.

Chapter 2

Cuntz comparison and hereditary C^*-subalgebras

The Cuntz semigroup of a C^*-algebra can be equivalently defined in terms of equivalence classes of positive elements in its stabilization (see Section 1.1) or in terms of equivalence classes of countably generated Hilbert modules over it (see [22]). Alternatively, as pointed out in [22] on page 2, one could consider singly generated hereditary C^*-subalgebras in the stabilization.
With this in mind, we observe that the right analogue of a properly infinite full element (see Definition 2.1.1) in the stabilization of a C^*-algebra is a hereditary large C^*-subalgebra (see Definition 2.1.4 and Proposition 2.2.3).
The notion of largeness for hereditary C^*-subalgebras was introduced by Rørdam in [63] Definition 2.4. Using this concept he proved that the positive cone and the scale of the K_0-group of a stable C^*-algebra coincide (see the discussion after Definition 2.4 and at the beginning of Section 3 in [63]), the point being that a stable C^*-algebra is large in its stabilization.
In the first section we study largeness for hereditary C^*-subalgebras of general C^*-algebras, giving a characterization in terms of Cuntz equivalence in Proposition 2.1.9 and we observe that largeness does not pass to the stabilization with an example (see Proposition 2.1.13 and Example 2.1.15)
In the second section we focus on the case of hereditary singly generated C^*-subalgebras of stable σ-unital C^*-algebras. In this case largeness passes to the stabilization and we find, using Brown's result, an expression for the approximants appearing in the definition of largeness in terms of elements of the starting large subalgebra in Proposition 2.2.2.
In the third section we investigate the connection between largeness in stable C^*-algebras and stability in the σ-unital case under assumptions on the stable rank (see

Proposition 2.3.10 and Proposition 2.3.11). In view of this we will be able to give a partial answer to a question by Kirchberg and Rørdam (Question 2.3.1) in Corollary 2.3.12.
In the last section of this chapter, we try to retrieve information about the hereditary C^*-subalgebras of the stabilization of a C^*-algebra that are not large, for a particular class of C^*-algebras.
We see that under some extra assumptions on a C^*-algebra A which include the presence of a unique lower semicontinuous 2-quasitrace, we can prove in Theorem 2.4.12 that any singly generated hereditary C^*-subalgebra of $A \otimes \mathbb{K}$ is either algebraically simple or stable.
The counterpart of this result at the level of Hilbert modules is Theorem 2.4.19; namely, every countably generated Hilbert module over $A \otimes \mathbb{K}$ is either isomorphic to $A \otimes \mathbb{K}$ (as Hilbert modules over $A \otimes \mathbb{K}$) or is compactly contained in it (see Definition 2.3.6).
We also obtain a description of the Pedersen ideal of such C^*-algebras in terms of the unique lower semicontinuous 2-quasitrace in Proposition 2.4.11.

2.1 Largeness for hereditary C^*-subalgebras

In this section we recall a concept of largeness for hereditary C^*-subalgebras that was introduced in [63] in order to prove that for a stable C^*-algebra A its scale $D_0(A)$ is equal to the positive part of $K_0(A)$. We will see a characterization in terms of Cuntz equivalence (Proposition 2.1.9) and recall a property of large hereditary C^*-subalgebras in Proposition 2.1.11. By Proposition 2.1.13, in the σ-unital case, the condition that a hereditary C^*-subalgebra B of a C^*-algebra A is large in $A \otimes \mathbb{K}$ is equivalent to the condition that every strictly positive element in B is properly infinite. In particular largeness does not pass in general to the stabilization by Example 2.1.15.

We recall two concepts that will be important in the following sections

Definition 2.1.1 ([44] Definition 3.2). *A positive element a in a C^*-algebra A is called properly infinite if $a \oplus a \precsim a$.*

Lemma 2.1.2 (Proposition 3.5 of [44]). *Let a be a properly infinite element in a C^*-algebra A. Then $a \otimes 1_n \precsim a$ for every $n \in \mathbb{N}$.*

Proof. If $a \otimes 1_n \precsim a$ for some $n \in \mathbb{N}$, then

$$a \otimes 1_{n+1} = a \oplus (a \otimes 1_n) \precsim a \oplus a \precsim a.$$

Hence the conclusion follows by induction. □

The following is contained in [44] and gives a link between two different concept of infiniteness; we will see in the following sections that under some assumptions these two concepts coincide. We recall that a unital C^*-algebra A is said to be properly infinite if $1 \in A$ is a properly infinite element (see [44] 3).

Lemma 2.1.3 ([44] Proposition 3.3)**.** *Let A be a C^*-algebra whose multiplier algebra $M(A)$ is properly infinite and let h be a strictly positive element in A. Then h is properly infinite.*

Proof. Note that for every $z \in A$ we have $zz^* \sim zz^*zz^* \precsim z^*z$ and $z^*z \sim z^*zz^*z \precsim zz^*$, thus $zz^* \sim z^*z$. Let s_1 and s_2 be isometries in $M(A)$ with orthogonal ranges and set $x := s_1h^{1/2}$, $y := s_2h^{1/2}$. Then we have

$$\begin{aligned}
\begin{pmatrix} h & 0 \\ 0 & h \end{pmatrix} &= \begin{pmatrix} x^*x & x^*y \\ y^*x & y^*y \end{pmatrix} \\
&= \begin{pmatrix} x^* & 0 \\ y^* & 0 \end{pmatrix} \begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix} \\
&\sim \begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x^* & 0 \\ y^* & 0 \end{pmatrix} \\
&= \begin{pmatrix} s_1hs_1^* + s_2hs_2^* & 0 \\ 0 & 0 \end{pmatrix} \\
&\precsim \begin{pmatrix} h & 0 \\ 0 & 0 \end{pmatrix}. \quad \square
\end{aligned}$$

The other fundamental concept of this chapter is the following

Definition 2.1.4 ([63] Definition 2.4)**.** *Let A be a C^*-algebra and B a hereditary C^*-subalgebra of A. B is large in A if for every positive element $a \in A$ and every $\epsilon > 0$ there exists $x \in A$ such that $\|a - xx^*\| < \epsilon$ and x^*x belongs to B.*

The following result makes clear why in the two examples of hereditary large C^*-subalgebra that we will give, the fullness assumption will be necessary. It is stated without proof in the discussion following Definition 2.4 of [63].

Lemma 2.1.5. *Let B be a hereditary large C^*-subalgebra of a C^*-algebra A. Then B is full in A.*

Proof. Let a be a positive element in A and $\epsilon > 0$. Take $\epsilon' > 0$ such that $\epsilon'(2\|a^{1/2}\| + \epsilon') < \epsilon$ and $x \in A$ such that $\|xx^* - a^{1/2}\| < \epsilon'$. Then, since $\|xx^*\| < \|a^{1/2}\| + \epsilon'$, we have

$$\begin{aligned}\|x(x^*x)x^* - a\| &= \|xx^*(xx^* - a^{1/2}) + (xx^* - a^{1/2})a^{1/2}\| \\ &\leqslant \|xx^*\|\|xx^* - a^{1/2}\| + \|xx^* - a^{1/2}\|\|a^{1/2}\| \\ &\leqslant (\|a^{1/2}\| + \epsilon')\epsilon' + \|a^{1/2}\|\epsilon' < \epsilon.\end{aligned}$$

The result follows since every element in A is a linear combination of positive elements. □

Two examples of large subalgebras are the following:

Example 2.1.6. *[[63] Lemma 2.5] Any full, hereditary C^*-subalgebra of a purely infinite C^*-algebra;*

Example 2.1.7. *[[63] Lemma 2.6] Any full, stable, hereditary C^*-subalgebra of a C^*-algebra.*

Note that in both cases, if we assume σ-unitality, any strictly positive element in the large subalgebras would be properly infinite:

- If A is a purely infinite C^*-algebra, then any positive element in A is properly infinite by [44] Theorem 4.16; hence this is true in particular for any strictly positive element in any full hereditary C^*-subalgebra.

- If B is a stable C^*-algebra, then its multiplier algebra is properly infinite, since it contains the C^*-algebra of bounded operators on an infinite dimensional Hilbert space as a unital C^*-subalgebra; hence any strictly positive element in B is properly infinite in virtue of Lemma 2.1.3.

These two examples suggest a link between the property of a hereditary full C^*-subalgebra of a C^*-algebra A to be large and the property that any strictly positive element in B be properly infinite. The proper infiniteness of an element is a property that depends only on the C^*-algebra B and not on the possibly larger C^*-algebra A. Hence the two properties are not likely to be equivalent in general. We will see in Corollary 2.1.14 that the existence of a properly infinite full element in B is enough to guarantee largeness.

In the following we will often need to specify when the hereditary C^*-algebras we consider are σ-unital. By the following Lemma, this is the case exactly when B is singly generated.

Lemma 2.1.8. *Let B be a hereditary σ-unital C^*-subalgebra of a C^*-algebra A and let h be a strictly positive element for B. Then B is the hereditary C^*-subalgebra of A generated by h and h is full in A if and only if B is full in A.*
Moreover, if B is a singly generated hereditary C^*-subalgera of A, then it is σ-unital.

Proof. From the sequence of inclusions

$$B = BAB = \overline{hBh}A\overline{hBh} \subset \overline{hAh} \subset BAB = B$$

we see that $B = \mathrm{her}(h)$.
Suppose now that B is full in A. Let a be an element in A and $\epsilon > 0$; there are elements a_i, c_i in A and b_i in B, $i = 1, ..., n$ such that $\|a - \sum_{i=1}^n a_i b_i c_i\| < \epsilon/2$. For every $1 \leqslant i \leqslant n$ there is a $d_i \in B$ such that $\|b_i - hd_ih\| \leqslant \epsilon/(2\|a_i\|\|c_i\|)$. Hence we have

$$\begin{aligned}
\|a - \sum_{i=1}^n a_i h(d_i h c_i)\| &\leqslant \|a - \sum_{i=1}^n a_i b_i c_i\| + \|\sum_{i=1}^n a_i b_i c_i - \sum_{i=1}^n a_i h d_i h c_i\| \\
&\leqslant \|a - \sum_{i=1}^n a_i b_i c_i\| + \sum_{i=1}^n \|a_i\|\|b_i - \sum_{i=1}^n h d_i h\|\|c_i\| \\
&\leqslant \epsilon.
\end{aligned}$$

Thus h is full in A. Clearly if h is full in A then B is full in A.
Suppose that B is the hereditary C^*-subalgebra of A generated by h. Using an approximate unit for A we see that h^2 belongs to B and since B is a C^*-algebra, also h belongs to B. Furthermore, $h^{1/n}$ is an approximate unit for B. □

Now we pass to the characterization of largeness in terms of Cuntz equivalence.

Proposition 2.1.9. *Let B be a hereditary C^*-subalgebra of a C^*-algebra A. The following conditions are equivalent:*

(i) *B is large in A.*

(ii) *For every positive a in A and any $\epsilon > 0$ there is an element $a_\epsilon \in B$ such that $(a - \epsilon)_+ \sim a_\epsilon$.*

(iii) $(A_+)_\sim = \{[a] = \sup_n [a_n] \ : \ a_n \in B_+\}$.

If B is σ-unital and h is a strictly positive element for B, the above conditions are equivalent to.

(iv) $a \precsim h$ for every positive a in A.

Proof. (i)⇒(ii): Let a be a positive element in A and take $\epsilon > 0$. Since B is large in A, there exists $x \in A$ such that $\|a - x^*x\| \leqslant \epsilon$ and xx^* belongs to B; hence there is an $r \in A$ such that

$$(a - \epsilon)_+ = r^*x^*xr \sim xrr^*x^* \leqslant \|r\|^2 xx^* \in B.$$

Since B is hereditary the element xrr^*x^* belongs to B and (ii) follows.
(ii)⇒(iii): Suppose that (ii) hols and let a be a positive element in A. Since we can write $[a] = \sup_n[(a - 1/n)_+]$ and for each $n \in \mathbb{N}$ there is a positive element $a_{1/n}$ in B such that $(a - 1/n)_+ \sim a_{1/n}$, it follows that $[a] = \sup_n[a_{1/n}]$.
(iii)⇒(i): Let a be a positive element in A; by assumption we can write $[a] = \sup_n[a_n]$ with $a_n \in B_+$ for every $n \in \mathbb{N}$. Let $\epsilon > 0$; choose n large enough in order to have $1/n < \epsilon$ and $\epsilon' > 0$ small enough in order to have $1/n + \epsilon' = \epsilon$. Since $[(a - 1/n)_+] \ll [a]$, we see that, by definition of compact containment, there is an $m \in \mathbb{N}$ such that $(a - 1/n)_+ \precsim a_m$; in particular there is an element $x \in A$ such that $\|(a - 1/n)_+ - xa_mx^*\| < \epsilon'$ and so we can find $s \in A$ such that $((a - 1/n)_+ - \epsilon')_+ = (a - (1/n + \epsilon'))_+ = (a - \epsilon)_+ = (sr)a_m(sr)^*$. The element $x \coloneqq (sr)a_m^{1/2}$ is such that $\|a - xx^*\| < \epsilon$ and $x^*x = a_m^{1/2}(sr)^*(sr)a_m^{1/2} \leqslant \|sr\|^2 a_m$ belongs to B.
(ii)⇒(iv): Since B is the hereditary C^*-algebra generated by h (see Lemma 2.1.8), it follows that any element in B is Cuntz subequivalent to h. Thus, given a positive in A, since $(a-\epsilon)_+ \precsim h$ for any $\epsilon > 0$, it follows by Rørdam Lemma 1.2.4 that $a \precsim h$.
(iv)⇒(i): Let a be a positive element in A and let $\epsilon > 0$ be given. Since $a \precsim h$, there is an element $r \in A$ such that $\|a - rhr^*\| < \epsilon$; put $x = h^{1/2}r^*$. Then $\|a - x^*x\| < \epsilon$ and $xx^* = h^{1/2}r^*rh^{1/2}$ belongs to B. □

Corollary 2.1.10. *Let A be a stable C^*-algebra and B a hereditary large C^*-subalgebra. Then for every positive element a in A, there exist a sequence of positive elements b_n in B and a sequence of unitaries u_n in $\tilde{A}$ such that*

$$a = \lim_n u_n^* b_n u_n.$$

Proof. Using [70] Corollary 2.56 we see that if B is large in A, for every $n \in \mathbb{N}$ there exist a unitary u_n in $\tilde{A}$ and an element b_n in B such that $u_n(a-1/n)_+u_n^* = b_n$. Thus

$a = \lim_n (a - 1/n)_+ = \lim_n u_n^* b_n u_n$. □

The following property of hereditary large C^*-subalgebras was shown by Rørdam in [63] in the discussion after Definition 2.4 and at the beginning of chapter 3.

Proposition 2.1.11. *Let A be a C^*-algebra and B be a large hereditary C^*-subalgebra of $A \otimes \mathbb{K}$. Then*

$$K_0(A)^+ = \{[p]_0 \ : \ p \in B\}.$$

Proof. We reproduce the proof given by Rørdam. Let $p \in A \otimes \mathbb{K}$ be a projection and let $x \in A \otimes \mathbb{K}$ be such that $\|x^*x - p\| < 1/2$ and $xx^* \in B$. Then there is $z \in A \otimes \mathbb{K}$ such that $z^*x^*xz = p$ and $xzz^*x^* \in B$; thus every projection in $A \otimes \mathbb{K}$ is Murray-von Neumann equivalent to a projection in B. □

We will see now that for a hereditary C^*-subalgebra, the property of admitting properly infinite strictly positive elements is at least as strong as the property of being large. Before we state the following

Lemma 2.1.12. *Let B be a full hereditary C^*-subalgebra of a C^*-algebra A and p a rank-one projection in $\mathbb{K}$. Then $B \otimes p$ is full in $A \otimes \mathbb{K}$.*

Proof. Let a be an element in $A \otimes \mathbb{K}$ and let $\epsilon > 0$. There are k, $n \in \mathbb{N}$ and $a_1, ..., a_k \in A$, $x_1, ..., x_k \in M_n$ such that $\|a - \sum_{i=1}^k a_i \otimes x_i\| < \epsilon/2$. For every $1 \leqslant i \leqslant k$ there are $m_i \in \mathbb{N}$, $s_i \in \mathbb{N}$ and y_{l_i}, z_{l_i} in M_n, $l_i = 1, ..., m_i$, c_{j_i}, d_{j_i} in A, b_{j_i} in B, $j_i = 1, ..., s_i$ such that $x_i = \sum_{i=1}^{m_i} y_{l_i} p z_{l_i}$ and $\|a_i - \sum_{j=1}^{s_i} c_{j_i} b_{j_i} d_{j_i}\| < \epsilon/(2\|x_i\|)$. Therefore

$$\begin{aligned}
&\|a - \sum_{i=1}^k \sum_{j=1}^{s_i} \sum_{l=1}^{m_i} (c_{j_i} \otimes y_{l_i})(b_{j_i} \otimes p)(d_{j_i} \otimes z_{l_i})\| \\
&\qquad \leqslant \|a - \sum_{i=1}^k a_i \otimes x_i\| + \|\sum_{i=1}^k a_i \otimes x_i - \sum_{i=1}^k (\sum_{j=1}^{s_i} c_{j_i} b_{j_i} d_{j_i}) \otimes x_i\| \\
&\qquad < \epsilon/2 + \sum_{i=1}^k \|a - \sum_{j=1}^{s_i} c_{j_i} b_{j_i} d_{j_i}\| \|x_i\| \\
&\qquad < \epsilon/2 + \epsilon/2 = \epsilon.
\end{aligned}$$

The Lemma is proved. □

A characterization of proper infiniteness for strictly positive elements in full hereditary C^*-subalgebras is given by the following

Proposition 2.1.13. *Let B be a full hereditary σ-unital C^*-subalgebra of a C^*-algebra A and let p be a rank-one projection in $\mathbb{K}$. Then $B \otimes p$ is a full hereditary C^*-subalgebra of $A \otimes \mathbb{K}$ and the following are equivalent:*

(i) $B \otimes p$ is a hereditary large C^*-subalgebra of $A \otimes \mathbb{K}$.

(ii) Every strictly positive element in B is properly infinite.

Proof. If B is hereditary in A, then $B \otimes p$ is hereditary in $A \otimes \mathbb{K}$ since $B \otimes p = (B \otimes p)A \otimes \mathbb{K}(B \otimes p)$. By Lemma 2.1.12 $B \otimes p$ is full in $A \otimes \mathbb{K}$. Let h be a strictly positive element in B.
(i)⇒(ii): By (i)⇒(iv) in Proposition 2.1.9 $h \oplus h \precsim h$.
(ii)⇒(i): From Lemma 2.1.8 we see that $h(\otimes p)$ is full in $A \otimes \mathbb{K}$ and so if a is an element in $(A \otimes \mathbb{K})_+$ and $\epsilon > 0$, there are $y_1, ..., y_n, z_1, ..., z_n$ such that $\|a - \sum_{i=1}^n y_i h z_i^*\| < \epsilon$. Using the polarization identity, we can find elements $x_1, ..., x_k$ such that $\|a - \sum_{i=1}^k x_i h x_i^*\| < \epsilon$. Thus there is an element $r \in A \otimes \mathbb{K}$ such that $(a - \epsilon)_+ = \sum_{i=1}^k (r x_i) h (x_i^* r^*)$ and then

$$(a - \epsilon)_+ \precsim h \otimes 1_k \precsim h,$$

where we have used Lemma 2.1.2. It follows by Rørdam Lemma that $a \precsim h$ and then by (iii)⇒(i) of Proposition 2.1.9 that $B \otimes p$ is large in $A \otimes \mathbb{K}$. □

Corollary 2.1.14. *Let B be a full hereditary σ-unital C^*-subalgebra of a C^*-algebra A and h a strictly positive element in B. If h is properly infinite, then B is large in A.*

Proof. By Proposition 2.1.13 $B \otimes p$ is large in $A \otimes \mathbb{K}$ and so in particular (using (i)⇒(iii) of Proposition 2.1.9) for every positive element a in A we have $a \precsim_{(A \otimes \mathbb{K})} h$. Now the result follows since $A \otimes p$ is a hereditary C^*-subalgebra of $A \otimes \mathbb{K}$ and so Cuntz equivalence is implemented in $A(\otimes p)$ by Proposition 1.2.1. □

Note that in both examples 2.1.6 and 2.1.7 any strictly positive element in the large hereditary C^*-subalgebra is properly infinite and by Corollary 2.1.14 the condition that a strictly positive element of a full hereditary C^*-subalgera is properly infinite is enough to guarantee largeness. It is natural to ask wether there are σ-unital hereditary large C^*-subalgebras not admitting properly infinite strictly positive elements. As one might guess, any unital infinite C^*-algebra that is not properly infinite contains such large hereditary C^*-subalgebras, as we see in the following Example. In particular, in virtue of Proposition 2.1.13, largeness does not pass in general to the stabilization.

Example 2.1.15. *Let A be a unital infinite C^*-algebra that is not properly infinite, then there is an element $s \in A$ and a projection $p \in A$ such that $1 \neq p = s^*s$ and $ss^* = 1$. Then the hereditary C^*-subalgebra pAp of A is large since $1 \sim p$ (see Proposition 2.1.9) and 1, thus p, is not properly infinite.*
Note that the class of infinite unital C^*-algebras that are not properly infinite is not empty, since the Toeplitz algebra $\mathcal{T}$ belongs to this class. To see this, observe that it is infinite since it is the universal unital C^*-algebra generated by an isometry. Furthermore, if $\pi : \mathcal{T} \to C(S^1)$ is the quotient map and if p and q are orthogonal projections in $\mathcal{T}$ such that $p \sim q$, then we should have $\pi(p) = \pi(q) = \pi(1) = 1$ since they are Murray-von Neumann equivalent and $\pi(p) \perp \pi(q)$, entailing $\pi(p) = \pi(q) = 0$, a contradiction.

2.2 Largeness in stable C^*-algebras

If A is a σ-unital stable C^*-algebra and B is a σ-unital hereditary C^*-subalgebra, using Cuntz equivalence it is easy to show that B is large in A (in the sense of [63] Definition 2.4) if and only if any strictly positive element for B is properly infinite. In what follows we want to give another proof of this result using Brown's Lemma 2.4 of [15]. The main ingredient for this is Proposition 2.2.2, where, for a hereditary σ-unital large C^*-subalgebra B of a stable σ-unital C^*-algebra A, we give a concrete expression for the approximants appearing in the definition of largeness of a positive element in $A \otimes \mathbb{K}$ by means of unitaries in the multiplier algebra $M(A \otimes \mathbb{K})$.

Lemma 2.2.1. *Let A and B be C^*-algebras and $\phi : A \to B$ a surjective $*$-homomorphism. Suppose C is a full hereditary large C^*-subalgebra of A. Then $\phi(C)$ is a full hereditary large C^*-subalgebra of B.*

Proof. Since $CAC = C$, then $\phi(C)B\phi(C) = \phi(C)\phi(A)\phi(C) = \phi(A) = B$ and $\phi(C)$ is hereditary in B.
Let b be a positive element in B and $\epsilon > 0$. Since ϕ is surjective, there is a positive element a in A such that $b = \phi(a)$. There exists $x \in A$ such that $\|a - xx^*\| < \epsilon$ and x^*x belongs to C. Thus $\phi(x)$ is such that $\|b - \phi(x)\phi(x^*)\| < \epsilon$ and $\phi(x^*)\phi(x)$

belongs to $\phi(C)$. The proof is complete. □

Proposition 2.2.2. *Let B be a hereditary large σ-unital C^*-subalgebra of a stable σ-unital C^*-algebra A and let p be a rank-one projection in $\mathbb{K}$. Then there is a unitary $v \in M(A \otimes \mathbb{K})$ such that for every $a \in (A \otimes \mathbb{K})_+$ and $\epsilon > 0$ we can find an element $x \in A \subset A \otimes \mathbb{K}$ with*

$$\|a - (vx)(vx)^*\| < \epsilon, \qquad (vx)^*(vx) = x^*x \in B.$$

In particular, $B \otimes p$ is large in $A \otimes \mathbb{K}$.

Proof. Let $q := 1 \otimes p$ in $M(A \otimes \mathbb{K})$. By Lemma 2.1.13 $A \otimes p \otimes \mathbb{K}$ is a full corner in $A \otimes \mathbb{K} \otimes \mathbb{K}$ and by Lemma 2.4 of [15] (see Chapter 1) we can exhibit an element u in $M(A \otimes \mathbb{K} \otimes \mathbb{K})$ such that

$$uu^* = 1, \qquad u^*u = q \otimes 1.$$

The map

$$\psi : \begin{array}{ccc} A \otimes p \otimes \mathbb{K} & \to & A \otimes \mathbb{K} \otimes \mathbb{K} \\ a & \mapsto & uau^* \end{array}$$

is a $*$-isomorphism.
Let $\phi : A \to A \otimes \mathbb{K}$ be a $*$-isomorphism and let $\iota_{1,3} : A \otimes \mathbb{K} \to A \otimes p \otimes \mathbb{K}$ be the $*$-isomorphism defined on elementary tensors by $\iota_{1,3}(a \otimes k) = a \otimes p \otimes k$ for elements a in A and k in $\mathbb{K}$. Define $\phi_{1,3} := \iota_{1,3} \circ \phi$; this is a $*$-isomorphism since both ϕ and $\iota_{1,3}$ are.
Since B is a hereditary large C^*-subalgebra of A and $\phi_{1,3}$ is a $*$-isomorphism, it follows from Lemma 2.2.1 that $\phi_{1,3}(B)$ is a hereditary large C^*-subalgebra of $A \otimes p \otimes \mathbb{K}$.
Let now a be a positive element in $A \otimes \mathbb{K} \otimes \mathbb{K}$ and $\epsilon > 0$. There exist a positive element a' in $A \otimes p \otimes \mathbb{K}$ such that $a = ua'u^*$ and an element x in $A \otimes p \otimes \mathbb{K}$ such that

$$\|a' - xx^*\| < \epsilon \qquad \text{and} \qquad x^*x \in \phi_{1,3}(B).$$

Hence

$$\|a - uxx^*u^*\| = \|ua'u^* - uxx^*u^*\| < \epsilon$$

and

$$x^*u^*ux = x^*(q \otimes 1)x = x^*x \in \phi_{1,3}(B).$$

Thus $\phi_{1,3}(B)$ is a hereditary large C^*-subalgebra of $A \otimes p \otimes \mathbb{K}$.

Define the $*$-isomorphism $\eta : A\otimes\mathbb{K}\otimes\mathbb{K} \to A\otimes\mathbb{K}$ given on elementary tensors by $\eta(z\otimes k\otimes h) = \phi_{1,3}^{-1}(z\otimes p\otimes h)\otimes k$ for z in A and k, h in $\mathbb{K}$. Applying again Lemma 2.2.1 we obtain that $\eta(\phi_{1,3}(B)) = B\otimes p$ is a hereditary large C^*-subalgebra of $A\otimes\mathbb{K}$. Let now a be a positive element in $A\otimes\mathbb{K}$. From the above there exist $c \in (A\otimes\mathbb{K}\otimes\mathbb{K})_+$ and $d \in (A\otimes p\otimes\mathbb{K})_+$ such that $a = \eta(c)$ and

$$\|c-(ud)(ud)^*\| < \epsilon, \qquad (ud)^*(ud) \in \phi_{1,3}(B).$$

Let $\bar{\eta} : M(A\otimes\mathbb{K}\otimes\mathbb{K}) \to M(A\otimes\mathbb{K})$ be the $*$-isomorphism obtained by extension of the $*$-isomorphism η and set

$$y := \eta(d) \in A\otimes p, \qquad v := \bar{\eta}(u).$$

Then we have

$$\|a-(vy)(vy)^*\| < \epsilon, \qquad (vy)^*(vy) \in B\otimes p. \qquad \square$$

Note that any two strictly positive elements in a C^*-algebra are Cuntz equivalent.

Proposition 2.2.3. *Let B be a full hereditary σ-unital C^*-subalgebra of a stable σ-unital C^*-algebra A. The following are equivalent:*

(i) B is large in A.

(ii) Every strictly positive element in B is properly infinite.

Proof. It follows from Proposition 2.1.13, Corollary 2.1.14 and Proposition 2.2.2 that (i)$\Leftrightarrow$(ii). $\square$

2.3 On a question by Kirchberg and Rørdam

In this section we recall the definition of weak and almost stable rank 1 (Definition 2.3.2). We obtain in Proposition 2.3.5, as a consequence of Lemma 3.2 of [62], an equivalent characterization of stability for σ-unital C^*-algebras with weak stable rank 1 in terms of infiniteness of the mutiplier algebra. Furthermore, using results from [22] and [51], we see in Proposition 2.3.11 that in the case the stabilization of A has almost stable rank 1, stability is also equivalent to largeness of A in $A\otimes\mathbb{K}$. This gives a partial answer to a question reaised by Kirchberg and Rørdam in [44].

In [44] the authors asked the following

Question 2.3.1 ([44] Question 3.4)**.** *Let A be a C^*-algebra and $a \in A_+$ a properly infinite element. Is $M(\mathrm{her}(a))$ properly infinite?*

To answer this question in a very specific situation we will use the characterization of Cuntz equivalence given by Cowards, Elliott and Ivanescu in terms of countably generated Hilbert modules that appears in [22] in the particular case the C^*-algebra $\overline{aAa}$ has "low dimension" in the following sense:

Definition 2.3.2 ([70] Definition 2.50, [59] Definition 3.1)**.** *Let A be a C^*-algebra. We say that A has weak stable rank 1 if every element in A can be approximated by invertible elements in its minimal unitization $A \subset \overline{GL(\tilde{A})}$.*
A has almost stable rank 1 if every hereditary C^*-subalgebra of A has weak stable rank 1.

Remark 2.3.3. Recall that a C^*-algebra has stable rank 1 if $\tilde{A} \subset \overline{GL(\tilde{A})}$. In particular for separable C^*-algebras we have

$$\mathrm{sr}\,1 \Rightarrow \mathrm{asr}\,1 \Rightarrow \mathrm{wsr}\,1.$$

Moreover every element in a dense subset of a stable C^*-algebra can be written as the product of two nihilpotent elements and so every stable C^*-algebra has weak stable rank 1. Since infinite simple unital C^*-algebras do not have stable rank 1 and having stable rank 1 is a stable property, it is easy to see that

$$\mathrm{wsr}\,1 \nRightarrow \mathrm{sr}\,1.$$

The difference between stable rank 1 and almost stable rank 1 is more subtle and has been explored in [71] Example 6.6, where it is shown that having almost stable rank 1 is not a stable property; hence

$$\mathrm{asr} \nRightarrow \mathrm{sr}\,1.$$

We already observed in Lemma 2.1.3 that if A is a σ-unital C^*-algebra whose multiplier algebra is properly infinite, then any strictly positive element in A is properly infinite. To get the other implication, we need to see how these two conditions are related to stability. The following Proposition was proved by Rørdam in [62] Lemma 3.2 under the assumption of stable rank 1, but the same proof applies ad litteram if we just assume *weak* stable rank 1. We write the proof for completeness.

Proposition 2.3.4. *Let A be a σ-unital C^*-algebra with weak stable rank 1 and suppose that $M(A)$ is properly infinite. Then A is stable. The same is true if A is* *simple and $M(A)$ is infinite.*

Proof. We will see that for any $a \in A_+$ and any $\epsilon > 0$ there exist elements b, $c \in A_+$ and $z \in A$ with $zz^* = b$, $z^*z = c$, $bc = 0$ and $\|a - b\| \leqslant \epsilon$. This will imply that condition (iii) in Theorem 3.1 of [62] is satisfied.
So let $a \in A_+$, $\epsilon > 0$ and suppose $M(A)$ is properly infinite. Pick u, $v \in M(A)$ such that $u^*u = v^*v = 1$ and $vv^* \perp uu^*$. Let $b := uau^*$, $c := vav^*$ and $x := ua^{1/2}$, $y := ua^{1/2}v^*$ so that

$$xx^* = yy^* = b, \qquad x^*x = a, \qquad y^*y = c.$$

Since A has weak stable rank 1, we can find invertible elements x_n in $\tilde{A}$ such that $x_n \to x$. Write $x_n = u_n|x_n|$ for the polar decomposition of x_n and note that u_n belongs to $\tilde{A}$. We have

$$u_n|x_n|^2u_n^* \to b, \quad |x_n|^2 \to a$$

and so $b_n := u_n^*bu_n \to a$. Set $c_n := u_n^*cu_n$ and $z_n := u_n^*yu_n$. Then for every n we have

$$b_nc_n = 0, \quad z_nz_n^* = b_n, \quad z_n^*z_n = c_n$$

and for n large enough

$$\|b_n - a\| < \epsilon.$$

For the case in which A is simple and $M(A)$ is infinite,the result follows from Lemma 3.3 of [62] by applying the same procedure as above. □

In [63] Proposition 3.5 it is proved that for a σ-unital C^*-algbra with stable rank 1, the property of being stable is equivalent to the property that its multiplier algebra is properly infinite. The point now is that while there are stable C^*-algebras not having stable rank 1, every stable C^*-algebra has *weak* stable rank 1. So we are lead to the following

Proposition 2.3.5. *Let A be a σ-unital C^*-algebra. The following are equivalent*

(i) A is stable,

(ii) A has weak stable rank 1 and $M(A)$ is properly infinite.

If A is simple, they are equivalent to

(iii) A has weak stable rank 1 and $M(A)$ is infinite.

Proof. $(i) \Rightarrow (ii)$: If A is stable, it has weak stable rank 1 by [70] Lemma 2.51 and $M(A)$ is properly infinite.
$(ii) \Rightarrow (i)$ and $(iii) \Rightarrow (i)$ are consequences of Proposition 2.3.4.□

Now we need to spend some words about the equivalent characterizations of Cuntz equivalence that we mentioned at the beginning of this section.

Let A be a σ-unital C^*-algebra. Every countably generated (right) Hilbert module for its stabilization $A \otimes \mathbb{K}$ is in fact singly generated and is isomorphic to the right ideal generated by a positive element in $A \otimes \mathbb{K}$. The definition of Cuntz equivalence for Hilbert modules is a translation of Rørdam Lemma in this context.
In what follows, if X is a Hilbert module over a C^*-algebra A, we denote by $\mathbb{K}(X)$ the C^*-algebra of compact operators over X (see [57] Definition 2.24).

