A MINIMIZATION APPROACH TO THE WAVE EQUATION ON
TIME-DEPENDENT DOMAINS

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ABSTRACT. We prove the existence of weak solutions to the homogeneous wave equation
on a suitable class of time-dependent domains. Using the approach suggested by De Giorgi
and developed by Serra and Tilli, such solutions are approximated by minimizers of suitable
functionals in space-time.

KEYWORDS: wave equation, time-dependent domains, minimization

AMS SUBJECT CLASSIFICATIONS: 35L15, 49J10, 35Q74, 74R10, 35L90

INTRODUCTION

Several problems in dynamic fracture mechanics lead to the study of the wave equation in
time-dependent domains (see [6, 7, 3]). The main difficulty is that at every time $t$ the solution
belongs to a different function space $V_t$. It is not restrictive to assume that all spaces $V_t$ are
embedded in a given Hilbert space $H$.

In the case of fracture mechanics, a common situation is $V_t = H^1(\Omega \setminus \Gamma_t)$ and $H = L^2(\Omega)$,
where $\Omega$ is a domain in $\mathbb{R}^d$ and $\Gamma_t$ is a closed $(d-1)$-dimensional subset of $\Omega$, representing
the crack at time $t$. A natural assumption on $\Gamma_t$ is that it is monotonically increasing with respect
to $t$, thus encoding the fact that, once created, a crack cannot disappear. As a consequence, the spaces $V_t$ are increasing in time too.

To deal with possibly irregular cracks a more general increasing family of spaces has been
considered in [2]: $V_t = GSBV^2(\Omega, \Gamma_t)$, defined as the space of functions $u \in GSBV(\Omega)$ such
that $u \in L^2(\Omega)$, $\nabla u \in L^2(\Omega; \mathbb{R}^d)$, and $J_u \subset \Gamma_t$ (see [1] for the definition and properties of
these spaces and for the definition of the approximate gradient $\nabla u$ and of the jump set $J_u$).

Given $u^0 \in V_0$ and $u^1 \in H$, the Cauchy problem we are interested in is formally written as

\begin{equation}
\left\{
\begin{array}{ll}
  u''(t) + Au(t) = 0 & \text{for a.e. } t > 0, \\
  u(t) \in V_t & \text{for a.e. } t > 0, \\
  u(0) = u^0, \quad u'(0) = u^1,
\end{array}
\right.
\end{equation}

where $'$ denotes the time derivative and $A$ is a continuous and coercive linear operator ($A = -\Delta$ with homogeneous Neumann boundary conditions in the examples considered above).

The existence of a solution for (0.1) has already been proven in [2], through a time-discrete
approach, by solving suitable incremental minimum problems and then passing to the limit
as the time step tends to zero.

The purpose of this paper is to prove that a solution of (0.1) can be approximated by
global minimizers of suitable energy functionals defined as integrals on $[0, \infty)$ with respect to
time. On the one hand this shows a link between solutions of the hyperbolic problem (0.1)
and solutions of minimum problems for integral functionals on the same time domain. On
the other hand this result provides a new proof of the existence of a solution to (0.1).

The seminal idea of this approximation process goes back to a conjecture by De Giorgi [5]
on the nonlinear wave equation. Such a conjecture has been proven by Serra and Tilli in [8]
and, in a more general setting, in [9].

In our paper we extend their result to the case of time-dependent domains. To illustrate
the global minimization approach in our setting, we focus on the model case and, in a more general setting, in [9].

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This functional is to be minimized, for every fixed

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satisfying the initial conditions \( u(0) = u^0 \) and \( u'(0) = u^1 \) and the time-dependent constraint

u(t) \in V_t for a.e. \( t > 0 \). Once the existence of a minimizer \( u_\varepsilon \) is proven, the Euler-Lagrange

equation of (0.2) formally reads as

\[
\varepsilon^2 u''_\varepsilon(t) - 2\varepsilon u'''_\varepsilon(t) + u''_\varepsilon(t) - \Delta u_\varepsilon(t) = 0 \quad \text{in} \quad \Omega \setminus \Gamma_t,
\]

and hence, letting \( \varepsilon \to 0 \), one formally obtains a solution to the wave equation in (0.1).

As mentioned above, a quite general scheme to pass to the limit rigorously has been intro-
duced by Serra and Tilli in [9] when time-dependent constraint \( u(t) \in V_t \) is not present. The
proof consists in finding suitable estimates on the minimizers \( u_\varepsilon \) of the functionals \( F_\varepsilon \) and to
exploit these estimates in order to obtain, by compactness, the convergence of \( u_\varepsilon \) to a weak
solution \( u \) to the wave equation.

In this paper we implement this scheme in the case of time-dependent domains. This
requires some changes in the proof, since all competitors of the minimum problem for (0.2)
must satisfy the constraint \( u(t) \in V_t \) for a.e. \( t > 0 \).

The main change is in the proof of the key estimate for \( u_\varepsilon(t) \), which is obtained in [9] by
using an inner variation \( u_\varepsilon(\varphi_\delta(t)) \) for a suitable function \( \varphi_\delta : [0, \infty) \to [0, \infty) \).
Since in our case we have to require that \( u_\varepsilon(\varphi_\delta(t)) \in V_t \) for a.e. \( t > 0 \), this variation is admissible only if
\( \varphi_\delta(t) \leq t \) for a.e. \( t > 0 \). By the technical definition of \( \varphi_\delta \), this leads to the constraint \( \delta > 0 \).
Therefore the standard comparison between the functional on \( u_\varepsilon(\varphi_\delta(t)) \) and on the minimizer
\( u_\varepsilon(t) \), in the limit as \( \delta \to 0^+ \), gives only an inequality, instead of the equality proven in [9, formula (4.7)]. This inequality, however, turns out to be enough to obtain the other estimates
of [9] with minor changes.

A further difficulty appears when proving that the limit \( u \) of \( u_\varepsilon \) is a weak solution of (0.1),
since also the test functions \( \eta \) must satisfy the constraint \( \eta(t) \in V_t \) for a.e. \( t > 0 \). Therefore,
to adapt the proof of [9], we have to approximate an arbitrary test function \( \eta \) satisfying the
constraint \( \eta(t) \in V_t \) for a.e. \( t > 0 \) by sums of functions of the form \( \varphi(t) v \) with \( v \in V_s \) and
\( \varphi \in C^2(\mathbb{R}) \) with \( \text{supp}(\varphi) \subset [s, \infty) \), which still satisfy the constraint.

