Physics Letters B 785 (2018) 95–104

Contents lists available at ScienceDirect

Physics Letters B

www.elsevier.com/locate/physletb

On neutrino mixing in matter and CP and T violation effects in neutrino oscillations

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ARTICLE INFO

Article history:
Received 14 July 2018
Received in revised form 9 August 2018
Accepted 17 August 2018
Available online 20 August 2018
Editor: A. Ringwald

ABSTRACT

Aspects of 3-neutrino mixing and oscillations in vacuum and in matter with constant density are investigated working with a real form of the neutrino Hamiltonian. We find the (approximate) equalities $\theta^m_{23} = \theta_{23}$ and $\delta^m = \delta$, θ_{23} (θ^m_{23}) and δ (δ^m) being respectively the atmospheric neutrino mixing angle and the Dirac CP violation phase in vacuum (in matter) of the neutrino mixing matrix, which are shown to represent excellent approximations for the conditions of the T2K (T2HK), T2HKK, NOνA and DUNE neutrino oscillation experiments. A new derivation of the known relation $\sin 2\theta^m_{23} \sin\delta^m = \sin 2\theta_{23} \sin\delta$ is presented and it is used to obtain a correlation between the shifts of θ_{23} and δ due to the matter effect. A derivation of the relation between the rephasing invariants which determine the magnitude of CP and T violating effects in 3-flavour neutrino oscillations in vacuum, $J_{\rm CP}$, and of the T violating effects in matter with constant density, $J^m_{\rm T} \equiv J^m$, reported in [1] without a proof, is presented. It is shown that the function F which appears in this relation, $J^m = J_{\rm CP}\, F$, and whose explicit form was given in [1], coincides with the function $\tilde{F}$ in the similar relation $J^m = J_{\rm CP}\tilde{F}$ derived in [2], although F and $\tilde{F}$ are expressed in terms of different sets of neutrino mass and mixing parameters and have completely different forms.

1. Introduction and preliminary remarks

It was shown in 1988 in ref. [1] that in the case of what is currently referred to as the reference 3-neutrino mixing (see, e.g., [3]), the magnitude of the CP and T violating (T violating) effects in neutrino oscillations in vacuum (in matter with constant density) are controlled by the rephasing invariant $J_{\rm CP}$ ($J^m_{\rm T} \equiv J^m$) associated with the Dirac CP violation phase present in the Pontecorvo, Maki, Nakagawa and Sakata (PMNS) [4,5] neutrino mixing matrix:

$$J_{\rm CP}(J^m) = {\rm Im}\left(\left(U^{(m)}_{e2}\right)\left(U^{(m)}_{\mu 3}\right)\left(U^{(m)}_{e3}\right)^*\left(U^{(m)}_{\mu 2}\right)^*\right), \quad (1)$$

where $U^{(m)}_{li}$, $l = e, \mu, \tau$, $i = 1, 2, 3$, are the elements of the PMNS matrix in vacuum (in matter) $U^{(m)}$. The CP violating asymmetries in the case of neutrino oscillations in vacuum, for example,

$$A^{(l,l')}_{\rm CPvac} = P^{\rm vac}(\nu_l \to \nu_{l'}) - P^{\rm vac}(\bar{\nu}_l \to \bar{\nu}_{l'}), \quad l \neq l' \ \text{ and } \ l, l' = e, \mu, \tau, \quad (2)$$

$P^{\rm vac}(\nu_l \to \nu_{l'})$ and $P^{\rm vac}(\bar{\nu}_l \to \bar{\nu}_{l'})$ being the probabilities of respectively $\nu_l \to \nu_{l'}$ and $\bar{\nu}_l \to \bar{\nu}_{l'}$ oscillations, were shown to be given by [1]:

$$A^{(e,\mu)}_{\rm CPvac} = A^{(\mu,\tau)}_{\rm CPvac} = -A^{(e,\tau)}_{\rm CPvac} = 4\, J_{\rm CP}\, \Phi^{\rm vac}_{\rm osc}, \quad (3)$$

with

$$\Phi^{\rm vac}_{\rm osc} = \sin\left(\frac{\Delta m^2_{21} L}{2E}\right) + \sin\left(\frac{\Delta m^2_{32} L}{2E}\right) + \sin\left(\frac{\Delta m^2_{13} L}{2E}\right), \quad (4)$$

where $\Delta m^2_{ij} = m^2_i - m^2_j$, $i \neq j$, m_i, $i = 1, 2, 3$, is the mass of the neutrino ν_i with definite mass in vacuum, E is the neutrino energy and L is the distance travelled by the neutrinos. In [1] similar results were shown to be valid for the T-violating asymmetries in oscillations in vacuum (in matter), $A^{(l',l)}_{\rm Tvac(m)} = P^{vac(m)}(\nu_l \to \nu_{l'}) - P^{vac(m)}(\nu_{l'} \to \nu_l)$:

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https://doi.org/10.1016/j.physletb.2018.08.025

$$A^{(\mu,e)}_{\rm Tvac(m)} = A^{(\tau,\mu)}_{\rm Tvac(m)} = -A^{(e,\tau)}_{\rm Tvac(m)} = 4\, J^{(m)}_{\rm CP(T)}\, \Phi^{\rm vac(m)}_{\rm osc}\,, \tag{5}$$

where $\Phi^{\rm m}_{\rm osc}$ has the same form as $\Phi^{\rm vac}_{\rm osc}$ in eq. (4) with Δm^2_{ij} replaced by mass splitting in the matter $\Delta M^2_{ij} = M^2_i - M^2_j$, and M_i, $i = 1, 2, 3$, are neutrino mass-eigenvalues in the matter. In vacuum the T violating asymmetries in antineutrino oscillations, $\bar{A}^{(l',l)}_{\rm Tvac} = P^{\rm vac}(\bar{\nu}_l \to \bar{\nu}_{l'}) - P^{\rm vac}(\bar{\nu}_{l'} \to \bar{\nu}_l)$, are related to those in neutrino oscillations owing to the CPT invariance: $\bar{A}^{(l',l)}_{\rm Tvac} = -A^{(l',l)}_{\rm Tvac}$. In ordinary matter (Earth, Sun)[2] the presence of matter causes CP and CPT violating effects in neutrino oscillations [6] and $|\bar{A}^{(l',l)}_{\rm Tm}| \neq |A^{(l',l)}_{\rm Tm}|$. However, in ordinary matter with constant density or with density profile which is symmetric relative to the middle point, like the matter of the Earth, the matter effects preserve the T symmetry and do not generate T violating effects in neutrino oscillations [1]. Thus, T violating effects in the flavour neutrino oscillations taking place when the neutrinos traverse, e.g., the Earth mantle or the Earth core can be caused in the case of 3-neutrino mixing only by the Dirac phase in the PMNS matrix.

The $J_{\rm CP}$-factor in the expressions for $A^{(l,l')}_{\rm CP(T)\ vac}$, $l \neq l'$, is analogous to the rephasing invariant associated with the Dirac phase in the Cabibbo–Kobayashi–Maskawa (CKM) quark mixing matrix, introduced in [7]. In the standard parametrization of the PMNS mixing matrix (see, e.g., [3]) it has the form:

$$J_{\rm CP} = \frac{1}{8}\cos\theta_{13}\sin 2\theta_{12}\,\sin 2\theta_{23}\,\sin 2\theta_{13}\,\sin\delta\,, \tag{6}$$

where θ_{12}, θ_{23} and θ_{13} are the solar, atmospheric and reactor neutrino mixing angles and δ is the Dirac CP violation phase. The expression for the $J_{\rm CP}$-factor is the same in the parametrisation of the PMNS matrix $U_{\rm PMNS} \equiv U$ employed in [1]:

$$U = R_{23}(\theta_{23})\,P_{33}(\delta)\,R_{13}(\theta_{13})\,R_{12}(\theta_{12})\,, \tag{7}$$

where

$$R_{23}(\theta_{23}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_{23} & \sin\theta_{23} \\ 0 & -\sin\theta_{23} & \cos\theta_{23} \end{pmatrix},$$

$$P_{33}(\delta) = {\rm diag}(1, 1, e^{i\delta})\,, \tag{8}$$

and

$$R_{13}(\theta_{13}) = \begin{pmatrix} \cos\theta_{13} & 0 & \sin\theta_{13} \\ 0 & 1 & 0 \\ -\sin\theta_{13} & 0 & \cos\theta_{13} \end{pmatrix},$$

$$R_{12}(\theta_{12}) = \begin{pmatrix} \cos\theta_{12} & \sin\theta_{12} & 0 \\ -\sin\theta_{12} & \cos\theta_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{9}$$

The expression of the PMNS matrix in the standard parametrisation U^{sp}, is related to the expression in the parametrisation in eq. (7) as follows: $U^{sp} = U\,P^*_{33}(\delta)$.

In eq. (7) the two CP violation (CPV) Majorana phases present in $U_{\rm PMNS}$ in the case of massive Majorana neutrinos [8] were omitted since, as was shown in [8,6], the probabilities of flavour neutrino oscillations of interest for the study performed in [1] and for the present study, do not depend on the Majorana phases. Thus, the results presented in [1] and the new results derived in the present article are valid for both Dirac and Majorana neutrinos with definite masses in vacuum.