Definition 2.3.6 ([22], 1)**.** *Let $X \subset Y$ be an inclusion of Hilbert modules (over a C^*-algebra A). We say that X is compactly contained in Y and write $X \Subset Y$ if there is a selfadjoint element $x \in \mathbb{K}(Y)$, that is the identity on X, $x|_X = \mathrm{id}|_X$.*
We say that X is Cuntz subequivalent to Y and write $X \precsim Y$ if every compactly contained Hilbert submodule of X is isomorphic to a submodule of Y.
Two Hilbert modules X and Y are Cuntz equivalent, $X \sim Y$, if $X \precsim Y$ and $Y \precsim X$.

In [22] the authors proved that when passing to Cuntz equivalence classes, Cuntz subequivalence induces a partial order that is compatible with direct sum of Hilbert modules, making the set of equivalence classes a partially ordered semigroup. Furthermore in [22] Appendix 6 it is proved that this semigroup coincides with the Cuntz semigroup $Cu(A)$ and the correspondence sends the Cuntz equivalence class of a positive element in $A \otimes \mathbb{K}$ to the Cuntz equivalence class of the right ideal generated by this element in $A \otimes \mathbb{K}$.
What will be important for us is that in this picture, in the case of two hereditary C^*-subalgebras of $A \otimes \mathbb{K}$ of the form $\overline{x(A \otimes \mathbb{K})x}$ and $\overline{y(A \otimes \mathbb{K})y}$ for some x and y positive elements in $A \otimes \mathbb{K}$, Cuntz equivalence for x and y is the same as isomorphism of the modules $\overline{x(A \otimes \mathbb{K})}$ and $\overline{y(A \otimes \mathbb{K})}$, in the case $A \otimes \mathbb{K}$ has almost stable rank 1.
In what follows, if A is a C^*-algebra and x is a positive element in $A \otimes \mathbb{K}$, we will write E_x for the right ideal generated by x in $A \otimes \mathbb{K}$, considered as a right Hilbert module over $A \otimes \mathbb{K}$. The next result was originally proved in Theorem 3 of [22] for C^*-algebras with stable rank 1, but, as already noticed by Robert in [59], the same proof applies to the case in which $A \otimes \mathbb{K}$ has almost stable rank 1.

Theorem 2.3.7. *Let A be a C^*-algebra and x, y two positive elements in $A \otimes \mathbb{K}$.*

Suppose that $A \otimes \mathbb{K}$ has almost stable rank 1 and that the Hilbert modules E_x and E_y are Cuntz equivalent. Then $E_x \simeq E_y$ as Hilbert modules.
Furthermore $E_x \precsim E_y$ if and only if E_x is isomorphic to a sub-Hilbert module of E_y.

Sketch of the proof. Every Hilbert module in the equivalent description of $Cu(A)$ can be written as an increasing sequence of compactly contained submodules. Let $X \coloneqq \overline{x(A \otimes \mathbb{K})}$ and $Y \coloneqq \overline{y(A \otimes \mathbb{K})}$. The hypothesis that they are Cuntz equivalent implies that we can find sequences

$$\begin{array}{ccccccccc} X_1 & & \Subset & & X_2 & & \Subset & \dots & \subset X \\ & \overset{\phi_1}{\searrow} & & \overset{\psi_1}{\nearrow} & & \overset{\phi_2}{\searrow} & & & \\ & & Y_1 & & \Subset & & Y_2 & \Subset \; \dots & \subset Y \end{array}$$

where the arrows ϕ_n, ψ_n are isomorphisms on their ranges. The idea of the proof is the following:
Choose finite subsets F_n of $X_n \subset X$ such that $F_n \cup (\psi_n \circ \phi_n) F_n \subset F_{n+1}$ and $\bigcup_n F_n$ is dense in X. In the same way define subsets G_n of Y with the extra assumption that $\phi_1(F_1) \subset G_1$.
Using [47] Proposition 1.3 approximate $\psi_n \circ \phi_n \in \mathbb{B}(X_n, \psi_n \circ \phi_n(X_n))$ within F_n by an element η_n in $\mathbb{K}(X_n, \psi_n \circ \phi_n(X_n))$ of norm one and extend this to an element $x_n \in \mathbb{K}(X_{n+1})$ (the existence of such extension can be proved using an approximate identity for $\mathbb{K}(X_n)$). In the same way define elements $y_n \in \mathbb{K}(Y)$ from $\phi_{n+1} \circ \psi_n \in \mathbb{B}(Y_n, \phi_{n+1} \circ \psi_n(Y_n))$.
Since both $\mathbb{K}(X_{n+1})$ and $\mathbb{K}(Y_{n+1})$ have weak stable rank 1, we can choose invertible elements $\tilde{x}_n \in \mathbb{K}(X_{n+1})^\sim$ and $\tilde{y}_n \in \mathbb{K}(Y_{n+1})^\sim$ that are arbitrarily close to x_n and y_n respectively.
Since $\tilde{x}_n$ approximately preserves the inner product on F_n (this is exactly preserved by $\psi_n \circ \phi_n$), it follows that $|x_n|$ is close on F_n to the identity and hence x_n is close to its unitary part u_n on F_n and since x_n is invertible, u_n belongs to $\mathbb{K}(X_{n+1})^\sim$. In the same way one finds unitaries v_n in $\mathbb{K}(Y_{n+1})^\sim$ such that y_n is close to v_n on G_n.
So, up to tolerances ϵ_n and δ_n one has

$$\psi_n \circ \phi_n \sim_{\epsilon_n} u_n \quad \text{on } F_n, \qquad \phi_{n+1} \circ \psi_n \sim_{\delta_n} v_n \quad \text{on } G_n.$$

Modifying the above maps up to inner automorphisms, we can write

$$\tilde{\psi}_n \circ \tilde{\phi}_n \sim_{\epsilon_n} \mathrm{id} \quad \text{on } F_n, \qquad \tilde{\phi}_{n+1} \circ \tilde{\psi}_n \sim_{\delta_n} \mathrm{id} \quad \text{on } G_n.$$

Hence, choosing the tolerances small enough (see [29]), we can suppose

$$\tilde{\phi}_{n+1} \circ \tilde{\psi}_n \circ \tilde{\phi}_n \sim_{2^{-n}} \tilde{\phi}_n \quad \text{on } F_n, \qquad \tilde{\psi}_{n+1} \circ \tilde{\phi}_{n+1}\tilde{\psi}_n \sim_{2^{-(n+1)}} \tilde{\psi}_n \quad \text{on } G_n$$

and

$$\tilde{\phi}_{n+1} \circ \tilde{\psi}_n \circ \tilde{\phi}_n \sim_{2^{-(2n-1)}} \tilde{\phi}_{n+1} \quad \text{on } F_n, \qquad \tilde{\psi}_{n+1} \circ \tilde{\phi}_{n+1} \circ \tilde{\psi}_n \sim_{2^{-2n}} \tilde{\psi}_n \quad \text{on } G_n.$$

Thus

$$\tilde{\phi}_{n+1} \sim_{2^{-2n+2}} \tilde{\phi}_n \quad \text{on } F_n, \qquad \tilde{\psi}_{n+1} \sim_{2^{-2n+1}} \tilde{\psi}_n \quad \text{on } G_n.$$

Now, using a variation of Theorem 2.2 of [28] and Theorem 3 of [29], the result follows.
Similarly, if $E_x \precsim E_y$, one can use a one-sided approximate intertwining argument to see that E_x is isomorphic to a sub-Hilbert module of E_y. □

The following two Propositions are contained in Proposition 4.3 and Proposition 4.6 of [51] and are an application of Theorem 2.3.7.

Proposition 2.3.8. *Let A be a C^*-algebra and x, y positive elements in $A \otimes \mathbb{K}$ and suppose that $A \otimes \mathbb{K}$ has almost stable rank 1. Then the following are equivalent:*

(i) $x \sim y$,

(ii) there exists $z \in A \otimes \mathbb{K}$ *such that* $\mathrm{her}(x) = \mathrm{her}(z^*z)$ *and* $\mathrm{her}(y) = \mathrm{her}(zz^*)$,

(iii) there exists $z = u|z| \in A \otimes \mathbb{K}$ *such that* $x = z^*z$ *and the map*

$$\mathrm{her}(x) \to \mathrm{her}(y)$$

$$a \mapsto uau^*$$

is a $*$-isomorphism,

(iv) $E_x \simeq E_y$.

Proof. (i)⇔(iv): As already observed, by [22] Theorem 3, in this case $x \sim y$ if and only if $E_x \simeq E_y$.
(iv)⇔(ii): This is contained in Proposition 4.3 of [51].
(ii)⇒(iii): Suppose there is an element $z \in A \otimes \mathbb{K}$ such that $\mathrm{her}(x) = \mathrm{her}(z^*z)$ and $\mathrm{her}(y) = \mathrm{her}(zz^*)$. Consider the polar decomposition $z = u|z|$ with $u \in (A \otimes \mathbb{K})^{**}$. Then the map $a \mapsto uau^*$ from $\mathrm{her}(x)$ to $\mathrm{her}(y)$ defines a $*$-isomorphism by Lemma 2.37 of [70].

(iii)⇒(ii): If the unitary $u \in (A \otimes \mathbb{K})^{**}$ appearing in the polar decomposition of $z = u|z|$ implements a $*$-isomorphism $a \mapsto uau^*$ from $\mathrm{her}(x)$ to $\mathrm{her}(y)$, then in particular uxu^* is a strictly positive element for $\mathrm{her}(y)$ and so $\mathrm{her}(y) = \mathrm{her}(uxu^*) = \mathrm{her}(zz^*)$ and $\mathrm{her}(x) = \mathrm{her}(z^*z)$. □

There is also an analogous statement for Cuntz subequivalence in place of Cuntz equivalence.

Proposition 2.3.9. *Let A be a C^*-algebra and x, y two positive elements in $A \otimes \mathbb{K}$ and suppose that $A \otimes \mathbb{K}$ has almost stable rank 1. The following are equivalent:*

(i) $x \precsim y$,

(ii) there exists a Hilbert module E' such that $E_x \simeq E' \subset E_y$,

(iii) there exists $z \in A \otimes \mathbb{K}$ such that $\mathrm{her}(x) = \mathrm{her}(z^*z)$, $\mathrm{her}(zz^*) \subset \mathrm{her}(y)$,

(iv) there exists $z = u|z| \in A \otimes \mathbb{K}$ such that $x = z^*z$ and $u\,\mathrm{her}(x)u^* \subset \mathrm{her}(y)$.

Proof. (i)⇔(ii) is contained in Theorem 2.3.7.
(ii)⇔(iii) is contained in Proposition 4.6 of [51].
(iii)⇒(iv): Consider the polar decomposition $z = u|z|$. Then the map $a \mapsto uau^*$ from $\mathrm{her}(x) = \mathrm{her}(z^*z)$ to $\mathrm{her}(zz^*) \subset \mathrm{her}(y)$ is a $*$-isomorphism.
(iv)⇒(iii): If $x = z^*z$, then $u\,\mathrm{her}(x)u^* = u\,\mathrm{her}(z^*z)u^* = \mathrm{her}(uz^*zu^*) = \mathrm{her}(zz^*) \subset \mathrm{her}(y)$. □

As a consequence, we obtain the following characterization of hereditary large C^*-subalgebras of stable C^*-algebras with almost stable rank 1.

Proposition 2.3.10. *Let A be a σ-unital stable C^*-algebra with almost stable rank 1 with a strictly positive element h and B a hereditary σ-unital C^*-subalgebra. The following are equivalent*

(i) B is large in A,

(ii) there exists an element $z = u|z| \in A$ such that $h = z^*z$ and $B = uAu^*$,

(iii) B is full and $*$-isomorphic to A.

Proof. (i)⇔(ii): Let x be a strictly positive element for B. Since by Proposition 2.1.9 B is large in A if and only if $x \sim h$, this follows from Proposition 2.3.8.
(ii)⇒(iii): We only need to show that B is full. Let $z := uh^{1/2}$, this element belongs

to A by [70] Lemma 2.12; then $zz^* = uhu^*$ belongs to B and $z^*(zz^*)z = (z^*z)^2$ belongs to the algebraic ideal generated by B. Hence $z^*z = h$ belongs to the closed two-sided ideal generated by B and since h is strictly positive for A, it follows that it contains A.
(iii)⇒(i): If B is $*$-isomorphic to A, it is stable and hence x is properly infinite and full, entailing largeness by Proposition 2.2.3. □

In virtue of the above observations, it is now clear that in the case of a C^*-algebra whose stabilization has almost stable rank 1, stability has the following equivalent characterization.

Proposition 2.3.11. *Let A be a σ-unital C^*-algebra such that $A \otimes \mathbb{K}$ has almost stable rank 1 and let h be a strictly positive element in A. The following are equivalent*

(i) $M(A)$ is properly infinite,

(ii) A is stable,

(iii) h is properly infinite.

If A is simple, the above conditions are equivalent to

(iv) $M(A)$ is infinite.

Proof. If A is stable, then $M(A)$ is properly infinite as already pointed out, thus $(ii) \Rightarrow (i)$.
By Lemma 2.1.3 $(i) \Rightarrow (iii)$.
By Proposition 2.3.10 $(iii) \Rightarrow (ii)$ since in this case A is large in $A \otimes \mathbb{K}$.
Trivially $(i) \Rightarrow (iv)$.
If A is simple and $M(A)$ is infinite, then A is stable by [62] Theorem 3.5 and so $(iv) \Rightarrow (ii)$. □

Hence, we obtain the following partial answer to Kirchberg and Rørdam's question 2.3.1.

Corollary 2.3.12. *Let A be a C^*-algebra and $a \in A_+$ a properly infinite element such that $\mathrm{her}(a) \otimes \mathbb{K}$ has almost stable rank 1. Then $M(\mathrm{her}(a))$ is properly infinite.*

Proof. By Corollary 2.3.11 we only have to show that a belongs to $\mathrm{her}(a)$ and that it is strictly positive (in $\mathrm{her}(a)$). This fact is contained in Lemma 2.1.8. □

2.4 Comparison for certain C^*-algebras

In this section we want to retrieve information on the structure of a particular class of C^*-algebras by looking at their Cuntz semigroup. We will see in Proposition 2.4.1 that for C^*-algebras whose stabilization has almost stable rank 1 the Pedersen ideal has a description in terms of compact containment in their Cuntz semigroup. It turns out that in the case A is simple, with almost unperforated Cuntz semigroup and with a unique nontrivial lower semicontinuous 2-quasitrace, the Pedersen ideal coincides with the ideal associated to the corresponding functional on $Cu(A)$ (Proposition 2.4.11). Using this, we are able to say in Theorem 2.4.12 that the C^*-algebras belonging to this class are either algebraically simple or stable. We also see the counterpart of this result at the level of Cuntz equivalence classes of positive elements in the stabilization and correspondingly at the level of the associated Hilbert modules in Theorem 2.4.19.

If A is a C^*-algebra, we denote by $\mathrm{Ped}(A)$ its Pedersen ideal, i.e. the dense ideal that is contained in every other dense ideal and refer to [53] 5.6 for its construction and properties.
We begin with the characterization of the Pedersen ideal in terms of compact containment for a certain class of C^*-algebras. A more general result, that appears as a Remark on page 4 of [72], was suggested to hold by Elliott without the assumption that the stabilization of the C^*-algebra has almost stable rank 1. Fruitful discussions with the colleague and friend André Schemaitat lead to the following weaker version:

Proposition 2.4.1. *Let A be a C^*-algebra such that $A \otimes \mathbb{K}$ has almost stable rank 1. Then*

$$\mathrm{Ped}(A) = \{a \in A \ : \ \exists b \in M_\infty(A)_+ \ \text{ such that } \ [|a|] \ll [b]\}.$$

Proof. Let a be an element of A and write $a = u|a|$ for its polar decomposition. If $|a|$ belongs to $\mathrm{Ped}(A)$, then by [53] Proposition 5.6.2 $|a|^\alpha$ belongs to $\mathrm{Ped}(A)$ for every $\alpha > 0$; note now that $u|a|^{1-\alpha}$ belongs to A for every $0 < \alpha < 1$ (cfr. [70] Lemma 2.12) and so $u|a| = u|a|^{1-\alpha}|a|^\alpha$ belongs to $\mathrm{Ped}(A)$. Thus we are left to prove that the set

$$J_+ \coloneqq \{a \in A_+ \ : \ \exists b \in M_\infty(A)_+ \ \text{ such that } \ [a] \ll [b]\}$$

is contained in $\mathrm{Ped}(A)$ and that

$$J \coloneqq \{a \in A \ : \ \exists b \in M_\infty(A)_+ \ \text{ such that } \ [|a|] \ll [b]\}$$

is an ideal and that it is dense.
Let a be a positive element in A and suppose there is $n \in \mathbb{N}$ and a positive element b in $M_n(A)$ such that $[a] \ll [b]$. Then there is $\epsilon > 0$ such that $a \precsim (b-\epsilon)_+$ in $M_n(A)$. Since $A \otimes \mathbb{K}$ has almost stable rank 1, by the proof of [22] Theorem 3 and by Proposition 4.6 of [51] (see the discussion in the previous section), there is an element y in $M_n(A)$ such that $\mathrm{her}(a) = \mathrm{her}(yy^*)$, $y^*y \in \mathrm{her}((b-\epsilon)_+)$; in particular by [53] Proposition 5.6.2 y^*y belongs to $\mathrm{Ped}(M_n(A))$ and the same is true for yy^*.
Now we want to show that for any $n \in \mathbb{N}$, $\mathrm{Ped}(M_n(A)) = \mathrm{Ped}(A) \otimes_{alg} M_n$. First of all notice that $\mathrm{Ped}(A) \otimes_{alg} M_n$ is a dense ideal in $M_n(A)$ and so $\mathrm{Ped}(M_n(A)) \subset \mathrm{Ped}(A) \otimes_{alg} M_n$. Secondly, if $x = (x_{i,j})$ is an element in $M_n(A)$ such that $x_{i,j}$ belongs to $\mathrm{Ped}(A)$ for every $1 \leqslant i,j \leqslant n$, we can write each $x_{i,j}$ as a linear combination of positive elements $y_{i,j}^{(k)}$, for each of which there are a natural number $m_{i,j}^{(k)}$ and elements $z_{i,j}^{(k,l)}$, $\tilde{z}_{i,j}^{(k,l)}$ such that $y_{i,j}^{(k)} \leqslant \sum_{l=1}^{m_{i,j}^{(k)}} z_{i,j}^{(k,l)}$ and $z_{i,j}^{(k,l)} \tilde{z}_{i,j}^{(k,l)} = z_{i,j}^{(k,l)}$. Denoting by $e_{i,j}$ the matrix units for M_n, it follows that the same is true if we replace the elements $x_{i,j}$, $y_{i,j}^{(k)}$, $z_{i,j}^{(k,l)}$ and $\tilde{z}_{i,j}^{(k,l)}$ with $x_{i,j} \otimes e_{i,j}$, $y_{i,j}^{(k)} \otimes e_{i,j}$, $z_{i,j}^{(k,l)} \otimes e_{i,j}$ and $\tilde{z}_{i,j}^{(k,l)} \otimes e_{i,j}$ respectively. It follows that for every $1 \leqslant i,j \leqslant n$ $x_{i,j} \otimes e_{i,j}$ belongs to $\mathrm{Ped}(M_n(A))$ and so $x \in \mathrm{Ped}(M_n(A))$. Thus $\mathrm{Ped}(M_n(A)) = M_n(\mathrm{Ped}(A))$. In particular $A \cap \mathrm{Ped}(M_n(A)) = A \cap (M_n(\mathrm{Ped}(A)) = \mathrm{Ped}(A)$.
Thus $\mathrm{her}(yy^*) = \mathrm{her}(a)$ belongs to $A \cap M_n(\mathrm{Ped}(A)) = \mathrm{Ped}(A)$ and so a belongs to $\mathrm{Ped}(A)$.
Now we show that J is an ideal; for the proof of this fact we use a technique from the proof of [44] Lemma 3.12, that we recall. Let a belong to J and b be an element in A, then

$$|ab| \precsim |a|, \qquad |ba| \precsim |a|.$$

For note that $abb^*a^* \leqslant \|b\|^2 aa^* \sim |a|^2 \sim |a|$ and $baa^*b^* \sim a^*b^*ba$.
Note that if a belongs to J, also λa belongs to J for any $\lambda \in \mathbb{C}$; hence we need to show that if a and b both belong to J, then also their sum $a+b$ is in J. So let a', $b' \in M_\infty(A)_+$ and $\delta > \epsilon > 0$ be such that

$$|a| \precsim (a'-\epsilon)_+, \qquad |b| \precsim (b'-\delta)_+.$$

For any elements a and b in A we have $(a-b)(a-b)^* = aa^* + bb^* - ab^* - ba^*$, which entails $ab^* + ba^* \leqslant |a|^2 + |b|^2$ and then $|a+b| \sim |a+b|^2 \leqslant 2(|a|+|b|)(|a|+|b|) \precsim |a|+|b|$. Thus

$$|a+b| \precsim |a| + |b| \precsim |a| \oplus |b| \precsim$$
$$(a'-\epsilon)_+ \oplus (b'-\delta)_+ \precsim (a'-\epsilon)_+ \oplus (b'-\epsilon)_+ = (a' \oplus b' - \epsilon)_+,$$

where, in the least equality, we have used the fact that $C(a \oplus b) = C(a) \oplus C(b)$ canonically.
Since every element in A is a linear combination of positive elements and J is a linear space, we just have to prove that J_+ is dense in A_+. Let $a \in A_+$, then $\|a - (a-\epsilon)_+\| < \epsilon$ and $[(a-\epsilon)_+] \ll [a]$.
Hence J is a dense two-sided ideal and has to be equal to $\mathrm{Ped}(A)$. □

Corollary 2.4.2. *Let A be a stable C^*-algebra with almost stable rank 1. Then*

$$\mathrm{Ped}(A) = \{a \in A \; : \; \exists b \in A_+ \quad \textit{such that} \;\; [|a|] \ll [b]\}.$$

Proof. Since A is stable, it is large in $A \otimes \mathbb{K}$. Suppose there are $a \in A$, $b \in M_n(A)_+$ such that $[|a|] \ll [b]$, then there is $\epsilon > 0$ such that $[|a|] \leqslant [(b-\epsilon)_+] \ll [(b-\epsilon/2)_+]$; by largeness, there is $a_{\epsilon/2} \in A_+$ such that $[(b-\epsilon/2)_+] = [a_{\epsilon/2}]$ and thus $[|a|] \ll [a_{\epsilon/2}]$. The result follows from Proposition 2.4.1. □

We want to obtain another characterization of the Pedersen ideal in the case a C^*-algebra admits a unique lower semicontinuous 2-quasitrace. For that, we need to impose some further hypothesis. The first one is stable finiteness (see definition below); indeed this property gives to the Cuntz semigroup a very specific structure in the simple case.

Definition 2.4.3 ([4] Definition 5.3.1, Sections 2.1.1, 5.2.2)**.** *Let A be a C^*-algebra and $Cu(A)$ its Cuntz semigroup. An element $[a] \in Cu(A)$ is said to be soft if for every $[a'] \in Cu(A)$ such that $[a'] \ll [a]$ there exists $n \in \mathbb{N}$ such that $(n+1)[a'] \leqslant n[a]$.*
An element $[a] \in Cu(A)$ is said to be compact if $[a] \ll [a]$.
An element $[a] \in Cu(A)$ is said to be finite if whenever there exists $[b] \in Cu(A)$ such that $[a] + [b] = [a]$ then $[b] = 0$.

Stable finiteness for unital simple C^*-algebras can be equivalently characterized by the existence of a nontrivial 2-quasitrace. In view of Proposition 2.4.8, the algebraic counterpart of this concept at the level of the Cuntz semigroup is the following

Definition 2.4.4 ([4] Section 5.2.2)**.** *Let A be a C^*-algebra and $Cu(A)$ its Cuntz semigroup. We say that $Cu(A)$ is stably finite if, whenever there are elements $[a]$, $[a']$ in $Cu(A)$ such that $[a] \ll [a']$, then $[a]$ is finite.*

The next Lemma, concerning the nature of soft elements in stably finite Cuntz semigroups, should be compared with Lemma 5.3.8 of [4], where a similar statement is proved in the category of algebraic Cuntz semigroups.

Lemma 2.4.5 (see [4] Lemma 5.3.8)**.** *Let A be a simple C^*-algebra such that $Cu(A)$ is stably finite. Then an element $[a] \in Cu(A)$ is soft if and only if 0 is not an isolated point of the spectrum of any representative of $[a]$ in $A \otimes \mathbb{K}$.*

Proof. Let $[a]$ be a soft element in $Cu(A)$. Pick $[a'] \in Cu(A)$ such that $[a'] \ll [a]$. In particular there is $\epsilon > 0$ such that for every representative a' of $[a']$ we have $a' \precsim (a - \epsilon)_+$; using functional calculus we can construct an element $x \in (A \otimes \mathbb{K})_+$ such that $a' \oplus x \precsim a$, for example we can take $x = f(a)$ with

$$f(t) = \begin{cases} (3/\epsilon)t & \text{for } 0 \leqslant t \leqslant \epsilon/3 \\ 2 - (3/\epsilon)t & \text{for } \epsilon/3 < t \leqslant (2/3)\epsilon \\ 0 & \text{for } (2/3)\epsilon \leqslant t \end{cases} .$$

By definition of stable finiteness $[a']$ is finite. Suppose that $[x] = 0$, then $[a] \ll [a]$ and by the assumption that $[a]$ is soft, there exists $n \in \mathbb{N}$ such that $(n+1)[a] \leqslant n[a]$; but since $[a]$ is finite, this entails $n[a] = 0$ and then $[a] = 0$.
Suppose now that a is a representative of $[a]$ and that 0 is not an isolated point in the spectrum of a. Let $[a'] \ll [a]$; then there is an element $[x] \neq 0$ such that $[a'] + [x] \leqslant [a]$. Since A is simple, $[a] \leqslant \sup_n n[x]$. But since $[a'] \ll [a]$, it follows that there is a $k \in \mathbb{N}$ such that $[a'] \leqslant k[x]$ and

$$(k+1)[a'] = k[a'] + [a'] \leqslant k[a'] + k[x] \leqslant k[a]$$

and $[a]$ is soft. □

Conversely, if A is a C^*-algebra and its Cuntz semigroup is stably finite, then the spectrum of every representative of a compact element in $Cu(A)$ contains 0 as an isolated point. In particular, every compact element in $Cu(A)$ is the class of a projection. A unital C^*-algebra is stably finite if and only if its Cuntz semigroup is stably finite by Proposition 6.22 of [70]; hence in this case the thesis of Proposition 2.4.6 is already proved in Theorem 3.5 of [16]. The following result has also to be compared with [31] Proposition 6.4 (iv), in which it is proved, using similar techniques, that any compact element in the Cuntz semigroup of a simple C^*-algebras is either infinite or the class of a finite projection.

Proposition 2.4.6. *Let A be a C^*-algebra such that $Cu(A)$ is stably finite. If $[a] \in Cu(A)$ is compact, then 0 is an isolated point in the spectrum of every representative of $[a]$ in $A \otimes \mathbb{K}$. In this case there is a projection $p \in A \otimes \mathbb{K}$ such that $[a] = [p]$.*

Proof. Let $\epsilon > 0$ be such that $[a] = [(a-\epsilon)_+]$ and suppose that in the spectrum of a representative a for $[a]$, 0 is not an isolated point. Then we can find a function f defined on the spectrum of a whose support is contained in $(0, \epsilon)$ and such that $f(a) \neq 0$. Since $f(a)$ is orthogonal to $(a-\epsilon)_+$ we have $[(a-\epsilon)_+] + [f(a)] \leqslant [a]$, but $[a] = [(a-\epsilon)_+]$ and so $[a] = [(a-\epsilon)_+] + [f(a)]$ is an infinite element. This is a contradiction since $Cu(A)$ is assumed to be stably finite. Thus $f(a) = 0$ and 0 is an isolated point of the spectrum of a and using functional calculus we can find a projection $p \in A \otimes \mathbb{K}$ such that $a \sim p$. □

As a consequence we immediately obtain the following

Proposition 2.4.7 ([4] Proposition 5.3.16)**.** *Let A be a simple C^*-algebra such that $Cu(A)$ is stably finite. Then every element $[a] \in Cu(A)$ is either soft or compact.*

Stable finiteness of the Cuntz semigroup of a C^*-algebra, which seems to be a difficult property to check directly, turns out to have an easy characterization in the simple case, that we will recall in a moment.
Following the notation of [4] page 51, if λ is a functional on $Cu(A)$, we say that λ is *trivial* if it only takes values 0 and ∞ and is *nontrivial* otherwise.

Proposition 2.4.8 ([4] Proposition 5.2.10)**.** *Let A be a simple C^*-algebra. Then $Cu(A)$ is stably finite if and only if there exists a nontrivial functional in $F(Cu(A))$.*

In analogy with the case of functionals on a Cuntz semigrop, if A is a C^*-algebra and τ a lower semicontinuous 2-quasitrace on A, we say that τ is *trivial* if it only takes the values 0 and ∞; otherwise τ is said to be *nontrivial*. Denote by τ_0 the trivial lower semicontinuous 2-quasitrace on A defined by

$$\tau_0(a) = 0 \qquad \text{for every } a \in A_+$$

and by τ_∞ the one defined by

$$\tau_\infty(a) = \begin{cases} 0 & \text{for } a = 0 \\ \infty & \text{for } a \in A_+ - \{0\}. \end{cases}.$$

In the same way we define the functionals on $Cu(A)$ λ_0 defined by

$$\lambda_0([a]) = 0 \qquad \text{for all } [a] \in Cu(A)$$

and

$$\lambda_\infty([a]) = \begin{cases} 0 & \text{for } [a] = 0 \\ \infty & \text{for } [a] \in Cu(A) - \{0\} \end{cases}.$$

Lemma 2.4.9. *Let A be a simple C^*-algebra. Then the only possible trivial lower semicontinuous 2-quasitraces on A are τ_0 and τ_∞ and the only possible trivial functionals on $Cu(A)$ are λ_0 and λ_∞.*
Furthermore, A admits a unique nontrivial lower semicontinuous 2-quasitrace (up to a scalar) if and only if $Cu(A)$ admits a unique nontrivial functional (up to a scalar).

Proof. Let λ be a trivial functional on $Cu(A)$. We prove that the set

$$N_\lambda := \{a \in A \otimes \mathbb{K} \ : \ \lambda([|a|]) = 0\}$$

is an algebraic ideal. From the proof of Proposition 2.4.1 we know that for any a and b in $A \otimes \mathbb{K}$, $|ab|$, $|ba| \precsim |a|$ and $|a + b| \precsim |a| \oplus |b|$. Thus if a is in N_λ and b is in $A \otimes \mathbb{K}$, $\lambda([|ab|]) \leqslant \lambda([|a|]) = 0$ and if a and b are both in N_λ, then $\lambda([|a+b|]) \leqslant \lambda([|a|]) + \lambda([|b|]) = 0$. Since $|a| \sim |\alpha a|$ for every $a \in A \otimes \mathbb{K}$ and $\alpha \in \mathbb{C}^\times$, it follows that N_λ is an ideal in $A \otimes \mathbb{K}$ and by simplicity it is either the set $\{0\}$ or is dense in $A \otimes \mathbb{K}$.
If $N_\lambda = \{0\}$, then $\lambda = \lambda_\infty$; hence suppose $N_\lambda \neq 0$. Let $[a]$ be an element of $Cu(A)$ and note that $\lambda([a]) = \sup_{\epsilon > 0} \lambda([(a - \epsilon)_+])$. Since N_λ is dense, $\mathrm{Ped}(A \otimes \mathbb{K}) \subset N_\lambda$ and so $\lambda([(a - \epsilon)_+]) = 0$ for every $\epsilon > 0$. Thus $\lambda([a]) = 0$ and $\lambda = \lambda_0$.
Using the bijective correspondence of Theorem 1.3.2, the 2-quasitrace associated to a functional λ is $\tau_\lambda(a) = \int_0^\infty \lambda([(a - \epsilon)_+])$ for $a \in A_+$. Hence the functional λ_0 corresponds to the lower semicontinuous 2-quasitrace τ_0 and λ_∞ corresponds to τ_∞.
Since the functional on $Cu(A)$ associated to a lower semicontinuous 2-quasitrace τ is given by $d_\tau([a]) = \sup_n \tau(f_n(a))$ (see Chapter 1.1 for the definition of f_n), the functional associated to $r\tau$, $r \in \mathbb{R}_+$ is rd_τ and so, in virtue of the bijection between lower semicontinuous 2-quasitraces on A and functionals on $Cu(A)$, A admits a unique nontrivial lower semicontinuous 2-quasitrace if and only if $Cu(A)$ admits a unique nontrivial functional, up to scalars. □

The other property we need to impose is almost unperforation for the Cuntz semigroup, which, when coupled with stable finiteness, permits comparison among all the non-compact elements, in virtue of the following

Theorem 2.4.10 ([4] Theorem 5.3.12)**.** *Let A be a C^*-algebra such that $Cu(A)$ is almost unperforated. If $[a]$ and $[b]$ are elements in $Cu(A)$ such that $[a]$ is soft and $\lambda([a]) \leqslant \lambda([b])$ for every functional $\lambda \in F(Cu(A))$, then $[a] \leqslant [b]$.*

Now we are ready to give the aforementioned characterization of the Pedersen ideal.

