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Abstract. We consider the extension problem for Lie algebroids over schemes over a field. Given a locally free Lie algebroid $\mathscr{Q}$ over a scheme X, and a sheaf of finitely generated Lie $\mathscr{O}_X$-algebras $\mathscr{L}$, we first consider the case when $\mathscr{L}$ is abelian, and classify the equivalence classes of extensions $0 \to \mathscr{L} \to \mathscr{E} \to \mathscr{Q} \to 0$ in terms of a suitable Lie algebroid hypercohomology group. In the nonabelian case, we first construct the obstruction to the extension problem, and then, assuming that the obstruction vanishes, we reduce the classification theorem to the abelian case. In the preliminary sections we study free Lie algebroids and recall some basic facts about Lie algebroid hypecohomology.

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1. Introduction

Let X be a noetherian separated scheme over a field $\Bbbk$. Let $\mathscr{Q}$ be a locally free Lie algebroid on X, and $\mathscr{L}$ a sheaf of finitely generated Lie $\mathscr{O}_X$-algebras (definitions will be given in Section 2). An exact sequence of Lie algebroids

$$0 \to \mathscr{L} \to \mathscr{E} \to \mathscr{Q} \to 0 \tag{1}$$

is called an extension of $\mathscr{Q}$ by $\mathscr{L}$. Any such extension defines a morphism $\alpha\colon \mathscr{Q} \to \mathscr{O}ut(\mathscr{L})$, where $\mathscr{O}ut(\mathscr{L})$ is the Lie algebroid of outer derivations of $\mathscr{L}$, by letting

$$\alpha(x)(y) = \{x', y\}_{\mathscr{E}}$$

where x' is any counterimage of x in $\mathscr{E}$. It also induces a representation of $\mathscr{Q}$ on the centre $Z(\mathscr{L})$ of $\mathscr{L}$, i.e., a morphism $\alpha\colon \mathscr{Q} \to \mathscr{D}er(Z(\mathscr{L}))$.

Any two extensions $\mathscr{E}_1$, $\mathscr{E}_2$ are considered to be equivalent if there is a morphism $\mathscr{E}_1 \to \mathscr{E}_2$ such that the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathscr{L} & \longrightarrow & \mathscr{E}_1 & \longrightarrow & \mathscr{Q} & \longrightarrow & 0 \\ & & \| & & \downarrow & & \| & & \\ 0 & \longrightarrow & \mathscr{L} & \longrightarrow & \mathscr{E}_2 & \longrightarrow & \mathscr{Q} & \longrightarrow & 0 \end{array} \tag{2}$$

commutes.

In this paper we study the problem of finding extensions of Lie algebroids as in (1) such that the induced $\mathscr{Q}$-module structure of $Z(\mathscr{L})$ coincides with that given by α. This problem was already studied in [2] by realizing the hypercohomology groups of a Lie algebroid in terms of Čech complexes. Here, following [1], we adopt an intrinsic approach.

If $\mathscr{L}$ is abelian, the problem is unobstructed, as α defines an action of $\mathscr{Q}$ on $\mathscr{L}$, and one can define the semidirect product Lie algebroid

$$\mathscr{E} = \mathscr{L} \rtimes_\alpha \mathscr{Q},$$

where $\mathscr{E} = \mathscr{L} \oplus \mathscr{Q}$ as $\mathscr{A}$-modules, with bracket

$$\{(\ell, x),\, (\ell', x')\} = (\alpha(x)(\ell') - \alpha(x')(\ell), \{x, x'\})$$

and anchor $a\colon \mathscr{E} \to \mathscr{D}er_{\Bbbk}(\mathscr{A})$ given by the anchor b of $\mathscr{Q}$, i.e., $a((\ell, x)) = b(x)$. On the other hand, if $\mathscr{L}$ is not abelian, α does not define an action of $\mathscr{Q}$ on $\mathscr{L}$, and the problem of finding an extension of $\mathscr{Q}$ by $\mathscr{L}$ is obstructed by a class $\mathbf{ob}(\alpha)$ in the group

$$\mathbb{H}^3(\mathscr{Q}; Z(\mathscr{L}))^{(1)} = \mathbb{H}^3(X, \tau^{\geq 1}\mathscr{L} \otimes \Lambda^\bullet \mathscr{Q}^*),$$

i.e., the third hypercohomology group of a truncation of the Chevalley-Eilenberg-de Rham complex of $\mathscr{Q}$ with coefficients in $\mathscr{L}$.

When the obstruction is zero (which, as we have seen, is always the case when $\mathscr{L}$ is abelian), the equivalence classes of extensions of $\mathscr{Q}$ by $\mathscr{L}$, inducing on $Z(\mathscr{L})$ the $\mathscr{Q}$-action given by α, are classified by the group

$$\mathbb{H}^2(\mathscr{Q}; Z(\mathscr{L}))^{(1)} = \mathbb{H}^2(X, \tau^{\geq 1}\mathscr{L} \otimes \Lambda^\bullet \mathscr{Q}^*).$$

So we have the following theorem.

Theorem 1.1. *Given a locally free Lie algebroid $\mathscr{Q}$, a sheaf $\mathscr{L}$ of finitely generated Lie $\mathscr{O}_X$-algebras, and a morphism $\alpha\colon \mathscr{Q} \to \mathscr{O}ut(\mathscr{L})$, the problem of finding an extension of $\mathscr{Q}$ by $\mathscr{L}$ inducing on $Z(\mathscr{L})$ the $\mathscr{Q}$-action given by α is obstructed by a class $\mathbf{ob}(\alpha) \in \mathbb{H}^3(\mathscr{Q}; Z(\mathscr{L}))^{(1)}$. If $\mathbf{ob}(\alpha) = 0$, the space of equivalence classes of extensions is a torsor over the group $\mathbb{H}^2(\mathscr{Q}; Z(\mathscr{L}))^{(1)}$.*

Remark 1.2. In the abelian case the space of equivalence classes of extensions is naturally identified with $\mathbb{H}^2(\mathscr{Q}; Z(\mathscr{L}))^{(1)}$, with the zero element of the latter space being identified with the semidirect product extension. △

The contents of this paper are as follows. In Section 2 we review the fundamentals about Lie algebroid cohomology, stressing a few facts that will be needed later on in the paper. Since some arguments will involve the use of free Lie algebroids, in Section 3 we develop their basic theory. In Section 4 we prove the classification Theorem in the abelian case.

In Section 5 we first construct the obstruction to the extension problem for Lie algebroids, and then, assuming that the obstruction vanishes, we reduce the classification theorem to the abelian case.

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2. Lie algebroids and their hypercohomology

In this section, basically following [15] and [1], we recall some basic facts about the cohomology of Lie-Rinehart algebras, and the hypercohomology of Lie algebroids over schemes.

2.1. **Lie-Rinehart algebras.**

As Lie algebroids are in a sense Lie-Rinehart algebras with coefficients, we start with some issues about Lie-Rinehart algebras.

Let A be a finitely generated commutative, associative unital algebra over a field $\Bbbk$. A $(\Bbbk, A)$-Lie-Rinehart algebra is a pair (L, a), where L is an A-module equipped with a $\Bbbk$-linear Lie algebra bracket $\{\ ,\ \}$, and $a\colon L \to \operatorname{Der}_{\Bbbk}(A)$ a representation of L in $\operatorname{Der}_{\Bbbk}(A)$ (the anchor) that satisfies the Leibniz rule

$$\{s, ft\} = f\{s,t\} + a(s)(f)\, t$$

where $s, t \in L$ and $f \in A$.

We consider a useful class of Lie-Rinehart algebras. Let $\Bbbk$ be a field, and A a commutative associative algebra over $\Bbbk$. Let V be a $\Bbbk$-vector space, and define

$$L = A \otimes_{\Bbbk} V.$$

Let $G^\bullet(L)$ be the graded algebra generated by L over A, with A in degree 0 and L in degree one. We have [5]

Proposition 2.1. *Lie-Rinehart algebra structures on L are in a one-to-one correspondence with degree -1, graded-symmetric $\Bbbk$-Lie brackets on*

$$G^0(L) \oplus G^1(L) = A \oplus L$$

that satisfy the Leibniz rule

$$[\alpha s, s'] = \alpha[s, s'] + [\alpha, s']s$$

for $\alpha \in A$, $s, s' \in L$.

Note that the bracket is required to satisfy a graded Jacobi identity, which implies the usual Jacobi identity for L. Moreover, the anchor of L is given by the map

$$a\colon L \to \operatorname{Der}_{\Bbbk} A, \qquad a(x)(\alpha) = [\alpha, x].$$

A useful class of examples is provided by taking $V = \mathfrak{g}$, where $\mathfrak{g}$ is a Lie algebra over $\Bbbk$ equipped with a $\Bbbk$-Lie algebra homomorphism $a\colon \mathfrak{g} \to \operatorname{Der}_{\Bbbk} A$ (the Lie-Rinehart algebras obtained in this way are called *transformation Lie-Rinehart algebras* [11]). The bracket on $G^0(L) \oplus G^1(L)$ is defined as

$$[\alpha \otimes \xi, \beta] = \alpha\, a(\xi)(\beta)$$

$$[\alpha \otimes \xi, \beta \otimes \eta] = \alpha\beta\,[\xi, \eta] + \alpha\, a(\xi)(\beta)\,\eta - \beta\, a(\eta)(\alpha)\,\xi$$

for $\alpha, \beta \in A$, $\xi, \eta \in \mathfrak{g}$. (Note that the bracket of two elements in $G^0(L)$ is always zero as the bracket is supposed to have degree -1.) The anchor a_L of L is given by

$$a_L(\alpha \otimes \xi)(\beta) = \alpha\, a(\xi)(\beta).$$

2.2. **Lie algebroid cohomology.** We consider now Lie algebroids. All schemes will be noetherian and separated. Let X be a scheme over a field $\Bbbk$ (the same results hold in the holomorphic category). We shall denote by $\mathscr{O}_X$ the sheaf of regular functions on X, by $\Bbbk_X$ the constant sheaf on X with stalk $\Bbbk$, and by Θ_X the tangent sheaf of X (the sheaf of derivations of the structure sheaf $\mathscr{O}_X$), which is a sheaf of $\Bbbk_X$-Lie algebras. A Lie algebroid $\mathscr{C}$ on X is a coherent $\mathscr{O}_X$-module $\mathscr{C}$ equipped with:

- a $\Bbbk$-linear Lie bracket defined on sections of $\mathscr{C}$, satisfying the Jacobi identity;
- a morphism of $\mathscr{O}_X$-modules $a\colon \mathscr{C} \to \Theta_X$, called the *anchor* of $\mathscr{C}$, which is also a morphism of sheaves of $\Bbbk$-Lie algebras.

The Leibniz rule

$$\{s, ft\} = f\{s, t\} + a(s)(f)\,t$$

is required to hold for all sections s, t of $\mathscr{C}$ and f of $\mathscr{O}_X$ (actually the Leibniz rule and the Jacobi identity imply that the anchor is a morphism of $\Bbbk_X$-Lie algebras).

A morphism $(\mathscr{C}, a) \to (\mathscr{C}', a')$ of Lie algebroids defined over the same scheme X is a morphism of $\mathscr{O}_X$-modules $f\colon \mathscr{C} \to \mathscr{C}'$, which is compatible with the brackets defined in $\mathscr{C}$ and in $\mathscr{C}'$, and such that $a' \circ f = a$. Note that this implies that the kernel of a morphism of Lie algebroids has a trivial anchor, i.e., it is a sheaf of $\mathscr{O}_X$-Lie algebras.