1. Description of the problem

1.1. Setting. To study the wave equation in time-dependent domains we adopt the functional
setting introduced in [4]. Let \( H \) be a separable Hilbert space and let \( (V_t)_{t \in [0, \infty)} \) be a family
of separable Hilbert spaces with the following properties

(H1) for every \( t \in [0, \infty) \) the space \( V_t \) is contained and dense in \( H \) with continuous
embedding;
(H2) for every $s, t \in [0, \infty)$, with $s < t$, $V_s$ is a closed subspace of $V_t$ with the induced scalar product.

The scalar product in $H$ is denoted by $(\cdot, \cdot)$ and the corresponding norm by $\| \cdot \|$. The norm in $V_t$ is denoted by $\| \cdot \|_t$. By (H2) for every $0 \leq s < t$ we have $\| v \|_s = \| v \|_t$ for every $v \in V_s$.

The dual of $H$ is identified with $H^*$, while for every $t \in [0, T]$ the dual of $V_t$ is denoted by $V_t^*$.

Note that the adjoint of the continuous embedding of $V_t$ into $H$ provides a continuous embedding of $H$ into $V_t^*$ and that $H$ is dense in $V_t^*$. Let $\langle \cdot, \cdot \rangle_t$ be the duality product between $V_t^*$ and $V_t$ and let $\| \cdot \|_t^*$ be the corresponding dual norm. Note that $\langle \cdot, \cdot \rangle_t$ is the unique continuous bilinear map on $V_t^* \times V_t$ satisfying

$$\langle h, v \rangle_t = (h, v) \quad \text{for every } h \in H \text{ and } v \in V_t.$$ 

Let $V_\infty := \bigcup_{t \geq 0} V_t$ and let $a: V_\infty \times V_\infty \rightarrow \mathbb{R}$ be a bilinear symmetric form satisfying the following conditions:

(H3) continuity: there exists $M_0 > 0$ such that

$$|a(u, v)| \leq M_0 \| u \|_t \| v \|_t \quad \text{for every } t \geq 0 \text{ and every } u, v \in V_t;$$

(H4) coercivity: there exist $\lambda_0 \geq 0$ and $\nu_0 > 0$ such that

$$a(u, u) + \lambda_0 \| u \|^2 \geq \nu_0 \| u \|^2_t \quad \text{for every } t \geq 0 \text{ and every } u \in V_t;$$

(H5) positive semidefiniteness:

$$a(u, u) \geq 0 \quad \text{for every } u \in V_\infty .$$

For every $\tau, t \in [0, \infty)$ let $A_t^\tau : V_t \rightarrow V_t^*$ be the continuous linear operator defined by

$$\langle A_t^\tau u, v \rangle : = a(u, v) \quad \text{for every } u \in V_t \text{ and } v \in V_t.$$ 

Note that

$$\| A_t^\tau u \|_{V_t}^* \leq M_0 \| u \|_t \quad \text{for every } u \in V_t.$$ 

Finally, we set $Q(u) := a(u, u)$ for every $u \in V_\infty$.

**Definition 1.1.** Given $T > 0$, we define $\mathcal{W}^{0,1}_T := L^2((0,T);V_T) \cap H^1((0,T);H)$, with the Hilbert space structure induced by the scalar product

$$\langle u, v \rangle_{\mathcal{W}^{0,1}_T} = \langle u, v \rangle_{L^2((0,T);V_T)} + \langle u', v' \rangle_{L^2((0,T);H)},$$

where $u'$ and $v'$ denote the distributional derivatives. The norm induced by the scalar product $(\cdot, \cdot)_{\mathcal{W}^{0,1}_T}$ is denoted by $\| \cdot \|_{\mathcal{W}^{0,1}_T}$. Moreover, we define

$$\mathcal{V}^{0,1}_T := \{ u \in \mathcal{W}^{0,1}_T : u(t) \in V_t \text{ for a.e. } t \in (0,T) \},$$

and note that it is a closed subspace of $\mathcal{W}^{0,1}_T$.

Analogously, we define $\mathcal{W}^{0,2}_T := L^2((0,T);V_T) \cap H^2((0,T);H)$, with the Hilbert space structure induced by the scalar product

$$\langle u, v \rangle_{\mathcal{W}^{0,2}_T} = \langle u, v \rangle_{L^2((0,T);V_T)} + \langle u', v' \rangle_{L^2((0,T);H)} + \langle u'', v'' \rangle_{L^2((0,T);H)},$$

and the space

$$\mathcal{V}^{0,2}_T := \{ u \in \mathcal{W}^{0,2}_T : u(t) \in V_t \text{ for a.e. } t \in (0,T) \},$$

which is a closed subspace of $\mathcal{W}^{0,2}_T$.

Finally, $\mathcal{V}^{0,1}$ (resp. $\mathcal{V}^{0,2}$) is defined as the space of functions $u: (0, +\infty) \rightarrow H$ whose restrictions to $(0, T)$ belong to $\mathcal{V}^{0,1}_T$ (resp. $\mathcal{V}^{0,2}_T$) for every $T > 0$. 
Remark 1.2. It is well known that every function $u \in H^1((0,T);H)$ (resp. $u \in H^2((0,T);H)$) admits a representative, still denoted by $u$, which belongs to the space $C^0([0,T];H)$ (resp. $C^1([0,T];H)$). With this convention we have $V_T^{0,1} \subset C^0([0,T];H)$ (resp. $V_T^{0,2} \subset C^1([0,T];H)$) for every $T > 0$.

Definition 1.3. We say that $u$ is a weak solution of the equation
\begin{equation}
\tag{1.7}
u''(t) + A(t)u(t) = 0, \quad u(t) \in V_t \quad \text{for } t \in [0,\infty)
\end{equation}
if $u \in V_T^{0,1}$ and for every $T > 0$
\begin{equation}
\tag{1.8}\int_0^T (u'(t), \psi'(t)) \, dt = \int_0^T a(u(t), \psi(t)) \, dt
\end{equation}
for every $\psi \in V_T^{0,1}$ with $\psi(0) = \psi(T) = 0$.

For every Banach space $X$ let $C_w([0,T];X)$ be the space of functions $u: [0,T] \to X$ that are continuous for the weak topology of $X$.

Remark 1.4. If $u$ is a weak solution of (1.7) with $u \in L^\infty((0,T);V_T)$ and $u' \in L^\infty((0,T);H)$ for every $T > 0$, then [4, Theorem 2.17 and Proposition 2.18] imply that, after a modification on a set of measure zero, $u \in C_w([0,T];V_T)$ and $u' \in C_w([0,T];H)$ for every $T > 0$.