Proposition 2.4.11. *Let A be a simple C^*-algebra with almost unperforated Cuntz semigroup such that $A \otimes \mathbb{K}$ has almost stable rank 1 and assume that A admits a unique lower semicontinuous nontrivial 2-quasitrace τ (up to scalar multiples). Let d_τ be the corresponding functional on $Cu(A)$. Then*

$$\mathrm{Ped}(A) = \{a \in A \ : \ d_\tau([|a|]) < \infty\}.$$

Proof. Let τ be the unique nontrivial 2-quasitrace on A, up to scalars. By Lemma 2.4.9 d_τ is a nontrivial functional on $Cu(A)$ and is the unique such functional, up to scalars.
We want to prove that

$$J \coloneqq \{a \in A \ : \ d_\tau([|a|]) < \infty\} = \mathrm{Ped}(A).$$

First of all notice that J is an ideal (the proof goes as in Proposition 2.4.1 and Lemma 2.4.9 using the fact that d_τ is a functional on $Cu(A)$) and it is obviously dense since A is simple.
Secondly, note that by Proposition 2.4.8 $Cu(A)$ is stably finite and hence by Proposition 2.4.7 every element in $Cu(A)$ is either compact or soft.
Let $a \in A$ be such that $0 \neq d_\tau([|a|]) < \infty$; if $[|a|]$ is compact, then a belongs to $\mathrm{Ped}(A)$ by Proposition 2.4.1, thus we can suppose that $[|a|]$ is soft.
Since $0 < d_\tau([|a|]) < \infty$ and $d_\tau([|a|]) = \sup_n d_\tau([(|a| - 1/n)_+])$, there exists $m \in \mathbb{N}$ such that $0 < d_\tau([(|a| - 1/m)_+]) < \infty$ and so there is a $k \in \mathbb{N}$ such that $kd_\tau([(|a|-1/m)_+]) \geqslant d_\tau([|a|])$. Using comparison for soft elements (Theorem 2.4.10) and the fact that the only possible functionals on $Cu(A)$ are λ_0, λ_∞ and d_τ by Lemma 2.4.9, we see that $[|a|] \leqslant [(|a|-1/m)_+ \otimes 1_k] \ll [(|a|-1/2m)_+ \otimes 1_k]$ and so a belongs to $\mathrm{Ped}(A)$ by Proposition 2.4.1. The proof is complete. □

With this characterization of the Pedersen ideal we can prove that the σ-unital C^*-algebras satisfying the hypothesis of Proposition 2.4.11 are either algebraically simple or stable.

Theorem 2.4.12. *Let A be a simple σ-unital C^*-algebra with almost unperforated Cuntz semigroup such that $A \otimes \mathbb{K}$ has almost stable rank 1 and suppose that A admits a unique lower semicontinuous nontrivial 2-quasitrace τ (up to scalar multiples). Then either*

- *A is stable, or*
- *A is algebraically simple.*

If h is a strictly positive element for A, the case that A is stable corresponds to $d_\tau(h) = \infty$ and the case that A is algebraically simple corresponds to $d_\tau(h) < \infty$.

Proof. Since A is σ-unital, it admits a strictly positive element h. We distinguish different cases.
If $[h]$ is compact, then h belongs to $\mathrm{Ped}(A) \subset \mathrm{Ped}(A \otimes \mathbb{K})$ and so $A = \mathrm{her}(h)$ is algebraically simple.
Suppose now that $[h]$ is soft. If $d_\tau([h]) < \infty$, then by Proposition 2.4.11 h again belongs to $\mathrm{Ped}(A) \subset \mathrm{Ped}(A \otimes \mathbb{K})$ and so $A = \mathrm{her}(h)$ is algebraically simple.
If $d_\tau([h]) = \infty$, then by comparison $h \oplus h \sim h$ and so by Proposition 2.2.3 $A = \mathrm{her}(h)$ is large in $A \otimes \mathbb{K}$ and since $A \otimes \mathbb{K}$ has almost stable rank 1, Proposition 2.3.10 applies and $A \simeq A \otimes \mathbb{K}$.
Suppose now that A is both algebraically simple and stable, then it follows from [11] Theorem 1.2 that it cannot admit a nontrivial dimension function on $\mathrm{Ped}(A)$, contradicting the hypothesis. □

Remark 2.4.13. There are examples of non-algebraically simple, non-stable simple σ-unital C^*-algebras with stable rank 1 whose Cuntz semigroup is almost unperforated and stably finite. They were constructed by Blackadar in [10]; namely, in Corollary 4.5 it is proved that for any simple unital AF-algebra A and any F_σ subset of its space of bounded traces, it is possible to find an hereditary AF-subalgebra B of $A \otimes \mathbb{K}$ containing A such that the bounded traces on B are exactly the extensions coming from the F_σ set chosen. It is clear that this construction does not work if A has just one trace.

Example 2.4.14. *The C^*-algebra $\mathbb{K}$ of compact operators on a separable Hilbert space has stable rank 1 and a unique nontrivial trace, up to scalars. Its Cuntz semigroup is $Cu(\mathbb{K}) = \bar{\mathbb{N}}$, which is almost unperforated. By Theorem 2.4.12 any hereditary C^*-subalgebra of $\mathbb{K}$ is either algebraically simple (actually unital) or isomorphic to $\mathbb{K}$.*

Example 2.4.15. *The Jiang-Su algebra $\mathcal{Z}$ (see [41]) has stable rank 1, a unique trace and almost unperforated Cuntz semigroup equal to $Cu(\mathcal{Z}) = \mathbb{N} \sqcup (0, \infty]$. The hereditary C^*-subalgebras of $\mathcal{Z} \otimes \mathbb{K}$ generated by compact elements (corresponding to $\mathbb{N} \subset Cu(A)$) are unital; the ones corresponding to elements in $(0, \infty)$ are nonunital and algebraically simple.*

Example 2.4.16. *The C^*-algebra $\mathcal{W}$ introduced by Jacelon in [40] has stable rank* 1*, a unique trace and its Cuntz semigroup is $Cu(\mathcal{W}) = [0, \infty]$, which is almost unperforated (one could also just notice that $\mathcal{W}$ is $\mathcal{Z}$-stable). Thus any σ-unital C^*-algebra that is Morita-equivalent to $\mathcal{W}$ is either algebraically simple (in correspondence of elements in $(0, \infty) \subset Cu(\mathcal{W})$) or stable.*

Example 2.4.17. *In the following we will see another example: let Γ be a cocompact discrete subgroup of $\mathrm{SL}(2, \mathbb{R})$, then every hereditary C^*-subalgebra of the crossed product C^*-algebra of the horocycle flow on $\Gamma \backslash \mathrm{SL}(2, \mathbb{R})$ (see Chapter 3) is either algebraically simple or stable. Its Cuntz semigroup is $[0, \infty]$.*

In the situation of Theorem 2.4.12, the Cuntz semigroup has the following property:

Proposition 2.4.18. *Let A be a simple C^*-algebra admitting a unique nontrivial 2-quasitrace τ such that $A \otimes \mathbb{K}$ has almost stable rank* 1*. For any element $[a] \in Cu(A)$ we have*

$$\begin{aligned} [a] \quad \textit{is finite} \; &\Leftrightarrow \exists [b] \in Cu(A) \; : \; [a] \ll [b] \\ &\Leftrightarrow \exists [b] \in Cu(A) \; : \; [a] < [b]. \end{aligned}$$

Furthermore

$$\mathrm{Ped}(A \otimes \mathbb{K}) = \{ a \in A \otimes \mathbb{K} \; : \; [|a|] \;\; \textit{is finite} \; \}.$$

Proof. Note that the same argument used in Proposition 2.4.11 can be used to state that

$$\mathrm{Ped}(A \otimes \mathbb{K}) = \{ a \in A \otimes \mathbb{K} \; : \; d_\tau([|a|]) < \infty \}. \tag{2.1}$$

Using this, we will prove that an element $[a] \in Cu(A)$ is not finite if and only if $d_\tau([a]) = \infty$.
For suppose that $[a]$ is infinite. Then there is an element $[b] \neq 0$ in $Cu(A)$ such that $[a] + [b] = [a]$. Then $d_\tau([b]) + d_\tau([a]) = d_\tau([a])$. Since $[a] + n[b] = [a]$ for every $n \in \mathbb{N}$, we also have $\sup_n n[b] \leqslant [a] + \sup_n n[b] = \sup_n([a] + n[b]) = [a]$ and by simplicity this last term is Cuntz subequivalent to $\sup_n n[b]$. By definition of functional it follows that $d_\tau([a]) = \sup_n n d_\tau([b]) = \infty$.
On the other hand, if $d_\tau([a]) = \infty$, then $[a]$ is not compact (otherwise it would belong to the Pedersen ideal of $A \otimes \mathbb{K}$) and using again stable finiteness, we deduce that $[a]$ is soft. Then by comparison $[a] + [a] = [a]$ and $[a]$ is infinite.
Thus, for $a \in A \otimes \mathbb{K}$ we have

$$[|a|] \;\; \text{is not finite} \;\Leftrightarrow d_\tau([|a|]) = \infty \Leftrightarrow a \notin \mathrm{Ped}(A \otimes \mathbb{K}).$$

Otherwise stated

$$[|a|] \text{ is finite } \Leftrightarrow d_\tau([|a|]) < \infty \Leftrightarrow a \in \mathrm{Ped}(A \otimes \mathbb{K}).$$

Using Corollary 2.4.2 and (2.1) we obtain

$$[|a|] \text{ is finite } \Leftrightarrow \exists [b] \in Cu(A) \ : \ [|a|] \ll [b].$$

So we are left to prove that the way below relation on different elements is just Cuntz subequivalence.
Suppose we are given $[a]$ and $[b]$ in $Cu(A)$ with $[a] < [b]$. As usual we distinguish two cases.
If at least one between $[a]$ and $[b]$ is compact, then clearly $[a] \ll [b]$.
Hence suppose $[a]$ and $[b]$ are soft. We have $d_\tau([a]) < d_\tau([b])$. Recall that $d_\tau([b]) = \sup_{\epsilon>0} d_\tau([(b-\epsilon)_+])$ and observe that $d_\tau([(b-\epsilon)_+]) < d_\tau([b])$ for every $\epsilon > 0$, since otherwise we would have $[b] \leqslant [(b-t)_+]$ for some t. In particular there is $s > 0$ such that $d_\tau([a]) < d_\tau([(b-s)_+])$ and so $[a] \leqslant [(b-s)_+] \ll [b]$. □

As a consequence of the correspondence between Cuntz classes and countably generated Hilbert modules and of Proposition 2.4.18, we obtain the following

Theorem 2.4.19. *Let A be a σ-unital stable simple C^*-algebra with almost unperforated Cuntz semigroup, with almost stable rank 1 and admitting a unique lower semicontinuous nontrivial 2-quasitrace τ. Let x be a positive element in A, then we have*

$$[x] \ \textit{is infinite} \ \Leftrightarrow \mathrm{her}(x) \ \textit{is stable} \ \Leftrightarrow E_x \simeq A$$

and

$$[x] \ \textit{is finite} \ \Leftrightarrow \mathrm{her}(x) \ \textit{is algebraically simple} \ \Leftrightarrow E_x \Subset A.$$

Furthermore, if y is a positive element in A, then $[x] < [y]$ if and only if there exist elements $z \in A$ and e self adjoint in $\mathrm{her}(y)$ such that $\mathrm{her}(x) = \mathrm{her}(z^*z)$, $\mathrm{her}(zz^*) \subset \mathrm{her}(y)$, $ea = a$ for every $a \in \mathrm{her}(zz^*)$. This condition is also equivalent to the existence of a Hilbert module E over A such that $E_x \simeq E \Subset E_y$.

Proof. If A is simple, by [4] Lemma 5.2.8 a positive element $[x] \in A$ is infinite if and only if it is properly infinite, indeed, if there is a positive element $y \in A$ such that $[x] + [y] \leqslant [x]$, then $[x] + [x] \leqslant \sup_n([x] + n[y]) \leqslant [x]$.
We have already noticed that, as a consequence of [22] Theorem 3, $[x]$ is properly infinite if and only if $\mathrm{her}(x)$ is stable (see Proposition 2.3.11) and that, since A has almost stable rank 1 and x is Cuntz equivalent to any strictly positive element for

A, $E_x \simeq A$ as Hilbert modules. Also, again by Theorem 2.4.12, this is the case if and only if $d_\tau(x) = \infty$.
By Theorem 2.4.12 $\mathrm{her}(x)$ is algebraically simple if and only if $d_\tau(x) < \infty$ and this happens if and only if x belongs to $\mathrm{Ped}(A)$; by Proposition 2.4.18 this is equivalent to $[x]$ be finite. Note now that by Proposition 2.4.1 x belongs to $\mathrm{Ped}(A)$ if and only if $[x] \ll [h]$ for any strictly positive element h for A. The result follows from the equivalence between compact containment for classes of Hilbert modules and the way below relation (see [22] Theorem 1).
Suppose now that there is a positive element $y \in A$ such that $[x] < [y]$. By Proposition 2.4.18 this condition is equivalent to $[x] \ll [y]$, which in turn, by the equivalence between compact containment in the concrete and in the abstract sense proved in [22] Theorem 1, is equivalent to the existence of a Hilbert module E such that $E_x \precsim E \Subset E_y$. Thus, in the case A has almost stable rank 1, it follows by Theorem 3 of [22] (see Theorem 2.3.7 of the preceding section) that there exists a Hilbert module E' such that $E_x \simeq E' \subset E \Subset E_y$, hence in particular $E_x \simeq E' \Subset E_y$. Note that since A is σ-unital, x belongs to E_x and $E' = \overline{\Phi(x)A}$, with $\Phi(x) \in A$ since $E' \subset E_y$; it follows then from Proposition 4.3 of [51] that there is an element $z \in A$ such that $\mathrm{her}(x) = \mathrm{her}(z^*z)$ and $\mathrm{her}(\Phi(x)) = \mathrm{her}(zz^*)$. If now $e \in \mathbb{K}(E_y) = \mathrm{her}(y)$ is a self adjoint element such that $e|_{E_{\Phi(x)}} = \mathrm{id}|_{E_{\Phi(x)}}$, it follows that $ea = a$ for every $a \in \mathrm{her}(\Phi(x))$. Conversely, if there is a Hilbert module E such that $E_x \simeq E \Subset E_y$, it follows again by [22] Theorem 1 that $[x] \ll [y]$. Suppose now that there are an element $z \in A$ and e selfadjoint in $\mathrm{her}(y)$ such that $\mathrm{her}(x) = \mathrm{her}(z^*z)$, $\mathrm{her}(zz^*) \subset \mathrm{her}(y)$ and $ea = a$ for every $a \in \mathrm{her}(zz^*)$. Then again by Proposition 4.3 of [51] it follows that $E_x \simeq E_{zz^*}$ and since zz^* belongs to $\mathrm{her}(zz^*)$, it follows that $ezz^* = zz^*$, which implies $e|_{E_{zz^*}} = \mathrm{id}|_{E_{zz^*}}$ and hence $E_x \simeq E_{zz^*} \Subset E_y$. □

Remark 2.4.20. If in the previous Theorem we assume that A has stable rank 1 (and not just *almost* stable rank 1), then for an element $[x] \in Cu(A)$ it also holds that

$$[x] \text{ is compact } \Leftrightarrow \mathrm{her}(x) \text{ is unital } \Leftrightarrow E_x \text{ is finitely generated and projective .}$$

indeed, by Proposition 2.4.6 $[x]$ is compact if and only if $[x] = [p]$ for some projection $p \in A$ and by Proposition 2.3.8 in this case $\mathrm{her}(x)$ is unital. On the other hand, if $\mathrm{her}(x)$ is unital, then there is a strictly positive element in $\mathrm{her}(x)$, namely the identity $1_{\mathrm{her}(x)} \in \mathrm{her}(x)$, that is compact and since both x and $1_{\mathrm{her}(x)}$ are strictly positive elements in $\mathrm{her}(x)$, we have $1_{\mathrm{her}(x)} \sim x$ and $[x] = [1_{\mathrm{her}(x)}]$ is compact (note that we do not need to assume stable rank 1). The statement that $[x]$ is compact if and only if E_x is finitely generated and projective is contained in Corollary 5 of [22].

Chapter 3

Horocycle flow

In the first part of this chapter we review basic results about Hausdorfness and smoothness of quotients of spaces by group actions which can be found in [74] and [13] and then we recall the definition of horocycle and geodesic flow on quotients of $\mathrm{SL}(2,\mathbb{R})$ by cocompact discrete subgroups, including in Proposition 3.1.5 a proof of the freeness of the horocycle flow, following [27].
In the second section we recall the definition and some properties of hyperbolic Riemann surfaces, together with their connection to Fuchsian groups. References for the theory of Riemann surfaces are [35] and [32], while a standard reference for the theory of Fuchsian groups is [42].
In the third section we see what is the geometric counterpart of the geodesic and horocycle flow in the case the compact quotient of $\mathrm{SL}(2,\mathbb{R})$ is the unit tangent bundle of a hyperbolic Riemann surface; a good reference for this part is [26].
We end this chapter giving the statement of Hedlund's Theorem, concerning minimality of the horocycle flow and Furstenberg's Theorem, concerning the unique ergodicity of this flow. The proofs can be found respectively in [9] and [27].

3.1 Flows on compact quotients of $\mathrm{SL}(2,\mathbb{R})$

Let

$$\mathrm{SL}(2,\mathbb{R}) \coloneqq \{g \in M_2(\mathbb{R}) \mid \det(g) = 1\}$$

and endow it with the topology given by the inclusion $\mathrm{SL}(2,\mathbb{R}) \to \mathbb{R}^4$. With this topology, it is a Lie group.
We want to study properties of crossed product C^*-algebras arising from the action of a particular one-parameter subgroup of $\mathrm{SL}(2,\mathbb{R})$ on certain homogeneous spaces. We settle some notation that will be useful also in the following sections.

Following [9] II.3, we denote by A, N, N^- the following closed subgroups of $\mathrm{SL}(2,\mathbb{R})$:

$$A = \left\{ \begin{pmatrix} e^s & 0 \\ 0 & e^{-s} \end{pmatrix} \;:\; s \in \mathbb{R} \right\},$$

$$N = \left\{ \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \;:\; t \in \mathbb{R} \right\}, \qquad N^- = \left\{ \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} \;:\; t \in \mathbb{R} \right\}.$$

The spaces we want to consider will be quotients of $\mathrm{SL}(2,\mathbb{R})$ by discrete subgroups. In the following we will see that they are manifolds.

Lemma 3.1.1 ([74] Theorem 7.7)**.** *Let X be a topological Hausdorff space and $\sim$ an open equivalence relation (that is the quotient map $X \to X/_\sim$ is open). Then $R \coloneqq \{(x,y) \;:\; x \sim y\} \subset X \times X$ is closed if and only if $X/_\sim$ is Hausdorff.*

Proof. Suppose $X/\sim$ is Hausdorff. Pick $(x,y) \notin R$, that is $x \nsim y$. There are disjoint open sets $U \ni \pi(x)$ and $V \ni \pi(y)$. Let $\tilde{U} = \pi^{-1}(U) \ni x$, $\tilde{V} = \pi^{-1}(V) \ni y$. If the open set $\tilde{U} \times \tilde{V} \ni (x,y)$ intersects R, then it must contain a point (x',y') for which $x' \sim y'$ and so $\pi(x') = \pi(y')$ contrary to the assumption that $U \cap V = \emptyset$. Hence $\tilde{U} \times \tilde{V}$ does not intersect R and R is closed.
Conversely, suppose that R is closed, then given any two distinct points $\pi(x)$, $\pi(y) \in X/\sim$, there is an open set of the form $\tilde{U} \times \tilde{V}$ containing (x,y) and that does not contain points of R. Then $U = \pi(\tilde{U})$ and $V = \pi(\tilde{V})$ are disjoint sets and since $\sim$ is open, U and V are open sets. So $X/\sim$ is Hausdorff. □

The next Corollary is proved in [13] for the case of Lie groups, but the same proof applies to topological Hausdorff groups.

Corollary 3.1.2 ([13] Theorem III.7.12)**.** *Let G be a topological Hausdorff group and H a subgroup. Then the space $H\backslash G$ is Hausdorff if and only if H is closed in G.*

Proof. If $U \subset G$ is an open set, then $\pi^{-1}(\pi(U)) = \bigcup_{h\in H} hU$, that is open in G. By definition of quotient topology, this means that the quotient map is open.
Let H be a closed subgroup. We will check that $R = \{(g,g') \;:\; g = hg' \text{ for some } h \in H\}$ is a closed subset of $G \times G$. To this end, note that R is the inverse image of H under the continuous function

$$\begin{aligned} G \times G &\to G \\ (g,g') &\mapsto g{g'}^{-1} \end{aligned}$$

and the result follows by Lemma 3.1.1 since H is closed by assumption.
Conversely, if $H\backslash G$ is Haurdorff, then every point is closed and the same is true for its inverse image under the quotient map. Since G acts by homeomorphisms on itself, H is closed. □

The next Theorem is part of Theorem 8.3 in [13].

Theorem 3.1.3. *Let Γ be a discrete subgroup of a Lie group G. Then the quotient space $\Gamma\backslash G$ has the structure of a smooth manifold when endowed with the quotient topology.*

Proof. For every $x \in G$ we can choose a neighborhood $\tilde{U}$ such that $h\tilde{U} \cap \tilde{U} = \varnothing$ for $h \neq e$ (see [13] Definition 8.1 and the discussion that follows); hence $\pi_{\tilde{U}} := \pi|_{\tilde{U}}$ is a homeomorphism with its image U since π is both continuous and open. We can suppose that $\tilde{U}$ is a coordinate neghborhood $(\tilde{U}, \tilde{\phi})$. The map $\phi : U \to \tilde{\phi}(\tilde{U}) \subset \mathbb{R}^n$ is a homeomorphism; we call coordinate neighborhoods as the one just constructed *adimmsible.*
Suppose that $U = \pi(\tilde{U})$ and $V = \pi(\tilde{V})$ are overlapping admissible nighborhoods with coordinate maps $\phi = \tilde{\phi} \circ \pi_{\tilde{U}}^{-1}$ and $\psi = \tilde{\psi} \circ \pi_{\tilde{V}}^{-1}$. Altough $\tilde{U} \cap \tilde{V}$ may be empty, there must be an $h \in \Gamma$ such that $h\tilde{U} \cap \tilde{V} \neq \varnothing$. Since $\pi = \pi \circ h$ we may write $\psi^{-1} = \pi \circ \tilde{\psi}^{-1} = \pi \circ h \circ \tilde{\psi}^{-1}$ and so

$$\phi \circ \psi^{-1} = \tilde{\phi} \circ \pi_{\tilde{U}}^{-1} \circ \pi \circ h \circ \tilde{\psi}^{-1} = \tilde{\phi} \circ h \circ \tilde{\psi}^{-1}$$

and these maps are all C^∞. Of course the same holds for $\psi \circ \phi^{-1}$. Thus the admissible neighborhoods define a smooth manifold structure on $\Gamma\backslash G$. □

In virtue of Theorem 3.1.3, given a discrete subgroup of $\mathrm{SL}(2,\mathbb{R})$, the quotient space $\Gamma\backslash \mathrm{SL}(2,\mathbb{R})$ is a smooth manifold. There are two smooth flows associated to it that have been extensively studied in the past.

Definition 3.1.4 ([27] Section 11.3.1)**.** *Let Γ be a discrete subgroup of $\mathrm{SL}(2,\mathbb{R})$. The horocycle flow on the smooth manifold $M = \Gamma\backslash \mathrm{SL}(2,\mathbb{R})$ is the flow defined by*

$$\psi : \begin{array}{ccc} M \times N & \to & M \\ (\Gamma g, n) & \mapsto & \Gamma g n \end{array}.$$

The geodesic flow on $\Gamma\backslash \mathrm{SL}(2,\mathbb{R})$ is the flow defined by

$$\phi : \begin{array}{ccc} M \times A & \to & M \\ (\Gamma g, a) & \mapsto & \Gamma g a \end{array}.$$

Note that in the same way we could define the negative horocycle flow, as the one given by the action of the one-parameter subgroup N^-. This makes no difference from the dynamical point of view, since the map

$$\begin{aligned} \Gamma\backslash \mathrm{SL}(2,\mathbb{R}) &\to \Gamma\backslash \mathrm{SL}(2,\mathbb{R}) \\ \Gamma\gamma &\mapsto \Gamma\gamma \cdot \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \end{aligned}$$

is a topological conjugacy between the two flows for every discrete subgroup $\Gamma \subset \mathrm{SL}(2,\mathbb{R})$.
The geodesic and horocycle flows are related by the following formula:

$$\phi_{-s} \circ \psi_t \circ \phi_s = \psi_{te^{2s}}. \tag{3.1}$$

for every $s, t \in \mathbb{R}$. We now see a direct consequence of this relation. Since our interest will be focused on the horocycle flow on compact manifolds, we state the following result, which is a particular case of the more general result [27] Lemma 11.28.

Proposition 3.1.5. *Let* Γ *be a cocompact discrete subgroup of* $\mathrm{SL}(2,\mathbb{R})$*. Then the horocycle flow on* $\Gamma\backslash \mathrm{SL}(2,\mathbb{R})$ *is free.*

Proof. The flow

$$\begin{aligned} \mathrm{SL}(2,\mathbb{R}) &\to \mathrm{SL}(2,\mathbb{R}) \\ g &\mapsto g \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \end{aligned}$$

is free and since $\mathrm{SL}(2,\mathbb{R})$ is locally diffeomorphic to $\Gamma\backslash \mathrm{SL}(2,\mathbb{R})$, we can find an open cover of $\Gamma\backslash \mathrm{SL}(2,\mathbb{R})$ of open sets on which the geodesic flow is "locally" free. By compactness we can find a positive real number $r > 0$ such that $\psi_t(x) \neq x$ for every $x \in \Gamma\backslash \mathrm{SL}(2,\mathbb{R})$ and $0 < |t| < r$.
Suppose that $x_0 \in \Gamma\backslash \mathrm{SL}(2,\mathbb{R})$ is a periodic point for the horocycle flow with period t_0. Then, by (3.1), for every $s \in \mathbb{R}$ the point $\phi_{-s}(x_0)$ is periodic with period $r_0 e^{-2s}$, which is impossible. □

We will see in the following section that in particular cases the horocycle flow reduces to a flow on the unit tangent bundle of a Riemann surface.

3.2 Hyperbolic Riemann surfaces

A *Riemann surface* is a connected topological surface for which the local charts are holomorphic maps. If M and N are Riemann surfaces, we write $M \simeq N$ if there is a biholomorphism from M to N. We refer to [32] I.1 and [35] 1.1.1 for the definition and some examples.
The Riemann uniformization Theorem states that every simply connected Riemann surface is biholomorphic to one of the following (a proof can be found in [32] Chapter IV)

- the Riemann sphere $\mathbb{C} \cup \{\infty\}$,
- the complex plane $\mathbb{C}$,
- the hyperbolic plane $\mathbb{H}$,

where the local charts on $\mathbb{C} \cup \{\infty\}$ are given by $z \mapsto z$ on $\mathbb{C}$ and $z \mapsto 1/z$ on $(\mathbb{C} - \{0\}) \cup \{\infty\}$. The set underlying the hyperbolic plane $\mathbb{H}$ is $\{z \in \mathbb{C} \ : \ \Im(z) > 0\}$ and the local charts are given by $z \mapsto z$.
It follows that the universal (holomorphic) cover $\tilde{M}$ of a Riemann surface M has to be biholomorphic to one of $\mathbb{C} \cup \{\infty\}$, $\mathbb{C}$ or $\mathbb{H}$. Denote by $\mathrm{Aut}(\tilde{M})$ the group of biholomorphic maps from $\tilde{M}$ to itself; we have ([35] Proposition 1.27)

- $\mathrm{Aut}(\mathbb{C} \cup \{\infty\}) \simeq \mathrm{PSL}(2, \mathbb{C})$,
- $\mathrm{Aut}(\mathbb{C}) \simeq \mathrm{P}\Delta(2, \mathbb{C})$,
- $\mathrm{Aut}(\mathbb{H}) \simeq \mathrm{PSL}(2, \mathbb{R})$,

where

$$\mathrm{PSL}(2, \mathbb{C}) := \{\gamma \in \mathrm{Mat}_2(\mathbb{C}) \ : \ \det(\gamma) = 1\}/\{\pm 1\}$$

acts on $\mathbb{C} \cup \{\infty\}$ by Moebius transformations and $\mathrm{P}\Delta(2, \mathbb{C})$, $\mathrm{PSL}(2, \mathbb{R})$ are respectively the subgroup of upper-triangular complex matrices and of matrices with real entries; their action is given by restriction on the corresponding Riemann surfaces.
Now, given a Riemann surface M it turns out that, with few exceptions, its universal cover is $\mathbb{H}$:

Theorem 3.2.1 ([32] Theorem IV.6.3)**.** *If the universal cover of a Riemann surface M is $\mathbb{C} \cup \{\infty\}$ then $M \simeq \mathbb{C} \cup \{\infty\}$.*

Theorem 3.2.2 ([32] Theorem IV.6.4)**.** *If the universal cover of a Riemann surface M is $\mathbb{C}$, then M is biholomorphic to one of the following: $\mathbb{C}$, $\mathbb{C}^\times$ or the torus $\mathbb{T}^2$.*

Hence all the compact Riemann surfaces of genus greater than one admit as universal cover the hyperbolic plane $\mathbb{H}$. We will see that there is an abundance of representatives within this class.

Definition 3.2.3 ([8] page 2)**.** *A hyperbolic Riemann surface is a Riemann surface whose universal holomorphic cover is the hyperbolic plane* $\mathbb{H}$*.*

If M is a Riemann surface, its fundamental group $\pi_1(M)$ can be identified with the group of deck transformations of its (holomorphic) covering space $\mathrm{Deck}(\tilde{M}, \pi)$ in such a way that $M \simeq \pi_1(M)\backslash\tilde{M}$ ([35] Theorem 1.69). Thus, if M is hyperbolic, its fundamental group is a discrete subgroup of $\mathrm{Aut}(\mathbb{H}) = \mathrm{PSL}(2, \mathbb{R})$.
The condition for $\mathbb{H}$ to be the universal cover of a Riemann surface translates (as for any other notion of cover) into a prescription on the type of the fundamental group. We recall it in the following

Definition 3.2.4 ([42] 2.2)**.** *Let* Γ *be a discrete group acting on a topological space* X*. The action is properly discontinuous if for every* $x \in X$ *and* $K \subset X$ *compact, the set* $\{\gamma \in \Gamma \,:\, \gamma x \in K\}$ *is finite.*

At this point it is natural to ask which discrete subgroups of $\mathrm{PSL}(2, \mathbb{R})$ match this requirement.

Proposition 3.2.5 ([42] Theorem 2.2.6)**.** *Let* Γ *be a subgroup of* $\mathrm{PSL}(2, \mathbb{R})$*. Than* Γ *is discrete if and only if it acts properly discontinuously on the hyperbolic plane* $\mathbb{H}$*.*

Proof. Suppose Γ is not discrete. Then there is a sequence $\gamma_n \to e$ in Γ; hence $\gamma_n z \to z$ for every $z \in \mathbb{H}$ and the action cannot be properly discontinuous.
As to the converse, for any $z_0 \in \mathbb{H}$ and $K \subset \mathbb{H}$ compact, the set $E := \{\gamma \in \Gamma \,:\, \gamma z_0 \in K\}$ is closed since it is the inverse image of $K \cap \Gamma z_0$ under the map $\gamma \mapsto \gamma z_0$ from Γ to $\mathbb{H}$. As K is compact, there exist M, $\delta > 0$ such that

$$|z| \leqslant M, \qquad \Im(z) \geqslant \delta \qquad \forall z \in K.$$

Let

$$a = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in E.$$

Then

$$\left|\frac{az_0 + b}{cz_0 + d}\right| \leqslant M, \qquad \Im\left(\frac{az_0 + b}{cz_0 + d}\right) = \frac{\Im(z_0)}{|cz_0 + d|^2} \geqslant \delta.$$

Thus, setting $\alpha = \Im(z_0)$,

$$|cz_0 + d| \leqslant \sqrt{\alpha/\delta}, \qquad |az_0 + b| \leqslant M\sqrt{\alpha/\delta}$$

and the set of $(a, b, c, d) \in \mathbb{R}^4$ satisfying these inequalities is bounded. □

Remark 3.2.6 ([32] IV.5.3). The automorphism group of the sphere $\mathrm{Aut}(\mathbb{C} \cup \{\infty\}) = \mathrm{PSL}(2, \mathbb{C})$ contains a discrete subgroup whose action on $\mathbb{C} \cup \{\infty\}$ is not properly discontinuous. The Picard group

$$G_P = \{z \mapsto \frac{az + b}{cz + d} \ : \ ad - bc = 1 \ \text{ and } \ a, b, c, d \in \mathbb{Z}[i]\}$$

is discrete but every orbit is dense for its action on $\mathbb{C} \cup \{\infty\}$.

Definition 3.2.7 ([42] 2.2)**.** *A Fuchsian group is a discrete subgroup of* $\mathrm{PSL}(2, \mathbb{R})$.

If Γ_1 and Γ_2 are Fuchsian groups, we write $\Gamma_1 \sim_{\mathrm{PSL}(2,\mathbb{R})} \Gamma_2$ to mean that Γ_1 and Γ_2 are conjugated in $\mathrm{PSL}(2, \mathbb{R})$, that is, there is an element $\Phi \in PSL(2, \mathbb{R})$ such that $\Phi\Gamma_1\Phi^{-1} = \Gamma_2$.
One can show (see [35] Theorem 2.6) that for a *compact* hyperbolic Riemann surface of genus g, the fundamental group $\pi_1(M)$ has a presentation in terms of generators and relations

$$\pi_1(M) = \langle a_1, b_1, ..., a_g, b_g \mid \prod_{i=1}^{g}[a_i, b_i] = 1\rangle. \tag{3.2}$$

In particular, for fixed genus, they are all isomorphic. By the way they don't in general give rise to biholomorphic Riemann surfaces:

Proposition 3.2.8 ([35] Proposition 2.25)**.** *Let* $M = \Gamma_1\backslash\mathbb{H}$ *and* $N = \Gamma_2\backslash\mathbb{H}$ *be Riemann surfaces. Then* $M \simeq N$ *if and only if* $\Gamma_1 \sim_{\mathrm{PSL}(2,\mathbb{R})} \Gamma_2$.