Definition 2.2. *A representation of a Lie algebroid $\mathscr{C}$ is a pair $(\mathscr{M}, \rho)$, where $\mathscr{M}$ is a coherent $\mathscr{O}_X$-module, and ρ is an $\mathscr{O}_X$-linear morphism $\mathscr{C} \to End_{\Bbbk}(\mathscr{M})$ satisfying the conditions*

- $\rho(fs) = f\rho(s)$;
- $\rho(x)(fm) = f\rho(x)(m) + a(x)(f)\, m$

for all sections f, x and m of $\mathscr{O}_X$, $\mathscr{C}$ and $\mathscr{M}$, respectively.

A representation $\mathscr{M}$ of $\mathscr{C}$ will also be called a $\mathscr{C}$-module. We shall denote by $\mathrm{Rep}(\mathscr{C})$ the category of representations of a Lie algebroid $\mathscr{C}$. Given a representation $(\mathscr{M}, \rho)$, we define the *invariant submodule* $\mathscr{M}^{\mathscr{C}}$ of $\mathscr{M}$ as the sheaf of $\Bbbk_X$-modules

$$\mathscr{M}^{\mathscr{C}}(U) = \{m \in \mathscr{M}(U) \mid \rho(\mathscr{C})(m) = 0\}.$$

This is an $\mathscr{O}_X$-module when the anchor of $\mathscr{C}$ is trivial. In general, this defines a functor

$$(-)^{\mathscr{C}}\colon\ \mathrm{Rep}(\mathscr{C}) \to \Bbbk_X\text{-}\mathbf{mod}.$$

Assuming that $\mathscr{C}$ is locally free, we introduce the *Chevalley-Eilenberg-de Rham complex* of $\mathscr{C}$ with coefficients in a representation $(\mathscr{M}, \rho)$, which is a sheaf of differential graded algebras. This is $\mathscr{M} \otimes_{\mathscr{O}_X} \Lambda^\bullet_{\mathscr{O}_X}\mathscr{C}^*$ as a sheaf of $\mathscr{O}_X$-modules, with a product given by the wedge product, and a $\Bbbk$-linear differential $d_{\mathscr{C}}\colon \mathscr{M} \otimes_{\mathscr{O}_X} \Lambda^\bullet_{\mathscr{O}_X}\mathscr{C}^* \to \mathscr{M} \otimes_{\mathscr{O}_X} \Lambda^{\bullet+1}_{\mathscr{O}_X}\mathscr{C}^*$ defined by the formula

$$\begin{aligned}(d_{\mathscr{C}}\xi)(s_1, \ldots, s_{p+1}) \;&=\; \sum_{i=1}^{p+1}(-1)^{i-1}\rho(s_i)(\xi(s_1, \ldots, \hat{s}_i, \ldots, s_{p+1})) \\ &+\; \sum_{i<j}(-1)^{i+j}\xi([s_i, s_j], \ldots, \hat{s}_i, \ldots, \hat{s}_j, \ldots, s_{p+1})\end{aligned}$$

for $s_1, \ldots, s_{p+1}$ sections of $\mathscr{C}$, and ξ a section of $\mathscr{M} \otimes_{\mathscr{O}_X} \Lambda^p_{\mathscr{O}_X}\mathscr{C}^*$.

The hypercohomology of the complex $(\mathscr{M}\otimes_{\mathscr{O}_X}\Lambda^\bullet_{\mathscr{O}_X}\mathscr{C}^*, d_{\mathscr{C}})$, denoted $\mathbb{H}^\bullet(\mathscr{C};\mathscr{M})$, is called the *Lie algebroid cohomology* of $\mathscr{C}$ with coefficients in $(\mathscr{M},\rho)$. If X is affine, the hypercohomology $\mathbb{H}^\bullet(\mathscr{C};\mathscr{M})$ reduces the cohomology of the $(\Bbbk,\mathscr{O}_X(X))$-Lie-Rinehart algebra $\mathscr{C}(X)$ with coefficients in $\mathscr{M}(X)$ [15].

2.3. **Cohomology of transformation Lie algebroids.** If $\mathscr{L}$ is a locally free sheaf of $\Bbbk_X$-Lie algebras, and $b\colon \mathscr{L}\to \mathscr{D}er_{\Bbbk}(\mathscr{O}_X)$ is a morphism of Lie $\Bbbk_X$-algebras, one can, in analogy with the case of Lie-Rinehart algebras, define the *transformation Lie algebroid* $\mathscr{C}=\mathscr{O}_X\otimes_{\Bbbk}\mathscr{L}$, with anchor $a(f\otimes\xi)=f\otimes b(\xi)$. Let $\mathscr{M}$ be a representation of $\mathscr{C}$ which is locally free as an $\mathscr{O}_X$-module; then $\mathscr{M}$ is also a representation of $\mathscr{L}$, and each fibre $\mathscr{M}_x$ is a representation of the Lie algebra $\mathscr{L}_x$ (the fibre of $\mathscr{L}$ at $x\in X$. Then one immediately has an isomorphism of $\Bbbk$-vector spaces

$$\mathbb{H}^\bullet(\mathscr{C};\mathscr{M})\simeq\mathbb{H}^\bullet(\mathscr{L};\mathscr{M}).$$

Moreover there is a spectral sequence converging to these groups whose second page is

$$E_2^{p,q}=H^p(X,\mathscr{H}^q(\mathscr{L};\mathscr{M})).$$

Here $\mathscr{H}^q(\mathscr{L};\mathscr{M})$ is a vector bundle whose fibre at $x\in X$ is the Chevalley-Eilenberg cohomology $H^q(\mathscr{L}_x;\mathscr{M}_x)$ of the Lie algebra $\mathscr{L}_x$ with coefficients in the vector space $\mathscr{M}_x$.

2.4. **Lie algebroid cohomology as a derived functor.** Given a locally free algebroid $\mathscr{C}$, we consider the functor

$$I^{\mathscr{C}}\colon \operatorname{Rep}(\mathscr{C})\to \Bbbk\textbf{-mod},\qquad \mathscr{M}\mapsto\Gamma(X,\mathscr{M}^{\mathscr{C}}).$$

This is left-exact, and since $\operatorname{Rep}(\mathscr{C})$ has enough injectives [1], we can take its derived functors. It was shown in [1] that these derived functors are isomorphic to the hypecohomology functors, that is, for every representation $\mathscr{M}$ of $\mathscr{C}$ there are functorial isomorphisms

$$R^iI^{\mathscr{C}}(\mathscr{M})\simeq\mathbb{H}^i(\mathscr{C};\mathscr{M}),\quad i\geq 0.$$

In the same way, the derived functors of the functor $(-)^{\mathscr{C}}$ applied to a representation $\mathscr{M}$ give the cohomology sheaves of the Chevalley-Eilenberg-de Rham complex with coefficients in $\mathscr{M}$:

$$R^i\mathscr{M}^{\mathscr{C}}\simeq\mathscr{H}^i(\mathscr{C};\mathscr{M}),\quad i\geq 0.$$

2.5. **A local-to-global spectral sequence.** One has $I^{\mathscr{C}} = \Gamma \circ (-)^{\mathscr{C}}$. Moreover, when $\mathscr{I}$ is an injective object in $\operatorname{Rep}(\mathscr{C})$, one has $\mathscr{H}^i(\mathscr{C}; \mathscr{I}) = 0$ for $i > 0$ [1]. As a result, there is a spectral sequence, converging to $\mathbb{H}^\bullet(\mathscr{C}; \mathscr{M})$, whose second term is

$$E_2^{pq} = H^p(X, \mathscr{H}^q(\mathscr{C}; \mathscr{M})).$$

2.6. **A Hochshild-Serre spectral sequence.** Let us consider an extension of Lie algebroids as in (1). As we already noticed, $\mathscr{L}$ is a sheaf of $\mathscr{O}_X$-Lie algebras, i.e., it has a vanishing anchor. Thus, if $\mathscr{M}$ is a representation of $\mathscr{E}$, the $\mathscr{L}$-invariant submodule $\mathscr{M}^{\mathscr{L}}$ is an $\mathscr{O}_X$-module, and moreover, it is a representation of $\mathscr{Q}$. One has a commutative diagram of functors

$$\begin{array}{ccc} \operatorname{Rep}(\mathscr{E}) & \xrightarrow{(-)^{\mathscr{L}}} & \operatorname{Rep}(\mathscr{Q}) \\ & {\scriptstyle I^{\mathscr{E}}} \searrow & \downarrow {\scriptstyle I^{\mathscr{Q}}} \\ & & \mathbb{k}\textbf{-mod} \end{array}$$

The functors $(-)^{\mathscr{L}}$ and $I^{\mathscr{Q}}$ are left-exact, and moreover, $(-)^{\mathscr{L}}$ maps injective objects of $\operatorname{Rep}(\mathscr{E})$ to $I^{\mathscr{Q}}$-acyclic objects of $\operatorname{Rep}(\mathscr{Q})$ ([9, Prop. 2.4.6 (vii)],[1]), so that there is a Grothendieck spectral sequence converging to $R^\bullet I^{\mathscr{E}}(\mathscr{M})$, whose second page is $E_2^{pq} = R^p I^{\mathscr{Q}}(R^q \mathscr{M}^{\mathscr{L}})$ [6]. This generalizes the Hochschild-Serre spectral sequence one has for extensions of Lie algebras [7, 2]. Changing notation, we have [1, 2]:

Theorem 2.3. *For every representation $\mathscr{M}$ of $\mathscr{E}$ there is a spectral sequence converging to $\mathbb{H}^\bullet(\mathscr{E}; \mathscr{M})$, whose second page is*

$$E_2^{pq} = \mathbb{H}^p(\mathscr{Q}; \mathscr{H}^q(\mathscr{L}; \mathscr{M})).$$

It may be useful to record the explicit form of the five-term sequence of this spectral sequence:

$$0 \to \mathbb{H}^1(\mathscr{Q}; \mathscr{M}^{\mathscr{L}}) \to \mathbb{H}^1(\mathscr{E}; \mathscr{M}) \to \\ \mathbb{H}^0(\mathscr{Q}; \mathscr{H}^1(\mathscr{L}; \mathscr{M})) \to \mathbb{H}^2(\mathscr{Q}; \mathscr{M}^{\mathscr{L}}) \to \mathbb{H}^2(\mathscr{E}; \mathscr{M}) \,.$$

2.7. **The universal enveloping algebroid.** The universal enveloping algebra $\mathfrak{U}(L)$ of a $(\Bbbk, A)$-Lie-Rinehart algebra L was defined in [15]. It can be described as the quotient of the tensor algebra $T(L)$ of L over A by the ideal $J(L)$ generated by the elements

$$s \otimes t - t \otimes s - [s,t] \quad \text{with} \quad s,t \in L$$

$$f \otimes s - s \otimes f + a(s)(f) \quad \text{with} \quad s \in L,\ f \in A$$

(here $a\colon L \to \mathrm{Der}_\Bbbk(A)$ is the anchor morphism). $\mathfrak{U}(L)$ is an A-module; there are canonical monomorphisms $A \to \mathfrak{U}(L)$ and $\iota\colon L \to \mathfrak{U}(L)$, and a morphism $\mathfrak{U}(L) \to \mathfrak{U}(L)/I = A$ (the augmentation morphism) where I is the ideal generated by $\iota(L)$.

The construction of the universal enveloping algebra $\mathfrak{U}(L)$ of a $(\Bbbk, A)$-Lie-Rinehart algebra is functorial, and therefore one can define the universal enveloping algebra $\mathfrak{U}(\mathscr{C})$ of a Lie algebroid $\mathscr{C}$ by applying the previous definition to every $(\Bbbk, \mathscr{O}_X(U))$-Lie-Rinehart algebra $\mathscr{C}(U)$, where U runs over the open sets in X [1].