1.2. Main results. Throughout the paper we fix $u^0 \in V_0$, $u^1 \in H$, and a sequence $\{u^\varepsilon\}_0 \subset V_0$ such that
\begin{equation}
\tag{1.9}\|u^\varepsilon - u^1\|_H \to 0 \quad \text{as } \varepsilon \to 0^+ \quad \text{and} \quad \varepsilon\|u^1\|_0 \leq C_1,
\end{equation}
for some constant $C_1 > 0$. For every $\varepsilon > 0$ we consider the functional
\begin{equation}
\tag{1.10}\mathcal{F}_\varepsilon(u) := \frac{1}{2} \int_0^\infty e^{-t/\varepsilon} \left(\varepsilon^2 \|u''(t)\|^2 + Q(u(t))\right) \, dt,
\end{equation}
defined on the set
\begin{equation}
\tag{1.11}V_T^{0,2}(u^0, u^1) := \{u \in V_T^{0,2} : u(0) = u^0, u'(0) = u^1\},
\end{equation}
which is well-defined in view of Remark 1.2.

We now state our main results, which are proven in Sections 2, 3, and 4.

Theorem 1.5. For every $\varepsilon \in (0,1)$ the functional $\mathcal{F}_\varepsilon$ admits a unique global minimizer $u^\varepsilon$ in the set $V_T^{0,2}(u^0, u^1)$. Moreover,
\begin{equation}
\tag{1.12}\mathcal{F}_\varepsilon(u^\varepsilon) \leq \bar{C}_\varepsilon,
\end{equation}
for some constant $\bar{C} > 0$ depending only on $\|u^0\|_0$ and $C_1$.

In particular, if $\varepsilon\|u^1\|_0 \to 0$ as $\varepsilon \to 0^+$, then
\begin{equation}
\tag{1.13}\mathcal{F}_\varepsilon(u^\varepsilon) \leq \varepsilon \left(\frac{1}{2} Q(u^0) + r_{\varepsilon}\right),
\end{equation}
where $r_{\varepsilon} \to 0$ as $\varepsilon \to 0^+$.

Theorem 1.6. There exists a constant $C > 0$ such that for every $\varepsilon \in (0,1)$ the minimizer $u^\varepsilon$ of $\mathcal{F}_\varepsilon$ in $V_T^{0,2}(u^0, u^1)$ satisfies the estimates:
\begin{equation}
\tag{1.14}\int_t^{t+\tau} Q(u^\varepsilon(s)) \, ds \leq C \tau \quad \text{for every } t \geq 0, \tau \geq \varepsilon,
\end{equation}
\begin{equation}
\tag{1.15}\|u^\varepsilon(t)\|^2 \leq C(1 + t^2) \quad \text{for every } t \geq 0,
\end{equation}
\begin{equation}
\tag{1.16}\|u_{\varepsilon}'(t)\| \leq C \quad \text{for every } t \geq 0.
\end{equation}
Theorem 1.7. For every $\varepsilon \in (0,1)$ let $u_{\varepsilon}$ be the minimizer of $F_{\varepsilon}$ in $\mathcal{V}^{0,2}(u^0,u^1_{\varepsilon})$. Then for every sequence $\{\varepsilon_n\} \subset (0,1)$, with $\varepsilon_n \to 0$ as $n \to \infty$, there exist a subsequence, not relabeled, and a weak solution $u$ of (1.7) such that $u_{\varepsilon_n} \rightharpoonup u$ weakly in $\mathcal{V}^0_T$ for every $T > 0$. Moreover the following properties hold:

(a) weak continuity: $u \in C_w([0,T];V_T)$ and $u' \in C_w([0,T];H)$ for every $T > 0$;

(b) initial conditions: $u(0) = u^0$ and $u'(0) = u^1$.

If, in addition, $\varepsilon\|u^1_{\varepsilon}\|_0 \to 0$ as $\varepsilon \to 1^+$, then the following energy inequality holds:

\[
\|u'(t)\|^2 + Q(u(t)) \leq \|u^1\|^2 + Q(u^0) \quad \text{for every } t > 0.
\]

2. Proof of Theorem 1.5

Before proving our results we introduce a change of variables that will be useful throughout the paper.

Remark 2.1. For every $\varepsilon > 0$ and every $T > 0$ we set

$$
\mathcal{W}^{0,2}_{\varepsilon,T} := L^2((0, T); V_{\varepsilon,T}) \cap H^2((0, T); H),
$$

$$
\mathcal{V}^{0,2}_{\varepsilon,T} := \{v \in \mathcal{W}^{0,2}_{\varepsilon,T} : v(t) \in V_{\varepsilon,t} \text{ for a.e. } t \in (0, T)\}.
$$

Note that $\mathcal{W}^{0,2}_{\varepsilon,T}$ is a Hilbert space with the scalar product

$$
(u, v)_{\mathcal{W}^{0,2}_{\varepsilon,T}} = (u, v)_{L^2((0, T); V_{\varepsilon,T})} + (u', v')_{L^2((0, T); H)} + (u'', v'')_{L^2((0, T); H)},
$$

and $\mathcal{V}^{0,2}_{\varepsilon,T}$ is a closed subspace of $\mathcal{W}^{0,2}_{\varepsilon,T}$. Furthermore, $\mathcal{V}^{0,2}_{\varepsilon,T}$ denotes the space of functions $u: [0, \infty) \to H$ whose restrictions to $(0, T)$ belong to $\mathcal{V}^{0,2}_{\varepsilon,T}$ for every $T > 0$. By Remark 1.2 every $u \in \mathcal{W}^{0,2}_{\varepsilon,T}$ admits a representative, still denoted by $u$, which belongs to $C^1([0, T]; H)$. With this convention we have $\mathcal{V}^{0,2}_{\varepsilon,T} \subset C^1([0, T]; H)$ for every $T > 0$. Finally, we define

$$
\mathcal{V}_{\varepsilon,T}^{0,2}(u^0, u^1_{\varepsilon}) := \{v \in \mathcal{V}^{0,2}_{\varepsilon,T} : v(0) = 0, v'(0) = \varepsilon u^1_{\varepsilon}\}.
$$

It is easy to see that if $u \in \mathcal{V}_{\varepsilon}^{0,2}(u^0, u^1_{\varepsilon})$, then the function $v$ defined by

\[
v(t) := u(\varepsilon t)
\]

belongs to $\mathcal{V}^{0,2}_{\varepsilon,T}(u^0, u^1_{\varepsilon})$ and

\[
\mathcal{F}_\varepsilon(u) = \varepsilon \mathcal{G}_\varepsilon(v),
\]

where

\[
\mathcal{G}_\varepsilon(v) := \frac{1}{2} \int_0^\infty e^{-t} \left(\frac{\|v''(t)\|^2}{\varepsilon^2} + Q(v(t))\right) dt.
\]

In view of Remark 2.1, Theorem 1.5 is a consequence of the following result for the functional $\mathcal{G}_\varepsilon$.