Proof. Let $\phi : M \to N$ be a biholomorphism and $x \in M$, $y \in N$ such that $\phi(x) = n$. Let also $(\mathbb{H}, \pi_M)$ and $(\mathbb{H}, \pi_N)$ be the universal covers of M and N respectively. Pick $\tilde{x}$ and $\tilde{y}$ in $\mathbb{H}$ such that $\pi_M(\tilde{x}) = x$ and $\pi_N(\tilde{y}) = y$. By the universal property of the universal cover ([35] Theorem 1.69 (ii)), there is a unique biholomorphism $\Phi \in \mathrm{Aut}(\mathbb{H})$ such that $\Phi(\tilde{x}) = \tilde{y}$. Let now $\delta \in \Gamma_1$. We have

$$\pi_N \circ \Phi \circ \delta \circ \Phi^{-1} = \phi \circ \pi_M \circ \delta \circ \Phi^{-1} = \phi \circ \pi_M \circ \Phi^{-1} = \pi_N$$

and so $\hat{\delta} := \Phi \circ \delta \circ \Phi^{-1}$ is a deck transformation for $(\mathbb{H}, \pi_N)$.
To see that every deck transformation is obtained in this way, observe that we can

use the same argument to show that, given $\hat{\eta} \in \Gamma_2$, then $\Phi^{-1} \circ \hat{\eta} \circ \Phi = \eta$ belongs to Γ_1 and $\hat{\eta} = \Phi \circ \eta \circ \Phi^{-1}$.
Conversely, if there is $\Phi \in \mathrm{PSL}(2, \mathbb{R})$ such that $\Phi\Gamma_1\Phi^{-1} = \Gamma_2$, then the map

$$\begin{aligned} \Gamma_1\backslash\mathbb{H} &\to \Gamma_2\backslash\mathbb{H} \\ \Gamma_1\tilde{x} &\mapsto \Gamma_2\Phi(\tilde{x}) \end{aligned}$$

is a biholomorphism. □

Now, using 3.2 we can establish how many equivalence classes of biholomorphic compact hyperbolic Riemann surfaces are there.
Fix a genus $g > 1$. The generators of a cocompact Fuchsian group in the presentation given above belong to $\mathrm{PSL}(2, \mathbb{R})$, hence each of them correspond to three real parameters. The condition on the product of the commutators imposes three relations: this gives $6g - 3$ real parameters. Now we have to consider all the possible subgroups of $\mathrm{PSL}(2, \mathbb{R})$ which are conjugated to a certain Fuchsian group Γ, that is the ones of the form $\gamma\Gamma\gamma^{-1}$ for $\gamma \in \mathrm{PSL}(2, \mathbb{R})$. This condition reduces the number of real parameters by 3. Thus there are $6g - 6$ real parameters needed to count the classes of $\mathrm{PSL}(2, \mathbb{R})$-conjugated cocompact Fuchsian groups acting freely on $\mathbb{H}$. Hence the same is true for the corresponding classes of Riemann surfaces.

We give now a slight generalization of Proposition 3.2.8 for general holomorphic coverings. It will be used to interpret the construction of the C^*-algebras that we will define in a functorial way.

Proposition 3.2.9. *Let Γ, Γ' be Fuchsian groups acting freely on $\mathbb{H}$. There exists $\Phi \in \mathrm{PSL}(2, \mathbb{R})$ such that*

$$\Phi\Gamma\Phi^{-1} \leqslant \Gamma'$$

if and only if there is a holomorphic cover

$$\phi : \Gamma\backslash\mathbb{H} \to \Gamma'\backslash\mathbb{H}.$$

Furthermore, the group $\Phi\Gamma\Phi^{-1}$ has index n in Γ' if and only if p is an n-sheeted cover.

Proof. Suppose there is $\Phi \in \mathrm{PSL}(2, \mathbb{R})$ such that $\Phi\Gamma\Phi^{-1} \leqslant \Gamma'$.
From Proposition 3.2.8 there is a biholomorphism

$$\Gamma\backslash\mathbb{H} \simeq \Phi\Gamma\Phi^{-1}\backslash\mathbb{H}$$

and so we can suppose $\Gamma \leqslant \Gamma'$.
Denote by $\pi_\Gamma : \mathbb{H} \to \Gamma\backslash\mathbb{H}$ and $\pi_{\Gamma'} : \mathbb{H} \to \Gamma'\backslash\mathbb{H}$ the quotient maps.
The map

$$\pi_{\Gamma,\Gamma'} : \begin{array}{ccc} \Gamma\backslash\mathbb{H} & \to & \Gamma'\backslash\mathbb{H} \\ \Gamma\tilde{x} & \mapsto & \Gamma'\tilde{x} \end{array}$$

is surjective. Let $U \subset \mathbb{H}$ be small enough such that $\pi_{\Gamma'}|_U : U \to \pi_{\Gamma'}(U)$ is a biholomorphism; then the map $\pi_\Gamma(U) \to \pi_{\Gamma'} \circ \pi_\Gamma|^{-1}_{\pi_\Gamma(U)}(\pi_\Gamma(U)) = \pi_{\Gamma,\Gamma'}(\pi_\Gamma(U))$ is a biholomorphism and so $\pi_{\Gamma,\Gamma'}$ is holomorphic. Furthermore, the elements of the fiber of a point $\Gamma'\tilde{x}$ under $\pi_{\Gamma,\Gamma'}$ are in correspondence with the Γ-orbits of the translated of $\tilde{x}$ by elements of Γ'; hence it is a covering map and if Γ has index n in Γ', then $\pi_{\Gamma,\Gamma'}$ is an n-sheeted cover.
On the other hand, suppose that we are given a holomorphic cover

$$p : \Gamma\backslash\mathbb{H} \to \Gamma'\backslash\mathbb{H}.$$

From the lifting property of the coverings (see the discussion after Example 1.66 of [35]), there exists $\Phi \in \mathrm{Aut}(\mathbb{H}) = \mathrm{PSL}(2, \mathbb{R})$ such that

$$\pi_{\Gamma'} \circ \Phi = p \circ \pi_\Gamma.$$

Hence, for every $\gamma \in \Gamma$ we have

$$\begin{aligned} \pi_{\Gamma'} \circ \Phi \circ \gamma \circ \Phi^{-1} &= p \circ \pi_\Gamma \circ \gamma \circ \Phi^{-1} \\ &= p \circ \pi_\Gamma \circ \Phi^{-1} \\ &= \pi_{\Gamma'}, \end{aligned}$$

from which it follows that $\Phi\Gamma\Phi^{-1} \leqslant \Gamma'$. Note now that from the relation $\pi_{\Gamma'} \circ \Phi = p \circ \pi_\Gamma$ it also follows that for every $\tilde{x} \in \mathbb{H}$ the fiber $\{p^{-1}(\Gamma'\tilde{x})\}$ is in bijection with the set $\{(\Phi\Gamma\Phi^{-1})\gamma'\tilde{x}\}_{\gamma' \in \Gamma'}$ and so if p is an n-sheeted covering, $\Phi\Gamma\Phi^{-1}$ has index n in Γ'. □

3.3 Flows on the unit tangent bundle

Holomorphic maps are analytic, so infinitely complex differentiable and thus infinitely real differentiable. Hence, if M is a Riemann surface, it is in particular a smooth manifold and we can consider its tangent bundle $\mathrm{T}\,M$. Using this we will show how to construct a metric on M whose induced topology coincides with the

topology induced by the complex structure.
Define on the tangent space at $z = x + iy \in \mathbb{H}$ the Hermitian structure

$$\langle \xi^z, \eta^z \rangle_z := \frac{\xi^z_x \eta^z_x + \xi^z_y \eta^z_y}{y^2},$$

where ξ^z_x, ξ^z_y are respectively the x and y components of $\xi^z \in \mathrm{T}_z M \simeq \mathbb{R}^2$ and the same for η^z. This gives for every pair of smooth sections ξ, $\eta : M \to \mathrm{T} M$ a smooth function $z \mapsto \langle \xi^z, \eta^z \rangle_z$, which gives M the structure of a Riemannian manifold.
The induced metric (distance function), called *hyperbolic metric* (see [26] I.1), is given by

$$d_{\mathbb{H}}(z_1, z_2) := \inf_{l \in P^{z_2}_{z_1}} \int \langle l'(t), l'(t) \rangle^{1/2}_{l(t)},$$

where $P^{z_2}_{z_1}$ is the set of smooth paths from z_1 to z_2. It can be shown that the infimum appearing in the definition of this metric is realized by smooth curves called *geodesics*; furthermore they are always either segments of half-circles centered on the real line, or segments of straight lines perpendicular to the real axes ([26] proposition 1.4).
The topology induced by this metric coincides with the Euclidean topology (see [42] Theorem 1.3.3 and Exercise 1.11) and hence with the topology induced by the complex structure as well.
The relationship between the complex structure and this metric is even deeper. The following is proved in [26] Section I.1.

Proposition 3.3.1. *The action of the group of holomorphic automorphisms of the hyperbolic plane* $\mathrm{PSL}(2, \mathbb{R})$ *is isometric and preserves the angles.*

Proof. Let $\gamma \in \mathrm{PSL}(2, \mathbb{R})$, $z \in \mathbb{H}$ and $\xi \in \mathrm{T}_z \mathbb{H}$. Then, denoting by $\mathrm{d}\gamma$ the differential of γ,

$$\mathrm{d}\gamma_z \xi = \frac{\xi}{(cz + d)^2} \quad \text{in } \mathrm{T}_{\gamma(z)} \mathbb{H}$$

and

$$\Im(\gamma(z)) = \frac{\Im(z)}{|cz + d|^2}.$$

So it is straightforward to verify that, for $\eta \in \mathrm{T}_z \mathbb{H}$,

$$\begin{aligned} |\langle \mathrm{d}\gamma_z \xi, \mathrm{d}\gamma_z \eta \rangle_{\gamma(z)}| &= \frac{|\xi_x \eta_x + \xi_y \eta_y|}{|cz + d|^4} \cdot \frac{|cz + d|^4}{(\Im z)^2} \\ &= |\langle \xi, \eta \rangle_z| \end{aligned}$$

and
$$\frac{\langle \mathrm{d}\gamma_z\xi, \mathrm{d}\gamma_z\eta\rangle_{\gamma(z)}}{\langle \mathrm{d}\gamma_z\xi, \mathrm{d}\gamma_z\xi\rangle^{1/2}\langle \mathrm{d}\gamma_z\eta, \mathrm{d}\gamma_z\eta\rangle^{1/2}} = \frac{\langle \xi, \eta\rangle_z}{\langle \xi, \xi\rangle^{1/2}\langle \eta, \eta\rangle^{1/2}}. \qquad \square$$

Even more is true. In fact, $\mathrm{PSL}(2,\mathbb{R})$ is *exactly* the group of isometries of $\mathbb{H}$ that preserve the angles by the proof of [42] Theorem 1.3.1.
Note now that the hyperbolic plane $\mathbb{H}$ is parallelizable $\mathrm{T}\,\mathbb{H} \simeq \mathbb{R}^2 \times \mathbb{H}$ and so also its unit tangent bundle

$$\mathrm{T}_1\,\mathbb{H} := \{(z,\zeta) \in \mathrm{T}\,\mathbb{H} \ : \ \langle \zeta, \zeta\rangle = 1\}$$

is trivial: $\mathrm{T}_1\,\mathbb{H} \simeq S^1 \times \mathbb{H}$. Since the action of $\mathrm{PSL}(2,\mathbb{R})$ on $\mathbb{H}$ is isometric, there is a well defined action on the unit tangent bundle $\mathrm{T}_1\,\mathbb{H}$. By the way we can also consider $\mathrm{PSL}(2,\mathbb{R})$ as a homogeneus space over itself:

Proposition 3.3.2 ([9] Lemma II.3.1)**.** *There is an isomorphism of* $\mathrm{PSL}(2,\mathbb{R})$*-spaces*
$$\mathrm{PSL}(2,\mathbb{R}) \simeq \mathrm{T}_1\mathbb{H}.$$

Proof. Since the maps $z \mapsto az + b$ with $a > 0$, $b \in \mathbb{R}$, for $z \in \mathbb{H}$ are elements of $\mathrm{PSL}(2,\mathbb{R})$, the action of $\mathrm{PSL}(2,\mathbb{R})$ on $\mathbb{H}$ is transitive. We want to show that its action on $\mathrm{T}_1\,\mathbb{H}$ is simply transitive, i.e. it is transitive and the stabilizer of each point is trivial.
Let (z_1,ζ_1) and (z_2,ζ_2) be two elements of $\mathrm{T}_1\,\mathbb{H}$ and choose elements γ_1, γ_2 in $\mathrm{PSL}(2,\mathbb{R})$ such that $\gamma_1(z_1) = \gamma_2(z_2) = i$. The stabilizer of i is the group

$$K := \{\gamma(\theta) \in \mathrm{PSL}(2,\mathbb{R}) \ : \ \gamma(\theta)(z) = \frac{z\cos\theta - \sin\theta}{z\sin\theta + \cos\theta}, \quad \theta \in \mathbb{R}\}$$

and for $(i,\zeta) \in \mathrm{T}_{1,i}\,\mathbb{H}$, the action of an element $\gamma(\theta) \in K$ reads $\gamma(\theta)(i,\zeta) = (i, \mathrm{d}_{\gamma(\theta)}\zeta) = (i, e^{-2i\theta}\zeta)$. In particular, we can find an element $\gamma(\theta) \in K$ such that $(\gamma_2^{-1} \circ \gamma(\theta) \circ \gamma_1)(z_1,\zeta_1) = (z_2,\zeta_2)$ and so the action of $\mathrm{PSL}(2,\mathbb{R})$ on $\mathrm{T}_1\,\mathbb{H}$ is transitive.
Suppose that $\eta \in \mathrm{PSL}(2,\mathbb{R})$ fixes a point $(z,\zeta) \in \mathrm{T}_1\,\mathbb{H}$. As before, we can find an element $\gamma \in \mathrm{PSL}(2,\mathbb{R})$ such that $\gamma(z) = i$. Let $\eta \in \mathrm{T}_{1,i}\,\mathbb{H}$ be such that $\gamma(z,\zeta) = (i, \mathrm{d}_\gamma\zeta) = (i,\eta)$. Thus $\gamma \circ \eta \circ \gamma^{-1}$ fixes (i,η) and so it has to be an element of K of the form $\gamma(\theta)$ for some $\theta \in \mathbb{R}$. Since $\eta = \mathrm{d}_{\gamma(\theta)}\eta = e^{-2i\theta}\eta$, it turns out that θ is a multiple of π and so $\gamma(\theta) = \mathrm{id}$. $\square$

When defining the geodesic and horocycle flow on the unit tangent bundle of a Riemann surface, we will use an explicit form of the homeomorphism $\mathrm{T}_1\,\mathbb{H} \simeq$

$\mathrm{PSL}(2,\mathbb{R})$; namely, the one sending an element $(z,\zeta) \in \mathrm{T}_1\,\mathbb{H}$ to the unique element $\gamma^{(z,\zeta)}$ belonging to $\mathrm{PSL}(2,\mathbb{R})$ such that $\gamma^{(z,\zeta)}(i,\zeta_0) = (z,\zeta)$, where ζ_0 is the unit vector pointing upwards.
From Proposition 3.3.2 it immediately follows the following

Proposition 3.3.3. *Let M be a hyperbolic Riemann surface with corresponding Fuchsian group Γ. Then there is a homeomorphism*

$$\mathrm{T}_1\,M \simeq \Gamma\backslash\,\mathrm{PSL}(2,\mathbb{R}),$$

where the Riemannian structure on $\mathrm{T}\,M$ is the one induced by the quotient map $\mathbb{H} \to M$.

Proof. Consider the quotient map

$$\pi_\Gamma : \mathbb{H} \to M.$$

Its differential induces for every $z \in \mathbb{H}$ the equivalence

$$\Gamma\backslash\,\mathrm{T}_z\,\mathbb{H} = \mathrm{T}_{\pi_\Gamma(z)}(\Gamma\backslash\mathbb{H}).$$

The hermitian structure on $\mathrm{T}\,M$ is the one induced by π_Γ and so

$$\Gamma\backslash\,\mathrm{T}_{1,z}\,\mathbb{H} = \mathrm{T}_{1,\pi_\Gamma(z)}(\Gamma\backslash\mathbb{H}).$$

Since the homeomorphism $\mathrm{T}_1\,\mathbb{H} \to \mathrm{PSL}(2,\mathbb{R})$ commutes with the left action of elements of $\mathrm{PSL}(2,\mathbb{R})$, we obtain a homeomorphism

$$\Gamma\backslash\,\mathrm{PSL}(2,\mathbb{R}) \simeq \Gamma\backslash\,\mathrm{T}_1\,\mathbb{H} = \mathrm{T}_1(\Gamma\backslash\mathbb{H}). \quad \square$$

We will be concerned with two flows on the unit tangent bundle $\mathrm{T}_1\,\mathbb{H}$ of a compact hyperbolic Riemann surface M. They are the *geodesic* and *horocycle* flow (as will be clear the choice of the names is consistent with the situation studied in the previous section). We will see how they look in both descriptions of $\mathrm{T}_1\,\mathbb{H}$ and then we will push them forward on the compact Riemann surface via the universal covering map.
References for the following discussion are [26] Chapters III and IV, [9] Chapters II and IV.
Let $\pi_p : \mathrm{SL}(2,\mathbb{R}) \to \mathrm{PSL}(2,\mathbb{R})$ be the quotient map.

The geodesic flow[[9] II, II.3]: Let $(z, \zeta) \in \mathrm{T}_1\,\mathbb{H}$. It determines a unique geodesic l passing through $z \in \mathbb{H}$ in the direction ζ. Let z_t be the point at (hyperbolic) distance t from z in the direction of ζ and let ζ_t be the unit tangent vector to l in z_t in the same direction. The map

$$(z, \zeta) \mapsto (z_t, \zeta_t)$$

defines a flow, which we call the geodesic flow and write

$$\phi_t(z, \zeta) := (z_t, \zeta_t).$$

Proposition 3.3.4 ([9] Proposition II.3.2). *The geodesic flow on* $\mathrm{T}_1\,\mathbb{H}$ *corresponds to the right multiplication*

$$\begin{aligned} \mathrm{PSL}(2, \mathbb{R}) &\to \mathrm{PSL}(2, \mathbb{R}) \\ \gamma &\mapsto \gamma \cdot \pi_p(g_t), \end{aligned}$$

where

$$g_t := \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix}.$$

Proof. Consider the point $(i, \zeta_0) \in \mathrm{T}_1\,\mathbb{H}$, where, as in the discussion after Theorem 3.3.2, ζ_0 is the unit tangent vector at i pointing in the positive direction of the imaginary axes. Then

$$\phi_t(i, \zeta_0) = g_t(i, \zeta_0).$$

Given $(z, \zeta) \in \mathrm{T}_1\,\mathbb{H}$, let $\gamma^{(z,\zeta)}$ be the unique element of $\mathrm{PSL}(2, \mathbb{R})$ such that

$$(z, \zeta) = \gamma^{(z,\zeta)}(i, \zeta_0).$$

Since $\gamma^{(z,\zeta)}$ is an isometry, in particular it maps geodesics to geodesics and preserves the distance; hence we have

$$\phi_t(\gamma^{(z,\zeta)}(i, \zeta_0)) = \gamma^{(z,\zeta)}(\phi_t(i, \zeta_0)) = (\gamma^{(z,\zeta)} \cdot g_t)(i, \zeta_0). \qquad \square$$

The horocycle flow[[9], IV.1]: Let $(z, \zeta) \in \mathrm{T}_1\,\mathbb{H}$ and let l be the unique geodesic determined by (z, ζ). For $t \in \mathbb{R}$ let C_t be the hyperbolic circle with center z_t and passing through z. When $t \to \infty$ (respectively $t \to -\infty$), C_t converges to a circle tangent to the x-axes or to a line parallel to the x-axes and having ζ as inward (respectively outward) normal vector ζ.
This "limit circle" $C_{+\infty(-\infty)}$ is the positive (respectively negative) horocycle deter-

mined by (z, ζ). We define a flow on $\mathrm{T}_1\,\mathbb{H}$ as follows.
Let $z^t_{+(-)}$ be the point on the horocycle $C_{+\infty(-\infty)}$ at a distance t from z with respect to the positive (respectively negative) orientation of the circle. Let $\zeta^t_{+(-)}$ be the unit inward (respectively outward) normal vector to $C_{+\infty(-\infty)}$ at $z^t_{+(-)}$. The one-parameter group of transformations

$$\psi_t^{+(-)} : \begin{array}{ccc} \mathrm{T}_1\,\mathbb{H} & \to & \mathrm{T}_1\,\mathbb{H} \\ (z,\zeta) & \mapsto & (z^t_{+(-)}, \zeta^t_{+(-)}) \end{array}$$

is the positive (negative) horocycle flow.

Proposition 3.3.5 ([26] Theorem IV.1.1)**.** *Under the identification* $\mathrm{T}_1\,\mathbb{H} = \mathrm{PSL}(2,\mathbb{R})$*, the flows* $\psi^{+(-)}$ *correspond to*

$$\psi^{+(-)} : \begin{array}{ccc} \mathrm{PSL}(2,\mathbb{R}) & \to & \mathrm{PSL}(2,\mathbb{R}) \\ \gamma & \mapsto & \gamma \cdot \pi_p(h_t^{\pm}), \end{array}$$

where

$$h_t^+ := \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, \qquad h_t^- := \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}.$$

Proof. Horocycles are preserved under the action of $\mathrm{PSL}(2,\mathbb{R})$ and so, as in the case of the geodesic flow, it is sufficient to check the assertion on the point $(i, \zeta_0) \in \mathrm{T}_1\,\mathbb{H}$, where ζ_0 is the unit tangent vector at i pointing in the positive direction of the imaginary axes. For $(z^t_+, \zeta^t_+) = \psi^+_t(i, \zeta_0)$ we have $z^t_+ = i + t$, $\zeta^t_+ = \zeta_0$ and so $(z^t_+, \zeta^t_+) = h^+_t(i, \zeta_0)$. Similarly, the point $h^-_t i = i/(it+1)$ belongs to the euclidean circle $C_{i/2,1/2}$ of radius $1/2$ centered in $i/2$ and the vector $i/(i+t)^2 = (\mathrm{d}h^-_t)_i \zeta_0$ is the outward normal vector to $C_{i/2,1/2}$ in the point $h^-_t i$ and so $\psi^-_t(i, \zeta_0) = h^-_t(i, \zeta_0)$. $\square$

Both the geodesic and (positive and negative) horocycle flows on $\mathrm{PSL}(2,\mathbb{R})$ induce flows on the unit tangent bundle of any compact hyperbolic Riemann surface in the obvious way.
As in the previous section, we can restrict ourselves to the positive horocycle flow and denote it simply by ψ_t.

Applying the same proof of Proposition 3.1.5 we obtain

Proposition 3.3.6. *Let* M *be a compact hyperbolic Riemann surface. Then the horocycle flow on its unit tangent bundle is free.*

Let $\pi_p : \mathrm{SL}(2,\mathbb{R}) \to \mathrm{PSL}(2,\mathbb{R})$ be the quotient map and Γ_p be a discrete subgroup of $\mathrm{PSL}(2,\mathbb{R})$. Denote by $\pm\Gamma_p := \pi_p^{-1}(\Gamma_p) \subset \mathrm{SL}(2,\mathbb{R})$ the *symmetrization* of Γ_p. We will see in the following Lemma that we can identify the quotient space $\Gamma_p\backslash \mathrm{PSL}(2,\mathbb{R})$ with $\pm\Gamma_p\backslash \mathrm{SL}(2,\mathbb{R})$ (as $\mathrm{SL}(2,\mathbb{R})$-spaces). In particular Γ_p is cocompact in $\mathrm{PSL}(2,\mathbb{R})$ if and only if $\pi_p^{-1}(\Gamma_p)$ is cocompact in $\mathrm{SL}(2,\mathbb{R})$ and the two dynamical systems $(\pm\Gamma_p\backslash \mathrm{SL}(2,\mathbb{R}), \mathbb{R}, \psi)$ and $(\Gamma_p\backslash \mathrm{PSL}(2,\mathbb{R}), \mathbb{R}, \psi)$ are isomorphic.

Lemma 3.3.7. *Let G be a topological Hausdorff group and N a discrete normal subgroup. Let*

$$\pi : G \to G/N$$

be the quotient map. If H is a discrete subgroup of G/N, then $\pi^{-1}(H)$ is a discrete subgroup of G.
Furthermore, there is a G-homeomorphism $\phi : G/\pi^{-1}(H) \to (G/N)/H$ such that if we denote by $\rho_H : G/N \to (G/N)/H$ and $\rho_{\pi^{-1}(H)} : G \to G/\pi^{-1}(H)$ the quotient maps, we have

$$\phi \circ \rho_{\pi^{-1}(H)} = \rho_H \circ \pi.$$

Proof. Since N is normal, the quotient map is a group homomorphism and then the inverse image of a subgroup of G/N is a subgroup of G. Thus $\pi^{-1}(H)$ is a subgroup of G.
$\pi^{-1}(H)$ is closed in G since π is continuous and H is closed in G/N.
By definition of discreteness, given an element $x \in G/N$, there exists an open set U containing x such that $U \cap H = \{x\}$. Then $\pi^{-1}(x) = \pi^{-1}(U) \cap \pi^{-1}(H)$ and since $\pi^{-1}(H)$ is a discrete subgroup of G, for every $\tilde{x} \in \pi^{-1}(x)$ there exists an open set V containing $\tilde{x}$ such that $\pi^{-1}(x) \cap V = \{\tilde{x}\}$. Thus $(V \cap \pi^{-1}(U)) \cap \pi^{-1}(N) = \{\tilde{x}\}$ and $\pi^{-1}(H)$ is discrete.
Note that $\pi^{-1}(H) = NH$. The map

$$\begin{aligned} G/\pi^{-1}(H) &\to (G/N)/H \\ g(NH) &\mapsto (gN)H \end{aligned}$$

defines a homeomorphism that commutes with left multiplication by elements of G.
□

We will not go into the proofs of other dynamical properties of the horocycle flow, for which we refer to Chapter 11 of [27] for the case of discrete subgroups of $\mathrm{SL}(2,\mathbb{R})$ and [26] for the case of the unit tangent bundle of hyperbolic Riemann surfaces.
In the next chapter we will make use of the following two results.

Theorem 3.3.8 (Hedlund). *If* Γ *is a discrete cocompact subgroup of* $\mathrm{SL}(2,\mathbb{R})$*, then the horocycle flow on* X_Γ *is minimal.*

A proof of Hedlund's Theorem can be found in [9] Chapter 4 Theorem 1.9.

Theorem 3.3.9 (Furstenberg). *If* Γ *is a cocompact discrete subgroup of* $\mathrm{SL}(2,\mathbb{R})$*, then the horocycle flow on* X_Γ *is uniquely ergodic.*

The original proof of Furstenberg's Theorem can be found in [33]; a different approach (due tu Margulis) can be found in [27] Theorem 11.27.

Chapter 4

The C^*-algebras

The results of the previous chapter confine the crossed product C^*-algebras associated to the horocycle flow on compact quotients of $\mathrm{SL}(2,\mathbb{R})$ by discrete subgroups to the class of C^*-algebras studied in the last section of Chapter 2. Hence we can apply the results contained there to this particular situation and this is the content of Theorem 4.1.2, Theorem 4.1.4 and Theorem 4.1.5.
In the second section of this chapter we study the functoriality of the crossed product construction associated to the horocycle flow, exhibiting both a covariant and a contravariant functor, the first induced by inclusion modulo conjugation of discrete subgroups of $\mathrm{SL}(2,\mathbb{R})$ and the second one reflecting the properness of surjective maps between homogeneous spaces given by inclusions of groups with finite index. This is the content of Proposition 4.2.10.
In the last section of this chapter we specialize to the case of compact hyperbolic Riemann surfaces and the induced contravariant functor defined in Proposition 4.3.5 sends finite sheeted holomorphic coverings to injective $*$-homomorphisms.

4.1 Properties of the crossed product C^*-algebra of the horocycle flow

If G is a connected Lie group acting smoothly on a manifold M, then the manifold is foliated by the orbits of G (see [49] Definition 2.1 for the definition of a foliated manifold) and one can associate to this foliation a groupoid, called the *holonomy groupoid* and the corresponding reduced C^*-algebra ([49] Definition 6.3). If the action of the group is free, the holonomy groupoid coincides with the groupoid $M \times G$ and in virtue of [49] Proposition 6.5 the reduced C^*-algebra of this groupoid coin-

cides with the reduced crossed product C^*-algebra $C_0(M) \rtimes_r G$. Furthermore, in the case M is compact, every Radon measure on M that is invariant under G gives an invariant transverse Radon measure (see [21] 5.α) and by Theorem 6.30 of [49] the invariant transverse Radon measures are in bijection with the lower semicontinuous densely defined traces on the C^*-algebra $C(M) \rtimes_r G$. In particular, in the uniquely ergodic case, there is only such a trace (up to a scalar).
Assuming that $C(M) \rtimes_r G$ is exact, then by [43] every lower semicontinuous 2-quasitrace on it is a trace and then in the uniquely ergodic case, $C(M) \rtimes_r G$ admits just one lower semicontinuous 2-quasitrace.
In virtue of Proposition 3.3.6, Hedlund's Theorem 3.3.8 and Furstenberg's Theorem 3.3.9, what we have just said applies to the case of the action of $\mathbb{R}$ given by the horocycle flow on a homogeneus space of the form $\Gamma\backslash \mathrm{SL}(2,\mathbb{R})$, with Γ a cocompact subgroup of $\mathrm{SL}(2,\mathbb{R})$. So the C^*-algebra $C(\Gamma\backslash \mathrm{SL}(2,\mathbb{R})) \rtimes \mathbb{R}$ (we will always avoid writing the action explicitly since we will only consider the one induced on $C(\Gamma\backslash \mathrm{SL}(2,\mathbb{R}))$ by the horocycle flow on $\Gamma\backslash \mathrm{SL}(2,\mathbb{R})$) is simple and with a unique lower semicontinuous trace (up to scalars).
Furthermore, it follows from [38] Corollary 9.1 that the horocycle flow ψ has finite Rokhlin dimension and then by [38] Theorem 3.5 that $C(\Gamma\backslash \mathrm{SL}(2,\mathbb{R})) \rtimes \mathbb{R}$ has finite nuclear dimension. In particular, by [77] it is $\mathcal{Z}$-stable and by [64] its Cuntz semigroup is almost unperforated. Then we see that all the hypothesis of Theorem 2.4.12 are satisfied, except possibly the hypothesis that its stabilization has almost stable rank 1. Note however that by Corollary 6.7 of [38] we can infer that this C^*-algebra is stable and so we would "only" need to check that it has almost stable rank 1 (rather then its stabilization).

To simplify notation, in the following, whenever we are given a discrete subgroup Γ of $\mathrm{SL}(2,\mathbb{R})$, we will denote the corresponding homogeneous space $\Gamma\backslash \mathrm{SL}(2,\mathbb{R})$ by X_Γ.

Proposition 4.1.1. *Let Γ be a cocompact discrete subgroup of $\mathrm{SL}(2,\mathbb{R})$. The C^*-algebra $C(X_\Gamma) \rtimes \mathbb{R}$ has almost stable rank 1.*

Proof. By [59] Corollary 3.2 it is enough to check that $C(X_\Gamma) \rtimes \mathbb{R}$ is projectionless. The proof of this fact is due to A. Connes (see [21] page 129): Let τ be the unique trace on $C(X_\Gamma) \rtimes \mathbb{R}$ associated to the ψ-invariant measure on X_Γ. Using relation (3.1) we see that the geodesic flow ϕ induces a flow $\check{\phi}$ on $C(X_\Gamma) \rtimes \mathbb{R}$ given by

$$\check{\phi}_s(f)(t)(x) = e^{-2s} f(e^{-2s}t)(\phi_{-s}(x))$$

for $f \in C_c(\mathbb{R}, C(X_\Gamma))$, $s, t \in \mathbb{R}$ and $x \in X_\Gamma$. Suppose now that there is a non-zero projection $p \in C(X_\Gamma) \rtimes \mathbb{R}$; the map $p \mapsto \check{\phi}_s(p)$ is norm-continuous and so $\tau(p) = \tau \circ \check{\phi}_s(p) = e^{-2s}\tau(p)$ for every $s \in \mathbb{R}$, which entails $\tau(p) = 0$. Since τ is faithful, we have $p = 0$. □

As a consequence of Proposition 2.4.11 and Theorem 2.4.12 we obtain

Theorem 4.1.2. *Let Γ be a cocompact discrete subgroup of* $\mathrm{SL}(2, \mathbb{R})$ *and τ be the lower semicontinuous trace associated to the unique invariant probability measure for the horocycle flow on X_Γ. Then*

$$\mathrm{Ped}(C(X_\Gamma) \rtimes \mathbb{R}) = \{a \in C(X_\Gamma) \rtimes \mathbb{R} \ : \ d_\tau([|a|]) < \infty\}.$$

Every hereditary C^*-subalgebra of $C(X_\Gamma) \rtimes \mathbb{R}$ is either algebraically simple or isomorphic to $C(X_\Gamma) \rtimes \mathbb{R}$.

Proof. Since X_Γ is a manifold (see Theorem 3.1.3) and $\mathbb{R}$ is second countable, $C(X_\Gamma) \rtimes \mathbb{R}$ is separable. Thus every hereditary C^*-subalgebra is singly generated. Nevertheless, since it admits a unique nontrivial lower semicontinuous trace, the same is true for every hereditary C^*-subalgebra (since a lowersemicontinuous trace extends in a unique way to the stabilization). By [43] and nuclearity of $C(X_\Gamma) \rtimes \mathbb{R}$ (and since nuclearity is a stable property) every lower semicontinuous 2-quasitrace on every hereditary C^*-subalgebra is actually a lower semicontinuous trace, hence every hereditary C^*-subalgebra of $C(X_\Gamma) \rtimes \mathbb{R}$ satisfies the hypothesis of Theorem 2.4.12. □

We proceed now to present another dynamical description of the C^*-algebra $C(X_\Gamma) \rtimes \mathbb{R}$.
As already observed by Furstenberg in [33], the homogeneous space $\mathrm{SL}(2, \mathbb{R})/N$ can be identified with the space $\mathbb{E} := \mathbb{R}^2 - \{0\}$ and the left action of $\mathrm{SL}(2, \mathbb{R})$ on $\mathrm{SL}(2, \mathbb{R})/N$ translates into the action given by matrix multiplication of vectors. To see this, choose an orthonormal basis $\{e_1, e_2\}$ on $\mathbb{R}^2$ and consider the action of $\mathrm{SL}(2, \mathbb{R})$ on $\mathbb{E} := \mathbb{R}^2 - \{0\}$ given by matrix multiplication of vectors with respect to the chosen basis

$$g \cdot v = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} av_1 + bv_2 \\ cv_1 + dv_2 \end{pmatrix},$$

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{R}), \ v = v_1 e_1 + v_2 e_2 \in \mathbb{E}.$$

The stabilizer of the point $(1,0)^T$ is N and so the homogeneus space $\mathrm{SL}(2,\mathbb{R})/N$ can be identified with $\mathbb{E}$.
Let Γ be a discrete subgroup of $\mathrm{SL}(2,\mathbb{R})$; the action of $\mathrm{SL}(2,\mathbb{R})$ on $\mathbb{E}$ restricts to an action of Γ and in virtue of Proposition 1.5.4 there is a Morita equivalence between the full crossed products

$$C_0(X_\Gamma) \rtimes \mathbb{R} \simeq_{M.e.} C_0(\mathbb{E}) \rtimes \Gamma,$$

where, as for the case of the reals acting on $C(X_\Gamma)$, we avoid writing explicitly the action of Γ on $C_0(\mathbb{E})$, since it will be always understood that it is the action induced by the matrix multiplication of vectors.
The next Lemma will permit us to show that the above Morita equivalence induces a $*$-isomorphism.