2.8. **Lie algebroid hypercohomology and derivations.** We assume that the Lie algebroid $\mathscr{C}$ is locally free. The category $\mathrm{Rep}(\mathscr{C})$ and the category of $\mathfrak{U}(\mathscr{C})$-modules are equivalent, and the functors $\mathbb{H}^i(\mathscr{C};-)$ and $\mathrm{Ext}^i_{\mathfrak{U}(\mathscr{C})}(\mathscr{O}_X,-)$ are isomorphic as functors $\mathrm{Rep}(\mathscr{C}) \to \Bbbk$**-mod** [1] . Let $\mathscr{I}$ be the kernel of the augmentation morphism $\mathfrak{U}(\mathscr{C}) \to \mathscr{O}_X$. By applying the functor $\mathrm{Hom}_{\mathfrak{U}(\mathscr{C})}(-,\mathscr{M})$ to the exact sequence of $\mathfrak{U}(\mathscr{C})$-modules

$$0 \to \mathscr{I} \to \mathfrak{U}(\mathscr{C}) \to \mathscr{O}_X \to 0,$$

we obtain the exact sequence

$$0 \to I^{\mathscr{C}}(\mathscr{M}) \to \Gamma(X,\mathscr{M}) \to \mathrm{Hom}_{\mathfrak{U}(\mathscr{C})}(\mathscr{I},\mathscr{M}) \to \mathbb{H}^1(\mathscr{C};\mathscr{M}) \to H^1(X,\mathscr{M})$$

So every element in $\mathbb{H}^1(\mathscr{C};\mathscr{M})$ which goes to zero in $H^1(X,\mathscr{M})$ (for instance, this will always happen if X is affine) is represented by a morphism of $\mathfrak{U}(\mathscr{C})$-modules $\phi\colon \mathscr{I} \to \mathscr{M}$. This in turn induces a morphism $D_\phi\colon \mathscr{C} \to \mathscr{M}$ by letting

$$D_\phi(x) = \phi(i(x))$$

where i is the natural inclusion $\mathscr{C} \to \mathscr{I}$. This is a derivation of $\mathscr{C}$ with values in $\mathscr{M}$, as one has

$$D_\phi(\{x,y\}) = \phi(i(\{x,y\})) = \phi(i(x)i(y) - i(y)i(x)) = x(D_\phi(y)) - y(D_\phi(x)). \tag{3}$$

The morphism $\phi \mapsto D_\phi$ establishes indeed an isomorphism $\mathrm{Hom}_{\mathfrak{U}(\mathscr{C})}(\mathscr{I}, \mathscr{M}) \simeq \mathscr{D}er(\mathscr{C}, \mathscr{M})$.

2.9. **The truncated complex.** Consider in general a Lie algebroid $\mathscr{C}$ and a $\mathscr{C}$-module $\mathscr{M}$. The truncated Chevalley-Eilenberg complex $\tau^{\geq 1}\Lambda^\bullet\mathscr{C}^* \otimes \mathscr{M}$ is a quotient of the complex $\Lambda^\bullet\mathscr{C}^* \otimes \mathscr{M}$, and one has an exact sequence of complexes

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \mathscr{M} & \xrightarrow{d^0} & \mathrm{Im}\, d^0 & \longrightarrow & 0 & & \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathscr{M} & \xrightarrow{d^0} & \mathscr{C}^* \otimes \mathscr{M} & \longrightarrow & \Lambda^2\mathscr{C}^* \otimes \mathscr{M} & \longrightarrow & \dots \\
 & & \downarrow & & \downarrow & & \| & & \\
 & & 0 & \longrightarrow & \mathrm{coker}\, d^0 & \longrightarrow & \Lambda^2\mathscr{C}^* \otimes \mathscr{M} & \longrightarrow & \dots \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}
$$

We denote by $\mathbb{H}(\mathscr{C}; \mathscr{M})^{(1)}$ the hypecohomology of the truncated complex. Since the complex $0 \to \mathscr{M} \to \mathrm{Im}\, d^0 \to 0$ is quasi isomorphic to the complex $0 \to \ker d^0 \simeq \mathscr{M}^{\mathscr{C}} \to 0$, we have a long exact sequence

$$
\dots \to H^i(X, \mathscr{M}^{\mathscr{C}}) \to \mathbb{H}^i(\mathscr{C}; \mathscr{M}) \to \mathbb{H}^i(\mathscr{C}; \mathscr{M})^{(1)} \to H^{i+1}(X, \mathscr{M}^{\mathscr{C}}) \to \dots \qquad (4)
$$

Remark 2.4. If we analogously denote by $\mathscr{H}^\bullet(\mathscr{C}; \mathscr{M})^{(1)}$ the cohomology sheaves of the truncated complex, we have of course

$$
\mathscr{H}^0(\mathscr{C}; \mathscr{M})^{(1)} = 0, \qquad \mathscr{H}^i(\mathscr{C}; \mathscr{M})^{(1)} = \mathscr{H}^i(\mathscr{C}; \mathscr{M}) \quad \text{for} \quad i > 0.
$$

△

Remark 2.5. The hypercohomology $\mathbb{H}^\bullet(\mathscr{C}; \mathscr{M})^{(1)}$ can be computed as the cohomology of the total complex of the double complex $H^{p,q} = \check{C}^p(\mathfrak{U}, \tau^{\geq 1}\Lambda^q\mathscr{C}^* \otimes \mathscr{M})$, where $\mathfrak{U}$ is any affine cover of X, and $\check{C}^\bullet$ denotes the associated Čech complex. Since $H^{0,0} = 0$, we have $\mathbb{H}^0(\mathscr{C}; \mathscr{M})^{(1)} = 0$. Note that this implies $\mathbb{H}^0(\mathscr{C}; \mathscr{M}) \simeq H^0(X, \mathscr{M}^{\mathscr{C}})$, which is indeed correct as the functors $\mathbb{H}^i(\mathscr{C}; -)$ are the derived functors of the functor $H^0(X, (-)^{\mathscr{C}}) \colon \mathrm{Rep}(\mathscr{C}) \to$ $\mathbb{k}$**-mod**. △

2.10. **A spectral sequence for truncated Chevalley-Eilenberg complexes.** We check how the spectral sequence described in Section 2.6 is modified if we consider truncated complexes. Let $(\mathscr{M},\rho)$ be a n $\mathscr{C}$-module. The Lie algebra bundle $\mathscr{L}$ acts on the truncated complex $Tr^\bullet_{\mathscr{C}} = \tau^{\geq 1}\Lambda^\bullet\mathscr{C}^*\otimes\mathscr{M}^{\mathscr{L}}$ by inner multiplication; note in particular that it acts on $\operatorname{coker} d^0$. Indeed if ξ is section of $\operatorname{coker} d^0$, one represents it with a section τ of $\mathscr{C}^*\otimes\mathscr{M}^{\mathscr{L}}$, and if s is a section of $\mathscr{L}$, we let $s\rfloor\xi = s\rfloor\tau$. This is well defined, because if $\tau = d^0 m$, then

$$s\rfloor d^0 m = \rho(s)(m) = 0.$$

The need that this equation holds is the reason why we take coefficients in $\mathscr{M}^{\mathscr{L}}$ rather than $\mathscr{M}$. Also, note that $\mathscr{M}^{\mathscr{L}}$ is both a $\mathscr{C}$- and $\mathscr{Q}$-module.

We introduce a filtration $\mathscr{F}^\bullet_\bullet$ of $Tr^\bullet_{\mathscr{C}}$ by defining $\mathscr{F}^q_p$ as the subsheaf of $Tr^q_{\mathscr{C}}$ whose sections are annihilated by the inner product with $q-p+1$ sections of $\mathscr{L}$. As in Lemma 4.2 of [2], we have that the corresponding graded module is

$$\operatorname{gr}_p Tr^{p+q}_{\mathscr{C}} \simeq Tr^p_{\mathscr{Q}}\otimes\Lambda^q\mathscr{L}^*\otimes\mathscr{M}^{\mathscr{L}}$$

where $Tr^\bullet_{\mathscr{Q}}$ is defined similarly as $Tr^\bullet_{\mathscr{C}}$. As in [2] and [1], one gets a spectral sequence converging to $\mathbb{H}(\mathscr{C};\mathscr{M}^{\mathscr{L}})^{(1)}$ whose second term is

$$E_2^{pq} \simeq \mathbb{H}^p(\mathscr{Q};\mathscr{H}^q(\mathscr{L};\mathscr{M}^{\mathscr{L}}))^{(1)}.$$

In view of Remark 2.5, one has $E_2^{0,q} = 0$, so that the corresponding five term sequence splits to give

$$\mathbb{H}^1(\mathscr{Q};\mathscr{M}^{\mathscr{L}})^{(1)} \simeq \mathbb{H}^1(\mathscr{C};\mathscr{M}^{\mathscr{L}})^{(1)}, \qquad 0\to\mathbb{H}^2(\mathscr{Q};\mathscr{M}^{\mathscr{L}})^{(1)}\to\mathbb{H}^2(\mathscr{C};\mathscr{M}^{\mathscr{L}})^{(1)}\,.$$

3. Free Lie algebroids

3.1. **Free Lie and associative algebras over a set.** We start by recalling the construction of the free Lie algebra over a set and it universal enveloping algebra ([14], see also [16]). Let $\Bbbk$ be a field and A a commutative, associative $\Bbbk$-algebra with unit. We briefly remind the construction of a free Lie algebra over a set, and its universal enveloping algebra. If S

is a set, and M_S the associated magma, the vector space $A_S = \Bbbk[M_S]$ freely generated by M_S over $\Bbbk$ is an algebra over $\Bbbk$, with product given by the product in the magma.

Let I_S be the two-sided ideal ideal generated in A_S by the elements

$$xx, \quad x \in A_S \quad \text{and} \quad (xy)z + (zx)y + (yz)x, \quad x, y, z \in A_S.$$

The quotient A_S/I_S is a $\Bbbk$-Lie algebra — the free $\Bbbk$-Lie algebra over S — that we denote $\mathrm{Lie}_{\Bbbk,S}$.

Similarly, let V_S be the vector space freely generated by S over $\Bbbk$, and let $\mathrm{Ass}_{\Bbbk,S}$ be its tensor algebra — the free associative $\Bbbk$-algebra over the set S. $\mathrm{Ass}_{\Bbbk,S}$ turns out to be isomorphic to the universal enveloping algebra of $\mathrm{Lie}_{\Bbbk,S}$ [14, 16].

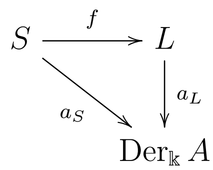
Any Lie algebra $\mathfrak{g}$ over $\Bbbk$ can be realized as the quotient of a free Lie algebra. If $S = \{x_i\}$ is a set of generators of $\mathfrak{g}$, one has indeed a surjection $\mathrm{Lie}_{\Bbbk,S} \to \mathfrak{g}$.

3.2. **The free Lie-Rinehart algebra over a set.** Let $\Bbbk$ be a field, and A a commutative associative algebra over $\Bbbk$. Let S be a set equipped with a map $a_S \colon S \to \mathrm{Der}_\Bbbk A$ such the induced map $\mathrm{Lie}_{\Bbbk,S} \to \mathrm{Der}_\Bbbk A$ is a morphism of $\Bbbk$-Lie algebras (that we denote by the same symbol).[1] With this data, we can make

$$L_{A,S} = A \otimes_\Bbbk \mathrm{Lie}_{\Bbbk,S}$$

into a $(\Bbbk, A)$-Lie-Rinehart algebra. We call this the *free* $(\Bbbk, A)$*-Lie-Rinehart algebra* over the pair (S, a_S). It has a natural map $S \to L_{A,S}$.