Theorem 2.2. For every $\varepsilon \in (0, 1)$ the functional $\mathcal{G}_\varepsilon$ admits a unique global minimizer $v_\varepsilon$ in the class $\mathcal{V}^{0,2}_{\varepsilon}(u^0, u^1_{\varepsilon})$. Moreover,

\[
\mathcal{G}_\varepsilon(v_\varepsilon) \leq \bar{C},
\]

for some constant $\bar{C} < \infty$ depending only on $\|u^0\|_0$ and $C_1$.

Furthermore $u_\varepsilon(t) := v_{\varepsilon}(\frac{t}{\varepsilon})$ is the unique global minimizer of $\mathcal{F}_\varepsilon$ in $\mathcal{V}^{0,2}(u^0, u^1_{\varepsilon})$ and satisfies (1.12).
Finally, if $\varepsilon \|u_0^1\|_0 \to 0$ as $\varepsilon \to 0^+$, then

$$G_{\varepsilon}(v_{\varepsilon}) \leq \frac{1}{2} Q(u^0) + r_\varepsilon,$$

where $r_\varepsilon \to 0$ as $\varepsilon \to 0$ and $u_\varepsilon$ satisfies (1.13).

Proof. Fix $\varepsilon > 0$ and set $v(t) := u^0 + \varepsilon u_1^1$ for every $t \geq 0$. Note that $v \in V_{\varepsilon,2}^1(u^0, \varepsilon u_1^1)$, since $u^0, u_1^1 \in V_0 \subset V_t$ for every $t \geq 0$. By (H3) and by (1.9), we have

$$G_{\varepsilon}(v) = \frac{1}{2} \int_0^\infty e^{-t} Q(v(t)) \, dt \leq \frac{1}{2} Q(u^0) + M_0 \varepsilon \|u_1^1\|_0(\varepsilon) + \|u^0\|_0) \leq \tilde{C},$$

where $\tilde{C}$ is a constant depending only on $C_1$ and $\|u_0\|_0$. Note that, if $\varepsilon \|u_1^1\|_0 \to 0$ as $\varepsilon \to 0^+$, then by (2.3) it follows that

$$G_{\varepsilon}(v) \leq \frac{1}{2} Q(u^0) + r_\varepsilon,$$

where $r_\varepsilon \to 0$ as $\varepsilon \to 0$.

In particular, $G_{\varepsilon}$ has a finite infimum and (2.3) (as well as (2.4)) follows as soon as $G_{\varepsilon}$ has an absolute minimizer $v_{\varepsilon}$. To show this, consider a minimizing sequence $\{v_{\varepsilon,n}\} \subset V_{0,2}^1(u^0, \varepsilon u_1^1)$ and fix $T > 0$. By the very definition of $G_{\varepsilon}$ and by (2.5),

$$\int_0^T \|v_{\varepsilon,n}(t)\|^2 \, dt \leq e^T \int_0^T e^{-t} \|v_{\varepsilon,n}(t)\|^2 \, dt \leq 2\varepsilon^2 e^T G_{\varepsilon}(v_{\varepsilon,n}) \leq \varepsilon^2 C_T,$$

for some constant $C_T > 0$. The bound (2.7), together with the boundary conditions

$$v_{\varepsilon,n}(0) = u^0 \quad \text{and} \quad v_{\varepsilon,n}'(0) = \varepsilon u_1^1,$$

implies

$$\|v_{\varepsilon,n}\|_{H^2((0,T);H)} \leq C_{T,\varepsilon}$$

for some constant $C_{T,\varepsilon} > 0$ independent of $n$. Moreover, by (H2) and (H4), for $t \in [0,T]$ we have

$$\nu_0 \|v_{\varepsilon,n}(t)\|_{T}^2 = \nu_0 \|v_{\varepsilon,n}(t)\|_{L^2(0,T)}^2 \leq \lambda_0 \|v_{\varepsilon,n}(t)\|_{L^2(0,T)}^2 + Q(v_{\varepsilon,n}(t))$$

from which, using (2.5) and (2.9), we get

$$\nu_0 \|v_{\varepsilon,n}\|_{L^2((0,T);V_T)}^2 \leq \lambda_0 \|v_{\varepsilon,n}\|_{L^2((0,T);H)}^2 + \int_0^T Q(v_{\varepsilon,n}(t)) \, dt \leq \tilde{C}_{T,\varepsilon}$$

for some constant $\tilde{C}_{T,\varepsilon} > 0$ independent of $n$. It follows that $\|v_{\varepsilon,n}\|_{W_{T,2}}^{1,2}$ is uniformly bounded and hence, up to a subsequence, $v_{\varepsilon,n} \rightharpoonup v_\varepsilon$ in $W_{\varepsilon,2}^{0,2}$ as $n \to \infty$, for some $v_\varepsilon \in W_{\varepsilon,2}^{0,2}$. Moreover, since $V_{\varepsilon,T}^{0,2}$ is closed, $v_\varepsilon \in V_{\varepsilon,T}^{0,2}$. By the arbitrariness of $T$ we have $v_\varepsilon \in V_{\varepsilon}^{0,2}$ and by (2.8) we get $v_\varepsilon \in V_{\varepsilon}^{0,2}(u^0, \varepsilon u_1^1)$. Finally, since $G_{\varepsilon}$ is lower semi-continuous and strictly convex by (H5), $v_\varepsilon$ is the unique minimizer of $G_{\varepsilon}$ in $V_{\varepsilon}^{0,2}(u^0, \varepsilon u_1^1)$. The statements about $u_\varepsilon(t)$ follow from Remark 2.1. \[\Box\]
3. Proof of Theorem 1.6

We first introduce some notations. Let \( v_\varepsilon \) be the minimizer of \( G_\varepsilon \) in \( V_0^{0,2}(u^0, \varepsilon u_1^1) \) and let \( L_\varepsilon \) be the corresponding Lagrangian defined as
\[
L_\varepsilon(t) := D_\varepsilon(t) + Q_\varepsilon(t),
\]
where
\[
D_\varepsilon(t) := \frac{\|v''_\varepsilon(t)\|^2}{2\varepsilon^2} \quad \text{and} \quad Q_\varepsilon(t) := \frac{Q(v_\varepsilon(t))}{2}.
\]
Moreover, we define the kinetic energy function \( K_\varepsilon \) as
\[
K_\varepsilon(t) := \frac{\|v'_\varepsilon(t)\|^2}{2\varepsilon^2}.
\]

We shall use the following result, which can be proven as in [9, Lemma 3.4].