Lemma 4.1.3. *Let A be a C^*-algebra and G be a discrete group acting on A. Suppose that there is a densely defined lower semicontinuous G-invariant trace τ on A such that whenever $\{x_n\}$ is a sequence of positive elements with bounded trace converging to a positive element x with bounded trace, then $\tau(x_n) \to \tau(x)$, i.e. τ is continuous on its domain. Then there is a densely defined lower semicontinuous trace on $A \rtimes_r G$.*

Proof. Let $E : A \rtimes_r G \to A$ be the faithful conditional expectation sending an element $x = \sum_{g\in G} x_g u_g$ belonging to $C_c(G, A)$ to x_e. If τ is a densely defined lower semicontinuous trace on A, then $\mathrm{dom}(\tau) = \{a \in A_+ : \tau(a) < \infty\}$ is dense in A. Let $x = \sum_{g\in G} x_g u_g$ be an element in $C_c(G, A)$ and $\epsilon > 0$; take $x_\epsilon \in \mathrm{dom}(\tau)$ such that $\|x_\epsilon - x_e\|_{A\rtimes_r G} = \|x_\epsilon - x_e\|_A < \epsilon$. The element $y = \sum_{g\in G} y_g u_g$ defined by

$$y_g := \begin{cases} y_g = x_g & \text{for } g \neq e \\ y_e = x_\epsilon \end{cases}$$

is such that

$$\|x - y\|_{A\rtimes_r G} = \|x_\epsilon - x_e\|_A < \epsilon.$$

This proves that $E^{-1}(\mathrm{dom}\,\tau)$ is dense in $A \rtimes_r G$.
Let now x be an element in $A \rtimes_r G$ such that x^*x belongs to $E^{-1}(\mathrm{dom}(\tau))$; since $\mathrm{dom}(\tau)$ is dense in A_+, if we take approximants x_n for x in $C_c(G, \mathrm{Ped}(A))$, it follows that both $x_n^*x_n$ and $x_nx_n^*$ belong to $C_c(G, \mathrm{dom}(\tau))$. By the G-invariance of τ, for each approximant x_n we have $\tau \circ E(x_n^*x_n) = \tau \circ E(x_nx_n^*)$ and since $x_n^*x_n$ converges to x^*x, then $E(x_n^*x_n)$ converges to $E(x^*x)$ and so by hypothesis

$\tau \circ E(x_n x_n^*) = \tau \circ E(x_n^* x_n) \to \tau \circ E(x^* x)$. Note now that by lower semicontinuity $\tau \circ E(xx^*) \leqslant \lim \tau \circ E(x_n x_n^*) < \infty$ and so $\tau \circ E(x_n x_n^*) \to \tau \circ E(xx^*)$. Hence $\tau \circ E(x^* x) = \tau \circ E(xx^*)$.
Suppose now that $\tau \circ E(x^* x) = \infty$. Then by lower semicontinuity $\tau \circ E(x^* x) \leqslant \lim \tau \circ E(x_n^* x_n) = \lim \tau \circ E(x_n x_n^*) = \infty$. Suppose that $\tau \circ E(xx^*) < \infty$, then we should have $\tau \circ E(x_n x_n^*) \to \tau \circ E(xx^*)$, a contradiction. □

Theorem 4.1.4. *Let Γ be a discrete cocompact subgroup of $\mathrm{SL}(2,\mathbb{R})$. Then*

$$C(X_\Gamma) \rtimes \mathbb{R} \simeq C_0(\mathbb{E}) \rtimes \Gamma.$$

Proof. By stability of $C(\Gamma \backslash \mathrm{SL}(2,\mathbb{R})) \rtimes \mathbb{R}$ we can identify $C_0(\mathbb{E}) \rtimes \Gamma$ with a hereditary C^*-subalgebra of it. The Lebesgue measure μ_L on $\mathbb{E}$ is invariant for the action of $\mathrm{SL}(2,\mathbb{R})$ and by the Lebesgue dominated convergence Theorem, the associated trace on $C_0(\mathbb{E})$ satisfies the hypothesis of Lemma 4.1.3; hence it induces a lower semicontinuous trace on $C_0(\mathbb{E}) \rtimes \Gamma$ and hence a lower semicontinuous trace τ on $C(\Gamma \backslash \mathrm{SL}(2,\mathbb{R})) \rtimes \mathbb{R}$. But $C_0(\mathbb{E}) \rtimes \Gamma$ contains $C_0(\mathbb{E})$ as a C^*-subalgebra and we can find an element $f \in C_0(\mathbb{E})$ such that $d_\tau(f) = \mu_L(\mathrm{supp}(f)) = \infty$, where we have used Proposition 1.3.4. By simplicity, if h is a strictly positive element in $C_0(\mathbb{E}) \rtimes \Gamma$, then $d_\tau([h]) = \infty$ and so $C_0(\mathbb{E}) \rtimes \Gamma$ is stable by Theorem 1.3 of [59]. □

As a consequence of Theorem 4.1.4 and Theorem 2.4.19, there is a particularly easy description of the Hilbert modules for $C(\Gamma \backslash \mathrm{SL}(2,\mathbb{R})) \rtimes \mathbb{R}$.

Theorem 4.1.5. *Let Γ be a cocompact discrete subgroup of $\mathrm{SL}(2,\mathbb{R})$. Every countably generated right Hilbert module for $C(\Gamma \backslash \mathrm{SL}(2,\mathbb{R})) \rtimes \mathbb{R} \simeq C_0(\mathbb{E}) \rtimes \Gamma$ is isomorphic to one of the form*

$$\overline{f \cdot (C_0(\mathbb{E}) \rtimes \Gamma)}, \qquad f \in C_0(\mathbb{E}).$$

For two such Hilbert modules we have

$$\overline{f \cdot (C_0(\mathbb{E}) \rtimes \Gamma)} \simeq \overline{g \cdot (C_0(\mathbb{E}) \rtimes \Gamma)} \qquad \Leftrightarrow \qquad \mu_L(\mathrm{supp}(f)) = \mu_L(\mathrm{supp}(g))$$

and there exists a Hilbert module E such that $\overline{f \cdot (C_0(\mathbb{E}) \rtimes \Gamma)} \simeq E \Subset \overline{g \cdot (C_0(\mathbb{E}) \rtimes \Gamma)}$ if and only if $\mu_L(\mathrm{supp}(f)) < \mu_L(\mathrm{supp}(g))$.

Proof. As already pointed out in Proposition 4.1.4 the C^*-algebra $C_0(\mathbb{E}) \rtimes \Gamma$ has almost stable rank 1 and is projectionless; thus by Proposition 2.4.6 its Cuntz semigroup does not contain compact elements. Hence the countably generated Hilbert modules correspond to soft elements and Cuntz equivalence of soft elements is implemented by the unique (up to scalar multiples) nontrivial functional associated to the unique (up to scalar multiples) lower semicontinuous trace τ. By Lemma 4.1.3 and using $\mathrm{SL}(2,\mathbb{R})$-invariance of the Lebesgue measure on $\mathbb{E}$ it follows that all the possible values in the range of the dimension function are obtained by Cuntz equivalence classes of elements in $C_0(\mathbb{E})$ since, for every $f \in C_0(\mathbb{E})$, we have $d_\tau(f) = \mu_L(\mathrm{supp}(f))$. The result follows from Theorem 2.4.19. □

4.2 Functoriality

Consider the categories $\mathfrak{G}_{d,f}(2,\mathbb{R})$, $\mathfrak{G}_d(2,\mathbb{R})$, $\mathfrak{G}_d^c(2,\mathbb{R})$, $\mathfrak{C}$, $\mathfrak{C}_i$ and $\mathfrak{C}_i^c$, where

- $\mathfrak{G}_d^{(c)}(2,\mathbb{R})$ is the category whose objects are discrete (cocompact) subgroups of $\mathrm{SL}(2,\mathbb{R})$ and the morphisms are inclusions modulo conjugation in $\mathrm{SL}(2,\mathbb{R})$,
- $\mathfrak{G}_{d,f}(2,\mathbb{R})$ is the category whose objects are discrete subgroups of $\mathrm{SL}(2,\mathbb{R})$ and the morphisms are inclusions of finite index modulo conjugation in $\mathrm{SL}(2,\mathbb{R})$,
- $\mathfrak{C}_{(i)}$ is the category whose objects are separable nuclear C^*-algebras in the UCT class and a morphism $\phi : A \to B$ between objects of $\mathfrak{C}_{(i)}$ is a(n injective) $*$-homomorphism from A to B,
- $\mathfrak{C}_i^c$ is the full subcategory of $\mathfrak{C}_i$ whose objects are separable, simple, stable, projectionless C^*-algebras with a unique trace that are in the UCT class.

Note that if $H \subset K \subset G$ are group inclusions and the index of H in K is n, the index of K in G is m, then the index of H in G is at most nm, hence $\mathfrak{G}_{d,f}(2,\mathbb{R})$ is indeed a category.

In Lemma 4.2.1 and Lemma 4.2.2 we check functoriality for general dynamical systems and together with Lemma 4.2.4 and Lemma 4.2.5 they will be used in Proposition 4.2.10 to interpret the crossed product construction coming from the horocycle flow as a contravariant functor $\mathfrak{G}_{d,f}(2,\mathbb{R}) \to \mathfrak{C}_\mathrm{i}$.
We also want to exibhit covariant functors $\mathfrak{G}_d(2,\mathbb{R}) \to \mathfrak{C}$ and $\mathfrak{G}_d^c(2,\mathbb{R}) \to \mathfrak{C}_i^c$; to this end we prove Lemma 4.2.6 and Lemma 4.2.8.

The following two Lemmata are well known and for completeness we give a hint of their proof.

Lemma 4.2.1. *Let (A, G, α) and (B, G, β) be two C^*-dynamical systems and let $\phi : A \to B$ be an equivariant $*$-homomorphism:*

$$\phi \circ \alpha_t = \beta_t \circ \phi \qquad \text{for all } t \in G.$$

Then there exists a $*$-homomorphism

$$\phi \rtimes G : A \rtimes G \to B \rtimes G,$$

If ϕ is either injective or surjective, the same is true for $\phi \rtimes G$.
If ϕ is a $*$-isomorphism, then $\phi \rtimes G$ is a $*$-isomorphism.

Proof. The existence of such a $*$-homomorphism is stated in [75] Corollary 2.48. It is the extension of the algebraic $*$-homomorphism $\Phi : C_c(G, A) \to C_c(G, B)$ defined by $\Phi(f)(s) = \phi(f(s))$.
If ϕ is injective, then $\ker(\phi \rtimes G) = (\ker \phi) \rtimes G$ by [75] Proposition 3.19 and so also $\phi \rtimes G$ is injective. If ϕ is surjective, then $\phi \rtimes G$ maps $C_c(G, A)$ onto $C_c(G, B)$ and since a $*$-homomorphism is automatically closed, $\phi \rtimes G$ is surjective.
It follows that if ϕ is a $*$-isomorphism, $\phi \rtimes G$ is a $*$-isomorphism. □

Lemma 4.2.2. *Let (X, α, G) and (Y, β, G) be dynamical systems with X, Y locally compact and Hausdorff and G a locally compact group. Denote by $\hat{\alpha}$ and $\hat{\beta}$ the corresponding actions on $C_0(X)$ and $C_0(Y)$ respectively. Let $\phi : X \to Y$ be an equivariant proper continuous map. Then there is a $*$-homomorphism*

$$\hat{\phi} \rtimes G : C_0(Y) \rtimes G \to C_0(X) \rtimes G.$$

It is injective if ϕ is surjective and it is a $*$-isomorphism if ϕ is a topological conjugacy.

Proof. Since ϕ is proper, the map $\hat{\phi} : C_0(Y) \to C_0(X)$ sending f to $f \circ \phi$ is a $*$-homomorphism, which is injective if ϕ is surjective. Furthermore for every $g \in G$ and $f \in C_0(Y)$ we have

$$\begin{aligned}
(\hat{\alpha}_g \circ \hat{\phi})(f) &= \hat{\alpha}_g(f \circ \phi) \\
&= f \circ \phi \circ \alpha_{g^{-1}} \\
&= f \circ \beta_{g^{-1}} \circ \phi \\
&= \hat{\phi}(f \circ \beta_{g^{-1}}) \\
&= (\hat{\phi} \circ \hat{\beta}_g)(f)
\end{aligned}$$

and thus $\hat{\phi}$ is an equivariant $*$-homomorphism from the C^*-dynamical system $(C_0(Y), G, \hat{\beta})$ to the C^*-dynamical system $(C_0(X), G, \hat{\alpha})$. The result follows by applying Lemma 4.2.1. □

The following three Lemmata concern homogeneus spaces and the purpose here is to prove that given two discrete subgroups H and K of $\mathrm{SL}(2,\mathbb{R})$ and an element $g \in \mathrm{SL}(2,\mathbb{R})$ such that $gHg^{-1} \subset K$, the map $H\backslash \mathrm{SL}(2,\mathbb{R}) \to K\backslash \mathrm{SL}(2,\mathbb{R})$ sending the coset Hs to Kgs for $s \in \mathrm{SL}(2,\mathbb{R})$ is proper. These results, together with Lemma 4.2.1 and Lemma 4.2.2 will be used in Proposition 4.2.10 to construct the aforementioned contravariant functor from $\mathfrak{G}^{(c)}_{d,f}(2,\mathbb{R})$ to $\mathfrak{C}^{(c)}_i$.

Lemma 4.2.3. *Let H be a discrete cocompact subgroup of a locally compact group G and g an element of G. Then gHg^{-1} is a cocompact discrete subgroup of G.*

Proof. Since the action of G on itself is given by homeomorphisms, if x is an element of H and U is an open set in G such that $U \cap H = \{x\}$, then $gUg^{-1} \cap gHg^{-1} = \{gxg^{-1}\}$ and gUg^{-1} is open. Then gHg^{-1} is a discrete subgroup of G.
Let $\{U_\alpha\}_{\alpha\in\Lambda}$ be an open cover of $gHg^{-1}\backslash G$. Then $\{g^{-1}U_\alpha\}_{\alpha\in\Lambda}$ is an open cover of $H\backslash G$ and thus gives a finite cover $\{V_i\}_{i=1}^n$. Then $\{gV_i\}_{i=1}^n = \{U_i\}_{i=1}^n$ is a finite cover of $gHg^{-1}\backslash G$. □

In the same spirit of Proposition 3.2.8 we have the following

Lemma 4.2.4. *Let H be a closed subgroup of a locally compact group G and let $g \in G$. The map*

$$\Phi_g : \begin{array}{ccc} H\backslash G & \to & (gHg^{-1})\backslash G \\ Hs & \mapsto & (gHg^{-1})gs \end{array}$$

is an isomorphism of G-spaces.

Proof. The map Φ_g is well defined and bijective.
Let U be an open subset of $(gHg^{-1})\backslash G$. Then $\Phi_g^{-1}(U) = g^{-1}U$ is open in $H\backslash G$. Thus Φ_g is continuous.
Let s be an element of G. The following diagram

$$\begin{array}{ccc} H\backslash G & \xrightarrow{\Phi_g} & (gHg^{-1})\backslash G \\ \big\downarrow{\scriptstyle \cdot(s^{-1})} & & \big\downarrow{\scriptstyle \cdot(s^{-1})} \\ H\backslash G & \xrightarrow{\Phi_g} & (gHg^{-1})\backslash G \end{array}$$

is commutative.
The same applies to

$$\Phi_g^{-1}: \begin{array}{ccc} (gHg^{-1})\backslash G & \to & H\backslash G \\ (gHg^{-1})k & \mapsto & Hg^{-1}k \end{array}$$

and so we have proved the claim. □

Let $H \subset K$ be an inclusion of closed subgroups of a locally compact group G. Note that the surjective map $\phi : H\backslash G \to K\backslash G$ that sends a coset Hg to Kg is continuous and respects the right action of G.
The problem of determining wether the crossed product construction is a functor from $\mathfrak{G}_{d,f}(2,\mathbb{R})$ to $\mathfrak{C}_i$ is closely related to the problem wether given an inclusion of discrete subgroups $\Gamma' \subset \Gamma$ of $\mathrm{SL}(2,\mathbb{R})$, the map $\Gamma'\backslash \mathrm{SL}(2,\mathbb{R}) \to \Gamma\backslash \mathrm{SL}(2,\mathbb{R})$ sending $\Gamma' g$ to Γg for $g \in \mathrm{SL}(2,\mathbb{R})$ is proper and the answer relies on the possibility to cover a coset Kg, viewed as a subset of $\mathrm{SL}(2,\mathbb{R})$, with a finite number of cosets Hg_i, $i = 1, ..., n$ (compare with Proposition 3.2.9).

Lemma 4.2.5. *Let $H \subset K$ be closed subgroups of a locally compact group G and suppose that H has finite index in K. Then the surjective map*

$$\phi : H\backslash G \to K\backslash G$$
$$Hg \mapsto Kg$$

is proper.

Proof. We want to show that ϕ is a finite-sheeted covering map.
First we prove that ϕ is open. Let $\rho : G \to H\backslash G$ and $\pi : G \to K\backslash G$ be the quotient maps. Let $U \subset H\backslash G$ be an open set, $U = \{Hg_\alpha\}_{g_\alpha \in \Lambda}$ for a given index set $\Lambda \subset G$. The image under ϕ is $\phi(U) = \{Kg_\alpha\}_{g_\alpha \in \Lambda}$. Now the set $\rho^{-1}(U) = H\Lambda$ is open by definition and since the quotient map π is given by a group action, it is open and $\pi(\rho^{-1}(U)) = \{Kg\}_{g \in H\Lambda} = \{Kg_\alpha\}_{g_\alpha \in \Lambda}$ is open.
By definition of index, there exist $k_1, ..., k_n \in K$ such that

$$K = Hk_1 \cup ... \cup Hk_n.$$

Hence, given $g \in G$, the elements $k_i g$, $i = 1, ..., n$ are such that

$$H(k_1 g) \cup ... \cup H(k_n g) = Kg.$$

Let now Kg be an element of $K\backslash G$ and consider the set $\phi^{-1}(Kg) = \{Hg' : Kg' = Kg\}$. We want to show that $\#\phi^{-1}(Kg) \leqslant n$. For let $Hg' \in \phi^{-1}(Kg)$; then $Kg' =$

$Kg = Hk_1g \cup ... \cup Hk_ng$. In particular there exist $k \in K$ such that $g' = kg$ and $h \in H$, $i \in \{1, ..., n\}$ such that $kg = hk_ig$. Thus

$$Hg' = Hkg = Hhk_ig = Hk_ig.$$

Since Hg' was an arbitrary element of $\phi^{-1}(Kg)$, we can conclude that

$$\phi^{-1}(Kg) = H(k_1g) \cup ... \cup H(k_ng).$$

Now let $H(k_ig) \in \phi^{-1}(Kg)$. Using the Hausdorff property, we can find an open subset U of G containing g such that $Hk_iU \cap Hk_jU = \emptyset$ for all $j \neq i$, $1 \leqslant j \leqslant n$. The map $\phi|_{Hk_iU} : Hk_iU \to KU$ is a homeomorphism. Thus ϕ is a finite sheeted covering map, hence proper. □

Now we come to the necessary observations for the construction of covariant functors from $\mathfrak{G}_d$ to $\mathfrak{C}$ and from $\mathfrak{G}_d^c$ to $\mathfrak{C}_i^c$.
The functoriality of both the reduced and full group C^*-algebra constructions with respect to inclusions of discrete groups are proved in [18] Section 2.5. For our purposes, we will need to give an analogous statement at the level of full crossed products.

Lemma 4.2.6. *Let A be a C^*-algebra and Γ, Γ' be discrete groups acting on A by means of actions α^Γ and $\alpha^{\Gamma'}$ respectively. Suppose that $\Gamma \subset \Gamma'$ as groups and that $\alpha^{\Gamma'}|_\Gamma = \alpha^\Gamma$. Then there is a $*$-homomorphism*

$$\Phi : A \rtimes \Gamma \to A \rtimes \Gamma'.$$

Φ is unital if A is a unital C^*-algebra.

Proof. Let f and g be elements in $C_c(\Gamma, A)$; write them as $f = \sum_{\gamma \in \Gamma} f_\gamma u_\gamma$ and $g = \sum_{\eta \in \Gamma} g_\eta u_\eta$ respectively, where f_γ and g_γ are elements in A and they are different from zero only for finitely many γ's in Γ. Define the map

$$\begin{aligned} \Phi_c : C_c(\Gamma, A) &\to C_c(\Gamma', A) \\ \sum_{\gamma \in \Gamma} f_\gamma u_\gamma &\mapsto \sum_{\gamma'} \hat{f}_{\gamma'} v_{\gamma'}, \end{aligned}$$

where

$$\hat{f}_{\gamma'} = \begin{cases} f_\gamma & \text{for } \gamma = \gamma' \in \Gamma \\ 0 & \text{for } \gamma' \notin \Gamma \end{cases}.$$

Note that Φ_c is linear. We now see that it is an algebraic $*$-homomorphism.

$$\begin{aligned}\Phi_c(f * g) &= \Phi_c(\sum_{\gamma,\eta\in\Gamma} f_\gamma \alpha^\Gamma_\gamma(g_\eta) u_{\gamma\eta}) \\ &= \Phi_c(\sum_{\zeta\in\Gamma} h_\zeta u_\zeta) \\ &= \sum_{\zeta'\in\Gamma'} \hat{h}_{\zeta'} v_{\zeta'},\end{aligned}$$

where $h_\zeta \coloneqq \sum_{\gamma\eta=\zeta} f_\gamma \alpha^\Gamma_\gamma(g_\eta)$.

On the other hand

$$\begin{aligned}\Phi_c(f) * \Phi_c(g) &= \sum_{\gamma'\in\Gamma'} \hat{f}_{\gamma'} \alpha^{\Gamma'}_{\gamma'}(\hat{g}_{\eta'}) v_{\gamma'\eta'} \\ &= \sum_{\zeta'\in\Gamma'} \bar{h}_{\zeta'} v_{\zeta'},\end{aligned}$$

where $\bar{h}_{\zeta'} \coloneqq \sum_{\gamma'\eta'=\zeta'} \hat{f}_{\gamma'} \alpha^{\Gamma'}_{\gamma'}(\hat{g}_{\eta'})$. Note now that for each term in the sum we have

$$\begin{aligned}\hat{f}_{\gamma'} \alpha^{\Gamma'}_{\gamma'}(\hat{g}_{\eta'}) &= \begin{cases} f_\gamma \alpha^{\Gamma'}_\gamma(g_\eta) & \text{for } \gamma' = \gamma, \eta' = \eta \in \Gamma \\ 0 & \text{for } \gamma' \text{ or } \eta' \notin \Gamma \end{cases} \\ &= \begin{cases} f_\gamma \alpha^{\Gamma}_\gamma(g_\eta) & \text{for } \gamma' = \gamma, \eta' = \eta \in \Gamma \\ 0 & \text{for } \gamma' \text{ or } \eta' \notin \Gamma \end{cases}.\end{aligned}$$

It follows that $\bar{h} = \hat{h}$ and hence $\Phi_c(f * g) = \Phi_c(f) * \Phi_c(g)$.
We check now that Φ_c is involutive:

$$\begin{aligned}(\Phi_c(f))^* &= (\sum_{\gamma'\in\Gamma'} \hat{f}^*_{\gamma'} v_{\gamma'})^* \\ &= \sum_{\gamma'\in\Gamma'} \alpha^{\Gamma'}_{(\gamma')^{-1}}(\hat{f}^*_{\gamma'}) v_{(\gamma')^{-1}} \\ &= \sum_{\gamma'\in\Gamma'} \alpha^{\Gamma'}_{\gamma'}(\hat{f}^*_{(\gamma')^{-1}}) v_{\gamma'}.\end{aligned}$$

On the other hand we have

$$\begin{aligned}\Phi_c(f^*) &= \Phi_c(\sum_{\gamma\in\Gamma} \alpha^{\Gamma}_{\gamma^{-1}}(f^*_{\gamma})u_{\gamma^{-1}}) \\ &= \Phi_c(\sum_{\gamma\in\Gamma} \alpha^{\Gamma}_{\gamma}(f^*_{\gamma^{-1}})u_{\gamma}) \\ &= \sum_{\gamma'\in\Gamma'} (\alpha^{\Gamma}_{\gamma'}(f^*_{(\gamma')^{-1}}))^{\wedge} v_{\gamma'}\end{aligned}$$

and the result follows since for every $\gamma' \in \Gamma'$

$$\begin{aligned}(\alpha^{\Gamma}_{\gamma'}(f^*_{(\gamma')^{-1}}))^{\wedge} &= \begin{cases}\alpha^{\Gamma}_{\gamma}(f^*_{\gamma^{-1}}) & \text{for } \gamma' = \gamma \in \Gamma \\ 0 & \text{for } \gamma' \notin \Gamma\end{cases} \\ &= \begin{cases}\alpha^{\Gamma'}_{\gamma}(f^*_{\gamma^{-1}}) & \text{for } \gamma' = \gamma \in \Gamma \\ 0 & \text{for } \gamma' \notin \Gamma\end{cases} \\ &= \alpha^{\Gamma'}_{\gamma'}(\hat{f}^*_{(\gamma')^{-1}}).\end{aligned}$$

Note that we have $\|\Phi_c(f)\|_1 = \|f\|_1$ and so we can extend Φ_c to a $*$-homomorphism of involutive Banach algebras from $L^1(\Gamma, A)$ to $L^1(\Gamma', A)$. If $\{h_\lambda\}$ is an approximate unit for A, then it is also an approximate unit for $L^1(\Gamma, A)$ and $\{\Phi_c(h_\lambda)\} = \{h_\lambda\}$ is an approximate unit for $L^1(\Gamma', A)$; thus every non-degenerate representation of $L^1(\Gamma', A)$ restricts to a non-degenerate representation of $L^1(\Gamma, A)$.
Hence Φ_c extends to a $*$-homomorphism

$$\Phi : A \rtimes \Gamma \to A \rtimes \Gamma'.$$

If A is unital, then $\Phi(1u_e) = 1v_e$ and Φ is unital. □

As a consequence, we have the following

Corollary 4.2.7. *Let $\Gamma' \subset \Gamma$ be an inclusion of discrete subgroups of* $\mathrm{SL}(2,\mathbb{R})$. *The map*

$$\Phi_c : \begin{array}{ccc} C_c(\Gamma, C_0(\mathbb{E})) & \to & C_c(\Gamma', C_0(\mathbb{E})) \\ \sum_{\gamma\in\Gamma} f_\gamma u_\gamma & \mapsto & \sum_{\gamma'\in\Gamma'} \hat{f}_{\gamma'} v_{\gamma'}\end{array}$$

extends to a $*$-homomorphism $\Phi : C_0(\mathbb{E}) \rtimes \Gamma \to C_0(\mathbb{E}) \rtimes \Gamma'$. In the case Γ' is cocompact, it is injective.

Proof. If Γ' is cocompact, then $C_0(\mathbb{E}) \rtimes \Gamma$ is simple. □

In the same spirit we have the following

Lemma 4.2.8. *Let $\alpha : G \to \mathrm{Aut}(A)$ be an action of a locally compact group G on a C^*-algebra A. Suppose that Γ and Γ' are discrete subgroups of G and that there is an element $g \in G$ such that $g\Gamma g^{-1} = \Gamma'$. Then there is a $*$-isomorphism*

$$\Psi : A \rtimes \Gamma \to A \rtimes \Gamma',$$

where the actions of Γ and Γ' on A are given by restriction of the action of G.

Proof. Denote by α^{Γ} and $\alpha^{\Gamma'}$ the actions of Γ and Γ' respectively on A. We keep using the same notation as in Lemma 4.2.6. Define the map

$$\begin{aligned} \Psi : C_c(\Gamma, A) &\to C_c(\Gamma', A) \\ \sum_{\gamma\in\Gamma} f_\gamma u_\gamma &\mapsto \sum_{\gamma\in\Gamma} \alpha_g(f_\gamma) v_{g\gamma g^{-1}} \end{aligned}$$

First we want to show that Ψ_c is an algebraic $*$-homomorphism. Let $f = \sum_{\gamma\in\Gamma} f_\gamma u_\gamma$ and $h = \sum_{\eta\in\Gamma} h_\eta u_\eta$ be elements of $C_c(\Gamma, A)$. We have

$$\begin{aligned} \Psi_c(f * h) &= \Psi_c\big((\sum_{\gamma\in\Gamma} f_\gamma u_\gamma)(\sum_{\eta\in\Gamma} h_\eta u_\eta)\big) \\ &= \Psi_c\big(\sum_{\gamma,\eta\in\Gamma} f_\gamma \alpha^{\Gamma}_\gamma(h_\eta) u_{\gamma\eta}\big) \\ &= \sum_{\gamma,\eta\in\Gamma} \alpha_g(f_\gamma)\alpha_{g\gamma}(h_\eta) v_{g\gamma\eta g^{-1}} \\ &= \big(\sum_{\gamma\in\Gamma} \alpha_g(f_\gamma) v_{g\gamma g^{-1}}\big)\big(\sum_{\eta} \alpha_g(h_\eta) v_{g\eta g^{-1}}\big) \\ &= \Psi_c(f) * \Psi_c(h). \end{aligned}$$

Ψ_c is also involutive:

$$\begin{aligned}
(\Psi_c(f))^* &= (\Psi_c(\sum_{\gamma\in\Gamma} f_\gamma u_\gamma))^* \\
&= (\sum_{\gamma\in\Gamma} \alpha_g(f_\gamma) v_{g\gamma g^{-1}})^* \\
&= \sum_{\gamma\in\Gamma} \alpha^{\Gamma'}_{g\gamma^{-1}g^{-1}}((\alpha_g(f_\gamma))^*) v_{g\gamma^{-1}g^{-1}} \\
&= \sum_{\gamma\in\Gamma} \alpha_{g\gamma^{-1}g^{-1}}(\alpha_g(f^*_\gamma)) v_{g\gamma^{-1}g^{-1}} \\
&= \sum_{\gamma\in\Gamma} \alpha_{g\gamma^{-1}}(f^*_\gamma) v_{g\gamma^{-1}g^{-1}} \\
&= \sum_{\gamma\in\Gamma} \alpha_g(\alpha_{\gamma^{-1}}(f^*_\gamma)) v_{g\gamma^{-1}g^{-1}} \\
&= \Psi_c(\sum_{\gamma\in\Gamma} \alpha^{\Gamma}_{\gamma^{-1}}(f^*_\gamma) u_{\gamma^{-1}}) \\
&= \Psi_c((\sum_{\gamma\in\Gamma} f_\gamma u_\gamma)^*) \\
&= \Psi_c(f^*).
\end{aligned}$$

Hence Ψ_c is an algebraic $*$-homomorphism. Note that $\|\Psi_c(f)\|_1 = \|f\|_1$; if $\{h_\lambda\}$ is an approximate unit for A, then it is also an approximate unit for both $L^1(\Gamma, A)$ and $L^1(\Gamma', A)$ and the same is true for $\alpha_g(h_\lambda)$. Thus every non-degenerate $*$-representation of $L^1(\Gamma', A)$ induces a non-degenerate $*$-representation of $L^1(\Gamma, A)$ and Ψ_c extends to a $*$-homomorphism Ψ, whose inverse is the extension of the algebraic $*$-homomorphism

$$\Psi_c^{-1} : \begin{array}{ccc} C_c(\Gamma', A) & \rightarrow & C_c(\Gamma, A) \\ \sum_{\gamma'\in\Gamma'} f_{\gamma'} v_{\gamma'} & \mapsto & \sum_{\gamma'\in\Gamma'} \alpha_{g^{-1}}(f_{\gamma'}) u_{g^{-1}\gamma' g}. \end{array}$$

To see this, note that for every $f \in C_c(\Gamma', A)$, we have that the full norm is given by

$$\|f\| = \|\Psi^{-1}\Psi(f)\| \leqslant \|\Psi(f)\| \leqslant \|f\|$$

and so if now f is an element in $A \rtimes \Gamma'$ and f_n is a sequence in $C_c(\Gamma', A)$ converging to f, we have

$$\|f\| - \|\Psi(f)\| \leqslant \|f - f_n\| + \|f_n\| - \|\Psi(f)\| =$$

$$\|f - f_n\| + \|\Psi(f_n)\| - \|\Psi(f)\| \to 0.$$

It follows that Ψ is a $*$-isomorphism. The proof is complete. □

Corollary 4.2.9. *Let Γ and Γ' be $\mathrm{SL}(2,\mathbb{R})$-conjugated discrete subgroups of $\mathrm{SL}(2,\mathbb{R})$ and let $g \in \mathrm{SL}(2,\mathbb{R})$ be such that $g\Gamma g^{-1} = \Gamma'$. The map*

$$\begin{aligned} \Psi : C_c(\Gamma, C_0(\mathbb{E})) &\to C_c(\Gamma', C_0(\mathbb{E})) \\ \sum_{\gamma\in\Gamma} f_\gamma u_\gamma &\mapsto \sum_{\gamma\in\Gamma} (f_\gamma \circ g^{-1}) v_{g\gamma g^{-1}} \end{aligned}$$

extends to a $*$-isomorphism $C_0(\mathbb{E}) \rtimes \Gamma \simeq C_0(\mathbb{E}) \rtimes \Gamma'$.