Proposition 3.1. *Let* (L, a_L) *be a* $(\Bbbk, A)$*-Lie-Rinehart algebra, and let* $f \colon S \to L$ *be a map such that the diagram*



[1]We could equivalently require the existence of the map $\mathrm{Lie}_{\Bbbk,S} \to \mathrm{Der}_\Bbbk A$, as the composition with the canonical map $S \to \mathrm{Lie}_{\Bbbk,S}$ yields the associated map $S \to \mathrm{Der}_\Bbbk A$.

commutes. There is a unique Lie-Rinehart algebra morphism $g\colon L_{A,S} \to L$ *such that the diagram*

$$\begin{array}{ccc} S & & \\ \downarrow & \searrow^{f} & \\ L_{A,S} & \xrightarrow[g]{} & L \end{array}$$

commutes.

Proof. By regarding L as a $\Bbbk$-Lie algebra, there is a map $\tilde{g}\colon \mathrm{Lie}_{\Bbbk,S} \to L$ making the diagram

$$\begin{array}{ccc} S & & \\ {\scriptstyle i_S}\downarrow & \searrow^{f} & \\ \mathrm{Lie}_{\Bbbk,S} & \xrightarrow[\tilde{g}]{} & L \end{array}$$

commutative. This defines the map g as $g(\alpha \otimes \xi) = \alpha\, \tilde{g}(\xi)$. The only thing we need to check is the compatibility between the anchors. If we set $\xi = i_S(s)$, we have indeed

$$a_L(g(\alpha \otimes \xi)) = \alpha\, a_L(\tilde{g}(i_S(s))) = \alpha\, a_L(f(s)) = \alpha\, a_S(s) = a_{L_{A,S}}(\alpha \otimes \xi).$$

□

The universal enveloping algebra of $L_{A,S}$ can be constructed as follows. Let $\mathrm{Ass}_{A,S}$ be the free associative algebra of S over A, and let $\iota_S\colon S \to \mathrm{Ass}_{A,S}$ be the natural map. Let J be the two-sided ideal in $\mathrm{Ass}_{A,S}$ generated by elements

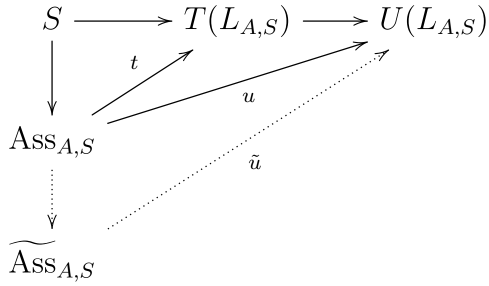
$$\alpha \otimes \iota_S(s) - \iota_S(s) \otimes \alpha + a_S(s)(\alpha) \tag{5}$$

for $\alpha \in A$ and $s \in S$, and define

$$\widetilde{\mathrm{Ass}}_{A,S} = \mathrm{Ass}_{A,S} / J.$$

Proposition 3.2. $\widetilde{\mathrm{Ass}}_{A,S}$ *is isomorphic to the universal enveloping algebra of the free* $(\Bbbk, A)$*-Lie-Rinehart algebra* $L_{A,S}$ *over* S*.*

Proof. Let us consider the diagram



where $T(L_{A,S})$ is the tensor algebra of $L_{A,S}$ over A; the universal enveloping algebra $U(L_{A,S})$ is the quotient of $T(L_{A,S})$ by the ideal generated by the elements

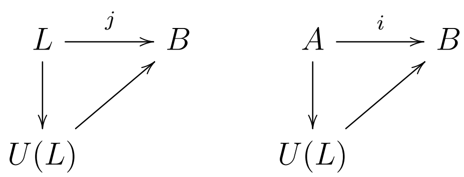
$$\alpha \otimes x - x \otimes \alpha + a_S(x)(\alpha)$$

for $\alpha \in A$ and $x \in L_{A,S}$. The arrows t and u exist because of the universal properties of the algebra $\mathrm{Ass}_{A,S}$, and the diagram formed by solid arrows is commutative, so that u is zero on the ideal $J \subset \mathrm{Ass}_{A,S}$ (see eq. (5)), and the arrow $\tilde{u}$ is well defined.

To define the inverse map, we recall the universal property of the universal enveloping algebra $U(L)$ of a $(\Bbbk, A)$-Lie-Rinehart (L, a) [12]. If B is a unital associative $\Bbbk$-algebra, we denote by B_{Lie} the algebra B regarded as a Lie algebra with the commutator bracket. The universal enveloping algebra $U(L)$ solves the following problem: for every algebra homomorphism $i \colon A \to B$ and every morphism of $\Bbbk$-Lie algebras $j \colon L \to B_{\mathrm{Lie}}$ such that for all $\alpha \in A$ and $x \in L$

$$i(\alpha)\, j(x) = j(\alpha x) \quad \text{and} \quad [j(x), i(\alpha)] = i(a(x)(\alpha))\,,$$

there is a unique morphism of A-modules $U(L) \to B$ making the diagrams



commutative (here B is an A-module via the map i).

In the case at hand, by Proposition 3.1 we have a map $L_{A,S} \to \widetilde{\mathrm{Ass}}_{A,S}$, and by the universality of the universal enveloping algebra we have recalled above, there is a unique compatible map $v\colon U(L_{A,S}) \to \widetilde{\mathrm{Ass}}_{A,S}$. It is easy to check that both $v \circ \tilde{u}$ and $\tilde{u} \circ v$ are the identity map. □

Let $K_{A,S}$ be the ideal generated in $U(L_{A,S})$ by the natural image of $L_{A,S}$. Then $U(L_{A,S})/K_{A,S} \simeq A$.

Corollary 3.3. *$K_{A,S} \simeq V_S \otimes_{\Bbbk} U(L_{A,S})$, so that $K_{A,S}$ is a free $U(L_{A,S})$-module. As a consequence the cohomology groups $H^i(L_{A,S}; M)$ vanish for $i \geq 2$ for any representation M of $L_{A,S}$.*

Proof. $K_{A,S}$ is the image into $U(L_{A,S})$ of the kernel of the map $\mathrm{Ass}_{A,S} \to A$, which is free over $\mathrm{Ass}_{A,S}$. An easy computation proves the first claim. Then, by applying the functor $\mathrm{Hom}_{U(L_{A,S})}(-, M)$ to the resolution

$$0 \to K_{A,S} \to U(L_{A,S}) \to A \to 0 \tag{6}$$

of A by free $U(L_{A,S})$-modules, one gets the second claim; one uses the isomorphism $H^i(L_{A,S}, -) \simeq \mathrm{Ext}^i_{U(L_{A,S})}(A, -)$ as functors $\mathrm{Rep}(L_{A,S}) \to \mathbf{Vect}_{\Bbbk}$ [15, 1]. Some more details are given in the proof of Proposition 3.7. □

Theorem 3.4. *Every $(\Bbbk, A)$-Lie-Rinehart algebra L is a quotient of the universal free $(\Bbbk, A)$-Lie-Rinehart algebra $L_{A,S}$ over some set S.*

Proof. If we look at L as a Lie algebra over $\Bbbk$, there is a surjective Lie algebra morphism $f\colon \mathrm{Lie}_{\Bbbk,S} \to L$ for some set S. We define $a_S\colon \mathrm{Lie}_{\Bbbk,S} \to \mathrm{Der}_{\Bbbk} A$ as $a_S = a \circ f$, and use this to define a free Lie algebroid $L_{A,S} = A \otimes \mathrm{Lie}_{\Bbbk,S}$, with a naturally defined map $\tilde{f}\colon L_{A,S} \to L$. One easily checks that this an A-linear morphism which is also a morphism of $\Bbbk$-Lie algebras, and satifies $a \circ \tilde{f} = a_S\colon L_{A,S} \to \mathrm{Der}_{\Bbbk} A$. □

Remark 3.5. (A functorial construction) The previous construction of free Lie-Rinheart algebras has an equivalent description in terms of adjoint functors [3, 8]. Let $\mathbf{Vect}_{\Bbbk}$ and $\mathbf{Lie}_{\Bbbk}$ be the categories of $\Bbbk$-vector spaces and $\Bbbk$-Lie algebras, respectively. Then the free Lie algebra functor $\mathbf{Flie}_{\Bbbk}\colon \mathbf{Vect}_{\Bbbk} \to \mathbf{Lie}_{\Bbbk}$ is the left adjoint to the obvious forgetful functor $\mathbf{Lie}_{\Bbbk} \to \mathbf{Vect}_{\Bbbk}$. If we consider the categories $\mathbf{Vect}_{\Bbbk}/\operatorname{Der}_{\Bbbk}(A)$ and $\mathbf{Lie}_{\Bbbk}/\operatorname{Der}_{\Bbbk}(A)$ of pairs (V, b), (L, a) respectively, with $b\colon V \to \mathrm{Der}_{\Bbbk}(A)$ a linear morphism, and $a\colon L \to \mathrm{Der}_{\Bbbk}(A)$ a morphism of Lie algebras, we obtain a functor

$$\mathbf{Vect}_{\Bbbk}/\operatorname{Der}_{\Bbbk}(A) \to \mathbf{Lie}_{\Bbbk}/\operatorname{Der}_{\Bbbk}(A)$$

which is again left adjoint to the obvious forgetful functor. Composing this with the tensor product functor $A \otimes -$ we get the free Lie-Rinehart functor

$$\mathbf{FreeLR}\colon \mathbf{Vect}_{\Bbbk}/\operatorname{Der}_{\Bbbk}(A) \to \mathbf{LR}_A \tag{7}$$

(where $\mathbf{LR}_A$ is the category of $(\Bbbk, A)$-Lie-Rinehart algebras).

Finally, let $\mathbf{Set}/\operatorname{Der}_{\Bbbk}(A)$ be the category of pairs (S, a_S), where S is a set, and a_S map $a_S\colon S \to \operatorname{Der}_{\Bbbk} A$ such that the induced map $\operatorname{Lie}_{\Bbbk,S} \to \operatorname{Der}_{\Bbbk} A$ is a morphism of $\Bbbk$-Lie algebras. By taking the free vector space over S this gives a functor

$$\mathbf{Set}/\operatorname{Der}_{\Bbbk}(A) \to \mathbf{Vect}_{\Bbbk}/\operatorname{Der}_{\Bbbk}(A).$$

By composing this functor with the functor (7) one obtains the functor $(S, a_S) \mapsto L_{A,S}$ we implicitly defined in Section 3.2. △

3.3. **Free Lie algebroids.** As all constructions in the previous sections are functorial, they can be sheafified. So, if $\mathscr{S}$ is a sheaf of sets on a scheme X, we can at first construct a sheaf $\mathscr{L}ie_{\Bbbk,\mathscr{S}}$ of $\Bbbk_X$-Lie algebras, whose space of sections over an open subset $U \subset X$ is the free Lie algebra over the set $\mathscr{S}(U)$. Let us assume that $a_{\mathscr{S}}\colon \mathscr{S} \to \Theta_X$ is a morphism of sheaves of sets such that the induced morphism $a_{\mathscr{S}}\colon \mathscr{L}ie_{\Bbbk,\mathscr{S}} \to \Theta_X$ is a morphism of sheaves of $\Bbbk$-algebras. For every open subset $U \subset X$ we can construct the free $(\Bbbk, \mathscr{O}_X(U))$-Lie-Rinehart algebra $L_{\mathscr{O}_X(U),\mathscr{S}(U)}$ over the set $\mathscr{S}(U)$ with anchor $a_{\mathscr{S}}(U)$; using the construction in Section 2.1, we can indeed make the $\mathscr{O}_X(U)$-module

$$\mathscr{L}(U) = \mathscr{O}_X(U) \otimes_{\Bbbk} \operatorname{Lie}_{\Bbbk,\mathscr{S}(U)}$$

into a Lie-Rinehart algebra. This defines a sheaf $\mathscr{L}_{\mathscr{S}}$ on X which is a Lie algebroid. We call it the *free -Lie algebroid over* $\mathscr{S}$ (The choice of the scheme X is understood. Moreover, although we do not record the choice of the morphism $a_{\mathscr{S}}\colon \mathscr{S} \to \mathscr{D}er_{\Bbbk}\mathscr{A}$ in the notation, we should remember that $\mathscr{L}_{\mathscr{S}}$ depends on it).