**Lemma 3.1.** There exists a constant \( C > 0 \) (depending only on \( \|u^0\|, \|u_1^1\| \), and \( C_1 \) in (1.9)) such that for every \( \varepsilon \in (0, 1) \) the minimizer \( v_\varepsilon \) of \( G_\varepsilon \) in \( V_0^{0,2}(u^0, \varepsilon u_1^1) \) satisfies
\[
\int_0^\infty e^{-t}D_\varepsilon(t)\,dt = \int_0^\infty e^{-t} \frac{\|v''_\varepsilon(t)\|^2}{2\varepsilon^2}\,dt \leq C,
\]
\[
\int_0^\infty e^{-t}K_\varepsilon(t)\,dt = \int_0^\infty e^{-t} \frac{\|v'_\varepsilon(t)\|^2}{2\varepsilon^2}\,dt \leq C.
\]

In particular, in view of Lemma 3.1, we have \( K_\varepsilon \in W^{1,1}(0, T) \) for all \( T > 0 \) and
\[
K'_\varepsilon(t) = \frac{1}{\varepsilon^2}(v'_\varepsilon(t), v''_\varepsilon(t)) \quad \text{for a.e. } t > 0.
\]

Following the approach in [9], we introduce the *average operator* \( A \), defined by
\[
(Af)(s) := \int_s^\infty e^{-(t-s)}f(t)\,dt, \quad s \geq 0.
\]
for every measurable function \( f: [0, \infty) \to [0, \infty] \).

We note that \( Af \) is well defined (possibly \( \infty \)) since \( f \geq 0 \). Moreover, the equality
\[
Af(0) = \int_0^\infty e^{-t}f(t)\,dt,
\]
implies that, if \( Af(0) < \infty \), then \( Af \) is absolutely continuous on all intervals \([0, T]\) and
\[
(Af)' = Af - f \quad \text{a.e. in } [0, \infty).
\]

In any case, since \( Af \geq 0 \), starting from \( f \geq 0 \) one can iterate \( A \), and a simple computation gives
\[
(A^2f)(s) = \int_s^\infty e^{-(t-s)(t-s)}f(t)\,dt,
\]
thus in particular
\[
(A^2f)(0) = \int_0^\infty e^{-t}f(t)\,dt.
\]

Finally, we define the approximate energy
\[
E_\varepsilon(t) := K_\varepsilon(t) + (A^2Q_\varepsilon)(t).
\]

The key ingredient in order to prove Theorem 1.6 is given by the following proposition.
Proposition 3.2. The function $E_\varepsilon$ is uniformly bounded and monotonically nonincreasing. More precisely, there exists $C'_1 > 0$, depending only on $\|u^0\|_0$, $\|u^1\|$, and $C_1$ in (1.9), such that

\begin{equation}
E_\varepsilon(t) \leq C'_1 \quad \text{for every } t \geq 0.
\end{equation}

Moreover, if $\varepsilon\|u^1_\varepsilon\|_0 \to 0$ as $\varepsilon \to 0^+$, then

\begin{equation}
E_\varepsilon(t) \leq \frac{1}{2}\|u^1_\varepsilon\|^2 + \frac{1}{2}Q(u^0) + \tilde{r}_\varepsilon,
\end{equation}

where $\tilde{r}_\varepsilon \to 0$ as $\varepsilon \to 0^+$.

Proof. The proof of Proposition 3.2 closely follows the strategy adopted in [9] to prove [9, Theorem 4.8]. We briefly sketch the main steps, underlining the main differences with respect to the case treated in [9]. The proof is divided into four steps.

Step 1. For every $g \in C^{1,1}([\mathbb{R}; [0, \infty])]$, with $g(0) = 0$ and $g(t)$ affine for $t$ sufficiently large, there exists a constant $C_1(g) > 0$, depending on $g$, $\|u^0\|_0$, and $C_1$ in (1.9), such that

\begin{equation}
\int_0^\infty e^{-s}(g'(s) - g(s))L_\varepsilon(s)\,ds - \int_0^\infty e^{-s}(4D_\varepsilon(s)g'(s) + K_\varepsilon'(s)g''(s))\,ds + R_\varepsilon \geq 0,
\end{equation}

where

\begin{equation}
R_\varepsilon := \varepsilon g'(0) \int_0^\infty e^{-s}s a(v_\varepsilon(s), u^1_\varepsilon)\,ds
\end{equation}

satisfies

\begin{equation}
|R_\varepsilon| < C_1(g).
\end{equation}

In particular, if $\varepsilon\|u^1_\varepsilon\|_0 \to 0$ as $\varepsilon \to 0^+$, then

\begin{equation}
|R_\varepsilon| \to 0 \quad \text{as } \varepsilon \to 0^+.
\end{equation}

Using the approximation argument in [9, Corollary 4.5], it is enough to prove (3.14) for $g \in C^2([\mathbb{R}; [0, \infty])]$ with $g(0) = 0$ and $g(t)$ constant for $t$ large enough.

For $\delta \geq 0$ small enough, the function $\varphi_\delta(t) := t - \delta g(t)$ is a $C^2$-diffeomorphism of $[0, \infty)$ into itself. We consider the function $v_{\varepsilon,\delta}(t) := v_\varepsilon(\varphi_\delta(t)) + t\delta g'(0)u^1_\varepsilon$. By construction $\varphi_\delta(t) \leq t$ so that, in view of (H2), $v_{\varepsilon,\delta} \in \mathcal{V}^{0,2}_\varepsilon$. Note that in the proof of this property the condition $\delta \geq 0$ is crucial. Moreover, $v_{\varepsilon,\delta}(0) = v_\varepsilon(0) = u^0$ and

\begin{equation}
v'_{\varepsilon,\delta}(t)|_{t=0} = v'_\varepsilon(0)(1 - \delta g'(0)) + \delta \varepsilon g'(0)u^1_\varepsilon = \varepsilon u^1_\varepsilon,
\end{equation}

whence $v_{\varepsilon,\delta} \in \mathcal{V}^{0,2}_\varepsilon(u^0, \varepsilon u^1_\varepsilon)$.