Proposition 4.2.10. *The crossed product constructions giving $C_0(X_\Gamma) \rtimes \mathbb{R}$ and $C_0(\mathbb{E}) \rtimes \Gamma$ define functors*

$$\begin{aligned} \mathfrak{f}^{(c)} : \mathfrak{G}^{(c)}_{d,f}(2,\mathbb{R}) &\to \mathfrak{C}^{(c)}_i \qquad \text{(contravariant)}\ , \\ \mathfrak{g} : \mathfrak{G}_d(2,\mathbb{R}) &\to \mathfrak{C} \qquad \text{(covariant)}\ , \\ \mathfrak{g}^c : \mathfrak{G}^c_d(2,\mathbb{R}) &\to \mathfrak{C}^c_i \qquad \text{(covariant)}\ . \end{aligned}$$

Proof. Let us define the functor $\mathfrak{f}$. By Lemma 4.2.1 if two discrete subgroups Γ' and Γ of $\mathrm{SL}(2,\mathbb{R})$ are conjugated, there is an induced $*$-isomorphism $C_0(X_\Gamma) \rtimes \mathbb{R} \simeq C_0(X_{\Gamma'}) \rtimes \mathbb{R}$, so we can restrict our attention to the case $\Gamma' \subset \Gamma$ of finite index. The map $\Gamma'\backslash \mathrm{SL}(2,\mathbb{R}) \to \Gamma\backslash \mathrm{SL}(2,\mathbb{R})$ sending a coset $\Gamma' g$ to Γg is a surjective proper map of $\mathrm{SL}(2,\mathbb{R})$-spaces and so it induces an injective $*$-homomorphism $C_0(X_\Gamma) \rtimes \mathbb{R} \to C_0(X_{\Gamma'}) \rtimes \mathbb{R}$.
Suppose now that we are given two morphisms ${}^g\iota_\Gamma^{\Gamma'}$ and ${}^{g'}\iota_{\Gamma'}^{\Gamma''}$ in $\mathfrak{G}_{d,f}(2,\mathbb{R})$ such that

$${}^g\iota_\Gamma^{\Gamma'} \qquad \text{corresponds to} \qquad \Gamma \sim_{\mathrm{SL}(2,\mathbb{R})} g\Gamma g^{-1} \subset \Gamma'$$

and

$${}^{g'}\iota_{\Gamma'}^{\Gamma''} \qquad \text{corresponds to} \qquad \Gamma' \sim_{\mathrm{SL}(2,\mathbb{R})} g'\Gamma' g'^{-1} \subset \Gamma''$$

Then the composition is given by the morphism

$${}^{g'g}\iota_\Gamma^{\Gamma''} \qquad \text{corresponding to} \qquad \Gamma \sim_{\mathrm{SL}(2,\mathbb{R})} g'g\Gamma(g'g)^{-1} \subset \Gamma''$$

Observe that the morphism

$${}_g\iota_\Gamma^{\Gamma'} \qquad \text{corresponding to} \qquad \Gamma \subset \Gamma' \sim_{\mathrm{SL}(2,\mathbb{R})} g\Gamma' g^{-1}$$

is equivalent to the morphism

$$ {}^{g}\iota_{\Gamma}^{g\Gamma' g^{-1}} \qquad \text{corresponding to} \qquad \Gamma \sim_{\mathrm{SL}(2,\mathbb{R})} g\Gamma g^{-1} \subset g\Gamma' g^{-1} $$

and thus to check functoriality we can restrict to morphisms of this form.
Start with a morphism of the form ${}^{g}\iota_{\Gamma}^{\Gamma'}$ and denote by $\mathfrak{f}({}^{g}\iota_{\Gamma}^{\Gamma'})$ the induced $*$-homomorphism from $C_0(X_{\Gamma'}) \rtimes \mathbb{R}$ to $C_0(X_\Gamma) \rtimes \mathbb{R}$. We can restrict ourselves to compactly supported functions. Let a morphism ${}^{g}\iota_{\Gamma}^{\Gamma'}$ be given; denote by $p_{g\Gamma g^{-1}}^{\Gamma'}$ the surjective proper continuous map defined in Lemma 4.2.5 and by Φ_g the isomorphism of $\mathrm{SL}(2,\mathbb{R})$-spaces defined in Lemma 4.2.4. Let $\hat{p}_{g\Gamma g^{-1}}^{\Gamma'}$ and $\hat{\Phi}_g$ be the correponding $*$-homomorphisms at the level of the crossed products as defined in Lemma 4.2.2. Then $\mathfrak{f}({}^{g}\iota_{\Gamma}^{\Gamma'}) = \hat{\Phi}_g \circ \hat{p}_{g\Gamma g^{-1}}^{\Gamma'} : C_0(\Gamma'\backslash \mathrm{SL}(2,\mathbb{R})) \rtimes \mathbb{R} \to C_0(\Gamma\backslash \mathrm{SL}(2,\mathbb{R})) \rtimes \mathbb{R}$. Let $f \in C_c(\mathbb{R}, C_0(\Gamma'\backslash \mathrm{SL}(2,\mathbb{R}))$ and $s \in \mathrm{SL}(2,\mathbb{R})$, we have

$$ \begin{aligned} \mathfrak{f}({}^{g}\iota_{\Gamma}^{\Gamma'})(f)(\Gamma s)(t) &= (\hat{\Phi}_g \circ \hat{p}_{g\Gamma g^{-1}}^{\Gamma'})(f)(\Gamma s)(t) \\ &= (\hat{p}_{g\Gamma g^{-1}}^{\Gamma'}(f))(\Phi_g(\Gamma s))(t) \\ &= (\hat{p}_{g\Gamma g^{-1}}^{\Gamma'}(f))((g\Gamma g^{-1})(gs))(t) \\ &= f(p_{g\Gamma g^{-1}}^{\Gamma'}((g\Gamma g^{-1})(gs)))(t) \\ &= f(\Gamma' gs)(t) \end{aligned} $$

for every $\Gamma s \in \Gamma\backslash \mathrm{SL}(2,\mathbb{R})$ and $t \in \mathbb{R}$. Analogously, if we are given a morphism ${}^{g'}\iota_{\Gamma'}^{\Gamma''}$, then

$$ \mathfrak{f}({}^{g'}\iota_{\Gamma'}^{\Gamma''})(f)(\Gamma' s)(t) = f(\Gamma'' g' s)(t) $$

for all $\Gamma' s \in \Gamma'\backslash \mathrm{SL}(2,\mathbb{R})$, $t \in \mathbb{R}$ and $f \in C_c(\mathbb{R}, C_0(\Gamma''\backslash \mathrm{SL}(2,\mathbb{R})))$. The composition reads

$$ \begin{aligned} ((\mathfrak{f}({}^{g}\iota_{\Gamma}^{\Gamma'}) \circ \mathfrak{f}({}^{g'}\iota_{\Gamma'}^{\Gamma''}))(f))(\Gamma s)(t) &= (\mathfrak{f}({}^{g'}\iota_{\Gamma'}^{\Gamma''})(f))(\Gamma' gs)(t) \\ &= f(\Gamma'' g' gs)(t) \end{aligned} $$

for $\Gamma s \in \Gamma\backslash \mathrm{SL}(2,\mathbb{R})$, $t \in \mathbb{R}$ and $f \in C_c(\mathbb{R}, C_0(\Gamma''\backslash \mathrm{SL}(2,\mathbb{R})))$. The composition ${}^{g'}\iota_{\Gamma'}^{\Gamma''} \circ {}^{g}\iota_{\Gamma}^{\Gamma'}$ is equal to the morphism ${}^{g'g}\iota_{\Gamma}^{\Gamma''}$ and

$$ (\mathfrak{f}({}^{g'g}\iota_{\Gamma}^{\Gamma''})(f))(\Gamma s)(t) = f(\Gamma'' g' gs)(t) $$

for all $\Gamma s \in \Gamma\backslash \mathrm{SL}(2,\mathbb{R})$, $t \in \mathbb{R}$ and $f \in C_c(\mathbb{R}, C_0(\Gamma''\backslash \mathrm{SL}(2,\mathbb{R})))$. Thus $\mathfrak{f}$ is a contravariant functor.
We want now to define the functor $\mathfrak{g}$. The map Ψ of Lemma 4.2.9 gives a $*$-isomorphism $\Psi : C_0(\mathbb{E}) \rtimes \Gamma' \to C_0(\mathbb{E}) \rtimes \Gamma$ in the case Γ and Γ' are $\mathrm{SL}(2,\mathbb{R})$-conjugated.

The map Φ of Corollary 4.2.7 gives a $*$-homomorphism from $C_0(\mathbb{E}) \rtimes \Gamma'$ to $C_0(\mathbb{E}) \rtimes \Gamma$ in the case $\Gamma' \subset \Gamma$ and their composition gives the desired morphism in $\mathfrak{C}$. Let ${}^g\iota_\Gamma^{\Gamma'}$ be a morphism in $\mathfrak{G}_d(2,\mathbb{R})$. The corresponding $*$-homomorphism reads

$$\mathfrak{g}({}^g\iota_\Gamma^{\Gamma'}) : \begin{array}{ccc} C_0(\mathbb{E}) \rtimes \Gamma & \to & C_0(\mathbb{E}) \rtimes \Gamma' \\ \sum_{\gamma\in\Gamma} f_\gamma u_\gamma & \mapsto & \sum_{\gamma'\in\Gamma'} (\hat{f}_{\gamma'} \circ g^{-1}) u_{\gamma'}, \end{array}$$

where

$$\hat{f}_{\gamma'} = \begin{cases} f_\gamma & \text{for } \gamma' = g\gamma g^{-1} \in g\Gamma g^{-1} \\ 0 & \text{for } \gamma' \notin g\Gamma g^{-1} \end{cases}.$$

Thus, if we are given two morphisms ${}^g\iota_\Gamma^{\Gamma'}$ and ${}^{g'}\iota_{\Gamma'}^{\Gamma''}$, the composition $\mathfrak{g}({}^{g'}\iota_{\Gamma'}^{\Gamma''}) \circ \mathfrak{g}({}^g\iota_\Gamma^{\Gamma'})$ reads

$$\begin{aligned} (\mathfrak{g}({}^{g'}\iota_{\Gamma'}^{\Gamma''}) \circ \mathfrak{g}({}^g\iota_\Gamma^{\Gamma'}))(\sum_{\gamma\in\Gamma} f_\gamma u_\gamma) &= \mathfrak{g}({}^{g'}\iota_{\Gamma'}^{\Gamma''})(\sum_{\gamma'\in\Gamma'} (\hat{f}_{\gamma'} \circ g^{-1}) u_{\gamma'}) \\ &= \sum_{\gamma''\in\Gamma''} (\hat{\hat{f}}_{\gamma''} \circ g^{-1} \circ (g')^{-1}) u_{\gamma''}, \end{aligned}$$

where

$$\begin{aligned} \hat{\hat{f}}_{\gamma''} &= \begin{cases} \hat{f}_{\gamma'} & \text{for } \gamma'' = g'\gamma'(g')^{-1} \in g'\Gamma'(g')^{-1} \\ 0 & \text{for } \gamma'' \notin g'\Gamma'(g')^{-1} \end{cases} \\ &= \begin{cases} f_\gamma & \text{for } \gamma'' = g'g\gamma(g'g)^{-1} \in g'g\Gamma(g'g)^{-1} \\ 0 & \text{for } \gamma'' \notin g'g\Gamma(g'g)^{-1} \end{cases}. \end{aligned}$$

Hence the morphism $\mathfrak{g}({}^{g'}\iota_{\Gamma'}^{\Gamma''}) \circ \mathfrak{g}({}^g\iota_\Gamma^{\Gamma'})$ is equal to

$$\mathfrak{g}({}^{g'g}\iota_\Gamma^{\Gamma''}) : \begin{array}{ccc} C_0(\mathbb{E}) \rtimes \Gamma & \to & C_0(\mathbb{E}) \rtimes \Gamma'' \\ \sum_{\gamma\in\Gamma} f_\gamma u_\gamma & \mapsto & \sum_{\gamma''\in\Gamma''} (\hat{f}_{\gamma''} \circ (g'g)^{-1}) u_{\gamma''}, \end{array}$$

where

$$\hat{f}_{\gamma''} = \begin{cases} f_\gamma & \text{for } \gamma'' = g'g\gamma(g'g)^{-1} \in g'g\Gamma(g'g)^{-1} \\ 0 & \text{for } \gamma'' \notin g'g\Gamma(g'g)^{-1} \end{cases}.$$

We have proved that $\mathfrak{g}$ is a covariant functor.
The functor $\mathfrak{g}$ restricts to the desired functor $\mathfrak{g}^c$. □

4.3 The case of hyperbolic Riemann surfaces

Let Γ_p be a cocompact Fuchsian group acting freely on $\mathbb{H}$ and let M be the corresponding compact hyperbolic Riemann surface. Let $\Gamma \coloneqq \pi_p^{-1}(\Gamma_p)$ be the corresponding discrete subgroup of $\mathrm{SL}(2,\mathbb{R})$ (see Lemma 3.3.7 and what follows). We can identify the $\mathrm{SL}(2,\mathbb{R})$-spaces $\Gamma\backslash\mathrm{SL}(2,\mathbb{R})$ and $\Gamma_p\backslash\mathrm{PSL}(2,\mathbb{R})$ and by Proposition 3.3.3 they are also equal to $\mathrm{T}_1 M$. In this case the C^*-algebra of the horocycle flow is $C(\mathrm{T}_1 M) \rtimes \mathbb{R}$.

Consider the categories $\mathfrak{R}_f$, $\mathfrak{R}_f^c$, $\mathbb{P}\mathfrak{G}_d(2,\mathbb{R})$ and $\mathbb{P}\mathfrak{G}_d^c(2,\mathbb{R})$ defined as:

- $\mathfrak{R}_f$ is the category whose objects are hyperbolic Riemann surfaces and the morphisms are finite sheeted holomorphic covering maps;
- $\mathfrak{R}_f^c$ is the full subcategory of $\mathfrak{R}$ whose objects are compact hyperbolic Riemann surfaces;
- $\mathbb{P}\mathfrak{G}_{d,f}(2,\mathbb{R})$ is the category whose objects are Fuchsian groups acting freely on the hyperblic upper half-plane $\mathbb{H}$; a morphism ${}^{\gamma}\iota_{\Gamma_p}^{\Gamma'_p} : \Gamma_p \to \Gamma'_p$ is an inclusion of finite index of Fuchsian groups modulo conjugation in $\mathrm{PSL}(2,\mathbb{R})$ (cfr. Section 4.2);
- $\mathbb{P}\mathfrak{G}_{d,f}^c(2,\mathbb{R})$ is the full subcategory of $\mathbb{P}\mathfrak{G}_d(2,\mathbb{R})$ whose objects are cocompact Fuchsian groups.

Following [48] I.3, given a category $\mathcal{C}$, a *congruence relation* $\sim$ on it is the data of an equivalence relation on every Hom-set such that if $f \sim f' : A \to B$ and $g \sim g' : B \to C$, then
$$g \circ f \sim g \circ f' \sim g' \circ f \sim g' \circ f'.$$
Given the congruence relation $\sim$ we can define the quotient category $\mathcal{C}_\sim$ with the same class of objects as $\mathcal{C}$ and in which the morphisms are equivalence classes of morphisms in $\mathcal{C}$.

Lemma 4.3.1. *Let $\mathcal{C}$ and $\mathcal{D}$ be two categories and $F : \mathcal{C} \to \mathcal{D}$ a functor. Suppose that there are congruence relations on $\mathcal{C}$ and $\mathcal{D}$ (both denoted by $\sim$) and that F respects the congruence relations:*

$$f \sim g : A \to B \Rightarrow F(f) \sim F(g) : F(A) \to F(B)$$

for every A, B objects in $\mathcal{C}$ and f, $g \in \mathrm{Hom}(A, B)$. Then there is a functor $F_\sim$: $\mathcal{C}_\sim \to \mathcal{D}_\sim$.

Proof. Let $f \sim g : A \to B$ in $\mathcal{C}$. By hypothesis $F(f) \sim F(g)$ in $\mathcal{D}$ and so the map

$$F_\sim^{A,B} : \begin{array}{ccc} \mathrm{Hom}(A,B)_\sim & \to & \mathrm{Hom}(F(A),F(B))_\sim \\ [f] & \mapsto & [F(f)] \end{array}$$

is well-defined. It also respects composition of morphisms since if $[f] : A \to B$ and $[g] : B \to C$ are morphisms in $\mathcal{C}_\sim$, then

$$F_\sim^{A,C} : [g \circ f] = [g] \circ [f] \mapsto [F(g)] \circ [F(f)] = [F(g) \circ F(f)] = [F(g \circ f)].$$

Clearly the identity in $\mathcal{C}_\sim$ is sent to the identity in $\mathcal{D}_\sim$ and so $F_\sim$ is a functor. □

Note now that if ${}^{g}\iota_\Gamma^{\Gamma'}$ and ${}^{g'}\iota_\Gamma^{\Gamma'}$ are morphisms in $\mathrm{Hom}_{\mathfrak{G}_{d,f}(2,\mathbb{R})}(\Gamma, \Gamma')$ such that $g(g')^{-1}$ belongs to Γ', then $\mathfrak{f}({}^{g}\iota_\Gamma^{\Gamma'}) = \mathfrak{f}({}^{g'}\iota_\Gamma^{\Gamma'}) : C(\Gamma' \backslash \mathrm{SL}(2,\mathbb{R})) \rtimes \mathbb{R} \to C(\Gamma \backslash \mathrm{SL}(2,\mathbb{R})) \rtimes \mathbb{R}$, the reason being that the maps $p_{g\Gamma g^{-1}}^{\Gamma'}$ and $p_{g'\Gamma(g')^{-1}}^{\Gamma'}$ defined in the proof of Proposition 4.2.10 coincide. Hence it is natural to identify two morphisms ${}^{g}\iota_\Gamma^{\Gamma'}$ and ${}^{g'}\iota_\Gamma^{\Gamma'}$ if $g(g')^{-1}$ belongs to Γ'. It is easy to check that this defines an equivalence relation on each Hom-set and that it actually gives a congruence relation on the category $\mathfrak{G}_{d,f}(2,\mathbb{R})$. We will denote this congruence relation simply by "$\sim$".

Lemma 4.3.2. *The functor $\mathfrak{f}^{(c)} : \mathfrak{G}_{d,f}^{(c)}(2,\mathbb{R}) \to \mathfrak{C}_i^{(c)}$ induces a functor*

$$\mathfrak{f}_\sim^{(c)} : \mathfrak{G}_{d,f}^{(c)}(2,\mathbb{R})_\sim \to \mathfrak{C}_i^{(c)}.$$

Proof. Consider on $\mathfrak{C}_i$ the trivial congruence relation, that is, two morphisms are identified if they are the same. Then the quotient category of $\mathfrak{C}_i$ by this congruence relation is clearly $\mathfrak{C}_i$. The result follows by the above discussion, applying Lemma 4.3.1. □

In the same way we can define a congruence relation on $\mathbb{P}\mathfrak{G}_d(2,\mathbb{R})$ by identifying two morphisms ${}^{\gamma}\iota_{\Gamma_p}^{\Gamma'_p}$ and ${}^{\gamma'}\iota_{\Gamma_p}^{\Gamma'_p}$ if $\gamma(\gamma')^{-1}$ belongs to Γ'_p. We will again denote this congruence relation by "$\sim$".

Lemma 4.3.3. *There are injective fully faithful functors*

$$\mathfrak{p}^{(c)} : \mathbb{P}\mathfrak{G}_{d,f}^{(c)}(2,\mathbb{R}) \to \mathfrak{G}_{d,f}^{(c)}(2,\mathbb{R}).$$

Hence $\mathbb{P}\mathfrak{G}^{(c)}_{d,f}(2,\mathbb{R})$ *is isomorphic to a full subcategory of* $\mathfrak{G}^{(c)}_{d,f}(2,\mathbb{R})$.
In particular there are injective fully faithful functors

$$\mathfrak{p}^{(c)}_{\sim} : \mathbb{P}\mathfrak{G}^{(c)}_{d,f}(2,\mathbb{R})_{\sim} \to \mathfrak{G}^{(c)}_{d,f}(2,\mathbb{R})_{\sim}.$$

Proof. The functor $\mathfrak{p}^{(c)}$ is defined in the following way:

- If Γ_p is a Fuchsian group, then $\mathfrak{p}^{(c)}(\Gamma_p) = \pi_p^{-1}(\Gamma_p)$;
- If ${}^{\gamma}\iota_{\Gamma_p}^{\Gamma'_p} : \Gamma_p \to \Gamma'_p$ is a morphism in $\mathbb{P}\mathfrak{G}^{(c)}_{d}(2,\mathbb{R})$, choose $g \in \pi_p^{-1}(\gamma)$ and define $\mathfrak{p}^{(c)}({}^{\gamma}\iota_{\Gamma_p}^{\Gamma'_p}) = {}^{g}\,\iota_{\pi_p^{-1}(\Gamma_p)}^{\pi_p^{-1}(\Gamma'_p)}$.

We have to spend some words about the definition of $\mathfrak{p}^{(c)}({}^{\gamma}\iota_{\Gamma_p}^{\Gamma'_p})$. First of all we have to show that this is actually a morphism; secondly (in order to have fullness) we want to show that any morphism $\pi_p^{-1}(\Gamma_p) \to \pi_p^{-1}(\Gamma'_p)$ in $\mathfrak{G}^{(c)}_{d}(2,\mathbb{R})$ arises in this way. We certainly have $\pi_p^{-1}(\gamma\Gamma_p\gamma^{-1}) \subset \pi_p^{-1}(\Gamma'_p)$; we will show that $\pi_p^{-1}(\gamma\Gamma_p\gamma^{-1}) = g\pi_p^{-1}(\Gamma_p)g^{-1}$ for every choice of $g \in \pi_p^{-1}(\gamma)$. Consider an element $h \in \pi_p^{-1}(\gamma\Gamma_p\gamma^{-1})$, then h is equal to one of the following

$$\begin{array}{ll}
\gamma^+\eta^+(\gamma^+)^{-1}, & \gamma^+\eta^+(\gamma^-)^{-1}, \\
\gamma^+\eta^-(\gamma^+)^{-1}, & \gamma^+\eta^-(\gamma^-)^{-1}, \\
\gamma^-\eta^+(\gamma^+)^{-1}, & \gamma^-\eta^+(\gamma^-)^{-1}, \\
\gamma^-\eta^-(\gamma^+)^{-1}, & \gamma^-\eta^-(\gamma^-)^{-1},
\end{array}$$

where $\{\gamma^+,\gamma^-\} = \pi_p^{-1}(\gamma)$ and $\{\eta^+,\eta^-\} = \pi_p^{-1}(\eta)$. Note now that we have

$$\begin{aligned}
\gamma^+\eta^+(\gamma^+)^{-1} &= \gamma^+\eta^-(\gamma^-)^{-1} \\
&= \gamma^-\eta^-(\gamma^+)^{-1} \\
&= \gamma^-\eta^+(\gamma^-)^{-1}
\end{aligned}$$

and

$$\begin{aligned}
\gamma^+\eta^-(\gamma^+)^{-1} &= \gamma^+\eta^+(\gamma^-)^{-1} \\
&= \gamma^-\eta^+(\gamma^+)^{-1} \\
&= \gamma^-\eta^-(\gamma^-)^{-1}.
\end{aligned}$$

Hence h has always the form $\gamma^+\eta^{\pm}(\gamma^+)^{-1}$ and since γ^+ was chosen arbitrarily

$$g\pi_p^{-1}(\Gamma_p)g^{-1} = \pi_p^{-1}(\gamma\Gamma_p\gamma^{-1}) \subset \pi_p^{-1}(\Gamma'_p)$$

for any choice of $g \in \pi_p^{-1}(\gamma)$ and ${}^g\iota_{\pi_p^{-1}(\Gamma_p)}^{\pi_p^{-1}(\Gamma'_p)}$ is a morphism.
$\mathfrak{p}^{(c)}$ is full: Let ${}^g\iota_{\pi_p^{-1}(\Gamma_p)}^{\pi_p^{-1}(\Gamma'_p)}$ be a morphism in $\mathfrak{G}_d^{(c)}(2,\mathbb{R})$ and let $\gamma = \pi_p(g) \in \mathrm{PSL}(2,\mathbb{R})$. Then ${}^\gamma\iota_{\Gamma_p}^{\Gamma'_p}$ is a morphism in $\mathbb{P}\mathfrak{G}_d^{(c)}(2,\mathbb{R})$; by the above computation we see that for any choice $\gamma^\pm$ of a representative of γ in $\pi_p^{-1}(\gamma)$, we have $\gamma^\pm h(\gamma^\pm)^{-1} = ghg^{-1}$ and so

$$
{}^g\iota_{\pi_p^{-1}(\Gamma_p)}^{\pi_p^{-1}(\Gamma'_p)} = \mathfrak{p}^{(c)}({}^\gamma\iota_{\Gamma_p}^{\Gamma'_p}).
$$

Thus $\mathfrak{p}^{(c)}$ if full.
$\mathfrak{p}^{(c)}$ is injective: Let $\mathfrak{p}^{(c)}(\Gamma_p) = \pi_p^{-1}(\Gamma_p)$ be equal to $\mathfrak{p}^{(c)}(\Gamma'_p)$; then $\Gamma_p = \pi_p(\pi_p^{-1}(\Gamma_p)) = \pi_p(\pi_p^{-1}(\Gamma'_p)) = \Gamma'_p$.
$\mathfrak{p}^{(c)}$ is faithful: If ${}^\gamma\iota_{\Gamma_p}^{\Gamma'_p}$ and ${}^{\gamma'}\iota_{\Gamma_p}^{\Gamma'_p}$ are two morphisms in $\mathbb{P}\mathfrak{G}_d^{(c)}(2,\mathbb{R})$ such that $\mathfrak{p}^{(c)}({}^\gamma\iota_{\Gamma_p}^{\Gamma'_p}) = \mathfrak{p}^{(c)}({}^{\gamma'}\iota_{\Gamma_p}^{\Gamma'_p})$, then, given $g \in \pi_p^{-1}(\gamma)$ and $g' \in \pi_p^{-1}(\gamma')$, we have $ghg^{-1} = g'h(g')^{-1}$ for every $h \in \pi_p^{-1}(\Gamma_p)$. Hence $\gamma\pi_p(h)\gamma^{-1} = \pi_p(ghg^{-1}) = \pi_p(g'h(g')^{-1}) = \gamma'\pi_p(h)(\gamma')^{-1}$ for every $\pi_p(h) \in \Gamma_p$; thus ${}^\gamma\iota_{\Gamma_p}^{\Gamma'_p} = {}^{\gamma'}\iota_{\Gamma_p}^{\Gamma'_p}$.
Note now that if $\gamma\Gamma_p\gamma^{-1} \subset \Gamma'_p$ has finite index n, then $\pi_p^{-1}(\gamma\Gamma_p\gamma^{-1}) \subset \Gamma'_p$ has index at most $2n$.
By Lemma 4.3.1 we need to check that this functor respects the congruence relations. This is a consequence of the definitions. □

Lemma 4.3.4. *There is an equivalence of categories between* $\mathfrak{R}_f^{(c)}$ *and* $\mathbb{P}\mathfrak{G}_{d,f}^{(c)}(2,\mathbb{R})_\sim$.

Proof. In virtue of Proposition 3.2.9 we can define the covariant functor

$$
F^{(c)} : \begin{array}{ccc} \mathbb{P}\mathfrak{G}_{d,f}^{(c)}(2,\mathbb{R}) & \to & \mathfrak{R}_f^{(c)} \\ \Gamma_p & \mapsto & \Gamma_p\backslash\mathbb{H} \\ {}^\gamma\iota_{\Gamma_p}^{\Gamma'_p} & \mapsto & \{\Gamma_p\tilde{x} \mapsto \Gamma'_p\gamma\tilde{x}\} \end{array}.
$$

Under this functor, two equivalent morphisms in $\mathfrak{G}_{d,f}(2,\mathbb{R})$ are mapped to the same holomorphic covering and by Lemma 3.2.9, the resulting covering is finite sheeted. By considering the trivial congruence relation on $\mathfrak{R}_f^{(c)}$ (i.e. equality) and applying Lemma 4.3.1 we obtain the desired functor. Denote this functor by $\mathfrak{p}^{(c)}$.
We want to show that $\mathfrak{p}^{(c)} : \mathbb{P}\mathfrak{G}_d^{(c)}(2,\mathbb{R})_\sim \to \mathfrak{R}_f^{(c)}$ is full, faithful and dense. The result will follow from [48] Proposition 10.1.
$\mathfrak{p}^{(c)}$ is full: By Proposition 3.2.9 for every Γ_p, Γ'_p Fuchsian groups acting freely on $\mathbb{H}$ and every holomorphic covering map $p : \Gamma_p\backslash\mathbb{H} \to \Gamma'_p\backslash\mathbb{H}$ there is at least one $\gamma \in \mathrm{PSL}(2,\mathbb{R})$ such that ${}^\gamma\iota_{\Gamma_p}^{\Gamma'_p}$ belongs to $\mathrm{Hom}(\Gamma_p,\Gamma'_p)$ in $\mathbb{P}\mathfrak{G}_{d,f}^{(c)}(2,\mathbb{R})$ and $p(\Gamma_p\tilde{x}) = \Gamma'_p(\gamma\tilde{x})$ for every $\tilde{x} \in \mathbb{H}$. Thus we obtain fulness.

$\mathfrak{p}^{(c)}$ is faithful: Suppose that we are given two finite sheeted holomorphic coverings $\{\Gamma_p\tilde{x} \mapsto \Gamma'_p\gamma\tilde{x}\}$ and $\{\Gamma_p\tilde{x} \mapsto \Gamma'_p\gamma'\tilde{x}\}$ in $\mathrm{Hom}_{\mathfrak{R}_f^{(c)}}(\Gamma_p\backslash\mathbb{H}, \Gamma'_p\backslash\mathbb{H})$ which coincide. Then, composing with the holomorphic covering $\mathbb{H} \to \Gamma_p\backslash\mathbb{H}$, $\tilde{x} \mapsto \Gamma_p\tilde{x}$, we see that also the compositions with this are the same and thus $\Gamma'_p\tilde{x} = \Gamma'_p\gamma(\gamma')^{-1}\tilde{x}$ for every $\tilde{x} \in \mathbb{H}$. This means that $\gamma(\gamma')^{-1}$ is a deck transformation for $\mathbb{H} \to \Gamma'_p\backslash\mathbb{H}$, $\tilde{x} \mapsto \Gamma'_p\tilde{x}$ and so it belongs to Γ'_p.
$\mathfrak{p}^{(c)}$ is dense: Let M be a (compact) hyperbolic Riemann surface. Then it is biholomorphic to a Riemann surface of the form $\Gamma_p\backslash\mathbb{H}$, where Γ_p is a Fuchsian group, that is cocompact if M is compact. □

Proposition 4.3.5. *The horocycle flow defines contravariant functors*

$$\mathfrak{r}^{(c)} : \mathfrak{R}_f^{(c)} \to \mathfrak{C}_i^{(c)},$$

such that if two Riemann surfaces are biholomorphic (that is isomorphic in $\mathfrak{R}_f^{(c)}$), then the associated C^*-algebras are $*$-isomorphic.

Proof. By Lemma 4.3.4 there is an equivalence of categories $\mathfrak{R}_f^{(c)} \to \mathbb{P}\mathfrak{G}_{d,f}^{(c)}(2,\mathbb{R})_\sim$. We can compose it with the fully faithful functor $\mathfrak{p}_\sim^{(c)} : \mathbb{P}\mathfrak{G}_{d,f}^{(c)}(2,\mathbb{R})_\sim \to \mathfrak{G}_{d,f}^{(c)}(2,\mathbb{R})_\sim$ and then with the functor $\mathfrak{f}_\sim^{(c)}$ defined in Lemma 4.3.2.
If two Riemann surfaces are isomorphic in $\mathfrak{R}_f^{(c)}$, then they are biholomorphic and the corresponding Fuchsian groups are $\mathrm{PSL}(2,\mathbb{R})$-conjugated; thus also the corresponding discrete subgroups of $\mathrm{SL}(2,\mathbb{R})$ are conjugated (in $\mathrm{SL}(2,\mathbb{R})$) and they give rise to $*$-isomorphic C^*-algebras. □

Remark 4.3.6. We have seen that two biholomorphic Riemann surfaces give rise to $*$-isomorphic C^*-algebras under the functor $\mathfrak{r}^{(c)}$. Note now that by the Thom-Connes isomorphism [19] Theorem 2, given a cocompact discrete subgroup Γ of $\mathrm{SL}(2,\mathbb{R})$, the K-theory of $C(X_\Gamma) \rtimes \mathbb{R}$ is completely determined by the complex K-theory of $C(X_\Gamma)$. Thus, if $\Gamma = \pm\pi_1(M_g)$ for a certain compact hyperbolic Riemann surface M_g of genus g, it reads

$$K_0(C(\mathrm{T}_1(M_g)) \rtimes \mathbb{R}) = \mathbb{Z}^{2g+1}, \qquad K_1(C(\mathrm{T}_1(M_g)) \rtimes \mathbb{R}) = \mathbb{Z}^{2g+1} \oplus \mathbb{Z}/(2g-2).$$

Both the order and the scale are trivial since $C(\mathrm{T}_1(M_g)) \rtimes \mathbb{R}$ is stable and projectionless.
Nevertheless, by [19] Corollary 2 the range of the pairing between K_0 and the unique trace is determined by the range of the Ruelle-Sullivan current associated to this flow

(see [21] 5.α), which in this case is trivial by [52]. Thus the Elliott invariant only contains information about the genus, or equivalently, the homeomorphic class of the Riemann surface.

Chapter 5

On stability of $C_0(\mathbb{E}) \rtimes \Gamma$

Let Γ be a discrete cocompact subgroup of $\mathrm{SL}(2,\mathbb{R})$. It is suggested by Theorem 4.1.4 that the properties of the crossed product C^*-algebra $C(\Gamma \backslash \mathrm{SL}(2,\mathbb{R})) \rtimes \mathbb{R}$ that we deduced from properties of the flow could be equivalently deduced from properties of the dynamical system $(\mathbb{E}, \Gamma)$, where $\mathbb{E} = \mathbb{R}^2 - \{0\}$.
In particular our attempt here is to establish stability of $C_0(\mathbb{E}) \rtimes \Gamma$ from dynamical considerations. To this end we focus on one equivalent characterization of stability contained in Proposition 2.3.5, that we recall in the following

Proposition. *Let A be a σ-unital C^*-algebra. The following are equivalent:*

- *A is stable,*
- *A has weak stable rank 1 and $M(A)$ is properly infinite,*

We then proceed in finding conditions under which an action of a discrete group gives rise to a crossed product C^*-algebra whose multiplier algebra is properly infinite. The more challenging question about the stable rank remains unanswered.