In ref. [1] the following relation between the rephasing invariants in vacuum and in matter with constant density, $J_{\rm CP}$ and $J^m_{\rm T} \equiv J^m$, has been reported:

$$J^m = J_{\rm CP}\,F(\theta_{12}, \theta_{13}, \Delta m^2_{21}, \Delta m^2_{31}, A)\,, \tag{10}$$

where $A = 2E\sqrt{2}\,G_{\rm F}\,N_e$ is the matter term [9–11], $G_{\rm F}$ and N_e being respectively the Fermi constant and the electron number density of matter. The function F in eq. (10) was given in the following explicit form in [1]:

$$F = \frac{F_1}{F_2\,F_3}\,D_{12}\,D_{13}\,D_{23}\,D_{32}\,, \tag{11}$$

where

$$D_{ij} \equiv m^2_i - M^2_j\,,\ \ i, j = 1, 2, 3\,, \tag{12}$$

$$\begin{aligned} F_1 = {} & D_{12}D_{13}D_{23}D_{32} + A\Big[D_{13}D_{23}\left(D_{32} - \Delta m^2_{31}\,|U_{e3}|^2\right) \\ & + D_{12}D_{32}\left(D_{23} - \Delta m^2_{21}\,|U_{e2}|^2\right)\Big] \\ & + A^2\Big[|U_{e1}|^2\,D_{32}\,D_{23} + |U_{e2}|^2\,D_{32}\,D_{13} + |U_{e3}|^2\,D_{12}\,D_{23}\Big]\,, \end{aligned} \tag{13}$$

$$\begin{aligned} F_2 = {} & |U_{e1}|^2\,(D_{12} + A)^2\,D^2_{32} + |U_{e2}|^2 D^2_{12} D^2_{32} \\ & + |U_{e3}|^2 D^2_{12}\,(D_{32} + A)^2 - A^2\,|U_{e1}|^2 |U_{e3}|^2 (\Delta m^2_{31})^2\,, \end{aligned} \tag{14}$$

$$\begin{aligned} F_3 = {} & |U_{e1}|^2\,(D_{13} + A)^2\,D^2_{23} + |U_{e3}|^2 D^2_{13} D^2_{23} \\ & + |U_{e2}|^2 D^2_{13}\,(D_{23} + A)^2 - A^2\,|U_{e1}|^2\,|U_{e2}|^2 (\Delta m^2_{21})^2\,. \end{aligned} \tag{15}$$

As was noticed in [1], the function F_3 can formally be obtained from the function F_2 by interchanging m^2_2 and m^2_3, M^2_2 and M^2_3, and $|U_{e2}|^2$ and $|U_{e3}|^2$. In the parametrisation (7) used in [1] and, thus in eqs. (13)–(15), U_{e1}, U_{e2} and U_{e3} are real quantities: $U_{e1} = c_{12}c_{13}$, $U_{e2} = s_{12}c_{13}$ and $U_{e3} = s_{13}$, where $c_{ij} \equiv \cos\theta_{ij}$ and $s_{ij} \equiv \sin\theta_{ij}$. Thus, $|U_{ei}|^2 = U^2_{ei}$, $i = 1, 2, 3$. The function $F(\theta_{12}, \theta_{13}, \Delta m^2_{21}, \Delta m^2_{31}, A)$ as defined by eqs. (11)–(15), depends, in particular, on the differences between the squares of the neutrino masses in vacuum and in matter, $D_{ij} = m^2_i - M^2_j$, $i \neq j$. However, as it follows from the form of the Hamiltonian of the neutrino system in matter with constant density, whose eigenvalues are $M^2_j/(2E)$ (see further), as well as from the explicit analytic expressions for M^2_j derived in [10], the mass squared differences D_{ij} of interest are functions of θ_{12}, θ_{13}, Δm^2_{21}, Δm^2_{31} and A and do not depend on θ_{23} and δ. As a consequence, the function F in eq. (10) is independent on θ_{23} and δ [1]: $F = F(\theta_{12}, \theta_{13}, \Delta m^2_{21}, \Delta m^2_{31}, A)$.

In deriving the relation (10), the following parametrisation of the neutrino mixing matrix in matter U^m was used:

$$\begin{aligned} & U^m = Q\;R_{23}(\theta^m_{23})\,P_{33}(\delta^m)\,R_{13}(\theta^m_{13})\,R_{12}(\theta^m_{12})\,, \\ & Q = {\rm diag}(1, e^{i\beta_2}, e^{i\beta_3})\,, \end{aligned} \tag{16}$$

where θ^m_{23}, θ^m_{13}, θ^m_{12}, δ^m are the neutrino mixing angles and the Dirac CPV phase in matter and the Majorana CPV phases were omitted. The phases β_2 and β_3 in the matrix Q are unphysical and do not play any role in the derivation of relation (10). They ensure that the matrix U^m can be cast in the form given in eq. (16) [12] (see also [13]). Obviously, the parametrisation of U^m in eq. (16) is analogous to the parametrisation (7) of the neutrino mixing matrix in vacuum.

[2] By "ordinary" we mean matter which does not contain antiprotons, antineutrons and positrons.

It follows from eqs. (11)–(15) that [1] in the case of oscillations in vacuum, i.e., for $N_e = 0$ ($A = 0$), one has

$$F(\theta_{12}, \theta_{13}, \Delta m^2_{21}, \Delta m^2_{31}, 0) = 1\,, \tag{17}$$

and that F is symmetric with respect to the interchange of m_2^2 and m_3^2, M_2^2 and M_3^2 and of $|U_{e2}|^2$ and $|U_{e3}|^2$.

The relation (10) between J^m and $J_{\rm CP}$ implies, in particular, that we can have $J^m \neq 0$ only if $J_{\rm CP} \neq 0$, i.e., T violation effects can be present in neutrino oscillations taking place in matter with constant density or density distributed symmetrically relative to the middle point (like in the Earth) only if CP and T violation effects are present in neutrino oscillations taking place in vacuum. It was shown also in [1] that the presence of matter can enhance somewhat $|J^m|$ with respect to its vacuum value $|J_{\rm CP}|$: in the example considered in [1] the enhancement was by a factor of 3. Taking the best fit values of neutrino oscillation parameters for neutrino mass spectrum with normal ordering (inverted ordering)[3] obtained in the global analysis in [14],

$$\theta_{12} = 33.62^\circ\,,\ \theta_{23} = 47.2^\circ\,(48.1^\circ)\,,\ \sin^2\theta_{13} = 8.54^\circ\,(8.58^\circ)\,,$$
$$\delta = 234^\circ\,(278^\circ)\,,\ \Delta m^2_{21} = 7.4 \times 10^{-5}{\rm eV}^2\,,$$
$$\Delta m^2_{31} = 2.494 \times 10^{-3}{\rm eV}^2\,(\Delta m^2_{32} = -2.465 \times 10^{-3}{\rm eV}^2)\,, \tag{18}$$

one always has for the ratio $|J^m/J_{\rm CP}| < 1.2$ [15]. This result persists even if we fix δ to its best fit value and vary the other neutrino oscillation parameters in their 3σ allowed ranges determined in [14]. Relaxing arbitrarily the 3σ experimental constraints on the allowed ranges of Δm^2_{21} and Δm^2_{31}, we find that indeed the maximal enhancement factor $|J^m/J_{\rm CP}|$ is 3.6 for neutrino mass spectrum with normal ordering (NO) and 2.9 for spectrum with inverted ordering (IO). In both cases, the maximal enhancement corresponds to J^m reaching its theoretical maximal value $\max(|J^m|) = 1/(6\sqrt{3})$.

In 1991 in [2] a relation similar to that given in eq. (10) was obtained:

$$J^m = J_{\rm CP}\,\tilde{F}\,. \tag{19}$$

The function $\tilde{F}$ was given in the following form:

$$\tilde{F} = \frac{\Delta m^2_{12}\,\Delta m^2_{23}\,\Delta m^2_{31}}{\Delta M^2_{12}\,\Delta M^2_{23}\,\Delta M^2_{31}}\,, \tag{20}$$

where $\Delta M^2_{ij} = M^2_i - M^2_j$.

The relation (10) between the rephasing invariants J^m and $J_{\rm CP}$ was presented in [1] without a proof. In the present article, after discussing certain aspects of neutrino mixing in matter, we provide a derivation of the relation (10). Further, we show that the function F in eq. (10), as defined in eqs. (13)–(15), coincides with the function $\tilde{F}$ in the relation (19) obtained in [2],

$$F = \tilde{F}\,, \tag{21}$$

i.e., that the function F is just another representation of the function $\tilde{F}$.