Theorem 3.4 immediately implies

Corollary 3.6. *Every Lie algebroid over X is the quotient of the free Lie algebroid $\mathscr{L}_{\mathscr{S}}$ for some sheaf of sets $\mathscr{S}$ on X and some morphism of sheaves of sets $a_{\mathscr{S}}\colon \mathscr{S} \to \Theta_X$.*

Also the functorial construction of Section 3.5 immediately generalizes to Lie algebroids. Analogously to Corollary 3.3, we obtain:

Proposition 3.7. *For every representation $\mathscr{M}$ of the free Lie algebroid $\mathscr{L}_{\mathscr{S}}$, the hypercohomology groups $\mathbb{H}^i(\mathscr{L}_{\mathscr{S}};\mathscr{M})$ vanish for $i \geq 2$.*

Proof. The proof is the same as for Proposition 3.3, but we give here a few more details. Analogously to equation (6), we have an exact sequence of $\mathfrak{U}(\mathscr{L}_{\mathscr{S}})$-modules

$$0 \to \mathscr{K} \to \mathfrak{U}(\mathscr{L}_{\mathscr{S}}) \to \mathscr{O}_X \to 0 \tag{8}$$

where $\mathscr{K}$ is free over $\mathfrak{U}(\mathscr{L}_{\mathscr{S}})$. Note that the category $\operatorname{Rep}(\mathscr{M})$ has enough projectives, and we can use the exact sequence (8) to compute the derived functors of the functor $F_{\mathscr{M}} = \operatorname{Hom}_{\mathfrak{U}(\mathscr{L}_{\mathscr{S}})}(-,\mathscr{M})$ applied to $\mathscr{O}_X$, obtaining $L^iF_{\mathscr{M}}(\mathscr{O}_X) = 0$ for $i \geq 2$. Moreover, $L^iF_{\mathscr{M}}(\mathscr{O}_X) \simeq \mathbb{H}^i(\mathscr{L}_{\mathscr{S}};\mathscr{M})$ for $i \geq 0$. □

Remark 3.8. If the sheaf of sets $\mathscr{S}$ is locally constant with a finite stalk, the sheaf of Lie algebras $\mathscr{L}ie_{\Bbbk,\mathscr{S}}$ is a locally free $\Bbbk_X$-module of finite rank. As a result, the free Lie algebroid $\mathscr{L}_{\mathscr{S}}$ is a locally free $\mathscr{O}_X$-module of finite rank. Corollary 3.6 can be strengthened to the claim that every locally free Lie algebroid of finite rank is the quotient of a free Lie algebroid which is a locally free $\mathscr{O}_X$-module of finite rank. △

4. Abelian extensions of Lie algebroids

4.1. **The extension problem.** Let $\mathscr{Q}$ be a locally free Lie algebroid on a scheme X, and $\mathscr{L}$ a bundle of Lie algebras over X, i.e., a locally free Lie algebroid with vanishing. Fix a morphism

$$\alpha\colon \mathscr{Q} \to \mathscr{O}ut(\mathscr{L}),$$

where $\mathscr{O}ut(\mathscr{L})$ is the sheaf of outer derivations of $\mathscr{L}$ (note that $\mathscr{O}ut(\mathscr{L})$ has a natural structure of Lie algebroid). Such a morphism gives the center $Z(\mathscr{L})$ of $\mathscr{L}$ a $\mathscr{Q}$-module structure, We shall denote by $\mathbb{H}^\bullet(\mathscr{Q};Z(\mathscr{L}))^{(1)}_\alpha$ the hypercohomology of the truncated complex $\tau^{\geq 1}\Lambda^\bullet\mathscr{Q}^*\otimes Z(\mathscr{L})$, with the $\mathscr{Q}$-module structure of $Z(\mathscr{L})$ given by α.

4.2. **Abelian extensions.** Consider an extension of Lie algebroids as in (1) where $\mathscr{L}$ is abelian and has trivial anchor, and is therefore a representation of all three Lie algebroids, $\mathscr{L}$, $\mathscr{E}$ and $\mathscr{Q}$ (moreover, $\mathscr{L}^{\mathscr{L}} = \mathscr{L}$). Thus we can take cohomology with coefficients in $\mathscr{L}$. We have a composition of morphisms

$$\mathbb{H}^0(\mathscr{Q};\mathscr{H}^1(\mathscr{L},\mathscr{L})) \to \mathbb{H}^2(\mathscr{Q};\mathscr{L}) \to \mathbb{H}^2(\mathscr{Q};\mathscr{L})^{(1)}. \tag{9}$$

As $\mathscr{H}^1(\mathscr{L},\mathscr{L})$ is the sheafication of the presheaf which to an open subset $U \subset X$ assigns the first cohomology of the Lie-Rinehart algebra $\mathscr{L}(U)$ with coefficients in $\mathscr{L}(U)$, from known results about Lie-Rinehart cohomology we get then that $\mathscr{H}^1(\mathscr{L};\mathscr{L})$ is isomorphic to the sheaf $\mathscr{E}nd_{\mathscr{O}_X}(\mathscr{L})$, and then we have

$$\mathbb{H}^0(\mathscr{Q};\mathscr{H}^1(\mathscr{L};\mathscr{L})) \simeq \mathrm{End}_{\mathscr{O}_X}(\mathscr{L})^{\mathscr{Q}}.$$

The identity morphism $\mathrm{id}_{\mathscr{L}}$ is obviously $\mathscr{Q}$-invariant, so it lies in $\mathrm{End}_{\mathscr{O}_X}(\mathscr{L})^{\mathscr{Q}}$.

As a result, by applying the morphism (9) to $\mathrm{id}_{\mathscr{L}}$, one has a set-theoretic map

$$\mathrm{Ext}_{\mathrm{LA}}(\mathscr{Q};\mathscr{L}) \to \mathbb{H}^2(\mathscr{Q};\mathscr{L})^{(1)} \tag{10}$$

where $\mathrm{Ext}_{\mathrm{LA}}(\mathscr{Q};\mathscr{L})$ is the set of equivalence classes of extensions of $\mathscr{Q}$ by $\mathscr{L}$ compatible with the morphism α, with the usual equivalence relation.

4.3. **Surjectivity of the extension map.** Our aim is to show that the map (10) is bijective; this will prove Theorem 1.1. Let us at first show that it is surjective. In view of Theorem 3.6, $\mathscr{Q}$ can be written as the quotient of the free Lie algebroid $\mathscr{L}_{\mathscr{S}}$ for a suitable sheaf of sets $\mathscr{S}$ and sheaf morphism $\mathscr{S} \to \mathscr{D}er_{\Bbbk}(\mathscr{O}_X)$. For simplicity we denote $\mathscr{F} = \mathscr{L}_{\mathscr{S}}$. So we have an exact sequence of Lie algebroids

$$0 \to \mathscr{T} \xrightarrow{f} \mathscr{F} \to \mathscr{Q} \to 0\,. \tag{11}$$

We have fixed a $\mathscr{Q}$-action on $\mathscr{L}$, so that there is also an $\mathscr{F}$-action via the morphism $\mathscr{F} \to \mathscr{Q}$, and the induced $\mathscr{T}$-action is trivial. We can therefore consider the cohomologies of the Lie algebroids in the exact sequence (11) with coefficients in $\mathscr{L}$.

Lemma 4.1. *There is a surjective morphism*

$$\mathrm{Hom}_{\mathscr{Q}}(\mathscr{T},\mathscr{L}) \to \mathbb{H}^2(\mathscr{Q};\mathscr{L})^{(1)}.$$

Proof. We we construct this morphism in three steps.

(1) If we define $\mathscr{T}' = \mathscr{T}/[\mathscr{T},\mathscr{T}]$ and $\mathscr{F}' = \mathscr{F}/[\mathscr{T},\mathscr{T}]$, we have an exact sequence

$$0 \to \mathscr{T}' \to \mathscr{F}' \to \mathscr{Q} \to 0 \tag{12}$$

and $\mathrm{Hom}_{\mathscr{Q}}(\mathscr{T}',\mathscr{L}) \simeq \mathrm{Hom}_{\mathscr{Q}}(\mathscr{T},\mathscr{L})$.

(2) As $\mathscr{T}'$ is abelian, and its action on $\mathscr{L}$ is trivial, we have

$$\mathbb{H}^0(\mathscr{Q};\mathscr{H}^1(\mathscr{T}',\mathscr{L})) \simeq \mathrm{Hom}_{\mathscr{Q}}(\mathscr{T}',\mathscr{L}).$$

(3) If we consider the natural morphism from the sequence (11) to the sequence (12), and for each sequence we write the relevant segment of the five-term sequence of the Hochschild-Serre spectral sequence, we get the diagram

$$\begin{array}{ccccc} \mathbb{H}^0(\mathscr{Q};\mathscr{H}^1(\mathscr{T},\mathscr{L})) & \longrightarrow & \mathbb{H}^2(\mathscr{Q};\mathscr{L}) & \longrightarrow & 0 \\ \downarrow & & \| & & \\ \mathbb{H}^0(\mathscr{Q};\mathscr{H}^1(\mathscr{T}',\mathscr{L})) & \longrightarrow & \mathbb{H}^2(\mathscr{Q};\mathscr{L}) & & \end{array}$$

(note that $\mathbb{H}^2(\mathscr{F};\mathscr{L}) = 0$, see Proposition 3.7). So the bottom horizontal arrow is surjective. □

Thus, given an element in $\mathbb{H}^2(\mathscr{Q};\mathscr{L})^{(1)}$, it comes from an element $\phi \in \mathrm{Hom}_{\mathscr{Q}}(\mathscr{T},\mathscr{L})$. This produces an extension as in (1), that is, an element in $\mathrm{Ext}_{\mathrm{LA}}(\mathscr{Q};\mathscr{L})$, by letting

$$\mathscr{E} = (\mathscr{L} \rtimes \mathscr{F})/\mathscr{K}\,. \tag{13}$$

Here one looks at $\mathscr{L}$ as an $\mathscr{F}$-module via $\mathscr{F} \to \mathscr{Q}$, and

$$\mathscr{K} = \mathrm{im}\,[\phi \times (-f) : \mathscr{T} \to \mathscr{L} \rtimes \mathscr{F}],$$

i.e.,

$$\mathscr{K} = \{(\phi(-t), t) \,|\, t \text{ a section of } \mathscr{T}\}.$$

It is clear by the construction that the map (10) sends this extension to the given element in $\mathbb{H}^2(\mathscr{Q};\mathscr{L})^{(1)}$, i.e., the map (10) is surjective.