We begin by giving the definition of contractive action (Definition 5.1.1) and prove in Proposition 5.1.3 that, in the case of a discrete group acting on a locally compact connected normal Hausdorff space by means of a contractive action, the multiplier algebras of the corresponding (reduced or full) crossed product C^*-algebras are infinite. Of course the idea is that the action on $\mathbb{E}$ of a certain class of discrete subgroups of $\mathrm{SL}(2,\mathbb{R})$, including the cocompact ones, should be contractive and by Proposition 5.2.1 this is actually the case for every discrete subgroup of $\mathrm{SL}(2,\mathbb{R})$ containing hyperbolic elements.
Note that if Γ is a cocompact subgroup of $\mathrm{SL}(2,\mathbb{R})$, by Lemma 5.2.3 also its image in $\mathrm{PSL}(2,\mathbb{R})$ is cocompact and so, by Lemma 5.3.2, Theorem 3.4.4 and Theorem

4.5.2 of [42], the set of axes of its hyperbolic elements is dense in $\mathbb{RP}^1$. We see in Proposition 5.2.2 that, as a consequence of Proposition 5.1.7, the multiplier algebra of the (reduced or full) crossed product C^*-algebra associated to an action of a discrete subgroup of $\mathrm{SL}(2,\mathbb{R})$ containing at least two hyperbolic elements with different axes is properly infinite.
In the last section, using the results contained in [45] Section 3 and in [46] and [3], we can prove that in the case of cocompact subgroups of $\mathrm{SL}(2,\mathbb{R})$ that are the lifts of Fuchsian groups not containing cyclic elements of order two, the multiplier algebra of the corresponding crossed product C^*-algebra contains a Kirchberg algebra in the UCT class as a unital C^*-subalgebra.

5.1 Contractive and paradoxical actions

Let X be a locally compact Hausdorff space and G a discrete group acting on it. We will see in the following that under certain assumptions we can infer infiniteness of the multiplier algebra of the crossed product C^*-algebra.

The concept of contractive action (see below) was already considered in [68] page 22 and has to be compared with the more restrictive Definition 2.1 of [3].

Definition 5.1.1. *Let G be a discrete group acting on a locally compact Hausdorff space X. The action is said to be contractive if there exists a non-empty open set U and an element $t \in G$ such that*

$$t\overline{U} \subsetneq U.$$

Let A be a C^*-algebra endowed with the action α of a locally compact group G and $I \subset A$ an α-invariant ideal. The inclusion map $\iota : I \to A$ is a covariant $*$-homomorphism and so by [75] Corollary 2.48 there is a $*$-homomorphism $\iota \rtimes G : I \rtimes G \to A \rtimes G$ that is the identity on $C_c(G, I)$. Now, as explained in [75] 3.4, since every nondegenerate covariant $*$-representation of I extends to a nondegenerate covariant $*$-representation of A, it follows that the closure of $C_c(G, I)$ in $A \rtimes G$ coincides with the image of $I \rtimes G$ under $\iota \rtimes G$.
In the case of the reduced crossed product, note that a faithful $*$-representation of A restricts to a faithful $*$-representation of I and the integrated form gives the desired inclusion $I \rtimes_r G \subset A \rtimes_r G$.
In particular, if A is any C^*-algebra and G a discrete group acting on it, then $A \rtimes_{(r)} G$ is an ideal in $M(A) \rtimes_{(r)} G$ (see [53] 3.12 for a discussion about multipliers of C^*-algebras), where the action of G on $M(A)$ is the extension of the action on A. Then

there is a unital $*$-homomorphism $\phi : M(A) \rtimes_{(r)} G \to M(A \rtimes_{(r)} G)$; if we identify $M(A \rtimes_{(r)} G)$ with the C^*-algebra of double centralizers on $A \rtimes_{(r)} G$, $\phi(x)y = xy$ for any x in $M(A) \rtimes_{(r)} G$ and y in $A \rtimes_{(r)} G$.

Recall the following

Definition 5.1.2 ([11] Definition 1.1)**.** *Let A be a C^*-algebra and x an element in A. x is called a scaling element if $x^*x(xx^*) = xx^*$ and $x^*x \neq xx^*$.*

Proposition 5.1.3. *Let G be a discrete group acting on a locally compact Hausdorff space X. Consider the following properties:*

(i) The action of G on X is contractive.

(ii) There exists a scaling elementary tensor in $C_b(X) \rtimes_{(r)} G$.

Then (ii)⇒(i). If X is a connected normal space, then (i)⇒(ii).

Proof. (ii)⇒(i): Let $x = u_t f$ be a scaling element in $C_c(G, C_b(X)) \subset C_b(X) \rtimes_{(r)} G$ and U the interior of supp(f). Since $x^*x = |f|^2$ and $xx^* = |f \circ t^{-1}|^2$, the condition $x^*xxx^* = xx^*$ implies $|f||_{t\overline{U}} = 1$; in particular $t\overline{U} \subset U$. Suppose that $t\overline{U} = U$. Then

$$|f||_{U^c} = 0, \qquad |f||_U = |f||_{t\overline{U}} = 1|_{t\overline{U}} = 1|_U$$

and

$$|f \circ t^{-1}||_{U^c} = |f \circ t^{-1}||_{(t\overline{U})^c} = 0, \qquad |f \circ t^{-1}||_U = |f \circ t^{-1}||_{t\overline{U}} = 1|_{t\overline{U}} = 1|_U.$$

This would mean that $|f| = |f \circ t^{-1}|$ and $x^*x = xx^*$. Hence $t\overline{U} \subsetneq U$.
Suppose now that X is a connected normal space and let $U \subset X$ be an open set, $t \in G$ be an element such that $t\overline{U} \subsetneq U$. By Urysohn Lemma (normality) there exists a continuous function $f : X \to [0, 1]$ such that $f = 1$ on $t\overline{U}$, $f = 0$ on U^c and by connectedness we can suppose that f is not identically 0 on $U - t\overline{U}$. Then $x := u_t f$ is such that $x^*x = f^2$, $xx^* = (f \circ t^{-1})^2$ and $x^*x(xx^*) = xx^*$. Since supp$(f \circ t^{-1}) \subsetneq$ supp(f), we have $x^*x \neq xx^*$. □

Corollary 5.1.4. *Let X be a locally compact connected normal Hausdorff space and G a discrete group acting on it by means of a contractive action. Then $M(C_0(X) \rtimes_{(r)} G)$ is infinite.*

Proof. Let x be as in Proposition 5.1.3 and $\phi : C_b(X) \rtimes_{(r)} G \to M(C_0(X) \rtimes_{(r)} G)$ be as in the discussion preceding Definition 5.1.2. We have to verify that $\phi(x)$ is a

scaling element. This is equivalent to showing that $\phi(x^*x) \neq \phi(xx^*)$.
Let $g \in C_0(X)$ be a positive nonzero function such that $\mathrm{supp}(g) \subset \mathrm{supp}(f) - t\overline{U}$ (this function exists by normality). Then

$$x^*xg = f^2g \neq 0, \qquad xx^*g = (f \circ t^{-1})^2 g = 0.$$

This entails $\phi(xx^*) \neq \phi(x^*x)$. The proof follows from Theorem 3.1 of [11]. □

A variation of the concept of contractive action is the following (see [68] Lemma 2.3.2) and is a slight variation of Definition 2.3.3 of [68].

Definition 5.1.5. *Let X be a locally compact Hausdorff space, G a discrete group acting on it and $U \subset X$ be a non-empty open set. We say that U is G-paradoxical, or just paradoxical, if there are positive natural numbers n, m, group elements $t_1, ..., t_{n+m}$ and non-empty open sets $U_1, ..., U_{n+m}$ such that*
$\bigcup_{i=1}^{n} U_i = \bigcup_{i=n+1}^{n+m} U_i = U$, $\bigcup_{i=1}^{n+m} t_i(U_i) \subset U$ and $t_i\overline{U_i} \cap t_j\overline{U_j} = \varnothing$ for every $i \neq j$.
We say that the action of G on X is paradoxical if X is a G-paradoxical open set.

Remark 5.1.6. Let G be a discrete group acting on a locally compact Hausdorff space X. Suppose that there are contractive open sets U_i associated to group elements t_i, $i = 1, ..., 4$ such that $t_i\overline{U_i} \subsetneq U_i$ and $t_iU_i \cap t_jU_j = \varnothing$ for every $i \neq j$. Suppose also that $U_1 \cup U_2 = U_3 \cup U_4 = X$, then X is a paradoxical open set. As we will see in Proposition 5.2.2 this is the case for the action of discrete subgroups of $\mathrm{SL}(2, \mathbb{R})$ containing two hyperbolic elements with different axes.

The connection between paradoxicality of an action on a locally compact Hausdorff space X and existence of properly infinite elements in the crossed product C^*-algebra associated to this action has been investigated in [68] Lemma 2.3.7, where it is shown that the characteristic function of a compact open paradoxical set is a properly infinite projection in the crossed product. Hence, in particular, if X is compact and the action is paradoxical, then the identity on X is properly infinite. Following this idea, we will show that in the case X is normal, connected and locally compact, the multiplier algebra of the crossed product C^*-algebra (and hence any strictly positive element for the crossed product C^*-algebra by Lemma 2.1.3) is properly infinite.

Proposition 5.1.7. *Let G be a discrete group acting on a locally compact connected normal Hausdorff space X. If the action is paradoxical, then $M(C_0(X) \rtimes_{(r)} G)$ is properly infinite.*

Proof. Let n, m, $t_1, ..., t_{n+m}$ and $U_1, ..., U_{n+m}$ be as in Definition 5.1.5 for the paradoxical set $U = X$. Taking unions and relabeling we can suppose $t_i \neq t_j$ for $i \neq j$. Let $F \coloneqq \{t_1, ..., t_n\}$, $F' \coloneqq \{t_{n+1}, ..., t_{n+m}\}$.
Since X is normal we can take a partition of unity $\{\phi_t\}_{t\in F}$ subordinated to $\{U_i\}_{i=1}^n$ and a partition of unity $\{\psi_s\}_{s\in F'}$ subordinated to $\{U_i\}_{i=n+1}^{n+m}$. Consider the extension of the action of G to $C_b(X)$ and the associated crossed product C^*-algebra $C_b(X) \rtimes_{(r)} G$.
Define $x \coloneqq \sum_{t\in F} u_t\phi_t^{1/2}$ and $y \coloneqq \sum_{t'\in F'} u_s\psi_s^{1/2}$. Then

$$x^*x = y^*y = 1.$$

Furthermore, using the same argument of [68] Theorem 2.3.4, take a point $p \in (\bigcup_{i=1}^n (t_iU_i))^c$. Let $E : C_b(X) \rtimes_r G \to C_b(X)$ denote the faithful conditional expectation and note that xx^* is an element in $C_c(G, C_b(X)) \subset C_b(X) \rtimes_r G$. We have $E(xx^*)(p) = 0$ and $E(1)(p) = 1(p) = 1$, entailing $xx^* \neq p$ and the same reasoning applies to yy^*. Note now that

$$x^*y = \sum_{t\in F, s\in F'} \phi_t^{1/2}(\psi_s^{1/2} \circ s^{-1}t)u_{t^{-1}s} = 0$$

and so $xx^* \perp yy^*$. Thus 1 is properly infinite in $C_b(X) \rtimes_{(r)} G$.
Let $\phi : C_b(X) \rtimes_{(r)} G \to M(C_0(X) \rtimes_{(r)} G)$ be as in Corollary 5.1.4. Take a positive function $f \in C_0(X)$ whose support is contained in $(\bigcup_{i=1}^n (t_iU_i))^c$. The existence of such a function is assured by normality since for every $i \neq j$, $t_i\overline{U_i} \cap t_j\overline{U_j} = \varnothing$ and X is connected. Then

$$xx^*f = \sum_{t\neq t'\in F} (\phi_t^{1/2} \circ t^{-1})(\phi_{t'}^{1/2} \circ t^{-1})u_{t(t')^{-1}}f$$

and $E(xx^*f) = 0$, where now E is the conditional expectation for $C_0(X) \rtimes_r G$. Hence $\phi(xx^*) \neq \phi(1) = 1$. The same applies to yy^* and so $1 \in M(C_0(X) \rtimes_{(r)} G)$ is properly infinite. □

5.2 The case of discrete subgroups of $\mathrm{SL}(2, \mathbb{R})$

It will follow from Proposition 5.2.1 that the results of the previous section apply to the case of discrete subgroups of $\mathrm{SL}(2, \mathbb{R})$ containing hyperbolic elements and by Lemma 5.2.3 this class of groups contain the discrete cocompact subgroups of

$\mathrm{SL}(2, \mathbb{R})$.

Recall from [42] 2.1 that an element of $\mathrm{SL}(2, \mathbb{R})$ is called *hyperbolic* if the absolute value of its trace is greater than 2.

Proposition 5.2.1. *Let Γ be a discrete subgroup of $\mathrm{SL}(2, \mathbb{R})$ containing a hyperbolic element. Then the action of Γ on $\mathbb{E}$ is contractive. In particular $M(C_0(\mathbb{E}) \rtimes \Gamma)$ is infinite.*

Proof. By [42] 2.1 every hyperbolic element γ in $\mathrm{SL}(2, \mathbb{R})$ is conjugated in $\mathrm{SL}(2, \mathbb{R})$ to an element of the form

$$\chi = \begin{pmatrix} \lambda & 0 \\ 0 & 1/\lambda \end{pmatrix}, \qquad \lambda > 0,$$

which clearly admits a contractive open set. Thus the same is true for γ. Since every metrizable space is normal, we can apply Corollary 5.1.4. □

As an application of Proposition 5.1.7 we obtain the following

Proposition 5.2.2. *Let Γ be a discrete subgroup of $\mathrm{SL}(2, \mathbb{R})$ containing two hyperbolic elements with different axes, i.e. with a different basis of eigenvectors. Then $M(C_0(X) \rtimes_{(r)} \Gamma)$ is properly infinite.*

Proof. Let γ_1 and γ_2 be hyperbolic elements as in the hypothesis. We can find open sets U_1, U_2 such that $\gamma_1 U_1 \subsetneq U_1$, $\gamma_1^{-1} U_2 \subsetneq U_2$ and $U_1 \cup U_2 = \mathbb{E}$. Similarly, we can find open sets U_3, U_4 such that $\gamma_2 U_3 \subsetneq U_3$, $\gamma_2^{-1} U_4 \subsetneq U_4$ and $U_3 \cup U_4 = \mathbb{E}$. Furthermore, by replacing the elements γ_1 and γ_2 by some powers if necessary, we can assume that $\gamma_1 U_1 \cap \gamma_1^{-1} U_2 = \gamma_1 U_1 \cap \gamma_2(U_3) = \gamma_1 U_1 \cap \gamma_2^{-1} U_4 = \gamma_1^{-1} U_2 \cap \gamma_2 U_3 = \gamma_1^{-1} U_2 \cap \gamma_2^{-1} U_4 = \gamma_2 U_3 \cap \gamma_2^{-2} U_4 = \varnothing$. Thus the action is paradoxical and we can apply Proposition 5.1.7. □

For every cocompact discrete subgroup of $\mathrm{SL}(2, \mathbb{R})$, the multiplier algebra of the C^*-algebra $C_0(\mathbb{E}) \rtimes \Gamma$ is properly infinite by stability. This property can be deduced, anticipating part of the discussion of the next section, from Proposition 5.2.2. Indeed by [42] Theorem 3.4.4 and Theorem 4.5.2 the set of fixed points of the hyperbolic transformations in a cocompact subgroup Γ_p of $\mathrm{PSL}(2, \mathbb{R})$ is dense in $\partial\mathbb{H}$ (see the discussion preceding Lemma 5.3.2 for the definition of $\partial\mathbb{H}$) and by Lemma 5.3.2 this set corresponds to the set of axes of the hyperbolic elements in $\pi_p^{-1}(\Gamma_p)$. By the following Lemma, if Γ is a cocompact subgroup of $\mathrm{SL}(2, \mathbb{R})$, then also its symmetrization $\pm\Gamma$ is cocompact and thus the same is true for its image under the quotient map $\pi_p : \mathrm{SL}(2,\mathbb{R}) \to \mathrm{PSL}(2,\mathbb{R})$. In particular Γ contains hyperbolic elements with different axes.

Lemma 5.2.3. *Let Γ be a discrete cocompact subgroup of $\mathrm{SL}(2,\mathbb{R})$, then $\pm\Gamma \coloneqq \pi_p^{-1}(\pi_p(\Gamma))$ is a discrete cocompact subgroup of $\mathrm{SL}(2,\mathbb{R})$.*

Proof. Let $\pm\Gamma = \{\pm\gamma \ : \ \gamma \in \Gamma\} = \Gamma \cup (-\Gamma)$. It is a closed subgroup of $\mathrm{SL}(2,\mathbb{R})$ since it is the union of two closed sets.
Suppose that $\pm\Gamma$ is not discrete; then there exists a sequence $\{\gamma_i\} \subset \pm\Gamma$ such that $\gamma_i \to \gamma$ for some $\gamma \in \pm\Gamma$ and $\{\gamma_i\}$ is not eventually constant. We distinguish three cases:

- The sequence $\{\gamma_i\}$ is eventually contained in Γ: then γ would be in Γ since Γ is closed and so Γ would not be discrete.
- The sequence $\{\gamma_i\}$ is eventually contained in $-\Gamma$: then the sequence $\{-\gamma_i\} \subset \Gamma$ would converge to $-\gamma$ in Γ, contradicting again the discreteness of Γ.
- For any $n \in \mathbb{N}$ there exists $l(n) > n$ such that $\gamma_{l(n)}$ belongs to $\{\gamma_i\} \cap \Gamma$. In this case $\{\gamma_{l(i)}\}$ would be a sequence in Γ (not eventually constant) converging to γ in Γ.

Hence $\pm\Gamma$ is discrete.
We now want to show that $\pm\Gamma$ is cocompact. The map $\rho : \mathrm{SL}(2,\mathbb{R})/\Gamma \to \mathrm{SL}(2,\mathbb{R})/(\pm\Gamma)$ is well defined and surjective. Let $U = \{g(\pm\Gamma)\}_{g\in\Lambda}$ be an open subset of $\mathrm{SL}(2,\mathbb{R})/(\pm\Gamma)$, where Λ is some index set. Denote by $\pi_\Gamma : \mathrm{SL}(2,\mathbb{R}) \to \mathrm{SL}(2,\mathbb{R})/\Gamma$ and $\pi_{\pm\Gamma} : \mathrm{SL}(2,\mathbb{R}) \to \mathrm{SL}(2,\mathbb{R})/(\pm\Gamma)$ the quotient maps. By definition of quotient topology U is open if and only if $\pi_{\pm\Gamma}^{-1}(U) = \{g\gamma'\}_{g\in\Lambda,\gamma'\in\pm\Gamma} = \{\pm g\gamma\}_{g\in\Lambda,\gamma\in\Gamma}$ is open in $\mathrm{SL}(2,\mathbb{R})$. Note now that $\rho^{-1}(U) = \{(\pm g)\Gamma\}_{g\in\Lambda}$ and thus $\pi_\Gamma^{-1}(\rho^{-1}(U)) = \{\pm g\gamma\}_{g\in\Lambda,\gamma\in\Gamma}$, that is open. Thus ρ is a continuous surjection and maps compact sets into compact sets. The proof is complete since Γ is assumed to be cocompact. □

5.3 Fuchsian groups without cyclic elements of order 2

We want now to use the results contained in [46] and [45] to extract more information about the the multiplier algebra of the crossed product C^*-algebra associated to the horocycle flow in some particular situations. The idea is just to observe that the projection map from $\mathbb{E} = \mathbb{R}^2 - \{0\}$ to $\mathbb{RP}^1$ (we recall the definition below) is equivariant with respect to the natural actions of $\mathrm{SL}(2,\mathbb{R})$ (Lemma 5.3.1) and that the action of $\mathrm{SL}(2,\mathbb{R})$ on the real projective line factors trough an action of $\mathrm{PSL}(2,\mathbb{R})$. Furthermore, for every Fuchsian group Γ_p of the first kind (see [42] page 67), the dynamical system $(\mathbb{RP}^1, \Gamma_p)$ is topologically conjugated to $(\partial\mathbb{H}, \Gamma_p)$ (by Lemma 5.3.2) , which is Example 2.2 of [46]. Hence combining results from [46] and [45] we can state Proposition 5.3.6.

We recall the definition of the real projective line $\mathbb{RP}^1$ and of the action of $\mathrm{SL}(2,\mathbb{R})$ on it.
As a topological space, $\mathbb{RP}^1$ is the quotient of $\mathbb{E}$ by the equivalence relation $\sim$

$$\mathbb{RP}^1 = \{(x,y) \in \mathbb{E}\}/\sim,$$

where $(x,y) \sim (x',y')$ if there exists $r \in \mathbb{R}$ such that $(x,y) = r(x',y')$. $\mathrm{SL}(2,\mathbb{R})$ acts on $\mathbb{RP}^1$ in the following way

$$g \cdot [x:y] = \begin{cases} [\frac{ax/y+b}{cx/y+d} : 1] & \text{for } cx \neq -dy,\ y \neq 0 \\ [\frac{a}{c} : 1] & \text{for } cx \neq -dy\ , y = 0 \\ [1:0] & \text{for } cx = -dy \end{cases},$$

where

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is an element of $\mathrm{SL}(2,\mathbb{R})$ and by $[x:y]$ we mean the equivalence class of the element $(x,y) \in \mathbb{E}$.
The following result can be proved by a direct computation.

Lemma 5.3.1. *The quotient map $p : \mathbb{E} \to \mathbb{RP}^1$ is equivariant for the action of $\mathrm{SL}(2,\mathbb{R})$, hence in particular for the action of any discrete subgroup Γ.*

$\mathrm{SL}(2,\mathbb{R})$ also admits an action on the one-point compactification of the real line $\partial\mathbb{H} := \mathbb{R} \cup \{\infty\}$ given by Moebius transformations:

$$g \cdot x := \begin{cases} \frac{ax+b}{cx+d} & \text{for } cx \neq -d,\ x \neq \infty \\ \infty & \text{for } cx = -d,\ x \neq \infty \\ \frac{a}{c} & \text{for } c \neq 0,\ x = \infty \\ \infty & \text{for } c = 0,\ x = \infty \end{cases},$$

where

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is an element of $\mathrm{SL}(2,\mathbb{R})$.

The following Lemma is well known, see [26] Exercise 1.14.

Lemma 5.3.2. *The homeomorphism*

$$\begin{array}{rrcl} & \partial\mathbb{H} & \to & \mathbb{RP}^1 \\ \phi : & \mathbb{R} \ni x & \mapsto & [x:1] \\ & \infty & \mapsto & [1:0] \end{array}$$

is an isomorphism of $\mathrm{SL}(2,\mathbb{R})$*-spaces.*

Let now Γ_p be a Fuchsian group of the first kind (see [42] page 67); note that the action we defined for discrete subgroups of $\mathrm{SL}(2,\mathbb{R})$ on $\partial\mathbb{H}$ factors through the quotient $\pi_p : \mathrm{SL}(2,\mathbb{R}) \to \mathrm{PSL}(2,\mathbb{R})$.
The crossed product C^*-algebra $C(\partial\mathbb{H}) \rtimes \Gamma_p$ has been studied in [46] and [3]. In the approach of [46] a key role is played by a property of this dynamical system that is encoded in the following

Definition 5.3.3. *Let X be a compact Hausdorff space and Γ a discrete group acting on it by homeomorphisms. The action is a strong boundary action if for every nonempty open sets U, $V \subset X$ there is a $t \in \Gamma$ such that $tU^c \subset V$, where U^c is the complement of U in X.*

By Lemma 3 of [46] the strong boundary condition implies existence of projections in the reduced crossed product C^*-algebra. If furthermore the action is topologically free, one obtains the following

Theorem 5.3.4 ([46] Theorem 5)**.** *Let Γ be a discrete group acting on a compact space X by means of a topologically free strong boundary action. The resulting reduced crossed product C^*-algebra $C(X) \rtimes_r \Gamma$ is simple and purely infinite.*

The action of Γ_p on $\partial\mathbb{H}$ satisfies the hypothesis of this theorem (this is a consequence of the fact that the fixed points of the hyperbolic elements of a Fuchsian group of the first kind form a dense subset of $\partial\mathbb{H}$ by [42] Theorem 3.4.4) and so $C(\partial\mathbb{H}) \rtimes_r \Gamma_p$ is a simple purely infinite C^*-algebra. As already pointed out in [46], this C^*-algebra is actually nuclear. To see this note that the group P of upper-triangular matrices with positive diagonal entries in $\mathrm{SL}(2,\mathbb{R})$ is a semidirect product $N \rtimes A$ (see [75] 3.3) and hence is amenable since both A and N are.

In particular, the groups $\pi_p(N)$ and $\pi_p(A)$ in $\mathrm{PSL}(2,\mathbb{R})$ are such that $\pi_p(P) = \pi_p(N)\pi_p(A)$, $\pi_p(N) \cap \pi_p(A) = \{e\}$ and $\pi_p(N)$ is normal in $\pi_p(P)$. Thus also $\pi_p(P)$ is isomorphic to a semidirect product of the form $\mathbb{R} \rtimes \mathbb{R}$ (since $\pi_p(N)$ and $\pi_p(A)$ are one-parameter subgroups of $\mathrm{PSL}(2,\mathbb{R})$). In particular, by [75] Proposition 3.11 and Proposition 1.5.4 we see that

$$\begin{aligned} C(\partial\mathbb{H}) \rtimes \Gamma_p &= C(\mathrm{PSL}(2,\mathbb{R})/\pi_p(NA)) \rtimes \Gamma_p \\ &\simeq_{M.e.} C(\Gamma_p \backslash \mathrm{PSL}(2,\mathbb{R})) \rtimes \pi_p(NA) \\ &= (C(\Gamma_p \backslash \mathrm{PSL}(2,\mathbb{R})) \rtimes \pi_p(N)) \rtimes \pi_p(A) \end{aligned}$$

for certain actions of $\pi_p(N)$ on $C(\Gamma_p \backslash \mathrm{PSL}(2,\mathbb{R}))$ and $\pi_p(A)$ on $C(\Gamma_p \backslash \mathrm{PSL}(2,\mathbb{R})) \rtimes \pi_p(N)$.
Now both nuclearity and the *UCT* are preserved under taking crossed products by $\mathbb{R}$ and so $C(\partial\mathbb{H}) \rtimes \Gamma_p$ is in fact classifiable by [54] Theorem 4.2.4.

As in the previous section, if G is a discrete group acting on a C^*-algebra A, we can consider the extension of this action to the multiplier algebra $M(A)$ and the corresponding full and reduced crossed products. For the proof of Proposition 5.3.6 we need the following

Lemma 5.3.5. *Let X be a locally compact Hausdorff space and Y be a compact Hausdorff space. Let G be a discrete group acting on them. If there is an equivariant surjective continuous map $p : X \to Y$, then there is a unital $*$-homomorphism*

$$\hat{p} \rtimes G : C(Y) \rtimes G \to C_b(X) \rtimes G.$$

Proof. The map

$$\hat{p} : \begin{array}{ccc} C(Y) & \to & C_b(X) \\ f & \mapsto & f \circ p \end{array}$$

is an injective $*$-homomorphism. Let α denote the action of G on X and β the action on Y; let also $\hat{\alpha}$ and $\hat{\beta}$ be the "dual" actions at the level of continuous functions (we keep denoting the action of a group on a C^*-algebra with the same symbol as its extension to the multiplier algebra) $\hat{\alpha}(f) = f \circ \alpha$, $\hat{\beta}(g) = g \circ \beta$, for every $f \in C(Y)$ and $g \in C_b(X)$.

The equivariance condition dualizes as

$$\begin{aligned}(\hat{\alpha}_t \circ \hat{p})(f) &= \hat{\alpha}_t(f \circ p) \\ &= f \circ p \circ \alpha_{-t} \\ &= f \circ \beta_{-t} \circ p \\ &= \hat{p}(f \circ \beta_{-t}) \\ &= (\hat{p} \circ \hat{\beta}_t)(f)\end{aligned}$$

for every $f \in C(Y)$ and $t \in \mathbb{R}$. If Y is compact, then $C(Y)$ has a unit 1 and if G is discrete $1(= 1u_e)$ is also a unit for $C(Y) \rtimes G$ and belongs to $C_c(G, C(Y))$; in this case also $C(Y) \rtimes G$ is unital. Then we have $\hat{p} \rtimes G(1) = 1 \circ p = 1 \in C_c(G, C_b(X))$. By Lemma 4.2.1 there is a unital $*$-homomorphism $\hat{p} \rtimes G : C(Y) \rtimes G \to C_b(X) \rtimes G$. $\square$

Let Γ be a cocompact discrete subgroup of $\mathrm{SL}(2, \mathbb{R})$ and $\Gamma_p = \pi_p(\Gamma)$ its image under the quotient map $\pi_p : \mathrm{SL}(2, \mathbb{R}) \to \mathrm{PSL}(2, \mathbb{R})$. By Lemma 5.2.3 Γ_p is a discrete cocompact subgroup of $\mathrm{PSL}(2, \mathbb{R})$ and by [45] Section 3, if Γ_p does not contain cyclic elements of order 2, there exists an injective group homomorphism $\Gamma_p \to \Gamma$. Now we can state the following

Proposition 5.3.6. *Let Γ be a cocompact discrete subgroup of* $\mathrm{SL}(2, \mathbb{R})$ *such that $\Gamma_p := \pi_p(\Gamma)$ does not contain cyclic elements of order* 2. *There is a unital (injective) $*$-homomorphism $C(\partial\mathbb{H}) \rtimes \Gamma_p \to M(C_0(\mathbb{E}) \rtimes \Gamma)$. In particular $M(C_0(\mathbb{E}) \rtimes \Gamma)$ contains a separable, simple, nuclear, purely infinite unital C^*-subalgebra in the UCT class as a unital C^*-subalgebra.*

Proof. By Lemma 5.3.1 and Lemma 5.3.5 there is a unital $*$-homomorphism

$$C(\mathbb{RP}^1) \rtimes \Gamma \to C_b(\mathbb{E}) \rtimes \Gamma.$$

Using Lemma 5.3.2 and Lemma 4.2.1 we see that $C(\mathbb{RP}^1) \rtimes \Gamma$ is $*$-isomorphic to $C(\partial\mathbb{H}) \rtimes \Gamma$ and thus we obtain a unital $*$-homomorphism

$$C(\partial\mathbb{H}) \rtimes \Gamma \to C_b(\mathbb{E}) \rtimes \Gamma.$$

Now $C_0(\mathbb{E})$ is an ideal in $C_b(\mathbb{E})$ and so $C_0(\mathbb{E}) \rtimes \Gamma$ is an ideal in $C_b(\mathbb{E}) \rtimes \Gamma$. As we have already argued in the previous sections, by the universal property of the multiplier algebra there exists a $*$-homomorphism

$$C_b(\mathbb{E}) \rtimes \Gamma \to M(C_0(\mathbb{E}) \rtimes \Gamma),$$

which is unital since $C_b(\mathbb{E}) \rtimes \Gamma$ is unital. Composing we obtain a unital $*$-homomorphism

$$\Psi : C(\partial\mathbb{H}) \rtimes \Gamma \to M(C_0(\mathbb{E}) \rtimes \Gamma).$$

The next step is to use Kra's result [45] Section 3 in order to construct a unital $*$-homomorphism from $C(\partial\mathbb{H}) \rtimes \Gamma_p$ to $C(\partial\mathbb{H}) \rtimes \Gamma$. The result of Kra is more general and concerns Klenian groups; the form that we need here is the following: Let Γ_p be a finitely generated Fuchsian group not containing cyclic elements of order 2, then there exists an injective group homomorphism $\phi : \Gamma_p \to \Gamma$, that is given by a choice of representatives for Γ_p in Γ.
Define

$$\Phi_c : \begin{array}{ccc} C_c(\Gamma_p, C(\partial\mathbb{H})) & \to & C_c(\Gamma, C(\partial\mathbb{H})) \\ \sum_{\gamma\in\Gamma_p} f_\gamma u_\gamma & \mapsto & \sum_{\gamma\in\Gamma_p} f_\gamma v_{\phi(\gamma)}. \end{array}$$

We want to check that Φ_c extends to a unital $*$-homomorphism and the crucial fact is that the action of Γ on $\partial\mathbb{H}$ factors through the action of Γ_p. Denote by α^p the action of Γ_p on $C(\partial\mathbb{H})$ and by α the action of Γ on $C(\partial\mathbb{H})$.
Let $f = \sum_{\gamma\in\Gamma_p} f_\gamma u_\gamma$ and $g = \sum_{\gamma'\in\Gamma_p} g_{\gamma'} u_{\gamma'}$ be elements of $C_c(\Gamma_p, C(\partial\mathbb{H}))$. Then

$$\begin{aligned} (\Phi_c(\sum_{\gamma\in\Gamma_p} f_\gamma u_\gamma))^* &= (\sum_{\gamma\in\Gamma_p} f_\gamma v_{\phi(\gamma)})^* \\ &= \sum_{\gamma\in\Gamma_p} \alpha_{\phi(\gamma)^{-1}}(\overline{f_\gamma}) v_{\phi(\gamma)^{-1}} \\ &= \sum_{\gamma\in\Gamma_p} (\overline{f_\gamma} \circ \phi(\gamma)) v_{\phi(\gamma)^{-1}} \\ &= \sum_{\gamma\in\Gamma_p} (\overline{f_\gamma} \circ \gamma) v_{\phi(\gamma)^{-1}} \end{aligned}$$

and

$$\begin{aligned} \Phi_c((\sum_{\gamma\in\Gamma_p} f_\gamma u_\gamma)^*) &= \Phi_c(\sum_{\gamma\in\Gamma_p} \alpha^p_{\gamma^{-1}}(\overline{f_\gamma}) u_{\gamma^{-1}}) \\ &= \Phi_c(\sum_{\gamma\in\Gamma_p} (\overline{f_\gamma} \circ \gamma) u_{\gamma^{-1}}) \\ &= \sum_{\gamma\in\Gamma_p} (\overline{f_\gamma} \circ \gamma) v_{\phi(\gamma)^{-1}}. \end{aligned}$$

For what concerns multiplicativity we have

$$\begin{aligned}\Phi_c((\sum_{\gamma\in\Gamma_p} f_\gamma u_\gamma)(\sum_{\gamma\in\Gamma_p} g_\gamma u_\gamma)) &= \Phi_c(\sum_{\gamma,\gamma'\in\Gamma_p} f_\gamma \alpha^p_\gamma(g_{\gamma'}) u_{\gamma\gamma'}) \\ &= \sum_{\gamma,\gamma'\in\Gamma_p} f_\gamma \alpha^p_\gamma(g_{\gamma'}) v_{\phi(\gamma)\phi(\gamma')}\end{aligned}$$

and

$$\begin{aligned}\Phi_c(\sum_{\gamma\in\Gamma_p} f_\gamma u_\gamma)\Phi_c(\sum_{\gamma'\in\Gamma_p} g_{\gamma'\in\Gamma_p} u_{\gamma'}) &= (\sum_{\gamma\in\Gamma_p} f_\gamma v_{\phi(\gamma)})(\sum_{\gamma'\in\Gamma_p} g_{\gamma'\in\Gamma_p} v_{\phi(\gamma')}) \\ &= \sum_{\gamma,\gamma'\in\Gamma_p} f_\gamma \alpha_{\phi(\gamma)}(g_{\gamma'}) v_{\phi(\gamma)\phi(\gamma')} \\ &= \sum_{\gamma,\gamma'\in\Gamma_p} f_\gamma \alpha^p_\gamma(g_{\gamma'}) v_{\phi(\gamma)\phi(\gamma')}\end{aligned}$$

and so Φ_c is an algebraic $*$-homomorphism, that is clearly unital. Now

$$\|\Phi_c(f)\|_1 = \|f\|_1$$

and thus Φ_c extends to a continuous unital $*$-homomorphism

$$\Phi_1 : L^1(\Gamma_p, C(\partial\mathbb{H})) \to L^1(\Gamma, C(\partial\mathbb{H})).$$

In particular

$$\|\Phi_1(g)\| = \sup_{\pi\rtimes V} \|\pi \rtimes V(\Phi_1(g))\| \leqslant \|g\|, \qquad \text{for any } g \in L^1(\Gamma_p, C(\partial\mathbb{H})),$$

where the supremum runs over all the covariant representations of $C_c(\Gamma, C(\partial\mathbb{H}))$. Hence Φ_c extends to an injective $*$-homomorphism $\Phi : C(\partial\mathbb{H}) \rtimes \Gamma_p \to C(\partial\mathbb{H}) \rtimes \Gamma$. The composition

$$\Psi \circ \Phi : C(\partial\mathbb{H}) \rtimes \Gamma_p \to M(C_0(\mathbb{E}) \rtimes \Gamma)$$

is a unital $*$-homomorphism, which is injective since $C(\partial\mathbb{H}) \rtimes \Gamma_p$ is simple. $\square$

Chapter 6

Spectral Triples on the Jiang-Su algebra

This chapter is based on a joint work with Professor Ludwik Dąbrowski ([7]).
According to the noncommutative differential geometry program [20, 21] both the topological and the metric information on a noncommutative space can be fully encoded as a *spectral triple* on the noncommutative algebra of coordinates on that space. Nowadays several noncommutative spectral triples have been constructed, with only a partial unifying scheme emerging behind some families of examples, e.g. quantum groups and their homogeneous spaces, like quantum spheres and quantum projective spaces. (see e.g., [24, 25, 50]) Also some preservation properties with respect to the product, inductive limits or extensions of algebras have been investigated.