2. On the 3-neutrino mixing in matter

In [1] the analysis was performed starting with the following Hamiltonian of the neutrino system in matter diagonalised with the help of the neutrino mixing matrix in matter U^m [6]:

$$\frac{1}{2E} U \left[\begin{pmatrix} m_1^2 & 0 & 0 \\ 0 & m_2^2 & 0 \\ 0 & 0 & m_3^2 \end{pmatrix} + U^\dagger \begin{pmatrix} A & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} U \right] U^\dagger$$
$$= \frac{1}{2E} U^m \begin{pmatrix} M_1^2 & 0 & 0 \\ 0 & M_2^2 & 0 \\ 0 & 0 & M_3^2 \end{pmatrix} (U^m)^\dagger\,. \tag{22}$$

It follows from the preceding equation[4] that the Hamiltonian of the neutrino system,

$$H = \frac{1}{2E} \begin{pmatrix} m_1^2 & 0 & 0 \\ 0 & m_2^2 & 0 \\ 0 & 0 & m_3^2 \end{pmatrix} + \frac{1}{2E} U^\dagger \begin{pmatrix} A & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} U \tag{23}$$

$$= \frac{1}{2E} \begin{pmatrix} m_1^2 + A|U_{e1}|^2 & AU^*_{e1}U_{e2} & AU^*_{e1}U_{e3} \\ AU^*_{e2}U_{e1} & m_2^2 + A|U_{e2}|^2 & AU^*_{e2}U_{e3} \\ AU^*_{e3}U_{e1} & AU^*_{e3}U_{e2} & m_3^2 + A|U_{e3}|^2 \end{pmatrix} \tag{24}$$

is diagonalised by the matrix $U^\dagger U^m$ and its eigenvalues are $M_i^2/(2E)$, $i = 1, 2, 3$. In the parametrisation (7) of the PMNS matrix U_{e1}, U_{e2} and U_{e3} are real quantities: $U_{e1} = c_{12}c_{13}$, $U_{e2} = s_{12}c_{13}$ and $U_{e3} = s_{13}$, where $c_{ij} \equiv \cos\theta_{ij}$ and $s_{ij} \equiv \sin\theta_{ij}$. As a consequence, the Hamiltonian H is a real symmetric matrix.[5] This implies that the matrix $U^\dagger U^m$, which diagonalises H, is a real orthogonal matrix:

$$U^\dagger U^m = O\,,\quad O^* = O\,,\quad O^T\,O = O\,O^T = {\rm diag}(1, 1, 1)\,. \tag{25}$$

Since H does not depend on θ_{23} and δ, $O = U^\dagger U^m$ should not depend on θ_{23} and δ either. The fact that the matrix O in eq. (25) is a real orthogonal matrix implies that in the parametrisations (7) and (16) of the PMNS matrix in vacuum and in matter, the matrix

$$\tilde{O} = R_{13}(\theta_{13})\,R_{12}(\theta_{12})\,O\,R^T_{12}(\theta^m_{12})\,R^T_{13}(\theta^m_{13})$$
$$= P^*_{33}(\delta)\,R^T_{23}(\theta_{23})\,Q\,\,R_{23}(\theta^m_{23})\,P_{33}(\delta^m) \tag{26}$$
$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & c^m_{23}c_{23}e^{i\beta_2} + s^m_{23}s_{23}e^{i\beta_3} & s^m_{23}c_{23}\,e^{i(\beta_2+\delta^m)} - c^m_{23}s_{23}\,e^{i(\beta_3+\delta^m)} \\ 0 & s_{23}c^m_{23}\,e^{i(\beta_2-\delta)} - c_{23}s^m_{23}\,e^{i(\beta_3-\delta)} & s^m_{23}s_{23}e^{i(\beta_2-\delta+\delta^m)} + c^m_{23}c_{23}e^{i(\beta_3-\delta+\delta^m)} \end{pmatrix}, \tag{27}$$

is a real orthogonal matrix.

The requirement of reality of the nondiagonal elements of $\tilde{O}$ leads to the conditions:

$$\cos\theta_{23}\sin\theta^m_{23}\,\sin(\beta_3 - \delta) = \sin\theta_{23}\cos\theta^m_{23}\,\sin(\beta_2 - \delta)\,,$$
$$\cos\theta_{23}\sin\theta^m_{23}\,\sin(\beta_2 + \delta^m) = \sin\theta_{23}\cos\theta^m_{23}\,\sin(\beta_3 + \delta^m)\,, \tag{28}$$

which imply, in particular:

$$\cos(2\,\beta_3 + \delta^m - \delta) = \cos(2\,\beta_2 + \delta^m - \delta)\,. \tag{29}$$

The last condition has two solutions:

$$\beta_3 = \beta_2 + k\pi\,,\quad k = 0, 1, 2, \ldots\,, \tag{30}$$
$$\beta_2 + \beta_3 = \delta - \delta^m + k'\pi\,,\quad k' = 0, 1, 2, \ldots\,. \tag{31}$$

[3] For a discussion of the different possible types of neutrino mass spectrum see, e.g., [3].

[4] The CPV Majorana phases α_{21} and α_{31}, enter into the expression for the PMNS matrix in vacuum through the diagonal matrix $P = {\rm diag}(1, e^{i\alpha_{21}/2}, e^{i\alpha_{31}/2})$ [8,16]: $U_{\rm PMNS} = UP$. It follows from the expression in the left hand side of eq. (22) that the Hamiltonian of neutrino system in matter, and thus the 3-flavour neutrino oscillations in matter, do not depend on the Majorana phases [6].

[5] Replacing the matrix U with $U^{sp} = UP^*_{33}(\delta)$ in eq. (22), it is easy to convince oneself that the Hamiltonian H has the form given in eq. (24) also in the standard parametrisation of the PMNS matrix with U_{e3} replaced by $|U_{e3}| = s_{13}$.

The requirement of reality of the diagonal elements of $\tilde{O}$ leads to:

$$\cos\theta_{23}\cos\theta^m_{23}\,\sin\beta_2 = -\,\sin\theta_{23}\sin\theta^m_{23}\,\sin\beta_3\,,$$
$$\cos\theta_{23}\cos\theta^m_{23}\,\sin(\beta_3-\delta+\delta^m) = -\,\sin\theta_{23}\sin\theta^m_{23}\,\sin(\beta_2-\delta+\delta^m)\,. \qquad (32)$$

These conditions also lead, in particular, to the constraint given in eq. (29) and to the solutions (30) and (31). It should be clear that satisfying the constraint (30) or (31) is not enough to ensure the reality of the matrix $\tilde{O}$.

Consider first the consequences of the constraint in eq. (30). Requiring in addition that the determinant of $\tilde{O}$ is a real quantity implies:

$$2\beta_2 = \delta - \delta^m + k'\pi\,,\quad k' = 0,1,2,\ldots \qquad (33)$$

The constraint in eq. (30) for $k=0$, for example, and the conditions of reality of the elements $(\tilde{O})_{23(32)}$ and $(\tilde{O})_{22(33)}$ of $\tilde{O}$ lead to:

$$\sin(\theta_{23}-\theta^m_{23})\,\sin(\beta_2-\delta) = 0\,,\quad \sin(\theta_{23}-\theta^m_{23})\,\sin(\beta_2+\delta^m) = 0\,. \qquad (34)$$

$$\cos(\theta_{23}-\theta^m_{23})\,\sin\beta_2 = 0\,,\quad \cos(\theta_{23}-\theta^m_{23})\,\sin(\beta_2-\delta+\delta^m) = 0\,. \qquad (35)$$

As a consequence of eq. (33) the second conditions in eqs. (34) and (35) are equivalent to the first conditions in eqs. (34) and (35). The constraint (30) for $k=0$ and the conditions (34) and (35) can be simultaneously satisfied if the following relations hold[6]:

$$\theta^m_{23} = \theta_{23}\,,\quad 0 < \theta_{23},\theta^m_{23} \le \pi/2\,, \qquad (36)$$
$$\delta^m = \delta + k'\pi\,,\quad k' = 0,1,2\,, \qquad (37)$$
$$\beta_2 = q\pi\,,\quad q = 0,1,2,\ldots\,. \qquad (38)$$

Numerical results on the dependence of θ^m_{23}, δ^m, $\overline{\theta}^m_{23}$ and $\overline{\delta}^m$ on the matter potential A, $\overline{\theta}^m_{23}$ and $\overline{\delta}^m$ being the corresponding antineutrino mixing angle and CP violation phase, show that the relations (36) and (37) cannot be exact. These relations are not the true solutions of the reality conditions of the matrix $\tilde{O}$ since they do not fully guarantee the reality of $\tilde{O}$ as given in eq. (27). Only if both $\theta_{23} = 45°$ and $\delta = \pm\pi/2$ hold in vacuum, the relations (36) and (37) are exact and are not violated by the effects of matter [17]. However, they are fulfilled with extremely high precision for the mixing of neutrinos (antineutrinos) in matter in the case of IO (NO) neutrino mass spectrum. We find that in this case for any $A/\Delta m^2_{21}$ and the best fit values of the neutrino oscillation parameters quoted in eq. (18) we have:

$$\left|\frac{\overset{(-)}{\theta}{}^m_{23}}{\theta_{23}} - 1\right| \lesssim 0.0004\ (0.0015)\,, \qquad (39)$$
$$\left|\frac{\overset{(-)}{\delta}{}^m}{\delta} - 1\right| \lesssim 0.0001\ (0.00006)\,. \qquad (40)$$

For the mixing of neutrinos (antineutrinos) in matter and spectrum with normal (inverted) ordering, eqs. (36) and (37) are fulfilled also with extremely high precision for $A/\Delta m^2_{21} < 30$:

$$\left|\frac{\overset{(-)}{\theta}{}^m_{23}}{\theta_{23}} - 1\right| \lesssim 0.006\ (0.0015)\,, \qquad (41)$$
$$\left|\frac{\overset{(-)}{\delta}{}^m}{\delta} - 1\right| \lesssim 0.0003\ (0.001)\,. \qquad (42)$$

For[7] $A/\Delta m^2_{21} \gtrsim 30$ and mixing of neutrinos (antineutrinos) in matter and NO (IO) neutrino mass spectrum we have:

$$\left|\frac{\overset{(-)}{\theta}{}^m_{23}}{\theta_{23}} - 1\right| \lesssim 0.07\ (0.016)\,, \qquad (43)$$
$$\left|\frac{\overset{(-)}{\delta}{}^m}{\delta} - 1\right| \lesssim 0.001\ (0.004)\,. \qquad (44)$$

Setting δ to its best fit value given in eq. (18) and varying the other neutrino oscillation parameters in their 3σ allowed ranges determined in [14] does not change significantly the results quoted in eqs. (40)–(44). Indeed, for mixing of neutrinos (antineutrinos) in matter in the case of IO (NO) neutrino mass spectrum and any $A/\Delta m^2_{21}$ we find that $|\overset{(-)}{\theta}{}^m_{23}/\theta_{23} - 1| \lesssim 0.0005$ (0.002) and $|\overset{(-)}{\delta}{}^m/\delta - 1| \lesssim 0.0002$ (0.0002). In the case of mixing of neutrinos (antineutrinos) in matter and NO (IO) spectrum and $A/\Delta m^2_{21} < 30$ we get $|\overset{(-)}{\theta}{}^m_{23}/\theta_{23} - 1| \lesssim 0.013$ (0.003) and $|\overset{(-)}{\delta}{}^m/\delta - 1| \lesssim 0.0013$ (0.003), while for $A/\Delta m^2_{21} \gtrsim 30$ we obtain $|\overset{(-)}{\theta}{}^m_{23}/\theta_{23} - 1| \lesssim 0.09$ (0.02) and $|\overset{(-)}{\delta}{}^m/\delta - 1| \lesssim 0.01$ (0.01).