Set $\psi_\delta(s) := \varphi_\delta^{-1}(s)$ for every $s \geq 0$. By the change of variables $t = \psi_\delta(s)$, it is straightforward to check that

\begin{equation}
G_\varepsilon(v_{\varepsilon,\delta}) = \frac{1}{2\varepsilon^2} \int_0^\infty \psi''_\delta(s)e^{-\psi_\delta(s)}\|v''_\varepsilon(s)\varphi''_\delta(\psi_\delta(s))\|^2 + v'_\varepsilon(s)\varphi''_\delta(\psi_\delta(s))\|s\| ds
\end{equation}

\begin{equation}
+ \frac{1}{2} \int_0^\infty \psi''_\delta(s)e^{-\psi_\delta(s)}Q(v_\varepsilon(s) + \delta \varepsilon g'(0)\psi_\delta(s)u^1_\varepsilon)\,ds.
\end{equation}

Notice that

\begin{equation}s = \varphi_\delta(\psi_\delta(s)) = \psi_\delta(s) - \delta g(\psi_\delta(s))
\end{equation}
so that, in view of the assumptions on \( g \), we have \( e^{-\psi(s)} \leq e^{\delta \| g \|_{L^\infty} e^{-s}} \). Moreover, since
\[
\psi'_\delta(s) = 1 + \delta g'(\psi_\delta(s)) \psi'_\delta(s) \quad \text{and} \quad \psi''_\delta(s) = \delta (g''(\psi_\delta(s))(\psi'_\delta(s))^2 + g'(\psi_\delta(s))\psi''_\delta(s)),
\]
for \( \delta \) sufficiently small both \( \psi'_\delta(s) \) and \( \psi''_\delta(s) \) are bounded uniformly with respect to \( s \). This fact, together with Lemma 3.1, implies that the first integral in (3.17) is finite. As for the second integral we have
\[
\frac{1}{2} \int_0^\infty \psi'_\delta(s)e^{-\psi_\delta(s)}Q(v_\varepsilon(s) + \delta\varepsilon g'(0)\psi_\delta(s)u^1_\varepsilon) \, ds \leq \frac{1}{2}\| \psi'_\delta \|_{L^\infty} e^{\delta \| g \|_{L^\infty}} (A_1 + A_2 + A_3),
\]
where
\[
A_1 := \int_0^\infty e^{-s}Q(v_\varepsilon(s)) \, ds,
\]
\[
A_2 := \delta^2 (g'(0))^2 \varepsilon^2 Q(u^1_\varepsilon) \int_0^\infty e^{-s}(\psi_\delta(s))^2 \, ds,
\]
\[
A_3 := 2\delta \varepsilon g'(0) \int_0^\infty e^{-s}\psi_\delta(s)a(v_\varepsilon(s), u^1_\varepsilon) \, ds.
\]

Now, \( A_1 < \infty \) by (2.3) and \( A_2 < +\infty \) in view of (3.18). Finally, by (H5) and the Cauchy inequality, we have \( A_3 \leq A_1 + A_2 < \infty \). It follows that \( G_{\varepsilon}(v_\varepsilon, \delta) < \infty \) for \( \delta \) sufficiently small. Analogously, one can show that differentiation under the integral sign in (3.17) is possible.

Since \( v_{\varepsilon,0} = v_\varepsilon \) and \( v_{\varepsilon,\delta} \in \mathbf{V}^0_{\varepsilon} (u^0_\varepsilon, \varepsilon u^1_\varepsilon) \) only for \( \delta \geq 0 \), the minimality of \( v_\varepsilon \) implies
\[
\left. \frac{d}{d\delta} G_{\varepsilon}(v_{\varepsilon,\delta}) \right|_{\delta = 0} \geq 0,
\]
while in [9] the equality holds. One can compute this derivative as in [9, pages 2031-2032] and one can check that it coincides with the left-hand side of (3.14).

As for \( R_\varepsilon \), by assumptions (H3) and (H5) and by (1.9) and (2.2), we have
\[
|R_\varepsilon| = \varepsilon |g'(0)| \int_0^\infty e^{-s} |a(v_\varepsilon(s), u^1_\varepsilon)| \, ds
\]
\[
\leq \varepsilon |g'(0)| \left( \int_0^\infty e^{-s} Q(v_\varepsilon(s)) \, ds + M_0 \| u^1_\varepsilon \|_0 \right) \int_0^\infty e^{-s} s^2 \, ds
\]
\[
\leq |g'(0)| (2\varepsilon G_{\varepsilon}(v_\varepsilon) + 2M_0 \| u^1_\varepsilon \|_0) \leq 2g'(0)(\varepsilon \tilde{C} + C_1) =: C_1(g),
\]
thus proving (3.15). By the last but one inequality in (3.20) and by (2.2), it follows that, if \( \varepsilon \| u^1_\varepsilon \|_0 \to 0 \) as \( \varepsilon \to 0^+ \), then \( R_\varepsilon \to 0 \) as \( \varepsilon \to 0^+ \).

**Step 2.** \( (A^2 L_\varepsilon)(0) \leq (AL_\varepsilon)(0) - 4(AD_\varepsilon)(0) + R_\varepsilon \).
The claim follows by applying (3.14) with \( g(t) = t \).

**Step 3.** \( K^t_\varepsilon(t) \leq (AL_\varepsilon)(t) - (A^2 L_\varepsilon)(t) - 4(AD_\varepsilon)(t) \) for almost every \( t > 0 \).
The proof closely resembles the one of [9, Corollary 4.7]. Fix \( t > 0 \) and for every \( \delta > 0 \) let \( g_{t,\delta} \) be defined by
\[
g_{t,\delta}(s) := \begin{cases} 0 & \text{if } s \leq t \\ \frac{(s-t)^2}{25} & \text{if } s \in [t, t + \delta] \\ s - t - \frac{\delta}{2} & \text{if } s \geq t + \delta. \end{cases}
\]
The claim follows by considering \( g = g_{t,\delta} \) in (3.14) and sending \( \delta \to 0 \).
Step 4. (3.12) holds true.

In view of Step 2 and (3.6), $A^2Q_\varepsilon$ and $K_\varepsilon$ are absolutely continuous on the intervals $[0,T]$ for every $T > 0$. Therefore, we can differentiate $E_\varepsilon$ and, using Step 3, (3.8), and the very definition of $L_\varepsilon$ in (3.1), we get

$$E_\varepsilon' = K_\varepsilon' + (A^2Q_\varepsilon)' = K_\varepsilon' + A^2Q_\varepsilon - AQ_\varepsilon$$

$$\leq AL_\varepsilon - A^2L_\varepsilon - 4AD_\varepsilon + A^2Q_\varepsilon - AQ_\varepsilon = -A^2D_\varepsilon - 3AD_\varepsilon \leq 0,$$

and hence $E_\varepsilon(t) \leq E_\varepsilon(0)$ for a.e. $t \geq 0$. Moreover, by the very definition of $E_\varepsilon$ and $L_\varepsilon$, together with (2.3), Step 2, and (3.15), it follows that

$$E_\varepsilon(0) = K_\varepsilon(0) + (A^2Q_\varepsilon)(0) - \frac{1}{2}\|u_\varepsilon\|^2 + (A^2Q_\varepsilon)(0)$$

finally, by using (3.16) and (2.4) in the last line in (3.22), we obtain that, if $\varepsilon\|u_\varepsilon\|_0 \to 0$ as $\varepsilon \to 0+$, then