4.4. **From Lie algebroid extensions to coherent sheaf extensions.** We may expect that there is a natural morphism

$$\varpi \colon \mathbb{H}^2(\mathscr{Q};\mathscr{L})^{(1)} \to \mathrm{Ext}^1_{\mathscr{O}_X}(\mathscr{Q},\mathscr{L})$$

whose composition with the map (10) is a forgetful morphism which only remembers that $\mathscr{E}$ is an extension of $\mathscr{Q}$ by $\mathscr{L}$ as $\mathscr{O}_X$-modules. We start by constructing the morphism. If $\mathscr{E}^\bullet$ is the complex of $\mathscr{O}_X$-modules with $\mathscr{Q}^* \otimes \mathscr{L}$ in degree 1 and 0 elsewhere, we have

$$\mathbb{H}^i(X,\mathscr{E}) \simeq H^{i-1}(X,\mathscr{Q}^* \otimes \mathscr{L}) \simeq \mathrm{Ext}^{i-1}_{\mathscr{O}_X}(\mathscr{Q},\mathscr{L}) \quad \text{for } i \geq 1.$$

Moreover, there is a morphism $\tau^{\geq 1}\mathscr{L} \otimes \Lambda^\bullet \mathscr{Q}^* \to \mathscr{E}^\bullet$ which is the inclusion $\operatorname{coker} d_0 \hookrightarrow \mathscr{Q}^* \otimes \mathscr{L}$ in degree 1, and 0 in all other degrees. Then ϖ is just the morphism induced in hypercohomology in degree 2. This is also easily described if we use Čech complexes to compute hypercohomology: ω is obtained by taking the (1,1) piece of a 2-cocycle of the truncated complex.

Lemma 4.2. *The composite morphism* $\mathrm{Ext}_{LA}(\mathscr{Q},\mathscr{L}) \to \mathrm{Ext}^1_{\mathscr{O}_X}(\mathscr{Q},\mathscr{L})$ *applied to the class of a Lie algebroid extension of* $\mathscr{Q}$ *by* $\mathscr{L}$ *yields the extension class of* $\mathscr{Q}$ *by* $\mathscr{L}$ *as* $\mathscr{O}_X$*-modules.*

Proof. If $\mathfrak{U}$ is an (affine) open cover of X, we can represent a class in $\mathbb{H}^2(\mathscr{Q};\mathscr{L})^{(1)}$ with a 2-cocyle ψ in the total complex of the double complex $\check{C}^\bullet(\mathfrak{U},\tau^{\geq 1}\Lambda^\bullet\mathscr{Q}^*\otimes\mathscr{L})$. Then ψ has two terms,

$$\psi^{1,1}\in\check{C}^1(\mathfrak{U},\operatorname{coker} d_0),\quad \psi^{0,2}\in\check{C}^0(\mathfrak{U},\Lambda^2\mathscr{Q}^*\otimes\mathscr{L})$$

which satisfy

$$\delta\psi^{1,1}=0,\quad \delta\psi^{0,2}+d\psi^{1,1}=0$$

where δ and d are the Čech differential and the differential of the complex $\tau^{\geq 1}\Lambda^\bullet\mathscr{Q}^*\otimes\mathscr{L}$, respectively. But $\psi^{1,1}$ can be regarded as a 1-cocycle in the Čech complex of the sheaf $\mathscr{Q}^*\otimes\mathscr{L}$. The analysis in [2] shows that $\psi^{1,1}$ is a cocycle representing the extension class of $\mathscr{Q}$ by $\mathscr{L}$ as $\mathscr{O}_X$-modules. □

4.5. **Injectivity of the extension map.** We show now that the map (10) is injective. We know from Lemma 4.2 that two classes in $\mathrm{Ext}_{\mathrm{LA}}(\mathscr{Q},\mathscr{L})$ which produce the same element in $\mathbb{H}^2(\mathscr{Q};\mathscr{L})^{(1)}$ are equivalent as extensions of $\mathscr{O}_X$-modules. Since the Lie algebroid structure is a local datum, this allows us to assume that X is affine.

Suppose that two extensions

$$0\to\mathscr{L}\to\mathscr{E}_i\to\mathscr{Q}\to 0,\qquad i=1,2$$

map to the same element in $H^2(\mathscr{Q};\mathscr{L})^{(1)}$. We can choose $\mathscr{F}$ so that it maps to both $\mathscr{E}_1$ and $\mathscr{E}_2$, compatibly with the morphism $\mathscr{F}\to\mathscr{Q}$. Denote $f_i\colon\mathscr{F}\to\mathscr{E}_i$. By the previous arguments, there are two classes φ_1, φ_2 in $\mathrm{Hom}_{\mathscr{Q}}(\mathscr{T}',\mathscr{L})$ that map to the same class in $H^2(\mathscr{Q};\mathscr{L})^{(1)}$, so that their difference $\varphi_2-\varphi_1$ comes from a class in $\mathbb{H}^1(\mathscr{F};\mathscr{L})$.

Now we use the fact that X is affine. From the discussion preceding eq. (3), there is a morphism $D\colon\mathscr{F}\to\mathscr{L}$ which satisfies the property (3). Let $g=g_1+D\colon\mathscr{F}\to\mathscr{E}_1$. This satisfies

$$g(\{x,y\})=[g(x),g(y)]$$

and is compatible with the anchors, so that it is a morphism of Lie algebroids. If we replace g_i with g, there exists $\varphi\colon \mathscr{T} \to \mathscr{L}$ such that the following diagram commutes:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathscr{T} & \longrightarrow & \mathscr{F} & \longrightarrow & \mathscr{Q} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle g_i} & & \| & & \\ 0 & \longrightarrow & \mathscr{L} & \longrightarrow & \mathscr{E}_i & \longrightarrow & \mathscr{Q} & \longrightarrow & 0 \\ & & & & \downarrow & & & & \\ & & & & 0 & & & & \end{array}$$

So we have

$$\mathscr{E}_1 \simeq \mathscr{F} / \ker \varphi \simeq \mathscr{E}_2.$$

The previous diagram also proves the commutativity of the diagram (2).

5. Nonabelian extensions

5.1. **The obstruction class.** As we already discussed, the problem of finding an extension of $\mathscr{Q}$ by $\mathscr{L}$ inducing a given morphism $\alpha\colon \mathscr{Q} \to \mathscr{D}er(Z(\mathscr{L}))$ is obstructed by a class in the group $\mathbb{H}^3(\mathscr{Q}; Z(\mathscr{L}))^{(1)}$. This obstruction class was already built in [2] using Čech resolutions. Here we want to give an more abstract construction, using the formalism so far developed in this paper. What we are going to do is essentially to generalize the treatment in [10] to Lie algebroids.

As we saw, $\mathscr{Q}$ can be represented as a quotient of a free Lie algebroid, which we denote $\mathscr{F}$. So we have an exact sequence of Lie algebroids as in (11). The epimorphism $\mathscr{F} \to \mathscr{Q}$ induces a epimorphism $\mathfrak{U}(\mathscr{F}) \to \mathfrak{U}(\mathscr{Q})$. Let $\mathscr{K}$ be the corresponding kernel, so that we have

$$0 \to \mathscr{K} \to \mathfrak{U}(\mathscr{F}) \to \mathfrak{U}(\mathscr{Q}) \to 0.$$

Moreover we denote by $\mathscr{J}$ the kernel of the augmentation morphism $\mathfrak{U}(\mathscr{F}) \to \mathscr{O}_X$. Note that $\mathscr{K}$ injects into $\mathscr{J}$, and $\mathscr{K}\mathscr{J}$ injects to $\mathscr{K}$. If we denote

$$\widetilde{\mathscr{K}}^i = \mathscr{K}^i / \mathscr{K}^{i+1}, \qquad \widetilde{\mathscr{J}}^i = \mathscr{K}^i \mathscr{J} / \mathscr{K}^{i+1} \mathscr{J}, \quad \text{for } i = 0, \ldots$$

(with $\mathscr{K}^0 = \mathfrak{U}(\mathscr{F})$), the sheaves $\widetilde{\mathscr{K}}^i$, $\widetilde{\mathscr{J}}^i$ are locally free $\mathscr{O}_X$-modules with a $\mathfrak{U}(\mathscr{Q})$-module structure. The previous injections define morphisms $\widetilde{\mathscr{K}}^i \to \widetilde{\mathscr{J}}^{i-1}$ and $\widetilde{\mathscr{J}}^i \to \widetilde{\mathscr{K}}^i$. Moreover, there is a morphism $\widetilde{\mathscr{J}}^0 \to \mathfrak{U}(\mathscr{Q})$.

Let $\mathscr{X}$ be a sheaf of sets, and $a_{\mathscr{F}}\colon \mathscr{X} \to \mathscr{D}er_{\Bbbk}(\mathscr{O}_X)$ a morphism such that the associated free Lie algebroid is isomorphic to $\mathscr{F}$. Analogously, let $\mathscr{Y}$ be a sheaf of sets whose associated sheaf of free Lie algebras is isomorphic to $\mathscr{T}$. Then $\mathscr{Y}$ generates $\mathscr{K}$ as a sheaf of free $\mathfrak{U}(\mathscr{Q})$-algebras. Moreover, products of i sections of $\mathscr{Y}$ mod $\mathscr{K}^{i+1}$ generate $\widetilde{\mathscr{K}}^i$, and products of i copies of $\mathscr{Y}$ times sections of $\mathscr{X}$ mod $\mathscr{K}^{i+1}\mathscr{J}$ generate $\widetilde{\mathscr{J}}^i$.

Lemma 5.1. *The sequence*

$$\cdots \to \widetilde{\mathscr{K}}^2 \to \widetilde{\mathscr{J}}^1 \to \widetilde{\mathscr{K}}^1 \to \widetilde{\mathscr{J}}^0 \to \mathfrak{U}(\mathscr{Q}) \to \mathscr{O}_X \to 0. \tag{14}$$

is a resolution of $\mathscr{O}_X$ by free $\mathfrak{U}(\mathscr{Q})$-modules.

Proof. Brute force diagram chasing. □

Moreover, we pick a lift $\tilde{\alpha}\colon \mathscr{F} \to \mathscr{D}er(\mathscr{L})$ of α, so that we have a commutative diagram

$$\begin{array}{ccccccccccc} & & 0 & \longrightarrow & \mathscr{T} & \longrightarrow & \mathscr{F} & \longrightarrow & \mathscr{Q} & \longrightarrow & 0 \\ & & & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \tilde{\alpha}} & & \downarrow{\scriptstyle \alpha} & & \\ 0 & \longrightarrow & Z(\mathscr{L}) & \longrightarrow & \mathscr{L} & \xrightarrow{\text{ad}} & \mathscr{D}er(\mathscr{L}) & \longrightarrow & \mathscr{O}ut(\mathscr{L}) & \longrightarrow & 0 \end{array} \tag{15}$$

where β is the induced morphism.

We define a morphism

$$o\colon \widetilde{\mathscr{J}}^1 \to Z(\mathscr{L}). \tag{16}$$

It is enough to define o on an element of the type yx, where x is a generator of $\mathscr{F}$, and y is a generator of $\mathscr{T}$. We let

$$o(yx) = \beta(\{x, y\}) - \tilde{\alpha}(x)(\beta(y)).$$

Note that if ℓ is section of $\mathscr{L}$, then

$$\{\beta(\{x, y\}), \ell\} = \{\{\beta(x), \beta(y)\}, \ell\} = \{\text{ad}(\beta(x))(\beta(y)), \ell\} = \{\tilde{\alpha}(x)(\beta(y)), \ell\}$$

so that o takes values in $Z(\mathscr{L})$.

We apply the functor $\operatorname{Hom}_{\mathfrak{U}(\mathscr{Q})}(-, Z(\mathscr{L}))$ to the resolution (14), obtaining

$$\begin{aligned} 0 \to I^{\mathscr{Q}}(Z(\mathscr{L})) \to \Gamma(X, Z(\mathscr{L})) &\xrightarrow{d_0} \operatorname{Hom}_{\mathfrak{U}(\mathscr{Q})}(\widetilde{\mathscr{J}}^0, Z(\mathscr{L})) \xrightarrow{d_1} \operatorname{Hom}_{\mathfrak{U}(\mathscr{Q})}(\widetilde{\mathscr{K}}^1, Z(\mathscr{L})) \\ &\xrightarrow{d_2} \operatorname{Hom}_{\mathfrak{U}(\mathscr{Q})}(\widetilde{\mathscr{J}}^1, Z(\mathscr{L})) \xrightarrow{d_3} \operatorname{Hom}_{\mathfrak{U}(\mathscr{Q})}(\widetilde{\mathscr{K}}^2, Z(\mathscr{L})) \to \ldots \end{aligned} \tag{17}$$

By general theory (Section 2 and [1]), the cohomology of this complex is isomorphic to $\mathbb{H}^\bullet(\mathscr{Q}; Z(\mathscr{L}))$. Note also that o is an element in $\operatorname{Hom}_{\mathfrak{U}(\mathscr{Q})}(\widetilde{\mathscr{J}}^1, Z(\mathscr{L}))$.