These results are illustrated in Figs. 1 and 2 (3 and 4) where the ratios $\theta^m_{23}/\theta_{23}$ and δ^m/δ (the ratios $\overline{\theta}^m_{23}/\theta_{23}$ and $\overline{\delta}^m/\delta$) are shown as functions of $A/\Delta m^2_{21}$ in the case of mixing of neutrinos (antineutrinos) and NO (left panel) and IO (right panel) neutrino mass spectrum. We used the best fit values of neutrino oscillation parameters Δm^2_{31}, Δm^2_{21}, θ_{12} and θ_{13} from [14] and the analytic expressions for M^2_i, $i = 1,2,3$, from [12].

The approximate ranges of values of $A/\Delta m^2_{21}$ relevant for the T2K (T2HK) [19], T2HKK [20], NOνA [21] and DUNE [22] long baseline neutrino oscillation experiments read, respectively: [0.266, 2.66], [0.306, 3.06], [2.90, 8.70] and [3.02, 12.10]. In obtaining these ranges we used the best fit value of $\Delta m^2_{21} = 7.4 \times 10^{-5}$ eV^2 and took into account i) that $A = 7.56 \times 10^{-5}$ eV^2 $(\rho/\text{g/cm}^3)\,(E/\text{GeV})$, where ρ is the matter density, ii) that the mean Earth density along the trajectories of the neutrinos in the T2K (T2HK), T2HKK, NOνA and DUNE long baseline neutrino oscillation experiments respectively is 2.60, 3.00, 2.84 and 2.96 g/cm^3, and iii) that in these experiments beams of neutrinos with energies $\sim$ (0.1–1.0) GeV (T2K, T2HK, T2HKK), $\sim$ (1–3) GeV (NOνA) and $\sim$ (1–4) GeV (DUNE) are being, or planned to be, used. At the peak neutrino energies at T2K (T2HK), T2HKK, NOνA and DUNE experiments of respectively 0.6 GeV, 0.6 GeV, 2.0 GeV and 2.6 GeV we have $A/\Delta m^2_{21} = 1.59$, 1.84, 5.80 and 7.86. Taking a wider neutrino energy interval for, e.g., NOνA and DUNE experiments of [1.0, 8.0] GeV, we get for the corresponding $A/\Delta m^2_{21}$ ranges: [2.90, 23.21] and [3.02, 24.20]. For all the intervals of values of $A/\Delta m^2_{21}$ quoted above, which are relevant for the T2K (T2HK), T2HKK, NOνA and DUNE experiments, the equalities (36) and (37) are excellent approximations.

Consider next the implications of the second condition (31) related to the requirement of reality of the matrix $\tilde{O}$. As can be easily shown, this condition alone i) ensures the reality of det$(\tilde{O})$,

[6] An alternative solution to the discussed constraints is $\theta^m_{23} \ne \theta_{23}$, $\beta_2 = q\pi$, $q = 0,1,2,\ldots$, $\delta = k'\pi$, $k' = 0,1,2$, $\delta^m = \tilde{k}'\pi$, $\tilde{k}' = 0,1,2$. It corresponds to CP (T) conserving values of δ (δ^m).

[7] For an analytic understanding of the results in eqs. (41) and (42) see [18].

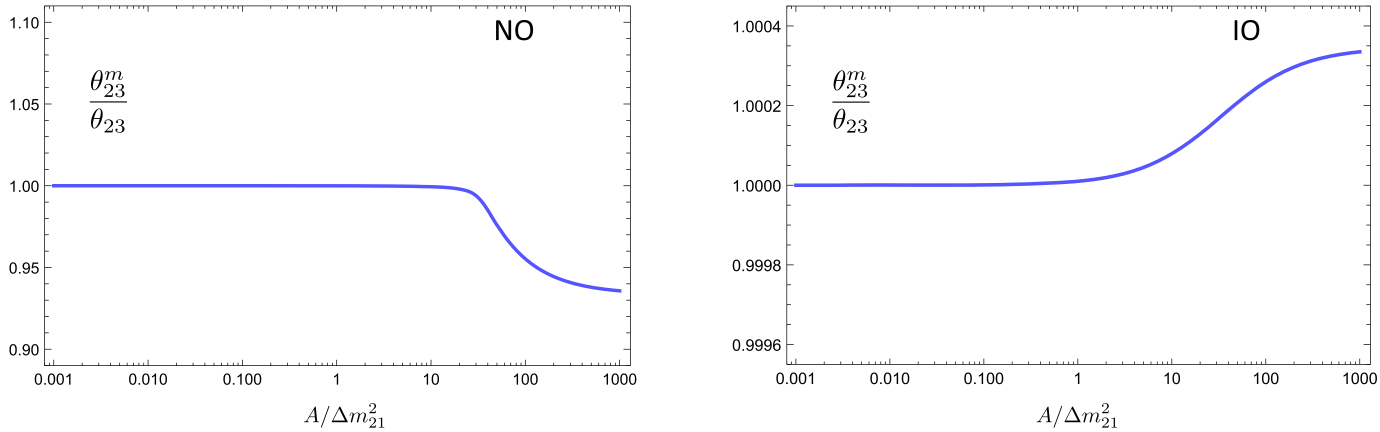

Fig. 1. The ratio $\theta_{23}^m/\theta_{23}$ as a function of $A/\Delta m^2_{21}$ in the case of mixing of neutrinos and NO (left panel) and IO (right panel) neutrino mass spectrum. See text for further details.

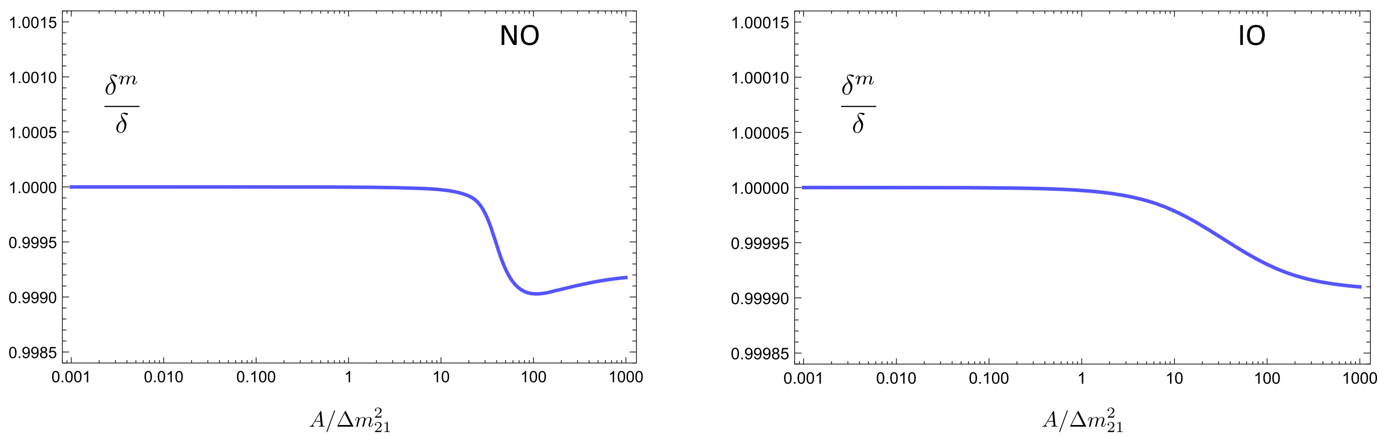

Fig. 2. The ratio δ^m/δ as a function of $A/\Delta m^2_{21}$ in the case of mixing of neutrinos and NO (left panel) and IO (right panel) neutrino mass spectrum. See text for further details.