$$E_\varepsilon(0) \leq \frac{1}{2}\|u_\varepsilon\|^2 + \frac{1}{2}Q(u^0) + r_\varepsilon + R_\varepsilon \leq \frac{1}{2}\|u_\varepsilon\|^2 + \frac{1}{2}Q(u^0) + \tilde{r}_\varepsilon,$$

where $C_1'$ depends on $\|u^0\|_0, \|u_\varepsilon\|$, and $C_1$ in (1.9). This concludes the proof of (3.12). Finally, by using (3.16) and (2.4) in the last line in (3.22), we obtain that, if $\varepsilon\|u_\varepsilon\|_0 \to 0$ as $\varepsilon \to 0+$, then

$$E_\varepsilon(0) \leq \frac{1}{2}\|u_\varepsilon\|^2 + \frac{1}{2}Q(u^0) + r_\varepsilon + R_\varepsilon \leq \frac{1}{2}\|u_\varepsilon\|^2 + \frac{1}{2}Q(u^0) + \tilde{r}_\varepsilon,$$

where $\tilde{r}_\varepsilon \to 0$ as $\varepsilon \to 0+$. Therefore also (3.13) holds true. \qed

4. PROOF OF THEOREM 1.7

Before proving Theorem 1.7, we introduce a suitable subset of $\mathcal{V}^{0,2}_{\varepsilon,T}$, which is dense in $\{\eta \in C^2_c((0,T);V_T) : \eta(t) \in V_t \text{ for every } t \in (0,T)\}$. For every $\varepsilon > 0$ and $T > 0$, we define $\mathcal{D}_T$ as the set of all functions $\eta \in C^2_c((0,T);V_T)$ of the form

$$\eta(t) = \sum_{i=2}^{N-2} \sum_{j=0}^2 \varphi_{i,j}(t)h_{i,j}$$

for some $N \in \mathbb{N}$, $0 = t_0 < t_1 < \ldots < t_N = T$, $\varphi_{i,j} \in C^2(\mathbb{R})$ with $\text{supp } \varphi_{i,j} \subset [t_{i-1},t_{i+1}]$, and $h_{i,j} \in V_{t_{i-1}}$ for $i = 2,\ldots,N-2$ and $j = 0,1,2$. By (H2) the last two conditions imply that $\eta(t) \in V_t$ for every $t \in [0,T]$. We now prove the density.

Lemma 4.1. Let $T > 0$. For every $\eta \in C^2_c((0,T);V_T)$, with $\eta(t) \in V_t$ for every $t \in (0,T)$, there exists a sequence $\{\eta_N\} \subset \mathcal{D}_T$ such that

$$\|\eta - \eta_N\|_{C^2((0,T);V_T)} \to 0 \quad \text{as } N \to \infty.$$  

Proof. Let $\eta \in C^2_c((0,T);V_T)$, with $\eta(t) \in V_t$ for every $t \in (0,T)$. In order to construct the approximating sequence $\{\eta_N\} \subset \mathcal{D}_T$ we make use of quintic Hermite interpolants, that we construct here through the Bernstein polynomials. Let $N \in \mathbb{N}$ and set $t_i = i\frac{T}{N}$ for $i = 0,1,\ldots,N$. Fix $i = 0,\ldots,N$. For $n \in \mathbb{N}$, we define the Bernstein polynomials in the interval $[t_i,t_{i+1}]$ as

$$B^i_{k,n}(t) := \begin{cases} \binom{n}{k} (t-t_i)^k(t_{i+1}-t)^{n-k} & \text{for } k = 0,\ldots,n, \\ 0 & \text{for } k < 0 \text{ or } k > n, \end{cases}$$
and we define the polynomials of the spline basis as follows

\[ \psi_{i,0,+}(t) := \frac{N^5}{T^5} (B_{0,5}^i(t) + B_{1,5}^i(t) + B_{2,5}^i(t)), \quad \psi_{i,0,-}(t) := \frac{N^5}{T^5} (B_{3,5}^i(t) + B_{4,5}^i(t) + B_{5,5}^i(t)), \]

\[ \psi_{i,1,+}(t) := \frac{N^4}{5T^4} (B_{1,5}^i(t) + 2B_{2,5}^i(t)), \quad \psi_{i,1,-}(t) := -\frac{N^4}{5T^4} (2B_{3,5}^i(t) + B_{4,5}^i(t)), \]

\[ \psi_{i,2,+}(t) := \frac{N^3}{20T^3} B_{2,5}^i(t), \quad \psi_{i,2,-}(t) := \frac{N^3}{20T^3} B_{3,5}^i(t). \]

By construction, it is easy to see that

\[ \psi_{i,0,+}(t) + \psi_{i,0,-}(t) = 1 \quad \text{for} \quad t \in [t_i, t_{i+1}]. \]

Moreover, by using that

\[ \frac{d}{dt} B_{k,n}^i(t) = n(B_{k-1,n-1}^i(t) - B_{k,n-1}^i(t)), \]

one can easily show that

\[ -\frac{T}{N} \psi_{i,0,+}'(t) + \psi_{i,1,+}'(t) + \psi_{i,1,-}'(t) = 1, \]

\[ -\frac{T^2}{2N^2} \psi_{i,0,+}''(t) + \frac{T}{N} \psi_{i,1,+}''(t) + \psi_{i,2,+}''(t) + \psi_{i,2,-}''(t) = 1. \]

For every \( i = 1, \ldots, N - 1 \) and \( j = 0, 1, 2 \) we set

\[ \varphi_{i,j}(t) := \begin{cases} \psi_{i-1,j-}(t) & \text{if} \quad t \in [t_{i-1}, t_i], \\ \psi_{i,j}(t) & \text{if} \quad t \in [t_i, t_{i+1}], \\ 0 & \text{elsewhere}. \end{cases} \]

Finally, we define the function

\[ \eta_N(t) := \sum_{i=2}^{N-2} (\varphi_{i,0}(t)\eta(t_{i-1}) + \varphi_{i,1}(t)\eta'(t_{i-1}) + \varphi_{i,2}(t)\eta''(t_{i-1})) . \]

By (H2) we have \( \eta(t_{i-1}) , \eta'(t_{i-1}) , \eta''(t_{i-1}) \in V_{t_{i-1}} \), hence \( \eta_N \in D_T \) for every \( N \in \mathbb{N} \).