Lemma 5.2. $d_3(o) = 0$. *Moreover, the cohomology class of o in $\mathbb{H}^3(\mathscr{Q}; Z(\mathscr{L}))$ only depends on α.*

Proof. To prove that $d_3(o) = 0$ we need to show that $o(yx)$, as in equation (16), is zero when both y and x are sections of $\mathscr{T}$. But this follows from the commutativity of the diagram (15) (that is, from the definition of β). To prove that the cohomology class of o only depends on α means to show that this cohomology class vanishes when $\alpha = 0$. In this case, $\tilde{\alpha}$ takes values in the inner derivations, i.e., there is a morphism $\alpha\colon \hat{\mathscr{F}} \to \mathscr{L}$ such that $\tilde{\alpha}(x)(y) = \{\hat{\alpha}(x), y\}$. Moreover, $\hat{\alpha}_{|\mathscr{T}} = \beta$, so that

$$o(yx) = \hat{\alpha}(\{x, y\}) - \{\hat{\alpha}(x), \hat{\alpha}(y)\} = 0.$$

□

Remark 5.3. By computing the group $\mathbb{H}^3(\mathscr{Q}; Z(\mathscr{L}))$ using the Čech double complex $H^{p,q} = \check{C}^p(\mathfrak{U}, \Lambda^q \mathscr{Q}^* \otimes Z(\mathscr{L}))$, one sees that the 3-cocycle o has a vanishing $(3, 0)$ summand, so that its cohomology class is determined by its value in $\mathbb{H}^3(\mathscr{Q}; Z(\mathscr{L}))^{(1)}$. The same conclusion can be reached using the exact sequence (4). △

Definition 5.4. *We denote by* $\mathbf{ob}(\alpha)$ *the cohomology class induced in* $\mathbb{H}^3(\mathscr{Q}; Z(\mathscr{L}))^{(1)}$ *by o, and call it the* obstruction class *associated with α.*

Theorem 5.5. *Given a Lie algebroid $\mathscr{Q}$, a bundle $\mathscr{L}$ of Lie algebras over $\mathscr{O}_X$, and a morphism $\alpha\colon \mathscr{Q} \to \mathscr{O}ut(\mathscr{L})$, an extension of Lie algebroids as in* (1) *inducing on $Z(\mathscr{L})$ the $\mathscr{Q}$-module structure given by α exists if and only if* $\mathbf{ob}(\alpha) = 0$.

Proof. Assume that an extension as in (1) exists, inducing the given morphism α. Write $\mathscr{Q}$ as the quotient of a free algebroid $\mathscr{F}$, and lift the morphism $\mathscr{E} \to \mathscr{Q}$ to $\mathscr{F}$, obtaining a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathscr{T} & \longrightarrow & \mathscr{F} & \longrightarrow & \mathscr{Q} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \| & & \\ 0 & \longrightarrow & \mathscr{L} & \longrightarrow & \mathscr{E} & \longrightarrow & \mathscr{Q} & \longrightarrow & 0 \end{array}$$

where β is the induced morphism. Define $\tilde{\alpha}\colon \mathscr{F} \to \mathscr{D}er(\mathscr{L})$ by letting $\tilde{\alpha} = -\operatorname{ad} \circ \gamma$. Then $\tilde{\alpha}$ is a lift of α, and for all sections t of $\mathscr{T}$ and x of $\mathscr{F}$ one has

$$\beta(\{x, t\}) - \tilde{\alpha}(x)(\beta(t)) = 0 \tag{18}$$

so that the obstruction class $\mathbf{ob}(\alpha)$ vanishes.

Conversely, assume that $\mathbf{ob}(\alpha) = 0$, and take a lift $\tilde{\alpha}\colon \mathscr{F} \to \mathscr{D}er(\mathscr{L})$. Also in view of Remark 5.3, the corresponding cocycle lies in the image of the morphism d_2 in (17), so it defines a morphism $\beta\colon \mathscr{T} \to \mathscr{L}$, which satisfies the equation (18). We consider now the same semidirect product as in (13). This gives the desired extension. □

5.2. **Classifying extensions.** We assume now that the obstruction class $\mathbf{ob}(\alpha) = 0$ is zero, so that the set $\mathrm{Ext}_{\mathrm{LA}}(\mathscr{Q}, \mathscr{L})$ of equivalence classes of extensions of $\mathscr{Q}$ by $\mathscr{L}$ is not empty. We want to show that $\mathrm{Ext}_{\mathrm{LA}}(\mathscr{Q}, \mathscr{L})$ is a torsor on the group $\mathbb{H}^2(\mathscr{Q}; Z(\mathscr{L}))^{(1)}$. The idea is to reduce the problem to the abelian case. We shall be inspired by the treatment in [13] for the case of Lie algebras (actually, this is in turn an adaption to the case of Lie algebras of what was done by Eilenberg and Maclane for groups [4], and the Eilenberg-Maclane paper is much easier to read).

In particular, we shall prove the following result. For clarity, for every morphism $\alpha\colon \mathscr{Q} \to \mathscr{O}ut(\mathscr{L})$ we denote by α_0 the induced morphism $\alpha_0\colon \mathscr{Q} \to \mathscr{D}er(Z(\mathscr{L}))$.

Proposition 5.6. *The equivalence classes of extensions of $\mathscr{Q}$ by $\mathscr{L}$ inducing α are in a one-to-one correspondence with equivalence classes of extensions of $\mathscr{Q}$ by $Z(\mathscr{L})$ inducing α_0, and are therefore in a one-to-one correspondence with the elements of the group $\mathbb{H}^2(\mathscr{Q}; Z(\mathscr{L}))^{(1)}$.*

We need to develop some machinery to prove Proposition 5.6. Let $\mathscr{C}_1$, $\mathscr{C}_2$ be two Lie algebroids, and assume there are two surjective morphism $f_i\colon \mathscr{C}_i \to \mathscr{Q}$. The fibre product $\mathscr{C}_1 \times_{\mathscr{Q}} \mathscr{C}_2$ has a natural structure of Lie algebroid. Assuming that $\ker f_1$ and $\ker f_2$ have isomorphic centres, which we denote $\mathscr{Z}$, we define a product $\mathscr{C}_1 \star \mathscr{C}_2$ by letting

$$\mathscr{C}_1 \star \mathscr{C}_2 = \mathscr{C}_1 \times_{\mathscr{Q}} \mathscr{C}_2 / \mathscr{Z},$$

where $\mathscr{Z}$ is mapped to $\mathscr{C}_1 \times_{\mathscr{Q}} \mathscr{C}_2$ as $z \mapsto (z, -z)$.

Moreover, we shall consider pairs $(\mathscr{K}, \alpha)$, where $\mathscr{K}$ is a bundle of $\mathscr{O}_X$-Lie algebras on X, whose centre is isomorphic to a fixed bundle of abelian $\mathscr{O}_X$-Lie algebras $\mathscr{Z}$, and α is a morphism $\mathscr{Q} \to \mathscr{O}ut(\mathscr{K})$. We assume that the obstruction class $\mathbf{ob}(\alpha_0)$ vanishes, so that every for pair $(\mathscr{K}, \alpha)$ there are extensions

$$0 \to \mathscr{K} \to \mathscr{E} \to \mathscr{Q} \to 0$$

such that the induced morphism $\mathscr{Q} \to \mathscr{O}ut(\mathscr{K})$ coincides with α — i.e., every pair $(\mathscr{K}, \alpha)$ is extendible.

Given two pairs $(\mathscr{K}', \alpha')$, $(\mathscr{K}'', \alpha'')$, we define their product

$$(\mathscr{K}', \alpha') \star (\mathscr{K}'', \alpha'') = (\mathscr{K}' \oplus \mathscr{K}''/\mathscr{Z}, \alpha' \star \alpha'')$$

where $\mathscr{Z}$ is embedded as above, and $\alpha' \star \alpha'' \colon \mathscr{Q} \to \mathscr{O}ut(\mathscr{K}' \oplus \mathscr{K}')$ is the sum of α' and α'', which acts on the image of $\mathscr{Z}$ in $\mathscr{K}' \oplus \mathscr{K}''$ as the derivation $(\alpha_0, -\alpha_0)$. It is easy to check that if $\mathscr{E}'$ and $\mathscr{E}''$ are extensions of $\mathscr{Q}$ by $(\mathscr{K}', \alpha')$, $(\mathscr{K}'', \alpha'')$, respectively, then $\mathscr{E}' \star \mathscr{E}''$ is an extension of $\mathscr{Q}$ by $(\mathscr{K}', \alpha') \star (\mathscr{K}'', \alpha'')$. Since this is compatible with equivalence, in particular we have a map

$$\mathrm{Ext}^1(\mathscr{Q}, \mathscr{K}') \times \mathrm{Ext}^1(\mathscr{Q}, \mathscr{K}'') \to \mathrm{Ext}^1(\mathscr{Q}, \mathscr{K}' \star \mathscr{K}''). \tag{19}$$

Note that a derivation of a Lie algebra bundle always restricts to a derivation of its centre.

Lemma 5.7. *Given pairs $(\mathscr{K}, \alpha)$ and $(\mathscr{Z}, \beta)$, where β is the restriction of α to $\mathscr{Z}$, one has $(\mathscr{K}, \alpha) \star (\mathscr{Z}, \beta) \simeq (\mathscr{K}, \alpha)$.*

Proof. Direct computation. □

Remark 5.8. In this case, the map (19) becomes

$$\mathrm{Ext}^1(\mathscr{Q}, \mathscr{K}) \times \mathrm{Ext}^1(\mathscr{Q}, \mathscr{Z}) \to \mathrm{Ext}^1(\mathscr{Q}, \mathscr{K})$$

and on representing cocycles it is expressed by the sum of cocycles. △

We now fix a reference point in $\mathrm{Ext}_{\mathrm{LA}}(\mathscr{Q}, \mathscr{L})$, that is, we fix an extension $\mathscr{E}$ of $\mathscr{Q}$ by $\mathscr{L}$. The following two Lemmas provide a proof of Proposition 5.6.