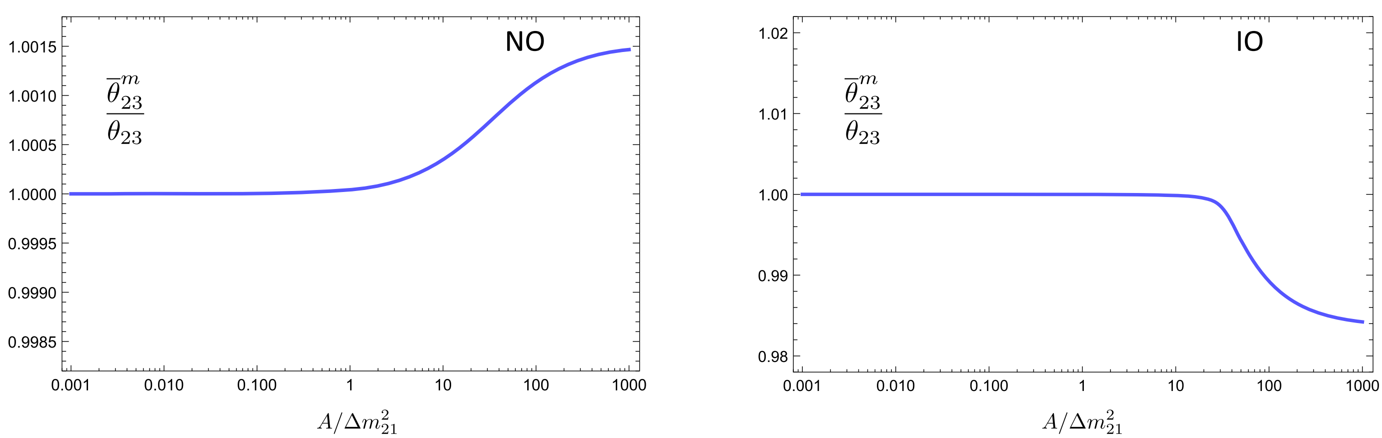

Fig. 3. The same as in Fig. 1 but for the ratio $\overline{\theta}_{23}^m/\theta_{23}$. See text for further details.

and ii) makes identical the two conditions in eq. (28) and the two conditions in eq. (32). Thus, after using condition (31) there are still two independent conditions to be satisfied to ensure the reality of the matrix $\tilde{O}$. We will derive next a condition that can substitute one of the required two conditions. The second condition then can be either the condition in eq. (28) or the condition in eq. (32) (after eq. (31) has been used).

The condition of orthogonality of $\tilde{O}$, $\tilde{O}(\tilde{O})^T = \mathrm{diag}(1,1,1)$, as can be shown, leads to the following additional constraints:

$$(c_{23}^m)^2 \sin(2\beta_2 - \delta) + (s_{23}^m)^2 \sin(2\beta_2 - \delta + 2\delta^m) = -\frac{c_{23}^m s_{23}^m}{c_{23} s_{23}} \cos 2\theta_{23} \sin\delta^m \,, \tag{45}$$

$$\begin{aligned} &\sin 2\theta_{23} \sin 2\theta_{23}^m \sin\delta \sin\delta^m \\ &\quad + (c_{23}^m)^2 \cos(2\beta_2) + (s_{23}^m)^2 \cos(2\beta_2 + 2\delta^m) \\ &= 1 - 2 s_{23}^2 \sin\delta \,[(c_{23}^m)^2 \sin(2\beta_2 - \delta) \\ &\quad + (s_{23}^m)^2 \sin(2\beta_2 - \delta + 2\delta^m)] \,, \end{aligned} \tag{46}$$

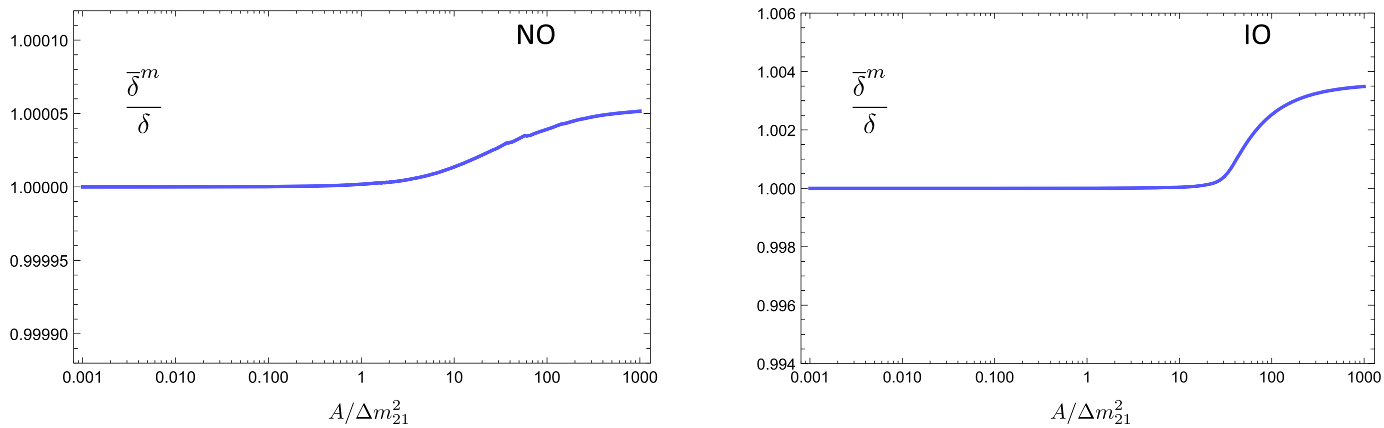

Fig. 4. The same as Fig. 2 but for the ratio $\overline{\delta}^m/\delta$. See text for further details.

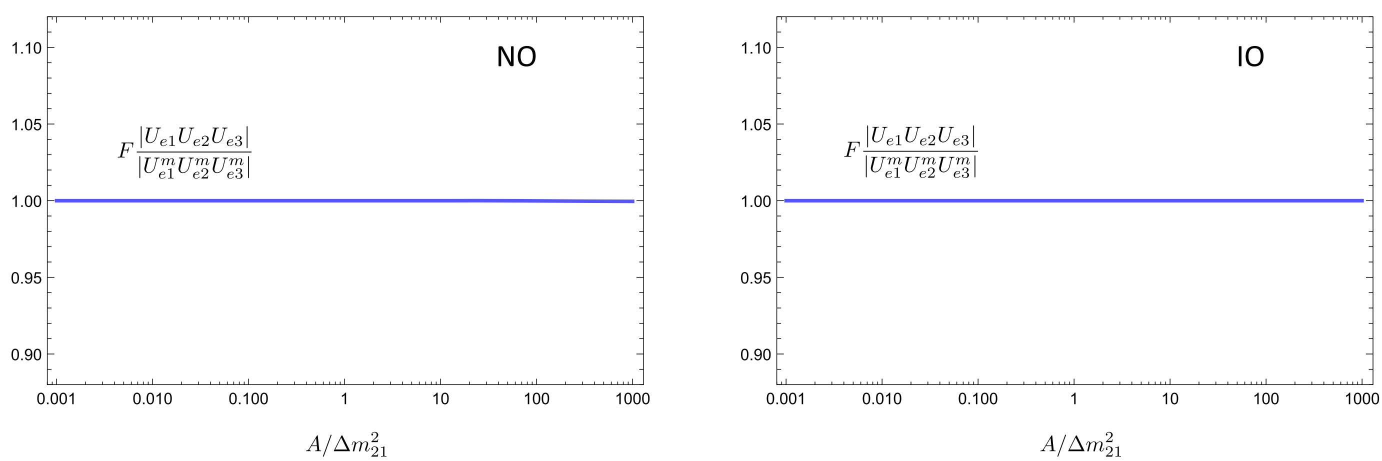

Fig. 5. The ratio $\sin 2\theta^m_{23}\sin\delta^m/(\sin 2\theta_{23}\sin\delta)$, eq. (49), for mixing of neutrinos as a function of $A/\Delta m^2_{21}$ in the cases of NO (left panel) and IO (right panel) neutrino mass spectra. See text for further details.

$$
\begin{aligned}
&- \sin 2\theta_{23}\, \sin 2\theta^m_{23}\, \cos\delta\, \sin\delta^m + (c^m_{23})^2\, \sin(2\beta_2) \\
&\quad + (s^m_{23})^2\, \sin(2\beta_2 + 2\delta^m) \\
&= 2\, s^2_{23}\cos\delta\, [(c^m_{23})^2\, \sin(2\beta_2 - \delta) \\
&\quad + (s^m_{23})^2\, \sin(2\beta_2 - \delta + 2\delta^m)]\,, \qquad (47)
\end{aligned}
$$

where we have used the relation in eq. (31). Conditions (45), (46) and (47) follow from the requirements $(\tilde{O}(\tilde{O})^T)_{23(32)} = 0$, $\mathrm{Re}((\tilde{O}(\tilde{O})^T)_{22}) = 1$ and $\mathrm{Im}((\tilde{O}(\tilde{O})^T)_{22}) = 0$, respectively. Replacing $(c^m_{23})^2\, \sin(2\beta_2 - \delta) + (s^m_{23})^2\, \sin(2\beta_2 - \delta + 2\delta^m)$ in eqs. (46) and (47) with the right hand side of eq. (45), after certain simple algebra leads to the equality:

$$
\sin 2\theta^m_{23}\, \sin\delta^m = \sin 2\theta_{23}\, \sin\delta\,. \qquad (48)
$$

This result was derived in [23] (see also [13]) using the parameterisations (7) and (16) introduced in [1] but employing a different method.[8] The equality (48) implies that the product $\sin 2\theta_{23}\sin\delta$ does not depend on the matter potential, i.e., is the same for neutrino oscillations taking place in vacuum and in matter with constant density. It is valid for neutrino and antineutrino mixing in matter independently of the type of spectrum neutrino masses obey – with NO or IO. From eq. (10) using the parametrisations defined in eqs. (7) and (16) we find:

$$
\frac{\sin 2\theta^m_{23}\, \sin\delta^m}{\sin 2\theta_{23}\, \sin\delta} = F\, \frac{\cos\theta_{13}\, \sin 2\theta_{12}\, \sin 2\theta_{13}}{\cos\theta^m_{13}\, \sin 2\theta^m_{12}\, \sin 2\theta^m_{13}} = F\, \frac{U_{e1}\, U_{e2}\, U_{e3}}{U^m_{e1}\, U^m_{e2}\, U^m_{e3}}\,. \qquad (49)
$$