Most of these constructions are still awaiting however a proper analysis of such properties as smoothness, dimension (summability) and other conditions selected by Connes. As a testing ground for these and related matters as large as possible class of examples should be investigated, including some important new algebras.

In [67] a general way to construct a spectral triple on arbitrary quasi diagonal C^*-algebras was exhibited. However, in that case one cannot expect summability. Instead, summability was obtained in [1] for certain inductive family of coverings, and p-summability with arbitrary p for any AF-algebra through the construction in [5].

In the present paper we elaborate a construction that extends the latter mentioned approach to a wider class of particular inductive limits of matrix-valued function algebras whose connecting morphisms have a certain peculiar form. In particular this construction applies to the Jiang-Su algebra $\mathcal{Z}$ (cf. [41]), which was originally constructed in terms of an explicit particular inductive limit of dimension drop algebras. The aim therein was to obtain an example of an infinite-dimensional stably finite nuclear simple unital C^*-algebras with exactly one tracial state and with the same K-theory of the complex numbers. The importance of the Jiang-Su algebra $\mathcal{Z}$ stems from the fact that under some other hypothesis $\mathcal{Z}$-stability entails classification in terms of the Elliott invariant as proved in [76].

The organization of the paper is the following: In the first section we recall the definition of the Jiang-Su algebra and construct a particular AF-embedding for it. In the second section we compute the image of elements belonging to a dense subalgebra of the Jiang-Su algebra under the representation obtained by composing the aforementioned AF-embedding with the representation appearing in [5]. In the last section we use the above results to establish that some of the Dirac operators considered in [5] give rise to a spectral triple for the Jiang-Su algebra.

6.1 The AF-embedding

Let B be an inductive limit of C^*-algebras $B = \lim(B_i, \phi_i)$, with $B_0 = \mathbb{C}$ and where every B_i is a unital sub-C^*-algebra of the algebra of continuous-valued functions on the interval with values in M_{n_i} for some natural numbers n_i containing a dense $*$-subalgebra of Lipschitz functions. For $l > i$ natural numbers the connecting morphisms $\phi_{i,i+l}$ take the form

$$\phi_{i,i+l}(f) = u_{i,i+l} \begin{pmatrix} f \circ \xi_{i,1}^{i+l} \otimes 1_{N_{i,1}^{i+l}} & & 0 \\ & \ddots & \\ 0 & & f \circ \xi_{i,k_i^{i+l}}^{i+l} \otimes 1_{N_{i,k_i^{i+l}}^{i+l}} \end{pmatrix} u_{i,i+l}^*$$

for some natural numbers k_i^{i+l}, $N_{i,1}^{i+l}$,...,$N_{i,k_i^{i+l}}^{i+l}$, a unitary $u_{i,i+l} \in C([0,1], M_{n_{i+l}})$ and some paths $\xi_{i,1}^{i+l}, \dots, \xi_{i,k_i^{i+l}}^{i+l} : [0,1] \to [0,1]$ satisfying

$$|\xi_{i,r}^{i+l}(x) - \xi_{i,r}^{i+l}(y)| \leqslant \frac{1}{2^l}, \qquad \text{for } 1 \leqslant r \leqslant k_i^{i+l},\ x, y \in [0,1]. \tag{6.1}$$

The operators $u_{i,i+l}$ are unitaries in $C([0,1], M_{n_{i+l}})$.

We will take advantage of the original construction in [41] of the Jiang-Su algebra $\mathcal{Z}$ as an inductive limit of prime dimension drop algebras Z_i, which are C^*-algebras of continuous functions from $[0,1]$ to $M_{p_i} \otimes M_{q_i}$ for some p_i and q_i coprime such that $f(0)$ belongs to $M_{p_i} \otimes 1_{q_i}$ and $f(1)$ belongs to $1_{p_i} \otimes M_{q_i}$ for every $f \in Z_i$. There it was proven that given $p_i, q_i, n_i = p_i q_i$ defining the prime dimension drop algebra Z_i, there are numbers $N_{i,1}^{i+1}$, $N_{i,2}^{i+1}$ and $N_{i,3}^{i+1}$ a unitary $u_{i,i+1} \in C([0,1], M_{n_{i+1}})$ such that

$$Z_i \to Z_{i+1}$$

$$\phi_i : f \mapsto u_{i,i+1} \begin{pmatrix} f \circ \xi_{i,1}^{i+1} \otimes 1_{N_{i,1}^{i+1}} & & 0 \\ & f \circ \xi_{i,2}^{i+1} \otimes 1_{N_{i,2}^{i+1}} & \\ 0 & & f \circ \xi_{i,3}^{i+1} \otimes 1_{N_{i,3}^{i+1}} \end{pmatrix} u_{i,i+1}^*$$

is a connecting morphism for $\xi_1 = x/2$, $\xi_2 = 1/2$ and $\xi_3 = (x+1)/2$.
As a consequence, given a natural number l, the connecting morphism $Z_i \to Z_{i+l}$ has the form

$$\phi_{i,i+l}(f) = u_{i,i+l} \begin{pmatrix} f \circ \xi_{i,1}^{i+l} \otimes 1_{N_{i,1}^{i+l}} & & 0 \\ & \ddots & \\ 0 & & f \circ \xi_{i,k_i^{i+l}}^{i+l} \otimes 1_{N_{i,k_i^{i+l}}^{i+l}} \end{pmatrix} u_{i,i+l}^*$$

for some natural numbers k_i^{i+l}, $N_{i,1}^{i+l}$,...,$N_{i,k_i^{i+l}}^{i+l}$, a unitary $u_{i,i+l} \in C([0,1], M_{n_{i+l}})$ and some paths $\xi_{i,1}^{i+l}, ..., \xi_{i,k_i^{i+l}}^{i+l}$ that have the form

$$\xi_i^{i+l}(x) = \frac{x+r}{2^l} \qquad \text{for} \quad 0 \leqslant r \leqslant 2^l - 1$$

or

$$\xi_i^{i+l}(x) = \frac{s}{2^l} \qquad \text{for} \quad 1 \leqslant s \leqslant 2^l - 1,$$

It follows that the paths appearing in the connecting morphism $\phi_{l,m}$ satisfy equation (6.1) and $\mathcal{Z}$ belongs to the class of inductive limit C^*-algebras we want to consider.
Note that, given B as above, after reindexing the sequence B_i, for example sending $i \mapsto i^2$ we can always suppose that the paths appearing in the connecting morphisms satisfy

$$|\xi_{i,r}^{i+1}(x) - \xi_{i,r}^{i+1}(y)| \leqslant \frac{1}{2^i}$$

for any $1 \leqslant r \leqslant k_i^{i+1}$. This relation will be used for the proof of Lemma 6.1.1.

Fix a sequence of natural numbers n_i as above and consider the inductive limit $A = \lim(A_i, \phi_i^\circ)$, where $A_i = C([0,1], M_{n_i})$ and the connecting morphisms ϕ_i° are constructed in the same way as above, but they are considered as unital $*$-homomorphisms between the A_i's. For any $i, l \in \mathbb{N}$ denote by $\tilde{\phi}_{i,i+l}^\circ : A_i \to A_{i+l}$ the $*$-homomorphism

$$\tilde{\phi}^{\circ}_{i,i+l}(f) = \begin{pmatrix} f \circ \xi^{i+l}_{i,1} \otimes 1_{N^{i+l}_{i,1}} & & 0 \\ & \ddots & \\ 0 & & f \circ \xi^{i+l}_{i,k^{i+l}_i} \otimes 1_{N^{i+l}_{i,k^{i+l}_i}} \end{pmatrix}.$$

Let u_i be the unitary corresponding to the connecting morphism $A_1 \to A_i$ (or $B_1 \to B_i$). For any $f \in A_i$ (or B_i) there is a unique $\tilde{f} \in A_i$ such that $f = u_i \tilde{f} u_i^*$. In this way the connecting morphisms take the form

$$\phi^{\circ}_{i,i+l}(f) = u_{i,i+l}\tilde{\phi}^{\circ}_{i,i+l}(f)u^*_{i,i+l} = u_{i+l}\tilde{\phi}^{\circ}_{i,i+l}(\tilde{f})u^*_{i+l},$$

Let now $M = \lim(M_{n_i}, \psi_i)$, where $\psi_i(a) = a \otimes 1_{n_{i+1}/n_i}$.

Lemma 6.1.1. *There is a $*$-isomorphism*

$$\alpha : A \to M.$$

Let $\gamma \in (1, 2)$. A Lipschitz function $f \in A_i$ with Lipschitz constant $L_f < \gamma^i$ is sent to

$$\alpha(f) = \lim_{m\to\infty} \psi^{\infty}_m(\tilde{\phi}^{\circ}_{i,m}(\tilde{f})(0)).$$

Proof. Define $*$-homomorphisms

$$\alpha_i : A_i \to M_{n_{i+1}}, \quad f \mapsto \tilde{\phi}^{\circ}_i(\tilde{f})(0)$$

and

$$\beta_i : M_{n_i} \to A_i, \quad a \mapsto u_i \bar{a} u_i^*,$$

where $\bar{a} \in A_i$ is the constant matrix-valued function taking value $a \in M_{n_i}$. Let now $\gamma \in (1, 2)$ and take finite sets $F_i \subset A_i$ consisting of Lipschitz matrix-valued functions with Lipschitz constant less than γ^i and such that their union $\bigcup_i F_i$ is dense in A. For any $f \in F_i$ and $a \in M_{n_i}$ we have

$$\alpha_i \circ \beta_i(a) = \psi_i(a),$$

$$\|\beta_{i+1} \circ \alpha_i(f) - \phi^{\circ}_{i,i+1}(f)\| < \frac{\gamma^i}{2^i}.$$

Hence the result follows by [61] Proposition 2.3.2. □

6.2 The orthogonal decomposition

Let $\mathfrak{H}$ be the Hilbert space considered by Christensen and Antonescu in [5] corresponding to the GNS-representation induced by the unique trace τ on M. This trace is given on the finite-dimensional approximants relative to the inductive limit construction by the normalized trace on matrices. Following [5] we want to write $\mathfrak{H}$ as an infinite direct sum of the finite dimensional Hilbert spaces on which the M_{n_i}'s are represented.
Let $\mathfrak{H}_i = \overline{M_{n_i}}^{\tau}$ and let $v \in \mathfrak{H}_i$. We can consider v as a matrix of dimension n_i and for any $j < i$, we can write v as a matrix-valued matrix of the form

$$v = \begin{pmatrix} v^{j,i}_{1,1} & \cdots & v^{j,i}_{1,l^i_j} \\ \vdots & \ddots & \vdots \\ v^{j,i}_{l^i_j} & \cdots & v^{j,i}_{l^i_j,l^i_j} \end{pmatrix},$$

where $l^i_j = n_i/n_j$ is the multiplicity of M_{n_j} in M_{n_i} and $v^{j,i}_{k,l}$ are matrices in M_{n_j}. Below we will also use iteration of this procedure. With this notation, the projection $P_{i,j}$ from $\mathfrak{H}_i$ to $\mathfrak{H}_j$ reads

$$P_{i,j}(v) = \frac{1}{l^i_j} \sum_{k=1}^{l^i_j} v^{j,i}_{k,k} \quad \in M_{n_j}.$$

If $i > 1$, the projection R_j from $\mathfrak{H}_j$ to the orthogonal complement of $\mathfrak{H}_{j-1}$ in $\mathfrak{H}_j$ reads for $w \in \mathfrak{H}_j$:

$$R_j(w) = \begin{pmatrix} w^{j-1,j}_{1,1} - \frac{1}{l^j_{j-1}} \sum_{k=1}^{l^i_{i-1}} w^{j-1,j}_{k,k} & w^{j-1,j}_{1,2} & \cdots & w^{j-1,j}_{1,l^j_{j-1}} \\ w^{j-1,j}_{2,1} & w^{j-1,j}_{2,2} - \frac{1}{l^j_{j-1}} \sum_{k=1}^{l^i_{i-1}} w^{j-1,j}_{k,k} & & w^{j-1,j}_{2,l^j_{j-1}} \\ \vdots & & \ddots & \vdots \\ w^{j-1,j}_{l^j_{j-1},1} & \cdots & & w^{j-1,j}_{l^j_{j-1},l^j_{j-1}} - \frac{1}{l^j_{j-1}} \sum_{k=1}^{l^i_{i-1}} w^{j-1,j}_{k,k} \end{pmatrix}.$$

Hence, if we denote by $\mathfrak{K}_j = \mathfrak{H}_j \ominus \mathfrak{H}_{j-1}$, the projection $Q_j : \mathfrak{H} \to \mathfrak{K}_j$, when applied to an element $v \in \mathfrak{H}_i$ takes the form, for $1 \leqslant s,t \leqslant l^j_{j-1}$

$$(Q_j(v))^{j-1,j}_{s,t} = \begin{cases} \frac{1}{l^i_j} \sum_{k=1}^{l^i_j} (v^{j,i}_{k,k})^{j-1,j}_{s,s} - \frac{1}{l^i_{j-1}} \sum_{t=1}^{l^j_{j-1}} \sum_{k=1}^{l^i_j} (v^{j,i}_{k,k})^{j-1,j}_{t,t} & \text{for } s = t \\ \frac{1}{l^i_j} \sum_{k=1}^{l^i_j} (v^{j,i}_{k,k})^{j-1,j}_{s,t} & \text{for } s \neq t \end{cases},$$

where, with a slight abuse of notation, we identify the spaces $\mathfrak{H}_i$ with their images in $\mathfrak{H}$ and correspondingly consider the projections Q_j as operators from $\mathfrak{H}_i$ to $\mathfrak{K}_j$.

6.3 The commutators

Take $i < n-1 < m$ and $v \in \mathfrak{H}_m$, $f \in A_i$. We want to compute the elements $Q_n(\tilde{\phi}^\circ_{i,m}(\tilde{f})(0)v)$ and $\tilde{\phi}^\circ_{i,n}(\tilde{f})(0)Q_n v$. To this end we want to write $\tilde{\phi}^\circ_{i,m}(\tilde{f})$ as the composition $\tilde{\phi}^\circ_{n,m} \circ \tilde{\phi}^\circ_{n-1,n} \circ \tilde{\phi}^\circ_{i,n-1}(\tilde{f})$.
Let k^i_j be the amount of different paths appearing in the connecting morphism $\phi_{j,i}$. If $1 \leqslant j \leqslant k^n_{n-1}$, we denote by $\tilde{f} \circ [\xi^{n-1}_i] \circ \xi^n_{n-1,j} = \tilde{\phi}^\circ_{i,n-1}(\tilde{f}) \circ \xi^n_{n-1,j}$ the matrix-valued function

$$\begin{pmatrix} \tilde{f} \circ \xi^{n-1}_{i,1} \circ \xi^n_{n-1,j} \otimes 1_{N^{n-1}_{i,1}} & & 0 \\ & \ddots & \\ 0 & & \tilde{f} \circ \xi^{n-1}_{i,k^{n-1}_i} \circ \xi^n_{n-1,j} \otimes 1_{N^{n-1}_{i,k^{n-1}_i}} \end{pmatrix},$$

then we can write

$$\tilde{\phi}^\circ_{i,n}(\tilde{f}) = \tilde{\phi}^\circ_{n,n-1} \circ \tilde{\phi}^\circ_{i,n-1}(\tilde{f}) =$$

$$\begin{pmatrix} \tilde{f} \circ [\xi^{n-1}_i] \circ \xi^n_{n-1,1} \otimes 1_{N^n_{n-1,1}} & & 0 \\ & \ddots & \\ 0 & & \tilde{f} \circ [\xi^{n-1}_i] \circ \xi^n_{n-1,k^n_{n-1}} \otimes 1_{N^n_{n-1,k^n_{n-1}}} \end{pmatrix}.$$

For $1 \leqslant s, \leqslant l^n_{n-1}$, we denote by $\bar{\xi}^n_{n-1,s}$ the path

$$\bar{\xi}^n_{n-1,s} = \begin{cases} \xi^n_{n-1,1} & \text{for } 1 \leqslant s \leqslant N^n_{n-1,1} \\ \xi^n_{n-1,2} & \text{for } N^n_{n-1,1} < s \leqslant N^n_{n-1,1} + N^n_{n-1,2} \\ \vdots & \\ \xi_{n-1,k^n_{n-1}} & \text{for } \sum_{k=1}^{k^n_{n-1}-1} N^n_{n-1,k} < s \leqslant l^n_{n-1} \end{cases}.$$

Thus we obtain for $1 \leqslant s, t \leqslant l^n_{n-1}$,

$$(\tilde{\phi}^\circ_{n-1,n} \circ \tilde{\phi}^\circ_{i,n-1}(\tilde{f})(0)Q_n v)^{n-1,n}_{s,t} =$$

$$\frac{1}{l^m_n} \sum_{j=1}^{l^m_n} (\tilde{f} \circ [\xi^{n-1}_i] \circ \bar{\xi}^n_{n-1,s})(0)(v^{n,m}_{j,j})^{n-1,n}_{s,t} \qquad \text{for } s \neq t$$

and

$$\frac{1}{l_n^m}\sum_{j=1}^{l_n^m}(f\circ[\xi_i^{n-1}]\circ\bar{\xi}_{n-1,s}^n)(0)\left((v_{j,j}^{n,m})_{s,s}^{n-1,n}-\frac{1}{l_{n-1}^n}\sum_{k=1}^{l_{n-1}^n}(v_{j,j}^{n,m})_{k,k}^{n-1,n}\right)\quad \text{for } s=t.$$

In the same way, for $1\leqslant j\leqslant l_n^m$, we can define paths

$$\bar{\xi}_{n,j}=\begin{cases}\xi_{n,1}^m & \text{for } 1\leqslant j\leqslant N_{n,1}^m\\ \xi_{n,2}^m & \text{for } N_{n,1}^m<j\leqslant N_{n,1}^m+N_{n,2}^m\\ \vdots & \\ \xi_{n,k_n^m}^m & \text{for } \sum_{k=1}^{k_n^m-1}N_{n,k}^m<j\leqslant l_n^m\end{cases}$$

and compute for $1\leqslant s,t\leqslant l_{n-1}^n$,

$$(Q_n\tilde{\phi}_{i,m}^{\circ}(\tilde{f})(0)v)_{s,t}^{n-1,n}=(Q_n(\tilde{\phi}_{n,m}^{\circ}\circ\tilde{\phi}_{n-1,n}^{\circ}\circ\tilde{\phi}_{i,n-1}^{\circ})(\tilde{f})(0)v)_{s,t}^{n-1,n}=$$

$$\frac{1}{l_n^m}\sum_{j=1}^{l_n^m}(\tilde{f}\circ[\xi_i^{n-1}]\circ\bar{\xi}_{n-1,s}^n\circ\bar{\xi}_{n,j}^m)(0)(v_{j,j}^{n,m})_{s,t}^{n-1,n}\quad \text{for } s\neq t$$

and

$$\frac{1}{l_n^m}\sum_{j=1}^{l_n^m}[(\tilde{f}\circ[\xi_i^{n-1}]\circ\bar{\xi}_{n-1,s}^n\circ\bar{\xi}_{n,j}^m)(0)(v_{j,j}^{n,m})_{s,s}^{n-1,n}$$

$$-\frac{1}{l_{n-1}^n}\sum_{k=1}^{l_{n-1}^n}(\tilde{f}\circ[\xi_i^{n-1}]\circ\bar{\xi}_{n-1,k}^n\circ\bar{\xi}_{n,j}^m)(0)(v_{j,j}^{n,m})_{k,k}^{n-1,n})]\quad \text{for } s=t.$$

Thus we can write the commutators as follows

$$(Q_n(\tilde{\phi}_{i,m}^{\circ}(\tilde{f})(0)v)-\tilde{\phi}_{i,n}^{\circ}(\tilde{f})(0)Q_nv)_{s,t}^{n-1,n}=$$

$$\frac{1}{l_n^m}\sum_{j=1}^{l_n^m}(\tilde{f}\circ[\xi_i^{n-1}]\circ\bar{\xi}_{n-1,s}^n\circ\bar{\xi}_{n,j}^m-\tilde{f}\circ[\xi_i^{n-1}]\circ\bar{\xi}_{n-1,s}^n)(0)(v_{j,j}^{n,m})_{s,t}^{n-1,n}\quad \text{for } s\neq t$$

and

$$\frac{1}{l_n^m}\sum_{j=1}^{l_n^m}[(\tilde{f}\circ[\xi_i^{n-1}]\circ\bar{\xi}_{n-1,s}^n\circ\bar{\xi}_{n,j}^m-\tilde{f}\circ[\xi_i^{n-1}]\circ\bar{\xi}_{n-1,s}^n)(0)(v_{j,j}^{n,m})_{s,t}^{n-1,n}+$$

$$\frac{1}{l_{n-1}^n}\sum_{k=1}^{l_{n-1}^n}(\tilde{f}\circ[\xi_i^{n-1}]\circ\bar{\xi}_{n-1,s}^n-\tilde{f}\circ[\xi_i^{n-1}]\circ\bar{\xi}_{n-1,k}^n\circ\xi_{n,j}^m)(0)(v_{j,j}^{n,m})_{k,k}^{n-1,n}\circ\bar{\xi}_{n,j}^m]\quad \text{for } s=t.$$

Lemma 6.3.1. *Let $i < l < m \leqslant k$ be natural numbers and let ξ_i^l, ξ_l^m, ξ_l^k be paths on the interval $[0,1]$ such that*

$$|\xi_i^l(x) - \xi_i^l(y)| \leqslant \frac{1}{2^{l-i}}, \qquad \text{for any } x, y \in [0,1].$$

Then, given any $n > 0$ and any Lipschitz function in $C([0,1], M_n)$ with Lipschitz constant L_f, we have

$$\|(f \circ \xi_i^l \circ \xi_l^m)(0) - (f \circ \xi_i^l \circ \xi_l^k)(0)\| \leqslant \frac{2^i L_f}{2^l}.$$

Proof. This is a consequence of the fact that $|\xi_i^l(x) - \xi_i^l(y)| \leqslant \frac{1}{2^{l-i}}$ for every $x, y \in [0,1]$. □

6.4 The spectral triple

Note that if $D = \sum_n \alpha_n Q_n$ for a certain sequence of real numbers $\{\alpha_n\}$, then the domain of D, $\operatorname{dom}(D) = \{v \in \mathfrak{H} \ : \ \{\|\alpha_n Q_n v\|\} \in l^2(\mathbb{N})\}$ is left invariant under the action of any $f \in A$, thus in particular for every $f \in B$, and it makes sense to consider the (in general unbounded) operator $[D, f]$.
Moreover, if T is an unbounded operator on $\mathfrak{H}$ whose domain contains the algebraic direct sum $\oplus_{alg}\mathfrak{K}_i$ and $\|TP_n\|$ is uniformly bounded on n, then T extends (uniquely) to a bounded operator on the whole Hilbert space $\mathfrak{H}$.
Hence, to obtain boundedness of $[D, f]$, we will compute estimates for $\|[D, f]P_n\|$ for every n.
For every $i \in \mathbb{N}$ we will denote by LB_i the linear subspace of B_i consisting of Lipschitz functions with Lipschitz constant smaller than γ^i for some $\gamma \in (1, 2)$. Observe that $\phi^\circ|_{LB_i}$ is a linear map sending LB_i into LB_{i+1} and that the algebraic direct limit $\bigcup_i LB_i$ is a dense $*$-subalgebra of B.

Proposition 6.4.1. *Let $D = \sum_n \alpha_n Q_n$, with $\{\alpha_n\}$ a diverging sequence of real numbers satisfying $\alpha_0 = 0$, $|\alpha_n| \leqslant \beta^{(n-1)}$ with $\beta < 2$ and $n > 0$. Then $(\bigcup_i LB_i, \mathfrak{H}, D)$ is a spectral triple for B.*
It is p-summable whenever the sequences of numbers $\{\alpha_i\}$, $\{n_i\}$ satisfy

$$\sum_{i \geqslant 1} (1 + \alpha_i^2)^{-p/2} (n_i^2 - n_{i-1}^2) < \infty$$

for some $p > 0$.

Proof. After reindexing $i \mapsto i^2$, the $*$-isomorphism $\alpha : A \to M$ has the concrete description given in Lemma 6.1.1 and we will suppose that the index set is already reindexed, if necessary. Thus we can compose it with the GNS representation of M induced by the unique trace τ.
Let $l, m \in \mathbb{N}$ and $v \in \mathfrak{H}_l$ of norm one. Denote by $\beta^{\mathfrak{H}}_{l,m} : \mathfrak{H}_l \to \mathfrak{H}_m$ and $\beta^{\mathfrak{H}}_{l,\infty} : \mathfrak{H}_l \to \mathfrak{H}$ the connecting isometries. Note that for $i < m \in \mathbb{N}$ and $f \in LB_i$ the action of f on v reads

$$\lim_{m\to\infty} \beta^{\mathfrak{H}}_{m,\infty}\tilde{\phi}^{\circ}_{i,m}(\tilde{f})(0)\beta^{\mathfrak{H}}_{l,m}v.$$

Thus we can write

$$\|Q_n f v - f Q_n v\| = \|\beta^{\mathfrak{H},\infty}_{n} Q_n \lim_{m\to\infty} \tilde{\phi}^{\circ}_{i,m}(\tilde{f})(0)\beta^{\mathfrak{H}}_{l,m}v - \lim_{m\to\infty} \beta^{\mathfrak{H}}_{m,\infty}\tilde{\phi}^{\circ}_{i,m}(\tilde{f})(0)\beta^{\mathfrak{H}}_{n,m}Q_n v\|.$$

Since the sequence $\beta^{\mathfrak{H}}_{m,\infty}\tilde{\phi}^{\circ}_{i,m}(\tilde{f})(0)\beta^{\mathfrak{H}}_{l,m}v$ converges, there is an M such that

$$\|\beta^{\mathfrak{H}}_{k,\infty}\tilde{\phi}^{\circ}_{i,k}(\tilde{f})(0)\beta^{\mathfrak{H}}_{l,k}v - \lim_{m\to\infty} \beta^{\mathfrak{H}}_{m,\infty}\tilde{\phi}^{\circ}_{i,m}(\tilde{f})(0)\beta^{\mathfrak{H}}_{l,m}v\| \leqslant \frac{1}{2^{(n-1)}}$$

for any $k \geqslant M$. Moreover, by Lemma 6.3.1 and the discussion preceding it

$$\begin{aligned}\|[\beta^{\mathfrak{H}}_{n,m}\tilde{\phi}^{\circ}_{i,n}(\tilde{f})(0) &- \tilde{\phi}^{\circ}_{i,m}(\tilde{f})(0)\beta^{\mathfrak{H}}_{n,m}]Q_n v\| \\ &= \|(\beta^{\mathfrak{H}}_{n,m}\tilde{f}\circ[\xi^n_i](0) - \tilde{f}\circ[\xi^n_i]\circ[\xi^m_n](0)\beta^{\mathfrak{H}}_{n,m})Q_n v\| \leqslant \frac{2^i L_f}{2^{(n-1)}}\end{aligned}$$

for $m > n > i+1$ and

$$\|Q_n\tilde{\phi}^{\circ}_{i,M}(\tilde{f})(0)\beta^{\mathfrak{H}}_{l,M}v - \tilde{\phi}^{\circ}_{i,n}(\tilde{f})(0)Q_n\beta^{\mathfrak{H}}_{l,M}v\| \leqslant \frac{2^i L_f}{2^{(n-1)}}.$$

We can suppose $M > n$ and obtain

$$\begin{aligned}\|\beta^{\mathfrak{H}}_{n,\infty}Q_n \lim_{m\to\infty} &\beta^{\mathfrak{H}}_{m,\infty}\tilde{\phi}^{\circ}_{i,m}(\tilde{f})(0)\beta^{\mathfrak{H}}_{l,m}v - \lim_{m\to\infty}\beta^{\mathfrak{H}}_{m,\infty}\tilde{\phi}^{\circ}_{i,m}(\tilde{f})(0)\beta^{\mathfrak{H}}_{n,m}Q_n v\| \\ &\leqslant \|\beta^{\mathfrak{H}}_{n,\infty}Q_n[\beta^{\mathfrak{H}}_{M,\infty}\tilde{\phi}^{\circ}_{i,M}(\tilde{f})(0)\beta^{\mathfrak{H}}_{l,M}v - \lim_{m\to\infty}\beta^{\mathfrak{H}}_{m,\infty}\tilde{\phi}^{\circ}_{i,m}(\tilde{f})(0)\beta^{\mathfrak{H}}_{l,m}v]\| \\ &+ \|Q_n\tilde{\phi}^{\circ}_{i,M}(\tilde{f})(0)\beta^{\mathfrak{H}}_{l,M}v - \tilde{\phi}^{\circ}_{i,n}(\tilde{f})(0)Q_n\beta^{\mathfrak{H}}_{l,M}v\| \\ &+ \|\lim_{m\to\infty}\beta^{\mathfrak{H}}_{m,\infty}[\beta^{\mathfrak{H}}_{n,m}\tilde{\phi}^{\circ}_{i,n}(\tilde{f})(0) - \tilde{\phi}^{\circ}_{i,m}(\tilde{f})(0)\beta^{\mathfrak{H}}_{n,m}]Q_n\beta^{\mathfrak{H}}_{l,M}v\| \\ &\leqslant \frac{1+2^{i+1}L_f}{2^{(n-1)}}.\end{aligned}$$

Thus for $i < n-1$ we obtain

$$\|[\alpha_n Q_n, f]P_l\| \leqslant \frac{|\alpha_n|(1+2^{i+1}L_f)}{2^{(n-1)}} \leqslant (1+2^{i+1}L_f)(\beta/2)^{(n-1)}.$$

Hence

$$\begin{aligned}\|[D,f]\| &\leqslant \|[\sum_{n=1}^{i} \alpha_n Q_n, f\| + \|\sum_{n>i} \alpha_n Q_n, f]\| \\ &\leqslant 2\|f\| \sum_{n=1}^{i} |\alpha_n| + (1+2^{i+1}L_f)\sum_{n>i}(\beta/2)^{(n-1)} < \infty\end{aligned}$$

and $[D, f]$ extends to a bounded operator.
Moreover D has compact resolvent since it has discrete spectrum and its eigenvalues have finite multiplicity. Suppose we have sequences $\{\alpha_i\}$, $\{n_i\}$ and a real number $p > 0$ as in the statement. Then

$$\mathrm{Tr}((1+D^2)^{-p/2}) = 1 + \sum_{i\geqslant 1}(1+\alpha_i^2)^{-p/2}(n_i^2 - n_{i-1}^2) < \infty,$$

which concludes the proof. □

As the final comment we observe that by looking at the growth of the dimensions of the matrix algebras appearing in the original construction of the Jiang-Su algebra (cfr. [41]), it is clear that (6.4) can not be satisfied and the spectral triples exhibited above are not p-summable. Also, with the help of Stirling formula it can be seen that $\mathrm{Tr}\exp(-D^2)$ diverges and thus the θ-summability does not hold either.

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