It remains to prove (4.1). Let \( t \in \text{supp} \eta \). For \( N \in \mathbb{N} \) large enough there exists \( i = 2, \ldots, N - 3 \) such that \( t \in [t_i, t_{i+1}] \), so that by (4.2) and by the very definition of \( \eta_N, \psi_{i,1,\pm}, \) and \( \psi_{i,2,\pm} \), we have

\[ \| \eta_N(t) - \eta(t) \|_T \leq \| \psi_{i,0,+}(t)\eta(t_{i-1}) + \psi_{i,0,-}(t)\eta(t) - \eta(t) \|_T + O(1/N), \]

and hence \( \eta_N \) converges to \( \eta \) in \( V_T \) uniformly in \([0, T]\). Analogously, by (4.3), we obtain

\[ \| \eta_N'(t) - \eta'(t) \|_T \leq \| \psi_{i,0,+}'(t)\eta(t_{i-1}) + \psi_{i,0,-}'(t)\eta(t) + \frac{T}{N} \psi_{i,0,+}'(t)\eta'(t) \|_T \]

\[ + \| \psi_{i,1,+}' \|_{L^\infty} \| \eta(t_{i-1}) - \eta'(t) \|_T + \| \psi_{i,1,-}' \|_{L^\infty} \| \eta'(t_{i-1}) - \eta'(t) \|_T + O(1/N), \]

which, using that (by (4.2)) the first term on the right-hand side is bounded by

\[ \frac{T}{N} \| \psi_{i,0,+}'(t) \|_{L^\infty} \| \eta(t_{i-1}) - \eta'(t_{i-1}) \|_T + \eta'(t) \|_T, \]

implies that \( \eta_N' \) converges to \( \eta' \) in \( V_T \) uniformly in \([0, T]\). Analogously, using (4.2), (4.3), and (4.4), one can show that \( \eta_N'' \) converges uniformly to \( \eta'' \) in \([0, T]\). \( \square \)
Lemma 4.2. Let $\varepsilon > 0$ and $T > 0$. For every $\eta \in C^2_ε((0, T); V_T)$, with $\eta(t) \in V_t$ for every $t \in (0, T)$, we have

\begin{equation}
(4.5) \quad \int_0^T e^{-s/\varepsilon} \left( \varepsilon^2 \left( u''_\varepsilon(s), \eta''(s) \right) + a(u_\varepsilon(s), \eta(s)) \right) \, ds = 0.
\end{equation}

Proof. In view of Lemma 4.1, it is sufficient to prove (4.5) for $\eta \in D_T$. The proof is analogous to the one of [9, Lemma 5.1]. Let $\delta \in [-1, 1]$ and set $u_{\varepsilon, \delta} := u_\varepsilon + \delta \eta$. By construction, $u_{\varepsilon, \delta} \in V_T^{0,2}$ and, since $\eta$ has compact support, also the initial conditions are satisfied. Therefore $u_{\varepsilon, \delta} \in V_T^{0,2}(u^0, u^1_T)$, and, again by construction, $F_\varepsilon(u_{\varepsilon, \delta})$ is finite. Then the Euler-Lagrange equation (4.5) easily follows by differentiating $F_\varepsilon(u_{\varepsilon, \delta})$ with respect to $\delta$ at $\delta = 0$. \[\square\]

We are now in a position to prove Theorem 1.7.

Proof of Theorem 1.7. Let us fix a sequence $\{\varepsilon_n\} \subset (0, 1)$, with $\varepsilon_n \to 0$ as $n \to \infty$. We divide the proof into five steps.

\textbf{Step 1: There exist a subsequence, not relabeled, and a function $u \in V^{0,1}$ such that}

\begin{equation}
(4.6) \quad u_{\varepsilon_n} \rightharpoonup u \quad \text{in} \quad W_T^{0,1} \quad \text{for every} \quad T > 0.
\end{equation}

Moreover, $u' \in L^\infty((0, \infty); H)$ and $u \in L^\infty((0, T); V_T)$ for every $T > 0$.

Let $T > 0$. By (1.15) and (1.16),

$$\sup_{n \in \mathbb{N}} \|u_{\varepsilon_n}\|_{H^1((0, T); H)} < \infty.$$ 

This inequality, together with (H4) and (1.14), implies that there exists $C_T < \infty$ such that

$$\nu_0 \|u_{\varepsilon_n}\|_{L^2((0, T); V_T)}^2 \leq \int_0^T Q(u_{\varepsilon_n}(t)) \, dt + \lambda_0 \|u_{\varepsilon_n}\|_{L^2((0, T); H)}^2 \leq C_T.$$ 

As a result $\{u_{\varepsilon_n}\}$ is equibounded in $W_T^{0,1}$ and hence there exist a subsequence, not relabeled, and a function $u \in W_T^{0,1}$ such that $u_{\varepsilon_n} \rightharpoonup u$ weakly in $W_T^{0,1}$. Moreover, since $\{u_{\varepsilon_n}\} \subset V_T^{0,2} \subset V_T^{0,1}$ and $V_T^{0,1}$ is a closed subspace of $W_T^{0,1}$, we have that $u \in V_T^{0,1}$. By the arbitrariness of $T$, the function $u$ belongs to $V_T^{0,1}$ and (4.6) holds true. Furthermore, in view of (4.6), inequality (1.16) implies $u' \in L^\infty((0, \infty); H)$ and (1.15) gives $u \in L^\infty((0, T); V_T)$ for every $T > 0$.

\textbf{Step 2: Let $T > 0$. For every $\psi \in C^\infty_c((0, T); V_T)$, with $\psi(t) \in V_t$ for every $t \in (0, T)$, we have}

\begin{equation}
(4.7) \quad \int_0^T \left( u_{\varepsilon_n}'(t), \varepsilon_n^{-1} \psi''(t) + 2 \varepsilon_n \psi'(t) + \psi'(t) \right) \, dt = \int_0^T a(u_{\varepsilon_n}(t), \psi(t)) \, dt.
\end{equation}

The claim follows by considering $\eta(t) = e^{t/\varepsilon_n} \psi(t)$ in (4.5) and integrating by parts.

\textbf{Step 3: $u$ is a weak solution of (1.7).} By [4, Lemma 2.8], it is enough to prove the claim for $\psi \in C^\infty_c((0, T); V_T)$ with $\psi(t) \in V_t$ for every $t \in (0, T)$. In view of (4.6), one can pass to the limit as $n \to \infty$ in (4.7), thus obtaining (1.8).

\textbf{Step 4: $u$ satisfies (a) and (b).} Since $u' \in L^\infty((0, \infty); H)$ and $u \in L^\infty((0, T); V_T)$ for every $T > 0$ by Step 1, property (a) follows from Step 3, thanks to Remark 1.4. Claim (b) is obtained by combining (a), (1.9), and (4.6), together with the fact that $u_{\varepsilon_n} \in V_T^{0,1}(u^0, u^1_T)$.

\textbf{Step 5: $u$ satisfies the energy inequality (1.17).} By using [9, Lemma 6.1] and (3.13), one can argue as in [9, Section 6] to obtain that the energy inequality (1.17) is satisfied for almost every $t > 0$. Actually, in view of (a), this inequality is satisfied for every $t > 0$. \[\square\]
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