From this result and eq. (48) we obtain yet another equivalent representation of the function $F(\theta_{12}, \theta_{13}, \Delta m^2_{21}, \Delta m^2_{31}, A)$:

$$
F = \frac{U^m_{e1}\, U^m_{e2}\, U^m_{e3}}{U_{e1}\, U_{e2}\, U_{e3}}\,. \qquad (50)
$$

The ratio given in eq. (49) is shown graphically in Fig. 5 for mixing of neutrinos as a function of $A/\Delta m^2_{21}$ for the best fit values of neutrino oscillation parameters Δm^2_{31}, Δm^2_{21}, θ_{12} and θ_{13} from [14] and the analytic expressions for M^2_i, $i = 1, 2, 3$, from [12]. The numerical result presented in Fig. 5, as we have verified and could be expected, is valid not only for best fit values of the relevant neutrino oscillation parameters, but indeed holds for any values of these parameters, varied in their respective physical regions. The same result is valid for mixing of antineutrinos. Thus, the equality (48) is exact. It holds also in the standard parametrisations of the PMNS matrix (see footnote 4). In this case the ratio U_{e3}/U^m_{e3} in eqs. (49) and (50) has to be replaced by $|U_{e3}|/|U^m_{e3}|$.

It follows from eq. (48) that for the values of $\sin 2\theta_{23} = 1$ and $\delta = 3\pi/2$, which are perfectly compatible with the existing data,

[8] The method employed in [23] is based on the observation [24] that the parametrisation (7) allows to factor out the part $R_{23}(\theta_{23})P_{33}(\delta)$ in the neutrino mixing matrix in matter. In this case one works with the Hamiltonian $\hat{H} = P^*_{33}(\delta)R^T_{23}(\theta_{23})UHU^\dagger R_{23}(\theta_{23})P_{33}$, which is also a real symmetric matrix, where H is given in eq. (24).

we have $\sin 2\theta_{23}^m \sin\delta^m = -1$. This in turn implies $\sin 2\theta_{23}^m = 1$ and $\sin\delta^m = -1$, i.e., the vacuum values of $\theta_{23} = \pi/4$ and $\delta = 3\pi/2$ are not modified by the presence of matter [17].

Given the fact that, as we have seen, the corrections of θ_{23} and δ due to the matter effects are small, the relation (48) allows to relate the matter correction to θ_{23}, $\epsilon_{23}(A/\Delta m_{21}^2)$, with the matter correction to δ, $\epsilon_\delta(A/\Delta m_{21}^2)$. Working to leading order in $\epsilon_{23} \ll 1$ and $\epsilon_\delta \ll 1$ we get from eq. (48) using $\theta_{23}^m = \theta_{23} + \epsilon_{23}$ and $\delta^m = \delta + \epsilon_\delta$:

$$\epsilon_\delta(A/\Delta m_{21}^2)\cos\delta \cong -2\,\epsilon_{23}(A/\Delta m_{21}^2)\,\frac{\cos 2\theta_{23}}{\sin 2\theta_{23}}\,\sin\delta\,. \tag{51}$$

Thus, for $\theta_{23} = \pi/4$ and $\delta \neq 3\pi/2, \pi/2$, the leading order matter correction to δ vanishes, while for $\delta = 3\pi/2$ $(\pi/2)$ and $\theta_{23} \neq \pi/4$, the leading order matter correction to θ_{23} vanishes. For $\delta \neq q\pi/2$, $q = 0, 1, 2, 3, 4$, and $\theta_{23} \neq \pi/4$ the sign of ϵ_δ coincides with (is opposite to) the sign of ϵ_{23} provided $\cos 2\theta_{23}\cot\delta < 0$ $(\cos 2\theta_{23}\cot\delta > 0)$.

3. The relation between J^m and $J_{\rm CP}$

Equation (22) can be cast in the form:

$$\left[\begin{pmatrix} m_1^2 & 0 & 0 \\ 0 & m_2^2 & 0 \\ 0 & 0 & m_3^2 \end{pmatrix} + U^\dagger \begin{pmatrix} A & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} U\right](U^\dagger U^m) = (U^\dagger U^m)\begin{pmatrix} M_1^2 & 0 & 0 \\ 0 & M_2^2 & 0 \\ 0 & 0 & M_3^2 \end{pmatrix}. \tag{52}$$

One possible relatively simple way to derive the relation between J^m and $J_{\rm CP}$ given in eq. (19) and reported in [1] is to exploit the fact that the column matrices $((U^\dagger U^m)_{1i}\ (U^\dagger U^m)_{2i}\ (U^\dagger U^m)_{3i})^T$ are eigenvectors of the Hamiltonian H defined in eq. (24), corresponding to the eigenvalues $M_i^2/(2E)$, $i = 1, 2, 3$. Using this observation it is possible to derive from eq. (52) explicit expressions for the elements of the neutrino mixing matrix in matter U^m. They read:

$$U_{li}^m = \frac{1}{D_i}\left[N_i\,U_{li} - A\,U_{ei}\left(D_{ji}\,U_{ek}^*\,U_{lk} + D_{ki}\,U_{ej}^*\,U_{lj}\right)\right], \quad l = e, \mu, \tau\,, \tag{53}$$

where

$$N_i = D_{ji}\,D_{ki} + A\left(D_{ji}\,|U_{ek}|^2 + D_{ki}\,|U_{ej}|^2\right), \tag{54}$$

$$D_i^2 = N_i^2 + A^2\,|U_{ei}|^2\left(D_{ji}^2\,|U_{ek}|^2 + D_{ki}^2\,|U_{ej}|^2\right), \tag{55}$$

with $i, j, k = 1, 2, 3$, but $i \neq j \neq k \neq i$. For the elements of U^m of interest, U_{e2}^m, U_{e3}^m, $U_{\mu 2}^m$ and $U_{\mu 3}^m$ we get from eqs. (53)–(55):

$$U_{e2}^m = \frac{1}{D_2}\,U_{e2}\,D_{12}\,D_{32}\,, \tag{56}$$

$$U_{e3}^m = \frac{1}{D_3}\,U_{e3}\,D_{13}\,D_{23}\,, \tag{57}$$

$$U_{\mu 2}^m = \frac{1}{D_2}\left[N_2\,U_{\mu 2} - A\,U_{e2}\left(D_{12}\,U_{e3}^*\,U_{\mu 3} + D_{32}\,U_{e1}^*\,U_{\mu 1}\right)\right], \tag{58}$$

and

$$U_{\mu 3}^m = \frac{1}{D_3}\left[N_3\,U_{\mu 3} - A\,U_{e3}\left(D_{13}\,U_{e2}^*\,U_{\mu 2} + D_{23}\,U_{e1}^*\,U_{\mu 1}\right)\right]. \tag{59}$$

The function D_2^2, which, as it follows from eqs. (1) and (56)–(59), enters into the expression for J^m, is given by:

$$\begin{aligned} D_2^2 &= N_2^2 + A^2\,|U_{e2}|^2\left(D_{12}^2\,|U_{e3}|^2 + D_{32}^2\,|U_{e1}|^2\right) \\ &= D_{12}^2\,D_{32}^2 + 2\,A\,D_{12}\,D_{32}\left(D_{12}\,|U_{e3}|^2 + D_{32}\,|U_{e1}|^2\right) \\ &\quad + A^2\Big[D_{12}^2\,|U_{e3}|^2\left(|U_{e3}|^2 + |U_{e2}|^2\right) \\ &\quad + D_{32}^2\,|U_{e1}|^2\left(|U_{e1}|^2 + |U_{e2}|^2\right) + 2\,D_{12}\,D_{32}|U_{e1}|^2\,|U_{e3}|^2\Big] \\ &= D_{12}^2\,D_{32}^2 + 2\,A\,D_{12}\,D_{32}\left(D_{12}\,|U_{e3}|^2 + D_{32}\,|U_{e1}|^2\right) \\ &\quad + A^2\Big[D_{12}^2\,|U_{e3}|^2 + D_{32}^2\,|U_{e1}|^2 \\ &\quad - (D_{32} - D_{12})^2\,|U_{e1}|^2\,|U_{e3}|^2\Big]\,. \end{aligned} \tag{60}$$