Proposition 5.9. *Any extension $\mathscr{E}'$ of $\mathscr{Q}$ by $\mathscr{L}$ is equivalent to a product $\mathscr{E} \star \mathscr{D}$ of $\mathscr{E}$ by an extension $\mathscr{D}$ of $\mathscr{Q}$ by $Z(\mathscr{L})$.*

Proof. Choosing an open affine covering $\mathfrak{U} = \{U_i\}$ over which all bundles $\mathscr{L}$, $\mathscr{E}$ and $\mathscr{Q}$ trivialize, we may fix local splittings $s_i \colon \mathscr{Q}_{|U_i} \to \mathscr{E}_{|U_i}$. Then we can associate with $\mathscr{E}$ a triple of Čech cochains

$$\{\alpha_i\} \in \check{C}^0(\mathfrak{U}, \mathscr{Q}^* \otimes \mathscr{D}er(\mathscr{L}), \qquad \{\rho_i\} \in \check{C}^0(\mathscr{U}, \Lambda^2 \mathscr{Q}^* \otimes \mathscr{L}), \qquad \{\phi_{ij}\} \in \check{C}^1(\mathfrak{U}, \mathscr{Q}^* \otimes \mathscr{L})$$

by letting, for all sections x, y of $\mathscr{Q}$ and ℓ of $\mathscr{L}$,

$$\alpha_i(x)(\ell) = \{s_i(x), \ell\}, \qquad \rho(x, y) = \{s_i(x), s_i(y)\} - s_i(\{x, y\}), \qquad \phi_{ij} = s_{i|U_i \cap U_j} - s_{j|U_i \cap U_j}.$$

The α_i are local lifts of the morphism $\alpha\colon \mathscr{Q} \to \mathscr{O}ut(Z(\mathscr{L}))$, and satisfy the conditions

$$[\alpha_i(x), \alpha_i(y)] - \alpha_i(\{x, y\}) = \mathrm{ad}(\rho_i(x, y)), \qquad \alpha_i - \alpha_j = \mathrm{ad}\,\phi_{ij}. \tag{20}$$

Moreover, $\{\phi_{ij}\}$ is a cocycle which describes $\mathscr{E}$ as an extension of vector bundles. The cochain $\{\rho_i\}$ is closed under the Lie-Rinehart differential

$$d_{\alpha_i}\rho_i(x, y, z) = \alpha_i(x)(\rho_i(y, z)) - \rho_i(\{x, y\}, z) + \text{cycl. perm.} = 0$$

and describes $\mathscr{E}(U_i)$ as a Lie-Rinehart extension of $\mathscr{Q}(U_i)$ by $\mathscr{L}(U_i)$. Finally, these cochains satisfy a compatibility condition for the Lie-Rinehart algebra structures on $U_i \cap U_j$:

$$(\delta\rho)_{ij} = d_{\alpha_i}\phi_{ij}.$$

If $(\{\alpha_i'\}, \{\rho_i'\}, \{\phi_{ij}'\})$ is a triple describing the extension $\mathscr{E}'$, one can modify $\{\phi_{ij}'\}$ by adding a coboundary so that $\alpha_i' = \alpha_i$. If we define

$$\psi_{ij} = \phi_{ij}' - \phi_{ij} \tag{21}$$

then for all sections x of $\mathscr{Q}$ and ℓ of $\mathscr{L}$ one has $\{\psi_{ij}(x), \ell)\} = 0$, so that

$$\{\psi_{ij}\} \in \check{C}^1(\mathfrak{U}, \mathscr{Q}^* \otimes Z(\mathscr{L})).$$

We obtain therefore an extension of vector bundles

$$0 \to Z(\mathscr{L}) \to \mathscr{D} \to \mathscr{Q} \to 0.$$

Finally, we define a 2-cochain $\{\tau_i\} \in \check{C}^0(\mathfrak{U}, \Lambda^2\mathscr{Q}^* \otimes \mathscr{L})$

$$\tau_i = \rho_i' - \rho_i. \tag{22}$$

Since both ρ and ρ' satisfy the first condition in (20), $\{\tau_i\}$ takes values in the centre $Z(\mathscr{L})$, and moreover it satisfies the conditions

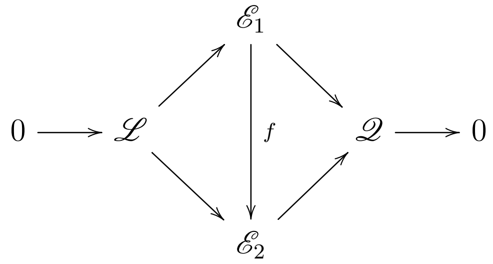
$$d_{\alpha_i}\tau_i = 0, \qquad (\delta\tau)_{ij} = d_{\alpha_i}\psi_{ij}.$$

So the triple $(\{\alpha_i\}, \{\tau_i\}, \{\psi_{ij}\})$ gives $\mathscr{D}$ a Lie algebroid structure.

Equations (21) and (22) express the fact that $\mathscr{E} \star \mathscr{D} \simeq \mathscr{E}'$. □

Proposition 5.10. *Given an extensions $\mathscr{E}$ of $\mathscr{Q}$ by $\mathscr{L}$ and two extensions $\mathscr{D}_1$, $\mathscr{D}_2$ of $\mathscr{Q}$ by $Z(\mathscr{L})$, the extensions $\mathscr{E}_1 = \mathscr{E} \star \mathscr{D}_1$ and $\mathscr{E}_2 = \mathscr{E} \star \mathscr{D}_2$ are equivalent if and only if $\mathscr{D}_1$ and $\mathscr{D}_2$ are equivalent.*

Proof. If $\mathscr{D}_1$ and $\mathscr{D}_2$ are equivalent, then $\mathscr{E}_1$ and $\mathscr{E}_2$ are certainly equivalent. Let us prove the converse. One has a Lie algebroid morphism $f\colon \mathscr{E}_1 \to \mathscr{E}_2$ such that the diagram



commutes. If $\{\phi_{ij}\}$ is a cocycle representing the extension class of $\mathscr{E}$, and $\{\psi_{ij}^{(1)}\}$ and $\{\psi_{ij}^{(2)}\}$ are cocycles representing the extension classes of $\mathscr{D}_1$ and $\mathscr{D}_2$ respectively, then, since $\mathscr{E}_1$ and $\mathscr{E}_2$ are isomorphic as vector bundles,

$$\phi_{ij} + \psi_{ij}^{(1)} = \phi_{ij} + \psi_{ij}^{(2)} + \chi_i - \chi_j$$

for some 0-cocycle χ. So the classes of $\{\psi_{ij}^{(1)}\}$ and $\{\psi_{ij}^{(2)}\}$ in $\operatorname{Ext}^1(\mathscr{Q}^*, Z(\mathscr{L}))$ coincide, i.e, $\mathscr{D}_1$ and $\mathscr{D}_2$ are equivalent as vector bundle extensions. Then we identify $\mathscr{D}_1$ and $\mathscr{D}_2$ as vector bundles.

We can introduce local splittings $\{s_i^1\}$, $\{s_i^2\}$ for $\mathscr{E}_1$ and $\mathscr{E}_2$ with the corresponding representing triples. We can again redefine the cocycle (say) $\{\phi_{ij}^{(2)}\}$ so that $\alpha_i^{(1)} = \alpha_i^{(2)}$ (and we shall denote this α_i). We also introduce $\{b_i\} \in \check{C}^0(\mathfrak{U}, \mathscr{Q}^* \otimes \mathscr{L})$ by letting $f \circ s_i^{(1)} = b_i + s_i^{(2)}$. As

$$\alpha_i(x)(\ell) = f(\alpha_i(x)(\ell)) = \{f(s^{(1)}(x)), \ell\} = \alpha_i(x)(\ell) + \{b_i(x), \ell\},$$

$\{b_i\}$ actually has values in $Z(\mathscr{L})$. The Lie algebroids $\mathscr{D}_1$, $\mathscr{D}_2$ are represented by the triples

$$(\{\alpha_i\},\ \{\tau_i^{(j)} = \rho_i^{(j)} - \rho_i\},\ \{\psi_{ij}^{(j)}\}), \quad j = 1, 2.$$

Moreover, one has the equalities

$$\begin{aligned} f(\rho_i^{(1)}(x,y)) &= \rho_i^{(2)}(x,y) + \{s^{(2)}(x), b_i(y)\} - \{s^{(2)}(y), b_i(x)\} - b_i(\{x,y\}) \\ &= \rho_i(x,y) + \tau_i^{(2)}(x,y) + (d_{\alpha_i} b_i)(x,y) \\ f(\rho_i^{(1)}(x,y)) &= \rho_i(x,y) + \tau_i^{(1)}(x,y) \end{aligned}$$

so that

$$\tau_i^{(2)} = \tau_i^{(1)} - d_{\alpha_i} b_i.$$

Therefore, $\mathscr{D}_1$ and $\mathscr{D}_2$ are equivalent. $\square$

Proof of Proposition 5.6. After fixing an extension $\mathscr{E}_0$, given any other extension $\mathscr{E}$ we can realize it as $\mathscr{E}_0 \star \mathscr{D}$; the extension $\mathscr{D}$ will give an element in $\mathbb{H}^2(\mathscr{Q}; Z(\mathscr{L}))^{(1)}$, which as a consequence of Proposition 5.10 only depends on the equivalence class of $\mathscr{E}$. The resulting map $\mathrm{Ext}_{\mathrm{LA}}(\mathscr{Q}, \mathscr{L}) \to \mathbb{H}^2(\mathscr{Q}; Z(\mathscr{L}))^{(1)}$ is bijective because it is so in the abelian case. □

Finally, it is clear that we have proved that $\mathrm{Ext}_{\mathrm{LA}}(\mathscr{Q}, \mathscr{L})$ is a torsor over $\mathbb{H}^2(\mathscr{Q}; Z(\mathscr{L}))^{(1)}$. This completes the proof of Theorem 1.1 in the nonabelian case.

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$$
\begin{array}{ccccccccccc}
0 & \longrightarrow & H^1(X,Z(\mathscr{L})) & \xrightarrow{\sim} & H^1(X,Z(\mathscr{L})) & \longrightarrow & H^0(X,\mathscr{H}^1(\mathscr{T},Z(\mathscr{L}))) & \longrightarrow & H^2(X,Z(\mathscr{L})) & \xrightarrow{\sim} & H^2(X,Z(\mathscr{L})) \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathbb{H}^1(\mathscr{Q};Z(\mathscr{L})) & \longrightarrow & \mathbb{H}^1(\mathscr{F};Z(\mathscr{L})) & \longrightarrow & \mathbb{H}^0(\mathscr{Q};\mathscr{H}^1(\mathscr{T},Z(\mathscr{L}))) & \longrightarrow & \mathbb{H}^2(\mathscr{Q};Z(\mathscr{L})) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathbb{H}^1(\mathscr{Q};Z(\mathscr{L}))^{(1)} & \longrightarrow & \mathbb{H}^1(\mathscr{F};Z(\mathscr{L}))^{(1)} & \longrightarrow & 0 & \longrightarrow & \mathbb{H}^2(\mathscr{Q};Z(\mathscr{L}))^{(1)} & \longrightarrow & \mathbb{H}^2(\mathscr{F};Z(\mathscr{L}))^{(1)} \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow\wr \\
0 & \longrightarrow & H^2(X,Z(\mathscr{L})) & \xrightarrow{\sim} & H^2(X,Z(\mathscr{L})) & \longrightarrow & H^1(X,\mathscr{H}^1(\mathscr{T},Z(\mathscr{L}))) & \longrightarrow & H^3(X,Z(\mathscr{L})) & \xrightarrow{\sim} & H^3(X,Z(\mathscr{L})) \\
 & & & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 & & & & 0 & \longrightarrow & \mathbb{H}^1(\mathscr{Q};\mathscr{H}^1(\mathscr{T},Z(\mathscr{L}))) & \xrightarrow{\sim} & \mathbb{H}^3(\mathscr{Q};Z(\mathscr{L})) & \longrightarrow & 0
\end{array}
$$

Table 1

This commutative diagram shows that

- $\mathbb{H}^1(\mathscr{F};Z(\mathscr{L}))^{(1)} = 0$;
- the morphism $\mathbb{H}^0(\mathscr{Q};\mathscr{H}^1(\mathscr{T},Z(\mathscr{L}))) \to \mathbb{H}^2(\mathscr{Q};Z(\mathscr{L}))$ is surjective
- $\mathbb{H}^1(\mathscr{Q};\mathscr{H}^1(\mathscr{T},Z(\mathscr{L}))) \simeq \mathbb{H}^3(\mathscr{Q};Z(\mathscr{L}))$.