It is easy to check that expression (60) for the function D_2^2 coincides with expression (14) for the function F_2, i.e., that we have

$$D_2^2 = F_2\,. \tag{61}$$

One can show in a similar way that the function D_3^2 coincides with the function F_3 given in eq. (15), i.e., that

$$D_3^2 = F_3\,. \tag{62}$$

The calculation of the rephasing invariant in matter J^m involves, in particular, the product $U_{\mu 3}^m (U_{\mu 2}^m)^* (U_{e3}^m)^* U_{e2}^m$ of elements of U^m. From eqs. (56)–(59) we have:

$$R \equiv \frac{D_2^2\,D_3^2}{D_{13}\,D_{23}\,D_{12}\,D_{32}}\,\frac{\mathrm{Im}\big((U_{\mu 2}^m)^*\,U_{\mu 3}^m\,(U_{e3}^m)^*\,U_{e2}^m\big)}{\mathrm{Im}\big(U_{\mu 2}^* U_{e2} U_{\mu 3} U_{e3}^*\big)} \tag{63}$$

$$\begin{aligned} &= \frac{1}{\mathrm{Im}\big(U_{\mu 2}^* U_{e2} U_{\mu 3} U_{e3}^*\big)}\,\mathrm{Im}\big([N_2^*\,U_{\mu 2}^*\,U_{e2} \\ &\quad - A\,|U_{e2}|^2\big(D_{12}\,U_{e3}\,U_{\mu 3}^* + D_{32}\,U_{e1}\,U_{\mu 1}^*\big)] \\ &\quad \times [N_3\,U_{\mu 3}\,U_{e3}^* - A\,|U_{e3}|^2\left(D_{13}\,U_{e2}^*\,U_{\mu 2} + D_{23}\,U_{e1}^*\,U_{\mu 1}\right)]\big)\,. \end{aligned} \tag{64}$$

Using the fact that

$$\begin{aligned} J_{\rm CP} &= \mathrm{Im}(U_{\mu 2}^* U_{e2} U_{\mu 3} U_{e3}^*) = \mathrm{Im}(U_{\mu 3}^* U_{e3} U_{\mu 1} U_{e1}^*) \\ &= -\,\mathrm{Im}(U_{\mu 2}^* U_{e2} U_{\mu 1} U_{e1}^*)\,, \end{aligned} \tag{65}$$

the function R in eq. (64), after some algebra, can be brought to the form:

$$\begin{aligned} R &= D_{13}\,D_{23}\,D_{12}\,D_{32} + A\big[D_{13}\,D_{23}\left(D_{32} - |U_{e3}|^2\,(D_{32} - D_{12})\right) \\ &\quad + D_{12}\,D_{32}\left(D_{23} - |U_{e2}|^2\,(D_{23} - D_{13})\right)\big] \\ &\quad + A^2\left[D_{23}\,D_{32}\,|U_{e1}|^2 + D_{13}\,D_{32}\,|U_{e2}|^2 + D_{12}\,D_{23}\,|U_{e3}|^2\right]. \end{aligned} \tag{66}$$

Equations (11), (10), (63) and the equalities $F_2 = D_2^2$ and $F_3 = D_3^2$ proven above, together with the equalities $D_{32} - D_{12} = \Delta m_{31}^2$ and $D_{23} - D_{13} = \Delta m_{21}^2$, imply that $R = F_1$. This completes the proof of the result reported in [1] and given in eqs. (10) and (11).

Two comments are in order. First, the function $F(\theta_{12}, \theta_{13}, \Delta m_{21}^2, \Delta m_{31}^2, A)$, as determined in eqs. (11) is positive. Indeed, it follows

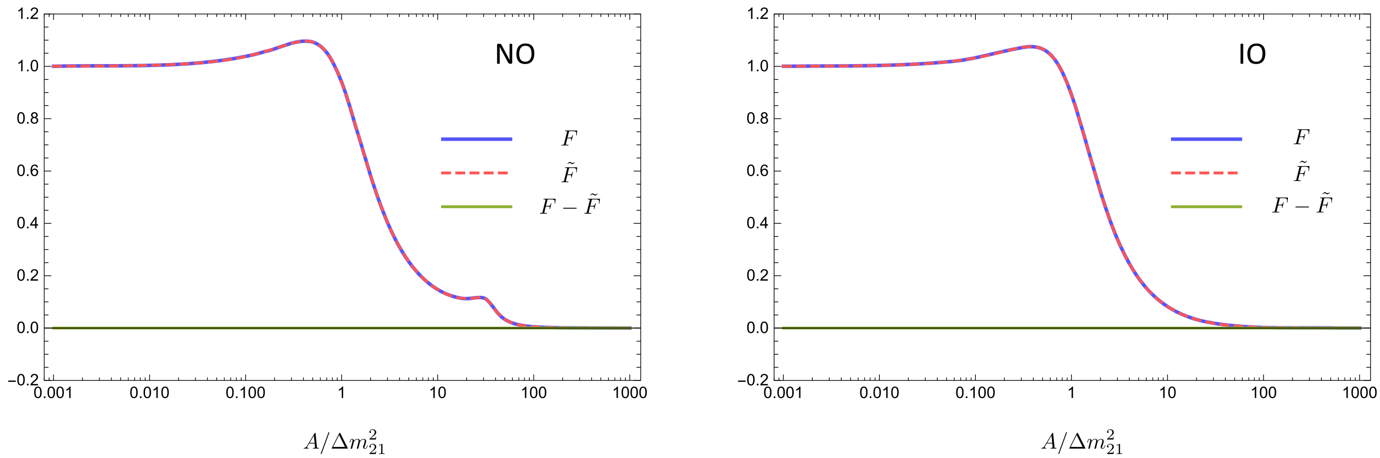

Fig. 6. The functions F (blue solid line), $\tilde{F}$ (red dashed line) and the difference $F - \tilde{F}$ (green line) versus $A/\Delta m^2_{21}$ for NO (left panel) and IO (right panel) neutrino mass spectrum. See text for further details. (For interpretation of the colours in the figure(s), the reader is referred to the web version of this article.)

from eqs. (61) and (62) that the functions F_2 and F_3 are positive. For $A = 0$, we have also $F_1 D_{12} D_{13} D_{23} D_{32} > 0$. One can show that this inequality holds also for $A \neq 0$, which leads to $F > 0$. This implies that the rephasing invariants in vacuum and in matter, $J_{\rm CP}$ and J^m, have the same sign:

$$\mathrm{sgn}(J^m) = \mathrm{sgn}(J_{\rm CP})\,. \tag{67}$$

This result is valid both for neutrino mass spectra with normal ordering ($\Delta m^2_{31} > 0$) and with inverted ordering ($\Delta m^2_{31} < 0$).

Second, the function $F(\theta_{12}, \theta_{13}, \Delta m^2_{21}, \Delta m^2_{31}, A)$ in eq. (10) has different equivalent representations. This should be clear from the fact that

$$\begin{aligned} J_{\rm CP}(J^m) &= \mathrm{Im}\left(\left(U^{(m)}_{\mu 2}\right)^* U^{(m)}_{e2}\, U^{(m)}_{\mu 3} \left(U^{(m)}_{e3}\right)^*\right) \\ &= \mathrm{Im}\left(\left(U^{(m)}_{\mu 3}\right)^* U^{(m)}_{e3}\, U^{(m)}_{\mu 1} \left(U^{(m)}_{e1}\right)^*\right) \\ &= \mathrm{Im}\left(U^{(m)}_{\mu 2} \left(U^{(m)}_{e2}\right)^* \left(U^{(m)}_{\mu 1}\right)^* U^{(m)}_{e1}\right) = \ldots\,, \end{aligned} \tag{68}$$

and the derivation presented above. Indeed, we can use the second or the third form of $J_{\rm CP}$ (J^m) in eq. (68) to obtain the relation given in eq. (10). The function F thus derived will differ in form from, but will be equal to, the function F defined in eqs. (11)–(15).

It follows from eqs. (10) and (19) that

$$\frac{J^m}{J_{\rm CP}} = F(\theta_{12}, \theta_{13}, \Delta m^2_{21}, \Delta m^2_{31}, A) = \tilde{F} = \frac{\Delta m^2_{12}\, \Delta m^2_{23}\, \Delta m^2_{31}}{\Delta M^2_{12}\, \Delta M^2_{23}\, \Delta M^2_{31}}\,, \tag{69}$$

i.e., that the function $F(\theta_{12}, \theta_{13}, \Delta m^2_{21}, \Delta m^2_{31}, A)$ found in [1] is another representation of the function $\tilde{F}$ found in [2]. The functions F and $\tilde{F}$ have very different forms. Nevertheless, as we have verified, they coincide numerically. This is illustrated in Fig. 6 where we show the functions F (eq. (11)), $\tilde{F}$ (eq. (20)) and the difference $(F - \tilde{F})$ versus $A/\Delta m^2_{21}$. We used the analytic expressions for M^2_i, $i = 1, 2, 3$, in terms of m^2_1, A and the neutrino oscillation parameters Δm^2_{31}, Δm^2_{21}, θ_{12} and θ_{13} derived in [12]. It should be clear from eq. (22) that, as we have already discussed, in the parametrisation (7) employed in [1] the mass parameters M^2_i, $i = 1, 2, 3$, do not depend on θ_{23} and δ. In Fig. 6, the neutrino oscillation parameters on which the functions F and $\tilde{F}$ depend were set to their best fit values found in the global analysis of the neutrino oscillation data in [14] in the cases of NO and IO neutrino mass spectra.

As is suggested by Fig. 6 and we have commented earlier, our numerical results show that the function F is positive.

The function F in eq. (10), as we have remarked earlier, does not depend on θ_{23} and δ. This implies that the ratio

$$\frac{J^m}{\sin 2\theta_{23} \sin\delta} = F\, \frac{J_{\rm CP}}{\sin 2\theta_{23} \sin\delta} = \frac{1}{8}\, F\, \cos\theta_{13}\, \sin 2\theta_{12}\, \sin 2\theta_{13}\,, \tag{70}$$

does not depend on θ_{23} and δ.

From eqs. (11), (61), (62) and (50), using $U^m_{e1} = U_{e1} D_{21} D_{31}/D_1$ we find a new expression for the function F_1 as well:

$$F_1 = \frac{D_2\, D_3}{D_1}\, D_{21}\, D_{31}\,. \tag{71}$$

4. The case of antineutrino mixing in matter

In the preceding Sections we have focused primarily on the mixing and oscillations in matter of flavour neutrinos. In this Section we will discuss briefly the case of mixing and oscillations in matter of flavour antineutrinos.

In ordinary matter (of, e.g., the Earth, the Sun) the mixing of antineutrinos in matter differs from the mixing of neutrinos in matter as a consequence of the fact that ordinary matter is not charge conjugation invariant: it contains protons, neutrons and electrons, but does not contain their antiparticles. This causes CP and CPT violating effects in the mixing and oscillations of neutrinos in matter [6]. As a consequence, the neutrino and antineutrino mixing angles, as well the masses of the respective neutrino mass-eigenstates, in matter differ. The expressions for the antineutrino mixing angles in matter, $\overline{\theta}_{ij}$, the neutrino masses in this case, $\overline{M}_k$, and the corresponding J-factor, $\overline{J}^m$, can be obtained from those corresponding to neutrino mixing in matter, as is well known, by replacing the potential A with $(-A)$.

Since the derivations of the results given in eqs. (69)–(71) do not depend on the sign of the matter term A, these results are valid also for mixing of antineutrinos in matter and for oscillations of antineutrinos $\overline{\nu}_l$ in matter with constant density. Thus, we have:

$$\overline{J}^m = J_{\rm CP}\, \overline{F}(\theta_{12}, \theta_{13}, \Delta m^2_{21}, \Delta m^2_{31}, A)\,, \tag{72}$$

$$\overline{F}(\theta_{12}, \theta_{13}, \Delta m^2_{21}, \Delta m^2_{31}, A) = F(\theta_{12}, \theta_{13}, \Delta m^2_{21}, \Delta m^2_{31}, -A)\,, \tag{73}$$

$$\overline{J}^m = \text{Im}\left(\left(\overline{U}^m_{e2}\right)\left(\overline{U}^m_{\mu 3}\right)\left(\overline{U}^m_{e3}\right)^*\left(\overline{U}^m_{\mu 2}\right)^*\right) = \frac{1}{8}\cos\overline{\theta}^m_{13}\sin 2\overline{\theta}^m_{12}\sin 2\overline{\theta}^m_{23}\sin 2\overline{\theta}^m_{13}\sin\overline{\delta}^m\,, \quad (74)$$

where $\overline{U}^m_{lj}$ are the elements of the antineutrino mixing matrix in matter $\overline{U}^m$, $\overline{\delta}^m$ is the Dirac phase present in $\overline{U}^m$, and $\overline{\theta}^m_{ij}$ are the antineutrino mixing angles in matter. We also have:

$$\sin 2\overline{\theta}^m_{23}\sin\overline{\delta}^m = \sin 2\theta_{23}\sin\delta\,, \quad (75)$$

$$\overline{F} = \frac{\overline{U}^m_{e1}\,\overline{U}^m_{e2}\,\overline{U}^m_{e3}}{U_{e1}\,U_{e2}\,U_{e3}} = \frac{\Delta m^2_{12}\,\Delta m^2_{23}\,\Delta m^2_{31}}{\Delta\overline{M}^2_{12}\,\Delta\overline{M}^2_{23}\,\Delta\overline{M}^2_{31}}\,, \quad (76)$$

with $\overline{M}^2_{ij} = \overline{M}^2_i - \overline{M}^2_j$. From the exact relations (48) and (75) we get

$$\sin 2\theta^m_{23}\sin\delta^m = \sin 2\overline{\theta}^m_{23}\sin\overline{\delta}^m = \sin 2\theta_{23}\sin\delta\,, \quad (77)$$

while eqs. (69) and (76) imply:

$$\overline{F} = F\,\frac{\Delta M^2_{12}\,\Delta M^2_{23}\,\Delta M^2_{31}}{\Delta\overline{M}^2_{12}\,\Delta\overline{M}^2_{23}\,\Delta\overline{M}^2_{31}}\,. \quad (78)$$

Finally, as in the neutrino mixing in matter case, the equalities

$$\overline{\theta}^m_{23} = \theta_{23}\,,\quad 0 < \theta_{23}, \overline{\theta}^m_{23} \leq \pi/2\,, \quad (79)$$

$$\overline{\delta}^m = \delta\,, \quad (80)$$

although not exact, represent excellent approximations for the ranges of values of $A/\Delta m^2_{21}$ relevant for the T2K (T2HK), T2HKK, NOνA and DUNE neutrino oscillation experiments.

5. Summary

In the present article we have analysed aspects of 3-neutrino mixing in matter and of CP and T violation in 3-flavour neutrino oscillations in vacuum and in matter with constant density. The analyses have been performed in the parametrisation of the PMNS neutrino mixing matrix $U_{\text{PMNS}} \equiv U$ specified in eq. (7) and introduced in [1]. However, as we have shown, the results obtained in our study are valid (in some cases with trivial modifications) also in the standard parametrisation of the PMNS matrix (see, e.g., [3]).

Investigating the case of 3-neutrino mixing in matter with constant density we have derived first the relations $\theta^m_{23} = \theta_{23}$ and $\delta^m = \delta$, θ_{23} (θ^m_{23}) and δ (δ^m) being respectively the atmospheric neutrino mixing angle and the Dirac CP violation phase in vacuum (in matter) present in the PMNS neutrino mixing matrix. Performing a detailed numerical analysis we have shown that although these equalities are not exact, they represent excellent approximations for the ranges of values of $A/\Delta m^2_{21} < 30$ relevant for the T2K (T2HK), T2HKK, NOνA and DUNE neutrino oscillation experiments, the deviations from each of the two relations not exceeding respectively 1.3×10^{-2} and 1.3×10^{-3} (Figs. 1 and 2)). Similar conclusion is valid for the corresponding parameters $\overline{\theta}^m_{23}$ and $\overline{\delta}^m$ in the case of mixing of antineutrinos (Figs. 3 and 4)).

We have derived next the relation $\sin 2\theta^m_{23}\sin\delta^m = \sin 2\theta_{23}\sin\delta$, and have shown numerically that it is exact (Fig. 5). The relation is well known in the literature (see [23,13]). We have presented a new derivation of this result. Using the indicated relation and the fact that the deviations of θ^m_{23} from θ_{23}, $\epsilon_{23}(A/\Delta m^2_{21})$, and of δ^m from δ, $\epsilon_\delta(A/\Delta m^2_{21})$, are small, $|\epsilon_{23}|, |\epsilon_\delta| \ll 1$, we have derived a relation between ϵ_{23} and ϵ_δ working in leading order in these two parameters (eq. (51)). It follows from this relation, in particular, that for $\theta_{23} = \pi/4$ and $\delta \neq 3\pi/2, \pi/2$, the leading order matter correction to δ vanishes, while for $\delta = 3\pi/2$ ($\pi/2$) and $\theta_{23} \neq \pi/4$, the leading order matter correction to θ_{23} vanishes.

We have discussed further the relation between the rephasing invariants, associated with the Dirac phase in the neutrino mixing matrix, which determine the magnitude of CP and T violating effects in 3-flavour neutrino oscillations in vacuum, J_{CP}, and of the T violating effects in matter with constant density, $J^m_{\text{T}} \equiv J^m$, obtained in [1]: $J^m = J_{\text{CP}}\,F$. F is a function whose explicit form in terms of the squared masses in vacuum and in matter of the mass-eigenstate neutrinos, of the solar and reactor neutrino mixing angles and of the neutrino matter potential (eq. (11)) was given in [1]. The quoted relation between J^m and J_{CP} was reported in [1] without a proof. We have presented a derivation of this relation. We have shown also that the function $F = F(\theta_{12}, \theta_{13}, \Delta m^2_{21}, \Delta m^2_{31}, A)$ i) is positive, $F > 0$, which implies that J^m and J_{CP} have the same sign, $\text{sgn}(J^m) = \text{sgn}(J_{\text{CP}})$, and that ii) it can have different forms. We have proven also that the function F as given in [1] is another representation of the so-called called "Naumov factor" (Fig. 6): $F = \Delta m^2_{12}\Delta m^2_{23}\Delta m^2_{31}(\Delta M^2_{12}\Delta M^2_{23}\Delta M^2_{31})^{-1}$, where $\Delta m^2_{ij} = m^2_i - m^2_j$, $\Delta M^2_{ij} = M^2_i - M^2_j$, m_i and M_i, $i = 1, 2, 3$, being the masses of the three mass-eigenstate neutrinos in vacuum and in matter.

Finally, we have considered briefly the case of antineutrino mixing in matter and have shown that results similar to those derived for the mixing of neutrinos in matter are valid also in this case.

The results of the present study contribute to the understanding of the neutrino mixing in matter and flavour neutrino oscillations in matter with constant density, widely explored in the literature on the subject. They could be useful for the studies of neutrino oscillations in long baseline neutrino oscillation experiments T2K (T2HK), T2HKK, NOνA and DUNE.

Acknowledgements

This work was supported in part by the INFN program on Theoretical Astroparticle Physics (TASP), by the European Union Horizon 2020 research and innovation programme under the Marie Sklodowska-Curie grants 674896 and 690575, by the World Premier International Research Center Initiative (WPI Initiative, MEXT), Japan (S.T.P.), as well as by the European Research Council under ERC grant "NuMass" FP7-IDEAS-ERC ERC-CG 617143 (Y.L.Z.).

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